Dynamic Programming for Solving Optimization Problems

(Chain Matrix Multiplication Problem)

Optimizations problem?

Steps in Development of Dynamic Algorithms

Why dynamic in optimization problem?

Introduction to Catalan numbers

Chain-Matrix Multiplication

Problem Analysis

- Brute Force approach
- Time Complexity

Conclusion

Optimization Problems

- If a problem has only ***one correct solution***, then optimization is not required
- For example, there is only one sorted sequence containing a given set of numbers.
- **Optimization problems** have ***many solutions***.
- We want to compute an ***optimal solution e. g.*** with minimal cost and maximal gain.
- There could be many solutions having optimal value
- Dynamic programming is very effective technique
- Development of dynamic programming algorithms can be broken into a sequence steps as in the next.

Steps in Development of Dynamic Algorithms

1. Characterize the structure of an optimal solution
2. Recursively define the value of an optimal solution
3. Compute the value of an optimal solution in a bottom-up fashion
4. Construct an optimal solution from computed information

Note: Steps 1-3 form the basis of a dynamic programming solution to a problem. Step 4 can be omitted only if the value of an optimal solution is required.

Why Dynamic Programming?

Dynamic programming, like divide and conquer method, solves problems by combining the solutions to sub-problems.

Divide and conquer algorithms:

- partition the problem into independent sub-problem
- Solve the sub-problem recursively and
- Combine their solutions to solve the original problem

In contrast, dynamic programming is applicable when the sub-problems are not independent.

Dynamic programming is typically applied to optimization problems.

Time Complexity in Dynamic Algorithms

Time complexity:

- If there are polynomial number of sub-problems.
- If each sub-problem can be computed in polynomial time.
- Then the solution of whole problem can be found in polynomial time.

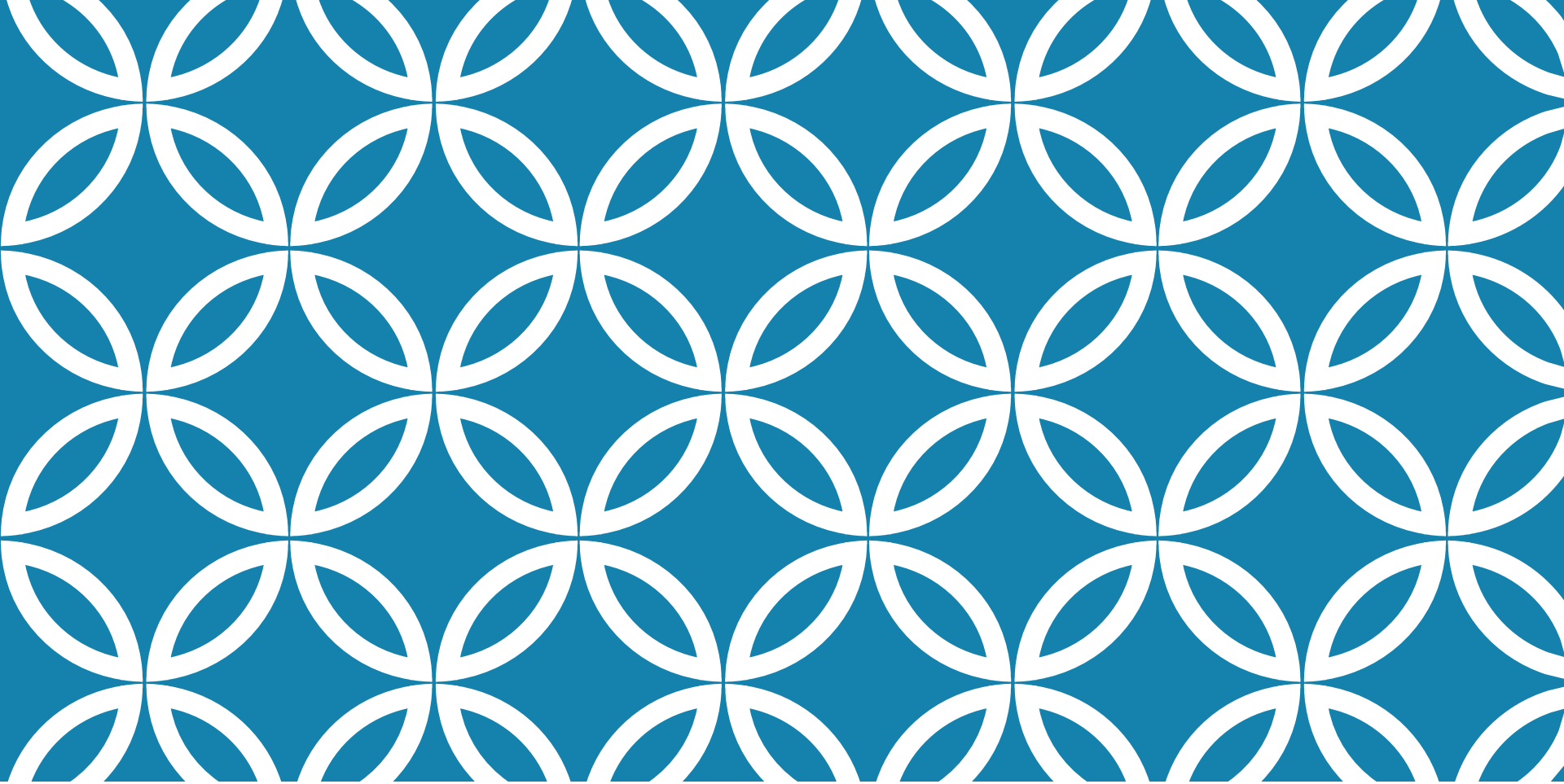
Remark:

Greedy also applies a top-down strategy but usually on one sub-problem so that the order of computation is clear

Time Complexity in Dynamic Algorithms

Polynomial Time

- An algorithm is said to be solvable in polynomial time if the number of steps required to complete the algorithm for a given input is O(n^k) for some nonnegative integer k, where n is the complexity of the input.
- Polynomial-time algorithms are said to be "fast.“
- Most familiar mathematical operations such as addition, subtraction, multiplication, and division, as well as computing square roots, powers, and logarithms, can be performed in polynomial time.



CATALAN NUMBERS

Multiplying n Numbers

Objective:

Find C(n), the number of ways to compute product $x_1 \cdot x_2 \cdots x_n$.

n	multiplication order
2	$(x_1 \cdot x_2)$
3	$(x_1 \cdot (x_2 \cdot x_3))$
	$((x_1 \cdot x_2) \cdot x_3)$
4	$(x_1 \cdot (x_2 \cdot (x_3 \cdot x_4)))$
	$(x_1 \cdot ((x_2 \cdot x_3) \cdot x_4))$
	$((x_1 \cdot x_2) \cdot (x_3 \cdot x_4))$
	$((x_1 \cdot (x_2 \cdot x_3)) \cdot x_4)$
	$(((x_1 \cdot x_2) \cdot x_3) \cdot x_4)$

Multiplying n Numbers – small n

n	C(n)
1	1
2	1
3	2
4	5
5	14
6	42
7	132

Multiplying n Numbers - small n

Recursive equation:

where is the last multiplication?

$$C(n) = \sum_{k=1}^{n-1} C(k) \cdot C(n-k)$$

Catalan numbers: $$C(n) = \frac{1}{n}\binom{2n-2}{n-1}.$$

Asymptotic value: $$C(n) \approx \frac{4^n}{n^{3/2}}$$

$$\frac{C(n)}{C(n-1)} \to 4 \quad \text{for } n \to \infty$$

CHAIN-MATRIX MULTIPLICATION

Problem Statement: Chain Matrix Multiplication

Statement: The chain-matrix multiplication problem can be stated as below:

Given a chain of $[A_1, A_2, \ldots, A_n]$ of n matrices where for $i = 1, 2, \ldots, n$, matrix A_i has dimension $p_{i-1} \times p_i$, find the order of multiplication which minimizes the number of scalar multiplications.

Note:

Order of A_1 is $p_0 \times p_1$,

Order of A_2 is $p_1 \times p_2$,

Order of A_3 is $p_2 \times p_3$, etc.

Order of $A_1 \times A_2 \times A_3$ is $p_0 \times p_3$,

Order of $A_1 \times A_2 \times \ldots \times A_n$ is $p_0 \times p_n$

Objective is to find order not multiplication

Given a sequence of matrices, we want to find a most efficient way to multiply these matrices

It means that problem is not actually to perform the multiplications, but decide the order in which these must be multiplied to reduce the cost.

This problem is an optimization type which can be solved using dynamic programming.

The problem is not limited to find an efficient way of multiplication of matrices, but can be used to be applied in various purposes.

But how to transform the original problem into chain matrix multiplication, this is another issue, which is common in systems modeling.

Why this problem is of Optimization Category?

If these matrices are all square and of same size, the multiplication order will not affect the total cost.

If matrices are of different sizes but compatible for multiplication, then order can make big difference.

Brute Force approach

- The number of possible multiplication orders are exponential in n, and so trying all possible orders may take a very long time.

Dynamic Programming

- To find an optimal solution, we will discuss it using dynamic programming to solve it efficiently.

Assumptions (Only Multiplications Considered)

We really want is the minimum cost to multiply

But we know that cost of an algorithm depends on how many number of operations are performed i.e.

We must be interested to minimize number of operations, needed to multiply out the matrices.

As in matrices multiplication, there will be addition as well multiplication operations in addition to other

Since cost of multiplication is dominated over addition therefore we will minimize the number of multiplication operations in this problem.

In case of two matrices, there is only one way to multiply them, so the cost fixed.

Chain Matrix Multiplication

(Brute Force Approach)

Brute Force Chain Matrix Multiplication Algorithm

If we wish to multiply two matrices:

$A = a[i, j]_{p, q}$ and $B = b[i, j]_{q, r}$

Now if $C = AB$ then order of C is $p \times r$.

Since in each entry $c[i, j]$, there are q number of scalar of multiplications

Total number of scalar multiplications in computing C = Total entries in C x Cost of computing a single entry = $p \cdot r \cdot q$

Hence the computational cost of $AB = p \cdot q \cdot r$

The formula to calculate single entry.

$$C[i, j] = \sum_{k=1}^{q} A[i, k] B[k, j]$$

Brute Force Chain Matrix Multiplication

Example

Given a sequence $[A_1, A_2, A_3, A_4]$

Order of $A_1 = 10 \times 100$

Order of $A_2 = 100 \times 5$

Order of $A_3 = 5 \times 50$

Order of $A_4 = 50 \times 20$

Compute the order of the product $A_1 . A_2 . A_3 . A_4$ in such a way that minimizes the total number of scalar multiplications.

Brute Force Chain Matrix Multiplication

There are five ways to parenthesize this product

Cost of computing the matrix product may vary, depending on order of parenthesis.

All possible ways of parenthesizing

$(A_1 \cdot (A_2 \cdot (A_3 \cdot A_4)))$

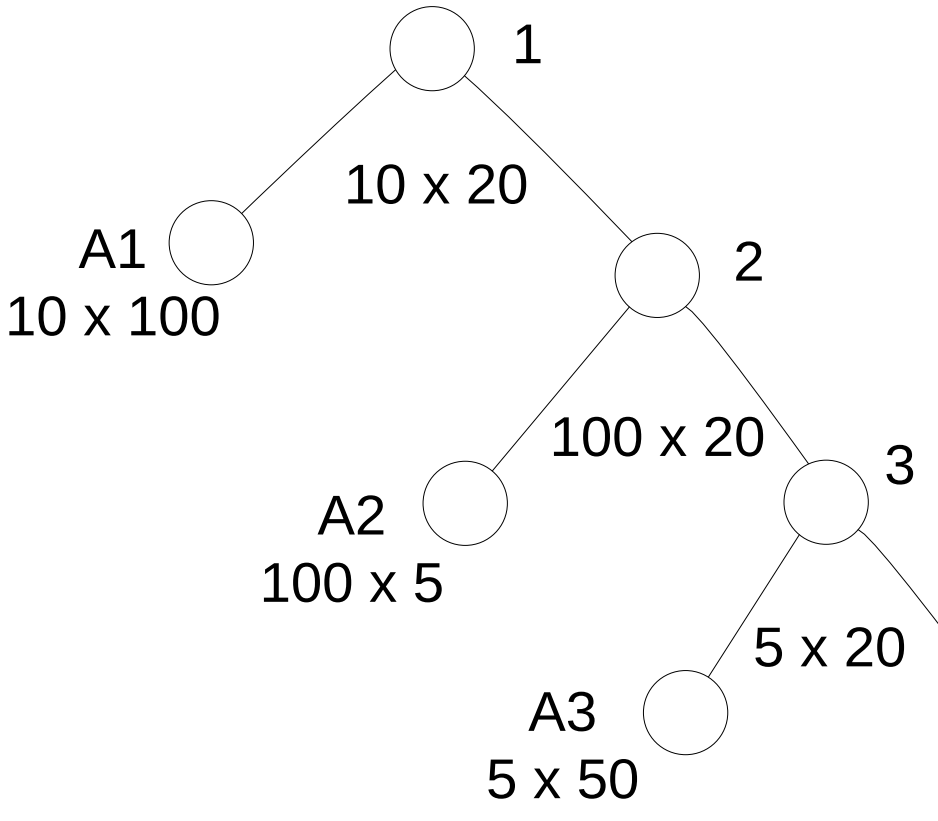
$(A_1 \cdot ((A_2 \cdot A_3) \cdot A_4))$

$((A_1 \cdot A_2) \cdot (A_3 \cdot A_4))$

$((A_1 \cdot (A_2 \cdot A_3)) \cdot A_4)$

$(((A_1 \cdot A_2) \cdot A_3) \cdot A_4)$

Kinds of problems solved by algorithms



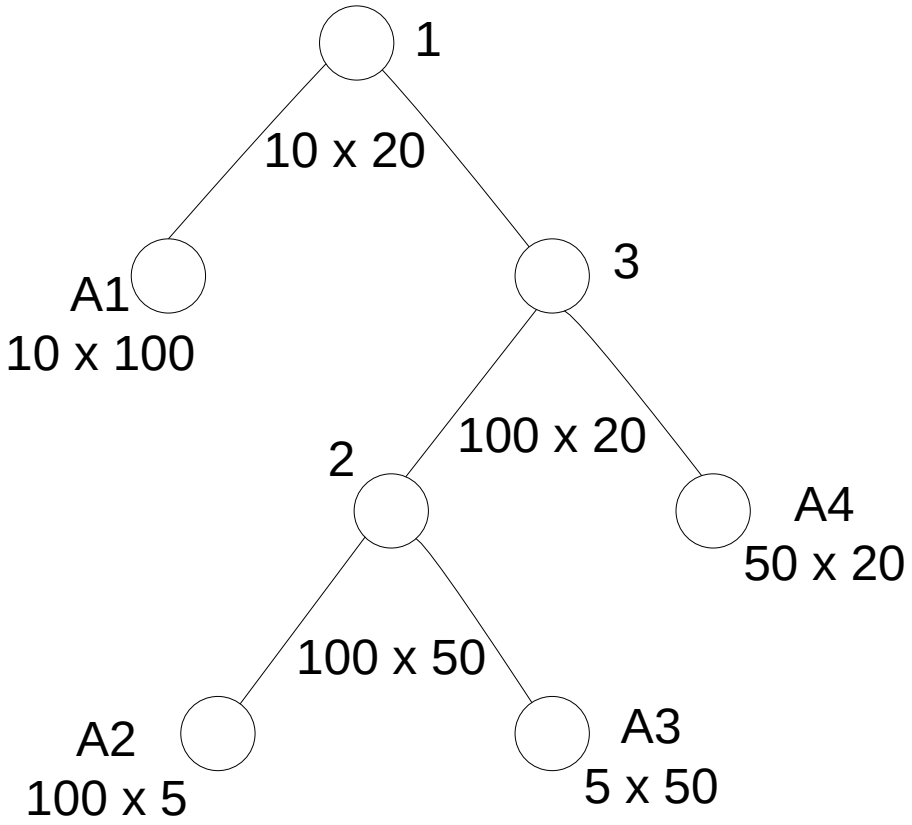
10 x 100 x 20 = 20000

100 x 5 x 20 = 10000

5 x 50 x 20 = 5000

Total Cost = 35000

Second Chain : (A1 · ((A2 . A3). A4))



10 x100 x 20 = 20000

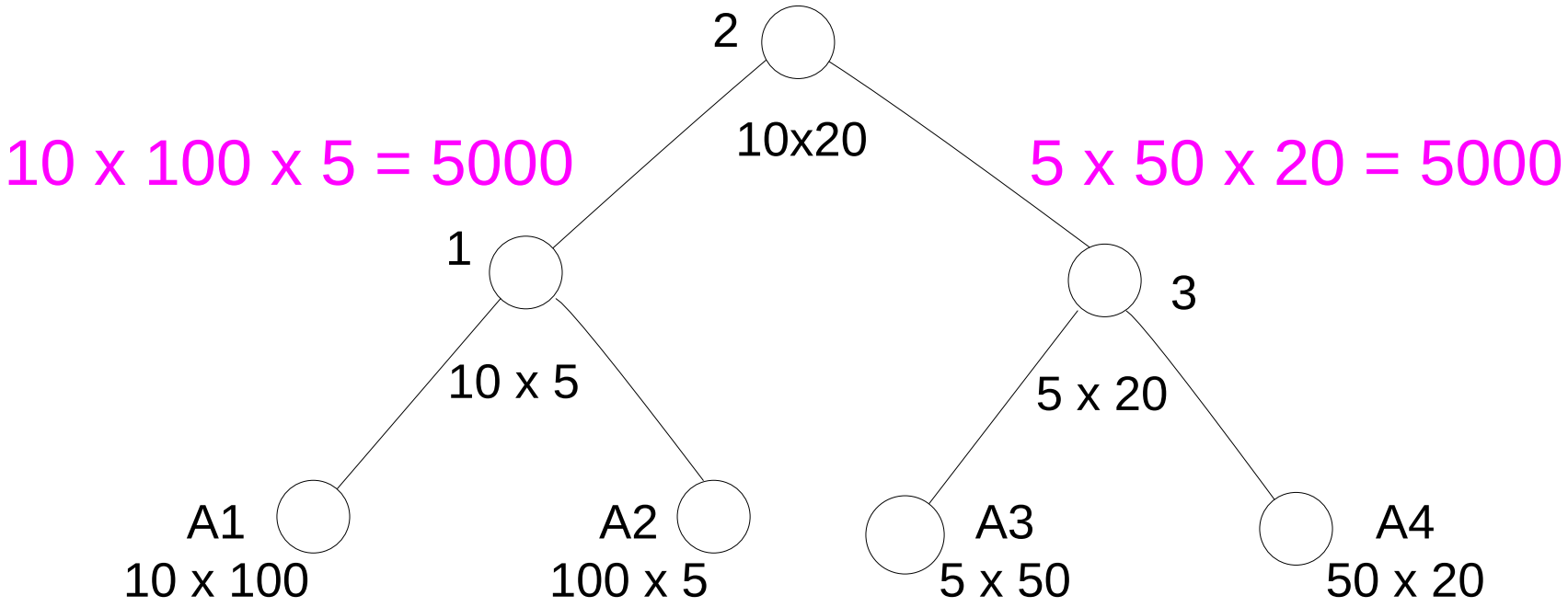
100 x 50 x 20 = 100000

100 x 5 x 50 = 25000

Total Cost = 145000

Third Chain : ((A1 · A2). (A3 . A4))

10 x 5 x 20 = 1000



Total Cost = 11000

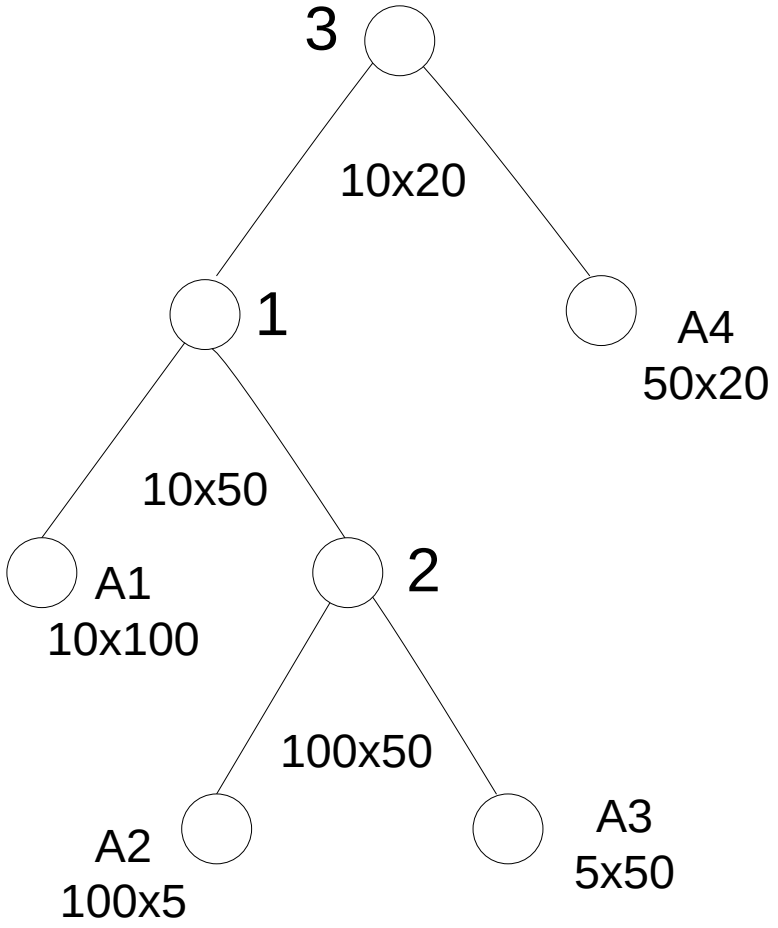
Fourth Chain : ((A1 · (A2 . A3)). A4)

10 x 50 x 20 = 10000

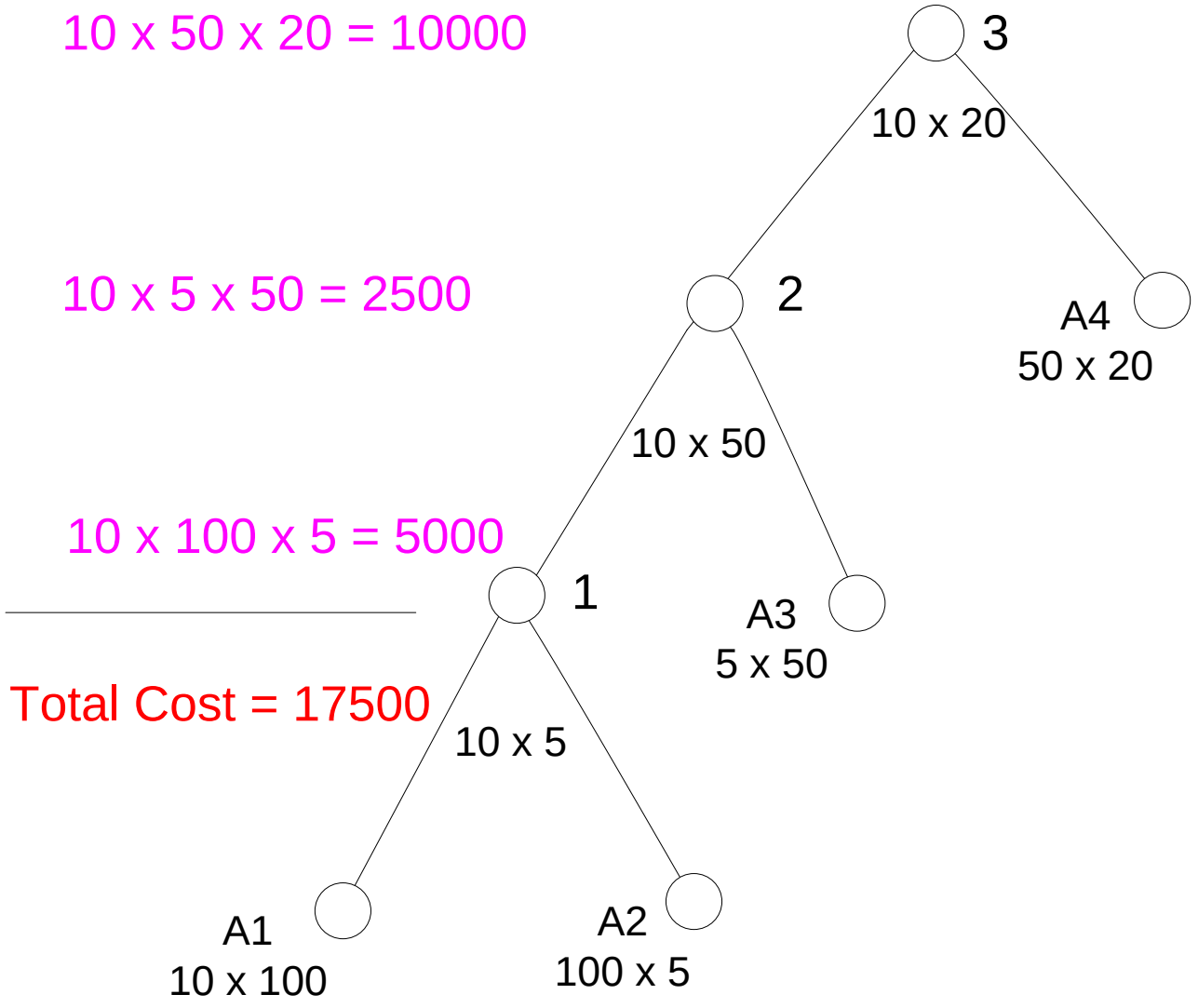
10 x 100 x 50 = 50000

100 x 5 x 50 = 25000

Total Cost = 85000



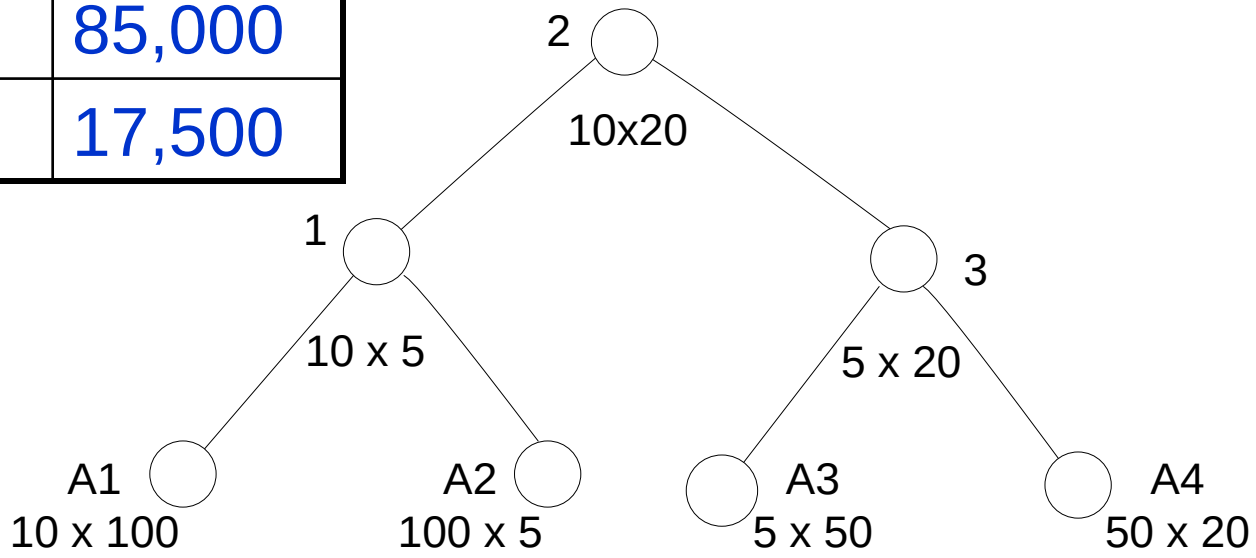
Fifth Chain : (((A1 · A2). A3). A4)



Chain Matrix Cost

First Chain	35,000
Second Chain	145,000
Third Chain	11,000
Fourth Chain	85,000
Fifth Chain	17,500

((A1 · A2). (A3 . A4))



Generalization of Brute Force Approach

If there is sequence of n matrices, $[A_1, A_2, \ldots, A_n]$

A_i has dimension p_{i-1} x p_i, where for $i = 1, 2, \ldots, n$

Find order of multiplication that minimizes number of scalar multiplications using brute force approach

Recurrence Relation: After k^{th} matrix, create two sub-lists, one with k and other with n - k matrices i.e.

$$(A_1 A_2 A_3 A_4 A_5 \ldots A_k)\ (A_{k+1} A_{k+2} \cdots A_n)$$

Let P(n) be the number of different ways of parenthesizing n items

$$P(n) = \begin{cases} 1 & \text{if } n = 1, \\ \sum_{k=1}^{n-1} P(k)P(n-k) & \text{if } n \geq 2 \end{cases}$$

Generalization of Brute Force Approach

If n = 2

P(2) = P(1).P(1) = 1.1 = 1

If n = 3

P(3) = P(1).P(2) + P(2).P(1) = 1.1 + 1.1 = 2

$(A_1 A_2 A_3) = ((A_1 . A_2). A_3)$ OR $(A_1 . (A_2. A_3))$

If n = 4

P(4) = P(1).P(3) + P(2).P(2) + P(3).P(1) = 1.2 + 1.1 + 2.1 = 5

$$P(n) = \begin{cases} 1 & \text{if } n = 1, \\ \sum_{k=1}^{n-1} P(k)P(n-k) & \text{if } n \geq 2 \end{cases}$$

Why Brute Force Approach not Economical

This is related to a famous function in combinatorics called the Catalan numbers.

Catalan numbers are related with the number of different binary trees on n nodes.

$P(n) \in (4^n / n^{3/2})$

The dominating term is the exponential 4^n thus P(n) will grow large very quickly.

And hence this approach is not economical.

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Note:

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Order of A_2 is $p_1 \times p_2$,

Order of A_3 is $p_2 \times p_3$, etc.

Order of $A_1 \times A_2 \times A_3$ is $p_0 \times p_3$,

of $A_1 \times A_2 \times \ldots \times A_n$ is $p_0 \times p_n$

Dynamic Programming Solution

Why Dynamic Programming in this problem?

- Problem is of type optimization
- Sub-problems are dependent
- Optimal structure can be characterized and
- Can be defined recursively
- Solution for base cases exits
- Optimal solution can be constructed
- Hence here is dynamic programming

Dynamic Programming Formulation

- Let $A_{i..j} = A_i \cdot A_{i+1} \cdots A_j$
- Order of $A_i = p_{i-1} \times p_i$, and
- Order of $A_j = p_{j-1} \times p_j$,
- Order of $A_{i..j}$ = rows in A_i x columns in $A_j = p_{i-1} \times p_j$
- At the highest level of parenthesisation,
- $A_{i..j} = A_{i..k} \times A_{k+1..j}$ $\quad i \leq k < j$
- Let m[i, j] = minimum number of multiplications needed to compute $A_{i..j}$, for $1 \leq i \leq j \leq n$
- Objective function = finding minimum number of multiplications needed to compute $A_{1..n}$

i.e. to compute m[1, n]

Mathematical Model

$A_{i..j} = (A_i . A_{i+1} \cdots A_k) . (A_{k+1} . A_{k+2} \cdots A_j) = A_{i..k} \times A_{k+1..j}$
$i \leq k < j$

Order of $A_{i..k} = p_{i-1} \times p_k$, and order of $A_{k+1..j} = p_k \times p_j$,

m[i, k] = minimum number of multiplications needed to compute $A_{i..k}$

m[k+1, j] = minimum number of multiplications needed to compute $A_{k+1..j}$

Mathematical Model

$$m[i, i] = 0$$

$$m[i, j] = \min_{i \leq k < j} (m[i, k] + m[k + 1, j] + p_{i-1} p_k p_j)$$

Example: Dynamic Programming

Problem: Compute optimal multiplication order for a series of matrices given below

$$\frac{A_1}{10\times100}\cdot\frac{A_2}{100\times5}\cdot\frac{A_3}{5\times50}\cdot\frac{A_4}{50\times20}$$

$P_0 = 10$

$P_1 = 100$

$P_2 = 5$

$P_3 = 50$

$P_4 = 20$

m[1,1]	m[1,2]	m[1,3]	m[1,4]
	m[2,2]	m[2,3]	m[2,4]
		m[3,3]	m[3,4]
			m[4,4]

Main Diagonal

$$m[i,i] = 0, \forall i = 1,\ldots,4$$

$$m[i,j] = \min_{i \le k < j} (m[i,k] + m[k+1,j] + p_{i-1} \cdot p_k \cdot p_j)$$

Main Diagonal

m[1, 1] = 0

m[2, 2] = 0

m[3, 3] = 0

m[4, 4] = 0

Computing m[1, 2], m[2, 3], m[3, 4]

$$m[i, j] = \min_{i \le k < j} (m[i, k] + m[k+1, j] + p_{i-1} \cdot p_k \cdot p_j)$$

$$m[1,2] = \min_{1 \le k < 2} (m[1, k] + m[k+1, 2] + p_0 \cdot p_k \cdot p_2)$$

$$m[1,2] = \min (m[1,1] + m[2,2] + p_0 \cdot p_1 \cdot p_2)$$

m[1, 2] = 0 + 0 + 10 . 100 . 5

= 5000

s[1, 2] = k = 1

Computing m[2, 3]

$$m[i, j] = \min_{i \le k < j} (m[i, k] + m[k+1, j] + p_{i-1} \cdot p_k \cdot p_j)$$

$$m[2,3] = \min_{2 \le k < 3} (m[2, k] + m[k+1,3] + p_1 \cdot p_k \cdot p_3)$$

$$m[2,3] = \min (m[2,2] + m[3,3] + p_1 \cdot p_2 \cdot p3)$$

m[2, 3] = 0 + 0 + 100 . 5 . 50

= 25000

s[2, 3] = k = 2

Computing m[3, 4]

$$m[i, j] = \min_{i \le k < j} (m[i, k] + m[k+1, j] + p_{i-1} \cdot p_k \cdot p_j)$$

$$m[3,4] = \min_{3 \le k < 4} (m[3, k] + m[k+1,4] + p_2 \cdot p_k \cdot p_4)$$

$$m[3,4] = \min (m[3,3] + m[4,4] + p_2 \cdot p_3 \cdot p_4)$$

m[3, 4] = 0 + 0 + 5 . 50 . 20

= 5000

s[3, 4] = k = 3

Computing m[1, 3], m[2, 4]

$$m[i, j] = \min_{i \le k < j} (m[i,k] + m[k+1, j] + p_{i-1} \cdot p_k \cdot p_j)$$

$$m[1,3] = \min_{1 \le k < 3} (m[1,k] + m[k+1,3] + p_0 \cdot p_k \cdot p_3)$$

$$m[1,3] = \min (m[1,1] + m[2,3] + p_0 \cdot p_1 \cdot p_3,$$

$$m[1,2] + m[3,3] + p_0 \cdot p_2 \cdot p_3))$$

m[1, 3] = min(0+25000+10.100.50, 5000+0+10.5.50)

= min(75000, 2500) = 2500

s[1, 3] = k = 2

Computing m[2, 4]

$$m[i, j] = \min_{i \le k < j} (m[i, k] + m[k+1, j] + p_{i-1} \cdot p_k \cdot p_j)$$

$$m[2,4] = \min_{2 \le k < 4} (m[2, k] + m[k+1,4] + p_1 \cdot p_k \cdot p_4)$$

$$m[2,4] = \min (m[2,2] + m[3,4] + p_1 \cdot p_2 \cdot p_4,$$

$$m[2,3] + m[4,4] + p_1 \cdot p_3 \cdot p_4))$$

m[2, 4] = min(0+5000+100.5.20, 25000+0+100.50.20)

= min(15000, 35000) = 15000

s[2, 4] = k = 2

Computing m[1, 4]

$$m[i, j] = \min_{i \le k < j} (m[i, k] + m[k+1, j] + p_{i-1} \cdot p_k \cdot p_j)$$

$$m[1,4] = \min_{1 \le k < 4} (m[1, k] + m[k+1, 4] + p_0 \cdot p_k \cdot p_4)$$

$$m[1,4] = \min (m[1,1] + m[2,4] + p_0 \cdot p_1 \cdot p_4,$$

$$m[1,2] + m[3,4] + p_0 \cdot p_2 \cdot p_4, m[1,3] + m[4,4] + p_0 \cdot p_3 \cdot p_4)$$

m[1, 4] = min(0+15000+10.100.20, 5000+5000+
10.5.20, 2500+0+10.50.20)

= min(35000, 11000, 35000) = 11000

s[1, 4] = k = 2

Final Cost Matrix and Its Order of Computation

Final Cost Matrix

0	5000	2500	11000
	0	25000	15000
		0	5000
			0

Order of Computation

1	5	8	10
	2	6	9
		3	7
			4

K,s Values Leading Minimum m[i, j]

0	1	2	2
	0	2	2
		0	3
			0

Representing Order using Binary Tree

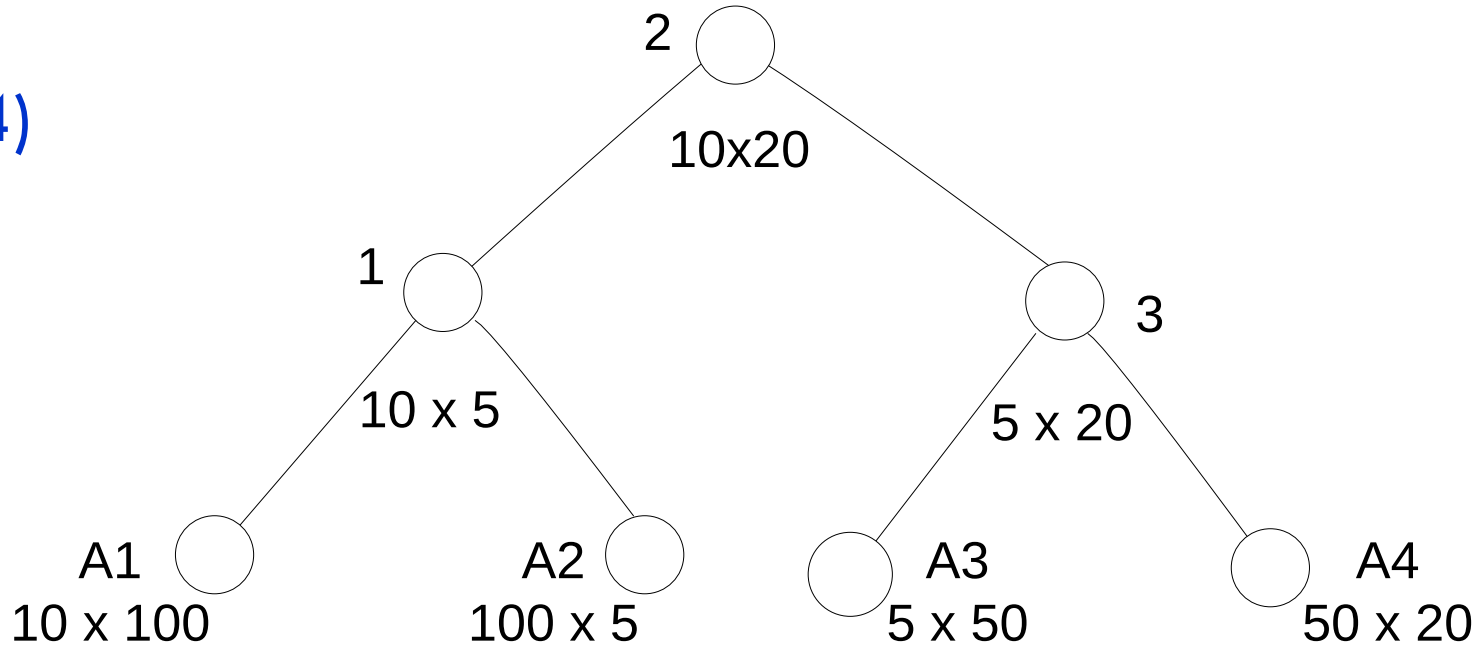
The above computation shows that the minimum cost for multiplying those four matrices is 11000.

The optimal order for multiplication is

$((A_1 . A_2) . (A_3 . A_4))$

For, m(1, 4)

k = 2



Chain-Matrix-Order(p)

```
1.  n ← length[p] – 1
2.  for i ← 1 to n
3.      do m[i, i] ← 0
4.  for l ← 2 to n,
5.      do for i ← 1 to n-l+1
6.          do j ← i+l-1
7.              m[i, j] ← ∞
8.              for k ← i to j-1
9.                  do q ← m[i, k] + m[k+1, j] + p_{i-1} . p_k . p_j
10.                     if q < m[i, j]
11.                         then m[i, j] = q
12.                             s[i, j] ← k
```

m[1,1]	m[1,2]	m[1,3]	m[1,4]
	m[2,2]	m[2,3]	m[2,4]
		m[3,3]	m[3,4]
			m[4,4]

Computational Cost

$$T(n) = n + \sum_{i=1}^{n} \sum_{j=i+1}^{n} (j - i) = \sum_{i=1}^{n} \sum_{k=1}^{n-i} k$$

$$T(n) = n + \sum_{i=1}^{n} \frac{(n-i)(n-i+1)}{2}$$

$$T(n) = n + \frac{1}{2} \sum_{i=1}^{n} (n^2 - 2ni + i^2 + n - i)$$

$$T(n) = n + \frac{1}{2} (\sum_{i=1}^{n} n^2 - \sum_{i=1}^{n} 2ni + \sum_{i=1}^{n} i^2 + \sum_{i=1}^{n} n - \sum_{i=1}^{n} i)$$

Computational Cost

$$T(n) = n + \frac{1}{2}\left(\sum_{i=1}^{n} n^2 - \sum_{i=1}^{n} 2ni + \sum_{i=1}^{n} i^2 + \sum_{i=1}^{n} n - \sum_{i=1}^{n} i\right)$$

$$T(n) = n + \frac{1}{2}\left(n^2 \sum_{i=1}^{n} 1 - 2n \sum_{i=1}^{n} i + \sum_{i=1}^{n} i^2 + n \sum_{i=1}^{n} 1 - \sum_{i=1}^{n} i\right)$$

$$T(n) = n + \frac{1}{2}\left(n^2 . n - 2n . \frac{n(n+1)}{2} + \frac{n(n+1)(2n+1)}{6} + n . n - \frac{n(n+1)}{2}\right)$$

Computational Cost

$$T(n) = n + \frac{1}{2}(n^2.n - 2n.\frac{n(n+1)}{2} + \frac{n(n+1)(2n+1)}{6} + n.n - \frac{n(n+1)}{2})$$

$$T(n) = n + \frac{1}{2}(n^3 - n^2(n+1) + \frac{n(n+1)(2n+1)}{6} + n^2 - \frac{n(n+1)}{2})$$

$$T(n) = n + \frac{1}{12}(6n^3 - 6n^3 - 6n^2 + 2n^3 + 3n^2 + n + 6n^2 - 3n^2 - 3n)$$

$$T(n) = \frac{1}{12}(12n + 2n^3 - 2n) = \frac{1}{12}(10n + 2n^3) = \frac{1}{6}(5n + n^3)$$

Cost Comparison Brute Force Dynamic Programming

Dynamic Programming

There are three loop

The most two loop for i, j, satisfy the condition: 1 ≤ i ≤ j ≤ n

Cost = ${}^{n}C_2 + n = n(n-1)/2 + n = \Theta(n^2)$

The third one most inner loop for k satisfies the condition, i ≤ k < j, in worst case, it cost n and

Hence total cost = $\Theta(n^2 . n) = \Theta(n^3)$

Brute Force Approach

$P(n) = C(n - 1)\ C(n) \in (4^n/n^{3/2})$

Generalization: Sequence of Objects

Although this algorithm applies well to the problem of matrix chain multiplication

Many researchers have noted that it generalizes well to solving a more abstract problem

- given a linear sequence of objects
- an associative binary operation on those objects hold
- the objective to find a way to compute the cost of performing that operation on any two given objects
- and finally computing the minimum cost for grouping these objects to apply the operation over the entire sequence.

It is obvious that this problem can be solved using chain matrix multiplication, because there is a one to one correspondence between both problem

Generalization: String Concatenation

One common special case of chain matrix multiplication problem is string concatenation.

For example, we are give a list of strings.

- The cost of concatenating two strings of length m and n is for example O(m + n)
- Since we need O(m) time to find the end of the first string and O(n) time to copy the second string onto the end of it.
- Using this cost function, we can write a dynamic programming algorithm to find the fastest way to concatenate a sequence of strings
- It is possible to concatenate all in time proportional to sum of their lengths, but here we are interested to link this problem with chain matrix multiplication

Generalization: Parallel Processors

Another generalization is to solve the problem when many parallel processors are available.

In this case, instead of adding the costs of computing each subsequence, we just take the maximum, because we can do them both simultaneously.

This can drastically affect both the minimum cost and the final optimal grouping

But of course more balanced groupings that keep all the processors busy is more favorable solution

There exists some more sophisticated approaches to solve this problem

Conclusion

Created some notations to describe mathematical model of the chain matrix multiplication problem

A recursive model was described

Based on this model dynamic programming algorithm was designed

An example was taken for applying model to solve dynamically, constructing optimal solution based on the given information

Time complexity was computed for the Algorithm

Applications of chain matrix problem are discussed