Algorithmic Methods for Mathematical Models (AMMM)

Greedy Randomized Adaptive Search Procedure (GRASP)

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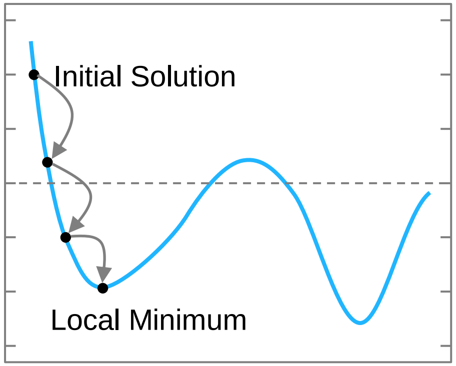
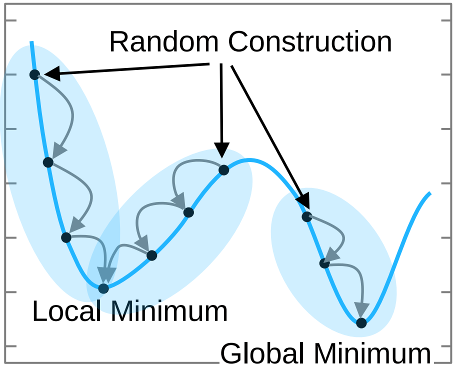
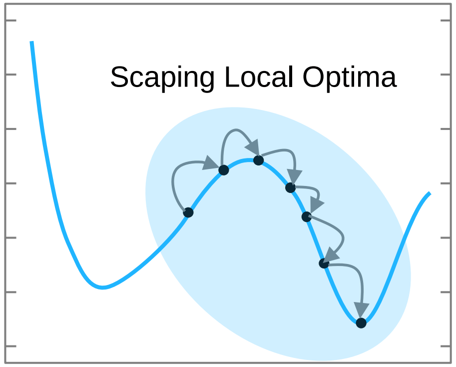
Meta-heuristics



- Classical methodology limitations:
  - Deterministic
  - No global minimum is reached (just local minimum)
- Meta-heuristics go beyond heuristics by:
  - adding **variability** (randomize)
  - allowing **escaping from local optima**, at risk of cycling





Meta-heuristics

- Some well-known meta-heuristics are:
  - **GRASP** (Feo and Resende): a multi-start meta-heuristic for combinatorial problems.
  - Evolutionary algorithms (genetics). **BRKGA** (M. Resende)
  - **Simulated Annealing**, probabilistic meta-heuristic often used when the search space is discrete
  - **Tabu Search** (Fred W. Glover): Enhances the performance of local search by using memory structures.
  - **Ant colony**: probabilistic technique (Marco Dorigo)
  - **Path relinking**: an *intensification* method.

Greedy Randomized Adaptive Search Procedures (GRASP)

- GRASP* is a meta-heuristic for **combinatorial problems**.
- **Each GRASP iteration** consists basically of two phases: **construction** and **local search**.

Like in Greedy + LS

- The **construction phase** builds a feasible solution.
- Its neighborhood is investigated until a local minimum is found during the **local search** phase.
- The best overall solution is kept as the result.

$\omega_{best} \leftarrow \{\}$

for k=1..Max_Iterations **do** ← Randomized Selection

$\boxed{\omega = \text{doConstructionPhase ()}}$

$\boxed{\omega = \text{doLocalSearch } (\omega)}$ Deterministic

if $f(\omega) < f(\omega_{best})$ **then** $\omega_{best} \leftarrow \omega$

return ω_{best}

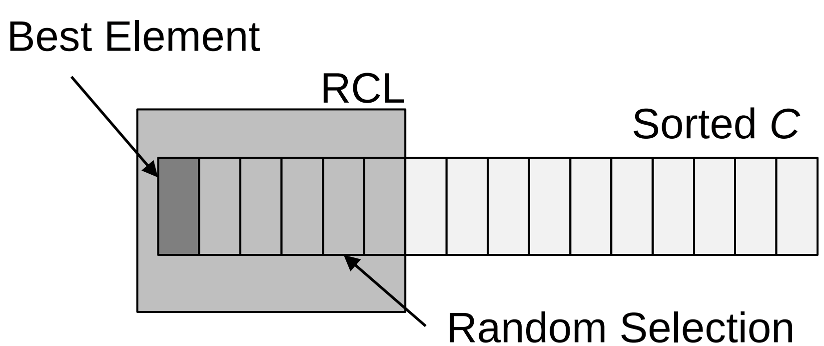
* M. Resende (http://mauricio.resende.info/)

GRASP Construction Phase

- At each iteration:
  - let the set of **candidate elements** be formed by all elements that can be incorporated to the partial solution under construction without destroying feasibility.
  - The selection of the next element to be added is determined by the evaluation of all candidate elements according to a **greedy function**.
  - A **restricted candidate list** (RCL) is created with the best elements, i.e. those whose incorporation to the current partial solution results in the **smallest incremental costs**.
- The element to be incorporated into the partial solution is **randomly selected** from those in the RCL.
- Once the selected element is incorporated to the partial solution, the candidate list is updated, and the incremental costs are reevaluated.

The Restricted Candidate List (RCL)

- The RCL contains “feasible” elements $c \in C$ that can be inserted into the partial solution without destroying feasibility.
- Elements in the RCL are those with the best (i.e., the smallest) incremental costs q(*c*).
- This list can be limited either by:
  - the **number of elements** (*cardinality-based*), the RCL is made up of the *p* elements with the best incremental costs, where *p* is a parameter.
  - their **quality** (*value-based*), the RCL is associated with a threshold parameter $\alpha \in [0,1]$.



Value-based RCL

- The RCL contains elements $c \in C$ whose quality is superior to the threshold value, i.e.,

$$q(c) \in [q^{min}, q^{min} + \alpha(q^{max} - q^{min})]$$

- The case **α=0** corresponds to a **pure greedy** algorithm
- The case **α=1** is equivalent to a **purely random** construction.

RCL (**α=0**)

9	**10**	**11**	**11**	**13**	**18**

$$q(c) \in [9, 9 + 0*(18 - 9)]$$

RCL (**α=1**)

9	**10**	**11**	**11**	**13**	**18**

$$q(c) \in [9, 9 + 1*(18 - 9)]$$

GRASP Constructive Phase

Greedy

Initialize C
$\omega \leftarrow \{\}$
while ω is not a solution **do**
  Evaluate $q(c)$ for all $c \in C$
  $c_{best} \leftarrow \text{argmax}\{q(c) \mid c \text{ in } C\}$
  $\omega \leftarrow \omega \cup \{c_{best}\}$
  update C
return ω

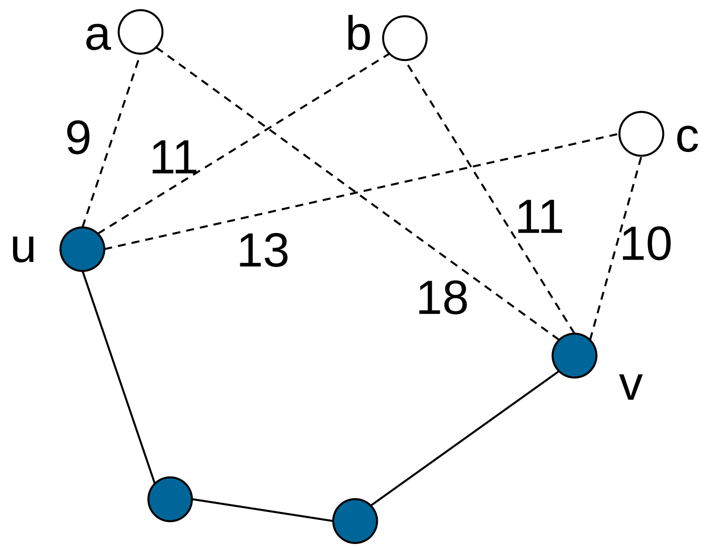
$c_{best} \leftarrow \text{argmin}\{q(c) \mid c \text{ in } C\}$

GRASP constructive phase

Initialize C
$\omega \leftarrow \{\}$
while S is not a solution **do**
  Evaluate $q(c)$ for all $c \in C$
  $q^{min} \leftarrow \min \{q(c) \mid c \in C\}$
  $q^{max} \leftarrow \max \{q(c) \mid c \in C\}$
  $RCL_{max} \leftarrow \{c \in C \mid q(c) \geq q^{max} - \alpha(q^{max} - q^{min})\}$
  Select $c \in$ RCL at random
  $\omega \leftarrow \omega \cup \{c\}$
  Update C
return ω

$RCL_{min} \leftarrow \{c \in C \mid q(c) \leq q^{min} + \alpha(q^{max} - q^{min})\}$

Example: TSP



α=0.3

q_{min}=9
q_{max}=18
q(e) ≤ 9+α(18-9)=11.7

RCL

9	**10**	**11**	**11**	**13**	**18**

Example: GRASP for Set Covering

M/P	p1	p2	p3	p4	p5	p6	p7	p8
1						X		
2			X	X			X	
3	X	X		X	X		X	
4	X			X	X	X		X
5					X	X		
cost	2	1	1	3	3	3	2	1

Let ω be the solution sub-family
Let R be the set of covered elements
Greedy function:

$q(p_j)=|p_j \cap (M \setminus R)| = |p_j \setminus (R \cap p_j)| \rightarrow$ Number of additional elements of p_j

If every p_j has its own associated cost c_j, the greedy function would be:

$q(p_j) = c_j / |p_j \cap (M \setminus R)|$

We use $\boldsymbol{\alpha}$**=0.5**

Set Covering: GRASP constructive

$\omega=\{\}$ compute $q(pj)\ \forall\ p_j \in P \setminus \omega$
$R=\{\}$ $RCL=\{p_j \mid q(c) \geq q^{max} - \alpha(q^{max} - q^{min})$

$q^{max} = 3$, $q^{min} = 1$, $q(p_j) \geq 2$, RCL={p1, p4, p5, p6, p7}
Get one element from the RCL at random: p1
$\omega=\{p1\}$
R={3, 4}

q(p1) = 2 q(p5) = 3
q(p2) = 1 q(p6) = 3
q(p3) = 1 q(p7) = 2
q(p4) = 3 q(p8) = 1

$q^{max} = 2$, $q^{min} = 0$, $q(p_j) \geq 1$, RCL={p3, p4, p5, p6, p7}
Get one element from the RCL at random: p6
$\omega=\{p1, p6\}$
R={1, 3, 4, 5}

q(p2) = 0 q(p6) = 2
q(p3) = 1 q(p7) = 1
q(p4) = 1 q(p8) = 0
q(p5) = 1

$q^{max} = 1$, $q^{min} = 0$, $q(p_j) \geq 0.5$, RCL={p3, p4, p7}
Get one element from the RCL at random: p7
$\omega=\{p1, p6, p7\}$
R=M

q(p2) = 0 q(p7) = 1
q(p3) = 1 q(p8) = 0
q(p4) = 1
q(p5) = 0

Cost: 7

Set covering: Local Search

M/P	p1	p2	p3	p4	p5	p6	p7	p8
1						X		
2			X	X			X	
3	X	X		X	X		X	
4	X			X	X	X		X
5					X	X		
cost	2	1	1	3	3	3	2	1

GASP Constructive Solution:
ω={p1, p6, p7} → Cost: 7

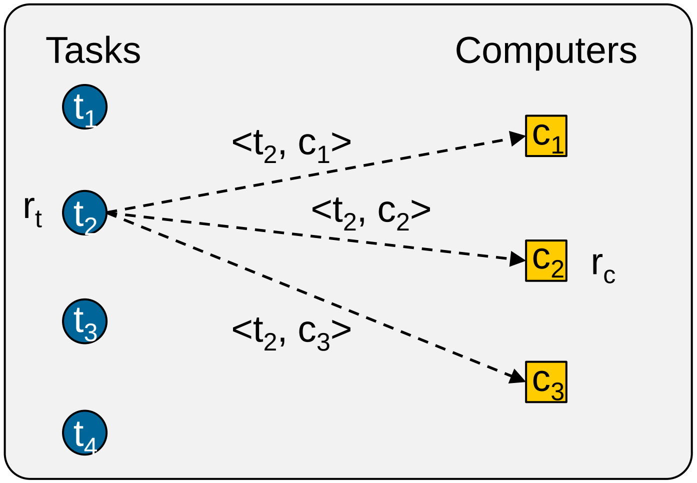
We can search **N_0(S)** and try to remove as many pj as possible from ω.

M	Times covered	Covered by
1	1	{p6}
2	1	{p7}
3	2	{p1, p7}
4	2	{p1, p6}
5	1	{p6}

We can remove p1 from S, as all the elements in p1 are covered more than once

Solution after local search:
ω={p6, p7} → Cost: 5

Example: Assign tasks to computers (lab session 2)



$$q(<\mathrm{t},\mathrm{c}>) = max\begin{Bmatrix} 1 - \dfrac{(residualCapaciy(c) - r_t)}{r_c} \\ 1 - \dfrac{residualCapaciy(c')}{r_{c'}} \mid c'in\ C, c' \neq c \end{Bmatrix}$$

Greedy

$\omega \leftarrow \emptyset$
$sortedT \leftarrow \mathrm{sort}(T, r_t, DESC)$
for each c in C **do** $residualCapacity(c) = r_c$
for each t in T **do**
    $C(t) \leftarrow \emptyset$
    for each c in C **do**
        if $r_t \leq residualCapacity(c)$ **then** $C(t) \leftarrow C(t) \cup \{c\}$
    if $|C(t)|=0$ **then return** INFEASIBLE
    $c_{best} \leftarrow \mathrm{argmin}\{q(<t,c>) \mid c \text{ in } C(t)\}$
    $residualCapacity(c_{best}) \leftarrow residualCapacity(c_{best}) - r_t$
    $\omega \leftarrow \omega \cup \{<t, c_{best}>\}$
return S

GRASP constructive phase

$\omega \leftarrow \emptyset$
$sortedT \leftarrow \mathrm{sort}(T, r_t, DESC)$
for each c in C **do** $residualCapacity(c) = r_c$
for each t in T **do**
    $C(t) \leftarrow \emptyset$
    for each c in C **do**
        if $r_t \leq residualCapacity(c)$ **then** $C(t) \leftarrow C(t) \cup \{c\}$
    if $|C(t)|=0$ **then return** INFEASIBLE
    $q^{min} \leftarrow \min\ \{q(<t,c>) \mid c \in C(t)\}$
    $q^{max} \leftarrow \max\ \{q(<t,c>) \mid c \in C(t)\}$
    $\mathrm{RCL}_{min} \leftarrow \{c \in C(t) \mid q(<t,c>) \leq q^{min} + \alpha(q^{max} - q^{min})\}$
    $c_{sel} \leftarrow$ select $c \in$ RCL at random
    $residualCapacity(c_{sel}) \leftarrow residualCapacity(c_{sel}) - r_t$
    $\omega \leftarrow \omega \cup \{<t, c_{sel}>\}$
return S

Assignment Tasks to computers: Iterative execution

α=0.3

Computers	c1	c2	c3	
rc	505.67	503.68	701.78	
Tasks	t1	t2	t3	t4
rt	261.27	560.89	310.51	105.8

sortedTasks	t2	t3	t1	t4

Computers	c1	c2	c3
residualCap	505.67	503.68	701.78

#1

task: t2	560.89
C(t2)	c3
RCL	c3

Computers	c1	c2	c3
residualCap	505.67	503.68	**140.89**
load	0	0	**0.799**
S	{**<t2,c3>**}		

#2

task: t3	310.51		
C(t2)	c1	c2	
Load if assignment			
c1	0.6141	qmin	
c2	0.6165	qmax	
RCL	q≤	0.6148	{c1}

Computers	c1	c2	c3
residualCap	**195.16**	503.68	140.89
load	**0.6141**	0	0.799
S	{<t2,c3>,**<t3,c1>**}		

#3

task: t1	261.27
C(t1)	c2
Load if assignment	
c2	0.5187
RCL	c2

Computers	c1	c2	c3
residualCap	195.16	**242.41**	140.89
load	0.6141	**0.5187**	0.799
S	{<t2,c3>,<t3,c1>,**<t1,c2>**}		

#4

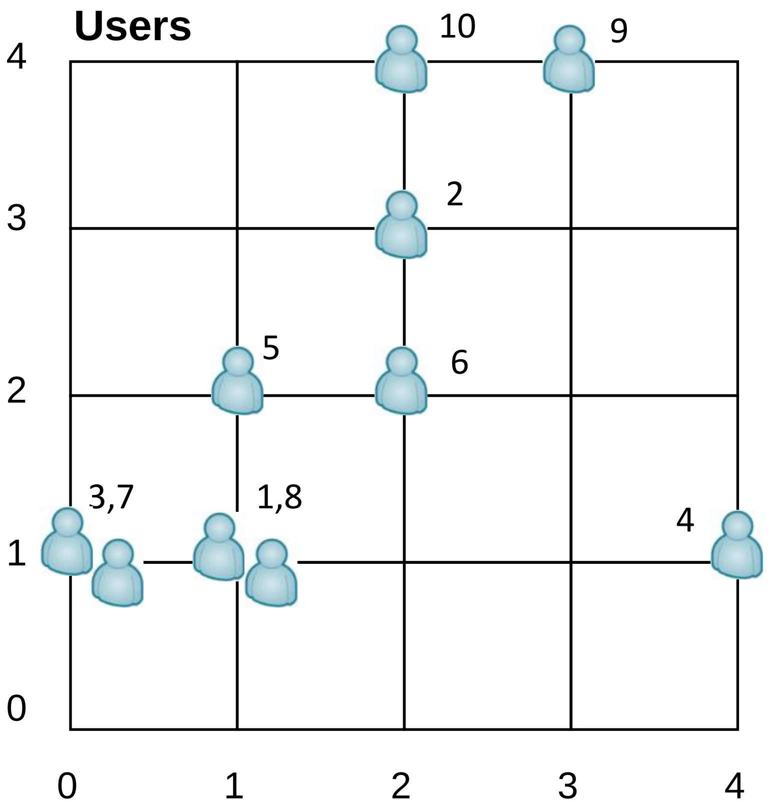
task: t4	105.8		
C(t4)	c1	c2	c3
Load if assignment			
c1	0.8233		
c2	0.7288	qmin	
c3	0.95	qmax	
RCL	q≤	0.8651	{c1,c2}

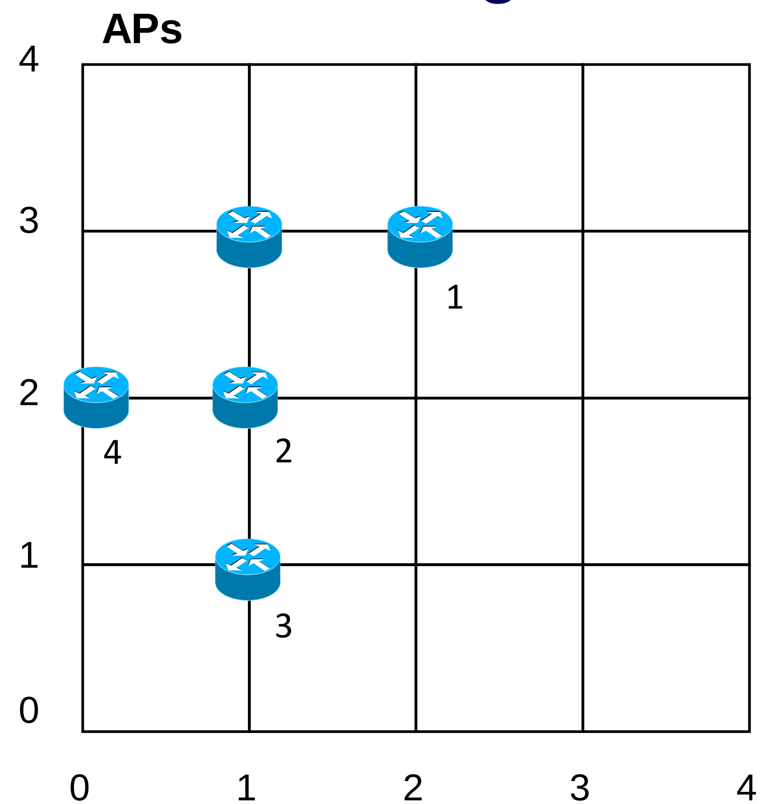
Computers	c1	c2	c3
residualCap	**89.4**	242.41	140.89
load	**0.823**	0.5187	0.799
S	{<t2,c3>,<t3,c1>,<t1,c2>,**<t4,c1>**}		

Solution

S	**{<t2,c3>,<t3,c1>,<t1,c2>,<t4,c1>}**
f(S)	**0.823**

Network Planning Problem





	R1	R2	R3
f	100	140	180
k	6	8	10
d	2	3	4

		u	1	2	3	4	5	6	7	8	9	10
d(u,a)		x	1	2	0	4	1	2	0	1	3	2
		y	1	3	1	1	2	2	1	1	4	4
1	2	3	2.2	0.0	2.8	2.8	1.4	1.0	2.8	2.2	1.4	1.0
2	1	2	1.0	1.4	1.4	3.2	0.0	1.0	1.4	1.0	2.8	2.2
3	1	1	0.0	2.2	1.0	3.0	1.0	1.4	1.0	0.0	3.6	3.2
4	0	2	1.4	2.2	1.0	4.1	1.0	2.0	1.0	1.4	3.6	2.8
5	1	3	2.0	1.0	2.2	3.6	1.0	1.4	2.2	2.0	2.2	1.4
a	**x**	**y**										

α=0.5

Network planning: Iterative execution (1/5)

#1

u	1	2	3	4	5	6	7	8	9	10
cr	2	3	4	1	2	2	1	2	3	4
q(u)	100	100	100	140	100	100	100	100	100	100
a	3	1	3	1	2	1	3	3	1	1
d(u,a)	0.0	0.0	1.0	2.8	0.0	1.0	1.0	0.0	1.4	1.0

qmin=100, qmax=140, q<=120

a	1	2	3	4	5
m			R1		
U(a)			{3}		
km-cr			2		

#2

u	1	2	3	4	5	6	7	8	9	10
cr	2	3	4	1	2	2	1	2	3	4
q(u)	0	40		40	0	0	0	0	80	80
a	3	3	3	3	3	3	3	3	3	3
d(u,a)	0.0	2.2	1.0	3.0	1.0	1.4	1.0	0.0	3.6	3.2

qmin=0, qmax=80, q<=40

a	1	2	3	4	5
m			R2		
U(a)			{3,4}		
km-cr			3		

Network planning: Iterative execution (2/5)

#3

u	1	2	3	4	5	6	7	8	9	10
cr	2	3	4	1	2	2	1	2	3	4
q(u)	0	0			0	0	0	0	40	40
a	3	3	3	3	3	3	3	3	3	3
d(u,a)	0.0	2.2	1.0	3.0	1.0	1.4	1.0	0.0	3.6	3.2

qmin=0, qmax=40, q<=20

a	1	2	3	4	5
m			R2		
U(a)			{1,3,4}		
km-cr			1		

#4

u	1	2	3	4	5	6	7	8	9	10
cr	2	3	4	1	2	2	1	2	3	4
q(u)		40			40	40	0	40	100	100
a	3	3	3	3	3	3	3	3	1	1
d(u,a)	0.0	2.2	1.0	3.0	1.0	1.4	1.0	0.0	1.4	1.0

qmin=0, qmax=100, q<=50

a	1	2	3	4	5
m			R2		
U(a)			{1,3,4,7}		
km-cr			0		

Network planning: Iterative execution (3/5)

#5

u	1	2	3	4	5	6	7	8	9	10
cr	2	3	4	1	2	2	1	2	3	4
q(u)		100			40	40		40	100	100
a	3	1	3	3	3	3	3	3	1	1
d(u,a)	0.0	0.0	1.0	3.0	1.0	1.4	1.0	0.0	1.4	1.0

qmin=40, qmax=100, q<=70

a	1	2	3	4	5
m			R3		
U(a)			{1,3,4,6,7}		
km-cr			0		

#6

u	1	2	3	4	5	6	7	8	9	10
cr	2	3	4	1	2	2	1	2	3	4
q(u)		100			100			100	100	100
a	3	1	3	3	2	3	3	2	1	1
d(u,a)	0.0	0.0	1.0	3.0	0.0	1.4	1.0	1.0	1.4	1.0

qmin=100, qmax=100, q<=100

a	1	2	3	4	5
m	R1		R3		
U(a)	{9}		{1,3,4,6,7}		
km-cr	3		0		

Network planning: Iterative execution (4/5)

#7

u	1	2	3	4	5	6	7	8	9	10
cr	2	3	4	1	2	2	1	2	3	4
q(u)		0			0			0		40
a	3	1	3	3	1	3	3	1	1	1
d(u,a)	0.0	0.0	1.0	3.0	1.4	1.4	1.0	2.2	1.4	1.0

qmin=0, qmax=40, q<=20

a	1	2	3	4	5
m	R1		R3		
U(a)	{2,9}		{1,3,4,6,7}		
km-cr	0		0		

#8

u	1	2	3	4	5	6	7	8	9	10
cr	2	3	4	1	2	2	1	2	3	4
q(u)					40			40		80
a	3	1	3	3	1	3	3	1	1	1
d(u,a)	0.0	0.0	1.0	3.0	1.4	1.4	1.0	2.2	1.4	1.0

qmin=40, qmax=80, q<=60

a	1	2	3	4	5
m	R2		R3		
U(a)	{2,5,9}		{1,3,4,6,7}		
km-cr	0		0		

Network planning: Iterative execution (5/5)

#9

u	1	2	3	4	5	6	7	8	9	10
cr	2	3	4	1	2	2	1	2	3	4
q(u)								40		100
a	3	1	3	3	1	3	3	1	1	5
d(u,a)	0.0	0.0	1.0	3.0	1.4	1.4	1.0	2.2	1.4	1.4

qmin=40, qmax=100, q<=70

a	1	2	3	4	5
m	R3		R3		
U(a)	{2,5,8,9}		{1,3,4,6,7}		
km-cr	0		0		

#10

u	1	2	3	4	5	6	7	8	9	10
cr	2	3	4	1	2	2	1	2	3	4
q(u)										100
a	3	1	3	3	1	3	3	1	1	5
d(u,a)	0.0	0.0	1.0	3.0	1.4	1.4	1.0	2.2	1.4	1.4

qmin=100, qmax=100, q<=100

a	1	2	3	4	5
m	R3		R3		R1
U(a)	{2,5,8,9}		{1,3,4,6,7}		{10}
km-cr	0		0		2

Solution Cost=460

Parameter tuning

RCL (**min** greedy cost)

9	**10**	**11**	**11**	**13**	**18**

α_{min}=0.3

q_{min}=9
q_{max}=18
$q(e) \leq 9+\alpha_{min}(18\text{-}9)=11.7$

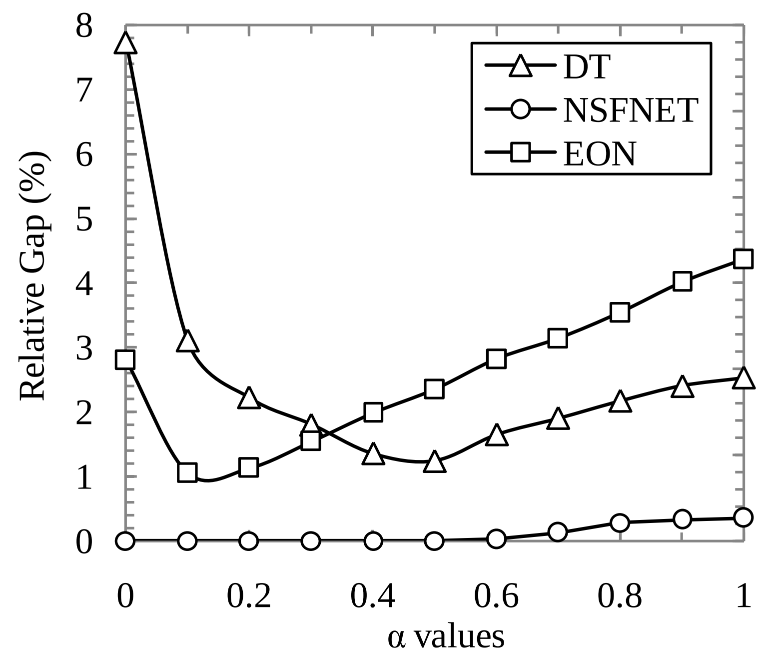
RCL (**max** greedy cost)

18	**13**	**11**	**11**	**10**	**9**

α_{max} =0.7

q_{min}=9
q_{max}=18
$q(e) \geq 18\text{-}\alpha_{max}(18\text{-}9)=11.7$

To decide the value of α that fits the best for our optimization problem, we have to solve one or more *test instances* using different values of α



Stop Criteria

- Different stop criteria can be devised:
  - Execution time
  - Number of iterations
  - Goodness of the solution (value of the incumbent w.r.t. a given value)
  - Time since last incumbent update
  - Iterations since last incumbent update
  - A mix of the above

$\omega_{best} \leftarrow \{\}$

Stop Criteria

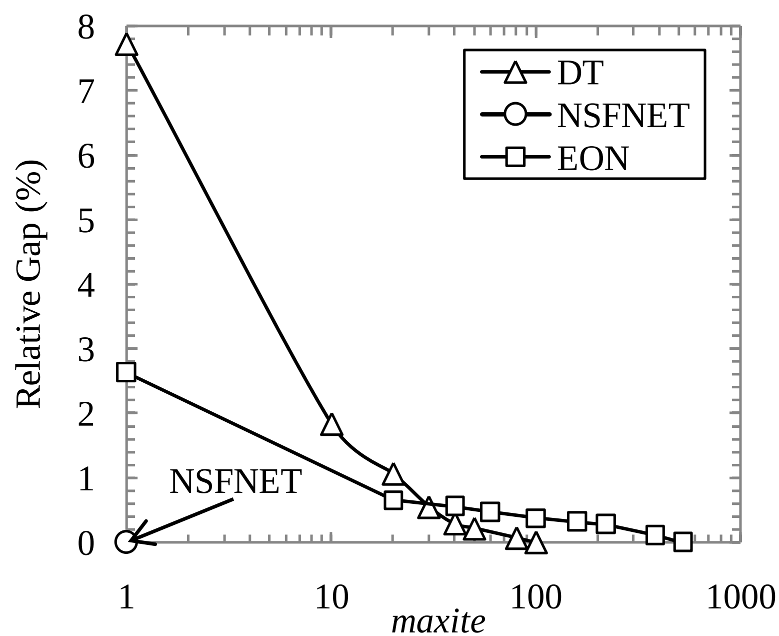
for $\boxed{k\text{=1..Max\_Iterations}}$ **do**

$\quad\omega$ = doConstructionPhase ()

$\quad\omega$ = doLocalSearch (ω)

$\quad$**if** f(ω) < f(ω_{best}) **then** $\omega_{best} \leftarrow \omega$

return ω_{best}



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