

# Encodings into SAT

## Combinatorial Problem Solving (CPS)

Enric Rodríguez-Carbonell

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# What is an encoding?

- Language of SAT solvers: **CNF propositional formulas**
- To solve combinatorial problems with SAT solvers, constraints have to be represented in this language
- An **encoding** of a constraint  $C$  into SAT is a CNF  $F$  that expresses  $C$ , so that there is a bijection

solutions to  $C \iff$  models of  $F$

# Examples: AMO constraints

- An **AMO constraint** is of the form  $x_0 + \dots + x_{n-1} \leq 1$  where each  $x_i$  is 0-1  
(**A**t **M**ost **O**ne of the variables can be true)
- **Quadratic encoding.**
  - ◆ Variables: the same  $x_0, \dots, x_{n-1}$
  - ◆ Clauses: for  $0 \leq i < j < n$ ,  $\overline{x_i} \vee \overline{x_j}$
  - ◆ Requires  $\binom{n}{2} = O(n^2)$  clauses
- Other encodings try to use fewer clauses, at the cost of introducing new variables

# Examples: AMO constraints

- **Logarithmic encoding.** Let  $m = \lceil \log_2 n \rceil$ . Then:
  - ◆ Variables: the  $x_i$  and new variables  $y_0, y_1, \dots, y_{m-1}$
  - ◆ Clauses: for  $0 \leq i < n$ ,  $0 \leq j < m$ 
    - $\overline{x_i} \vee y_j$  if the  $j$ -th digit in binary of  $i$  is 1
    - $\overline{x_i} \vee \overline{y_j}$  otherwise
  - ◆ Requires  $O(\log n)$  new variables,  $O(n \log n)$  clauses
- **Heule encoding.**
  - ◆ If  $n \leq 3$ , the encoding is the quadratic encoding.
  - ◆ If  $n \geq 4$ , introduce an auxiliary variable  $y$  and encode (recursively)  $x_0 + x_1 + y \leq 1$  and  $x_2 + \dots + x_{n-1} + \overline{y} \leq 1$ .
  - ◆ Requires  $O(n)$  new variables,  $O(n)$  clauses
- Other encodings exist (see next)

# Consistency and Arc-Consistency

- Let us consider an encoding of a constraint  $C$  such that there is a correspondence between assignments of the variables in  $C$  with boolean assignments of the variables in the encoding
- The encoding is **consistent** if whenever  $M$  is partial assignment inconsistent wrt  $C$  (i.e., can't be extended to a solution of  $C$ ), unit propagation leads to conflict
- The encoding is **arc-consistent** if
  - ◆ it is consistent, and
  - ◆ unit propagation discards arc-inconsistent values (i.e., values without a support)
- These are good properties for encodings:  
SAT solvers are very good at unit propagation!

# Consistency and Arc-Consistency

- In the case of an AMO constraint  $x_0 + \dots + x_{n-1} \leq 1$ :
- **Consistency**  $\equiv$  if there are two true vars  $x_i$  in  $M$  or more, then unit propagation should give a conflict
- **Arc-consistency**  $\equiv$  Consistency + if there is one true var  $x_i$  in  $M$ , then unit propagation should set all others  $x_j$  to false
- The quadratic, logarithmic and Heule encodings are all arc-consistent

# Cardinality Constraints

- A cardinality constraint is of the form  $x_1 + \dots + x_n \bowtie k$  where each  $x_i$  is 0-1 and  $\bowtie \in \{\leq, <, \geq, >, =\}$
- AMO are a particular case of card. constraints where  $k = 1$  and  $\bowtie$  is  $\leq$
- Without loss of generality we may assume  $\bowtie$  is  $<$ , i.e.,

$$x_1 + \dots + x_n < k$$

- **Naive encoding.**

- ◆ Variables: the same  $x_1, \dots, x_n$
- ◆ Clauses: for all  $1 \leq i_1 < i_2 < \dots < i_k \leq n$ ,

$$\overline{x_{i_1}} \vee \overline{x_{i_2}} \vee \dots \vee \overline{x_{i_k}}$$

- ◆ This is  $\binom{n}{k}$  clauses!

# Adders

- Again, other encodings try to use fewer clauses, at the cost of introducing new variables
- **Adder encoding.**  
Build an adder circuit by using bit-adders as building blocks:



$$s \leftrightarrow \text{XOR}(x, y, z)$$

$$c \leftrightarrow (x \wedge y) \vee (x \wedge z) \vee (y \wedge z)$$



# Adders

- Encodings of this kind are not arc-consistent.
- Consider  $x + y + z \leq 0$ .  
Then unit propagation should propagate  $\bar{x}, \bar{y}, \bar{z}$ .
- Let us encode the constraint with a full adder
- The encoding is the Tseitin transformation of  $\bar{s}, \bar{c}$  and
$$\begin{aligned}s &\leftrightarrow \text{XOR}(x, y, z) \\ c &\leftrightarrow (x \wedge y) \vee (x \wedge z) \vee (y \wedge z)\end{aligned}$$
- But unit propagation cannot propagate anything!

# Sorting Networks

## ■ Sorting Network encoding.

Pass  $x_1, \dots, x_n$  as inputs to a circuit that sorts (say, decreasingly)  $n$  bits.

Let  $y_1, \dots, y_n$  be the outputs of this circuit.

Then if the constraint to be encoded is

- ◆  $\sum_{i=1}^n x_i \geq k$ , then add clause  $y_k$
- ◆  $\sum_{i=1}^n x_i \leq k$ , then add clause  $\overline{y_{k+1}}$
- ◆  $\sum_{i=1}^n x_i = k$ , then add clauses  $y_k, \overline{y_{k+1}}$

# Sorting Networks

- How to build such a sorting circuit?
- A possibility is to implement mergesort
- In what follows: so-called odd-even sorting networks
- The basic block of odd-even sorting networks are 2-comparators

# 2-comparators

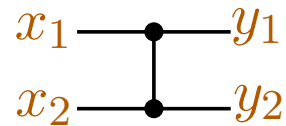
- A 2-comparator is a sorting network of size 2:
  - ◆ it has 2 input variables ( $x_1$  and  $x_2$ )
  - ◆ it has 2 output variables ( $y_1$  and  $y_2$ )
  - ◆  $y_1$  is true if and only if at least one of the input variables is true (i.e., it is the maximum or disjunction)
  - ◆  $y_2$  is true if and only if both two input variables are true (i.e., it is the minimum or conjunction)

# 2-comparators

## ■ Clauses:

$$\begin{array}{lll} x_1 \leftarrow y_2, & x_2 \leftarrow y_2, & x_1 \vee x_2 \leftarrow y_1, \\ x_1 \rightarrow y_1, & x_2 \rightarrow y_1, & x_1 \wedge x_2 \rightarrow y_2 \end{array}$$

## ■ Graphical representation:



## ■ Some simplifications are possible:

- ◆ For  $\geq$  constraints: **top three** clauses suffice
- ◆ For  $\leq$  constraints: **bottom three** clauses suffice
- ◆ For  $=$  constraints: **all** clauses needed

# Merge Networks

- From now on we assume that  $n$  is a power of two (if not, pad with variables set to false)
- A **merge network** takes as input two ordered sets of variables of size  $n$  and produces an ordered output of size  $2n$ .
- Let  $(x_1, \dots, x_n)$  and  $(x'_1, \dots, x'_n)$  be the inputs. We recursively define a merge network as follows:
- If  $n = 1$ , a merge network is a 2-comparator:

$$\text{Merge}(x_1; x'_1) := 2\text{-Comp}(x_1, x'_1).$$

# Merge Networks

■ For  $n > 1$ : Let us define

$$(z_1, z_3, \dots, z_{2n-1}) = \text{Merge}(x_1, x_3, \dots, x_{n-1}; x'_1, x'_3, \dots, x'_{n-1}),$$

$$(z_2, z_4, \dots, z_{2n}) = \text{Merge}(x_2, x_4, \dots, x_n; x'_2, x'_4, \dots, x'_n),$$

$$(y_2, y_3) = \text{2-Comp}(z_2, z_3),$$

$$(y_4, y_5) = \text{2-Comp}(z_4, z_5),$$

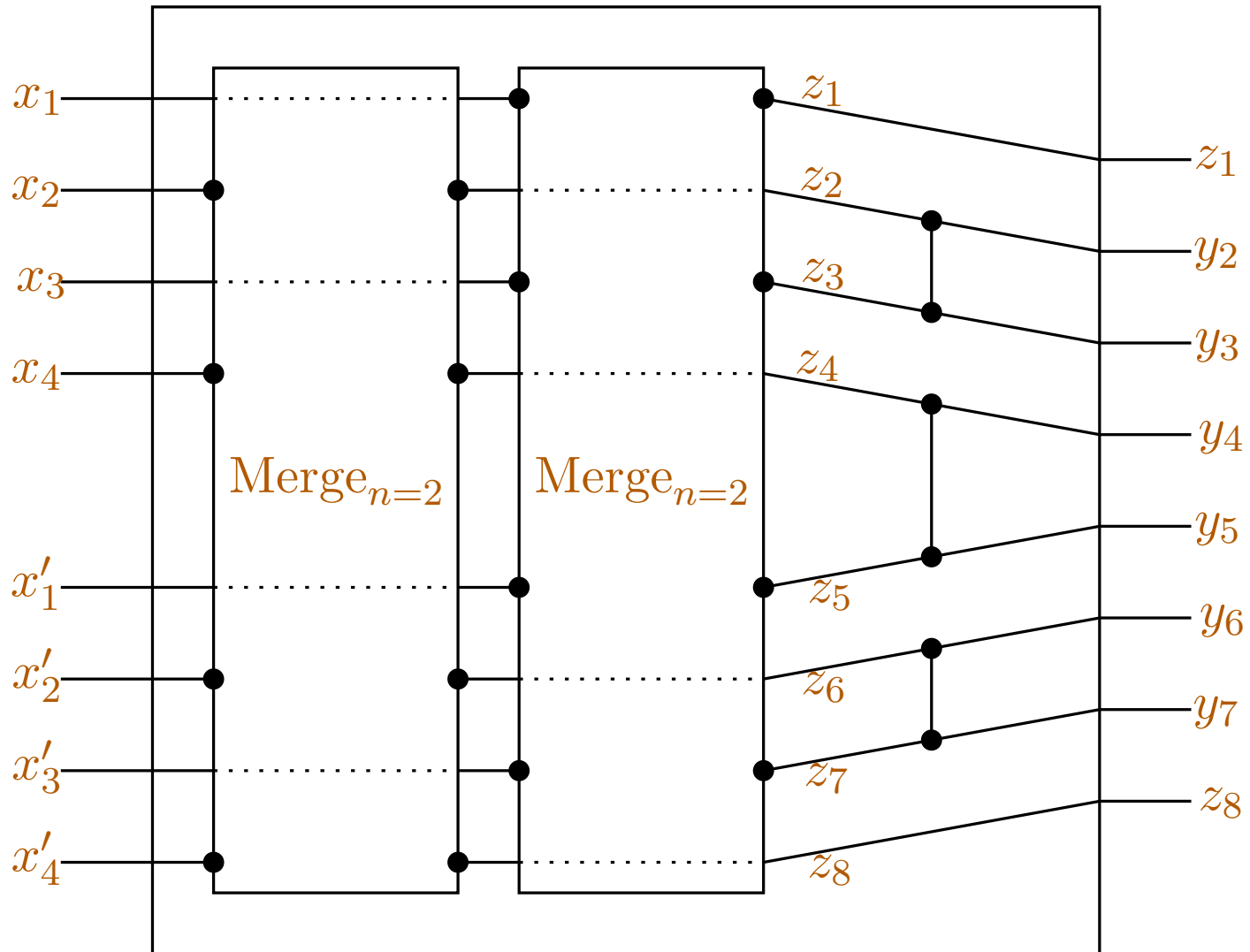
...

$$(y_{2n-2}, y_{2n-1}) = \text{2-Comp}(z_{2n-2}, z_{2n-1})$$

Then,

$$\text{Merge}(x_1, x_2, \dots, x_n; x'_1, x'_2, \dots, x'_n) := (z_1, y_2, y_3, \dots, y_{2n-1}, z_{2n})$$

# Merge Networks





# Merge Networks

Sketch of the proof of correctness of Merge:

By IH:  $\{x_1, x_3, \dots, x_{n-1}, x'_1, x'_3, \dots, x'_{n-1}\} = \{z_1, z_3, \dots, z_{2n-1}\}$

By IH:  $\{x_2, x_4, \dots, x_n, x'_2, x'_4, \dots, x'_n\} = \{z_2, z_4, \dots, z_{2n}\}$

Hence  $\{x_1, x_2, \dots, x_n, x'_1, x'_2, \dots, x'_n\} = \{z_1, z_2, \dots, z_{2n}\}$

And

$(y_2, y_3) = 2\text{-Comp}(z_2, z_3)$  implies  $\{y_2, y_3\} = \{z_2, z_3\}$

$(y_4, y_5) = 2\text{-Comp}(z_4, z_5)$  implies  $\{y_4, y_5\} = \{z_4, z_5\}$

...

$(y_{2n-2}, y_{2n-1}) = 2\text{-Comp}(z_{2n-2}, z_{2n-1})$  implies  $\{y_{2n-2}, y_{2n-1}\} = \{z_{2n-2}, z_{2n-1}\}$

So  $\{x_1, x_2, \dots, x_n, x'_1, x'_2, \dots, x'_n\} = \{z_1, y_2, y_3, \dots, y_{2n-2}, y_{2n-1}, z_{2n}\}$

# Merge Networks

Let us prove outputs are sorted decreasingly. For  $1 \leq i < n - 1$  let us see:

■  $z_{2i} \geq z_{2(i+1)+1}$ :

Let us see  $z_{2(i+1)+1} = 1$  implies  $z_{2i} = 1$

If  $z_{2(i+1)+1} = z_{2i+3} = z_{2(i+2)-1} = 1$  there are  $i + 2$  1's in odd  $x, x'$

Let  $p$  be the number of 1's in odd  $x$

Let  $q$  the number of 1's in odd  $x'$

Then  $p + q = i + 2$

As  $x, x'$  is ordered decreasingly,

there are  $p - 1$  1's in even  $x$ ,  $q - 1$  1's in even  $x'$

So altogether there are  $(p - 1) + (q - 1) = p + q - 2 = i$  1's in even  $x, x'$

Hence  $z_{2i} = 1$

# Merge Networks

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- $z_{2i} \geq z_{2(i+1)+1}$ : proved
- $z_{2i} \geq z_{2(i+1)}$ : by IH
- $z_{2i+1} \geq z_{2(i+1)+1}$ : by IH

# Merge Networks

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- $z_{2i} \geq z_{2(i+1)+1}$ : proved
- $z_{2i} \geq z_{2(i+1)}$ : by IH
- $z_{2i+1} \geq z_{2(i+1)+1}$ : by IH
- $z_{2i+1} \geq z_{2(i+1)}$ : similar to above

So  $\min(z_{2i}, z_{2i+1}) \geq \max(z_{2(i+1)}, z_{2(i+1)+1})$

But  $y_{2i+1} = \min(z_{2i}, z_{2i+1})$  and  $y_{2(i+1)} = \max(z_{2(i+1)}, z_{2(i+1)+1})$

So  $y_{2i+1} \geq y_{2(i+1)}$

And  $y_{2i} \geq y_{2i+1}$  for being outputs of 2-Comp

Altogether  $z_1, y_2, y_3, \dots, y_{2n-2}, y_{2n-1}, z_{2n}$  is sorted decreasingly

# Sorting Networks

- A **sorting network** of size  $n$  takes an input of size  $n$  and sorts it (decreasingly).
- We can build a sorting network by successively applying merge networks (as in mergesort).
- Let  $x_1, \dots, x_n$  be the inputs.  
We recursively define a sorting network as follows:
- If  $n = 2$ , a sorting network is a 2-comparator:

$$\text{Sorting}(x_1, x_2) := 2\text{-Comp}(x_1, x_2)$$

# Sorting Networks

■ For  $n > 2$ : Let us define

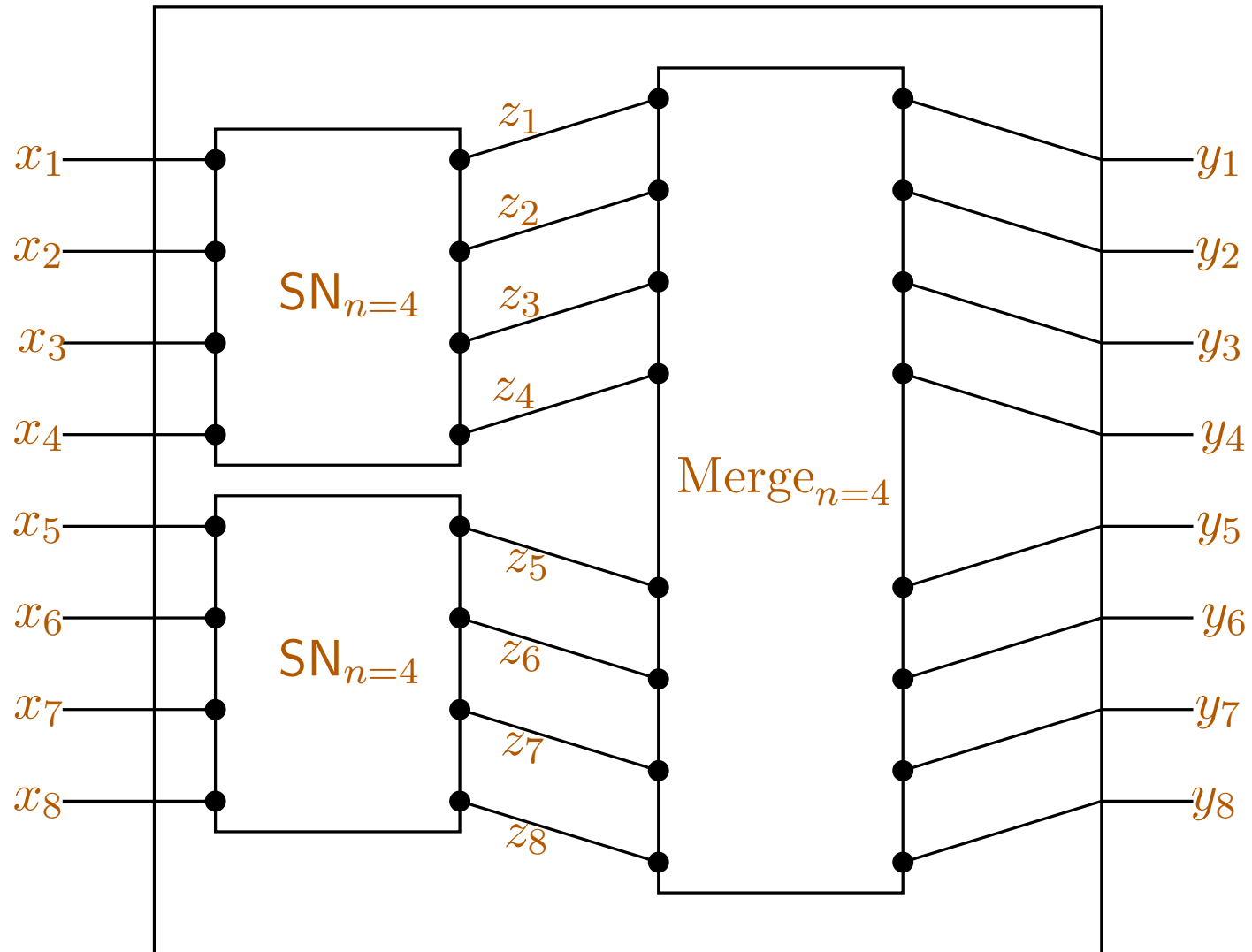
$$\begin{aligned}(z_1, z_2, \dots, z_{n/2}) &= \text{Sorting}(x_1, x_2, \dots, x_{n/2}), \\ (z_{n/2+1}, z_{n/2+2}, \dots, z_n) &= \text{Sorting}(x_{n/2+1}, x_{n/2+2}, \dots, x_n), \\ (y_1, y_2, \dots, y_n) &= \text{Merge}(z_1, z_2, \dots, z_{n/2}; z_{n/2+1} \dots, z_n)\end{aligned}$$

Then,

$$\text{Sorting}(x_1, x_2, \dots, x_n) := (y_1, y_2, \dots, y_n)$$



# Sorting Networks



# Sorting Networks

- This encoding of cardinality constraints is arc-consistent
- It uses  $O(n \log^2 n)$  new variables and  $O(n \log^2 n)$  clauses
- Several improvements are possible:
  - ◆ Only the first  $k$  outputs suffice:  
**cardinality networks** use  $O(n \log^2 k)$  vars and clauses
  - ◆ No need to assume that  $n$  is a power of two:  
merges can be defined for inputs of different sizes

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