Adv Comm, Hw#2, 810196391, [illegible]

Proakis 2.24. $Y = \sum_{i=1}^{n} x_i \quad , x_i \perp\!\!\!\perp x_j$

$$x_i = \begin{cases} 1 & p \\ 0 & (1-p) \end{cases}$$

a. $\Phi_Y(\omega) = E\{e^{j\omega Y}\} = \int_{\forall y} e^{j\omega y} f_Y(y)\, dy$

$$\Phi_Y(\omega) = E\{e^{j\omega(\sum_1^n x_i)}\} = E\{\prod_{i=1}^{n} e^{j\omega x_i}\} \overset{\text{independency}}{=} \prod_1^n E\{e^{j\omega x_i}\}$$

$$= \prod_{i=1}^{n} \Phi_{x_i}(\omega)$$

$$\Phi_{x_i}(\omega) = e^{j\omega(1)}(p) + e^{j\omega(0)}(1-p) = \underbrace{1-p}_{q} + pe^{j\omega} = q + pe^{j\omega}$$

$$\Rightarrow \Phi_Y(\omega) = (q + pe^{j\omega})^n \Rightarrow \begin{cases} X \text{ is Bernoulli Random Variable} \\ Y \text{ is Bionomial Random Variable} \end{cases}$$

b. $E\{Y\} = ?\,,\ E\{Y^2\} = ?$

$$\left.\frac{d\Phi_Y(\omega)}{d\omega}\right|_{\omega=0} = E\{jYe^{j\omega Y}\}\Big|_{\omega=0} = E\{jY\}$$

$$\Rightarrow \Phi_Y'(\omega) = n(q+pe^{j\omega})^{n-1} \times jpe^{j\omega} \Rightarrow \Phi_Y'(0) = jnp$$

$$\Rightarrow E\{Y\} = np$$

$$\left.\frac{d^2\Phi_Y(\omega)}{d\omega^2}\right|_{\omega=0} = E\{-Y^2 e^{j\omega Y}\}\Big|_{\omega=0} = E\{-Y^2\}$$

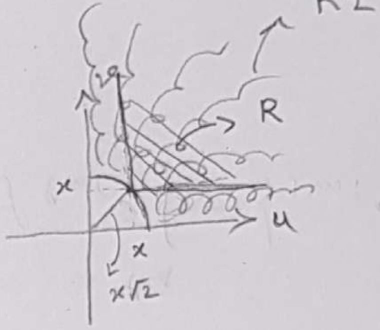
$$\Rightarrow \Phi_Y''(\omega) = n(n-1)(q+pe^{j\omega})^{n-2} \times jpe^{j\omega} \times jpe^{j\omega} + n(q+pe^{j\omega})^{n-1}(-1)pe^{j\omega}$$

$$\Rightarrow \Phi_Y''(0) = -n(n-1)p^2 - np \Rightarrow E\{Y^2\} = n^2p^2 - np^2 + np = n^2p^2 + np(1-p)$$

Proakis. 2.25 :

1- $u > x,\ v > x\ ,\ x > 0$

$$Q_x = \Pr\{N(0,1) > x\} = \int_x^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du$$



$$Q(x) = 1 - Q(-x) = 1 - \phi(x)$$

$$I = \iint_{(u,v)\in R} e^{-\frac{u^2+v^2}{2}} du\,dv$$

$$= \int_x^\infty \int_x^\infty e^{-\frac{u^2+v^2}{2}} du\,dv \leq \iint_{(r,\varphi)\in R_2} e^{-\frac{r^2}{2}} r\,dr\,d\varphi = \int_{r=x\sqrt{2}}^\infty \int_{\varphi=0}^{\frac{\pi}{2}} r e^{-\frac{r^2}{2}} d\varphi\, dr$$

$$= \frac{\pi}{2} \times \left(-e^{-\frac{r^2}{2}}\right)\Big|_{x\sqrt{2}}^{\infty} = \frac{\pi}{2}\left(e^{-x^2}\right) = \frac{\pi}{2} e^{-x^2}$$

$$\Rightarrow I = 2\pi \times \int_x^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du \times \int \frac{1}{\sqrt{2\pi}} e^{-\frac{v^2}{2}} dv = 2\pi\, Q^2(x)$$

$$\Rightarrow Q^2(x) \leqslant \frac{1}{4} e^{-x^2} \Rightarrow Q(x) \leqslant \frac{1}{2} e^{-\frac{x^2}{2}}$$

2- $$I = \int_x^\infty e^{-\frac{y^2}{2}} \times \frac{1}{y^2} dy \Rightarrow \begin{cases} u = e^{-\frac{y^2}{2}} \\ dv = \frac{dy}{y^2} \end{cases} \Rightarrow \begin{cases} du = -y e^{-\frac{y^2}{2}} \\ v = \frac{-1}{y} \end{cases}$$

$$\int u\,dv = uv - \int v\,du \quad , \quad \begin{matrix} u & dv \\ du & v \end{matrix}$$

$$\Rightarrow I = -\frac{1}{y} e^{-\frac{y^2}{2}}\Big|_x^\infty - \int_x^\infty e^{-\frac{y^2}{2}} dy = \frac{e^{-\frac{x^2}{2}}}{x} - \sqrt{2\pi}\, Q(x)$$

$\Rightarrow$ since the integrand is always positive, I is positive too

$$\Rightarrow I > 0 \Rightarrow \frac{1}{x} e^{-\frac{x^2}{2}} > \sqrt{2\pi}\, Q(x) \Rightarrow Q(x) < \frac{1}{\sqrt{2\pi}\, x} e^{-\frac{x^2}{2}}$$

$$\Rightarrow \int_x^\infty \frac{1}{y^2} e^{-\frac{y^2}{2}} dy = \frac{e^{-\frac{x^2}{2}}}{x} - \int_x^\infty e^{-\frac{y^2}{2}} dy \Rightarrow \int_x^\infty \left(1 + \frac{1}{y^2}\right) e^{-\frac{y^2}{2}} dy = \frac{e^{-\frac{x^2}{2}}}{x}$$

$$y \geqslant x \Rightarrow 1+\frac{1}{y^2} \leqslant 1+\frac{1}{x^2}$$

$$\Rightarrow e^{-\frac{x^2}{2}} \times \frac{1}{x} = \int_x^{\infty} \left(1+\frac{1}{y^2}\right) e^{-\frac{y^2}{2}} dy \leqslant \left(1+\frac{1}{x^2}\right) \int_x^{\infty} e^{-\frac{y^2}{2}} dy$$

$$\Rightarrow \frac{1}{x} e^{-\frac{x^2}{2}} \leqslant \frac{x^2+1}{x^2} \times \sqrt{2\pi}\, Q(x)$$

$$\Rightarrow \frac{x}{x^2+1} \times \frac{1}{\sqrt{2\pi}} \times e^{-\frac{x^2}{2}} \leqslant Q(x)$$

$$\Rightarrow \frac{x}{\sqrt{2\pi}\,(x^2+1)} \times e^{-\frac{x^2}{2}} \leqslant Q(x) \leqslant \frac{1}{\sqrt{2\pi}\, x} e^{-\frac{x^2}{2}}$$

3- if $x \to \infty \Rightarrow \frac{x}{x^2+1} \to \frac{1}{x}$

$\Rightarrow$ $Q(x)$ should converges to $\frac{1}{\sqrt{2\pi}\, x} e^{-\frac{x^2}{2}}$

$$\Rightarrow \lim_{x \to \infty} Q(x) \approx \frac{1}{x\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

2.26. X_i's are i.i.d.

$X_i \sim \text{Uniform}[0, A]$, $Y_n = \min\{X_1, \dots, X_n\} \Rightarrow f_{Y_n}(y) = ?$

a.

$$F_{Y_n}(y) = \Pr\{Y_n < y\} = \Pr\{\min\{X_1, \dots, X_n\} < y\}$$

$\Rightarrow$ if $y < 0 \Rightarrow F_{Y_n}(y) = 0$ & if $y > A \Rightarrow F_{Y_n}(y) = 1$

$$\Rightarrow 0 < y < A : \Pr\{\min\{X_1, \dots, X_n\} < y\} = 1 - \Pr\{\min\{X_1, \dots, X_n\} > y\}$$

$$= 1 - \Pr\{X_1 > y, X_2 > y, \dots, X_n > y\} = 1 - \Pr\{X_1 > y\} \times \dots \times \Pr\{X_n > y\}$$

$$= 1 - \left(\frac{A-y}{A}\right)^n \Rightarrow F_{Y_n}(y) = \begin{cases} 0 & y \leq 0 \\ -\left(\frac{A-y}{A}\right)^n + 1 & 0 \leq y \leq A \\ 1 & A \leq y \end{cases}$$

$$\Rightarrow f_{Y_n}(y) = \frac{\partial}{\partial y} F_{Y_n}(y) = -n\left(\frac{A-y}{A}\right)^{n-1} \times \left(-\frac{1}{A}\right) = \frac{n}{A} \times \left(\frac{A-y}{A}\right)^{n-1}$$

b.

$$\begin{cases} A \to \infty \\ n \to \infty \\ \frac{n}{A} \to \lambda \end{cases} \Rightarrow f_{Y_n}(y) = \frac{n}{A} \times \left(1 - \frac{y}{A}\right)^{n-1}$$

$$= \lim_{\substack{A \to \infty \\ n \to \infty}} \lambda \times \left(1 - \frac{y}{A}\right)^{n-1}$$

$$= \lim_{\substack{n \to \infty \\ A \to \infty}} \lambda \times e^{-y \times \frac{n-1}{A}} \quad \lambda$$

$$= \lambda e^{-\lambda y} \sim \text{Exp}(\lambda)$$

2.27. $X_i \sim N(0, \sigma_i^2)$, $E\{X_i X_j\} = C_{ij}$, $\underset{i\neq j}{i, j} \in \{1, 2, 3, 4\}$

$$\Rightarrow E\{X_1 X_2 X_3 X_4\} = C_{12}C_{34} + C_{13}C_{24} + C_{14}C_{23}$$

X_i's are jointly normal, so $Z \triangleq \omega_1 X_1 + \omega_2 X_2 + \omega_3 X_3 + \omega_4 X_4$ is Normal

$$\Rightarrow \Phi_{\underline{X}}(\omega_1, \omega_2, \omega_3, \omega_4) = E\{e^{j(\omega_1 X_1 + \omega_2 X_2 + \omega_3 X_3 + \omega_4 X_4)}\}$$

$$\Rightarrow \left.\frac{\partial^4 \Phi_{\underline{X}}(\underline{\omega})}{\partial\omega_4 \partial\omega_3 \partial\omega_2 \partial\omega_1}\right|_{\omega_1=\omega_2=\omega_3=\omega_4=0} = E\{(j)^4 X_1 X_2 X_3 X_4 e^{j\omega_1 X_1} e^{j\omega_2 X_2} e^{j\omega_3 X_3} e^{j\omega_4 X_4}\}\Big|_{\omega_1=\omega_2=\omega_3=\omega_4=0}$$

$$= E\{X_1 X_2 X_3 X_4\}$$

$$\Rightarrow \Phi_{\underline{X}}(\underline{\omega}) = E\{e^{jZ}\} = E\{e^{j\omega_Z Z}\}\Big|_{\omega_Z=1} = \Phi_Z(\omega_Z)\Big|_{\omega_Z=1}$$

$$\Rightarrow Z \sim N(\mu_Z, \sigma_Z^2) \Rightarrow \Phi_Z(\omega_Z) = e^{j\omega_Z \mu_Z - \frac{\sigma_Z^2}{2}\omega_Z^2}$$

$$\Rightarrow \mu_Z = E\{Z\} = \omega_1\mu_1 + \omega_2\mu_2 + \omega_3\mu_3 + \omega_4\mu_4 = 0$$

$$E\{Z^2\} = \sigma_Z^2 = \omega_1^2\sigma_1^2 + \omega_2^2\sigma_2^2 + \omega_3^2\sigma_3^2 + \omega_4^2\sigma_4^2 + 2\omega_1\omega_2 C_{12} + 2\omega_{13} C_{13}$$
$$+ 2\omega_{14} C_{14} + 2\omega_{23} C_{23} + 2\omega_{24} C_{24} + 2\omega_{34} C_{34}$$

$$\Rightarrow \Phi_{\underline{X}}(\underline{\omega}) = \Phi_Z(\omega_Z)\Big|_{\omega_Z=1} = e^{-\frac{\sigma_Z^2}{2}} = e^{-\frac{1}{2}(\omega_1^2\sigma_1^2 + \omega_2^2\sigma_2^2 + \sim)} = e^{u}$$

$$\boxed{u \triangleq -\frac{1}{2}\sigma_Z^2} \Rightarrow \frac{\partial \Phi_{\underline{X}}(\underline{\omega})}{\partial\omega_1} = e^{u} u_1 \quad \text{s.t. } u_1 \triangleq \frac{\partial u}{\partial \omega_1}$$

$$\Rightarrow \frac{\partial^2 \Phi_{\underline{X}}(\underline{\omega})}{\partial\omega_2 \partial\omega_1} = e^{u} u_2 u_1 + e^{u} u_{12} \quad \text{s.t. } u_{12} = \frac{\partial}{\partial\omega_2}\left(\frac{\partial}{\partial\omega_1} u\right) = -C_{12}$$

$$\Rightarrow \frac{\partial^3 \Phi_{\underline{X}}(\underline{\omega})}{\partial\omega_3 \partial\omega_2 \partial\omega_1} = e^{u} u_3 u_2 u_1 + e^{u} u_1 u_{23} + e^{u} u_2 u_{13} + e^{u} u_3 u_{12}$$

$$\Rightarrow \frac{\partial^4 \Phi_{\underline{x}}(\underline{\omega})}{\partial\omega_4 \partial\omega_3 \partial\omega_2 \partial\omega_1} = e^{u} u_4 u_3 u_2 u_1 + e^{u} u_2 u_1 u_{34} + e^{u} u_3 u_1 u_{24}$$

$$+ e^{u} u_3 u_2 u_{14} + e^{u} u_4 u_1 u_{23} + \underbrace{e^{u} u_{14} u_{23}}$$

$$+ e^{u} u_4 u_2 u_{13} + \underbrace{e^{u} u_{24} u_{13}} + e^{u} u_4 u_3 u_{12}$$

$$+ \underbrace{e^{u} u_{34} u_{12}}$$

$$\Rightarrow u_i = -\sigma_i^2 \omega_i - \sum_{\substack{j=1 \\ j\neq i}}^{4} C_{ij}\omega_j \Rightarrow u_i \Big|_{\omega_1=\omega_2=\omega_3=\omega_4=0} = 0$$

$$\Rightarrow u_{ij} = -C_{ij} \Rightarrow u_{ij}\Big|_{\underline{\omega}=0} = -C_{ij} \quad , \quad e^{u}\Big|_{\underline{\omega}=0} = 1$$

$$\Rightarrow \frac{\partial^4 \Phi_{\underline{x}}(\underline{\omega})}{\partial\omega_4 \partial\omega_3 \partial\omega_2 \partial\omega_1}\Bigg|_{\underline{\omega}=0} = u_{14}u_{23} + u_{24}u_{13} + u_{34}u_{12}$$

$$= C_{14}C_{23} + C_{24}C_{13} + C_{34}C_{12}$$

2.36. $\underline{Z}$ is proper $\Rightarrow$ $\underline{W} \triangleq \underline{\underline{A}}\,\underline{Z} + \underline{b}$ is proper

$$E\{\underline{W}\} = E\{\underline{\underline{A}}\,\underline{Z} + \underline{b}\} = \underline{\underline{A}}\,\underline{m}_z + \underline{b}$$

$$\Rightarrow \underline{Z} \text{ is proper} \equiv E\{(\underline{Z} - \underline{m}_z)(\underline{Z} - \underline{m}_z)^T\} = \tilde{C}_z = 0$$

$$\Rightarrow E\{(\underline{W} - \underline{m}_w)(\underline{W} - \underline{m}_w)^T\} = E\{(\underline{\underline{A}}\,\underline{Z} + \underline{b} - \underline{\underline{A}}\,\underline{m}_z - \underline{b})(\underline{\underline{A}}\,\underline{Z} + \underline{b} - \underline{\underline{A}}\,\underline{m}_z - \underline{b})^T\}$$

$$= E\{[\underline{\underline{A}}(\underline{Z} - \underline{m}_z)][\underline{\underline{A}}(\underline{Z} - \underline{m}_z)]^T\} = E\{\underline{\underline{A}}(\underline{Z} - \underline{m}_z)(\underline{Z} - \underline{m}_z)^T \underline{\underline{A}}^T\}$$

$$= \underline{\underline{A}}\,\tilde{C}_z \underline{\underline{A}}^T = \underline{\underline{0}} \Rightarrow \underline{W} \text{ is proper too.}$$

2.37. $X(t)$ and $Y(t)$ are jointly WSS.

$$\Rightarrow \begin{cases} R_X(t+\tau, t) = R_X(\tau) \\ R_Y(t+\tau, t) = R_Y(\tau) \end{cases} \quad \begin{cases} R_{XY}(t+\tau, t) = R_{XY}(\tau) \\ R_{YX}(t+\tau, t) = R_{YX}(\tau) \end{cases}$$

a. $Z(t) \triangleq X(t) + Y(t) \Rightarrow R_Z(t+\tau, t) = ?$

$$R_Z(t+\tau, t) = E\{Z(t+\tau) Z^*(t)\} = E\{[X(t+\tau) + Y(t+\tau)][X^*(t) + Y^*(t)]\}$$

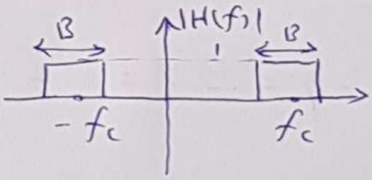
$$= R_X(\tau) + R_Y(\tau) + R_{XY}(\tau) + R_{YX}(\tau) = R_Z(\tau)$$

b. $R_{XY}(\tau) = R_{YX}(\tau) = 0 \Rightarrow R_Z(\tau) = R_X(\tau) + R_Y(\tau)$

c. $R_{XY}(\tau) = R_{YX}(\tau) = 0 \;\&\; m_X(t) = m_Y(t) = 0 \Rightarrow R_Z(\tau) = R_X(\tau) + R_Y(\tau)$

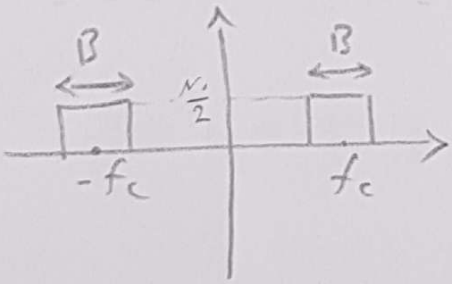
$m_Z(t) = 0$

2.38. $R_X(\tau) = \frac{1}{2} N_0 \delta(\tau)$,



$X(t) \longrightarrow \boxed{h(t)} \longrightarrow Y(t)$

$$S_X(f) = \mathcal{F}\{R_X(\tau)\} = \frac{N_0}{2} \Rightarrow S_Y(f) = S_X(f) \times |H(f)|^2$$

$\Rightarrow$ 

$$\Rightarrow P_Y = \langle E\{|Y(t)|^2\}\rangle_t = \langle E\{Y(t+\tau) Y^*(t)\}\rangle_t \Big|_{\tau=0}$$

$$= \langle R_Y(t+\tau, t)\rangle_t \Big|_{\tau=0} = R_Y(\tau)\Big|_{\tau=0} = \int_{\forall f} S_Y(f)\, df = 2 \times \left(\frac{N_0}{2} B\right) = N_0 B$$

2.40.

$$\underline{X} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \Rightarrow \underline{\underline{C}}_X = E\{(\underline{X}-\underline{\mu}_X)(\underline{X}-\underline{\mu}_X)^H\} = \begin{bmatrix} C_{11} & 0 & C_{13} \\ 0 & C_{22} & 0 \\ C_{31} & 0 & C_{33} \end{bmatrix}$$

$$\underline{Y} = \underline{\underline{A}}\,\underline{X}, \quad \underline{\underline{A}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix} \Rightarrow \underline{\underline{C}}_Y = E\{(\underline{Y}-\underline{\mu}_Y)(\underline{Y}-\underline{\mu}_Y)^H\}$$

$$= E\{A(\underline{X}-\underline{\mu}_X)(\underline{X}-\underline{\mu}_X)^H A^H\} = A\,\underline{\underline{C}}_X A^T$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} C_{11} & 0 & C_{13} \\ 0 & C_{22} & 0 \\ C_{31} & 0 & C_{33} \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} C_{11} & 0 & C_{13} \\ 0 & 2C_{22} & 0 \\ C_{11}+C_{31} & 0 & C_{13}+C_{33} \end{bmatrix} \times A^T$$

$$= \begin{bmatrix} C_{11} & 0 & C_{11}+C_{13} \\ 0 & 4C_{22} & 0 \\ C_{11}+C_{31} & 0 & C_{11}+C_{13}+C_{31}+C_{33} \end{bmatrix}$$

2.49. $v(t) = X\cos(2\pi f_c t) - Y\sin(2\pi f_c t)$

$v(t)$ is wss $\Longleftrightarrow \begin{cases} E\{X\} = E\{Y\} = 0 \\ E\{X^2\} = E\{Y^2\} \\ E\{XY\} = 0 \end{cases}$

$m_v(t) = \cos(2\pi f_c t)\,E\{X\} - \sin(2\pi f_c t)\,E\{Y\} = cte \Rightarrow E\{X\} = E\{Y\} = 0$

$R_v(t+\tau, t) = E\{v(t+\tau)\,v^*(t)\} = E\{[X\cos(2\pi f_c(t+\tau)) - Y\sin(2\pi f_c(t+\tau))]$

$\times [X\cos(2\pi f_c t) - Y\sin(2\pi f_c t)]\} = E\{X^2\} \times \frac{1}{2}[\cos(2\pi f_c(2t+\tau))$

$+ \cos(2\pi f_c \tau)] + E\{Y^2\} \times \frac{1}{2}[\cos(2\pi f_c \tau) - \cos(2\pi f_c(2t+\tau))]$

$$-E\{XY\} \times \frac{1}{2}\left[\sin(2\pi f_c(2t+\tau)) - \sin(2\pi f_c(\tau))\right]$$

$$-E\{XY\} \times \frac{1}{2}\left[\sin(2\pi f_c(2t+\tau)) + \sin(2\pi f_c \tau)\right]$$

$$= \frac{1}{2}\cos(2\pi f_c \tau)\left(E\{X^2\} + E\{Y^2\}\right)$$

$$+ \frac{1}{2}\cos(2\pi f_c(2t+\tau))\,\cancel{\left(E\{X^2\} - E\{Y^2\}\right)}^{0}$$

$$-\cancel{E\{XY\}}^{0}\sin(2\pi f_c(2t+\tau)) \quad \Rightarrow \quad E\{XY\} = 0\ ,\quad E\{X^2\} = E\{Y^2\}$$

2-54. $X(t) = A\sin(2\pi f_c t + \theta)$ s.t. $\theta \sim \text{Uniform}(0, 2\pi)$

$$R_X(t+\tau, t) = E\{A\sin(2\pi f_c(t+\tau) + \theta)\,A\sin(2\pi f_c t + \theta)\}$$

$$= \frac{A^2}{2}E\{\cos(2\pi f_c \tau)\} - \frac{A^2}{2}\underbrace{E\{\cos(2\pi f_c(2t+\tau) + 2\theta)\}}$$

$$= \frac{A^2}{2}\cos(2\pi f_c \tau) - \frac{A^2}{2}\,\mathrm{I}$$

$$\mathrm{I} = \int_0^{2\pi} \frac{1}{2\pi}\cos(2\pi f_c(\tau + 2t) + 2\theta)\,d\theta = \frac{1}{4\pi}\sin(2\pi f_c(2t+\tau) + 2\theta)\Big|_{\theta=0}^{2\pi}$$

$$= \left[\sin(2\pi f_c(2t+\tau) + 4\pi) - \sin(2\pi f_c(2t+\tau))\right] \times \frac{1}{4\pi} = 0$$

$$\Rightarrow R_X(\tau) = \frac{A^2}{2}\cos(2\pi f_c \tau)$$