Basics on Linear Programming

Algorithmic Methods for Mathematical Models

(AMMM)

Enric Rodríguez-Carbonell

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Vectors

- We will use vector/matrix notation
- A vector is a (vertically displayed) list of numbers. Examples:

$$c = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \qquad\qquad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

- To display it horizontally, a vector can be transposed. Example:

$$c^T = \begin{pmatrix} 1 & 0 & 2 \end{pmatrix}$$

- The scalar product of two vectors is the sum of products componentwise. Example:

$$c^T x = \begin{pmatrix} 1 & 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 1 \cdot x_1 + 0 \cdot x_2 + 2 \cdot x_3 = x_1 + 2x_3$$

Matrices (1)

- A matrix is a rectangular array of numbers. Example:

$$A = \begin{pmatrix} 0 & 1 & 2 \\ 2 & -1 & 0 \end{pmatrix}$$

- The product of a matrix and a vector is a vector consisting of the scalar products of the rows of the matrix and the vector. Example:

$$Ax = \begin{pmatrix} 0 & 1 & 2 \\ 2 & -1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \cdot x_1 + 1 \cdot x_2 + 2 \cdot x_3 \\ 2 \cdot x_1 - 1 \cdot x_2 + 0 \cdot x_3 \end{pmatrix} = \begin{pmatrix} x_2 + 2x_3 \\ 2x_1 - x_2 \end{pmatrix}$$

- A list of constraints can be written with vectors and matrices. Examples:

$$\begin{array}{l} x_1 \geq 0 \\ x_2 \geq 0 \\ x_3 \geq 0 \end{array} \quad \text{is equivalent to} \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{array}{l} x_2 + 2x_3 \leq 3 \\ 2x_1 - x_2 \leq 1 \end{array} \quad \text{is equivalent to} \quad \begin{pmatrix} 0 & 1 & 2 \\ 2 & -1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \leq \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

Matrices (2)

- Vectors & matrices can be used to write optimization problems compactly. Example:

$$\min \; x_1 + 2x_3$$
$$\begin{aligned} x_2 + 2x_3 &\leq 3 \\ 2x_1 - x_2 &\leq 1 \end{aligned}$$
$$x_1, x_2, x_3 \geq 0$$

can be written as

$$\min c^T x$$
$$\begin{aligned} Ax &\leq b \\ x &\geq 0 \end{aligned}$$

where

$$c^T = \begin{pmatrix} 1 & 0 & 2 \end{pmatrix} \quad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad A = \begin{pmatrix} 0 & 1 & 2 \\ 2 & -1 & 0 \end{pmatrix} \quad b = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

- Note that in $x \geq 0$, the 0 actually represents a vector of zeroes

Linear Programs (1)

- A linear program (LP) is an optimization problem in which:
  - a linear expression is to be minimized/maximized
  - constraints are linear equalities and inequalities
  - variables may take real numbers

- Example:

$$\min\ x_1 + 2x_3$$

$$x_2 + 2x_3 \leq 3$$
$$2x_1 - x_2 \leq 1$$

$$x_1, x_2, x_3 \geq 0$$

Linear Programs (2)

- More formally, a linear program is an optimization problem of the form

$$\min c^T x$$
$$A_1 x \leq b_1$$
$$A_2 x = b_2$$
$$A_3 x \geq b_3$$

- x is a vector of n variables that may take real values (i.e., $x \in \mathbb{R}^n$)
- c is a vector of n real numbers
- b_i are vectors of m_i real numbers for $i = 1, 2, 3$
- A_i are matrices of $m_i \times n$ real numbers for $i = 1, 2, 3$

- $c^T x$ is the objective function
- b_1, b_2, b_3 are the independent terms or right-hand sides
- $A_1 x \leq b_1$, $A_2 x = b_2$ and $A_3 x \geq b_3$ are the constraints

Linear Programs (3)

- Solving minimization or maximization problems is equivalent:

$$\max\{\ f(x) \mid x \in S\ \} \quad = \quad -\min\{\ -f(x) \mid x \in S\ \}$$

  In minimization problems $c^T x$ is also called cost function

- Sometimes we are only interested in a solution, there is no objective: satisfiability problems

  Satisfiability problems are also covered in this definition:
  take an arbitrary cost function, e.g., $c = 0$

Equivalent Forms of LP's (1)

- LP's as in the definition are not in a convenient form for algorithms
  WLOG we can transform such a problem as follows

1. Split $=$ constraints into $\geq$ and $\leq$ constraints

$$\begin{aligned} &\min c^T x \\ &A_1 x \leq b_1 \\ &A_2 x = b_2 \\ &A_3 x \geq b_3 \end{aligned} \quad \Longrightarrow \quad \begin{aligned} &\min c^T x \\ &A_1 x \leq b_1 \\ &A_2 x \leq b_2 \\ &A_2 x \geq b_2 \\ &A_3 x \geq b_3 \end{aligned}$$

Now all constraints are $\leq$ or $\geq$

Example:

$$\begin{aligned} &\min \;\; x + y + z \\ &x + y = 3 \\ &0 \leq x \leq 2 \\ &0 \leq y \leq 2 \end{aligned} \quad \Longrightarrow \quad \begin{aligned} &\min \;\; x + y + z \\ &x + y \leq 3 \\ &x + y \geq 3 \\ &0 \leq x \leq 2 \\ &0 \leq y \leq 2 \end{aligned}$$

Equivalent Forms of LP's (2)

2. Transform $\geq$ constraints into $\leq$ constraints by multiplying by -1

$$\begin{array}{l} \min c^T x \\ A_1 x \leq b_1 \\ A_2 x \geq b_2 \end{array} \quad \Longrightarrow \quad \begin{array}{l} \min c^T x \\ A_1 x \leq b_1 \\ -A_2 x \leq -b_2 \end{array}$$

Now all constraints are $\leq$

Example:

$$\begin{array}{l} \min \;\; x + y + z \\ x + y \leq 3 \\ x + y \geq 3 \\ 0 \leq x \leq 2 \\ 0 \leq y \leq 2 \end{array} \quad \Longrightarrow \quad \begin{array}{l} \min \;\; x + y + z \\ x + y \leq 3 \\ -x - y \leq -3 \\ 0 \leq x \leq 2 \\ 0 \leq y \leq 2 \end{array}$$

Equivalent Forms of LP's (3)

3. Replace variables x by $y - z$, where y, z are vectors of fresh variables, and add constraints $y \geq 0$, $z \geq 0$

$$\begin{array}{l} \min c^T x \\ Ax \leq b \end{array} \quad \Longrightarrow \quad \begin{array}{l} \min c^T y - c^T z \\ Ay - Az \leq b \\ y, z \geq 0 \end{array}$$

Now all constraints are $\leq$ and all variables have to be ≥ 0

Actually only needed for variables which are not already non-negative.
(in the example, only z)

Example:

$$\begin{array}{l} \min \ x + y + z \\ x + y \leq 3 \\ -x - y \leq -3 \\ 0 \leq x \leq 2 \\ 0 \leq y \leq 2 \end{array} \quad \Longrightarrow \quad \begin{array}{l} \min \ x + y + u - v \\ x + y \leq 3 \\ -x - y \leq -3 \\ 0 \leq x \leq 2 \\ 0 \leq y \leq 2 \\ u, v \geq 0 \end{array}$$

Equivalent Forms of LP's (4)

4. Add a slack variable to each $\leq$ constraint to convert it into $=$

$$\begin{aligned} &\min c^T x \\ &Ax \leq b \\ &x \geq 0 \end{aligned} \qquad \Longrightarrow \qquad \begin{aligned} &\min c^T x \\ &Ax + s = b \\ &x, s \geq 0 \end{aligned}$$

Now all constraints are $=$ and all variables have to be ≥ 0

Each equality has its own slack variable

Example:

$$\begin{aligned} &\min \; x + y + u - v \\ &x + y \leq 3 \\ &-x - y \leq -3 \\ &0 \leq x \leq 2 \\ &0 \leq y \leq 2 \\ &u, v \geq 0 \end{aligned} \qquad \Longrightarrow \qquad \begin{aligned} &\min \; x + y + u - v \\ &x + y + s_1 = 3 \\ &-x - y + s_2 = -3 \\ &x + s_3 = 2 \\ &y + s_4 = 2 \\ &x, y, u, v, s_1, s_2, s_3, s_4 \geq 0 \end{aligned}$$

Equivalent Forms of LP's (5)

Altogether:

$$
\begin{array}{l}
\min\ x+y+z \\
x+y=3 \\
0 \le x \le 2 \\
0 \le y \le 2
\end{array}
\quad \Longrightarrow \quad
\begin{array}{l}
\min\ x+y+u-v \\
x+y+s_1=3 \\
-x-y+s_2=-3 \\
x+s_3=2 \\
y+s_4=2 \\
x,y,u,v,s_1,s_2,s_3,s_4 \ge 0
\end{array}
$$

Equivalent Forms of LP's (6)

- In the end we get a problem in canonical form:

$$\begin{aligned} &\min c^T x \\ &Ax = b \\ &x \geq 0 \end{aligned}$$

$$c \in \mathbb{R}^n, b \in \mathbb{R}^m, A \in \mathbb{R}^{m \times n}, n \geq m, \operatorname{rank}(A) = m$$

- We say $Ax = b$ is the system of equations and $x \geq 0$ the sign constraints
- Condition $\operatorname{rank}(A) = m$ essentially ensures equalities are not redundant. (in mathematical terms: it means rows of A are linearly independent)

  For example, this linear program is not in canonical form:

$$\begin{aligned} &\min \ x + 2z \\ &y + 3z = 3 \\ &2x - y = 1 \\ &2x + 3z = 4 \\ &x, y, z, \geq 0 \end{aligned}$$

To get canonical form, just remove the redundancy: e.g., remove $2x + 3z = 4$

Equivalent Forms of LP's (7)

- The transformations guarantee that $n \geq m, \mathrm{rank}(A) = m$ since a slack variable is introduced for each inequality to convert it into equality

- The canonical form is actually not strictly necessary (the transformations increase the number of constraints and variables!), but is convenient in a first formulation of the algorithms

Equivalent Forms of LP's (8)

- Often variables are identified with columns of the matrix, and constraints are identified with rows

  Example:

$$\begin{aligned} &\min\ x + 2z \\ &y + 3z = 3 \\ &2x - y = 1 \\ &x, y, z, \geq 0 \end{aligned} \qquad A = \begin{pmatrix} 0 & 1 & 3 \\ 2 & -1 & 0 \end{pmatrix}$$

  - $y + 3z = 3 \rightsquigarrow \begin{pmatrix} 0 & 1 & 3 \end{pmatrix}$
  - $2x - y = 1 \rightsquigarrow \begin{pmatrix} 2 & -1 & 0 \end{pmatrix}$
  - $x \rightsquigarrow \begin{pmatrix} 0 \\ 2 \end{pmatrix}$
  - $y \rightsquigarrow \begin{pmatrix} 1 \\ -1 \end{pmatrix}$
  - $z \rightsquigarrow \begin{pmatrix} 3 \\ 0 \end{pmatrix}$

Methods for Solving LP's

- Simplex algorithms
- Interior-point algorithms

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- Simplex algorithms
- Interior-point algorithms

Basic Definitions (1)

- Let us consider an LP (in canonical form):

$$\min c^T x$$
$$Ax = b$$
$$x \geq 0$$

- Any x such that $Ax = b$ is called a solution
- A solution x satisfying $x \geq 0$ is called a feasible solution
- An LP with feasible solutions is called feasible, otherwise infeasible

$$\max x + 2y$$
$$x + y + s_1 = 3$$
$$x + s_2 = 2$$
$$y + s_3 = 2$$
$$x, y, s_1, s_2, s_3 \geq 0$$

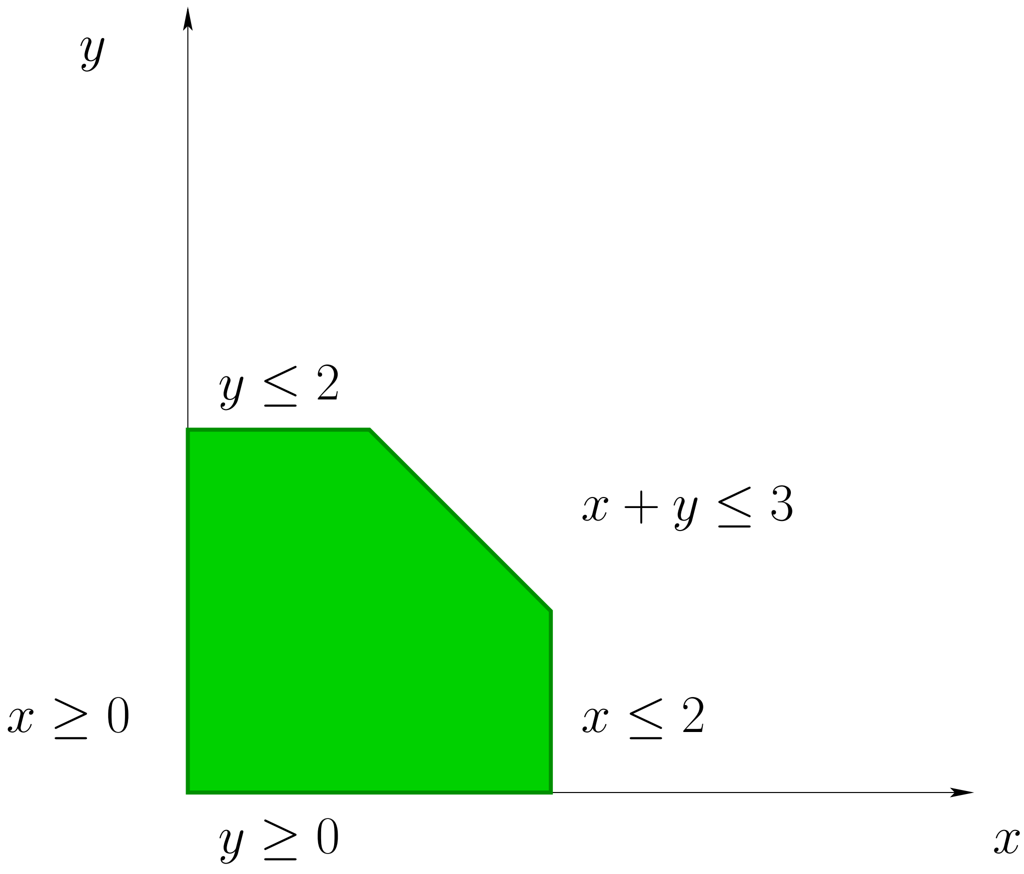
  - $(x, y, s_1, s_2, s_3) = (-1, -1, 5, 3, 3)$ is a solution but not feasible
  - $(x, y, s_1, s_2, s_3) = (1, 1, 1, 1, 1)$ is a feasible solution
  - This is an example of feasible LP

Basic Definitions (2)

- We can represent geometrically the set of feasible solutions (here, actually the projection onto the plane)

$$\max x + 2y$$
$$x + y + s_1 = 3$$
$$x + s_2 = 2$$
$$y + s_3 = 2$$
$$x, y, s_1, s_2, s_3 \geq 0$$

$$\max x + 2y$$
$$x + y \leq 3$$
$$x \leq 2$$
$$y \leq 2$$
$$x, y \geq 0$$



Basic Definitions (3)

- Let us consider an LP (in canonical form):

$$\begin{aligned} &\min c^T x \\ &Ax = b \\ &x \geq 0 \end{aligned}$$

- Any x such that $Ax = b$ is called a solution
- A solution x satisfying $x \geq 0$ is called a feasible solution
- An LP with feasible solutions is called feasible, otherwise infeasible

$$\begin{aligned} &\max x + 2y \\ &x + y + s_1 = -1 \\ &x + s_2 = 2 \\ &y + s_3 = 2 \\ &x, y, s_1, s_2, s_3 \geq 0 \end{aligned}$$

  - This is an example of infeasible LP

Basic Definitions (4)

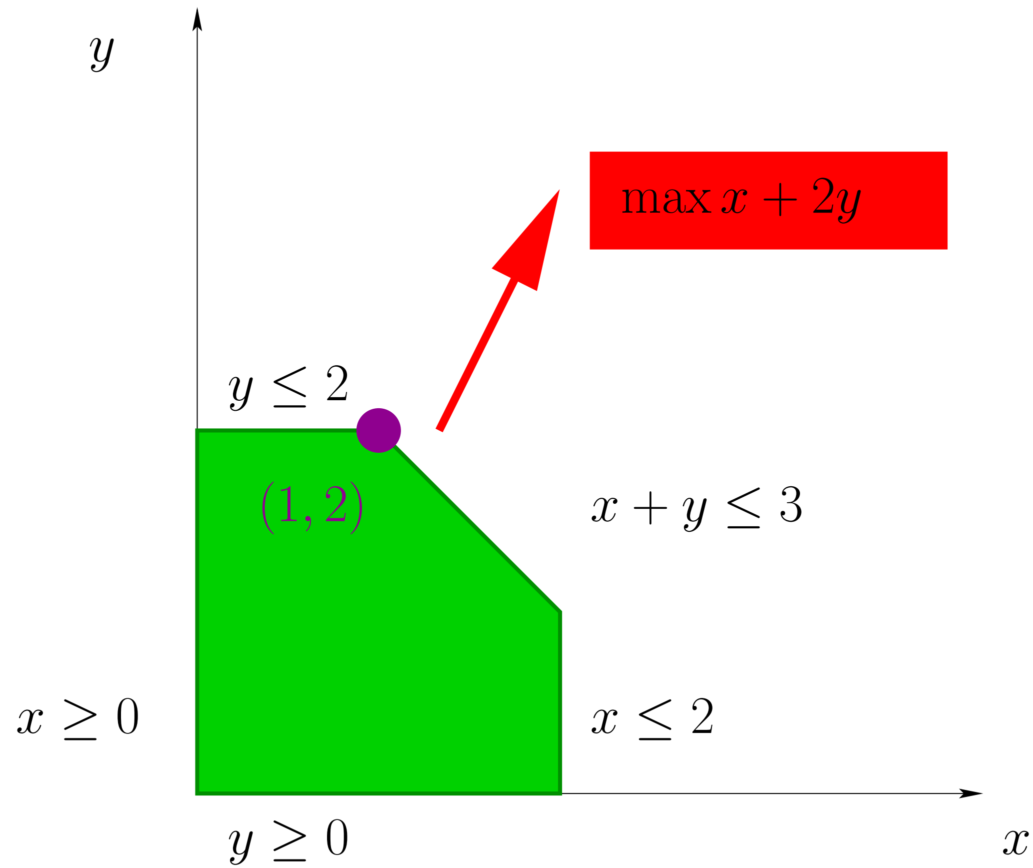
- A feasible solution x^* is called optimal if $c^T x^* \leq c^T x$ for all feasible solution x
- The next LP has a single optimal solution $(x, y, s_1, s_2, s_3) = (1, 2, 0, 1, 0)$:

$$\max x + 2y$$

$$x + y + s_1 = 3$$
$$x + s_2 = 2$$
$$y + s_3 = 2$$
$$x, y, s_1, s_2, s_3 \geq 0$$

$$\max x + 2y$$

$$x + y \leq 3$$
$$x \leq 2$$
$$y \leq 2$$
$$x, y \geq 0$$

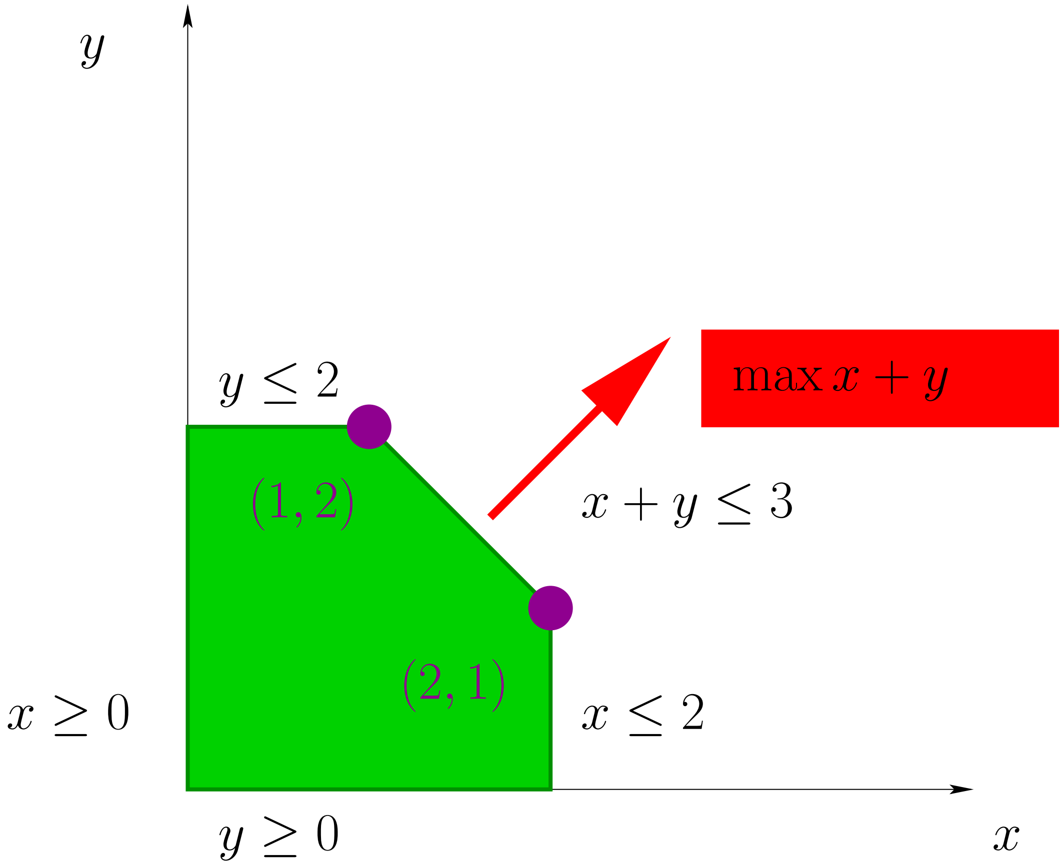


Basic Definitions (5)

- The next LP has more than one optimal solution:
  $\{(1,2,0,1,0),(2,1,0,0,1),(\frac{3}{2},\frac{3}{2},0,\frac{1}{2},\frac{1}{2})\},\ldots$

$$\max x+y$$
$$x+y+s_1=3$$
$$x+s_2=2$$
$$y+s_3=2$$
$$x,y,s_1,s_2,s_3\geq 0$$

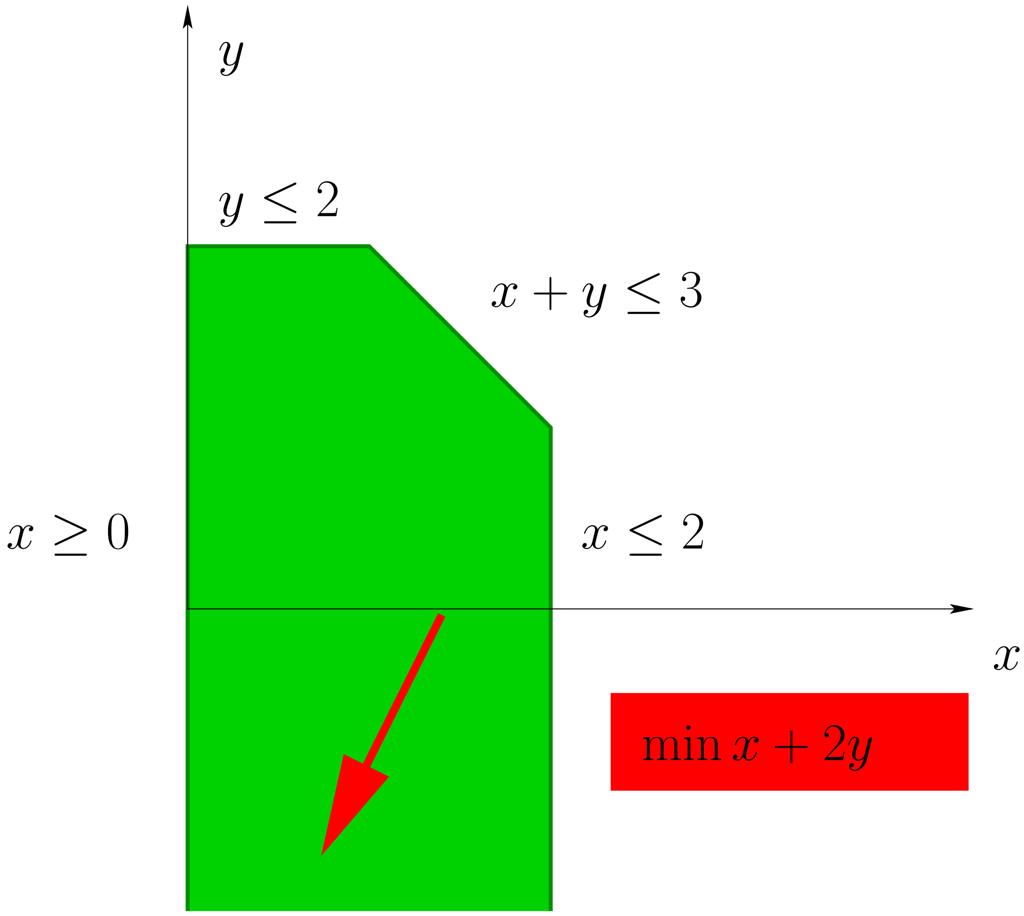
$$\max x+y$$
$$x+y\leq 3$$
$$x\leq 2$$
$$y\leq 2$$
$$x,y\geq 0$$



Basic Definitions (6)

- A feasible LP with no optimal solution is unbounded
- Example:

$$\min x + 2y$$
$$x + y \leq 3$$
$$0 \leq x \leq 2$$
$$y \leq 2$$



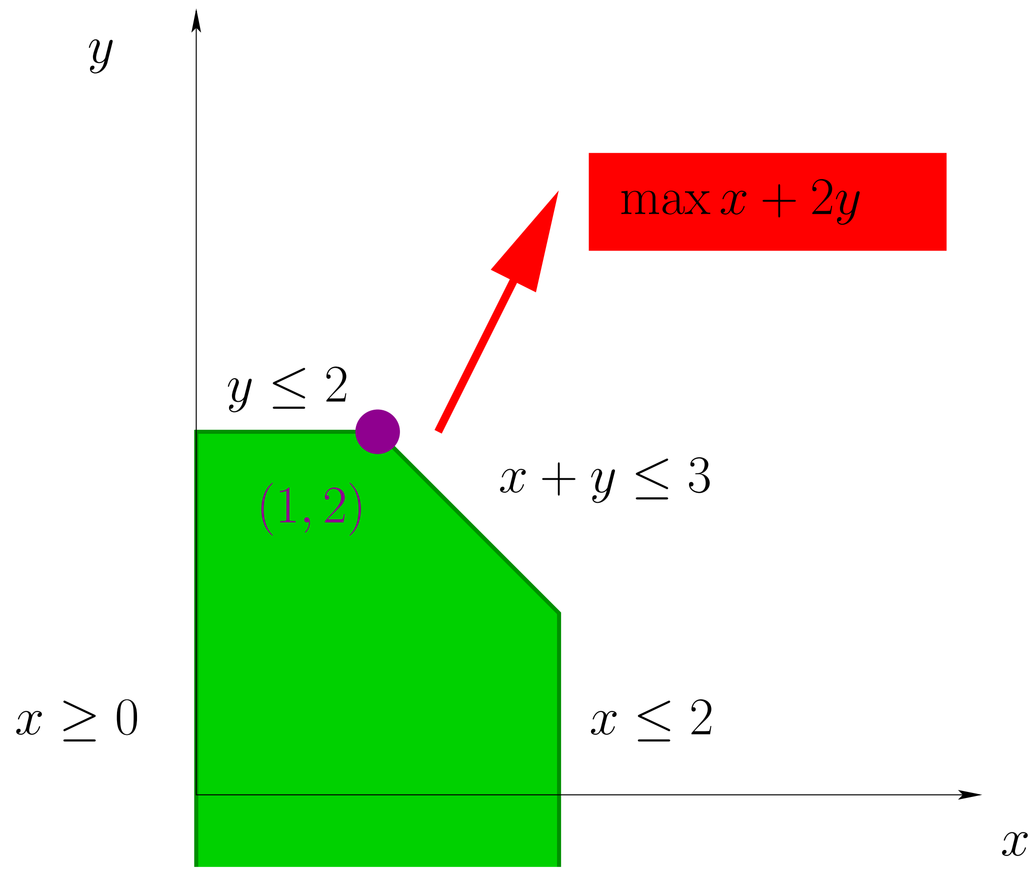
Basic Definitions (7)

- Warning! In mathematics, "unbounded" can mean different things
  - When applied to an LP: there is no optimal solution
  - When applied to a set of points: there is no bounding box including it
- If an LP is unbounded, then the set of feasible solutions is unbounded
- The converse does not hold:

$$\max x + 2y$$

$$x + y \leq 3$$
$$0 \leq x \leq 2$$
$$y \leq 2$$

Set of feasible solutions is unbounded,
but LP has optimal solutions



Bases (1)

- Recall that the matrix A has m rows and n columns, and $n \geq m, \ \mathrm{rank}(A) = m$.

- Let us denote by a_1, ..., a_n the columns of A

- Example:

$$
\begin{aligned}
&\max x + 2y \\
&x + y + s_1 = 3 \\
&x + s_2 = 2 \\
&y + s_3 = 2 \\
&x, y, s_1, s_2, s_3 \geq 0
\end{aligned}
$$

$$
A = \begin{array}{c} \begin{array}{ccccc} x & y & s_1 & s_2 & s_3 \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix} \end{array}
\qquad
\begin{array}{rcl} m & = & 3 \\ n & = & 5 \end{array}
$$

$$
a_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad a_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad a_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad a_4 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad a_5 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}
$$

Bases (2)

- If we choose m columns of A, we can join them and form a new matrix B
- Example: by choosing columns a_3, a_4, a_5 (i.e., variables s_1, s_2, s_3) we get

$$A = \begin{array}{c} \begin{array}{ccccc} x & y & s_1 & s_2 & s_3 \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix} \end{array} \quad B = (a_3\ a_4\ a_5) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- Note that B is a square matrix
- Such a matrix B is called a basis if it is invertible
- In that case, the variables associated to the columns of B are called basic
- Example:

$$B = (a_3\ a_4\ a_5) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ is a basis and variables s_1, s_2, s_3 are basic

Bases (3)

- Equivalently, B is a basis if
  we can isolate the variables associated to B from the system $Ax = b$

- Example:

$$
\begin{aligned}
&\max x + 2y \\
&x + y + s_1 = 3 \\
&x + s_2 = 2 \\
&y + s_3 = 2 \\
&x, y, s_1, s_2, s_3 \geq 0
\end{aligned}
\qquad \Longrightarrow \qquad
\begin{cases}
s_1 = 3 - x - y \\
s_2 = 2 - x \\
s_3 = 2 - y
\end{cases}
$$

- Bases exist if and only if condition $\operatorname{rank}(A) = m$ holds

Bases (4)

- The matrix obtained by joining the columns not in B is denoted by R (standing for the Rest of the variables)
- We denote by $x_{\mathcal{B}}$ the basic variables and by $x_{\mathcal{R}}$ the non-basic ones
- Example:

$$
\begin{aligned}
&\max x + 2y \\
&x + y + s_1 = 3 \\
&x + s_2 = 2 \\
&y + s_3 = 2 \\
&x, y, s_1, s_2, s_3 \geq 0
\end{aligned}
\qquad
A = \begin{array}{c} \\ \\ \end{array}\begin{array}{c} \begin{array}{ccccc} x & y & s_1 & s_2 & s_3 \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix} \end{array}
$$

(s_1, s_2, s_3) form a basis, and $x_{\mathcal{B}} = (s_1, s_2, s_3)$, $x_{\mathcal{R}} = (x, y)$

$$
B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad R = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}
$$

Bases (4)

- The matrix obtained by joining the columns not in B is denoted by R (standing for the Rest of the variables)
- We denote by $x_{\mathcal{B}}$ the basic variables and by $x_{\mathcal{R}}$ the non-basic ones
- Example:

$$\begin{aligned}
&\max x + 2y \\
&x + y + s_1 = 3 \\
&x + s_2 = 2 \\
&y + s_3 = 2 \\
&x, y, s_1, s_2, s_3 \geq 0
\end{aligned}
\qquad
\begin{array}{cc}
 & \begin{array}{ccccc} x & y & s_1 & s_2 & s_3 \end{array} \\
A = & \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix}
\end{array}$$

(x, s_1, s_2) do not form a basis:

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \text{ is not invertible}$$

Bases (5)

- If B is a basis, then the following holds

$$Bx_{\mathcal{B}} + Rx_{\mathcal{R}} = b$$

Hence:

$$x_{\mathcal{B}} = B^{-1}b - B^{-1}Rx_{\mathcal{R}}$$

We can isolate basic variables and express them in terms of non-basic ones

- Example: take $x_{\mathcal{B}} = (s_1 \ s_2 \ s_3)^T$ and $x_{\mathcal{R}} = (x \ y)^T$ and then

$$\begin{array}{l} \max x + 2y \\ x + y + s_1 = 3 \\ x + s_2 = 2 \\ y + s_3 = 2 \\ x, y, s_1, s_2, s_3 \geq 0 \end{array} \quad \Longrightarrow \quad \begin{cases} s_1 = 3 - x - y \\ s_2 = 2 - x \\ s_3 = 2 - y \end{cases}$$

$$B = B^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad B^{-1}b = b = \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix} \quad B^{-1}R = R = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Bases (6)

- In

$$x_{\mathcal{B}} = B^{-1}b - B^{-1}Rx_{\mathcal{R}}$$

  if non-basic variables are set to 0, then we get the solution

$$x_{\mathcal{R}} = 0, x_{\mathcal{B}} = B^{-1}b$$

  This solution is called the basic solution of basis B

- If the basic solution satisfies $x_{\mathcal{B}} \geq 0$ then
  it is called a basic feasible solution, and the basis is said to be feasible

Bases (7)

- Example: Consider basic variables (s_1, s_2, s_3) for the LP

  $$\begin{aligned}
  &\max x + 2y \\
  &x + y + s_1 = 3 \\
  &x + s_2 = 2 \\
  &y + s_3 = 2 \\
  &x, y, s_1, s_2, s_3 \geq 0
  \end{aligned}$$

  Then equations $x_{\mathcal{B}} = B^{-1}b - B^{-1}Rx_{\mathcal{R}}$ are $\begin{cases} s_1 = 3 - x - y \\ s_2 = 2 - x \\ s_3 = 2 - y \end{cases}$

  and then the basic solution is

  $$\begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix}, \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

  So basis (s_1, s_2, s_3) is feasible

Bases (8)

- Example: Consider basic variables (x, y, s_1) for the LP

$$\max x + 2y$$
$$x + y + s_1 = 3$$
$$x + s_2 = 2$$
$$y + s_3 = 2$$
$$x, y, s_1, s_2, s_3 \geq 0$$

Then equations $x_{\mathcal{B}} = B^{-1}b - B^{-1}Rx_{\mathcal{R}}$ are $\begin{cases} x = 2 - s_2 \\ y = 2 - s_3 \\ s_1 = -1 + s_2 + s_3 \end{cases}$

and then the basic solution is

$$\begin{pmatrix} x \\ y \\ s_1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}, \quad \begin{pmatrix} s_2 \\ s_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

So basis (x, y, s_1) is not feasible

Bases (9)

- A basis is called degenerate
  when at least one basic variable is assigned 0 in the basic solution

- Example: Consider basic variables (x, y, s_2) for the LP

  $$\begin{aligned} &\max x + 2y \\ &x + y + s_1 = 4 \\ &x + s_2 = 2 \\ &y + s_3 = 2 \\ &x, y, s_1, s_2, s_3 \geq 0 \end{aligned}$$

  Then equations $x_{\mathcal{B}} = B^{-1}b - B^{-1}Rx_{\mathcal{R}}$ are $\begin{cases} x = 2 + s_3 - s_1 \\ y = 2 - s_3 \\ s_2 = s_1 - s_3 \end{cases}$

  and then the basic solution is

  $$\begin{pmatrix} x \\ y \\ s_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} s_1 \\ s_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

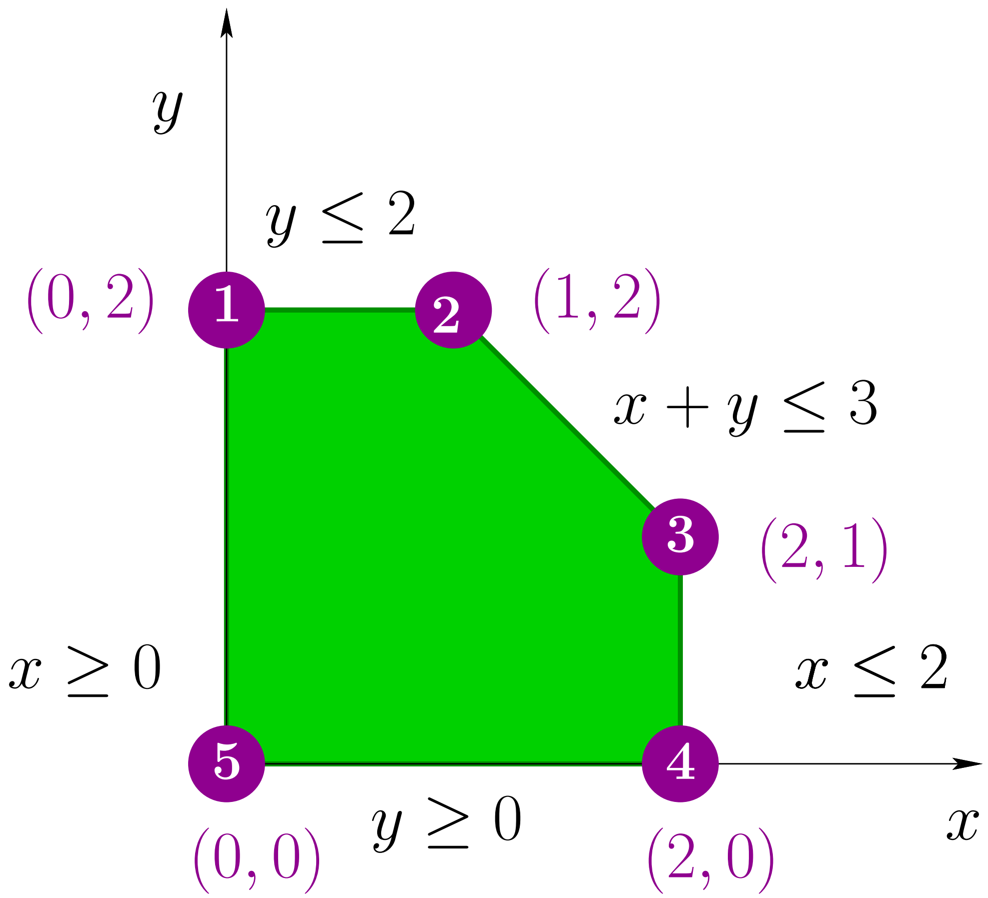
  So basis (x, y, s_1) is degenerate

Geometry of LP's (1)

- The set of feasible solutions of an LP is a convex polyhedron
- The basic feasible solutions are the vertices of the convex polyhedron

Geometry of LP's (2)

- $$\begin{aligned} &\max x + 2y \\ &x + y + s_1 = 3 \\ &x + s_2 = 2 \\ &y + s_3 = 2 \\ &x, y, s_1, s_2, s_3 \geq 0 \end{aligned}$$
- $x_{\mathcal{B}_1} = (y, s_1, s_2)$
- $x_{\mathcal{B}_2} = (x, y, s_2)$
- $x_{\mathcal{B}_3} = (x, y, s_3)$
- $x_{\mathcal{B}_4} = (x, s_1, s_3)$
- $x_{\mathcal{B}_5} = (s_1, s_2, s_3)$



Possible Outcomes of an LP (1)

- **Theorem (Fundamental Theorem of Linear Programming)**

  Let P be an LP.

  Then exactly one of the following holds:

  1. P is infeasible
  2. P is unbounded
  3. P has an optimal basic feasible solution

  It is sufficient to investigate basic feasible solutions!