Two-level Logic Synthesis and Optimization

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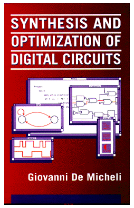


Module 1

- Objectives
  - Fundamentals of logic synthesis
  - Mathematical formulation
  - Definition of the problems

Combinational logic design
Background

- **Boolean Algebra**
  - **Quintuple** **$(B, +, \cdot, 0, 1)$**
  - **Binary Boolean algebra** **$B = \{ 0, 1 \}$**
- **Boolean function**
  - **Single output** **$f : B^n \rightarrow B$**
  - **Multiple output** **$f : B^n \rightarrow B^m$**
  - **Incompletely-specified:**
    - ***Don't care* symbol: ***
    - $f : B^n \rightarrow \{ 0, 1, * \}^m$

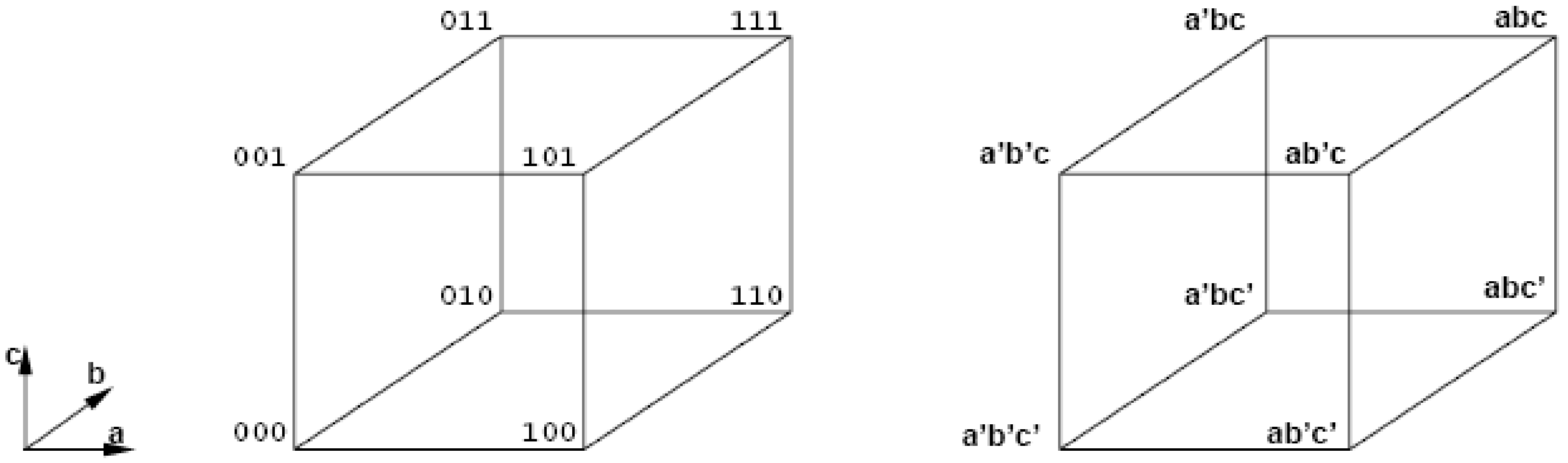
The *don't care* conditions

- We do not care about the value of a function
- Related to the environment
  - Input patterns that never occur
  - Input patterns such that some output is never observed
- Very important for synthesis and optimization

Definitions

- **Scalar function:**
  - **ON-set**
    - **Subset of the domain such that f is true**
  - **OFF-set**
    - **Subset of the domain such that f is false**
  - **DC-set**
    - **Subset of the domain such that f is a *don't care***
- **Multiple-output function:**
  - **ON, OFF, DC-sets defined for each component**

Cubical representation



Definitions

- Boolean **variables**
- Boolean **literals:**
  - Variables and their complement
- **Product** or **cube:**
  - Product of literals
- **Implicant:**
  - Product implying a value of the function (usually 1)
  - Hypercube in the Boolean space
- **Minterm:**
  - Product of all input variables implying a value of the function (usually 1)
  - Vertex in the Boolean space

Tabular representations

- **Truth table**
  - **List of all minterms of a function**
- **Implicant table or cover**
  - **List of implicants sufficient to define a function**
- **Note**
  - **Implicant tables are smaller in size as compared to truth tables**

Example of truth table

- $x = ab+a'c; \quad y = ab+bc+ac$

abc	xy
000	00
001	10
010	00
011	11
100	00
101	01
110	11
111	11

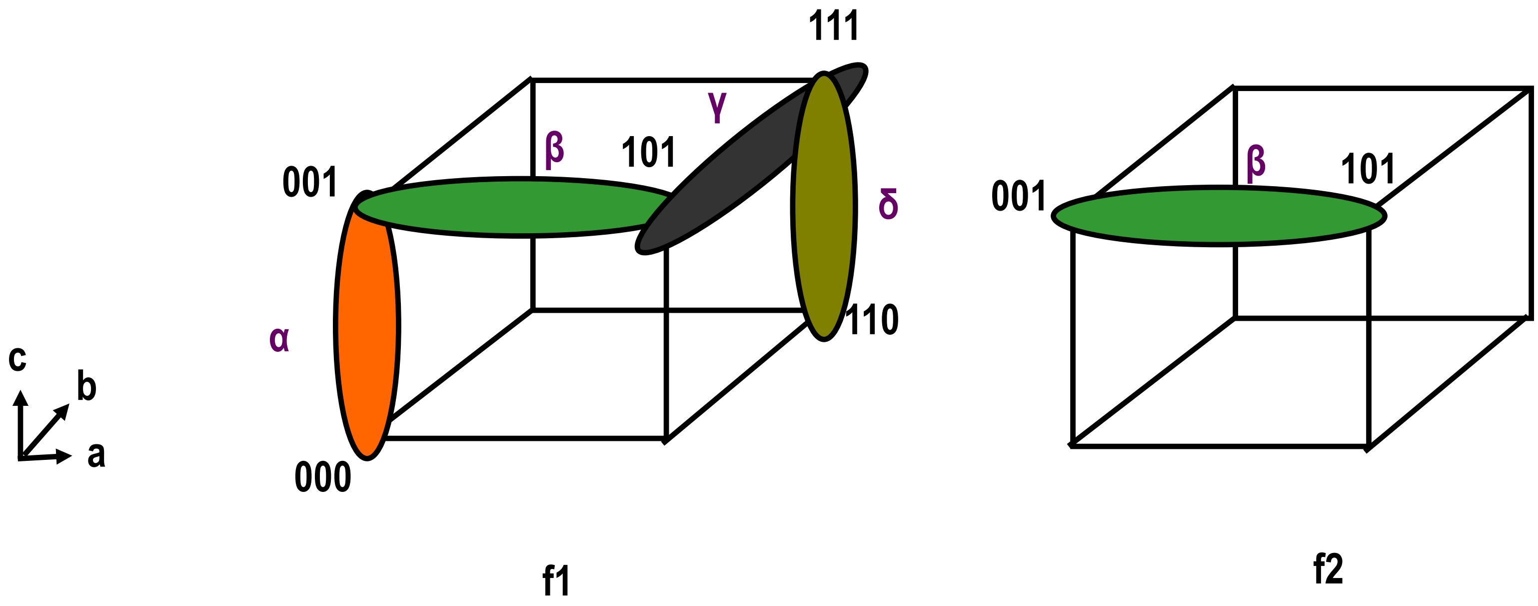
Example of implicant table

- x = ab+a'c;   y = ab+bc+ac

abc	xy
001	10
*11	11
101	01
11*	11

Cubical representation of minterms and implicants

- f_1 = a'b'c' + a'b'c + ab'c + abc +abc'
- f_2 = a'b'c + ab'c



Representations

- Visual representations
  - Cubical notation
  - Karnaugh maps
- Computer-oriented representations
  - Matrices
    - Sparse
    - Various encoding
  - Binary-decision diagrams
    - Address sparsity and efficiency

Module 2

- **Objectives**
  - **Two-level logic optimization**
  - **Motivation**
  - **Models**
  - **Exact algorithms for logic optimization**

Two-level logic optimization motivation

- Reduce size of the representation
- Direct implementation
  - PLAs reduce size and delay
- Other implementation styles
  - Reduce amount of information
  - Simplify local functions and connections

Programmable logic arrays

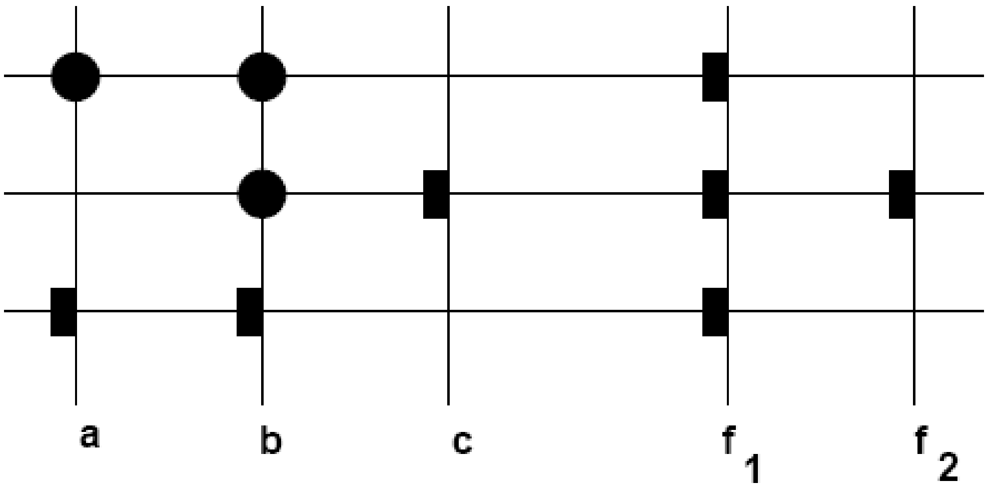
- Macro-cells with rectangular structure
  - Implement any multi-output function
  - Layout generated by module generators
  - Fairly popular in the seventies/eighties
- Advantages
  - Simple, predictable timing
- Disadvantages
  - Less flexible than cell-based realization
  - Dynamic operation
- Open issue
  - Will PLA structures be useful with new nanotechnologies? (e.g., nanowires)

Programmable logic array

- $f_1 = a'b' + b'c + ab; \quad f_2 = b'c$

00*	10
*01	11
11*	10

(a)



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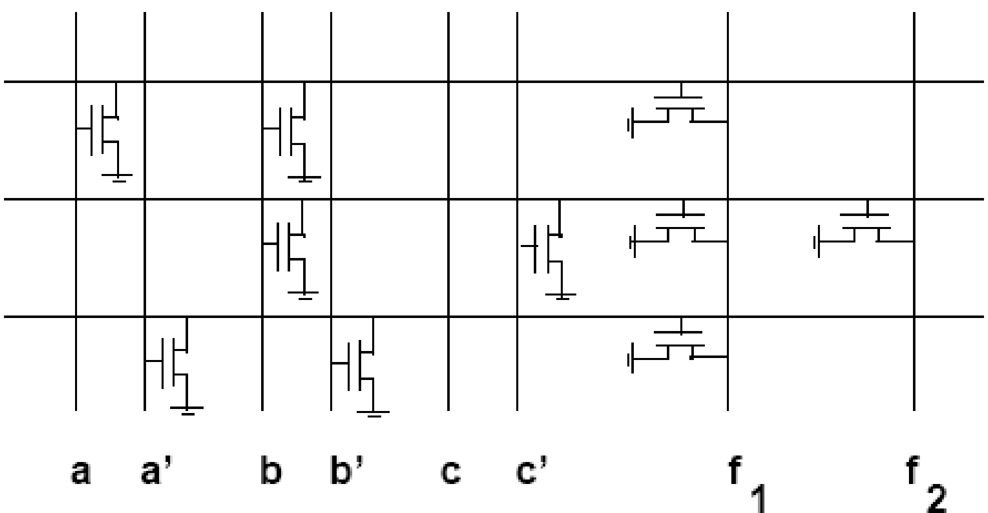

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Two-level minimization

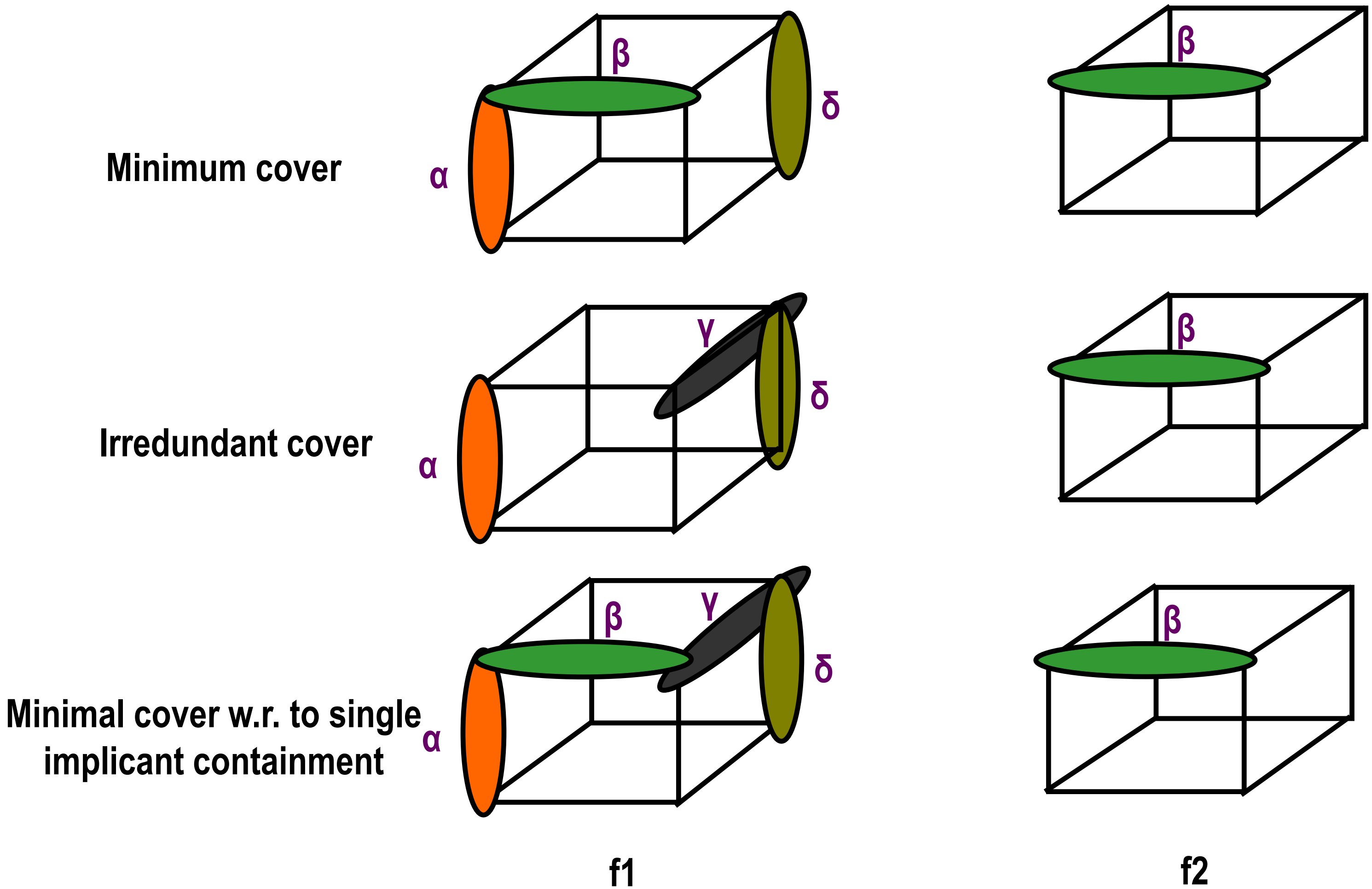
- **Assumptions**
  - **Primary goal is to reduce the number of implicants**
  - **All implicants have the same cost**
  - **Secondary goal is to reduce the number of literals**
- **Rationale**
  - **Implicants correspond to PLA rows**
  - **Literals correspond to transistors**

Definitions

- Minimum cover
  - Cover of a function with minimum number of implicants
  - Global optimum
- Minimal cover or irredundant cover
  - Cover of the function that is not a proper superset of another cover
  - No implicant can be dropped
  - Local optimum
- Minimal w.r.to 1-implicant containment
  - No implicant contained by another one
  - Weak local optimum

Example

- $f_1 = a'b'c' + a'b'c + ab'c + abc + abc'$; $f_2 = a'b'c + ab'c$



Definitions

- **Prime implicant**
  - Implicant not contained by any other implicant
- **Prime cover**
  - Cover of prime implicants
- **Essential prime implicant**
  - There exist some minterm covered only by that prime implicant
  - Needs to be included in the cover

Two-level logic minimization

- **Exact methods**
  - **Compute minimum cover**
  - **Often difficult/impossible for large functions**
  - **Based on Quine-McCluskey method**
- **Heuristic methods**
  - **Compute minimal covers (possibly minimum)**
  - **Large variety of methods and programs**
    - **MINI, PRESTO, ESPRESSO**

Exact logic minimization

- Quine's theorem:
  - There is a minimum cover that is prime
- Consequence
  - Search for minimum cover can be restricted to prime implicants
- Quine-McCluskey method
  - Compute prime implicants
  - Determine minimum cover

Prime implicant table

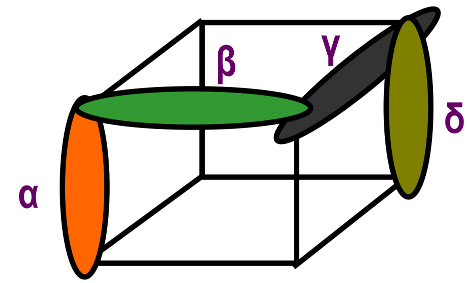
- Rows: minterms
- Columns: prime implicants
- Exponential size
  - 2^n minterms
  - Up to $3^n / n$ prime implicants
- Remarks
  - Some functions have much fewer primes
  - Minterms can be grouped together
  - Implicit methods for implicant enumeration

Example

- $f = a'b'c' + a'b'c + ab'c + abc + abc'$

- **Primes:**

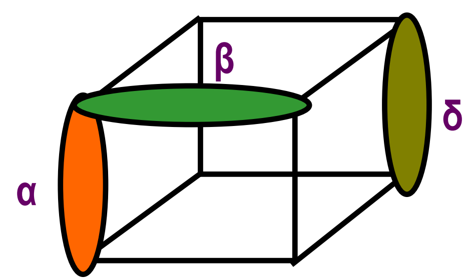
α	00*	1
β	*01	1
γ	1*1	1
δ	11*	1



Prime implicants of f

- **Table:**

	α	β	γ	δ
000	1	0	0	0
001	1	1	0	0
101	0	1	1	0
111	0	0	1	1
110	0	0	0	1



Minimum cover of f

Minimum cover
early methods

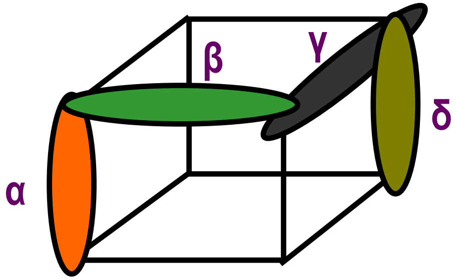
- **Reduce table**
  - **Iteratively identify essentials,**
    save them in the cover.
    Remove covered minterms
- **Petrick's method**
  - **Write covering clauses in *pos* form**
  - **Multiply out pos form into *sop* form**
  - **Select cube of minimum size**
- **Remark**
  - **Multiplying out clauses has exponential cost**

Example

- *pos* clauses
  - $(\alpha)(\alpha+\beta)\ (\beta+\gamma)(\gamma+\delta)(\delta) = 1$
- *sop* form:
  - $\alpha\beta\delta + \alpha\gamma\delta = 1$
- **Solutions:**

$\{\alpha\ \beta\ \delta\}$

$\{\alpha\ \gamma\ \delta\}$



Matrix representation

- View table as Boolean matrix: A
- Selection Boolean vector for primes: x
- Determine X such that
  - $A x \geq 1$
  - Select enough columns to cover all rows
- Minimize cardinality of x

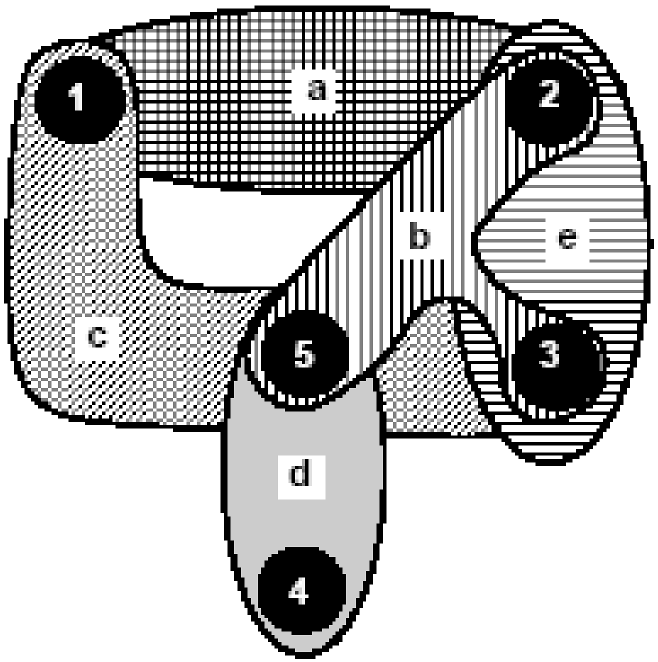
Example

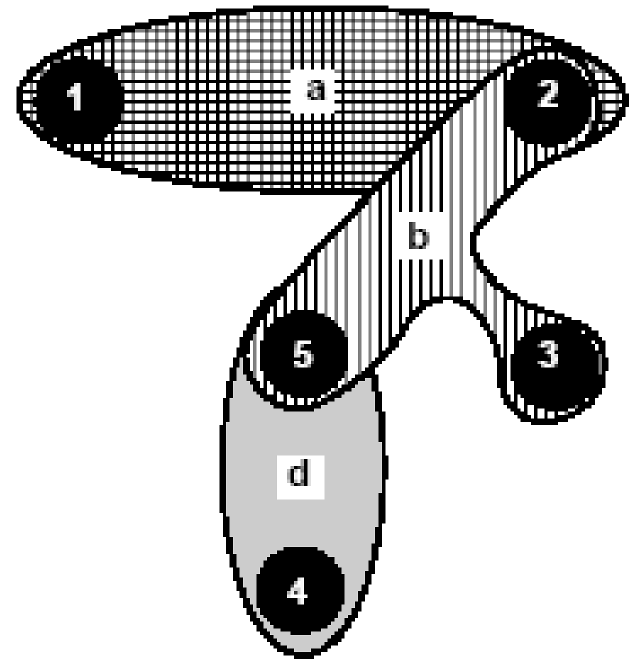
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Covering problem

- Set covering problem:
  - A set S -- minterm set
  - A collection C of subsets (implicant set)
  - Select fewest elements of C to cover S
- Computationally intractable problem
- Exact solution method
  - Branch and bound algorithm
- Several heuristic approximation methods

Example
Edge-cover of a hypergraph

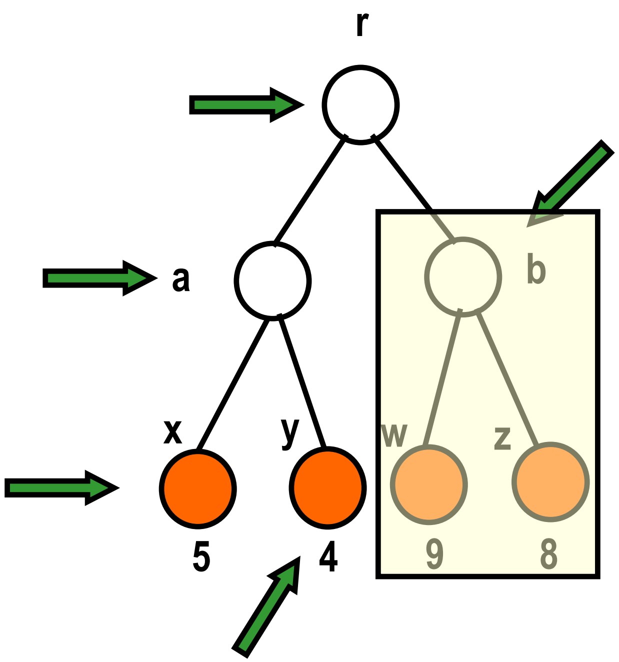




Branch and bound algorithm

- Tree search in the solution space
  - Potentially exponential
- Use bounding function:
  - If the lower bound on the solution cost that can be derived from a set of future choices exceeds the cost of the best solution seen so far, then kill the search
  - Bounding function should be fast to evaluate and accurate
- Good pruning may expedite the search

Example



Bound = 6

Kill sub-tree

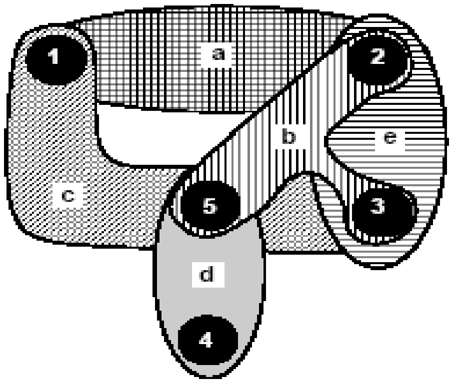
Branch and bound for logic minimization
Reduction strategies

- **Use matrix formulation of the problem**
- **Partitioning:**
  - **If A is block diagonal:**
    - **Solve covering problems for the corresponding blocks**
- **Essentials**
  - **Column incident to one (or more) rows with single 1**
    - **Select column**
    - **Remove covered row(s) from table**

Branch and bound for logic minimization Reduction strategies

- **Column (implicant) dominance:**
  - **If $a_{ki} \geq a_{kj}$ for all k**
    - **Remove column j (dominated)**
  - **Dominated implicant (j) has its minterms already covered by dominant implicant (i)**
- **Row (minterm) dominance:**
  - **If $a_{ik} \geq a_{jk}$ for all k**
    - **Remove row i (dominant)**
  - **When an implicant covers the dominated minterm, it also covers the dominant one**

Example



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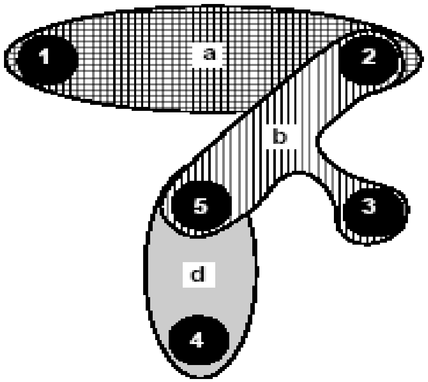


(b)

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

Example

- Fourth column is essential
- Fifth column is dominated
- Fifth row is dominant
- Matrix after reductions:

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

Branch and bound covering algorithm

```
EXACT_COVER(A,x,b) {
    Reduce matrix A and update corresponding x;
    if (current_estimate ≥ |b|) return (b);
    if (A has no rows) return(x);
    select a branching column c;
    x_c = 1;
    Ã = A after deleting c and rows incident to it;
    x~ = EXACT_COVER(Ã,x,b);
    if ( | x~| < |b| )
            b = x~;
    x_c = 0;
    Ã = A after deleting c;
    x~ = EXACT_COVER(Ã,x,b);
    if ( | x~| < |b| )
            b = x~;
    return(b);
}
```

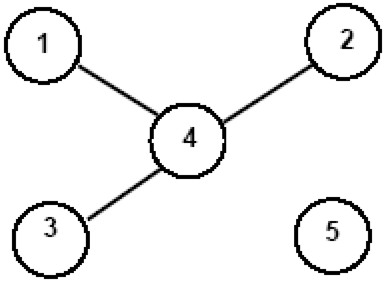
Bounding function

- Estimate lower bound on covers that can be derived from current solution vector **x**
- The sum of the 1s in **x**, plus bound of cover for local **A**
  - Independent set of rows
    - No 1 in the same column
    - Require independent implicants to cover
  - Construct graph to show pairwise independence
  - Find clique number
    - Size of the largest clique
  - Approximation (lower) is acceptable

Example

- Row 4 independent from 1,2,3
- Clique number and bound is 2

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$



Example

- There are no independent rows
  - Clique number is 1 (one vertex)
  - Bound is 1+1= 2
    - Because of the essential already selected

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

Example
Branching on the cyclic core

- Select first column
  - Recur with $\tilde{A} = [11]$
    - Delete one dominated column
    - Take other column (essential)
  - New cost is 3
- Exclude first column
  - Find another solution with cost equal to 3.
  - Discard

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

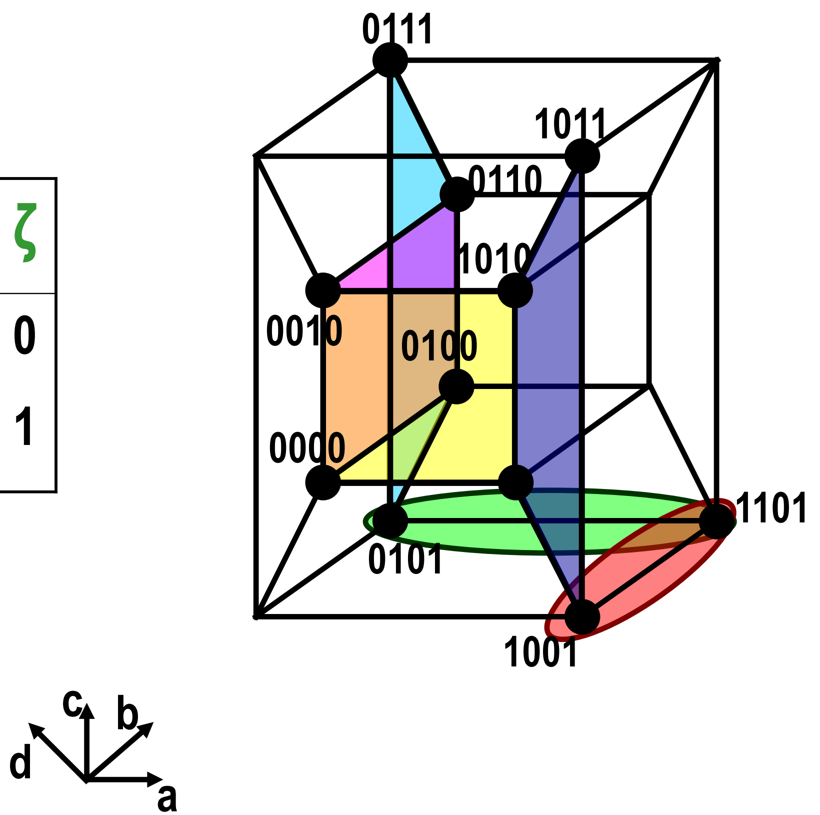
Espresso-exact

- Exact 2-level logic minimizer
- Exploits iterative reduction and branch and bound algorithm on cyclic core
- Compact implicant table
  - Rows represent groups of minterms covered by the same implicants
- Very efficient
  - Solves most benchmarks

Example

After removing the essentials

	α	β	ε	ζ
0000,0010	1	1	0	0
1101	0	0	1	1



α	0 * * 0	1
β	* 0 * 0	1
γ	0 1 * *	1
δ	1 0 * *	1
ε	1 * 0 1	1
ζ	* 1 0 1	1

Exact two-level minimization

- There are two main difficulties:
  - Storage of the implicant table
  - Solving the cyclic core
- Implicit representation of prime implicants
  - Methods based on binary decision diagrams
  - Avoid explicit tabulation
- Recent methods make 2-level optimization solve exactly almost all benchmarks
  - Heuristic optimization is just used to achieve solutions faster

Module 3

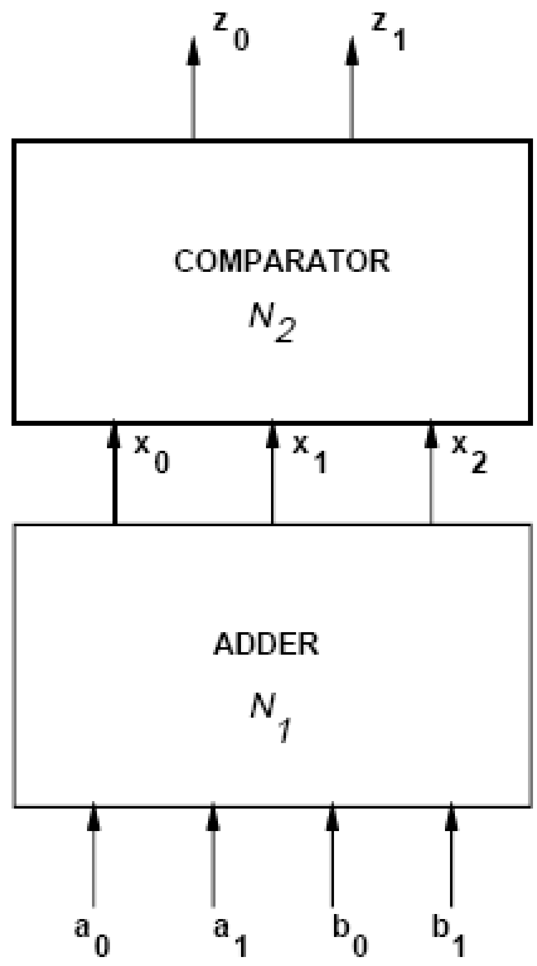
- **Boolean Relations**
  - **Motivation of using relations**
  - **Optimization of realization of Boolean relation**
  - **Comparisons to two-level optimization**

Boolean relations

- Generalization of Boolean functions
- More than one output pattern may correspond to an input pattern
  - Multiple-choice specifications
  - Model inner blocks of multi-level circuits
- Degrees of freedom in finding an implementation
  - More general than *don't care* conditions
- Problem:
  - Given a Boolean relation, find a minimum cover of a compatible Boolean function that can implement the relation

Example

- **Compare:**
  - a + b > 4 ?
  - a + b < 3 ?



Example

a_1	a_0	b_1	b_0	$\mathbf{x}$
0	0	0	0	{ 000, 001, 010 }
0	0	0	1	{ 000, 001, 010 }
0	0	1	0	{ 000, 001, 010 }
0	1	0	0	{ 000, 001, 010 }
1	0	0	0	{ 000, 001, 010 }
0	1	0	1	{ 000, 001, 010 }
0	0	1	1	{ 011, 100 }
0	1	1	0	{ 011, 100 }
1	0	0	1	{ 011, 100 }
1	0	1	0	{ 011, 100 }
1	1	0	0	{ 011, 100 }
0	1	1	1	{ 011, 100 }
1	1	0	1	{ 011, 100 }
1	0	1	1	{ 101, 110, 111 }
1	1	1	0	{ 101, 110, 111 }
1	1	1	1	{ 101, 110, 111 }

Example

- Circuit is no longer an adder

a_1	a_0	b_1	b_0	**x**
0	*	1	*	010
1	*	0	*	010
1	*	1	*	100
*	*	*	1	001
*	1	*	*	001

Minimization of Boolean relations

- Since there are many possible output values (for any input), there are many logic functions implementing the relation
  - Compatible functions
- Problem
  - Find a minimum compatible function
- Do not enumerate all compatible functions
  - Compute the primes of the compatible functions
    - C-primes
  - Derive a logic cover from the c-primes

Binate covering

- Covering problem is more complex
  - As compared to minimizing logic functions.
- In classic Boolean minimization we just need enough implicants to cover the minterm
  - Covering clause is unate in all variables
  - Any additional implicant does not hurt
- In Boolean relation optimization, we need to pick implicants to realize a compatible function
  - Some implicants cannot be taken together
  - Covering clause is binate (implicant mutual exclusion)
  - Non-compact Boolean space

Solving binate covering

- Binate cover can be solved with branch and bound
  - In practice much more difficult to solve, because it is harder to bound effectively
- Binate cover can be reduced to min-cost SAT
  - SAT solvers can be used
- Binate cover can be also modeled by BDDs
- Several approximation algorithms for binate cover

Boolean relations

- Generalization of Boolean functions
  - More degrees of freedom than don’t care sets
- Useful to represent multiple choice
- Useful to model internals of logic networks
- Elegant formalism, but computationally-intensive solution method