The Simplex Method

Algorithmic Methods for Mathematical Models

(AMMM)

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Global Idea

- **Theorem (Fundamental Theorem of Linear Programming)**
  Let P be an LP. Then exactly one of the following holds:

  1. P is infeasible
  2. P is unbounded
  3. P has an optimal basic feasible solution

- The theorem ensures it is sufficient to explore basic feasible solutions to find the optimum of a feasible and bounded LP

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  Let P be an LP. Then exactly one of the following holds:

  1. P is infeasible
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- The theorem ensures it is sufficient to explore basic feasible solutions to find the optimum of a feasible and bounded LP

- The simplex method moves from one basic feasible solution to another that does not worsen the objective function while

  - optimality or
  - unboundedness

  are not detected

Bases and Tableaux

- In what follows sign constraints are implicit and may be left out
- Given a basis B, its tableau is the system of equations

$$x_{\mathcal{B}} = B^{-1}b - B^{-1}Rx_{\mathcal{R}}$$

which expresses values of basic variables in terms of non-basic variables

- Example:

$$\begin{array}{l} \min -x - 2y \\ x + y + s_1 = 3 \\ x + s_2 = 2 \\ y + s_3 = 2 \\ x, y, s_1, s_2, s_3 \geq 0 \end{array} \quad \Longrightarrow \quad \left\{ \begin{array}{l} s_1 = 3 - x - y \\ s_2 = 2 - x \\ s_3 = 2 - y \end{array} \right.$$

$$x_{\mathcal{B}} = (s_1, s_2, s_3)$$
$$x_{\mathcal{R}} = (x, y)$$

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$$\begin{array}{l} x_{\mathcal{B}} = (x, y, s_2) \\ x_{\mathcal{R}} = (s_1, s_3) \end{array}$$

Basic Solution in a Tableau

- The basic solution can be easily obtained from the tableau by looking at independent terms

- Example:

$$\begin{cases} x = 1 + s_3 - s_1 \\ y = 2 - s_3 \\ s_2 = 1 - s_3 + s_1 \end{cases}$$

Recall that by definition of basic solution,
the values for non-basic variables are zero

Hence the solution is

Non-basic variables: $\begin{cases} s_1 = 0 \\ s_3 = 0 \end{cases}$ Basic variables: $\begin{cases} x = 1 \\ y = 2 \\ s_2 = 1 \end{cases}$

Objective Value in a Tableau

- Tableaux can be extended with
  the expression of the objective function in terms of non-basic variables

$$\begin{cases} \min -x - 2y \Longrightarrow \min -5 + s_1 + s_3 \\ x = 1 + s_3 - s_1 \\ y = 2 - s_3 \\ s_2 = 1 - s_3 + s_1 \end{cases}$$

given that
$$\begin{array}{ll} -x - 2y & = \\ -(1 + s_3 - s_1) - 2(2 - s_3) & = \\ -1 - s_3 + s_1 - 4 + 2s_3 & = \\ -5 + s_1 + s_3 & \end{array}$$

- Note that objective function $-x - 2y$ evaluates to -5 at basic solution

Non-basic variables: $\begin{cases} s_1 = 0 \\ s_3 = 0 \end{cases}$ Basic variables: $\begin{cases} x = 1 \\ y = 2 \\ s_2 = 1 \end{cases}$

- In general the value of the objective function at basic solution
  can be easily found by looking at independent term

Detecting Optimality

- Tableaux can be extended with
  the expression of the objective function in terms of non-basic variables

$$\begin{cases} \min -x - 2y \Longrightarrow \min -5 + s_1 + s_3 \\ x = 1 + s_3 - s_1 \\ y = 2 - s_3 \\ s_2 = 1 - s_3 + s_1 \end{cases}$$

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- Coefficients of non-basic variables in objective function after substitution are called reduced costs

- Sufficient condition for optimality: all reduced costs are ≥ 0

  The cost of any other feasible solution can't improve on the basic solution

  So the basic solution is optimal!

Detecting Optimality

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- Coefficients of non-basic variables in objective function after substitution are called reduced costs

- Sufficient condition for optimality: all reduced costs are ≥ 0

  The cost of any other feasible solution can't improve on the basic solution

  So the basic solution is optimal!

- The condition that all reduced costs are ≥ 0 is however not necessary for optimality: we may have an optimal basic solution, but we cannot see it

Improving the Basic Solution (1)

- What can we do if the tableau does not satisfy the optimality condition?

$$\begin{aligned} &\min -x - 2y \\ &x + y + s_1 = 3 \\ &x + s_2 = 2 \\ &y + s_3 = 2 \\ &x, y, s_1, s_2, s_3 \geq 0 \end{aligned}$$

$$\mathcal{B} = (s_1, s_2, s_3)$$

$$\begin{cases} \min \ -x - 2y \\ s_1 = 3 - x - y \\ s_2 = 2 - x \\ s_3 = 2 - y \end{cases}$$

- E.g. variable y has a negative reduced cost (namely, -2)
- If we can get a new solution where $y > 0$ and the rest of non-basic variables are still assigned to 0, we will get a better solution
- The algorithm tries to improve the objective value with this idea: increase the value of a non-basic variable with negative reduced cost while the rest of non-basic variables are frozen to 0

  E.g. increase y while keeping $x = 0$

Improving the Basic Solution (2)

- Recall: to have a feasible solution we must satisfy
  - the system of equations

    OK if basic variables are assigned values according to the tableau
  - the sign constraints of non-basic variables

    OK if we increase the value of the chosen non-basic variable (now 0) and leave all other non-basic variables frozen to 0
  - the sign constraints of basic variables

    This is what we have to be careful about.
- So let us increase the value of variable y
  while satisfying sign constraints on basic variables
- E.g., in the current tableau we have $s_1 = 3 - x - y$

  As $x = 0$, the value of s_1 changes as a function of y following $s_1 = 3 - y$

  Hence $s_1 \geq 0$ iff $3 - y \geq 0$ iff $y \leq 3$

  The sign constraint of s_1 limits the new value of y to ≤ 3

Improving the Basic Solution (3)

- Let us increase the value of variable y while satisfying sign constraints on basic variables

$$\begin{cases} s_1 = 3 - x - y & \text{Limits new value to } \leq 3 \\ s_2 = 2 - x & \text{Does not limit new value} \\ s_3 = 2 - y & \text{Limits new value to } \leq 2 \end{cases}$$

- Altogether the new value for y must satisfy $y \leq 3$ and $y \leq 2$

- So the best possible value for y is $\min(3, 2) = 2$

- The sign constraint of s_3 is tight, i.e., the inequality $s_3 \geq 0$ is the one that limits the new value for y

- Hence we get a solution that assigns

$$\text{Non-basic vars: } \begin{cases} y = 2 \\ x = 0 \end{cases} \qquad \text{Basic vars: } \begin{cases} s_1 = 3 - 0 - 2 = 1 \\ s_2 = 2 - 0 = 2 \\ s_3 = 2 - 2 = 0 \end{cases}$$

Improving the Basic Solution (4)

- The new solution is no longer the basic solution of the current basis.
  But it turns out it is the basic solution of another basis.
  We only need to change the current basis.

- When increasing the value of the improving non-basic variable, all basic variables for which the sign constraint is tight become 0

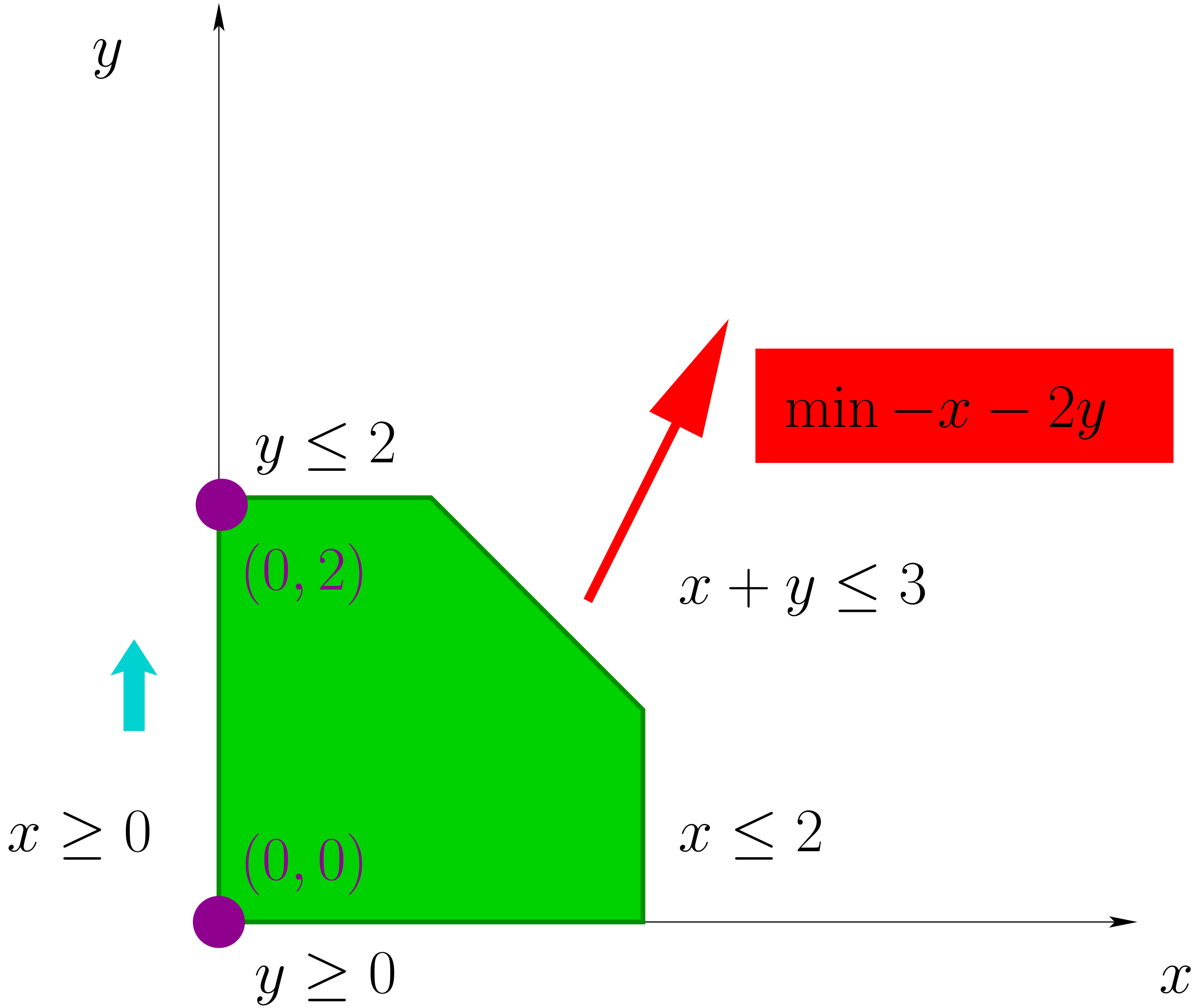
$$y = 2 \rightarrow s_3 = 0$$

- Choose a tight basic variable, here s_3, to be exchanged with the improving non-basic variable, here y
- The new basis is (s_1, s_2, y)
- Pivoting: we can get the tableau of the new basis by isolating the new non-basic variable and substituting everywhere:

$$s_3 = 2 - y \quad \Rightarrow \quad y = 2 - s_3$$

$$\begin{cases} \min \ -x - 2y \\ s_1 = 3 - x - y \\ s_2 = 2 - x \\ s_3 = 2 - y \end{cases} \quad \Longrightarrow \quad \begin{cases} \min \ -4 - x + 2s_3 \\ s_1 = 1 + s_3 - x \\ s_2 = 2 - x \\ y = 2 - s_3 \end{cases}$$

Improving the Basic Solution (5)



Improving the Basic Solution (6)

- Do we satisfy the optimality condition?
  No, variable x has negative reduced cost -1
- So let us now increase the value of variable x while keeping $s_3 = 0$

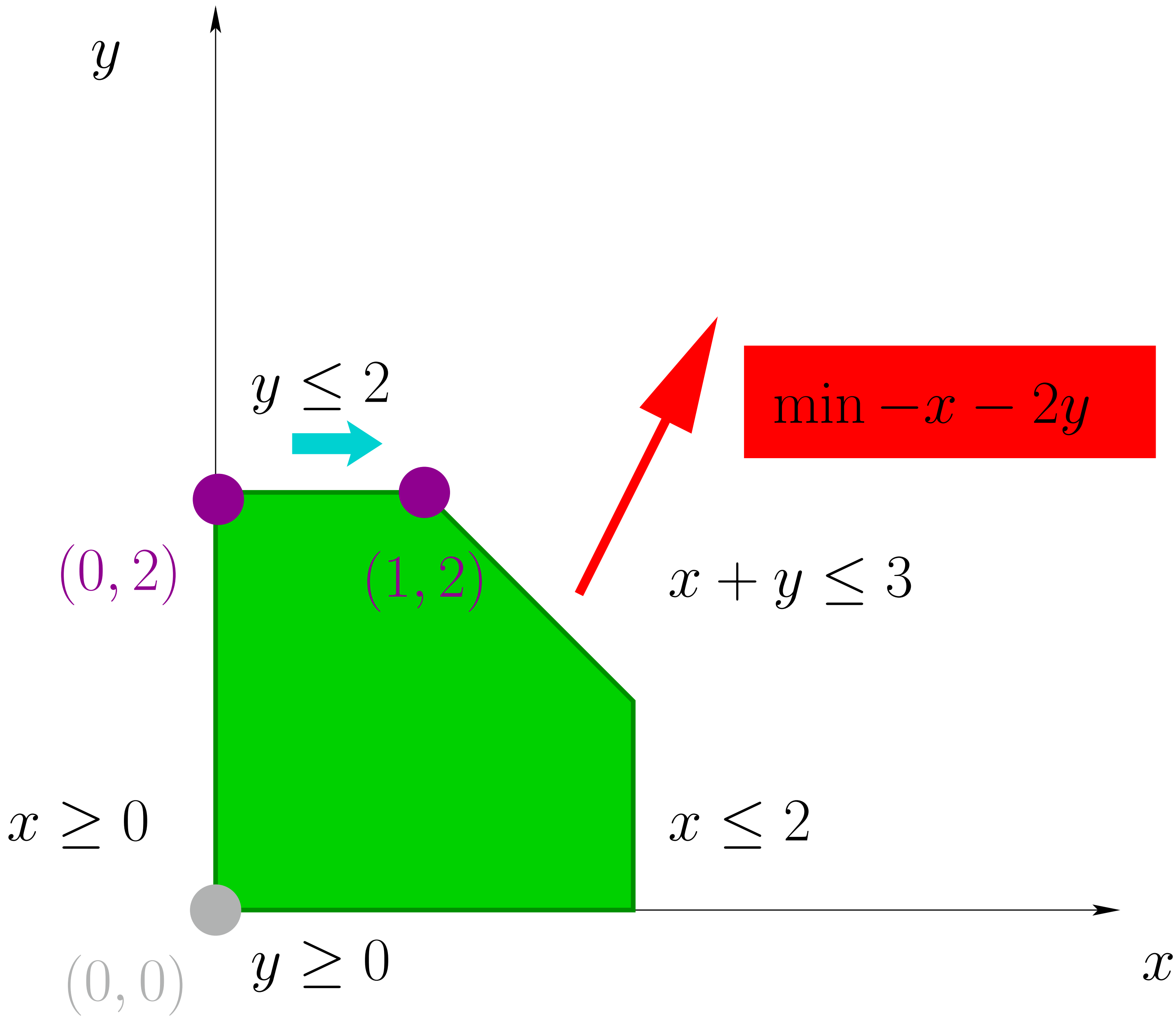
$$\begin{cases} \min \ -4 - x + 2s_3 & \\ s_1 = 1 + s_3 - x & \text{Limits new value to } \leq 1 \\ s_2 = 2 - x & \text{Limits new value to } \leq 2 \\ y = 2 - s_3 & \text{Does not limit new value} \end{cases}$$

- Best possible new value for x is $\min(2, 1) = 1$
- Variable s_1 is the only basic variable that is tight
- So variable s_1 leaves the basis and variable x enters

$$\begin{cases} \min \ -4 - x + 2s_3 \\ s_1 = 1 + s_3 - x \\ s_2 = 2 - x \\ y = 2 - s_3 \end{cases} \Longrightarrow \begin{cases} \min -5 + s_1 + s_3 \\ x = 1 + s_3 - s_1 \\ s_2 = 1 - s_3 + s_1 \\ y = 2 - s_3 \end{cases}$$

- Now we satisfy the optimality condition: optimum found!

Improving the Basic Solution (7)



Unboundedness (1)

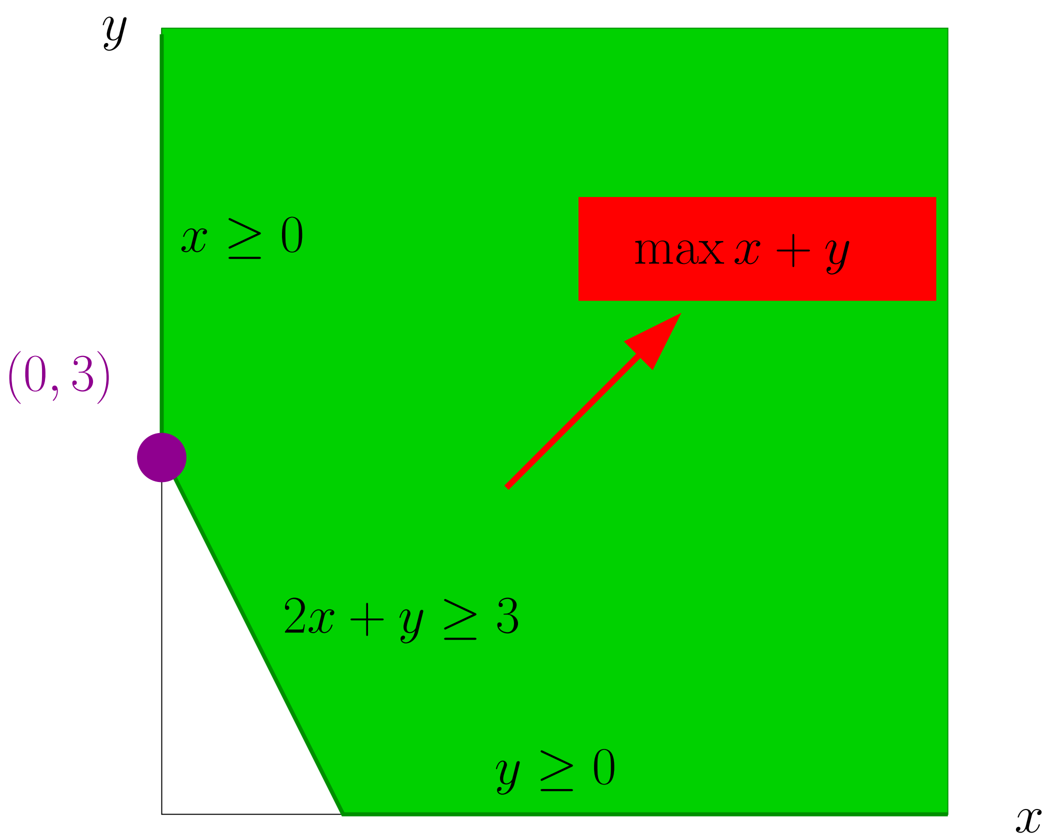
- We said that the simplex method moves from one basic feasible solution to another that does not worsen the objective function while
  - optimality or
  - unboundedness

  are not detected

- We have seen how optimality is detected
- But when is unboundedness is detected?

Unboundedness (2)

- We detect unboundedness when new value for chosen non-basic variable can grow indefinitely (i.e., the best value is not bounded)



$$\max \; x + y$$
$$2x + y \geq 3$$
$$x, y \geq 0$$

$$\Downarrow$$

$$\begin{cases} \min -x - y \\ 2x + y - s = 3 \end{cases}$$

$$\Downarrow$$

$$\begin{cases} \min -3 + x - s \\ y = 3 - 2x + s \end{cases}$$

- Non-basic variable s has a negative reduced cost -1
- We can make s larger and larger as it is compatible with constraint $y \geq 0$

Outline of the Simplex Algorithm

1. `Initialization:` If there exists a feasible basis,
   Then pick one.
   Else return INFEASIBLE.

2. `Pricing:` If all reduced costs are ≥ 0,
   Then return OPTIMAL.
   Else pick a non-basic variable with reduced cost < 0.

3. `Ratio test:` Compute the best value for the chosen non-basic variable that respects the sign constraints of basic variables.

   If best value is not bounded,
   Then return UNBOUNDED.
   Else pick tight basic variable to swap with the chosen non-basic variable

4. `Update:` Update the tableau and go to 2.

Finding an Initial Basis (1)

- Note that to solve an LP

$$\min c^T x$$
$$Ax = b$$
$$x \geq 0$$

initially we need a feasible basis at step 1.

- Step 1 (finding a feasible basis) is called phase I of the simplex algorithm

- Steps 2-4 (optimizing) are called phase II

- We'll see next how to get a feasible basis with the same simplex algorithm by solving another LP for which phase I is trivial

Finding an Initial Basis (2)

- For example

$$\begin{cases} \min \ -x - 2y \\ 1 \leq x + y \leq 3 \\ 0 \leq x \leq 2 \\ 0 \leq y \leq 2 \end{cases} \Rightarrow \begin{cases} \min -x - 2y \\ x + y + s_1 = 3 \\ x + y - s_2 = 1 \\ x + s_3 = 2 \\ y + s_4 = 2 \end{cases}$$

- An initial basis consisting of slacks is simple, as the inverse is the identity:

$$\begin{cases} s_1 = 3 - x - y \\ s_2 = -1 + x + y \\ s_3 = 2 - x \\ s_4 = 2 - y \end{cases}$$

- But in this example this basis turns out not to be feasible!

Finding an Initial Basis (3)

- Problem: the slack of constraint $x + y \geq 1$ has the "wrong" sign

$$x + y \geq 1 \quad \rightarrow \quad x + y - s_2 = 1 \quad \rightarrow \quad s_2 = -1 + x + y$$

- We can add an artificial variable z_1 to the equation with the "right" sign and use it in the basis instead of s_2:

$$\begin{cases} x + y + s_1 = 3 \\ x + y - s_2 + z_1 = 1 \\ x + s_3 = 2 \\ y + s_4 = 2 \end{cases} \Rightarrow \begin{cases} s_1 = 3 - x - y \\ z_1 = 1 - x - y + s_2 \\ s_3 = 2 - x \\ s_4 = 2 - y \end{cases}$$

- Variable z_1 measures how far we are from satisfying constraint $x + y \geq 1$

- In particular, if in a solution $z_1 = 0$ then $x + y \geq 1$ is satisfied

- Idea: minimize z_1 (temporarily forgetting the original objective function!) and see if we can get a solution where $z_1 = 0$

Finding an Initial Basis (4)

- So let us solve

$$\begin{cases} \min z_1 \\ x + y + s_1 = 3 \\ x + y - s_2 + z_1 = 1 \\ x + s_3 = 2 \\ y + s_4 = 2 \end{cases}$$

- Let us apply the simplex algorithm itself!
- Taking basic variables (s_1, z_1, s_3, s_4) we get the tableau:

$$\begin{cases} \min z_1 \\ s_1 = 3 - x - y \\ z_1 = 1 - x - y + s_2 \\ s_3 = 2 - x \\ s_4 = 2 - y \end{cases} \Longrightarrow \begin{cases} \min 1 - x - y + s_2 \\ s_1 = 3 - x - y \\ z_1 = 1 - x - y + s_2 \\ s_3 = 2 - x \\ s_4 = 2 - y \end{cases}$$

- Non-basic variables x, y do not meet the optimality condition: both have reduced cost -1
- E.g., let us choose x

Finding an Initial Basis (5)

- So let us now increase the value of variable x while keeping $y = s_2 = 0$

$$\begin{cases} \min 1 - x - y + s_2 & \\ s_1 = 3 - x - y & \text{Limits new value to } \leq 3 \\ z_1 = 1 - x - y + s_2 & \text{Limits new value to } \leq 1 \\ s_3 = 2 - x & \text{Limits new value to } \leq 2 \\ s_4 = 2 - y & \text{Does not limit new value} \end{cases}$$

- Altogether the new value for x must satisfy $x \leq 3$ and $x \leq 1$ and $x \leq 2$
- So the best possible value for x is $\min(3, 1, 2) = 1$
- Variable z_1 is the only basic variable that is tight
- So variable z_1 leaves the basis and variable x enters

$$\begin{cases} \min 1 - x - y + s_2 \\ s_1 = 3 - x - y \\ z_1 = 1 - x - y + s_2 \\ s_3 = 2 - x \\ s_4 = 2 - y \end{cases} \Rightarrow \begin{cases} \min z_1 \\ s_1 = 2 + z_1 - s_2 \\ x = 1 - z_1 - y + s_2 \\ s_3 = 1 + z_1 + y - s_2 \\ s_4 = 2 - y \end{cases}$$

Finding an Initial Basis (6)

- The tableau for current basic variables (s_1, x, s_3, s_4) is optimal:

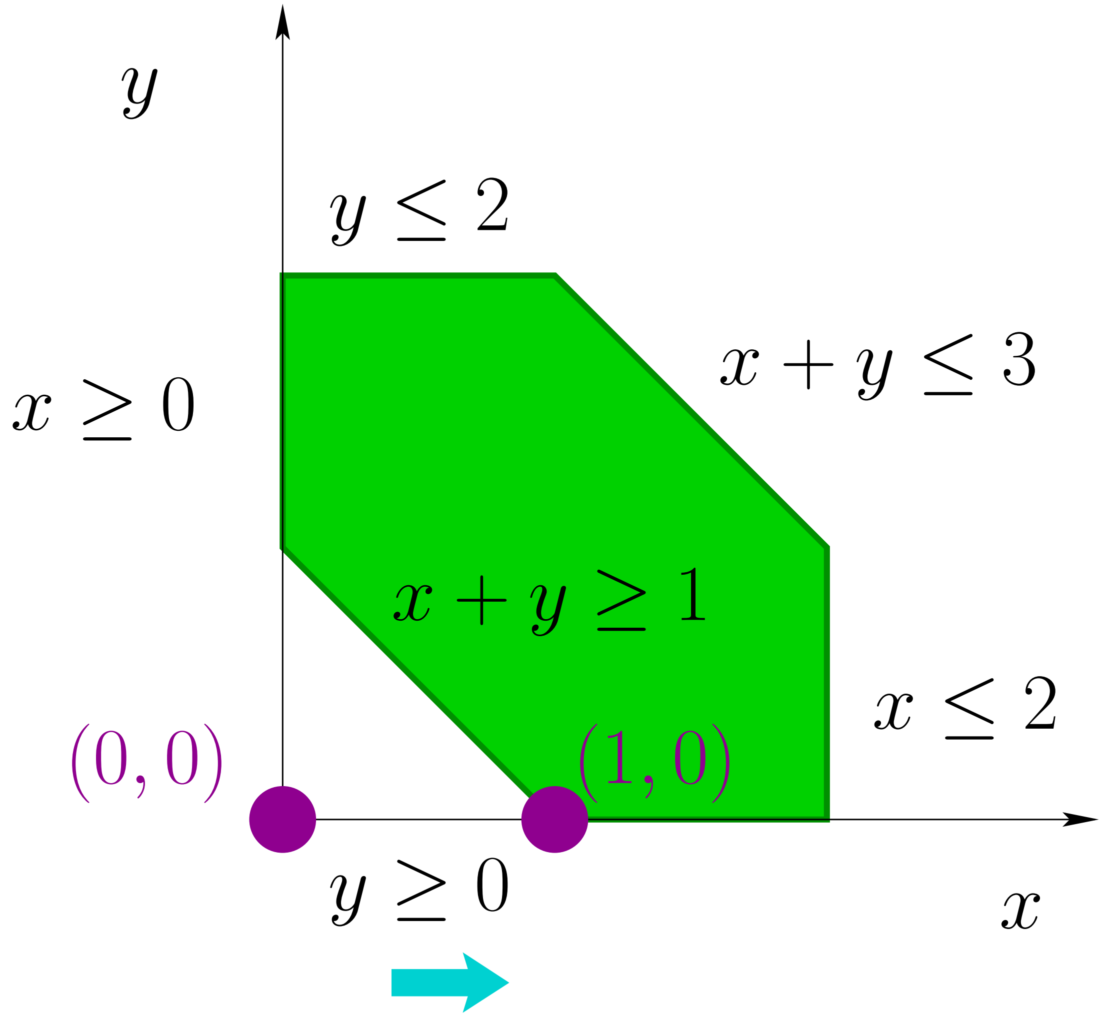
$$\begin{cases} \min z_1 \\ s_1 = 2 + z_1 - s_2 \\ x = 1 - z_1 - y + s_2 \\ s_3 = 1 + z_1 + y - s_2 \\ s_4 = 2 - y \end{cases}$$

- Dropping artificial variable z_1 yields a feasible basis for the original LP

$$\begin{cases} s_1 = 2 - s_2 \\ x = 1 - y + s_2 \\ s_3 = 1 + y - s_2 \\ s_4 = 2 - y \end{cases}$$

- This concludes phase I (step 1 of the simplex algorithm)

Finding an Initial Basis (7)



Finding an Initial Basis (8)

- Now we are ready to apply phase II (steps 2-4 of the simplex algorithm):

$$\begin{cases} \min -x - 2y \\ s_1 = 2 - s_2 \\ x = 1 - y + s_2 \\ s_3 = 1 + y - s_2 \\ s_4 = 2 - y \end{cases} \Longrightarrow \begin{cases} \min -1 - y - s_2 \\ s_1 = 2 - s_2 \\ x = 1 - y + s_2 \\ s_3 = 1 + y - s_2 \\ s_4 = 2 - y \end{cases}$$

- Non-basic variables y, s_2 do not meet the optimality condition: both have reduced cost -1

- ...

Finding an Initial Basis (9)

- In general, let us imagine that we want to get an initial feasible basis for

$$\begin{aligned} &\min c^T x \\ &Ax = b \\ &x \geq 0 \end{aligned}$$

- Let us assume wlog. that $b \geq 0$ (multiply equations by -1 if necessary)

- We introduce a new vector of artificial variables $z = (z_1, ..., z_m)$ and solve

$$\begin{aligned} &\min z_1 + \ldots + z_m \\ &Ax + z = b \\ &x, z \geq 0 \end{aligned}$$

- By taking the artificial variables z as basic variables
we get a basis with an easily computable tableau $z = b - Ax$
This is feasible as $b \geq 0$, so the simplex algorithm can be readily applied

- We may not need to add an artificial variable for each row
if the slack has already the right sign (as we did in the previous example)

Finding an Initial Basis (10)

- If the optimal cost of the phase I problem

$$\begin{aligned}&\min z_1 + \ldots + z_m\\&Ax + z = b\\&x, z \geq 0\end{aligned}$$

is 0 then a feasible basis for the original problem can be obtained by making all artificial variables non-basic and eventually dropping them

- Otherwise the cost is > 0 and the original problem is infeasible

Note that if the original problem is feasible
then the optimal cost of the phase I problem is 0.

Indeed, if there is x such that $Ax = b$ and $x \geq 0$ then $Ax + 0 = b$ and so $(x, 0)$ is a solution for

$$\begin{aligned}&\min z_1 + \ldots + z_m\\&Ax + z = b\\&x, z \geq 0\end{aligned}$$

with cost 0

Finding an Initial Basis (11)

- Let us apply phase I to an example of infeasible LP:

$$\begin{cases} \min \ -x-2y \\ 5 \le x+y \\ 0 \le x \le 2 \\ 0 \le y \le 2 \end{cases} \Rightarrow \begin{cases} \min \ z_1 \\ x+y-s_1+z_1=5 \\ x+s_2=2 \\ y+s_3=2 \end{cases}$$

- We take z_1, s_2, s_3 as basic variables of the initial basis:

$$\begin{cases} \min \ 5-x-y+s_1 \\ z_1=5-x-y+s_1 \\ s_2=2-x \\ s_3=2-y \end{cases}$$

- Non-basic variables x, y do not meet the optimality condition: both have reduced cost -1
- E.g., let us choose y

Finding an Initial Basis (12)

- So let us now increase the value of variable y while keeping $x = s_1 = 0$

$$\begin{cases} \min\ 5 - x - y + s_1 & \\ z_1 = 5 - x - y + s_1 & \text{Limits new value to} \leq 5 \\ s_2 = 2 - x & \text{Does not limit new value} \\ s_3 = 2 - y & \text{Limits new value to} \leq 2 \end{cases}$$

- The best possible value for y is $\min(5, 2) = 2$
- Variable s_3 is the only basic variable that is tight
- So variable s_3 leaves the basis and variable y enters

$$\begin{cases} \min\ 5 - x - y + s_1 \\ z_1 = 5 - x - y + s_1 \\ s_2 = 2 - x \\ s_3 = 2 - y \end{cases} \Rightarrow \begin{cases} \min\ 3 - x + s_3 + s_1 \\ z_1 = 3 - x + s_3 + s_1 \\ s_2 = 2 - x \\ y = 2 - s_3 \end{cases}$$

- Now non-basic variable x does not meet the optimality condition: it has reduced cost -1

Finding an Initial Basis (13)

- So let us now increase the value of variable x while keeping $s_1 = s_3 = 0$

$$\left\{\begin{array}{ll} \min\ 3 - x + s_3 + s_1 & \\ z_1 = 3 - x + s_3 + s_1 & \text{Limits new value to } \leq 3 \\ s_2 = 2 - x & \text{Limits new value to } \leq 2 \\ y = 2 - s_3 & \text{Does not limit new value} \end{array}\right.$$

- The best possible value for x is $\min(3, 2) = 2$
- Variable s_2 is the only basic variable that is tight
- So variable s_2 leaves the basis and variable x enters

$$\left\{\begin{array}{l} \min\ 3 - x + s_3 + s_1 \\ z_1 = 3 - x + s_3 + s_1 \\ s_2 = 2 - x \\ y = 2 - s_3 \end{array}\right. \Rightarrow \left\{\begin{array}{l} \min\ 1 + s_2 + s_3 + s_1 \\ z_1 = 1 + s_2 + s_3 + s_1 \\ x = 2 - s_2 \\ y = 2 - s_3 \end{array}\right.$$

- Optimality conditions are satisfied, so now this is an optimal basis.

  The optimal value of the objective is 1, so the original problem is infeasible

Big M Method

- Alternative to phase I + phase II approach
- LP is changed as follows, where M is a "big number"

$$\begin{array}{lcl} \min c^T x & & \min c^T x + M \cdot (z_1 + \ldots + z_m) \\ Ax = b & \Longrightarrow & Ax + z = b \\ x \geq 0 & & x, z \geq 0 \end{array} \qquad \text{where } b \geq 0$$

- Again, by taking the artificial variables we get an initial feasible basis
- Advantages:
  - Search of a feasible basis for original problem is not blind wrt. cost
- Problems:
  - If M is a fixed big number,
    then the algorithm becomes numerically unstable
  - If M is kept symbolically,
    then handling costs may become too expensive

Big M Method

$$\begin{cases} \min \ -x-2y \\ 1 \le x+y \le 3 \\ 0 \le x \le 2 \\ 0 \le y \le 2 \end{cases} \Rightarrow \begin{cases} \min -x-2y \\ x+y+s_1=3 \\ x+y-s_2=1 \\ x+s_3=2 \\ y+s_4=2 \end{cases} \Rightarrow \begin{cases} \min \ -x-2y+Mz \\ x+y+s_1=3 \\ x+y-s_2+z=1 \\ x+s_3=2 \\ y+s_4=2 \end{cases}$$

Let us solve

$$\begin{cases} \min \ -x-2y+Mz \\ x+y+s_1=3 \\ x+y-s_2+z=1 \\ x+s_3=2 \\ y+s_4=2 \end{cases}$$

starting with initial feasible basis (s_1, z, s_3, s_4)

Big M Method

$$\begin{cases} \min \ M + (-1 - M)x + (-2 - M)y + Ms_2 \\ s_1 = 3 - x - y \\ z = 1 - x - y + s_2 \\ s_3 = 2 - x \\ s_4 = 2 - y \end{cases}$$

$$\implies$$

$$\begin{cases} \min \ x - 2 - 2s_2 + (M + 2)z \\ s_1 = 2 + z - s_2 \\ y = 1 - x - z + s_2 \\ s_3 = 2 - x \\ s_4 = 1 + z + x - s_2 \end{cases}$$

Once z is non-basic we can drop it and continue the optimization:

$$\begin{cases} \min \ x - 2 - 2s_2 \\ s_1 = 2 - s_2 \\ y = 1 - x + s_2 \\ s_3 = 2 - x \\ s_4 = 1 + x - s_2 \end{cases}$$

Termination and Complexity

- A step of the simplex algorithm is degenerate
  if the increment of the chosen non-basic variable is 0

- At each step of the simplex algorithm:

  cost improvement = reduced cost · increment (of chosen non-basic var)

- If the step is degenerate then there is no cost improvement

- But degenerate steps can only happen with degenerate bases

- Assume no degenerate bases occur.

  Then there is a strict improvement from a base to the next one

  So simplex terminates, as bases cannot be repeated

  No. steps is at most exponential: there are $\leq \binom{n}{m}$ bases

  Tight bound for pathological cases (Klee-Minty cube)

  In practice the cost is polynomial

Termination and Complexity

- When there is degeneracy simplex may loop forever

- Termination guaranteed with anticycling rules, e.g. Bland's rule:

  Assume there is a fixed ordering of variables.

  Pricing: among non-basic vars with reduced cost < 0, take the least one

  Ratio test: among tight basic vars, take the least one

Pricing Strategies

1. Full pricing

   Choose the variable with the most negative reduced cost

2. Partial pricing

   Make a list with the indices of the P variables
   with the most negative reduced costs.

   In following iterations choose variables from the list
   until reduced costs of all variables in the list are ≥ 0

Pricing Strategies

3. Best-improvement pricing

   Let θ_k be the increment for a non-basic variable x_k with reduced cost $d_k < 0$. Choose the variable j such that

   $$|d_j| \cdot \theta_j = \max\{|d_k| \cdot \theta_k \ \text{ such that } \ d_k < 0, k \in \mathcal{R}\}$$

4. Normalized pricing.

   Let $n_k = ||\alpha_k||$ (in practice $n_k = \sqrt{1 + ||\alpha_k||^2}$)

   where α_k is the column in the tableau of variable x_k.

   Take criteria 1. or 2. but using $\frac{d_k}{n_k}$ instead of d_k

5. Other more sophisticate normalized pricing strategies:

   steepest edge, devex

Bounded Variables

- LP solvers implement a variant of the simplex algorithm that handles bounds more efficiently for LP's of the form

$$\begin{aligned} &\min c^T x \\ &Ax = b \\ &\ell \leq x \leq u \end{aligned}$$

- Bounds are incorporated into pricing and ratio test
- Now non-basic variables will take values at the lower or the upper bound

Bounded Variables

$$\begin{aligned} &\min -x - 2y \\ &x + y \le 3 \\ &0 \le x \le 2 \\ &0 \le y \le 2 \end{aligned} \Longrightarrow \begin{aligned} &\min -x - 2y \\ &x + y + s = 3 \\ &0 \le x \le 2 \\ &0 \le y \le 2 \\ &s \ge 0 \end{aligned} \Longrightarrow \begin{aligned} &\min -x - 2y \\ &s = 3 - x - y \\ &0 \le x \le 2 \\ &0 \le y \le 2 \\ &s \ge 0 \end{aligned}$$

- Initially non-basic variables x, y are at lower bound
- We choose variable x in pricing

Bounded Variables

$$\begin{cases} \min -x - 2y \\ s = 3 - x - y & \text{Limits new value to } \leq 3 \text{ as } s \geq 0 \\ 0 \leq x \leq 2 & \text{Limits new value to } \leq 2 \text{ as } x \leq 2 \\ 0 \leq y \leq 2 \\ s \geq 0 \end{cases}$$

- Best possible new value for x is $\min(3, 2) = 2$
- Bound flip: x is still non-basic, but is now at upper bound

$$\begin{cases} \min -x - 2y \\ s = 3 - x - y \\ 0 \leq x \leq 2 \\ 0 \leq y \leq 2 \\ s \geq 0 \end{cases}$$

Bounded Variables

- Pricing considers the bound status of non-basic variables
- A non-basic variable x_j with reduced cost d_j can improve the cost function
  - if x_j is at lower bound and $d_j < 0$; or
  - if x_j is at upper bound and $d_j > 0$
- Choose y in pricing:

$$\left\{\begin{array}{ll} \min -x - 2y & \\ s = 3 - x - y & \text{Limits new value to } \leq 1 \text{ as } s \geq 0 \\ 0 \leq x \leq 2 & \\ 0 \leq y \leq 2 & \text{Limits new value to } \leq 2 \text{ as } y \leq 2 \\ s \geq 0 & \end{array}\right.$$

- Best possible new value for y is $\min(1, 2) = 1$

Bounded Variables

- Usual pivoting step now:

$$s = 3 - x - y \quad \Rightarrow \quad y = 3 - x - s$$

$$\left\{\begin{array}{l} \min -x - 2y \\ s = 3 - x - y \\ 0 \le x \le 2 \\ 0 \le y \le 2 \\ s \ge 0 \end{array}\right. \quad \Longrightarrow \quad \left\{\begin{array}{l} \min -6 + x + 2s \\ y = 3 - x - s \\ 0 \le x \le 2 \\ 0 \le y \le 2 \\ s \ge 0 \end{array}\right.$$

Bounded Variables

- Choose x in pricing. To respect bounds for y:

$$0 \le y(x) \le 2$$
$$0 \le 3 - x \le 2$$
(since x decreases its value, $0 \le y(x)$ is OK)
$$3 - x \le 2$$
$$1 \le x$$

$$\begin{cases} \min -6 + x + 2s & \\ y = 3 - x - s & \text{Limits new value to } \ge 1 \\ 0 \le x \le 2 & \text{Limits new value to } \ge 0 \\ 0 \le y \le 2 & \\ s \ge 0 & \end{cases}$$

- Best possible new value for x is $\max(1, 0) = 1$

Bounded Variables

- Usual pivoting step now:

$$y = 3 - x - s \quad \Rightarrow \quad x = 3 - y - s$$

Bounded Variables

- Usual pivoting step now:

$$y = 3 - x - s \;\;\Rightarrow\;\; x = 3 - y - s$$

$$\left\{\begin{array}{l} \min -6 + x + 2s \\ y = 3 - x - s \\ 0 \le x \le 2 \\ 0 \le y \le 2 \\ s \ge 0 \end{array}\right. \quad \Longrightarrow \quad \left\{\begin{array}{l} \min -3 + s - y \\ x = 3 - y - s \\ 0 \le x \le 2 \\ 0 \le y \le 2 \\ s \ge 0 \end{array}\right.$$

- Since upper bound of y was tight, now y is set to its upper bound
- Optimal solution: $(x, y, s) = (1, 2, 0)$ with cost -5
- Now reading the basic solution and its cost is more involved!

Bounded Variables