Algorithmic Methods for Mathematical Models (AMMM)

Integer Linear Programming Modelling Exercises.

1. The production manager of a chemical plant is attempting to devise a shift pattern for his workforce. Each day of every working week is divided into three eight-hour shift periods (00:01-08:00, 08:01-16:00, 16:01-24:00) denoted by *night*, *day* and *late* respectively. The minimum number of workers required for each of these shifts over any working week is as below:

	Mon	**Tue**	**Wed**	**Thu**	**Fri**	**Sat**	**Sun**
Night	5	3	2	4	3	2	2
Day	7	8	9	5	7	2	5
Late	9	10	10	7	11	2	2

The union agreement governing acceptable shifts for workers is as follows:

(a) Each worker is assigned to work either a night shift or a day shift or a late shift and once a worker has been assigned to a shift they must remain on the same shift every day that they work.

(b) Each worker works four consecutive days during any seven day period.

In total there are currently 60 workers.

Formulate the production manager's problem as a MIP.

2. A food is manufactured by refining raw oils and blending them together. The raw oils come in two categories:

Vegetable oil:	VEG 1, VEG 2
Non-vegetable oil:	OIL 1, OIL 2, OIL 3

Each oil may be purchased for immediate delivery (January) or bought on the futures market for delivery in a subsequent month. Prices now and in the futures market are given below (in €/ton):

	VEG 1	VEG 2	OIL 1	OIL 2	OIL 3
January	110	120	130	110	115
February	130	130	110	90	115
March	110	140	130	100	95
April	120	110	120	120	125
May	100	120	150	110	105
June	90	100	140	80	135

The final product sells at 150 € per ton.

Vegetable oils and non-vegetable oils require different production lines for refining. In any month it is not possible to refine more than 200 tons of vegetable oils and more than 250 tons of non-vegetable oils. There is no loss of weight in the refining process and the cost of refining may be ignored.

Each month it is possible to store up to 1000 tons of each raw oil for later use. The cost of storage for vegetable and non-vegetable oil is 5 € per ton per month. The final product cannot be stored, nor can refined oils be stored.

There is a technical restriction relating to the hardness of the final product. In the units in which hardness is measured this must lie between 3 and 6. It is assumed that the hardness of the blend is the weighted average of the hardnesses of the raw oils, which are:

VEG 1	8.8
VEG 2	6.1
OIL 1	2.0
OIL 2	4.2
OIL 3	5.0

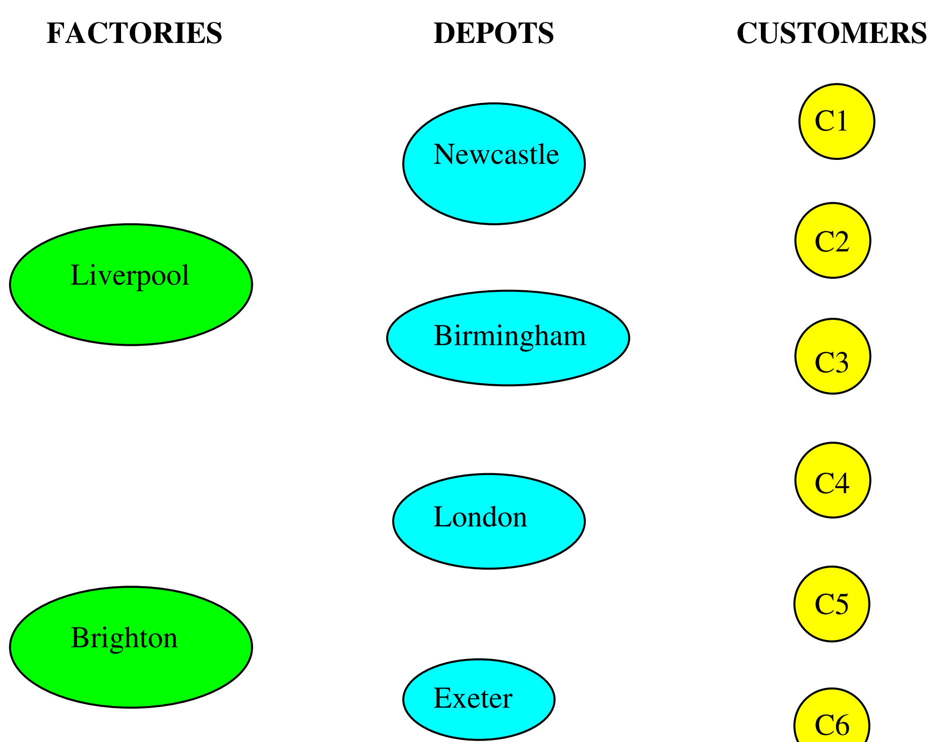
At present there are 500 tons of each type of raw oil in storage. It is required that exactly these stocks will also exist at the end of June.

(a) What buying and manufacturing policy should the company pursue in order to maximize profit?
Formulate the problem as an LP.

(b) It is wished to impose the following extra conditions:
- The food may never be made up of more than three oils in any month.
- If an oil is used in a month at least 20 tons must be used.
- If either of VEG 1 or VEG 2 is used in a month then OIL 3 must also be used.

Extend the food manufacture model to encompass these restrictions and find the new optimal solution. Can you formulate it as an LP?

3. A company has two factories, one at Liverpool and one at Brighton. In addition, it has four depots with storage facilities at Newcastle, Birmingham, London and Exeter. The company sells its product to six customers C1, C2, ..., C6. Customers can be supplied from either a depot or the factory directly.



Supplied to	Supplier					
	Liverpool factory	Brighton factory	Newcastle depot	Birmingham depot	London depot	Exeter depot
Depots						
Newcastle	0.5	—				
Birmingham	0.5	0.3				
London	1.0	0.5				
Exeter	0.2	0.2				
Customers						
C1	1.0	2.0	—	1.0	—	—
C2	—	—	1.5	0.5	1.5	—
C3	1.5	—	0.5	0.5	2.0	0.2
C4	2.0	—	1.5	1.0	—	1.5
C5	—	—	—	0.5	0.5	0.5
C6	1.0	—	1.0	—	1.5	1.5

The distribution costs are known; they are given in the table above (in £ per ton delivered). A dash in the table indicates the impossibility of certain suppliers for certain depots or customers.

Each factory has a monthly capacity given as follows, which cannot be exceeded:

Liverpool	150 000 tons
Brighton	200 000 tons

Each depot has a maximum monthly throughput given as follows, which cannot be exceeded:

Newcastle	70 000 tons
Birmingham	50 000 tons
London	100 000 tons
Exeter	40 000 tons

Each customer has a monthly requirement given as follows, which must be met:

C1	50 000 tons
C2	10 000 tons
C3	40 000 tons
C4	35 000 tons
C5	60 000 tons
C6	20 000 tons

(a) The company would like to determine what distribution pattern minimises overall cost. Formulate the problem as an LP.

(b) There is a possibility of opening new depots at Bristol and Northampton, as well as of enlarging the Birmingham depot.

It is not considered desirable to have more than four depots, and if necessary Newcastle or Exeter (or both) can be closed down.

The monthly costs (in interest charges) of the possible new depots and expansion at Birmingham are given in the table below together with the potential monthly throughputs.

	Cost (£1000)	Throughput (1000 tons)
Bristol	12	30
Northampton	4	25
Birmingham (expansion)	3	20

The monthly savings of closing down the Newcastle and Exeter depots are given in the following table:

	Savings (£1000)
Newcastle	10
Exeter	5

The distribution costs involving the new depots are given in the following tables (in £ per ton delivered):

Supplied to	**Supplier**	
	Liverpool factory	Brighton factory
New depots		
Bristol	0.6	0.4
Northampton	0.4	0.3

Supplied to	**Supplier**	
	Bristol depot	Northampton depot
Customers		
C1	1.2	—
C2	0.6	0.4
C3	0.5	—
C4	—	0.5
C5	0.3	0.6
C6	0.8	0.9

What would be the best resultant distribution pattern to minimise overall costs? Formulate the problem as a MIP. Could you formulate it as an LP?

4. An engineering factory makes seven products (PROD 1 to PROD 7) on the following machines: four grinders, two vertical drills, three horizontal drills, one borer and one planer. Each product yields a certain contribution to profit. These quantities (in €/unit) together with the production times (in hours/unit) required on each process are given below. A dash indicates that a product does not require a process.

	PROD 1	**PROD 2**	**PROD 3**	**PROD 4**	**PROD 5**	**PROD 6**	**PROD 7**
Contribution to profit	10	6	8	4	11	9	3
Grinding	0.5	0.7	—	—	0.3	0.2	0.5
Vertical drilling	0.1	0.2	—	0.3	—	0.6	—
Horizontal drilling	0.2	—	0.8	—	—	—	0.6
Boring	0.05	0.03	—	0.07	0.1	—	0.08
Planing	—	—	0.01	—	0.05	—	0.05

In the present month (January) and the five subsequent months, certain machines will be down for maintenance. These machines will be as follows:

January	1	Grinder
February	2	Horizontal drills
March	1	Borer
April	1	Vertical drill
May	1	Grinder and 1 Vertical drill
June	1	Planer and 1 Horizontal drill

There are upper limits on how much each product can be sold in each month. These are given in the following table:

	PROD 1	**PROD 2**	**PROD 3**	**PROD 4**	**PROD 5**	**PROD 6**	**PROD 7**
January	500	1000	300	300	800	200	100
February	600	500	200	0	400	300	150
March	300	600	0	0	500	400	100
April	200	300	400	500	200	0	100
May	0	100	500	100	1000	300	0
June	500	500	100	300	1100	500	60

It is possible to store up to 100 units of each product at a time at a cost of 0.5 € per unit per month. There are no stocks at present, but it is desired to have a stock of 50 of each type of product at the end of June.

It may be assumed that each month consists of only 24 working days, each with two shifts of 8 hours each day.

(a) When and what should the factory make in order to maximise the total profit? Formulate the problem as an LP.

(b) Instead of stipulating when each machine is down for maintenance, it is desired to find the best month for each machine to be down. Each machine must be down for maintenance in one month of the six apart from the grinding machines, only two of which need be down in any six months. Extend the model to allow it to make these extra decisions. Can you formulate it as an LP?

5. A mining company is going to plan how to continue operating in a certain area for the next five years. There are four mines in this area, but it can operate at most three in any one year. Although a mine may not operate in a certain year, it is still necessary to keep it "open", in the sense that royalties are payable, if it is to be operated in a future year. On the other hand, if a mine is not going to be worked again, it can be permanently closed down and no more royalties need be paid. The yearly royalties payable on each mine kept "open" are as follows:

Mine 1	5 million €
Mine 2	4 million €
Mine 3	4 million €
Mine 4	5 million €

There is an upper limit on the amount of ore that can be extracted from each mine in a year. These upper limits are as follows:

Mine 1	2×10^6 tons €
Mine 2	2.5×10^6 tons €
Mine 3	1.3×10^6 tons €
Mine 4	3×10^6 tons €

The ore from the different mines is of varying quality. The quality measurements for each mine are given in the following table:

Mine 1	1.0
Mine 2	0.7
Mine 3	1.5
Mine 4	0.5

Blending ores together results in an ore whose quality is the weighted average of the quality measurements of the ingredients. For example, if equal quantities of two ores were combined, the resultant ore would have a quality measurement half way between that of the ingredient ores.

In each year, it is necessary to combine the total outputs from each mine to produce a blended ore of exactly some stipulated quality. For each year, these qualities are as follows:

Year 1	0.9
Year 2	0.8
Year 3	1.2
Year 4	0.6
Year 5	1.0

The final blended ore sells for 10 € ton each year. Which mines should be operated each year and how much should they produce? Formulate the problem as a MIP.

6. A large company wishes to move some of its departments out of London. There are benefits to be derived from doing this (cheaper housing, government incentives, easier recruitment, etc.). Also, however, there will be greater costs of communication between departments. Where should each department be located so as to minimise overall yearly cost?

   The company comprises five departments (A, B, C, D and E). The possible cities for relocation are Bristol and Brighton. Alternatively, a department may be kept in London. None of these cities (including London) may be the location for more than three of the departments. Benefits to be derived from each relocation are given (in k€ per year) as follows:

	A	B	C	D	E
Bristol	10	15	10	20	5
Brighton	10	20	15	15	15

   Communication costs are determined by C_{ik} and D_{jl}, where C_{ik} is the quantity of communication between departments i and k per year and D_{jl} is the cost per unit of communication between cities j and l. Coefficients C_{ik} and D_{jl} are given in the following tables:

Quantities of communication C_{ik} (in thousands of units)

	A	B	C	D	E
A	—	0.0	1.0	1.5	0.0
B	—	—	1.4	1.2	0.0
C	—	—	—	0.0	2.0
D	—	—	—	—	0.7

Costs per unit of communication D_{jl} (in €)

	Bristol	Brighton	London
Bristol	5	14	13
Brighton	—	—	9
London	—	—	10

7. A number of power stations are committed to meeting the following electricity load demands over a day:

12 p.m.	to	6 a.m.	15000 MW
6 a.m.	to	9 a.m.	30000 MW
9 a.m.	to	3 p.m.	25000 MW
3 p.m.	to	6 p.m.	40000 MW
6 p.m.	to	12 p.m.	27000 MW

(the demands are for the whole indicated period, **not** per hour)

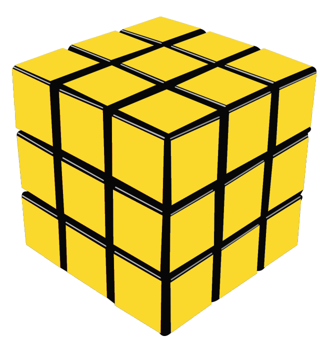
There are three types of generating units available: 12 of type 1, 10 of type 2, and 5 of type 3. Each generator has to work between a minimum and a maximum level. There is an hourly cost of running each generator at minimum level. In addition there is an extra hourly cost for each MW at which a unit is operated above minimum level. To start up a generator also involves a cost. All this information is given in the table below:

	Minimum level	Maximum level	Cost per hour at minimum	Cost per hour per MW above minimum	Start up cost
Type 1	850 MW	2000 MW	1000 €	2 €	2000 €
Type 2	1250 MW	1750 MW	2600 €	1.3 €	1000 €
Type 3	1500 MW	4000 MW	3000 €	3 €	500 €

In addition to meeting the estimated load demands there must be sufficient generators working at any time to make it possible to meet an increase in load of up to 15%. This increase would have to be accomplished by adjusting the output of generators already operating within their permitted limits.

Which generators should be working in which periods of the day to minimize total cost? Construct a MIP to answer this question.

8. Twenty-seven cells are arranged in a three-dimensional $3 \times 3 \times 3$ cube as shown below:



Three cells are regarded as lying in the same line if they are on the same horizontal or vertical line or the same diagonal. Diagonals exist on each horizontal and vertical section and connecting opposite vertices of the cube. (There are 49 lines altogether.)

Given 13 white balls and 14 black balls, arrange them, one to a cell, so as to minimise the number of lines with balls all of one colour.

9. Given this Sudoku:

5	3			7				
6			1	9	5			
	9	8					6	
8				6				3
4			8		3			1
7				2				6
	6					2	8	
			4	1	9			5
				8			7	9

formulate the problem of completing it as a MIP.

10. An outbreak of an infectious disease has been observed in a set N of locations.There exists a set M of teams capable of investigating these outbreaks. Team $i \in M$ can conclude its investigation of the outbreak in location $j \in N$ in t_{ij} hours. Each team can either investigate zero, one, or two outbreaks. If a team investigates two outbreaks, they must travel from one location to the next. The travel time from location $j_1 \in N$ to location $j_2 \in N$ is $d_{j_1 j_2}$. Once all outbreaks have been investigated, a disease control center can take action to combat the outbreak. The goal is to minimize the amount of time necessary to complete the investigation of all locations.

11. A medical company is attempting to acquire a certain drug from a set M of suppliers. The company wishes to have a stock of d_t units of this drug in month t, for $t = 1, \ldots, T$. Purchasing one unit of the drug from supplier $i \in M$ during time period $t \in \{1, \ldots, T\}$ costs c_{it} (in €). However, in order to purchase drugs from supplier $i \in M$, at any time period, the company must purchase a minimum of ℓ_i units of the drug during that time period. Fortunately, the company has room for h units of inventory, and so at most h units of the drug can be stored from one period to the next. If the company finds itself with too many units of the drug, it can simply throw away the extra supply. The goal is to minimize the cost required to purchase the drug for time periods $1, \ldots, T$.

12. A nurse is assigned a set N of patients. For each patient $i \in N$, the nurse must spend p_i minutes examining the patient. The nurse must then wait somewhere between ℓ_i and u_i minutes before checking up on the patient, which requires q_i minutes. The nurse, of course, cannot be in two places at the same time, although we will assume for simplicity that the travel time to walk from one patient's room to another is zero. The objective is to minimize the total amount of time required to tend to all patients.