Combinatorial Problem Solving (CPS)

2018-19 Spring Term. Final Exam: 3 hours
Publication of grades: 28/06
Revision of exams: 02/07 (office 113, Ω building, under request by e-mail: `erodri@cs.upc.edu`)

---

1. *(3 pts.)* Let $x_1, \ldots, x_n$ be integer variables. For each $1 \leq i \leq n$, the domain of variable x_i is the interval $[\ell_i, u_i]$, where $\ell_i, u_i \in \mathbb{Z}$ and $\ell_i \leq u_i$.

   Let C be a constraint of the form $a_1 x_1 + \cdots + a_n x_n \geq k$, where $a_i, k \in \mathbb{Z}$ and $a_i > 0$.

   Let us consider the CSP consisting of the single constraint C.

   (a) *(1 pt.)* Prove that, if x is a solution to the CSP, then for each $1 \leq i \leq n$ we have that

   $$x_i \geq \left\lceil \frac{k - \sum_{j=1, j\neq i}^{n} a_j u_j}{a_i} \right\rceil$$

   (b) *(1 pt.)* Let $S = \sum_{j=1}^{n} a_j u_j$. Prove that, if $S - \max_{1 \leq j \leq n}(a_j(u_j - l_j)) < k$, then the CSP is arc-inconsistent.

   (c) *(1 pt.)* Is the reverse implication of exercise (1b) true? If so, prove it. Otherwise, give a counterexample.

2. *(3 pts.)* Consider an integer linear program of the following form:

   $$\begin{array}{l} \min c^T x \\ Ax = b \\ x \geq 0 \quad x_i \in \mathbb{Z} \;\text{ for all } 1 \leq i \leq n. \end{array}$$

   Let β be a basic solution and

   $$x_i = \gamma - \sum_{j \in R} a_j x_j$$

   be the equation in the tableau of a basic variable x_i, where R are the indices of the non-basic variables and $\gamma, a_j \in \mathbb{R}$. Let us assume that $\beta_i \notin \mathbb{Z}$, where β_i is the value assigned by β to x_i.

   (a) *(1.5 pts.)* Prove that

   $$x_i + \sum_{j \in R} \lfloor a_j \rfloor x_j - \lfloor \gamma \rfloor = \gamma - \lfloor \gamma \rfloor - \sum_{j \in R} (a_j - \lfloor a_j \rfloor) x_j$$

   for all feasible solutions to the integer linear program.

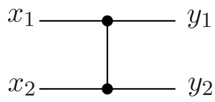
   (b) *(1.5 pts.)* Prove that

   $$\gamma - \lfloor \gamma \rfloor - \sum_{j \in R} (a_j - \lfloor a_j \rfloor) x_j \leq 0$$

   is a cut that cuts β away.

3. *(4 pts.)* Answer the following questions:

(a) *(1.5 pts.)* Using the standard representation of a 2-comparator with inputs x_1, x_2 and outputs y_1, y_2 (in decreasing order):



draw the circuit corresponding to a sorting network with 4 inputs v_1, v_2, v_3, v_4. Please indicate **clearly** the names of the auxiliary variables representing the wires of the circuit. Write also the set of clauses corresponding to the circuit.

Note: You can write the clauses as disjunctions or as implications.

(b) *(1.5 pts.)* A *pseudo-boolean constraint* is a constraint of the form $a_1x_1 + \cdots + a_mx_m \leq k$, where $k, a_1, \ldots, a_m$ are positive integers and $x_1, \ldots, x_m$ are boolean variables. Explain how to encode into SAT a pseudo-boolean constraint using sorting networks. Illustrate your method with the constraint $v_1 + 3v_2 \leq 2$ and give the resulting set of clauses.

(c) *(1 pt.)* Is your encoding of exercise (3b) arc-consistent? (that is, if a value of a variable does not have a support for the constraint, does unit propagation in the CNF propagate a literal that discards that value?)

If it is so, prove it. Otherwise, give a counterexample.