Mean field games and applications

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Mean field games

Framework

Mean field games (MFGs) are differential games with a continuum of players / agents, assumed to be rational, indistinguishable, and influenced only by the average behavior of other players.

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- Dynamics of a player: $\dot{\gamma}(t) = f(t, \gamma(t), u(t))$, where f is the same for all players, γ is the trajectory of the player in Ω, and u is a control each player can choose, with $u(t) \in U$.

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- Choice of u: each player wants to minimize some cost
$$\int_0^T L(t, \gamma(t), u(t), \rho_t)\, \mathrm{d}t + G(\gamma(T), \rho_T),$$
where $L : [0, T] \times \Omega \times U \times \mathcal{P}(\Omega) \to \mathbb{R}$ and $G : \Omega \times \mathcal{P}(\Omega) \to \mathbb{R}$ are the same for all players and $T \in (0, +\infty]$.

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Mean field games

A few examples

Example 1: Mexican wave (adapted from [Guéant, Lasry, Lions; 2011]).

Setting: people in a stadium doing a Mexican wave.

Mean field games

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Example 1: Mexican wave (adapted from [Guéant, Lasry, Lions; 2011]).

Setting: people in a stadium doing a Mexican wave.

- $\Omega = \mathbb{S}^1 \times [0, 1]$: $\theta \in \mathbb{S}^1$ represents position at the stadium, $z \in [0, 1]$ represents whether a person is seated (0), standing (1), or at an intermediate position.
- $\rho_0 \in \mathcal{P}(\Omega)$: initial distribution of people.
- $\begin{pmatrix} \dot\theta(t) \\ \dot z(t) \end{pmatrix} = \begin{pmatrix} 0 \\ u(t) \end{pmatrix}$, where $(\theta(t), z(t)) \in \Omega$ and $u(t) \in \mathbb{R}$.
- Each person wants to minimize

$$\int_0^T \Bigg[\underbrace{\frac{\lvert u(t)\rvert^2}{2}}_{\substack{\text{cost of}\\ \text{moving}}} + \alpha \underbrace{z(t)^p (1 - z(t))^q}_{\substack{\text{cost of intermediate}\\ \text{positions}}} + \beta \underbrace{\int_\Omega (z(t) - \zeta)^2 \chi(\vartheta - \theta(t))\, d\rho_t(\zeta, \vartheta)}_{\text{wish to behave as neighbors}}\Bigg] dt,$$

where $\alpha, \beta > 0$, $p, q \geq 1$, $\chi \geq 0$ on $\mathbb{S}^1$, $\int_{\mathbb{S}^1} \chi(\vartheta)\, d\vartheta = 1$, and χ is concentrated around zero.

Mean field games

A few examples

Example 2: Jet lag (adapted from the talk [Carmona; 2017]).

Setting: $\approx 10^4$ neuronal cells responsible for circadian rhythm.

Each cell behaves like an oscillator.

Mean field games

A few examples

Example 2: Jet lag (adapted from the talk [Carmona; 2017]).

Setting: $\approx 10^4$ neuronal cells responsible for circadian rhythm. Each cell behaves like an oscillator.

- $\Omega = \mathbb{S}^1$: $\theta \in \mathbb{S}^1$ represents oscillatory phase of a cell.
- $\rho_0 \in \mathcal{P}(\Omega)$: initial distribution of phases.
- $\dot\theta(t) = \omega_0 + u(t)$, where $\theta(t) \in \Omega$, $u(t) \in \mathbb{R}$, and $\omega_0 \approx \frac{2\pi}{24.5}$.
- Cost:

$$\limsup_{T\to+\infty} \frac{1}{T} \int_0^T \Bigg[\underbrace{\frac{|u(t)|^2}{2}}_{\substack{\text{cost of the}\\ \text{control}}} - \alpha \underbrace{\int_0^{2\pi} \cos\left(\theta(t) - \vartheta\right) d\rho_t(\vartheta)}_{\substack{\text{synchronization with}\\ \text{other cells}}} - \beta \underbrace{\cos\left(\theta(t) - \omega_S t - \varphi(t)\right)}_{\substack{\text{synchronization with}\\ \text{the Sun}}} \Bigg] dt,$$

where $\alpha, \beta > 0$, $\omega_S = \frac{2\pi}{24}$, and $\varphi(t)$ corresponds to the phase of the current time zone.

Mean field games

Main goals

Given a MFG model (i.e., Ω, $\rho_0 \in \mathcal{P}(\Omega)$, a control system $\dot{\gamma}(t) = f(t, \gamma(t), u(t))$, and an optimization criterion), the goals are to

- characterize equilibrium states, i.e., states where almost all agents satisfy their optimization criterion;
- obtain existence (and possibly uniqueness) results for the equations characterizing equilibria;
- obtain further properties of equilibria states (e.g. regularity);
- study convergence to equilibria;
- study convergence of games with N players to mean field games as $N \to \infty$;
- numerically approximate equilibria; etc.

Mean field games
A simple model

[Cardaliaguet, *Notes on Mean Field Games*; 2013], based on P.-L. Lions' lectures at Collège de France.

$$\Omega = \mathbb{R}^d; \qquad \rho_0 \in \mathcal{P}(\mathbb{R}^d); \qquad \dot\gamma(t) = u(t),\ u(t) \in \mathbb{R}^d;$$

$$\text{Minimize } \int_0^T \left[\frac{|u(t)|^2}{2} + F(\gamma(t), \rho_t)\right] dt + G(\gamma(T), \rho_T).$$

Equilibrium:

- Given $\rho : [0, T] \to \mathcal{P}(\Omega)$, compute optimal trajectories.
- Define an optimal flow: $t \mapsto \Phi(t, x)$ is an optimal trajectory starting from x.
- ρ is an equilibrium if Φ is well-defined and $\Phi(t, \cdot)_{\#}\rho_0 = \rho_t$ for every $t \in [0, T]$.

Mean field games
A simple model

Theorem

If F and G are continuous and satisfy some $\mathcal{C}^2$ bounds, and if ρ_0 is absolutely continuous with bounded and compactly supported density, then there exists an equilibrium ρ for this MFG.

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If F and G are continuous and satisfy some $\mathcal{C}^2$ bounds, and if ρ_0 is absolutely continuous with bounded and compactly supported density, then there exists an equilibrium ρ for this MFG. Moreover, there exists $\psi : [0, T] \times \mathbb{R}^d \to \mathbb{R}$ such that (ρ, ψ) solves the mean field equations

$$\begin{cases}
\partial_t \rho(t, x) - \operatorname{div}\left(\nabla_x \psi(t, x) \rho(t, x)\right) = 0, & (t, x) \in (0, T) \times \mathbb{R}^d, \\
-\partial_t \psi(t, x) + \dfrac{1}{2} \left|\nabla_x \psi(t, x)\right|^2 = F(x, \rho_t), & (t, x) \in (0, T) \times \mathbb{R}^d, \\
\rho(0, x) = \rho_0(x), & x \in \mathbb{R}^d, \\
\psi(T, x) = G(x, \rho_T), & x \in \mathbb{R}^d.
\end{cases}$$

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Coupled system of nonlinear PDEs: a continuity equation forward in time (in the sense of distributions) and a Hamilton–Jacobi equation backward in time (in the viscosity sense).

Mean field games
References

- Origins of MFGs: [Huang, Malhamé, Caines; 2006], [Lasry, Lions; 2006], [Lasry, Lions; 2006], [Huang, Caines, Malhamé; 2007], [Lasry, Lions; 2007].
- A tutorial on MFGs: [Cardaliaguet, *Notes on Mean Field Games*; 2013].
- More general models: [Guéant; 2012], [Feleqi; 2013], [Fornasier, Piccoli, Rossi; 2014], [Gomes, Saúde; 2014], [Cirant; 2015], [Carmona, Delarue, Lacker; 2016], [Moon, Başar; 2017].

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- Limits of games with N players: [Lasry, Lions; 2006], [Huang, Caines, Malhamé; 2007], [Bardi, Priuli; 2013], [Kolokoltsov, Troeva, Yang; 2014], [Fischer; 2017].
- Numerical approximations for MFGs: [Achdou, Camilli, Capuzzo-Dolcetta; 2012], [Guéant; 2012], [Achdou; 2013], [Carlini, Silva; 2014].
- Variational MFGs: [Mészáros, Silva; 2015], [Cardaliaguet, Mészáros, Santambrogio; 2016], [Prosinski, Santambrogio; 2016], [Benamou, Carlier, Santambrogio; 2017].
- Limit of large time horizon $T \to \infty$: [Cardaliaguet, Lasry, Lions, Porretta; 2013].
- MFGs on graphs and networks: [Gomes, Mohr, Souza; 2013], [Camilli, Carlini, Marchi; 2015], [Guéant; 2015], [Cacace, Camilli, Marchi; 2017].
- Master equation: [Bensoussan, Frehse, Yam; 2015], [Cardaliaguet, Delarue, Lasry, Lions; 2015].

Crowd motion
Framework

Goal: Provide mathematical models for the motion of a large number of people.

Shibuya Crossing, Tokyo, 2014.

Crowd motion
Framework

A vast literature, with several different models: [Henderson; 1971], [Gipps, Marksjö; 1985], [Borgers, Timmermans; 1986], [Yuhaski, Smith; 1989], [Løvås; 1994], [Helbing, Molnár; 1995], [Helbing, Farkas, Vicsek; 2000], [Blue, Adler; 2001], [Burstedde, Klauck, Schadschneider, Zittartz; 2001], [Hoogendoorn, Bovy; 2004], [Maury, Venel; 2009], [Maury, Roudneff-Chupin, Santambrogio, Venel; 2011], [Cristiani, Priuli, Tosin; 2015], ...

- Microscopic vs. macroscopic.
- Passive vs. active.
- Deterministic vs. stochastic.
- How to handle congestion.

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- Microscopic vs. macroscopic.
- Passive vs. active.
- Deterministic vs. stochastic.
- How to handle congestion.

Crowd motion

Mean field games with congestion penalization

Based on [Benamou, Carlier, Santambrogio; 2017]; parameters $\alpha > 0$, $m > 0$.

$$\Omega = \mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d; \qquad \rho_0 \in \mathcal{P}(\mathbb{T}^d); \qquad \dot{\gamma}(t) = u(t),\ u(t) \in \mathbb{R}^d;$$

$$\text{Minimize } \int_0^T \Bigg[\underbrace{\frac{\left|u(t)\right|^2}{2}}_{\substack{\text{speed}\\\text{penalization}}} + \alpha \underbrace{\Big(\rho(t, \gamma(t))\Big)^m}_{\substack{\text{congestion}\\\text{penalization}}} \Bigg] \mathrm{d}t + \underbrace{G(\gamma(T))}_{\text{target}}.$$

Crowd motion

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As before, equilibria characterized by the mean field equations

$$\begin{cases} \partial_t \rho(t, x) - \operatorname{div}(\nabla_x \psi(t, x) \rho(t, x)) = 0, & (t, x) \in (0, T) \times \mathbb{T}^d, \\ -\partial_t \psi(t, x) + \frac{1}{2} |\nabla_x \psi(t, x)|^2 = \alpha \rho(t, x)^m, & (t, x) \in (0, T) \times \mathbb{T}^d, \\ \rho(0, x) = \rho_0(x), & x \in \mathbb{T}^d, \\ \psi(T, x) = G(x), & x \in \mathbb{T}^d. \end{cases}$$

Crowd motion

Mean field games with congestion penalization

Equilibria also minimize a global energy: $(\rho, v = -\nabla\psi)$ minimize

$$\int_0^T \int_{\mathbb{T}^d} \left[\rho(t, x)\frac{\left|v(t, x)\right|^2}{2} + \frac{\alpha}{m+1}\rho(t, x)^{m+1}\right] \mathrm{d}x \, \mathrm{d}t + \int_{\mathbb{T}^d} G(x)\rho(T, x) \, \mathrm{d}x$$

among all (ρ, v) such that $\partial_t \rho + \operatorname{div}(v\rho) = 0$.

Crowd motion

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among all (ρ, v) such that $\partial_t \rho + \operatorname{div}(v\rho) = 0$.

Individual cost:

$$\int_0^T \left[\frac{|\dot\gamma(t)|^2}{2} + \alpha\rho(t,\gamma(t))^m\right] dt + G(\gamma(T)).$$

Crowd motion

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among all (ρ, v) such that $\partial_t \rho + \operatorname{div}(v\rho) = 0$.

Total cost:

$$\int_0^T \int_{\mathbb{T}^d} \left[\rho(t,x)\frac{|v(t,x)|^2}{2} + \alpha\rho(t,x)^{m+1}\right] \mathrm{d}x \, \mathrm{d}t + \int_{\mathbb{T}^d} G(x)\rho(t,x)\,\mathrm{d}x.$$

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Global energy and total cost are not the same!

Total cost at MFG equilibrium − Optimal total cost $=:$ Cost of anarchy.

Crowd motion

Mean field games with congestion penalization

Thanks to the fact that (ρ, v) minimizes a global energy, one can obtain more regularity on ρ.

Theorem (Benamou, Carlier, Santambrogio; 2017)

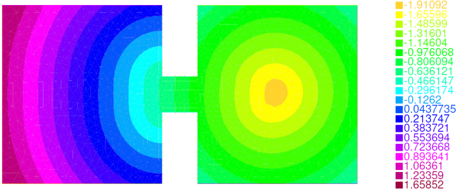
The equilibrium density ρ satisfies $\rho^{\frac{m+1}{2}} \in H^1_{\mathrm{loc}}((0, T) \times \mathbb{T}^d)$.

Proof based on duality arguments. See also [Brenier; 1999], [Cardaliaguet, Mészáros, Santambrogio; 2016].

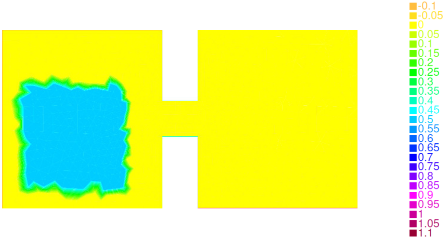
Crowd motion

Mean field games with congestion penalization

Simulations by [Benamou, Carlier, Santambrogio; 2017], $\alpha = 1$, $m = 5$.



Function G



Density ρ

Crowd motion

Mean field games with congestion constraint

Based on [Santambrogio; 2012], [Cardaliaguet, Mészáros, Santambrogio; 2016], and [Benamou, Carlier, Santambrogio; 2017].

Main idea: replace the congestion penalization $(\rho(t, x))^m$ in the cost by a constraint $\rho(t, x) \leq \rho_{\max}$.

Crowd motion

Mean field games with congestion constraint

Based on [Santambrogio; 2012], [Cardaliaguet, Mészáros, Santambrogio; 2016], and [Benamou, Carlier, Santambrogio; 2017].

Main idea: replace the congestion penalization $(\rho(t, x))^m$ in the cost by a constraint $\rho(t, x) \leq \rho_{\max}$.

Naive approach: every agent minimizes the non-penalized cost

$$\int_0^T \frac{\left|u(t)\right|^2}{2} \,\mathrm{d}t + G(\gamma(T))$$

while respecting the constraint $\rho(t, \gamma(t)) \leq \rho_{\max}$. But one extra agent does not violate the constraint...

Better idea: consider the minimization of the global energy

$$\int_0^T \int_{\mathbb{T}^d} \rho(t, x) \frac{\left|v(t, x)\right|^2}{2} \,\mathrm{d}x \,\mathrm{d}t + \int_{\mathbb{T}^d} G(x) \rho(T, x) \,\mathrm{d}x$$

among all (ρ, v) such that $\partial_t \rho + \operatorname{div}(v\rho) = 0$ and $\rho \leq \rho_{\max}$.

Crowd motion

Mean field games with congestion constraint

This corresponds to taking $\alpha = \frac{1}{\rho_{\max}^{m+1}}$ in the global energy

$$\int_0^T \int_{\mathbb{T}^d} \left[\rho(t,x)\frac{|v(t,x)|^2}{2} + \frac{\alpha}{m+1}\rho(t,x)^{m+1}\right] dx\, dt + \int_{\mathbb{T}^d} G(x)\rho(T,x)\, dx$$

and letting $m \to +\infty$ formally.

Previous mean field equations:

$$\begin{cases} \partial_t \rho(t,x) - \operatorname{div}(\nabla_x \psi(t,x)\rho(t,x)) = 0, & (t,x) \in (0,T)\times\mathbb{T}^d, \\ -\partial_t \psi(t,x) + \frac{1}{2}|\nabla_x \psi(t,x)|^2 = \left(\frac{\rho(t,x)}{\rho_{\max}}\right)^m, & (t,x) \in (0,T)\times\mathbb{T}^d, \\ \rho(0,x) = \rho_0(x), & x \in \mathbb{T}^d, \\ \psi(T,x) = G(x), & x \in \mathbb{T}^d. \end{cases}$$

Crowd motion

Mean field games with congestion constraint

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$$\int_0^T \int_{\mathbb{T}^d} \left[\rho(t,x)\frac{|v(t,x)|^2}{2} + \frac{\alpha}{m+1}\rho(t,x)^{m+1}\right] dx\,dt + \int_{\mathbb{T}^d} G(x)\rho(T,x)\,dx$$

and letting $m \to +\infty$ formally.

New mean field equations:

$$\begin{cases} \partial_t \rho(t,x) - \operatorname{div}(\nabla_x \psi(t,x)\rho(t,x)) = 0, & (t,x) \in (0,T)\times\mathbb{T}^d, \\ -\partial_t \psi(t,x) + \frac{1}{2}\left|\nabla_x \psi(t,x)\right|^2 = p(t,x), & (t,x) \in (0,T)\times\mathbb{T}^d, \\ \rho(0,x) = \rho_0(x), & x \in \mathbb{T}^d, \\ \psi(T,x) = G(x) + p(T,x), & x \in \mathbb{T}^d, \\ p(t,x) \geq 0, \quad \rho(t,x) \leq \rho_{\max}, & \\ p(t,x)\left[\rho_{\max} - \rho(t,x)\right] = 0, & (t,x) \in (0,T)\times\mathbb{T}^d. \end{cases}$$

p: pressure due to the density constraint. Similar to fluid mechanics.

Crowd motion

Mean field games with congestion constraint

Each agent now minimizes

$$\int_0^T \left[\frac{|u(t)|^2}{2} + p(t, \gamma(t))\right] \mathrm{d}t + G(\gamma(T)) + p(T, \gamma(T)).$$

p: price to discourage agents to go through saturated areas.

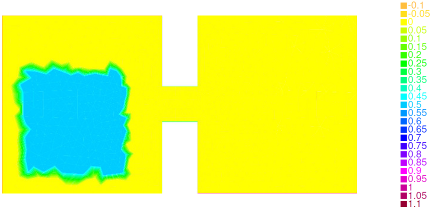
Results proved in [Cardaliaguet, Mészáros, Santambrogio; 2016]:

- existence of solutions to the mean field equations;
- convergence of solutions as $m \to +\infty$;
- regularity results.

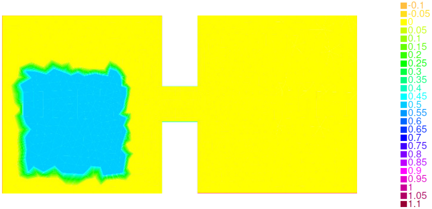
Crowd motion

Mean field games with congestion constraint

Simulations by [Benamou, Carlier, Santambrogio; 2017].



Penalization
$m = 5$



Constraint

Mean field games with velocity constraint

The model

Based on ongoing works with F. Santambrogio and S. Dweik.

Main ideas:

- Propose a model where the final time T is not fixed.
- Maximal speed of agents depends on local congestion.

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Model:

- $\Omega \subset \mathbb{R}^d$ non-empty, open, and bounded; $\rho_0 \in \mathcal{P}(\overline{\Omega})$.
- $\Gamma \subset \partial\Omega$ non-empty and closed.
- Maximal speed of the agents is bounded:
$$\dot{\gamma}(t) = f(\rho_t, \gamma(t))u(t), \qquad u(t) \in \overline{B}(0; 1) = \text{closed unit ball.}$$
- Optimization criterion: agents want to leave Ω through Γ in minimal time.
$$\begin{aligned}\inf\{T \geq 0 \mid \; & \dot{\gamma}(t) = f(\rho_t, \gamma(t))u(t),\ u : \mathbb{R}_+ \to \overline{B}(0, 1),\\ & \gamma(0) = x,\ \gamma(T) \in \Gamma,\ \gamma(t) \in \overline{\Omega} \text{ for } t \in [0, T],\\ & \dot{\gamma}(t) = 0 \text{ for } t > T\}.\end{aligned}$$

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Mean field games with velocity constraint

The Lagrangian approach

- Eulerian approach: $\rho : \mathbb{R}_+ \to \mathcal{P}(\overline{\Omega})$ is a curve on the set of measures. Motion is described by the density and the velocity field of the population.

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- Eulerian approach: $\rho : \mathbb{R}_+ \to \mathcal{P}(\overline{\Omega})$ is a curve on the set of measures. Motion is described by the density and the velocity field of the population.
- Lagrangian approach: $Q \in \mathcal{P}(\mathcal{C})$, where $\mathcal{C} = \mathcal{C}(\mathbb{R}_+, \overline{\Omega})$, is a measure on the set of curves. Motion is described by the trajectory of each agent.

Lagrangian framework for mean field games already used in the literature, cf. e.g. the survey in [Benamou, Carlier, Santambrogio; 2017].

Link between Eulerian and Lagrangian: $\rho_t = e_{t\#}Q$, where $e_t : \mathcal{C} \to \overline{\Omega}$ is the evaluation at time t of a curve, $e_t(\gamma) = \gamma(t)$.

Mean field games with velocity constraint

The Lagrangian approach

Definition

A measure $Q \in \mathcal{P}(\mathcal{C})$ is a Lagrangian equilibrium of the mean field game if $e_{0\#}Q = \rho_0$ and Q-almost every $\gamma \in \mathcal{C}$ is optimal for

$$\begin{aligned} \inf\{T \geq 0 \mid \dot\gamma(t) = f(e_{t\#}Q, \gamma(t))u(t),\ u : \mathbb{R}_+ \to \overline{B}(0, 1), \\ \gamma(0) = x,\ \gamma(T) \in \Gamma,\ \gamma(t) \in \overline{\Omega} \text{ for } t \in [0, T], \\ \dot\gamma(t) = 0 \text{ for } t > T\}. \end{aligned}$$

In the sequel, we consider

- the existence of a Lagrangian equilibrium;
- the characterization of equilibria by a system of mean field equations;
- some simulations.

Mean field games with velocity constraint

Existence of a Lagrangian equilibrium

We use the following assumptions on f and Ω.

Hypotheses

- $f : \mathcal{P}(\overline{\Omega}) \times \overline{\Omega} \to \mathbb{R}_+$ is Lipschitz continuous and
$$f_{\max} = \sup_{\substack{\mu \in \mathcal{P}(\overline{\Omega}) \\ x \in \overline{\Omega}}} f(\mu, x) < +\infty, \qquad f_{\min} = \inf_{\substack{\mu \in \mathcal{P}(\overline{\Omega}) \\ x \in \overline{\Omega}}} f(\mu, x) > 0.$$
- $\partial\Omega$ is $\mathcal{C}^{1,1}$ (can be generalized).

With no loss of generality (change in time scale): $f_{\max} = 1$.

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Distance in $\mathcal{P}(\overline{\Omega})$: Wasserstein distance
$$W_1(\mu, \nu) = \min_{\substack{\gamma \in \mathcal{P}(\overline{\Omega} \times \overline{\Omega}) \\ \pi_{1\#}\gamma = \mu,\ \pi_{2\#}\gamma = \nu}} \int_{\overline{\Omega} \times \overline{\Omega}} |x - y| \, d\gamma(x, y).$$

Mean field games with velocity constraint

Existence of a Lagrangian equilibrium

Theorem

Under the previous hypotheses, there exists a Lagrangian equilibrium $Q \in \mathcal{P}(\mathcal{C})$ for this game.

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Main ideas:

- For fixed $Q \in \mathcal{P}(\mathcal{C})$, compute the set $\boldsymbol{\Gamma}_Q \subset \mathcal{C}$ of all optimal trajectories for the measure Q.
  To study the optimal control problem, we introduce the value function
  $$\begin{aligned}\tau_Q(t_0, x_0) = \inf\{T \geq 0 \mid {} & \dot\gamma(t) = f(e_{t\#}Q, \gamma(t))u(t),\ u : \mathbb{R}_+ \to \overline{B}(0, 1),\\ & \gamma(t_0) = x_0,\ \gamma(t_0 + T) \in \Gamma,\ \gamma(t) \in \overline{\Omega} \text{ for } t \in [t_0, t_0 + T],\\ & \dot\gamma(t) = 0 \text{ for } t > t_0 + T\}.\end{aligned}$$
- Define $F(Q) = \{\widetilde{Q} \mid e_{0\#}\widetilde{Q} = \rho_0 \text{ and } \widetilde{Q}(\boldsymbol{\Gamma}_Q) = 1\}$. Equilibrium $\iff$ fixed point of the set-valued map F, i.e., $Q \in F(Q)$.
- Conclude by Kakutani fixed point theorem.

Mean field games with velocity constraint

Characterization of equilibria

We have proved the existence of a Lagrangian equilibrium to the minimal time mean field game.

- Advantage: easier than to prove than in the Eulerian approach. Application of Kakutani fixed point theorem requires fewer properties of the optimal trajectories.
- Drawback: we have no information on $\rho_t = e_{t\#}Q$.

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Goal: characterize τ_Q and ρ as solutions of a system of PDEs.

We will treat in the sequel only the case $\Gamma = \partial\Omega$.

Mean field games with velocity constraint

Characterization of equilibria

In addition to the previous hypotheses, we also assume:

Hypotheses

$f : \mathcal{P}(\overline{\Omega}) \times \overline{\Omega} \to \mathbb{R}_+^*$ is given by $f(\mu, x) = K[E(\mu, x)]$, with

$$E(\mu, x) = \int_{\overline{\Omega}} \chi(x - y)\eta(y)\, d\mu(y),$$

$K \in \mathcal{C}^{1,1}(\mathbb{R}_+, \mathbb{R}_+^*)$ is bounded, $\chi \in \mathcal{C}^{1,1}(\mathbb{R}^d, \mathbb{R}_+)$, and $\eta \in \mathcal{C}^{1,1}(\mathbb{R}^d, \mathbb{R}_+)$ with $\eta(x) = 0$ and $\nabla\eta(x) = 0$ for $x \in \partial\Omega$.

An agent looks around (according to the convolution kernel χ) to see the local concentration of agents, ignoring people who already left ($\eta(y) = 0$ on Γ, $\eta(y) = 1$ outside a neighborhood of Γ).

Mean field games with velocity constraint

Characterization of equilibria

Theorem

Under the previous assumptions, τ_Q and ρ solve the mean field equations

$$\begin{cases} \partial_t \rho(t, x) - \operatorname{div}_x \left[f(\rho_t, x) \dfrac{\nabla_x \tau_Q(t, x)}{\lvert \nabla_x \tau_Q(t, x) \rvert} \rho(t, x) \right] = 0, & \mathbb{R}_+ \times \Omega, \\ -\partial_t \tau_Q(t, x) + \lvert \nabla_x \tau_Q(t, x) \rvert f(\rho_t, x) - 1 = 0, & \mathbb{R}_+ \times \Omega, \\ \rho(0, x) = \rho_0(x), & \overline{\Omega}, \\ \tau_Q(t, x) = 0, & \mathbb{R}_+ \times \partial\Omega. \end{cases}$$

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Continuity equation satisfied in the sense of distributions,
Hamilton–Jacobi equation satisfied in the viscosity sense.

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Characterization of equilibria

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Continuity equation satisfied in the sense of distributions, Hamilton–Jacobi equation satisfied in the viscosity sense.

Hamilton–Jacobi equation can be obtained by standard techniques on optimal control (and does not use the previous assumptions).

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Continuity equation satisfied in the sense of distributions,
Hamilton–Jacobi equation satisfied in the viscosity sense.

Hamilton–Jacobi equation can be obtained by standard techniques on optimal control (and does not use the previous assumptions).
But the situation is much harder for the continuity equation.

Mean field games with velocity constraint

Characterization of equilibria

The continuity equation:

$$\partial_t \rho(t, x) - \operatorname{div}_x \left[f(\rho_t, x) \frac{\nabla_x \tau_Q(t, x)}{\lvert \nabla_x \tau_Q(t, x) \rvert} \rho(t, x) \right] = 0.$$

- If τ_Q is differentiable at a point $(t, \gamma(t))$ of an optimal trajectory, then $\nabla_x \tau_Q(t, \gamma(t)) \neq 0$ and the optimal control at this point is $u(t) = -\frac{\nabla_x \tau_Q(t, \gamma(t))}{\lvert \nabla_x \tau_Q(t, \gamma(t)) \rvert}$.
- τ_Q is Lipschitz, hence differentiable a.e., but it may be nowhere differentiable along a particular trajectory...
- We need more properties of τ_Q and the optimal trajectories

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The continuity equation:

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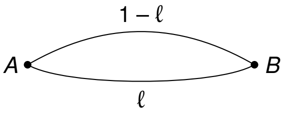
- τ_Q is Lipschitz, hence differentiable a.e., but it may be nowhere differentiable along a particular trajectory...
- We need more properties of τ_Q and the optimal trajectories obtained by
  - Pontryagin Maximum Principle $\implies$ optimal γ and u are $\mathcal{C}^{1,1}$;
    $\Downarrow$
  - proving semiconcavity of $\tau_Q \Longleftarrow (t, x) \mapsto f(\rho_t, x)$ is also $\mathcal{C}^{1,1}$ + [Cannarsa, Sinestrari; 2004];
  - using properties of semiconcave functions.

Mean field games with velocity constraint

Simulations

Simulation 1: going from A to B through two different paths.



Ω: the network;
$\Gamma = \{B\}$;
$\rho_0 = \delta_A$;
$\ell \in (0, 1)$.

$$\chi(x) = \begin{cases} \frac{1+\cos\left(\frac{\pi x}{\varepsilon}\right)}{2\varepsilon}, & \text{if } |x| < \varepsilon, \\ 0, & \text{if } |x| \geq \varepsilon, \end{cases}$$

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Mean field games with velocity constraint
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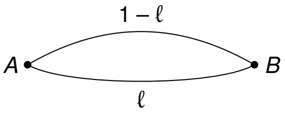

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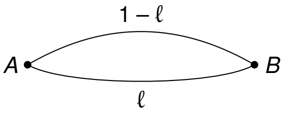

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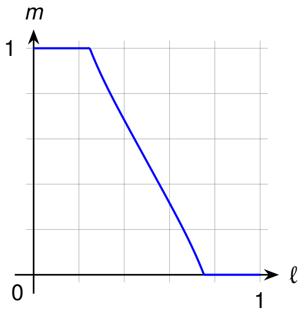

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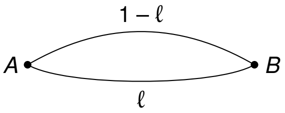

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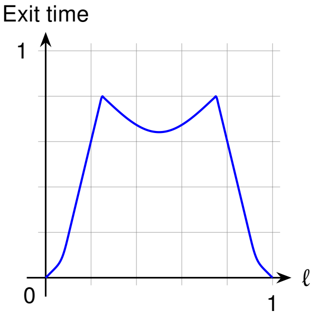
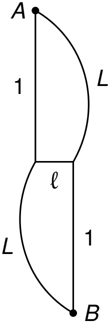


Mean field games with velocity constraint
Simulations

Simulation 2: going from A to B in a network.



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$L \in (1, +\infty)$, $\ell \in (0, 1)$;
χ, K, and ε as before;
$\eta(y) = 1$ for all $y \in \Omega$.

Mean field games with velocity constraint

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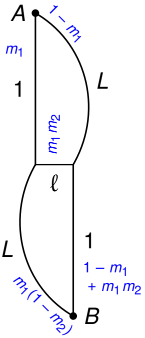

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If $0 < t < T_1$:

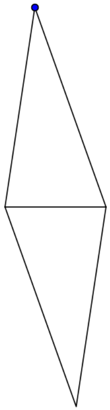
$$\rho_t = m_1 \delta_{x(t)} + (1 - m_1)\delta_{y(t)}.$$

If $T_1 < t < T_2$:

$$\rho_t = m_1(1 - m_2)\delta_{x(t)} + m_1 m_2 \delta_{z(t)} + (1 - m_1)\delta_{y(t)}.$$

If $t > T_2$:

$$\rho_t = m_1(1 - m_2)\delta_{x(t)} + (1 - m_1 + m_1 m_2)\delta_{y(t)}.$$



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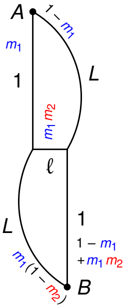

By symmetry,

$$m_1 = 1 - m_1 + m_1 m_2 \quad \text{and} \quad 1 - m_1 = m_1(1 - m_2)$$

$$\Longrightarrow \quad m_1 = \frac{1}{2 - m_2}.$$

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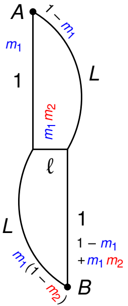

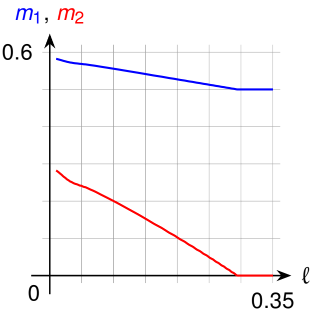

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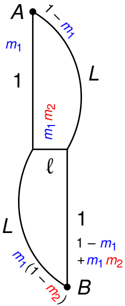

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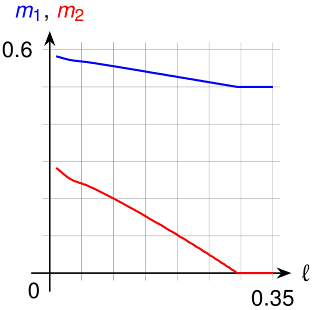

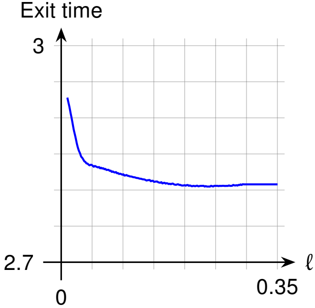

Mean field games with velocity constraint

Simulations

Simulation 3: exiting a circle.

$$\Omega = B(0;1)$$

$$\Gamma = \partial\Omega$$

$$\rho_0 = \delta_p$$

$$\chi(x) = \max\left(0, \frac{\varepsilon - |x|}{\varepsilon^2}\right)$$

$$\eta(x) = 1$$

$$K(x) = \frac{1}{1 + \left(\frac{x}{2}\right)^2}$$

$$p = (0.08, 0)$$

$$\varepsilon = 0.1$$