Linear regression: $y_{new} = \theta^T x_{new}$

Cost function: $$J(\theta) = \sum_{i=1}^{N} \left(y^{(i)} - h_\theta(x^{(i)})\right)^2$$

$$= \sum_{i=1}^{N} \left(y^{(i)} - \theta^T x^{(i)}\right)^2$$

also: $$J(\theta) = \| \underset{(N \times d+1)}{X} \; \underset{(d+1,1)}{\theta} - Y \|_2^2$$

$$\hat{\theta} = \arg\min_\theta J(\theta) \qquad \Downarrow \frac{\partial}{\partial \theta}$$

$$\Rightarrow \nabla J(\theta) = \frac{\partial J(\theta)}{\partial \theta} = 2X^T(X\theta - Y)$$

$$\Rightarrow X^T(X\theta - Y) = 0 \Rightarrow \boxed{\hat{\theta} = (X^T X)^{-1} X^T Y}$$

method I

method 2: (Gradient descent)

$$\theta^{(t)} = \theta^{(t-1)} - \rho \left.\frac{\partial J}{\partial \theta}\right|_{\theta^{(t-1)}}$$

ρ → learning rate

$$\Rightarrow \boxed{\theta^{(t)} = \theta^{(t-1)} - \rho\, X^T (X\theta^{(t-1)} - Y)}$$

method 3: stochastic Gradient descent:

X is a sample of original $[X]_{N \times d+1}$

$$X \Rightarrow \begin{bmatrix} \cdots & x & 1 \\ & & 1 \\ & & 1 \\ & & 1 \end{bmatrix}_{N(d+1)}$$

$$\theta = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_d \\ b \end{bmatrix}_{(d+1)\times 1}$$

Nonlinear features:

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_d \end{pmatrix} \Rightarrow \varphi(x) = \begin{pmatrix} x_1 \\ x_1^2 \\ x_2 \\ x_2^2 \\ 1 \end{pmatrix}$$

use $\varphi(x)$ instead of x

Generally: Cost function:

regularization: $$J(\theta) = \sum_{i=1}^{N} \ell\left(y^{(i)} - \theta^T x^{(i)}\right) + \lambda f(\theta)$$

where
- $\ell(e) = e^2$, $f(\theta) = \|\theta\|_2^2$ ⟶ Ridge
- $\ell(e) = e^2$, $f(\theta) = \|\theta\|_1$ ⟶ Lasso
- $\ell(e) = |e|$, ⟶ Robust

Ridge Regression:

$$\boxed{\hat{\theta}_\lambda = (X^T X + \lambda I)^{-1} X^T Y}$$

$$\frac{\partial J(\theta)}{\partial(\theta)} = 0$$

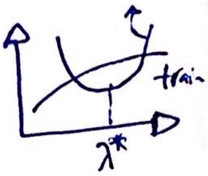
Set λ: K-fold cross validation

$D^{(1)}$ | $D^{(2)}$ | ... | $D^{(k)}$

every time one $D^{(i)}$ is hold-out set

rest is tr set. ...

find λ with min error:



Maximum Liklihood of observation

↓

Training

1) Coin problem:

$x \in \{0, 1\}$ $\quad P(x=1) = \theta$

$P(x=0) = 1-\theta$

$P(x) = \theta^x (1-\theta)^{1-x}$

$$\boxed{\max_\theta \log P(D|\theta) = \hat{\theta}_{ML}} \boxed{= \arg\max_\theta P(D|\theta)}$$

$$\ell = \log P(D|\theta) = \left(\sum_{i=1}^{N} x^{(i)}\right)\log\theta + \left(N - \sum_{i=1}^{N} x_i\right)\log(1-\theta)$$

$$\Rightarrow \frac{\partial L}{\partial \theta} = 0 \Rightarrow \boxed{\hat{\theta}_{ML} = \frac{\sum_{i=1}^{N} x_i}{N}}$$

2) linear regression:

$$P(y^{(i)} | x^{(i)}, \theta) = \mathcal{N}(y^{(i)}, \mu = x^{(i)T}\theta, \sigma^2)$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(y^{(i)} - x^{(i)T}\theta)^2 / 2\sigma^2}$$

$$\Rightarrow \max_\theta \log P(D|\theta) = \hat{\theta}_{ML}$$

$$\ell = \log P(D|\theta) \Rightarrow \hat{\theta} = \min_\theta \sum_{i=1}^{N} \left(y^{(i)} - \theta^T x^{(i)}\right)$$

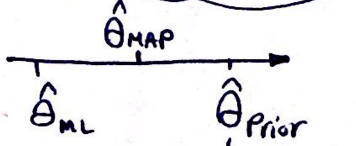
MAP Estimation (Baysian Est.)

$$\underbrace{P(\theta|D)}_{\text{Posterior}} \propto \underbrace{P(D|\theta)}_{\text{liklihood}}\ \underbrace{P(\theta)}_{\text{Prior}}$$

$$\underset{\substack{\text{Beta}\\ \alpha,\beta}}{P(\theta)} = \frac{\theta^{\alpha-1}(1-\theta)^{\beta-1}}{B(\alpha,\beta)} \Rightarrow \begin{cases} \text{mean} = \frac{\alpha}{\alpha+\beta} \\ \text{mode} = \frac{\alpha-1}{\alpha+\beta-2} \end{cases}$$

$$\hat{\theta}_{MAP} = \arg\max_{\theta} P(\theta|D)$$ → small N

$$\hat{\theta}_{ML} = \arg\max_{\theta} P(D|\theta)$$ → good for large N



$N \to 0 \Rightarrow \hat{\theta}_{MAP} = \hat{\theta}_{Prior}$

$N \to \infty \Rightarrow \hat{\theta}_{MAP} = \hat{\theta}_{ML}$

mode $P(\theta) = \frac{\alpha-1}{\alpha+\beta-2}$

for coin problem: $P(x) = \theta^{x}(1-\theta)^{1-x}$

$$\hat{\theta}_{MAP} = \frac{N}{N+\alpha+\beta-2}\underbrace{\left(\frac{\sum x_i}{N}\right)}_{\hat{\theta}_{ML}} + \frac{\alpha+\beta-2}{N+\alpha+\beta-2}\underbrace{\left(\frac{\alpha-1}{\alpha+\beta-2}\right)}_{\hat{\theta}_{Prior}}$$

Classification
- Generative Modeling: $P(x|y), P(y)$
- Discriminative Modeling: $P(y|x)$

1) Generative:

$$l(\theta) = \sum_{i=1}^{N} \log P(x^{(i)}, y^{(i)}|\theta) =$$

$$= \sum_{i=1}^{N} \log P(y^{(i)}|\theta) + \sum_{i=1}^{N} \log P(x^{(i)}|y^{(i)},\theta)$$

$$\Downarrow \qquad \hat{\theta}^{y}_{j} = \frac{\sum_{i=1}^{N} 1(y^{i}=j)}{\sum_{j=0}^{L-1}\sum_{i=1}^{N} 1(y^{i}=j)}$$

⇓ Naive base with Laplace Smoothing

$$\theta^{x_k|y}_{\bar{x}_k|j} = \frac{\left[\sum_{i}^{N} 1(y^{i}=j \wedge x_k^{i}=\bar{x}_k)\right]+t}{\left[\sum_{i}^{N} 1(y^{i}=j)\right]+mt}$$

Naier Baye: $P(x^{(i)}|y=j) = \prod_{l=1}^{k} P(x_l^{i}|y=j)$

2) Discriminative Modeling: logistic regression Model

$$P(y=1|x) = \frac{1}{1+e^{-w^T\phi(x)}}$$

$$P(y=0|x) = \frac{1}{1+e^{w^T\phi(x)}}$$

$$\Rightarrow w = \arg\max_{w} \underbrace{\sum_{i=1}^{N} \log P(y^{(i)}|x^{(i)},w)}_{l(w)}$$

$$\Rightarrow J(w) = -l(w)$$

$$\Rightarrow \frac{\partial J}{\partial w} = \sum_{i=1}^{N} -\phi(x^{(i)})\left[y^{(i)} - \frac{1}{1+e^{-\phi(x)^T w}}\right]$$

$$\frac{\partial J}{\partial w} = \sum -\phi(x^{(i)})\left[y^{(i)} - P(y^{(i)}=1|x^{(i)},w)\right]$$

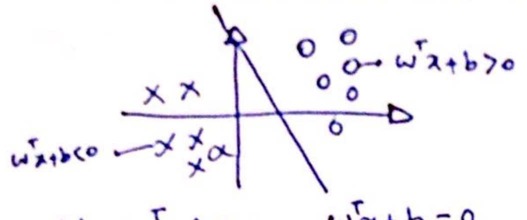
$$\Rightarrow \text{Gradient descent: } w^{t} = w^{t-1} - \rho\frac{\partial J}{\partial w}$$

Neural Network update:

$$\frac{\partial J}{\partial w_{ij}^{(l)}} = \frac{\partial J}{\partial z_i^{(l+1)}} \times \frac{\partial z_i^{(l+1)}}{\partial w_{ij}^{(l)}}$$

$$\frac{\partial J}{\partial b_i^{(l)}} = \frac{\partial J}{\partial z_i^{(l+1)}} \times \frac{\partial z_i^{(l+1)}}{\partial b_i^{(l)}}$$

SVM



$$g(w^Tx+b) = \begin{cases} +1 & w^Tx+b\geq 0 \\ -1 & w^Tx+b<0 \end{cases}$$

$\hat{\gamma}^{(i)} = y^{(i)}(w^Tx^{(i)}+b)$ → functional Margin

$\hat{\gamma} = \min_{i=1,\dots,N} \hat{\gamma}^{(i)}$ → overal functional margin

$\gamma^{(i)} = \left(\frac{w^Tx^{(i)}+b}{\|w\|_2}\right) y^{(i)}$ → geometric Margin

$\gamma = \min_{i=1,\dots,N} \gamma^{(i)}$ → ~~functional~~ overall geometric Margin

fix $\hat{\gamma}=1$

→ Primal SVM $\begin{cases} \min \|w\|_2^2 \\ \text{s.t. } y^{(i)}(w^Tx^{(i)}+b) \geq 1 \quad \forall i=1,2,\dots,N \end{cases}$

---

generally: $p^*: \begin{cases} \min f(w) \\ \text{s.t. } g_i(w) \leq 0 \quad i=1,2,\dots,k \\ h_i(w)=0, \quad i=1,2,\dots,L \end{cases}$

$\Rightarrow L(w,\alpha,\beta) \triangleq f(w) + \sum_{i=1}^{k} \underbrace{\alpha_i}_{\geq 0} g_i(w) + \sum_{i=1}^{L} \beta_i h_i(w)$

$\Rightarrow \max_{\beta,\,\alpha\geq 0} L(w,\alpha,\beta) = f(w)$ if (feasible) $g_i \leq 0$, $h_i=0$

$\Rightarrow \min_w f(w) = \min_w \max_{\alpha\geq 0,\beta} L(w,\alpha,\beta)$

$f_i(w)$, $g_i(w)$ Convex, $h_i(w)$ affine

$\Rightarrow \max_{\alpha\geq 0,\beta} \min_w L = \min_w \max_{\alpha\geq 0,\beta} L(w,\alpha,\beta)$

$\Rightarrow \boxed{\min_w f(w) = \max_{\alpha\geq 0,\beta} \min_w L(w,\alpha,\beta)}$

and $\frac{\partial L}{\partial w}=0$ $\quad \frac{\partial L}{\partial \alpha}=0$

$\frac{\partial L}{\partial \beta}=0$ $\quad \alpha_i g_i(w)=0$

$g_i(w)\leq 0$ $\quad h_i(w)=0$

back to SVM primal:

$\Rightarrow \frac{\partial L}{\partial w}=0 \Rightarrow w^* = \sum_{i=1}^{N} \alpha_i^* y^{(i)} x^{(i)}$

$\Rightarrow$ Dual SVM $\max_{\alpha_1,\alpha_2,\dots,\alpha_N} -\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} \alpha_i\alpha_j \langle x^{(i)}, x^{(j)}\rangle + \sum_{i=1}^{N}\alpha_i$

s.t. $\begin{cases} \alpha_i \geq 0 \quad \forall i=1,\dots,N \\ \sum_{i=1}^{N} \alpha_i y^i = 0 \end{cases}$

$$b^* = -\frac{\max_{y^i=-1} w^{*T}x^i + \min_{y^i=1} w^{*T}x^j}{2}$$

So for Test:

$g(x) = \text{sgn}(w^{*T}x+b^*)$ or $K(x^{(i)},x)$

or $g(x) = \text{Sgn}\left(\sum_{i=1}^{N}\alpha_i^* y^i \langle x^i, x\rangle + b^*\right)$

Kernels: $K(\text{Point1}, \text{Point2}) = \text{number}$

$K(x,z) = \langle \varphi(x), \varphi(z)\rangle = \varphi^T(x)\varphi(z)$

* if $\varphi(x) = \begin{pmatrix} x_1^2 \\ x_1x_2 \\ \vdots \\ x_k^2 \end{pmatrix} \rightarrow K(x,z) = (x^Tz)^2$

* if $\varphi(x) = \begin{pmatrix} x_1^n \\ x_1^{n-1}x_2 \\ \vdots \\ x_k^n \end{pmatrix}$ → all monomials of degree n

$\Rightarrow K(x,z) = (x^Tz)^n$

*if $\varphi(x) = \begin{pmatrix} x_1^n \\ x_1^{n-1}x_2 \\ \vdots \\ x_1 \\ x_2 \\ x_k \\ 1 \end{pmatrix}$ → all monomials of deg. n or less.

$K(x,z) = (x^Tz+c)^n$

*if $\varphi(x) = \infty$ dim $\Rightarrow K(x,z) = e^{-\|x-z\|_2^2/2\sigma^2}$

So No need to calculate $\varphi(x)$, $\varphi(z)$

Kernel Matrix: for $D = \{(x^{(i)}, y^{(i)})\}_{i=1}^{N}$

$\begin{bmatrix} K_{11} & \cdots & K_{1N} \\ K_{N1} & \cdots & K_{NN} \end{bmatrix} \rightarrow K_{ij} = K(x^{(i)}, x^{(j)})$

K is kernel Matrix iff&only if $\begin{cases} 1)\ K = K^T \\ 2)\ K \text{ is PSD} \\ z^TKz \geq 0 \ \forall z \end{cases}$

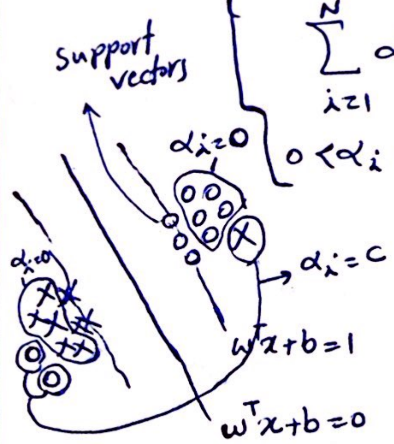
SVM - Soft Margin:

Primal: Soft Margin

$$\begin{cases} \min_{(w,b)} \ \frac{1}{2}\|w\|_2^2 + C\sum_{i=1}^{N} \xi_i \\ \text{s.t. } y^{(i)}(w^T x^{(i)} + b) \geq 1 - \xi_i \quad \forall i = 1,\dots,N \\ \xi_i \geq 0, \quad \forall i = 1,\dots,N \end{cases}$$

Dual: Soft Margin

$$\begin{cases} \max_{\alpha_1,\dots,\alpha_N} \ \sum_{i=1}^{N}\alpha_i - \frac{1}{2}\sum y^i y^j K(x^i, x^j)\alpha_i\alpha_j \\ \text{s.t. } 0 \leq \alpha_i \leq C, \quad \forall i = 1,\dots,N \\ \sum_{i=1}^{N}\alpha_i y^i = 0 \end{cases}$$



$0 < \alpha_i < C \Rightarrow y^i(w^T x^i + b) = 1$

$$* \begin{cases} y^i(w^T x^i + b) > 1 \Rightarrow \alpha_i = 0 \\ y^i(w^T x^i + b) < 1 \Rightarrow \alpha_i = C \end{cases}$$

Ensemble Method:

$x \to \varphi(x)$ causes overfitting
↳ ensemble methods are great!

Boosting:

* use weak classifier (h)

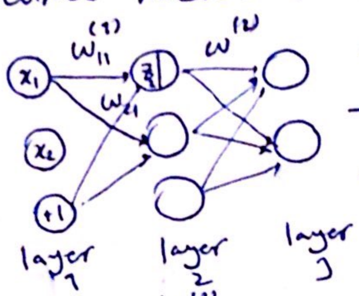
(1) For $t = 1,\dots,T$:
(2) if $t = 1$ then:
(3) Initialize weights $D_{t,i} = \frac{1}{N}$
(4) else
(5) set weights to $D_{t,i} \propto D_{t-1,i}\, e^{-\alpha_{t-1} y^{(i)} h_{t-1}(x^{(i)})}$
(6) End if
(7) Train a hypothesis h_t using D_t
(8) Evaluate Training Error $\varepsilon_t = \sum_{i=1}^{N} D_{t,i}\, 1(h_t(x^{(i)}) \neq y^{(i)})$
(9) Set $\alpha_t = \frac{1}{2}\ln\frac{1-\varepsilon_t}{\varepsilon_t}$
(10) End for
(11) $H(x) = \text{sign}\left(\sum_{t=1}^{T}\alpha_t h_t(x)\right) \Rightarrow$ ~~final~~ final classifier

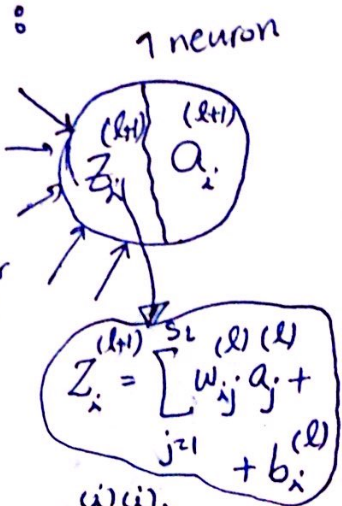
Bagging: ① No weight, random sampling with replacement
② uniform weight for hypothesis: $\alpha_t = \frac{1}{T}$

in Boosting: (1) $\varepsilon_t \to 0 \Rightarrow \alpha_t \to \infty$, $\varepsilon_t = 0.5 \Rightarrow \alpha_t \to 0$

(2) if $\begin{cases} h_{t-1}(x^{(i)})\, y^{(i)} > 0 \xrightarrow{\text{correct}} D_{t-1,i} \times \text{small weight} \\ h_{t-1}(x^{(i)})\, y^{(i)} < 0 \xrightarrow{\text{incorrect}} D_{t-1,i} \times \text{large weight} \end{cases}$

Neural Network:





$$J_\lambda(w,b) = \frac{1}{N}\sum_{i=1}^{N} J(w,b,x^{(i)},y^{(i)}) + \frac{\lambda}{2}\sum_{l=1}^{n-1}\|w^{(l)}\|_F^2$$

where $J(w,b,x^{(i)},y^{(i)}) = \|h_{w,b}(x^{(i)}) - y^{(i)}\|_2^2$

$\Rightarrow$ Gradient Descent:

$$w^{(l)} \leftarrow w^{(l)} - \alpha \left.\frac{\partial J_\lambda}{\partial w}\right|_{w^{(l)}}$$

$$b^{(l)} \leftarrow b^{(l)} - \alpha \left.\frac{\partial J_\lambda}{\partial w}\right|_{b^{(l)}}$$

Forward Propagation: $\{w^{(l)}, b^{(l)}\}_{l=1}^{n-1}$ fixed

$$\begin{cases} Z^{(l+1)} = w^{(l)} a^{(l)} + b^{(l)} \\ a^{(l+1)} = f(Z^{(l+1)}) \end{cases} ; \quad \forall l = 1,2,\dots,n-1$$

Back Propagation: $\{Z^{(l)}, a^{(l)}\}_{l=1}^{n}$ Computed & fixed

$\delta^{(n)} = (a^{(n)} - y) \odot f'(Z^{(n)})$

For $l = n-1, n-2, \dots, 2, 1$

① Set: $\delta^{(l)} = (w^{(l)T}\delta^{(l+1)}) \odot f'(Z^{(l+1)})$

② $\left.\frac{\partial J}{\partial w}\right|_{w^{(l)}} = \delta^{(l+1)} a^{(l)T}$

③ $\left.\frac{\partial J}{\partial b}\right|_{b^{(l)}} = \delta^{(l+1)}$

Update w, b:

$$w^{(l)} \leftarrow w^{(l)} - \alpha\left[\frac{1}{N}\left.\frac{\partial J}{\partial w}\right|_{w^{(l)}} + \lambda w^{(l)}\right]$$

$$b^{(l)} \leftarrow b^{(l)} - \alpha\left[\frac{1}{N}\left.\frac{\partial J}{\partial b}\right|_{b^{(l)}}\right]$$

Principal Componet Analysis (PCA)

(1) input : $D = \{x^{(i)}\}_{i=1}^{N} = [X]_{n\times N}$

(2) Output : $[U]_{n\times d}$, $[\mu]_{n\times 1}$, $[y]_{d\times N}$

(3) Compute $[\bar{x}] = \frac{1}{N}\sum_{i=1}^{N} x^{(i)}$

(4) Form : $X = [x^{(1)}-\bar{x}, x^{(2)}-\bar{x}, \dots x^{(N)}-\bar{x}]$

(5) Compute SVD of X

$$X = U_x \Sigma_x V_x^T \qquad U_x \in \mathbb{R}^{n\times n}$$

(6) Optimal $U_{n\times d} = U_x(:, 1:d)$

(7) Principal Componets : $[y] = U^T(\underbrace{x-\bar{x}}_{\text{original Data}})$

$$\text{PCA optz}\begin{cases} \min \frac{1}{2}\sum_{i=1}^{N} \|x^{(i)} - Uy^{(i)} - \mu\|_2^2 \\ \text{s.t. } \sum_{i=1}^{N} y^{(i)} = 0 \ , \ UU^T = I_d \end{cases}$$

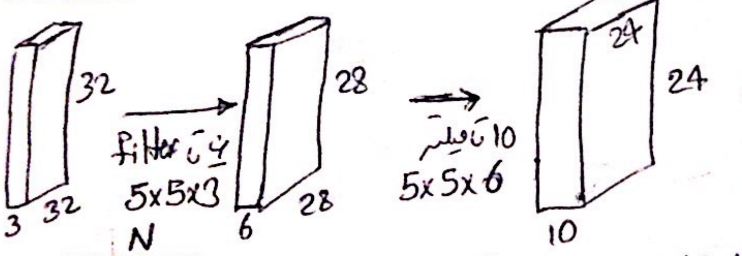
$\Rightarrow$ Then we write $L(\dots)$

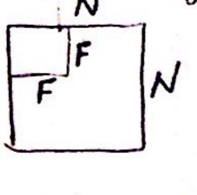
Then $\frac{\partial L}{\partial \mu} = 0 \Rightarrow \mu = \frac{1}{N}\sum_{i=1}^{N} x^{(i)}$

$\frac{\partial L}{\partial y_i} = 0 \Rightarrow y^{(i)} = U^T(x^{(i)} - \bar{x})$

Plug in $y^{(i)}$ & μ, ... optz $\Rightarrow$ ✓✓

Convolutional Neural Network





▲ output size $= (N-F)/\text{stride}+1$

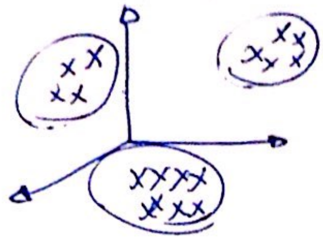
* for every 0 padding add 2 to N

$\rightarrow$ input 32x32x3

10 filter 5x5x3 $\Rightarrow$ # Parameters $= (\underbrace{5\times5\times3+1}_{\text{bias}})\times 10$

$= 760$

Clustering



Cost function :

$$J = \sum_{i=1}^{N} \min_{j=1,2,\dots,k} \|x^{(i)} - c_j\|_2^2$$

$$\Rightarrow \text{optz:}\begin{cases} \min \sum_{i=1}^{N}\sum_{j=1}^{K} z_{ij}\|x^{(i)} - c_j\|_2^2 \\ \text{s.t. } z_{ij} \in \{0,1\}, \forall i,j \ ; \ \sum_{j=1}^{K} z_{ij} = 1 \quad \forall i \end{cases}$$

where: $z_{ij} = \begin{cases} 1 & \text{if } x^{(i)} \text{ is assigned to } c_j \\ 0 & \text{else} \end{cases}$

Solve optz. with Coordinate Descent:

KMEANS

(1) Initialize $\{c_j^{(0)}\}_{j=1}^{K}$ and $\{z_{ij}\}_{i=1\dots N}^{j=1,\dots K}$

(2) For $t = 1, 2, \dots$ do :

▸ For every i

$$z_{ij}^{(t)} = \begin{cases} 1 & \text{if } j = \arg\min_{l=1,\dots,K} \|x^{(i)} - c_l^{(t-1)}\|_2^2 \\ 0 & \end{cases}$$

▸ For every j

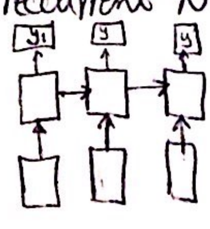
$$c_j^{(t)} = \frac{\sum_{i=1}^{N} 1(z_{ij}^{(t)} = 1)\, x^{(i)}}{\sum_{i=1}^{N} 1(z_{ij}^{(t)} = 1)}$$

(3) Output: $\{c_j^{(t)}\}, \{z_{ij}^{(t)}\} \quad \forall i,j$

K-means Converges (Cost function always decreasing.)

$$J(\{z_{ij}^{(t)}\}, \{c_j^{(t)}\}) \geq J(\{z_{ij}^{(t+1)}\}, \{c_j^{(t+1)}\})$$

reccurrent N.N



$h_t = f_W(h_{t-1}, x_t)$

ex:

$h_t = \tanh(W_{hh}h_{t-1} + W_{hx}x_t)$

$y_t = W_{hy}h_t$

HW5

① $P(A|B) = \frac{P(A\cap B)}{P(B)}$

② $P(A|B)\,P(B) = P(B|A)\,P(A) = \cancel{P(A\cap B)}$ $P(A\cap B)$

③ $P(A\cup B) = P(A) + P(B) - P(A\cap B)$

④ A, B indepe $\Rightarrow P(A\cap B) = P(A)\,P(B)$

Distributions

① Gaussian : $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$

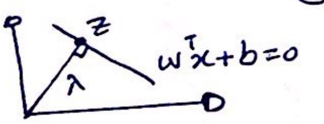
② Bernouli : $0<p<1$ $\quad f(x;p) = p^{x}(1-p)^{1-x}$ $\quad x\in\{0,1\}$

③ Uniform $\quad f(x;a,b) = \begin{cases} \frac{1}{b-a} & x\in[a,b] \\ 0 & \text{otherwise} \end{cases}$

④ Exponential Dist: $\quad f(x,\lambda) = \begin{cases} \lambda e^{-\lambda x} & x\geq 0 \;\; \lambda>0 \\ 0 & x<0 \end{cases}$

⑤ Poision Dist: $\quad f(x,\lambda) = \frac{\lambda^{x} e^{-\lambda}}{x!} \quad \lambda>0$

Distance from origin to hyperplane:



$\lambda = \frac{|b|}{\|w\|_2}$

$\vec{z} = \lambda \frac{w}{\|w\|_2}$

$k(x,z) = c_1 k_1(x,z) + c_2 k_2(x,z)$ ⇐ جمع

$k_1(x,z) = \phi_1^T(x)\,\phi_1(z)$

$k_2(x,z) = \phi_2^T(x)\,\phi_2(z)$

$\Rightarrow k(x,z) = c_1\phi_1^T(x)\phi_1(z) + c_2\phi_2^T(x)\,\phi_2(z)$

$k(x,z) = \phi^T(x)\;\phi(z)$

where $\phi = \begin{bmatrix}\sqrt{c_1}\,\phi_1(x) \\ \sqrt{c_2}\,\phi_2(x)\end{bmatrix}$ ✓✓

$K(x,z) = k_1(x,z)\,k_2(x,z)$ ⇐ ضرب

$k_1(x,z) \to \phi_1(x) = [f_1(x), f_2(x), \cdots]$

$k_2(x,z) \to \phi_2(x) = [g_1(x), g_2(x), \cdots]$

$K(x,z) = k_1(x,z)\,k_2(x,z) =$

$= \phi_1^T(x)\,\phi_1(z)\;\phi_2^T(x)\,\phi_2(z)$

$= \sum_{i=1}^{\infty} f_i(x) f_i(z) \sum_{j=1}^{\infty} g_j(x)\, g_j(z) =$

$= \sum_{i,j}^{\infty} (f_i(x)\, g_j(x))\,(f_i(z)\, g_j(z))$

$= \sum_{i,j} C_{i,j}(x)\; C_{i,j}(z)$

✓✓

Positive Semidefinit (A_{nn})

if $\boxed{z^TAz \geq 0}$ for $\forall z$

$\Rightarrow A = B^TB$

$P(X_1,\ldots,X_N) = \sum_{C_j} \prod_i P(X_i|C_j)\,P(C_j)$

Naieve base