Combinatorial Problem Solving (CPS)

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1. *(3 pts.)* Let x, y be variables with finite domains $D_x, D_y \subseteq \mathbb{Z}$. Let $C_1(x,y)$ and $C_2(x,y)$ be two constraints.

   (a) *(1.5 pts.)* If at least one of C_1, C_2 is arc-consistent, is it true that the constraint C_3 defined as $C_3(x,y) \equiv C_1(x,y) \vee C_2(x,y)$ is arc-consistent?

   If the answer is positive, prove so. If the answer is negative, give a counterexample.

   (b) *(1.5 pts.)* If C_1 and C_2 are both arc-consistent, is it true that the constraint C_4 defined as $C_4(x,y) \equiv C_1(x,y) \wedge C_2(x,y)$ is arc-consistent?

   If the answer is positive, prove so. If the answer is negative, give a counterexample.

2. *(3 pts.)* Let P_0 be a (minimization) linear program over $0-1$ variables $x = (x_1, ..., x_n)$ with cost function $c^T x$. Consider the following simplification of the branch-and-bound procedure for finding the minimum cost of P_0:

1 $S := \{P_0\}$
2 $UB := \infty$
3 **while** $(S \neq \emptyset)$ {
4     $P :=$ *choose_from*(S);
5     $S := S - \{P\}$
6     $\beta :=$ *basic_solution_of_optimal_basis_of* $(LP(P))$
7   **if** $(\beta \neq \perp \wedge c^T\beta < UB)$ {
8   **if** $(\forall x_i\ \beta(x_i) \in \{0,1\})$
9             $UB := c^T\beta$
10   **else** {
11             $x_j :=$ *choose_from*$(\{\ x_i \mid 0 < \beta(x_i) < 1\ \})$
12             $S := S \cup \{P \wedge x_j = 0\}\ \cup\ \{P \wedge x_j = 1\}$
13 } } }
14 **return** UB

where:

- function *choose_from*$(\cdot)$ returns an element of a given set;
- function $LP(\cdot)$ returns the linear programming relaxation of a given 0-1 linear program;
- function *basic_solution_of_optimal_basis_of* $(\cdot)$ calls the simplex algorithm on a given (real) linear program and returns the basic solution of an optimal basis ($\perp$ if infeasible or unbounded)

(a) *(0.5 pts.)* Can the linear programming relaxation $LP(P)$ at line 6 be an unbounded linear program? Justify your answer.

(b) *(0.5 pts.)* Does the branch-and-bound procedure terminate? Justify your answer.

(c) *(2 pts.)* At any iteration of the loop, the value of variable UB is an *upper* bound on the minimum cost of P_0. Explain the changes you would make to the above pseudo-code so that it also maintained a variable LB giving a *lower* bound on the minimum cost of P_0.

Note: The grade will depend on how accurate the lower bounds are.

3. *(4 pts.)* Let us consider $n = p \times q$ Boolean variables

$$\begin{matrix} x_{(0,0)} & x_{(0,1)} & \cdots & x_{(0,q-1)} \\ x_{(1,0)} & x_{(1,1)} & \cdots & x_{(1,q-1)} \\ \vdots & \vdots & \ddots & \vdots \\ x_{(p-1,0)} & x_{(p-1,1)} & \cdots & x_{(p-1,q-1)} \end{matrix}$$

Let $X = \{\, x_{(i,j)} \mid 0 \le i < p,\ 0 \le j < q \,\}$. The *product encoding* of the At-Most-One constraint

$$\mathrm{AMO}(X) \equiv \sum_{0 \le i < p,\ 0 \le j < q} x_{(i,j)} \le 1$$

is defined as follows. Let us introduce auxiliary variables $R = \{r_0, \ldots, r_{p-1}\}$ and $C = \{c_0, \ldots, c_{q-1}\}$ and the following set of clauses:

$$\{\, \neg x_{(i,j)} \vee r_i, \quad \neg x_{(i,j)} \vee c_j \mid 0 \le i < p,\ 0 \le j < q \,\}$$

together with the clauses resulting from encoding $\mathrm{AMO}(R)$ and $\mathrm{AMO}(C)$ (recursively, or with another encoding for AMO constraints).

(a) *(2 pts.)* Show that, if the encodings used for $\mathrm{AMO}(R)$ and $\mathrm{AMO}(C)$ are consistent, then the product encoding is consistent.

(b) *(2 pts.)* Show that, if the encodings used for $\mathrm{AMO}(R)$ and $\mathrm{AMO}(C)$ are arc-consistent, then the product encoding is arc-consistent.