Direction-of-arrival estimation with efficient graph spatial spectrum construction

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***Abstract*—Existing graph signal processing (GSP) based direction-of-arrival (DOA) estimation methods suffer from high computational complexity due to the way of graph spatial spectrum construction. In this paper, an efficient GSP-based DOA estimation method is proposed by incorporating a new cost function to construct the graph spatial spectrum. By performing the graph Fourier transform on the received signals that are represented with a weighted directed graph, the graph frequency spectrum are obtained and analyzed, from which a cost function is developed allowing efficient graph spatial spectrum construction. Then, the DOA is estimated by searching for peaks in the constructed graph spatial spectrum. Simulation results demonstrate improved estimation accuracy and efficiency compared to existing GSP-based DOA methods.**

***Index Terms*—DOA estimation, Graph signal processing, Graph Fourier transform, Graph spatial spectrum.**

I. Introduction

Graph signal processing (GSP) has now been a well-established field in signal processing [1], [2]. As a generalization of conventional discrete Fourier transform (DFT), graph Fourier transform (GFT) establishes a relationship between the signal vertex domain and the graph spectral domain [3], [4], and enables different graph Fourier basis derived from different graph topologies rather than fixed Fourier basis defined by the frequency to perform the transform [5]. Thus, GFT provides a flexible way to exploit the relationship among the signals on different sensors, making it a promising tool to realize good performance in complex signal processing scenarios. As a result, GSP and in particular DFT have been applied to different areas [6], [7], and most recently in array signal processing [8], [9].

As a canonical problem in array signal processing, direction-of-arrival (DOA) estimation has been widely studied in various applications, such as radar, sonar, wireless communications, etc [10]–[14]. With increasing complexity of the array system, such as space-air-ground-sea integrated network and distributed array system, conventional DOA estimation methods are facing new challenges in terms of both computational complexity and performance bottleneck [15]–[19].

Nowadays, DOA estimation with GSP is still at the early stage, and existing works mainly focus on exploring the basic GSP application with classic array structures. To name a few, a phase shift graph is introduced to express the relationship among different snapshots [9], where both the signal waveforms and the noises are sampled from stationary Gaussian stochastic processes and the phase relationship between the adjacent two snapshots is implicit. A joint temporal and spatial graph to express the multi-snapshots received signals by using Kronecker product in [20]. However, as the number of snapshots increases, the dimension of the adjacent matrix corresponding to such a graph increases accordingly, such that obtaining the graph Fourier basis from eigendecomposition of the adjacency matrix suffers from high computational complexity. Moreover, existing methods normally estimate the DOA from a peak searching process in graph spatial spectrum, which is constructed by the eigenvectors corresponding to noises, making the graph spatial spectrum sensitive to the effect of noises.

In this work, we further explore the GSP methodology and propose an efficient DOA estimation method by formulating the array manifold on a weighted directed graph, with the graph Fourier basis obtained by calculating the eigenvector of the corresponding adjacency matrix. Thus, GFT can be performed on the received signals to enable analysis in the graph frequency domain. According to the characteristic of the graph frequency spectrum with the matched graph Fourier basis and the received signal, we design a new cost function to construct the graph spatial spectrum, where the DOA is obtained by searching for peaks in the graph spatial spectrum. Benefiting from the new cost function, there is less noise interference in the obtained graph spatial spectrum compared with the existing GSP-based DOA estimation methods, leading to higher estimation accuracy. Moreover, the new method yields lower computational complexity over the existing methods. Advantages of the proposed method over existing GSP-based DOA estimation methods are demonstrated by computer simulations in terms of both estimation accuracy and computational complexity.

II. Fundamental And Model

Assume that there are N far-field narrowband signals impinging on a uniform linear array (ULA) from directions $\{\theta_0, \cdots, \theta_{N-1}\}$. The ULA consists of M sensors with half-wavelength inter-element spacing ($N < M$). The observation of the m-th sensor at time t is modeled as

$$x_m(t) = \sum_{i=0}^{N-1} s_i(t) e^{-j\tau_m(\theta_i)} + n_m(t) \quad m = 0, \cdots, M-1, \tag{1}$$

where $s_i(t)$ is the waveform of the i-th source, $\tau_m(\theta_i) = m\pi \sin\theta_i$ denotes the phase difference between the first sensor (i.e., sensor 0) and the m-th sensor, and $n_m(t)$ is the zero-mean additive white Gaussian noise on the m-th sensor. Thus, the array observation is modeled in a vector form as

$$\mathbf{x}(t) = \mathbf{A}(\boldsymbol{\theta})\mathbf{s}(t) + \mathbf{n}(t), \tag{2}$$

where $\mathbf{x}(t) = [x_0(t), \cdots, x_{M-1}(t)]^{\mathrm{T}}$ contains the observation of M sensors. $\mathbf{A}(\boldsymbol{\theta}) = [\boldsymbol{a}(\theta_0), \cdots, \boldsymbol{a}(\theta_{N-1})]$ is the steering matrix with

$$\boldsymbol{a}(\theta_i) = \left[1, e^{-j\tau_1(\theta_i)}, \cdots, e^{-j\tau_{M-1}(\theta_i)}\right]^{\mathrm{T}} \quad i = 0, \cdots, N-1, \tag{3}$$

and $\mathbf{s}(t) = [s_0(t), \cdots, s_{N-1}(t)]^{\mathrm{T}}$ contains the sources waveforms, and $\mathbf{n}(t)$ consists of the noise. Here, $[\cdot]^{\mathrm{T}}$ denotes the transpose.

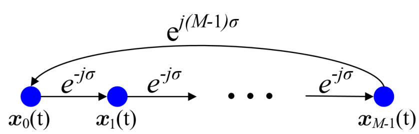


Fig. 1. Representation of a ULA with the weighted directed graph.

Different from the conventional signal processing based on the covariance matrix, we adopt a weighted directed graph to describe the ULA manifold [2] in Fig. 1. To be more specific, each vertex denotes a sensor, and the weight of each edge reflects the ideal phase difference of the received signals between the corresponding sensors. Here, $\sigma = \pi \sin\tilde{\theta}$, where $\tilde{\theta} \in [-90°, 90°]$ is the grid from a DOA dictionary. Accordingly, the adjacency matrix of the graph is

$$\mathbf{W} = \begin{bmatrix} \mathbf{0}_{M-1} & e^{j(M-1)\sigma} \\ e^{-j\sigma}\mathbf{I}_{M-1} & \mathbf{0}_{M-1}^{\mathrm{T}} \end{bmatrix}, \tag{4}$$

where $\mathbf{I}_{M-1}$ is a $(M-1)$-dimensional identity matrix, $\mathbf{0}_{M-1}$ is $(M-1)$-dimensional all-zero vector.

Similar to the DFT that utilizes a fixed orthonormal basis, GFT depends on the orthonormal basis consisting of the eigenvector of the adjacency matrix given the specific graph topology. Hence, The adjacency matrix in Eq. (4) enables us to adopt the GFT for DOA estimation.

III. Proposed Algorithm

After representing the signal with the graph, we adopt GFT to convert the signal to the graph frequency domain. To be more specific, by performing eigendecomposition on the matrix in Eq. (4), we have

$$\mathbf{W} = \mathbf{V}^{-1}\boldsymbol{\Sigma}\mathbf{V}, \tag{5}$$

where $\boldsymbol{\Sigma}$ is a diagonal matrix composed of eigenvalues, and $\mathbf{V}$ consists of eigenvectors.

Obviously, $\mathbf{V}$ is a unitary matrix due to the given graph topology, such that its column vectors constitute a set of orthonormal basis in the inner product space. Thus, the GFT matrix is given as $\mathbf{V}^{-1}$, and the graph frequency spectral of the received signal $\mathbf{x}$ is accordingly given as

$$\mathbf{x}_f = \mathbf{V}^{-1}\mathbf{x}. \tag{6}$$

For the single source case, we plot several graph frequency spectrum in Fig. 2. Obviously, only when $\tilde{\theta}$ in the graph matches the true DOA θ, i.e., $\tilde{\theta} = \theta$, there is a peak in the graph frequency spectral. Besides, compared with the graph frequency spectral of noiseless signals, the noise brings interference on all the graph frequency components. The reason is that, in noiseless scenario,

$$\dot{\boldsymbol{x}} = \mathbf{W}\dot{\boldsymbol{x}} \tag{7}$$

holds for $\tilde{\theta} = \theta$, where $\dot{\boldsymbol{x}} = \boldsymbol{a}(\theta)s$, such that $\dot{\boldsymbol{x}}$ is the eigenvector of matrix $\mathbf{W}$ corresponding to the unit eigenvalue and is orthogonal to other eigenvectors. Thus, only the eigenvector $\dot{\boldsymbol{x}}$ leads to non-zero graph frequency component while other eigenvectors bring all-zero frequency components in the graph frequency spectral. When the noise exists, other graph frequency components arise the non-zero values because none of the eigenvectors is strictly orthogonal to the noise. On the other hand, when $\tilde{\theta} \neq \theta$, Eq. (7) does not hold, i.e., there is no explicit relationship between the received signal and the eigenvectors, such that the power in the graph frequency spectral is not concentrated on certain frequency component and is distributed in all components in average.

In multi-sources case, when $\tilde{\theta}$ matches one of the true DOAs, e.g., $\tilde{\theta} = \theta_1$, the received signal from other DOAs can be treated as the interference, such that the peak in the graph frequency spectral still appears. Thus, the analysis for single-source case still holds for multi-sources case.

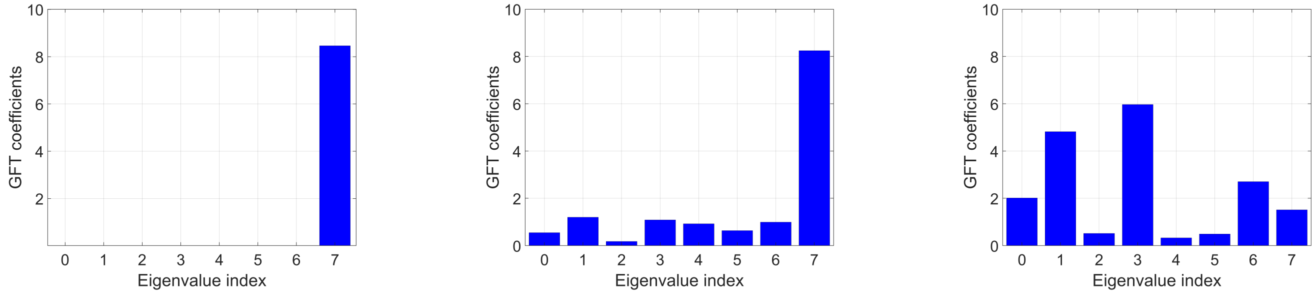
Benefiting from the different graph frequency spectrum for $\tilde{\theta} = \theta$ and $\tilde{\theta} \neq \theta$, we can construct a graph spatial spectrum and perform the peak searching process for DOA estimation, i.e.,

$$\hat{\theta} = \underset{\tilde{\theta}}{\operatorname{argmax}} f\left(\tilde{\theta}\right), \tag{8}$$

where

$$f\left(\tilde{\theta}\right) = \mathbf{v}\left(\tilde{\theta}\right)\mathbf{x}, \tag{9}$$

with $\mathbf{v}\left(\tilde{\theta}\right)$ denoting the eigenvector of $\mathbf{W}$ corresponding to the unit eigenvalue. According to the analysis of the graph frequency spectrum, we can see the eigenvector corresponding



(a) (b) (c)

Fig. 2. Graph frequency spectrum for single source case with the ULA $M = 8$. (a) $\tilde{\theta} = \theta$ without noise, (b) $\tilde{\theta} = \theta$ with noise, (c) $\tilde{\theta} \neq \theta$ with noise.

to unit eigenvalue plays a key role for DOA estimation. Thus, compared with the existing work [20] that uses other graph frequency components to formulate the graph spatial spectrum for DOA estimation, the proposed method suffers from less interference of noises. More specifically, the proposed method only utilizes one graph frequency component to formulate the graph spatial spectrum, while the existing work in [20] uses other $M-1$ graph frequency components. Moreover, the proposed method is more efficient than the existing work in [20] because we only use one graph frequency component in graph spatial spectrum calculation.

For the array observations with K snapshots, the DOA can be obtained by

$$\hat{\theta} = \underset{\tilde{\theta}}{\operatorname{argmax}}\, F\left(\tilde{\theta}\right), \tag{10}$$

where

$$F\left(\tilde{\theta}\right) = \frac{1}{K}\sum_{k=0}^{K-1} f_k\left(\tilde{\theta}\right), \tag{11}$$

with $f_k\left(\tilde{\theta}\right)$ is the cost function result of the k-th snapshot.

Obviously, the proposed method decomposes the $(M \times M)$-dimensional adjacency matrix for single snapshot rather than decomposes a huge $(KM \times KM)$-dimensional adjacency matrix as existing work [20] does, which leads to lower computational complexity.

IV. Simulation Results

In this section, we evaluate the proposed GSP-based DOA estimation algorithm compared with the existing GSP-based methods. We adopt a ULA with $M = 5$ sensors, where the DOAs are $36.8°$ and $[0°, 36.8°]$ for single source case and multi-sources case, respectively. Angle $\tilde{\theta}$ spans the range $[-90°, 90°]$ at intervals of $0.05°$. Here, we adopt the unconditional model to generate the signal waveforms, i.e., the signal waveforms are generated from a zero-mean stationary Gaussian stochastic process. The root mean square error (RMSE) is selected as the metric in our simulation, defined as

$$\text{RMSE} = \sqrt{\frac{1}{\mathcal{L}N}\sum_{l=1}^{\mathcal{L}}\sum_{i=0}^{N-1}(\hat{\theta}_{l,i} - \theta_{l,i})^2}, \tag{12}$$

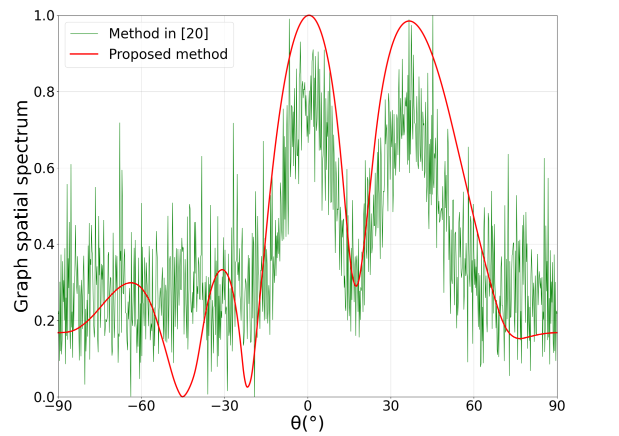
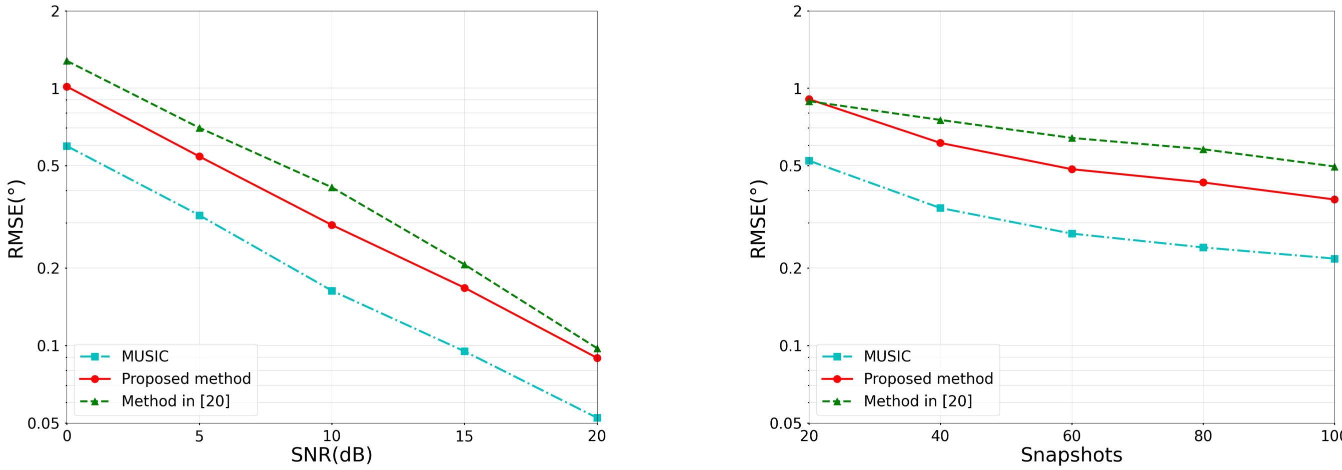


Fig. 3. Normalized graph spatial spectrum with $K = 512$ and SNR = 10 dB for two sources case.

where $\mathcal{L}$ is the number of Monte-Carlo trials, $\theta_{l,i}$ is the i-th angle in the l-th Monte-Carlo trial and $\hat{\theta}_{l,i}$ is its estimate. Here, $\mathcal{L} = 1000$ for each data point.

First, we plot the normalized graph spatial spectrum of the proposed algorithm and the existing work [20] under multi-sources case in Fig. 3, where the number of snapshots is 512 and the SNR for both sources is 10 dB. It can be observed that, the graph spatial spectrum of the proposed algorithm is much clearer than the existing work [20], indicating that the proposed method is less sensitive to noise. The reason is that, we use single graph frequency component rather than $M-1$ frequency components to construct the graph spatial spectrum.

Then, we evaluate the estimation accuracy of the proposed algorithm in single-source case with different SNR and different number of snapshots, where the existing GSP-based work [20] and the MUSIC algorithm is plotted as reference. It is seen from Fig. 4(a) and Fig. 4(b) that, with SNR and the number of snapshots increasing, the proposed algorithm outperforms the existing work [20], indicating that the proposed method is the most accurate algorithm in GSP-based DOA estimation algorithm. Although the accuracy of the proposed algorithm is not higher than well-accepted MUSIC algorithm, we have explored the basic idea of GSP for



(a) (b)

Fig. 4. RMSE of the proposed method, MUSIC and existing method in [20]. (a) RMSE versus SNR for single source case with K = 50. (b) RMSE versus snapshots for single source case with SNR = 5 dB.

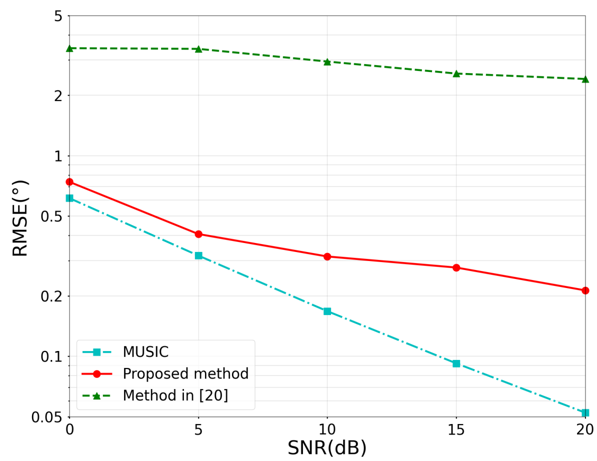


Fig. 5. RMSE versus SNR for two sources case with $K = 50$.

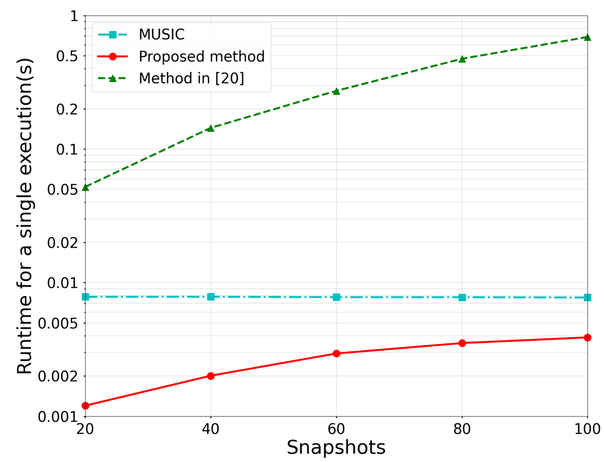


Fig. 6. Time consumption to run DOA estimation once versus number of snapshots.

DOA estimation and proposed a better graph spatial spectrum construction, which contributes to the GSP methodology for DOA estimation in more complex scenarios.

In addition to single source case, we also evaluate the estimation accuracy of the proposed method in multi-sources case under different SNRs. In Fig. 5, the RMSE of the proposed method is much better than that of the existing work [20] for all SNRs, and is comparable to MUSIC algorithm in low SNR situation. However, the gap between the proposed method and the MUSIC algorithm becomes larger as the SNR increases, indicating that the interference of the signal from unmatched DOA deteriorates the estimation. Finally, we compare the computational complexity of the proposed method, the existing work [20], and the MUSIC algorithm. In Table I, we demonstrate the run time of these three algorithms, where the number of snapshots is 20. For fair comparison, we calculate the matrices/vectors that can be pre-stored for all of these three methods off line, to accelerate the estimation as much as possible. From Table I, the proposed method took 1.302 seconds to run 1000 times, with a speed about 6 times that of MUSIC and almost 50 times that of the method in [20]. The reason is that, we only use a single eigenvector to calculate each point of the graph spatial spectrum, i.e., each point is calculated via a single multiplication among the eigenvector and the received signals.

TABLE I
TIME CONSUMPTION TO RUN DOA ESTIMATION 1000 TIMES FOR THREE ALGORITHMS

	Proposed method	**MUSIC**	**Method in [20]**
Time (s)	1.302	7.866	56.944

Further, we evaluate the run time with different number of snapshots in Fig. 6. It is observed that, the run time of the proposed method is shorter than that of the existing work [20]. Besides, the run time of the proposed algorithm grows with the number of snapshots increasing, but always lower than the MUSIC algorithm. Another observation is that, the run time of the existing work [20] grows faster than that of the proposed algorithm, indicating that the run time of the existing work is more sensitive to the number of snapshots.

V. CONCLUSIONS

In this paper, an efficient GSP-based DOA estimation method has been proposed. By representing the array received signals with a weighted directed graph and deriving the graph Fourier basis from the eigenvectors of the adjacency matrix, we perform GFT analysis of received signals in the graph frequency domain. Leveraging the characteristics of the graph frequency spectrum, we proposed a novel cost function that constructs the graph spatial spectrum, where the DOA is estimated by a peak searching process. Simulation results demonstrate that the proposed outperforms existing GSP-based DOA estimation method in terms of both estimation accuracy and computational complexity. In future work, we will further investigate the GSP-based DOA estimation methods in more complex scenarios beyond typical array configurations.

ACKNOWLEDGEMENT

This work was supported in part by National Natural Science Foundation of China (No. U21A20456), and the Fundamental Research Funds for Central Universities (No. 226-2023-00111, 226-2024-00004).

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