Local Consistency

Combinatorial Problem Solving (CPS)

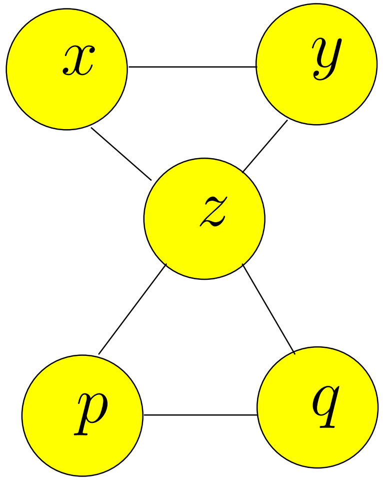
Enric Rodríguez-Carbonell (based on materials by Javier Larrosa)

February 14, 2020

Interaction Graph

- The interaction graph of a CSP is an undirected graph $G = (V, E)$ s.t.:
  - there is a vertex i associated to each variable x_i
  - there is an edge (i, j) if
    there exists some constraint having both x_i and x_j in its scope

CSP with Boolean variables x, y, z, p, q
and constraints: $x + y = z, |p - q| = z$



- For example, the interaction graph of graph coloring is the same graph!

Interaction Graph

- The interaction graph of a CSP is interesting to study
- E.g., connected components of the interaction graph can be solved independently (then just join the solutions)
- Here it is used to describe some propagation notions

Binary CSP's

- A CSP is binary if all its constraints have arity 2
- When considering binary CSP's,
  c_{ij} indicates the constraint between variables x_i and x_j
- In what follows we will focus on binary CSP's.

  We can do it wlog because of the following property:

  Theorem. Any CSP can be transformed into an equisatisfiable binary one.

Binary CSP's

- Consider the CSP with Boolean variables x, y, z, p, q and constraints: $x + y = z, |p - q| = z$.

- The equivalent binary CSP has:
  - *variables*: one for each original constraint

$$v_{x+y=z}, \qquad v_{|p-q|=z}$$

  - *domains*: the tuples that satisfy the original constraint

$$\begin{array}{lcl} d_{x+y=z} & = & \{(x, y, z) \in \{(0,0,0), (1,0,1), (0,1,1)\}\} \\ d_{|p-q|=z} & = & \{(p, q, z) \in \{(0,0,0), (1,0,1), (0,1,1), (1,1,0)\}\} \end{array}$$

  - *constraint*: satisfying tuples are consistent on common values

$$\begin{array}{ll} \{((0,0,\mathbf{0}), (0,0,\mathbf{0})), & ((0,0,\mathbf{0}), (1,1,\mathbf{0})), \\ ((1,0,\mathbf{1}), (1,0,\mathbf{1})), & ((1,0,\mathbf{1}), (0,1,\mathbf{1})), \\ ((0,1,\mathbf{1}), (1,0,\mathbf{1})), & ((0,1,\mathbf{1}), (0,1,\mathbf{1}))\} \end{array}$$

Binary CSP's

Proof: Let $P = (X, D, C)$ be the non-binary CSP.
An equisatisfiable binary one is $P' = (X', D', C')$ defined as follows:

- There is a variable associated to every constraint in P.
  Let x'_i be the variable associated to constraint $c_i \in C$.
- Let $S = (x_{i_1}, ..., x_{i_k})$ be the scope of c_i.
  The domain of x'_i is the set of tuples $\tau \in d_{i_1} \times ... \times d_{i_k}$ s.t. $c_i(\tau) = 1$.
- There is a binary constraint $c'_{ij} \in C'$ iff
  c_i and c_j have some common variable in their scopes.
- The constraint $c'_{ij}(\tau, \sigma)$ is true if
  τ and σ match in their common variables.
- This is known as the dual graph translation

- If σ is a solution to P, then a solution σ' to P' is obtained as follows:
  the value for x'_i is the projection of σ on the scope of c_i.
- If σ' is a solution to P', then a solution σ to P is obtained as follows:
  the value for x_j is the value assigned in σ' by any of the constraints where x_j appears

Filtering, Propagation

- Let $P = (X, D, C)$ be a (binary) CSP,
  $x_i \in X$ a variable and $a \in d_i$ a domain value
- $P[x_i \to a]$ is the CSP obtained from P by replacing d_i by $\{a\}$
- $a \in d_i$ is feasible if $P[x_i \to a]$ has solutions, unfeasible otherwise
- Example:
  - Let x, y be two integer variables with domains $[1, 10]$
  - Consider constraint $|x - y| > 5 \equiv (x - y > 5) \vee (y - x > 5)$
  - Values 5 and 6 for both variables are unfeasible

Filtering, Propagation

- Let $P = (X, D, C)$ be a (binary) CSP,
  $x_i \in X$ a variable and $a \in d_i$ a domain value
- $P[x_i \to a]$ is the CSP obtained from P by replacing d_i by $\{a\}$
- $a \in d_i$ is feasible if $P[x_i \to a]$ has solutions, unfeasible otherwise
- Example:
  - Let x, y be two integer variables with domains $[1, 10]$
  - Consider constraint $|x - y| > 5 \equiv (x - y > 5) \vee (y - x > 5)$
  - Values 5 and 6 for both variables are unfeasible
- Detecting unfeasible values is useful because
  they can be removed without losing solutions
- In general, detecting if a value is feasible is an NP-Complete problem
- But in some cases unfeasible values can be easily detected
- Filtering algorithms identify and remove unfeasible values efficiently
  Filtering is also called propagation (and filtering algorithms, propagators)

Local Consistency

- A clean way of designing filtering techniques is by means of the concept of local consistency
  - A local consistency property allows identifying unfeasible values: inconsistent values, i.e., not satisfying the property, are unfeasible
  - Local because only small pieces of the problem are considered (typically, one constraint)
  - Enforcing a local consistency property means propagating: removing the inconsistent values until the property is achieved
  - Enforcing local consistency should be cheap (polynomial time)

Local Consistency Properties

- The most important is Arc Consistency (AC) (aka Domain Consistency)
- Weaker than AC:
  - Directional AC (DAC)
  - Bounds Consistency (BC)
  - ...
- Stronger than AC:
  - Singleton AC (SAC)
  - Neighborhood Inverse Consistency (NIC)
  - ...

Arc Consistency

- Let $P = (X, D, C)$ be a (binary) CSP
- Value $a \in d_i$ of variable $x_i \in X$ is arc-consistent wrt. x_j if there is $b \in d_j$ (the support of a in x_j) s.t. $c_{ij}(a, b) = \mathrm{true}$
- The definition of arc-consistency is then lifted in the natural way:
  - Variable $x_i \in X$ is arc-consistent wrt. x_j if all values in its domain are arc-consistent wrt. x_j
  - Constraint $c_{ij} \in C$ is arc-consistent if x_i is arc-consistent wrt. x_j, and vice-versa
  - A CSP P is arc-consistent if all $c \in C$ are arc-consistent
- Notation: AC means arc-consistent
- If $a \in d_i$ is arc-inconsistent (not arc-consistent) wrt. some x_j, it is unfeasible. Hence, it can be safely removed!
- Enforcing AC means removing arc-inconsistent values until AC is achieved

Arc Consistency

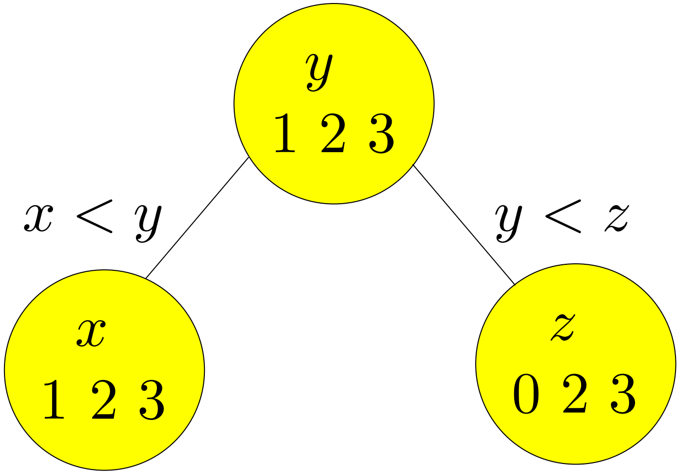
- The AC name comes from the arcs (constraints) of the interaction graph
- Example of enforcing AC.

  Consider the CSP with variables x, y, z,
  domains $d_x = d_y = \{1, 2, 3\}$ and $d_z = \{0, 2, 3\}$,
  and constraints $x < y, \quad y < z$
- Recall nodes are labelled with variables (here, also with their domains)
- Recall edges are labelled with constraints

Arc Consistency

- The AC name comes from the arcs (constraints) of the interaction graph
- Example of enforcing AC.

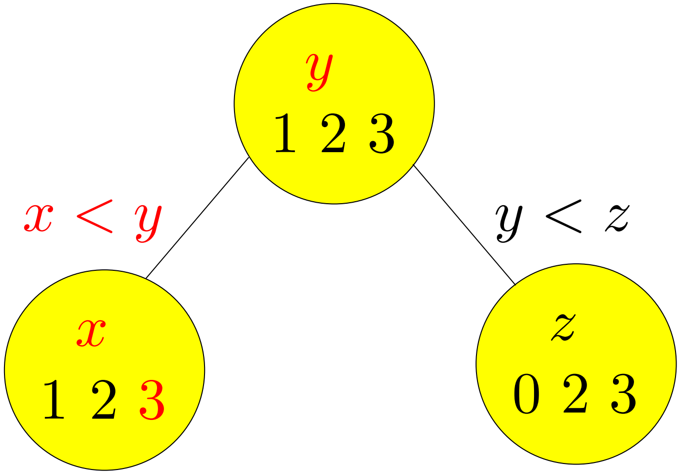
  Consider the CSP with variables x, y, z,
  domains $d_x = d_y = \{1, 2, 3\}$ and $d_z = \{0, 2, 3\}$,
  and constraints $x < y, \quad y < z$
- Recall nodes are labelled with variables (here, also with their domains)
- Recall edges are labelled with constraints



Arc Consistency

- The AC name comes from the arcs (constraints) of the interaction graph
- Example of enforcing AC.

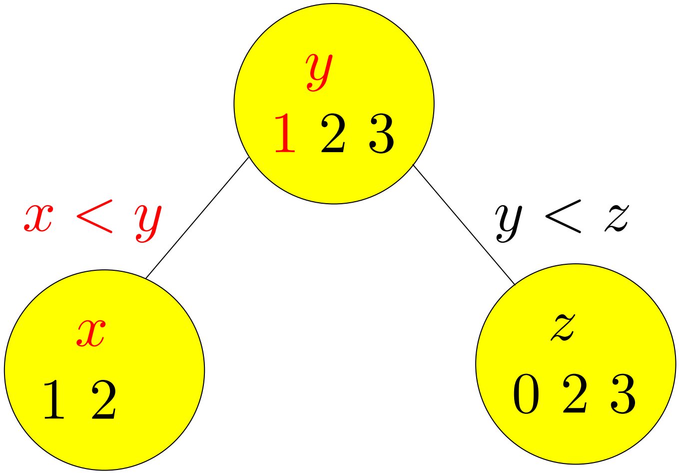
  Consider the CSP with variables x, y, z,
  domains $d_x = d_y = \{1, 2, 3\}$ and $d_z = \{0, 2, 3\}$,
  and constraints $x < y, \quad y < z$
- Recall nodes are labelled with variables (here, also with their domains)
- Recall edges are labelled with constraints



Arc Consistency

- The AC name comes from the arcs (constraints) of the interaction graph
- Example of enforcing AC.

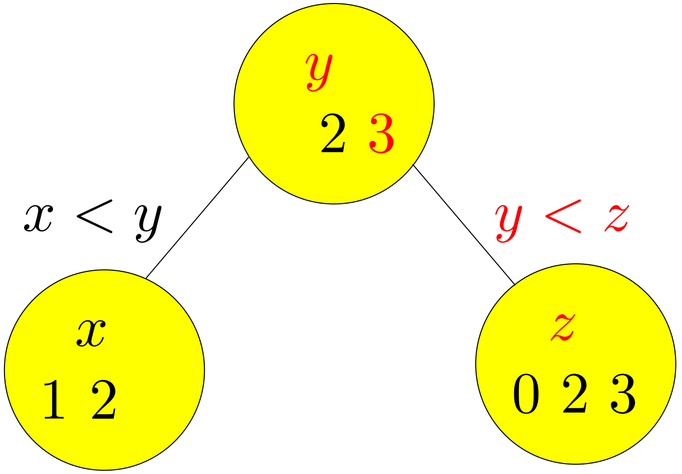
  Consider the CSP with variables x, y, z,
  domains $d_x = d_y = \{1, 2, 3\}$ and $d_z = \{0, 2, 3\}$,
  and constraints $x < y, \quad y < z$
- Recall nodes are labelled with variables (here, also with their domains)
- Recall edges are labelled with constraints



Arc Consistency

- The AC name comes from the arcs (constraints) of the interaction graph
- Example of enforcing AC.

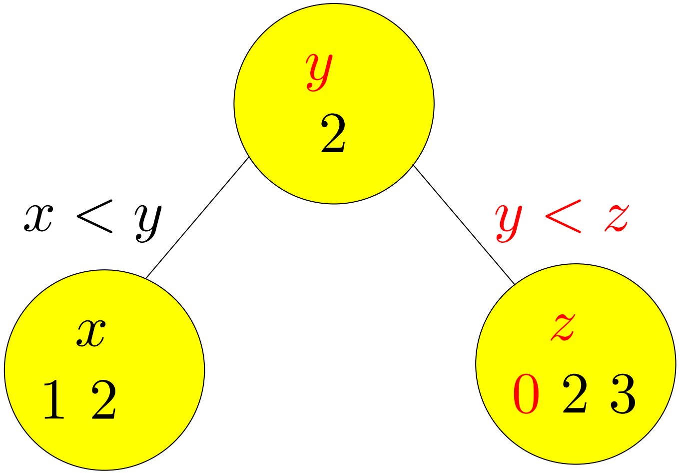
  Consider the CSP with variables x, y, z,
  domains $d_x = d_y = \{1, 2, 3\}$ and $d_z = \{0, 2, 3\}$,
  and constraints $x < y, \quad y < z$
- Recall nodes are labelled with variables (here, also with their domains)
- Recall edges are labelled with constraints



Arc Consistency

- The AC name comes from the arcs (constraints) of the interaction graph
- Example of enforcing AC.

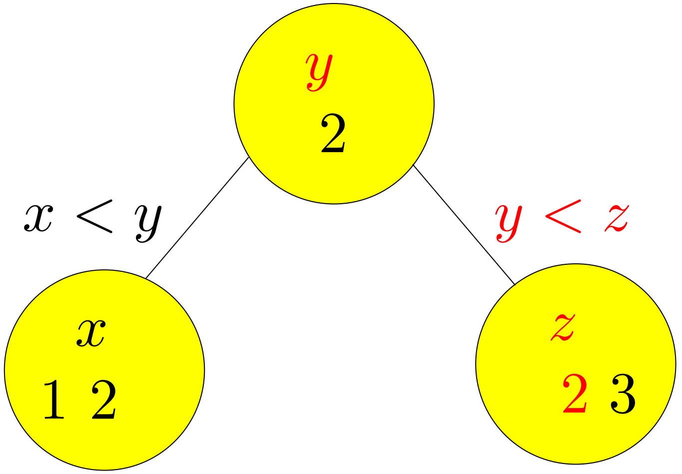
  Consider the CSP with variables x, y, z,
  domains $d_x = d_y = \{1, 2, 3\}$ and $d_z = \{0, 2, 3\}$,
  and constraints $x < y, \quad y < z$
- Recall nodes are labelled with variables (here, also with their domains)
- Recall edges are labelled with constraints



Arc Consistency

- The AC name comes from the arcs (constraints) of the interaction graph
- Example of enforcing AC.

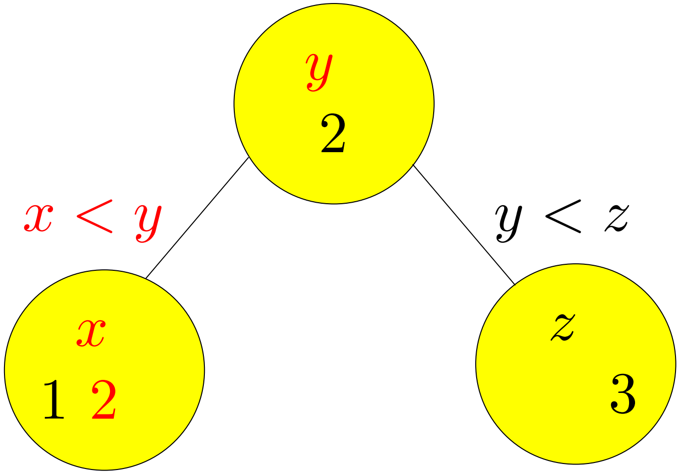
  Consider the CSP with variables x, y, z,
  domains $d_x = d_y = \{1, 2, 3\}$ and $d_z = \{0, 2, 3\}$,
  and constraints $x < y, \quad y < z$
- Recall nodes are labelled with variables (here, also with their domains)
- Recall edges are labelled with constraints



Arc Consistency

- The AC name comes from the arcs (constraints) of the interaction graph
- Example of enforcing AC.

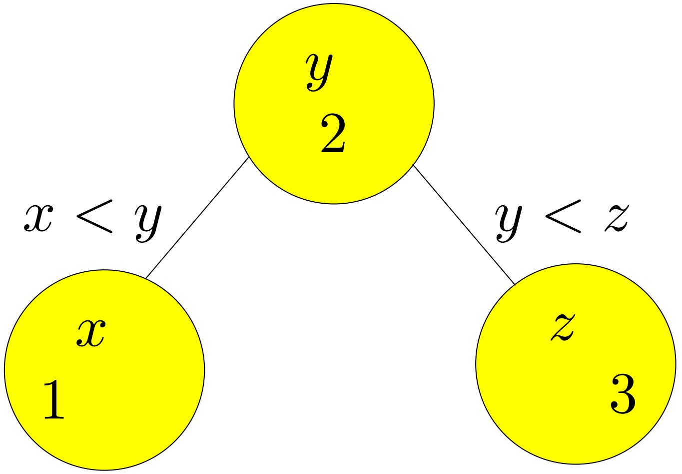
  Consider the CSP with variables x, y, z,
  domains $d_x = d_y = \{1, 2, 3\}$ and $d_z = \{0, 2, 3\}$,
  and constraints $x < y, \quad y < z$
- Recall nodes are labelled with variables (here, also with their domains)
- Recall edges are labelled with constraints



Arc Consistency

- The AC name comes from the arcs (constraints) of the interaction graph
- Example of enforcing AC.

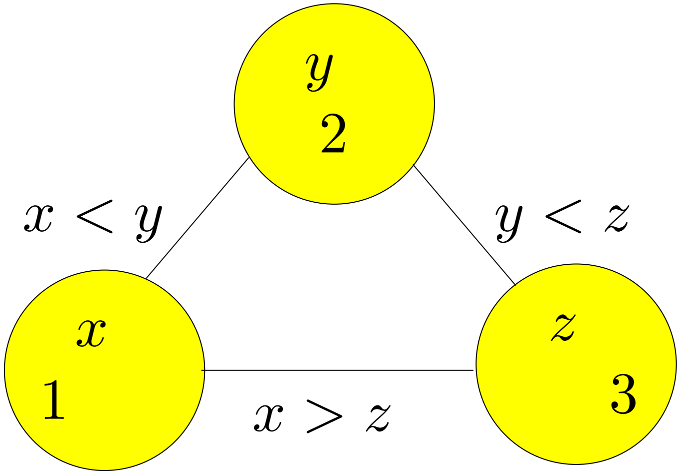
  Consider the CSP with variables x, y, z,
  domains $d_x = d_y = \{1, 2, 3\}$ and $d_z = \{0, 2, 3\}$,
  and constraints $x < y, \quad y < z$
- Recall nodes are labelled with variables (here, also with their domains)
- Recall edges are labelled with constraints



Arc Consistency

- Another example of enforcing AC.

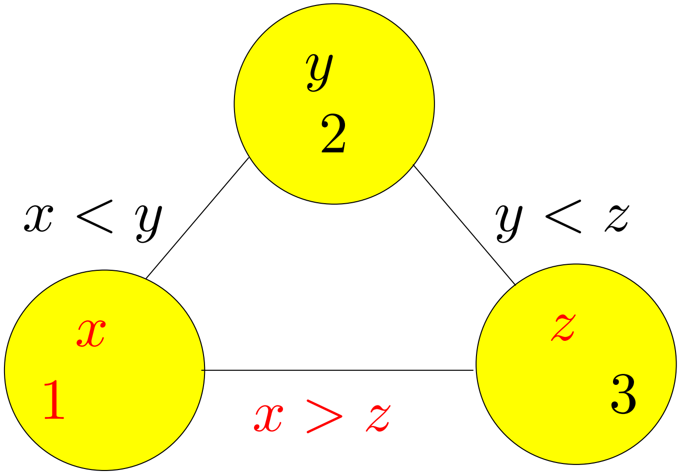
  Consider the CSP with variables x, y, z,
  domains $d_x = d_y = \{1, 2, 3\}$ and $d_z = \{0, 2, 3\}$,
  and constraints $x < y,\ y < z,\ x > z$



Arc Consistency

- Another example of enforcing AC.

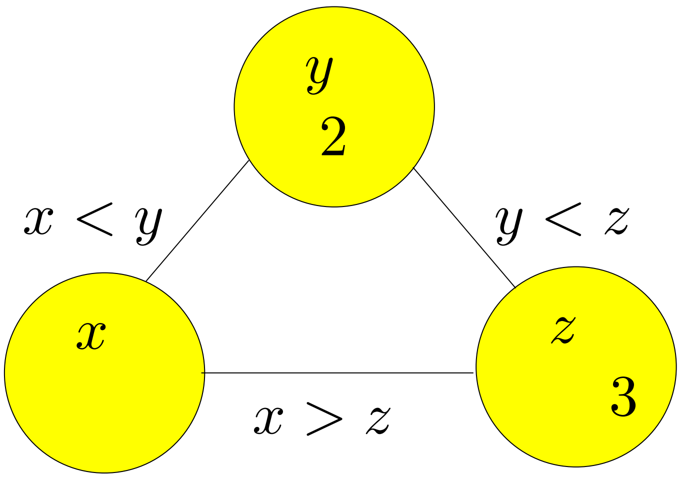
  Consider the CSP with variables x, y, z,
  domains $d_x = d_y = \{1, 2, 3\}$ and $d_z = \{0, 2, 3\}$,
  and constraints $x < y,\ y < z,\ x > z$



Arc Consistency

- Another example of enforcing AC.

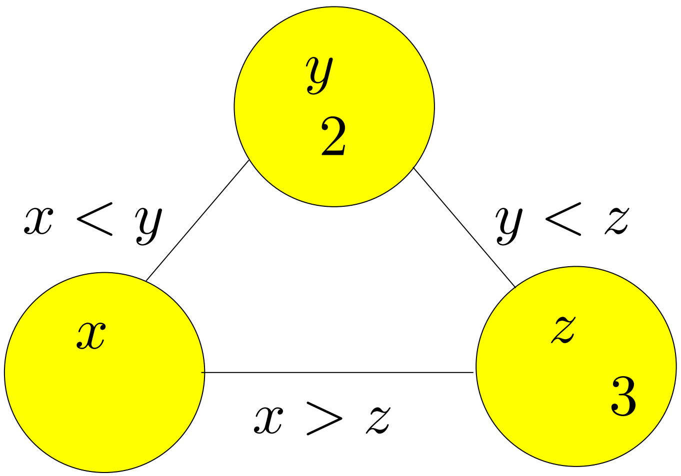
  Consider the CSP with variables x, y, z,
  domains $d_x = d_y = \{1, 2, 3\}$ and $d_z = \{0, 2, 3\}$,
  and constraints $x < y,\ y < z,\ x > z$



Arc Consistency

- Another example of enforcing AC.

  Consider the CSP with variables x, y, z,
  domains $d_x = d_y = \{1, 2, 3\}$ and $d_z = \{0, 2, 3\}$,
  and constraints $x < y,\ y < z,\ x > z$



- The domain of x has become empty!
- So the CSP cannot have any solution

Arc Consistency

- If while enforcing AC some domain becomes empty,
  then we know the original CSP has no solution

- Else, when there are no more arc-inconsistent values,
  we have a smaller equivalent CSP (no solution is lost)

- Uniqueness: the order of removal of arc-inconsistent values is irrelevant

- Now let us see algorithms for effectively enforcing AC

AC-1: Revise(i, j)

The simplest algorithm to enforce AC is called AC-1

Based on function `Revise`(i, j),
which removes values from d_i without support in d_j.
Returns true if some value is removed

function `Revise`(i, j)
    change $:= \mathrm{false}$
    for each $a \in d_i$ **do**
        if $\forall_{b \in d_j} \quad \neg c_{ij}(a, b)$ **then**
            change $:= \mathrm{true}$
            remove a **from** d_i
    return change

The time complexity of `Revise`(i, j) is $O(|d_i| \cdot |d_j|)$
(we assume that evaluating a binary constraint takes constant time)

AC-1

procedure `AC-1`(X, D, C)
    repeat
        change $:= \mathit{false}$
        for each $c_{ij} \in C$ **do**
            change $:=$ change $\vee$ `Revise`(i, j) $\vee$ `Revise`(j, i)
    until $\neg$ change

- The time complexity of AC-1 is $O(e \cdot n \cdot m^3)$,
  with $n = |X|$, $m = \max_i\{|d_i|\}$ and $e = |C|$ (note $e = O(n^2)$)
- Whenever a value has been removed,
  the domain should be checked if empty (not done here for simplicity)

AC-3

- A more efficient algorithm is AC-3, which only revises constraints with a chance of filtering domains
- AC-3 uses a set of pairs Q s.t. if $(i,j) \in Q$ then we can't ensure that all values in d_i have support in x_j

procedure $\texttt{AC-3}(X, D, C)$
  $Q := \{(i,j),(j,i) \mid c_{ij} \in C\}$ // each pair is added twice
  while $Q \neq \emptyset$ **do**
    $(i,j) := \texttt{Fetch}(Q)$ // selects and removes from Q
    if $\texttt{Revise}(i,j)$ **then**
      $Q := Q \cup \{(k,i) \mid c_{ki} \in C, k \neq j\}$

- Space complexity: $O(e)$
- Time complexity: $O(e \cdot m^3)$

Complexity of AC-3

$Q := \{(i,j),(j,i) | \;\; c_{ij} \in C\}$ // each pair is added twice
while $Q \neq \emptyset$ **do**
  $(i,j) :=$ `Fetch`(Q) // selects and removes from Q
  if `Revise`(i,j) **then**
    $Q := Q \cup \{(k,i) \mid c_{ki} \in C, k \neq j\}$

- (i,j) is in Q because d_j has been pruned.
  Therefore, it will be in Q at most m times.
  Consequently, the loop iterates at most $O(e \cdot m)$ times
- Without the red part, the cost of AC-3 would be $O(e \cdot m^3)$
- For a given i, `Revise`(i,j) will be true at most m times.
  Aggregated cost of red part due to i is

$$O(|\{k \,|\, c_{ki} \in C\}| \cdot m) = O(\deg(i) \cdot m)$$

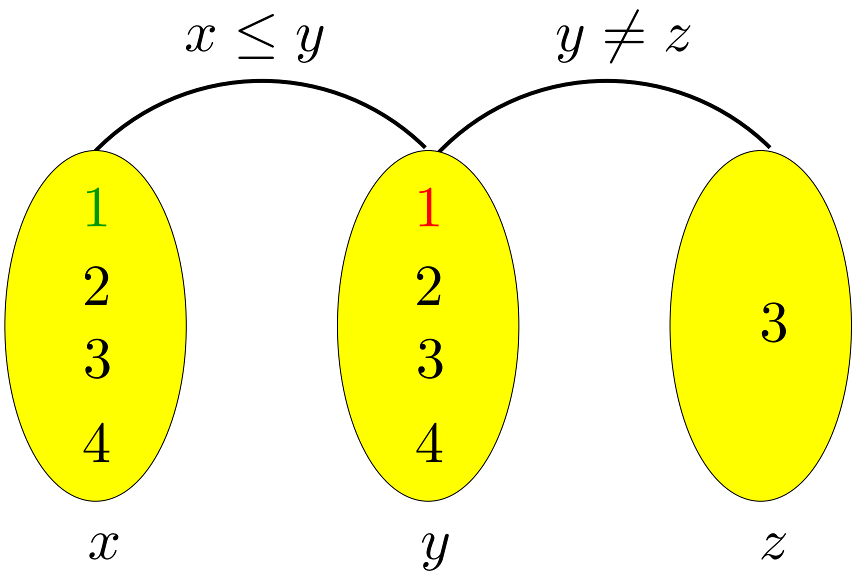
  So aggregated cost of red part due to all vars is $O(e \cdot m)$
- So the total cost in time is $O(e \cdot m^3)$

Example of AC-3

- Let x, y, z be vars with domains $d_x = d_y = \{1, 2, 3, 4\}$, $d_z = \{3\}$, and $c_1 \equiv x \leq y$, $c_2 \equiv y \neq z$

Example of AC-3

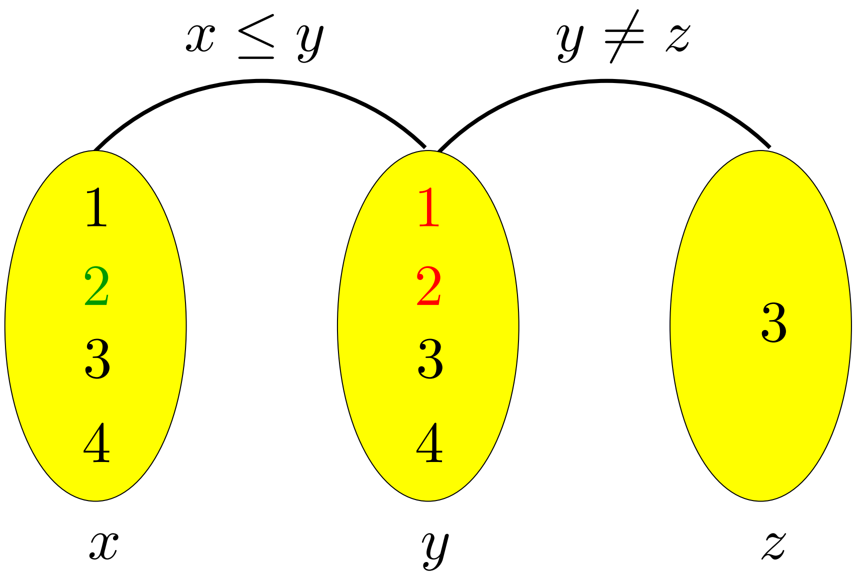
- Let x, y, z be vars with domains $d_x = d_y = \{1, 2, 3, 4\}$, $d_z = \{3\}$, and $c_1 \equiv x \leq y$, $c_2 \equiv y \neq z$



- Let us count the number of constraint checks of `Revise`(x, y)
  - Finding support for $(x, 1)$: 1

Example of AC-3

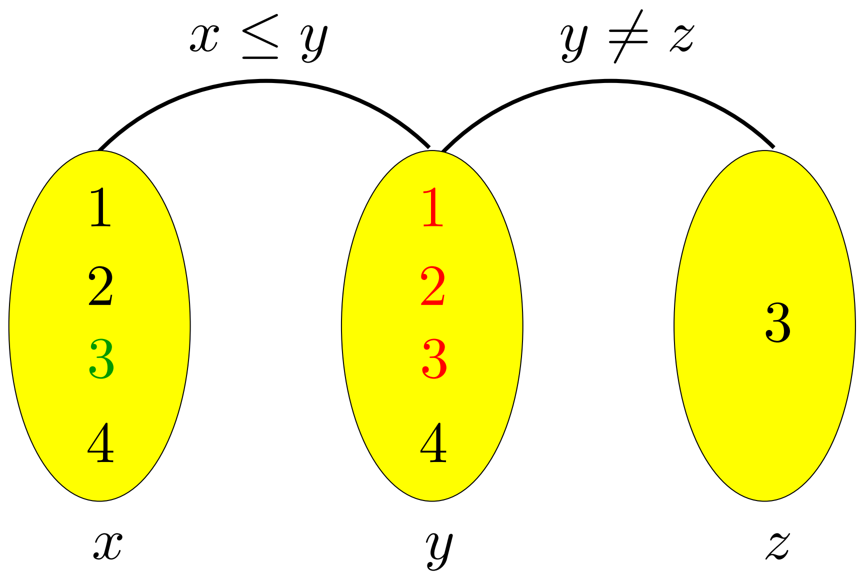
- Let x, y, z be vars with domains $d_x = d_y = \{1, 2, 3, 4\}$, $d_z = \{3\}$, and $c_1 \equiv x \leq y$, $c_2 \equiv y \neq z$



- Let us count the number of constraint checks of `Revise`(x, y)
    - Finding support for $(x, 2)$: 2

Example of AC-3

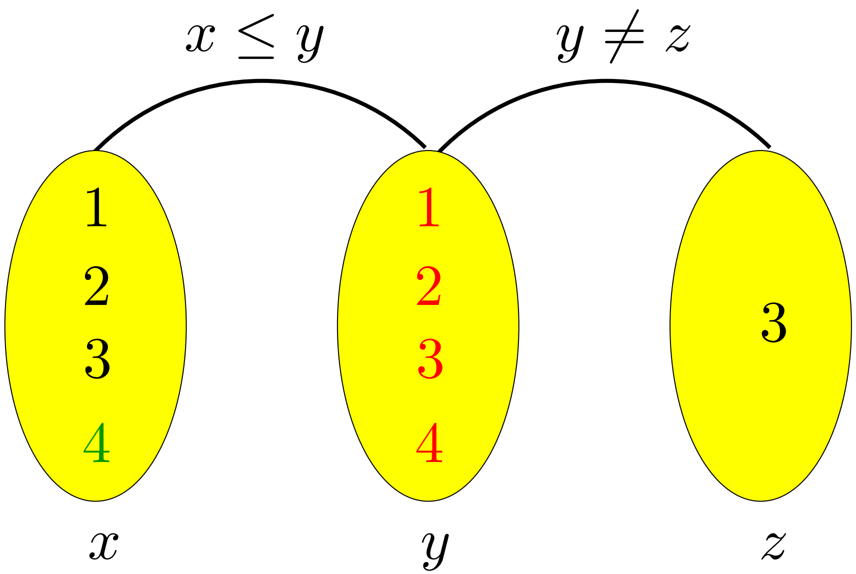
- Let x, y, z be vars with domains $d_x = d_y = \{1, 2, 3, 4\}$, $d_z = \{3\}$, and $c_1 \equiv x \leq y$, $c_2 \equiv y \neq z$



- Let us count the number of constraint checks of `Revise`(x, y)
  - Finding support for $(x, 3)$: 3

Example of AC-3

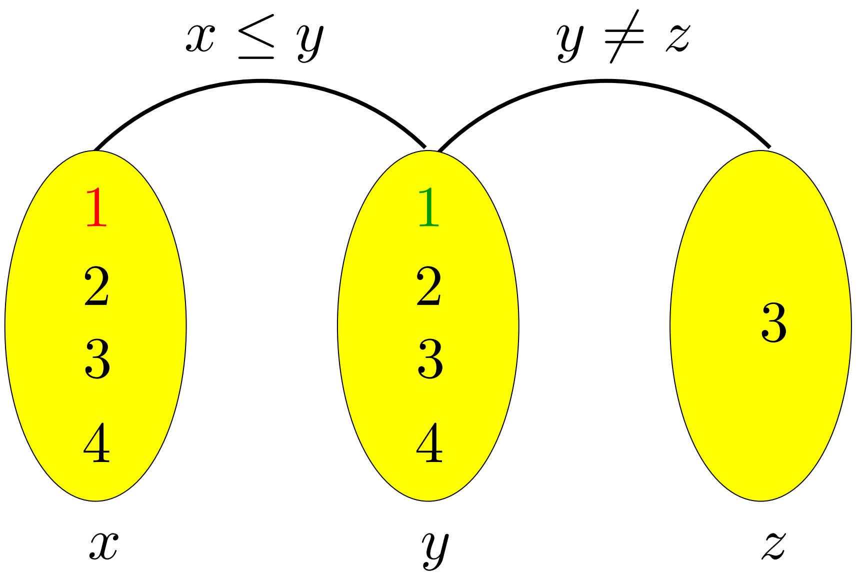
- Let x, y, z be vars with domains $d_x = d_y = \{1, 2, 3, 4\}$, $d_z = \{3\}$, and $c_1 \equiv x \leq y$, $c_2 \equiv y \neq z$



- Let us count the number of constraint checks of `Revise`(x, y)
  - Finding support for $(x, 4)$: 4

Example of AC-3

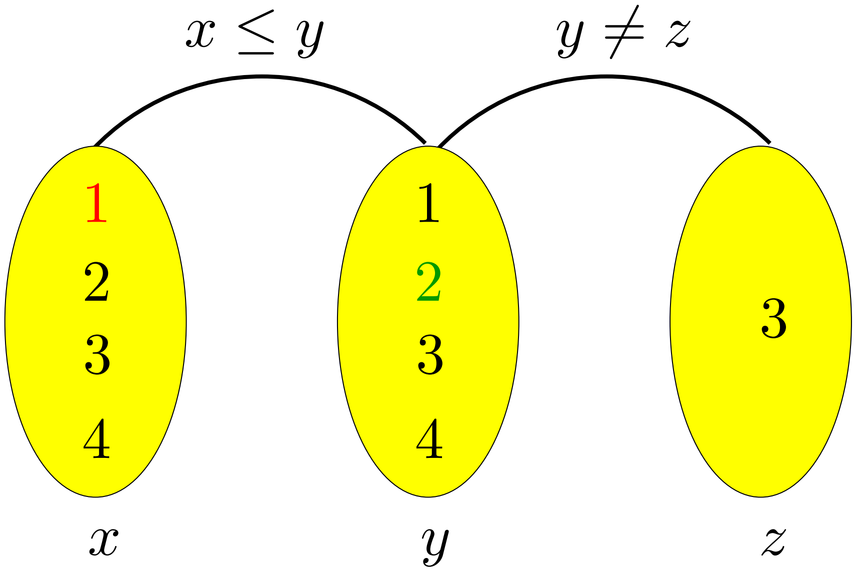
- Let x, y, z be vars with domains $d_x = d_y = \{1, 2, 3, 4\}$, $d_z = \{3\}$, and $c_1 \equiv x \leq y$, $c_2 \equiv y \neq z$



- Let us count the number of constraint checks of `Revise`(y, x)
  - Finding support for $(y, 1)$: 1

Example of AC-3

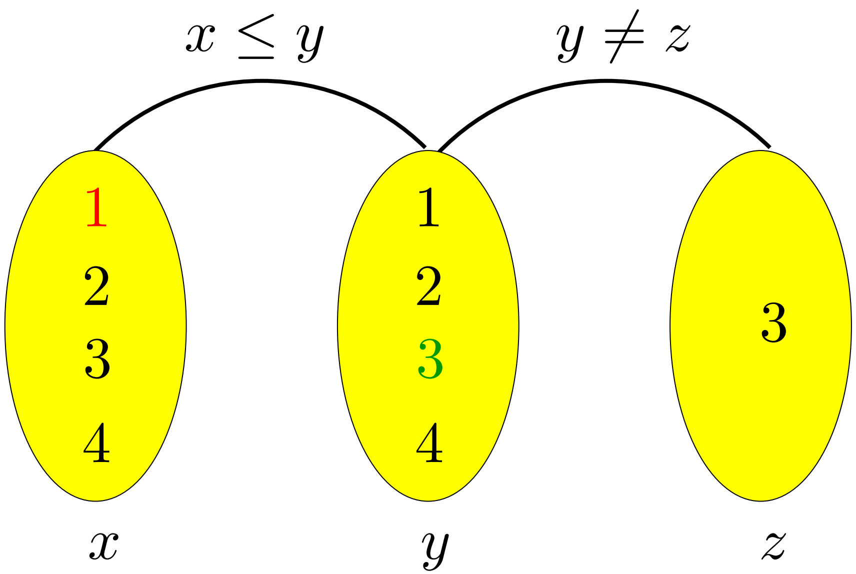
- Let x, y, z be vars with domains $d_x = d_y = \{1, 2, 3, 4\}$, $d_z = \{3\}$, and $c_1 \equiv x \leq y$, $c_2 \equiv y \neq z$



- Let us count the number of constraint checks of `Revise`(y, x)
  - Finding support for $(y, 2)$: 1

Example of AC-3

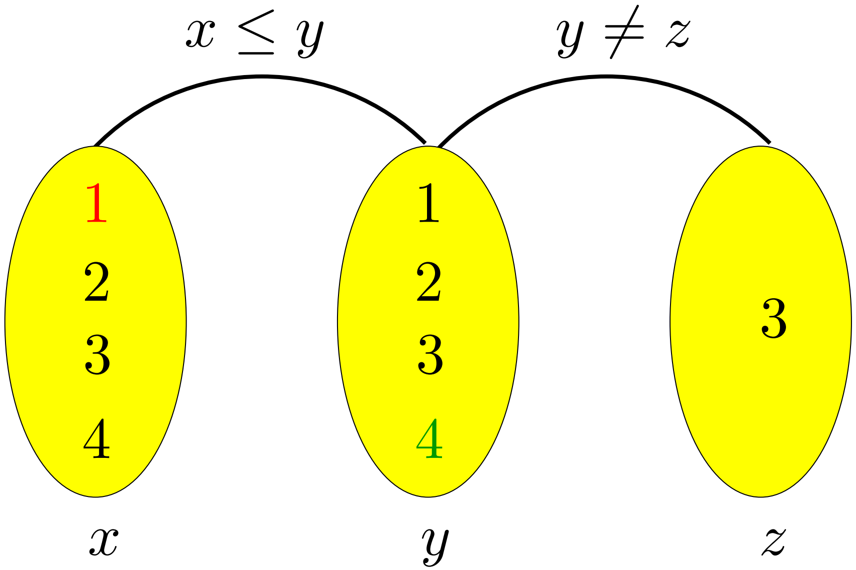
- Let x, y, z be vars with domains $d_x = d_y = \{1, 2, 3, 4\}$, $d_z = \{3\}$, and $c_1 \equiv x \leq y$, $c_2 \equiv y \neq z$



- Let us count the number of constraint checks of `Revise`(y, x)
  - Finding support for $(y, 3)$: 1

Example of AC-3

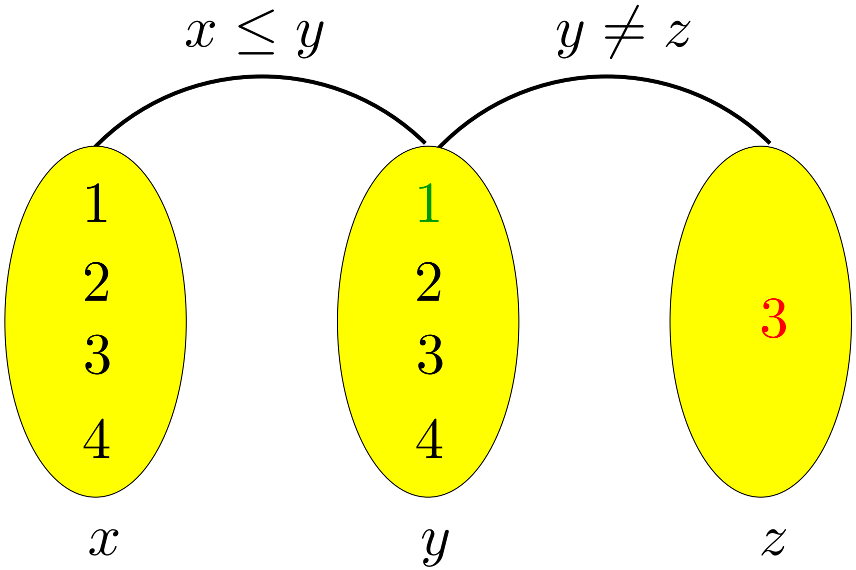
- Let x, y, z be vars with domains $d_x = d_y = \{1, 2, 3, 4\}$, $d_z = \{3\}$, and $c_1 \equiv x \leq y$, $c_2 \equiv y \neq z$



- Let us count the number of constraint checks of `Revise`(y, x)
  - Finding support for $(y, 4)$: 1

Example of AC-3

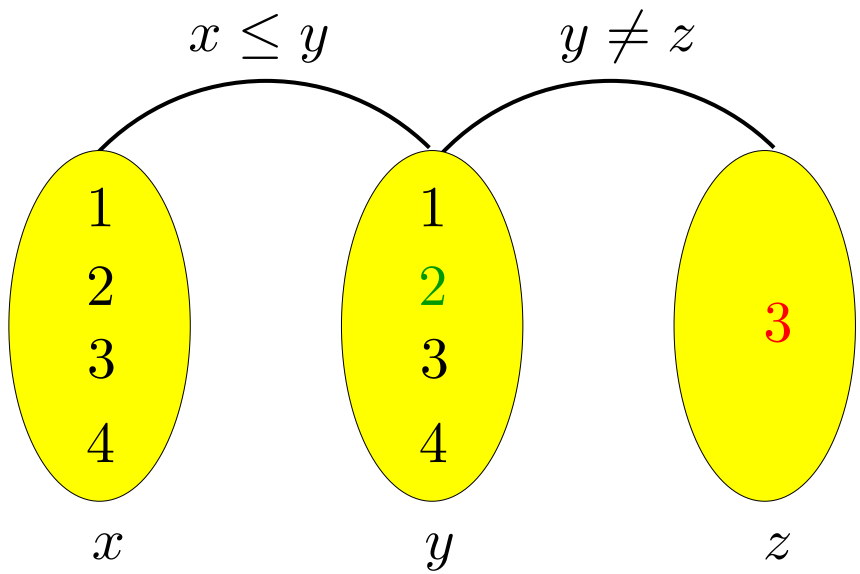
- Let x, y, z be vars with domains $d_x = d_y = \{1, 2, 3, 4\}$, $d_z = \{3\}$, and $c_1 \equiv x \leq y$, $c_2 \equiv y \neq z$



- Let us count the number of constraint checks of `Revise`(y, z)
  - Finding support for $(y, 1)$: 1

Example of AC-3

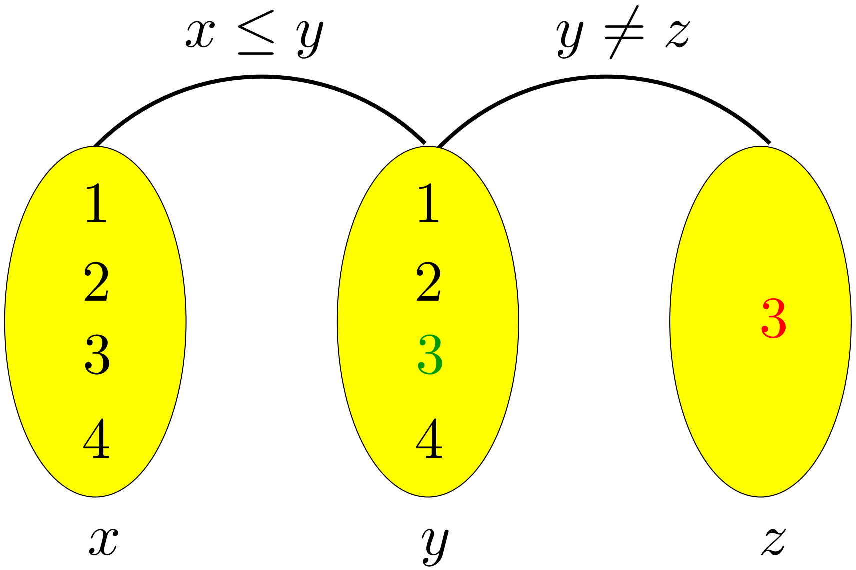
- Let x, y, z be vars with domains $d_x = d_y = \{1, 2, 3, 4\}$, $d_z = \{3\}$, and $c_1 \equiv x \leq y$, $c_2 \equiv y \neq z$



- Let us count the number of constraint checks of `Revise`(y, z)
  - Finding support for $(y, 2)$: 1

Example of AC-3

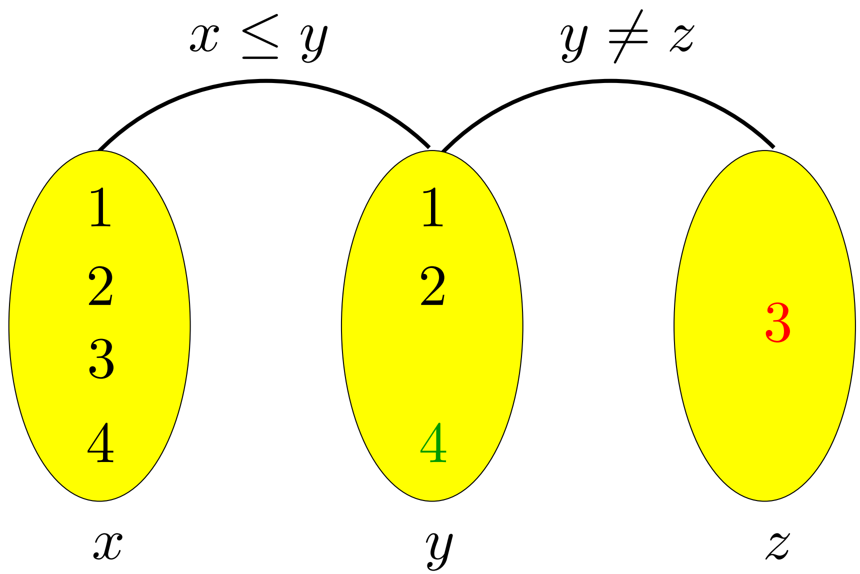
- Let x, y, z be vars with domains $d_x = d_y = \{1, 2, 3, 4\}$, $d_z = \{3\}$, and $c_1 \equiv x \leq y$, $c_2 \equiv y \neq z$



- Let us count the number of constraint checks of `Revise`(y, z)
  - Finding support for $(y, 3)$: 1

Example of AC-3

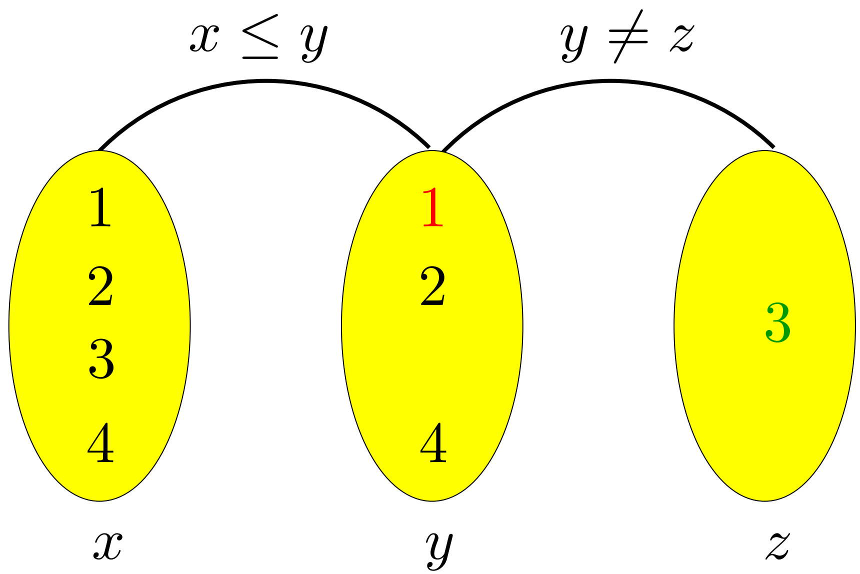
- Let x, y, z be vars with domains $d_x = d_y = \{1, 2, 3, 4\}$, $d_z = \{3\}$, and $c_1 \equiv x \leq y$, $c_2 \equiv y \neq z$



- Let us count the number of constraint checks of `Revise`(y, z)
  - Finding support for $(y, 4)$: 1

Example of AC-3

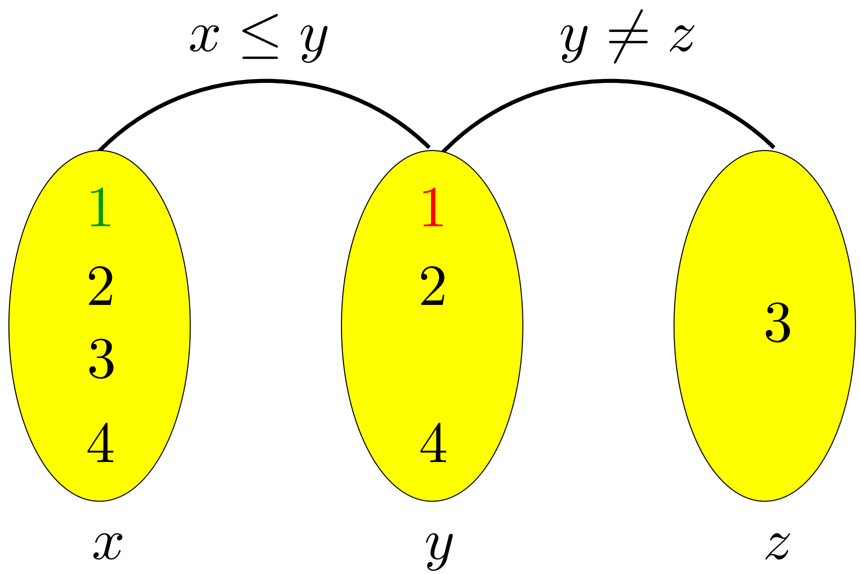
- Let x, y, z be vars with domains $d_x = d_y = \{1, 2, 3, 4\}$, $d_z = \{3\}$, and $c_1 \equiv x \leq y$, $c_2 \equiv y \neq z$



- Let us count the number of constraint checks of `Revise`(z, y)
  - Finding support for $(z, 3)$: 1

Example of AC-3

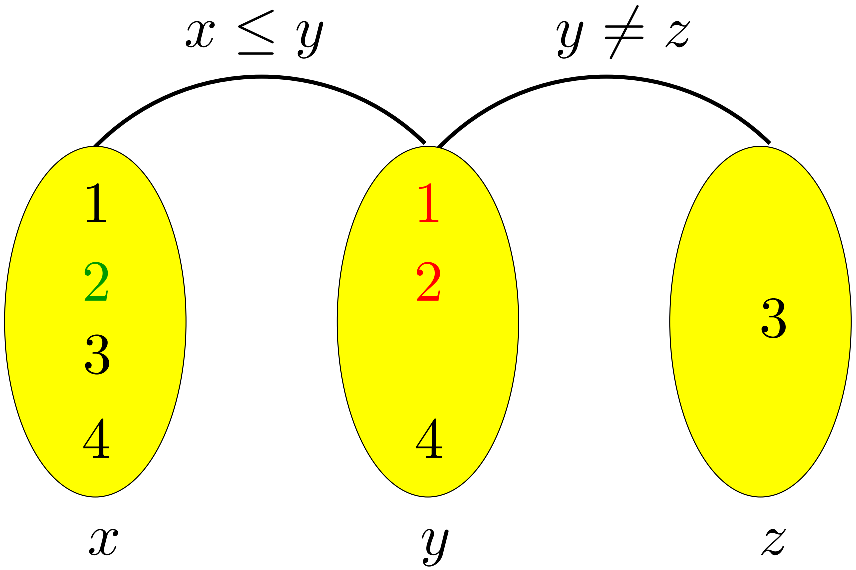
- Let x, y, z be vars with domains $d_x = d_y = \{1, 2, 3, 4\}$, $d_z = \{3\}$, and $c_1 \equiv x \leq y$, $c_2 \equiv y \neq z$



- Let us count the number of constraint checks of `Revise`(x, y)
    - Finding support for $(x, 1)$: 1

Example of AC-3

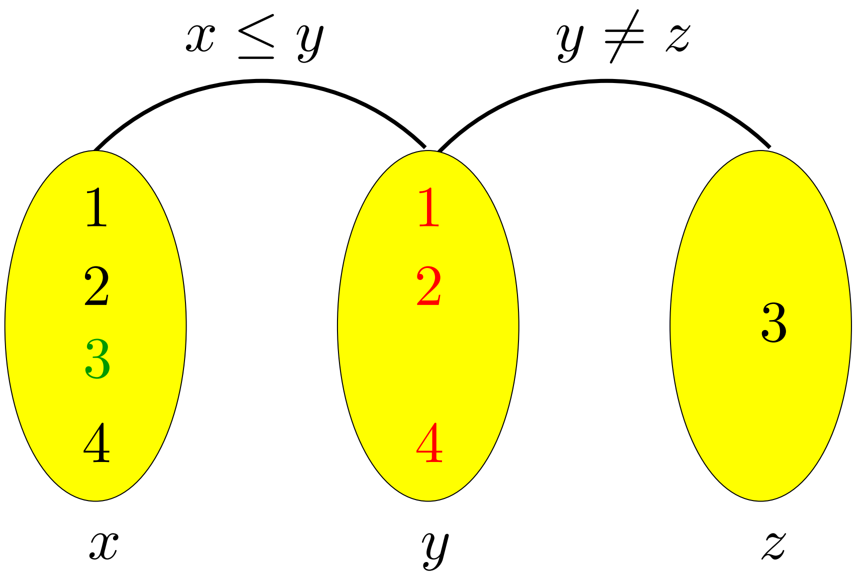
- Let x, y, z be vars with domains $d_x = d_y = \{1, 2, 3, 4\}$, $d_z = \{3\}$, and $c_1 \equiv x \leq y$, $c_2 \equiv y \neq z$



- Let us count the number of constraint checks of `Revise`(x, y)
    - Finding support for $(x, 2)$: 2

Example of AC-3

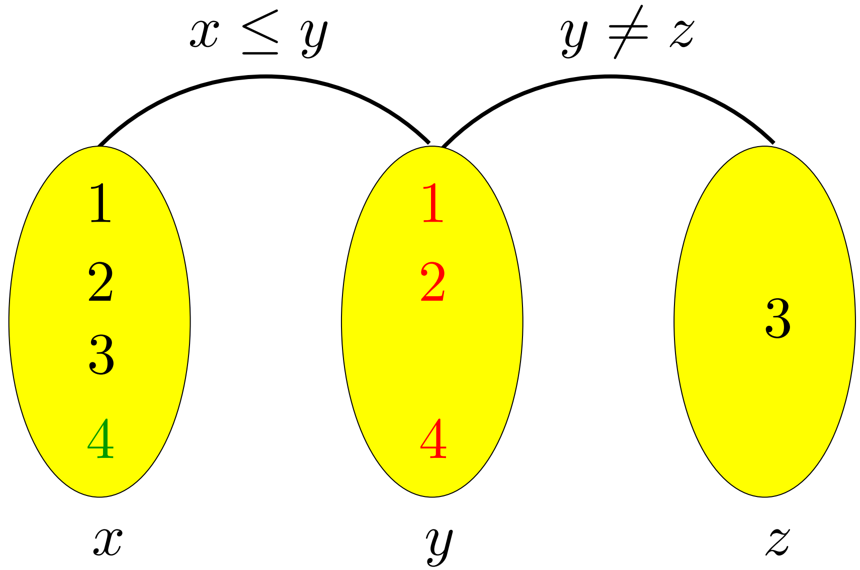
- Let x, y, z be vars with domains $d_x = d_y = \{1, 2, 3, 4\}$, $d_z = \{3\}$, and $c_1 \equiv x \leq y$, $c_2 \equiv y \neq z$



- Let us count the number of constraint checks of `Revise`(x, y)
    - Finding support for $(x, 3)$: 3

Example of AC-3

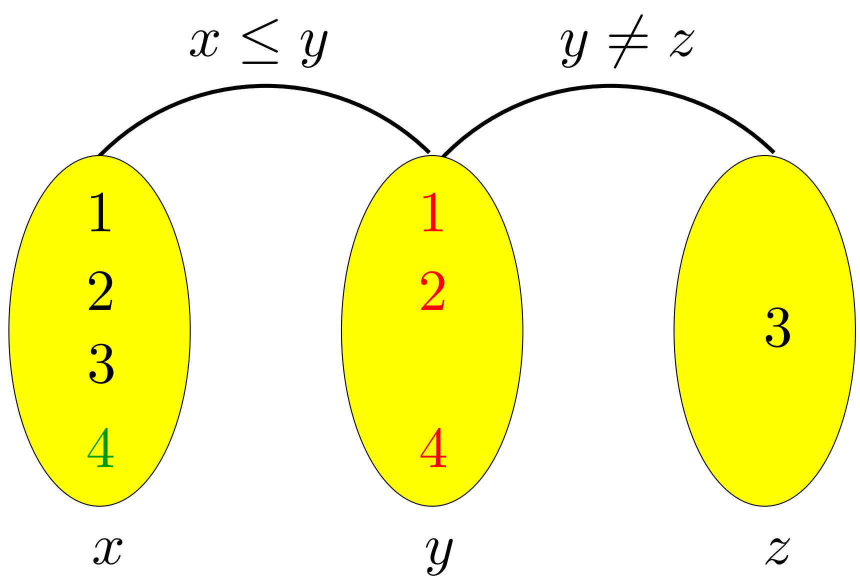
- Let x, y, z be vars with domains $d_x = d_y = \{1, 2, 3, 4\}$, $d_z = \{3\}$, and $c_1 \equiv x \leq y$, $c_2 \equiv y \neq z$



- Let us count the number of constraint checks of `Revise`(x, y)
  - Finding support for $(x, 4)$: 3

Example of AC-3

- Let x, y, z be vars with domains $d_x = d_y = \{1, 2, 3, 4\}$, $d_z = \{3\}$, and $c_1 \equiv x \leq y$, $c_2 \equiv y \neq z$



- Altogether $10 + 4 + 4 + 1 + 9 = 28$ constraint checks
- From last 9, the only new check is $((x, 3), (y, 4))$!
- Still a lot of work is repeated over and over again

AC-4

- AC-4 is an even more efficient algorithm. It uses:
  - $N[i,a,j] =$ "number of supports that $a \in d_i$ has in d_j"
  - $S[j,b] =$ "set of pairs (i,a) supported by $b \in d_j$"
  - $(i,a) \in Q$ means that a has been pruned from d_i and its effect has not been updated on the N array yet

procedure `AC-4`(X, D, C)

  // N is constructed full of 0's and S is constructed full of $\emptyset$'s

  // Initialization phase

  for each $c_{ij} \in C$, $a \in d_i$, $b \in d_j$ **such that** $c_{ij}(a,b)$ **do**

    // Value b in d_j is a support for value $a \in d_i$

    $N[i,a,j]++$

    $S[j,b] := S[j,b] \cup \{(i,a)\}$

  for each $c_{ij} \in C$, $a \in d_i$ **such that** $N[i,a,j] = 0$ **do**

    remove a **from** d_i

    $Q := Q \cup (i,a)$

  ...

AC-4

...
// Propagation phase
while $Q \neq \emptyset$ **do**
    $(j, b) := \texttt{Fetch}(Q)$
    for each $(i, a) \in S[j, b]$ **such that** $a \in d_i$ **do**
        $N[i, a, j] - -$
        if $N[i, a, j] = 0$ **then**
            remove a **from** d_i
            $Q := Q \cup (i, a)$

- Time complexity of AC-4: $O(e \cdot m^2)$ (provable optimal!)
    - the initialization phase has cost $O(e \cdot m^2)$
    - the propagation phase has cost $O(e \cdot m^2)$
- Space complexity of AC-4: $O(e \cdot m^2)$

Example of AC-4

- Let x, y, z be vars with domains $d_x = d_y = \{1, 2, 3, 4\}$, $d_z = \{3\}$, and $c_1 \equiv x \leq y$, $c_2 \equiv y \neq z$
- During initialization, AC-4 performs all possible constraint checks for every value in each domain
  - For c_1: $4 \cdot 4 = 16$ constraint checks
  - For c_2: $4 \cdot 1 = 4$ constraint checks
  - In total 20 constraint checks

Example of AC-4

- Let x, y, z be vars with domains $d_x = d_y = \{1, 2, 3, 4\}$, $d_z = \{3\}$, and $c_1 \equiv x \leq y$, $c_2 \equiv y \neq z$
- After initialization:

$$
\begin{array}{llll}
N[x,1,y]=4 & N[y,1,x]=1 & N[y,1,z]=1 & N[z,3,y]=3 \\
N[x,2,y]=3 & N[y,2,x]=2 & N[y,2,z]=1 & \\
N[x,3,y]=2 & N[y,3,x]=3 & N[y,3,z]=0 & \\
N[x,4,y]=1 & N[y,4,x]=4 & N[y,4,z]=1 &
\end{array}
$$

$$
\begin{array}{lr}
S[x,1] = \{(y,1),(y,2),(y,3),(y,4)\} & S[y,1] = \{(z,3),(x,1)\} \\
S[x,2] = \{(y,2),(y,3),(y,4)\} & S[y,2] = \{(z,3),(x,1),(x,2)\} \\
S[x,3] = \{(y,3),(y,4)\} & S[y,3] = \{(x,1),(x,2),(x,3)\} \\
S[x,3] = \{(y,4)\} & S[y,4] = \{(z,3),(x,1),(x,2),(x,3),(x,4)\}
\end{array}
$$

$$S[z,3] = \{(y,1),(y,2),(y,4)\}$$

Example of AC-4

- The only counter equal to zero is $N[y, 3, z]$
- $(y, 3)$ is removed, propagation loop starts with $(y, 3)$ in Q
- When $(y, 3)$ is picked, traverse $S[y, 3] = \{(x, 1), (x, 2), (x, 3)\}$
  - $N[x, 1, y], N[x, 2, y], N[x, 3, y]$ are decremented
- No counter becomes zero, so nothing is added to Q
- Propagation did not require any extra constraint check!
- However
  - Space complexity is very high
  - The initialization phase can be prohibitive
    (AC-4 has optimal worst-case complexity ...
    but always reaches this worst case)

AC-6

- Motivation: keep the optimal worst-case of AC-4
  but instead of counting all supports that a value has,
  just ensure that there is at least one
- AC-6 keeps the smallest support for each (x_i, a) on c_{ij}
- *Initialization phase*: cheaper than in AC-4
- *Propagation phase*: if value b of var x_j is removed
  - if it is not the current support of value a of var x_i:
    no work due to constraint c_{ij} has to be done
  - if it is the current support of value a of var x_i:
    a new support is sought,
    but starting from next value of b in d_j instead of $\min\{d_j\}$
- The algorithm uses:

  $S[j, b] =$"set of (i, a) s.t. b is the current support of value a of x_i wrt. x_j"

AC-6

procedure `AC-6`(X, D, C)
  $Q := \emptyset$; $S[j, b] := \emptyset$, $\forall x_j \in X$, $\forall b \in d_j$
  for each $x_i \in X,\ c_{ij} \in C$, $a \in d_i$ **do**
    $b :=$ smallest value in d_j s.t. $c_{ij}(a, b)$
    if $b \neq \perp$ **then** add (i, a) to $S[j, b]$
    else remove a from d_i and add (i, a) to Q
  while $Q \neq \emptyset$ **do**
    $(j, b) :=$ `Fetch`(Q)
    for each $(i, a) \in S[j, b]$ **such that** $a \in d_i$ **do**
      $c :=$ smallest value $b' \in d_j$ s.t. $b' > b \wedge c_{ij}(a, b')$
      if $c \neq \perp$ **then** add (i, a) to $S[j, c]$
      else remove a from d_i and add (i, a) to Q

- Time complexity: $O(e \cdot m^2)$
- Space complexity: $O(e \cdot m)$

Example of AC-6

- Let x, y, z be vars with domains $d_x = d_y = \{1, 2, 3, 4\}$, $d_z = \{3\}$, and $c_1 \equiv x \leq y$, $c_2 \equiv y \neq z$

- In initialization, AC-6 performs the same number of constraint checks as AC-3: 10+4 on c_1 and 4+1 on c_2

$$S[x,1] = \{(y,1),(y,2),(y,3),(y,4)\} \qquad S[y,1] = \{(z,3),(x,1)\}$$

$$S[x,2] = \{\} \qquad S[y,2] = \{(x,2)\}$$

$$S[x,3] = \{\} \qquad S[y,3] = \{(x,3)\}$$

$$S[x,3] = \{\} \qquad S[y,4] = \{(x,4)\}$$

$$S[z,3] = \{(y,1),(y,2),(y,4)\}$$

Example of AC-6

- Let x, y, z be vars with domains $d_x = d_y = \{1, 2, 3, 4\}$, $d_z = \{3\}$, and $c_1 \equiv x \leq y$, $c_2 \equiv y \neq z$
- In initialization, AC-6 performs the same number of constraint checks as AC-3: 10+4 on c_1 and 4+1 on c_2

$$S[x,1] = \{(y,1),(y,2),(y,3),(y,4)\} \qquad S[y,1] = \{(z,3),(x,1)\}$$
$$S[x,2] = \{\} \qquad S[y,2] = \{(x,2)\}$$
$$S[x,3] = \{\} \qquad S[y,3] = \{(x,3)\}$$
$$S[x,3] = \{\} \qquad S[y,4] = \{(x,4)\}$$
$$S[z,3] = \{(y,1),(y,2),(y,4)\}$$

- Q contains $(y,3)$, which has been removed

Example of AC-6

- Let x, y, z be vars with domains $d_x = d_y = \{1, 2, 3, 4\}$, $d_z = \{3\}$, and $c_1 \equiv x \leq y$, $c_2 \equiv y \neq z$
- In initialization, AC-6 performs the same number of constraint checks as AC-3: 10+4 on c_1 and 4+1 on c_2

$$S[x,1] = \{(y,1),(y,2),(y,3),(y,4)\} \qquad S[y,1] = \{(z,3),(x,1)\}$$
$$S[x,2] = \{\} \qquad S[y,2] = \{(x,2)\}$$
$$S[x,3] = \{\} \qquad S[y,3] = \{(x,3)\}$$
$$S[x,3] = \{\} \qquad S[y,4] = \{(x,4)\}$$
$$S[z,3] = \{(y,1),(y,2),(y,4)\}$$

- When AC-6 enters propagation loop it pops $(y, 3)$ from Q, $S[y,3] = \{(x,3)\}$ is traversed and a new support greater than 3 is sought for $(x, 3)$.

Example of AC-6

- Let x, y, z be vars with domains $d_x = d_y = \{1, 2, 3, 4\}$, $d_z = \{3\}$, and $c_1 \equiv x \leq y$, $c_2 \equiv y \neq z$

- In initialization, AC-6 performs the same number of constraint checks as AC-3: 10+4 on c_1 and 4+1 on c_2

$$S[x,1] = \{(y,1),(y,2),(y,3),(y,4)\} \qquad S[y,1] = \{(z,3),(x,1)\}$$
$$S[x,2] = \{\} \qquad S[y,2] = \{(x,2)\}$$
$$S[x,3] = \{\} \qquad S[y,3] = \{(x,3)\}$$
$$S[x,3] = \{\} \qquad S[y,4] = \{(x,4)\}$$
$$S[z,3] = \{(y,1),(y,2),(y,4)\}$$

- Just 1 extra constraint check:
as $c_1(3,4)$, we add $(x,3)$ to $S[y,4]$

Example of AC-6

- Let x, y, z be vars with domains $d_x = d_y = \{1, 2, 3, 4\}$, $d_z = \{3\}$, and $c_1 \equiv x \leq y$, $c_2 \equiv y \neq z$
- In initialization, AC-6 performs the same number of constraint checks as AC-3: 10+4 on c_1 and 4+1 on c_2

$$S[x,1] = \{(y,1),(y,2),(y,3),(y,4)\} \qquad S[y,1] = \{(z,3),(x,1)\}$$
$$S[x,2] = \{\} \qquad S[y,2] = \{(x,2)\}$$
$$S[x,3] = \{\} \qquad S[y,3] = \{(x,3)\}$$
$$S[x,3] = \{\} \qquad S[y,4] = \{(x,4)\}$$
$$S[z,3] = \{(y,1),(y,2),(y,4)\}$$

- In total 20 constraint checks

Weaker than AC

- Directional AC (DAC)
- Bounds Consistency (BC)

Directional Arc Consistency

- Given an order $\prec$ among variables,
  each x_i only needs supports with respect to greater variables in the order

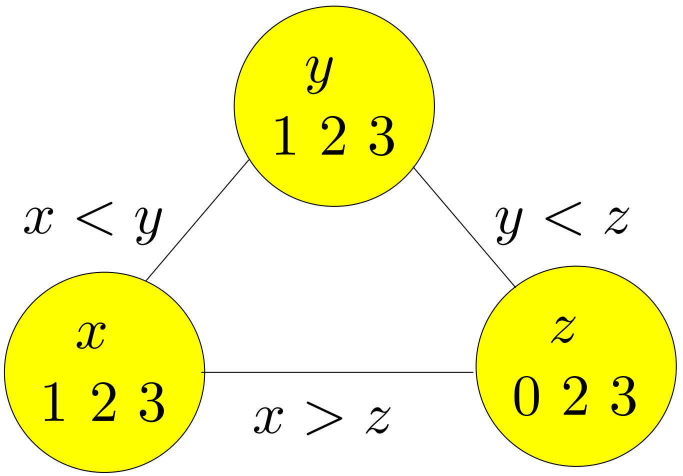
- Consider a CSP $P = (X, D, C)$.
  Constraint $c_{ij} \in C$ is directionally arc-consistent
  iff $x_i \prec x_j$ implies x_i is arc-consistent with respect to x_j

- The CSP P is directionally arc-consistent (DAC)
  iff all its constraints are directionally arc-consistent

- DAC is weaker than AC but is enforced more efficiently in practice

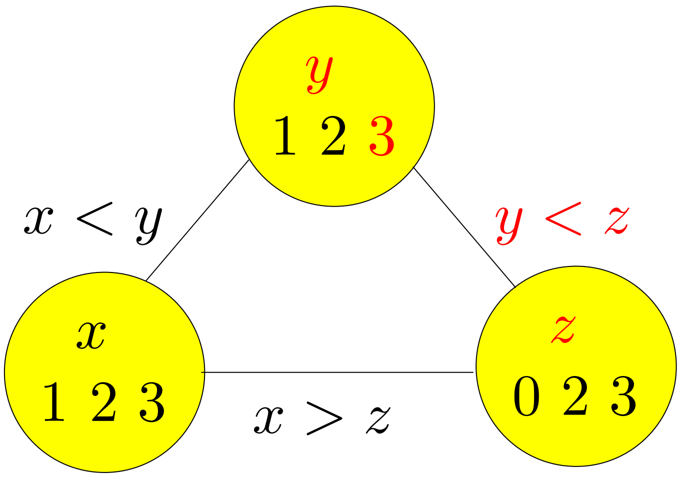
Directional Arc Consistency

- Consider CSP with vars $x \prec y \prec z$,
  domains $d_x = d_y = \{1, 2, 3\}$ and $d_z = \{0, 2, 3\}$,
  and constraints $x < y, \quad y < z, \quad x > z$



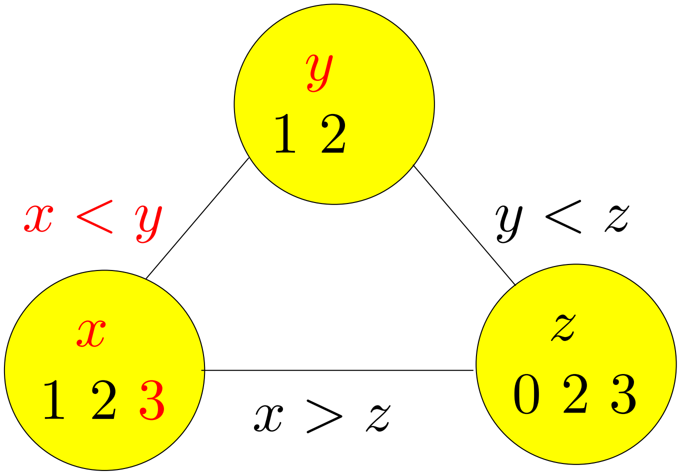
Directional Arc Consistency

- Consider CSP with vars $x \prec y \prec z$,
  domains $d_x = d_y = \{1, 2, 3\}$ and $d_z = \{0, 2, 3\}$,
  and constraints $x < y, \quad y < z, \quad x > z$



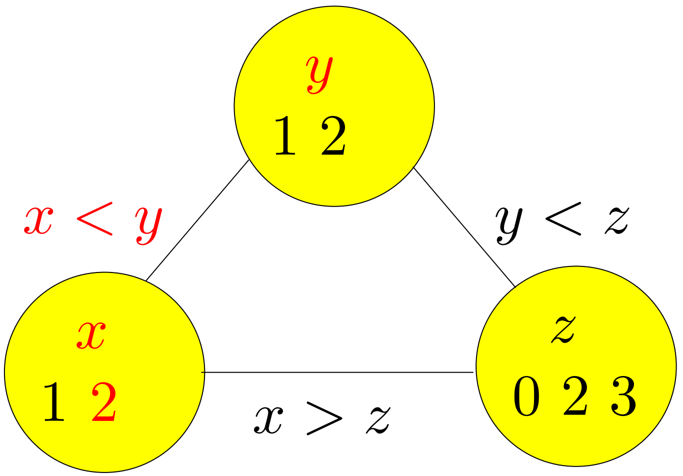
Directional Arc Consistency

- Consider CSP with vars $x \prec y \prec z$,
  domains $d_x = d_y = \{1, 2, 3\}$ and $d_z = \{0, 2, 3\}$,
  and constraints $x < y, \quad y < z, \quad x > z$



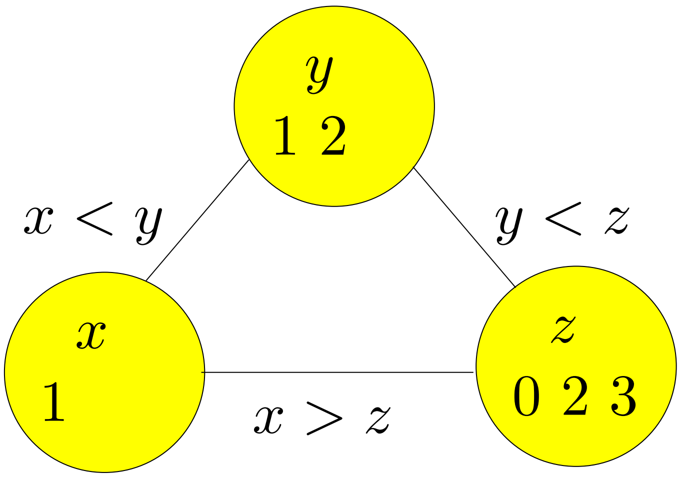
Directional Arc Consistency

- Consider CSP with vars $x \prec y \prec z$,
  domains $d_x = d_y = \{1, 2, 3\}$ and $d_z = \{0, 2, 3\}$,
  and constraints $x < y, \quad y < z, \quad x > z$



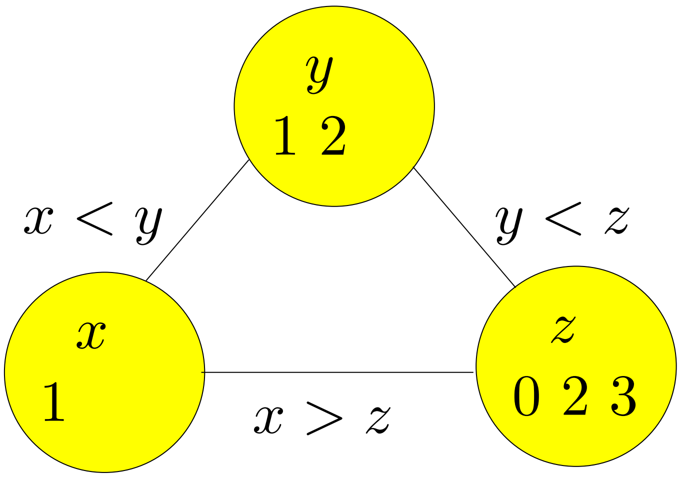
Directional Arc Consistency

- Consider CSP with vars $x \prec y \prec z$,
  domains $d_x = d_y = \{1, 2, 3\}$ and $d_z = \{0, 2, 3\}$,
  and constraints $x < y, \quad y < z, \quad x > z$



Directional Arc Consistency

- Consider CSP with vars $x \prec y \prec z$,
  domains $d_x = d_y = \{1, 2, 3\}$ and $d_z = \{0, 2, 3\}$,
  and constraints $x < y, \quad y < z, \quad x > z$



- Inconsistency is not detected by DAC!

DAC enforcing

procedure $\texttt{DAC}(X, D, C)$
  for each $i := n - 1$ **downto** 1 **do**
    for each c_{ij} s.t. $x_i \prec x_j$ **do** $\texttt{Revise}(i, j)$
endprocedure

- Only one pass is required
- Once x_i is made arc-consistent with respect to $x_j \succ x_i$, removing values from $x_k \prec x_i$ does not destroy the arc-consistency of x_i wrt. x_j
- Time complexity: $O(e \cdot m^2)$

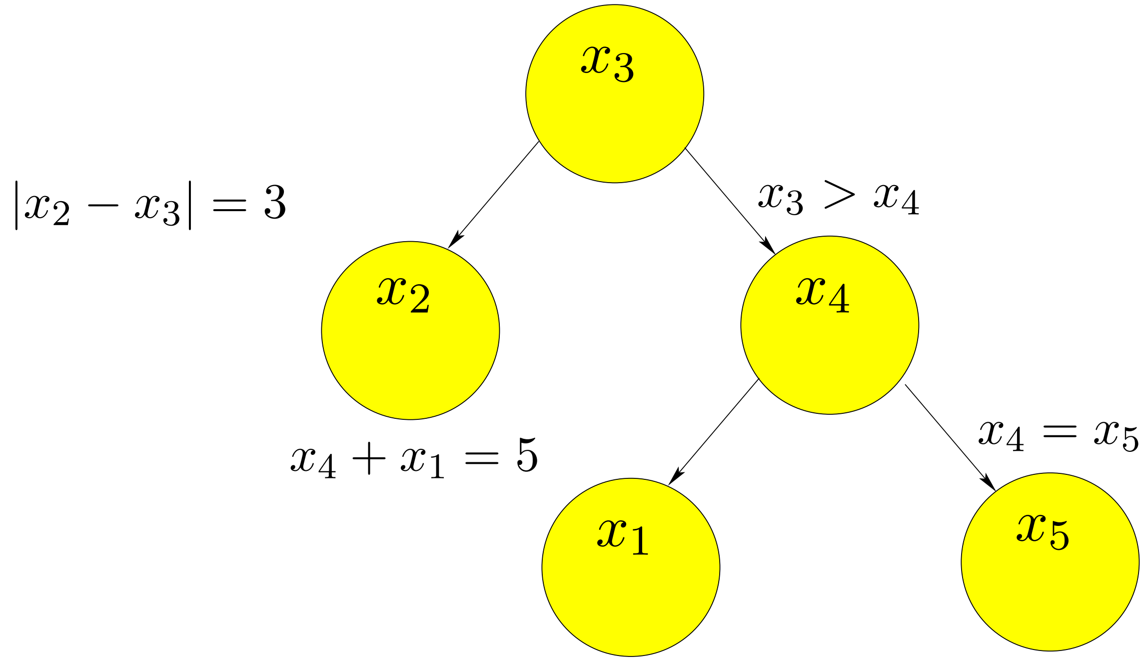
DAC and Acyclic CSP's

- A CSP is acyclic if its interaction graph has no loops
- **Theorem.** Acyclic CSP's can be solved in time $O(e \cdot m^2)$
- To prove this theorem we need some ingredients.
- Given an acyclic graph, i.e. a forest,
  let us choose a root for each tree and orient edges away from the roots
- Given a directed acyclic graph $G = (V, E)$,
  a topological ordering is a sequence of all vertices in V s.t.
  if $(u, v) \in E$ then u comes before v in the sequence

DAC and Acyclic CSP's

- Consider a CSP with 5 integer vars with domain $[1, 5]$, and constraints $|x_2 - x_3| = 3,\ x_3 > x_4,\ x_4 + x_1 = 5,\ x_4 = x_5$

  This CSP is acyclic, as its interaction graph is:



  - $(x_3, x_4, x_5, x_1, x_2)$ is a topological ordering
  - $(x_3, x_2, x_4, x_1, x_5)$ is a topological ordering
  - $(x_3, x_2, x_1, x_4, x_5)$ is not a topological ordering

DAC and Acyclic CSP's

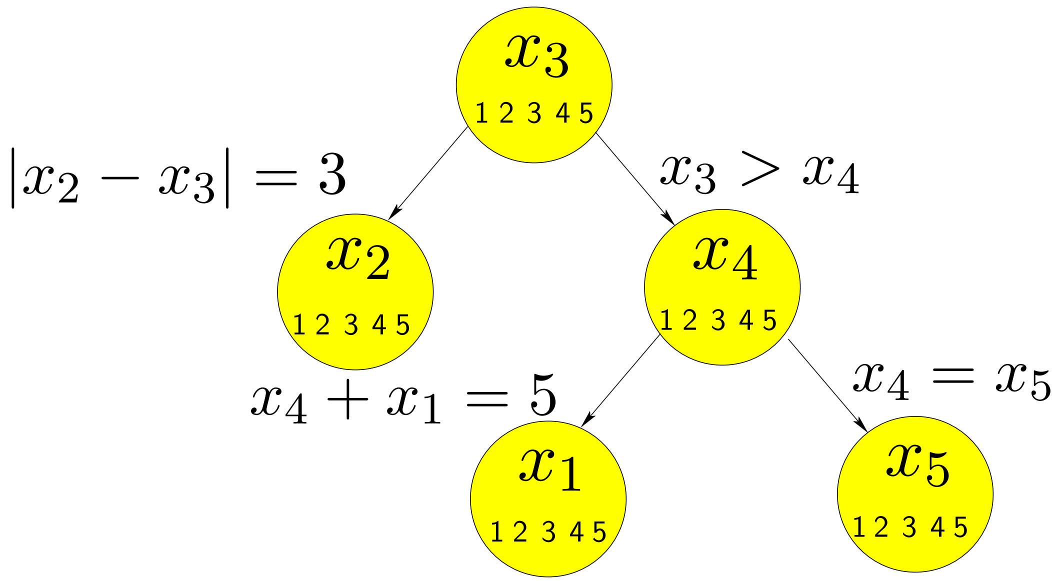
- **Theorem.** Acyclic CSP's can be solved in time $O(e \cdot m^2)$

  Proof. Consider the following algorithm:

// The graph of the CSP (X, D, C) is a tree rooted at x_1
// $(x_1, x_2, \ldots, x_n)$ is a topological ordering
procedure `AcyclicSolver`(X, D, C)
  $(X, D, C) := DAC(X, D, C)$ // enforce DAC wrt. the ordering
  $a :=$ any element from d_1
  $\mu := (x_1 \mapsto a)$
  for each $i := 2$ **to** n **do**
    // Any non-root node x_i of the tree has a parent
    $x_j :=$ parent(x_i) // x_j already assigned due to topological ordering
    $v :=$ any support of $\mu(x_j)$ from d_i // $\exists$ because DAC
    $\mu := \mu \circ (x_i \mapsto v)$

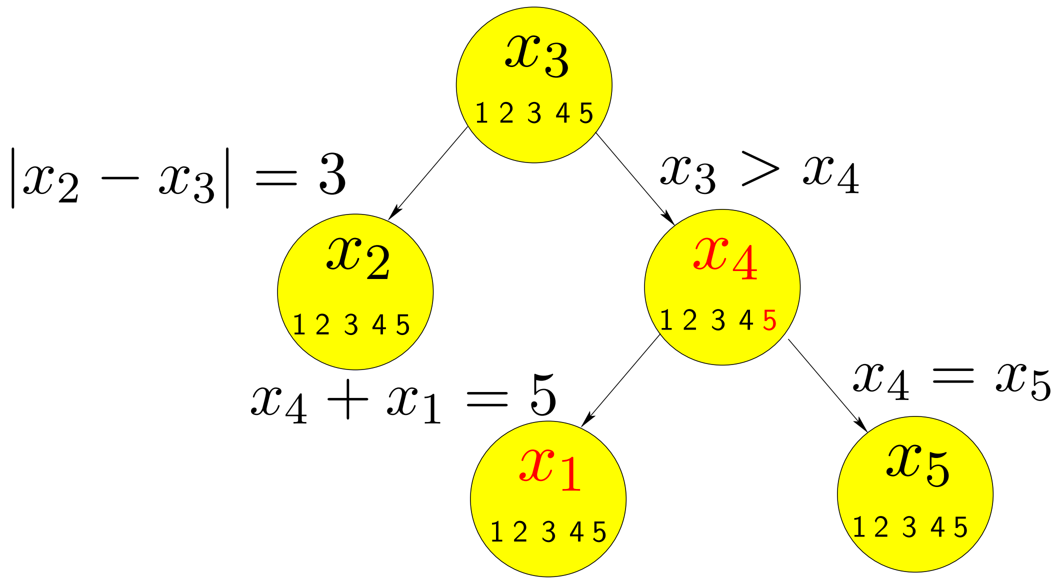
DAC and Acyclic CSP's

- Consider a CSP with 5 integer vars with domain $[1,5]$, and constraints $|x_2 - x_3| = 3,\ x_3 > x_4,\ x_4 + x_1 = 5,\ x_4 = x_5$
- Let us take the topological ordering $(x_3, x_4, x_5, x_1, x_2)$
- First we enforce DAC



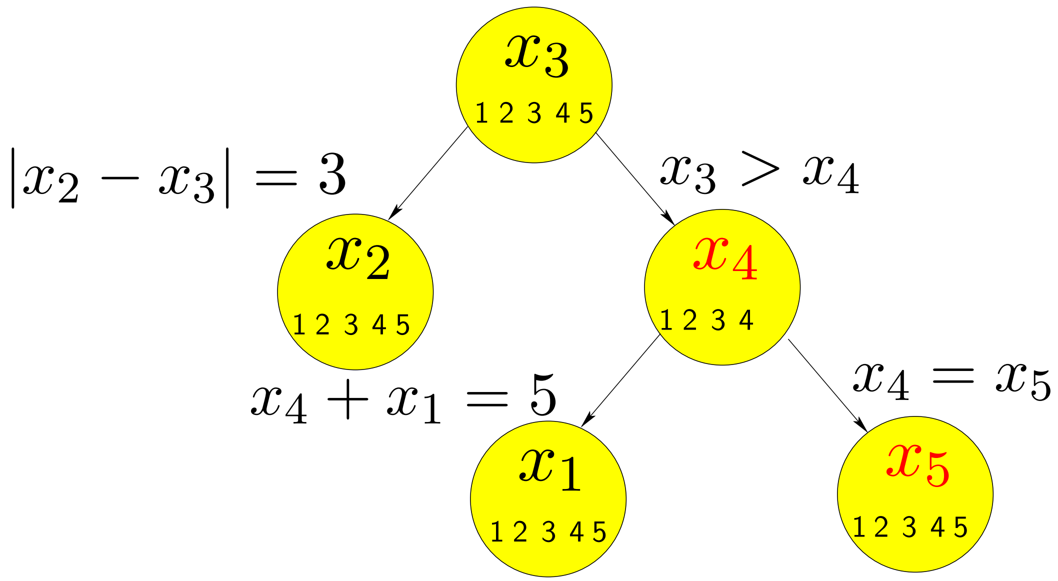
DAC and Acyclic CSP's

- Consider a CSP with 5 integer vars with domain $[1, 5]$, and constraints $|x_2 - x_3| = 3,\ x_3 > x_4,\ x_4 + x_1 = 5,\ x_4 = x_5$
- Let us take the topological ordering $(x_3, x_4, x_5, x_1, x_2)$
- First we enforce DAC



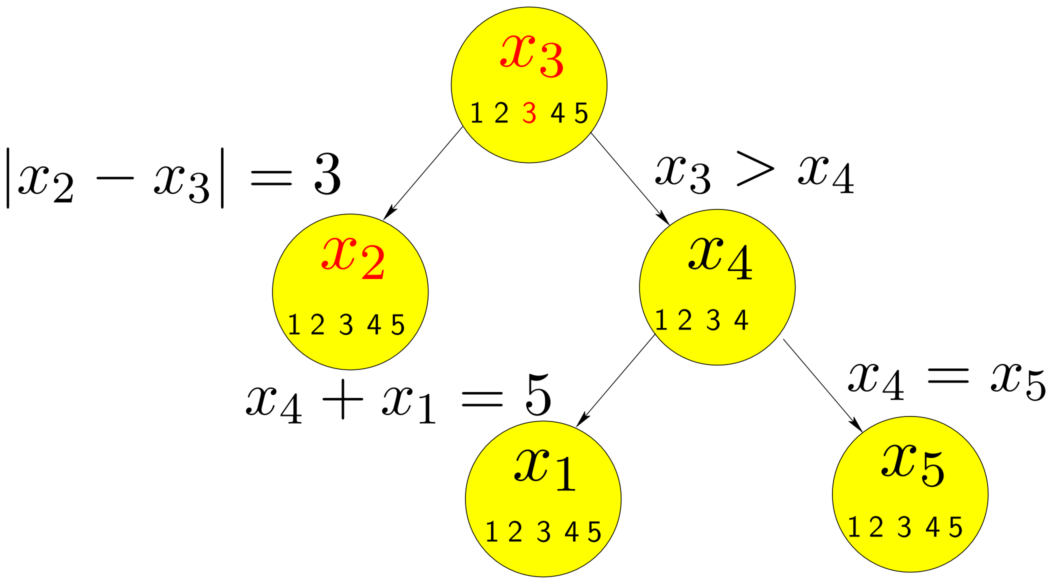
DAC and Acyclic CSP's

- Consider a CSP with 5 integer vars with domain $[1, 5]$, and constraints $|x_2 - x_3| = 3,\ x_3 > x_4,\ x_4 + x_1 = 5,\ x_4 = x_5$
- Let us take the topological ordering $(x_3, x_4, x_5, x_1, x_2)$
- First we enforce DAC



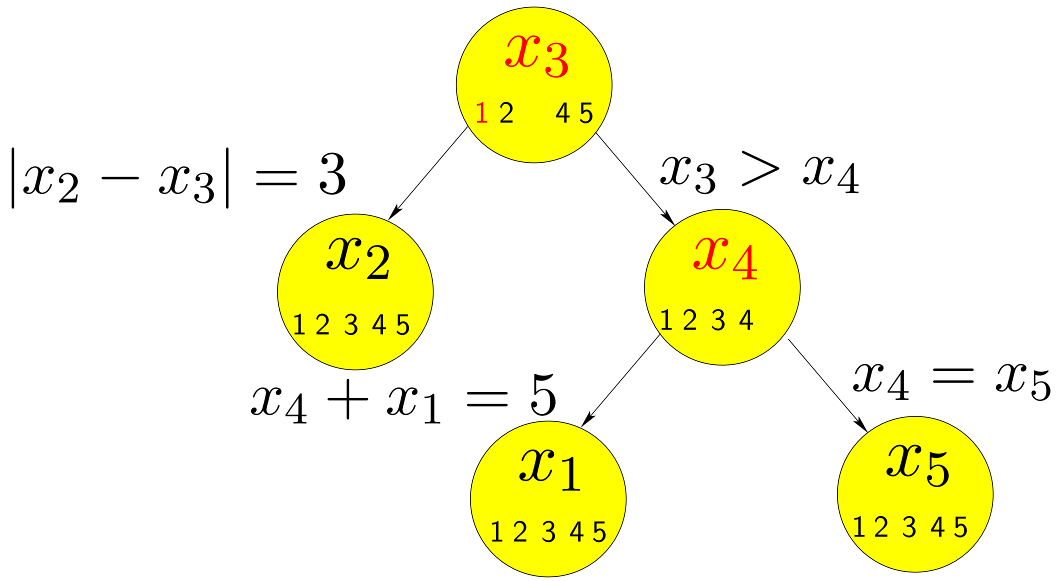
DAC and Acyclic CSP's

- Consider a CSP with 5 integer vars with domain $[1,5]$, and constraints $|x_2 - x_3| = 3,\ x_3 > x_4,\ x_4 + x_1 = 5,\ x_4 = x_5$
- Let us take the topological ordering $(x_3, x_4, x_5, x_1, x_2)$
- First we enforce DAC



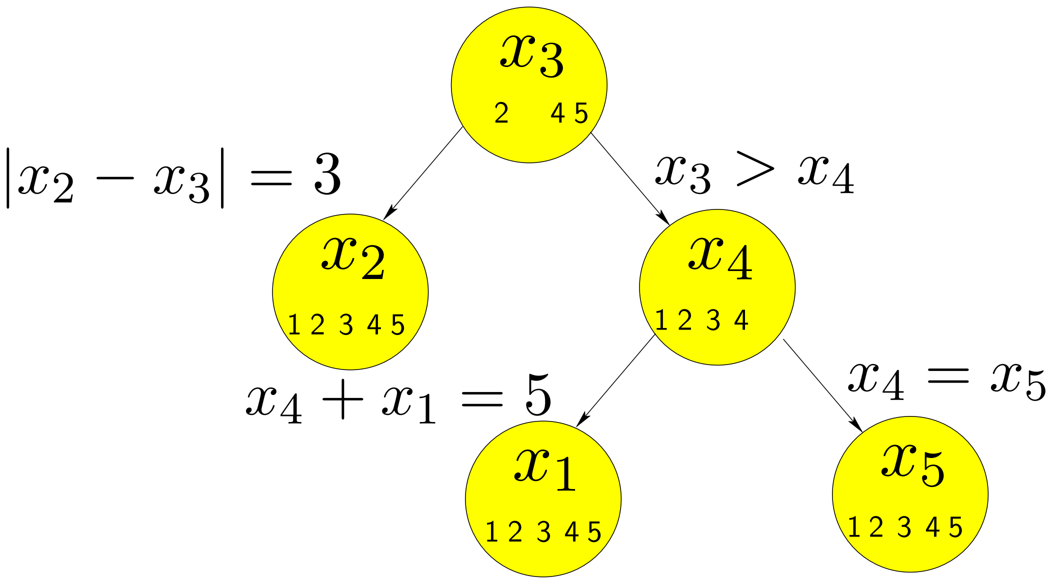
DAC and Acyclic CSP's

- Consider a CSP with 5 integer vars with domain $[1, 5]$, and constraints $|x_2 - x_3| = 3,\ x_3 > x_4,\ x_4 + x_1 = 5,\ x_4 = x_5$
- Let us take the topological ordering $(x_3, x_4, x_5, x_1, x_2)$
- First we enforce DAC



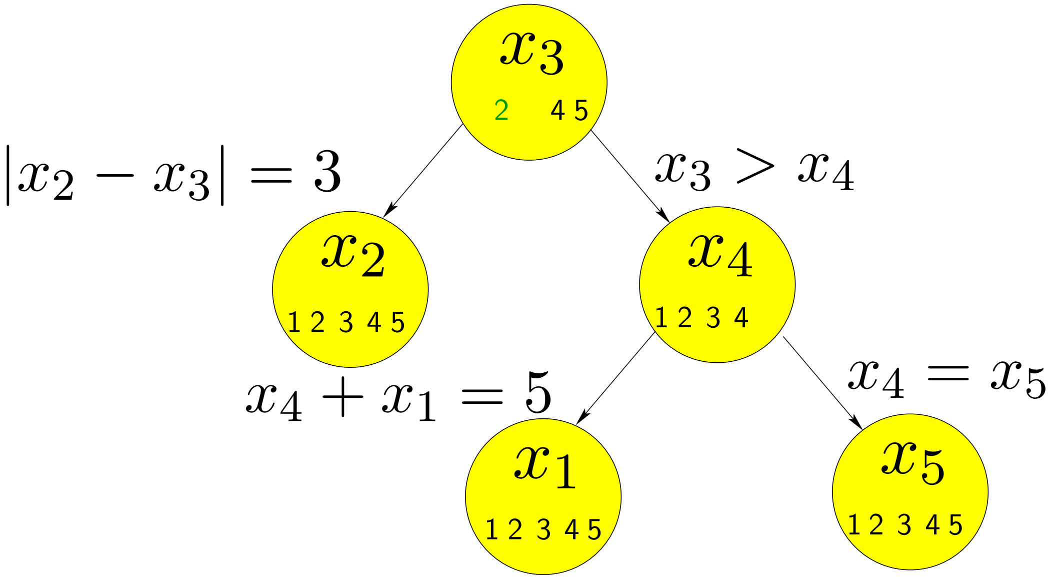
DAC and Acyclic CSP's

- Consider a CSP with 5 integer vars with domain $[1, 5]$, and constraints $|x_2 - x_3| = 3,\ x_3 > x_4,\ x_4 + x_1 = 5,\ x_4 = x_5$
- Let us take the topological ordering $(x_3, x_4, x_5, x_1, x_2)$
- First we enforce DAC



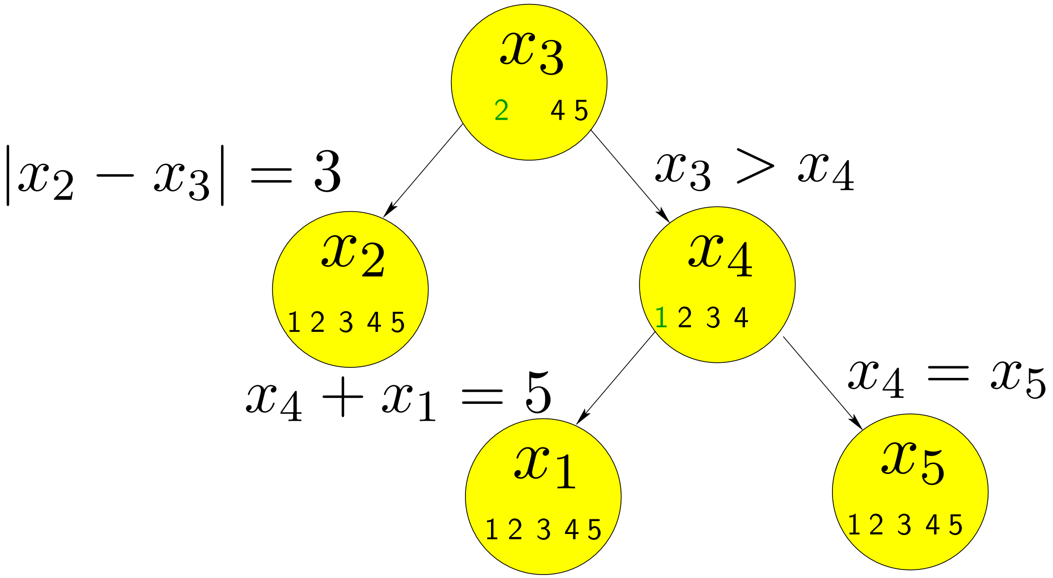
DAC and Acyclic CSP's

- Consider a CSP with 5 integer vars with domain $[1, 5]$, and constraints $|x_2 - x_3| = 3,\ x_3 > x_4,\ x_4 + x_1 = 5,\ x_4 = x_5$
- Let us take the topological ordering $(x_3, x_4, x_5, x_1, x_2)$
- Second we build the assignment



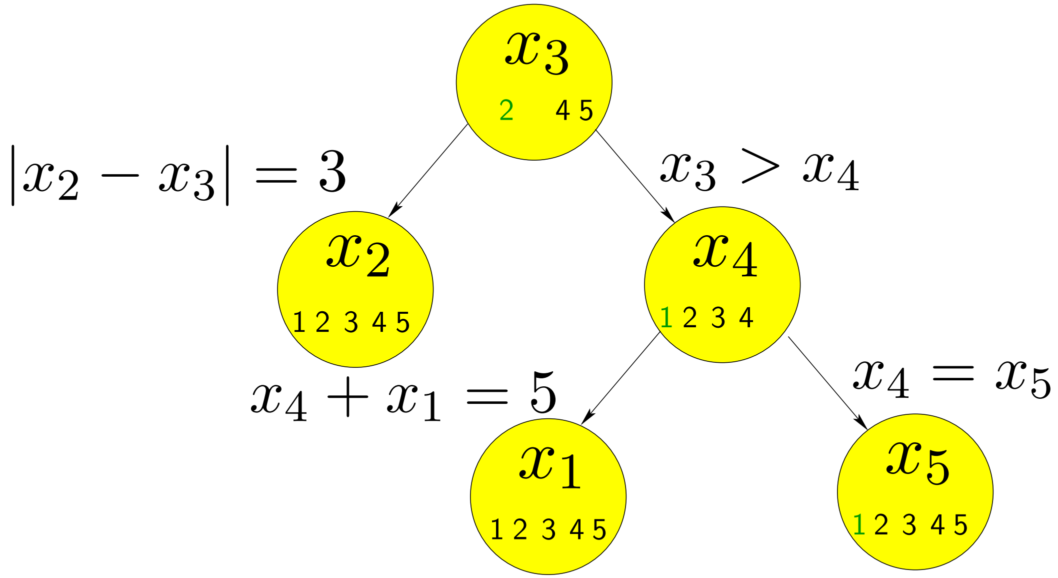
DAC and Acyclic CSP's

- Consider a CSP with 5 integer vars with domain $[1, 5]$, and constraints $|x_2 - x_3| = 3,\ x_3 > x_4,\ x_4 + x_1 = 5,\ x_4 = x_5$
- Let us take the topological ordering $(x_3, x_4, x_5, x_1, x_2)$
- Second we build the assignment



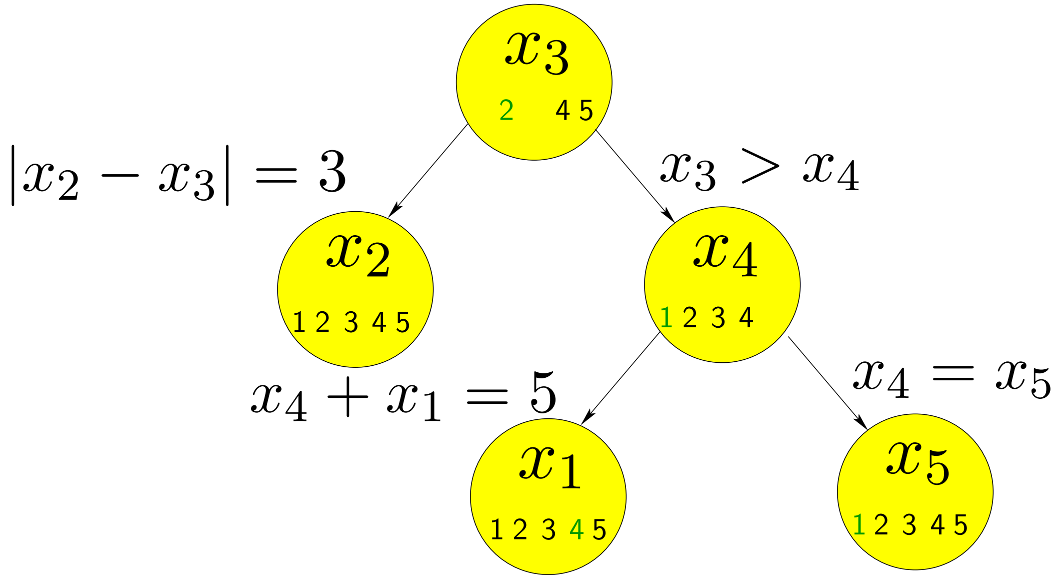
DAC and Acyclic CSP's

- Consider a CSP with 5 integer vars with domain $[1, 5]$, and constraints $|x_2 - x_3| = 3,\ x_3 > x_4,\ x_4 + x_1 = 5,\ x_4 = x_5$
- Let us take the topological ordering $(x_3, x_4, x_5, x_1, x_2)$
- Second we build the assignment



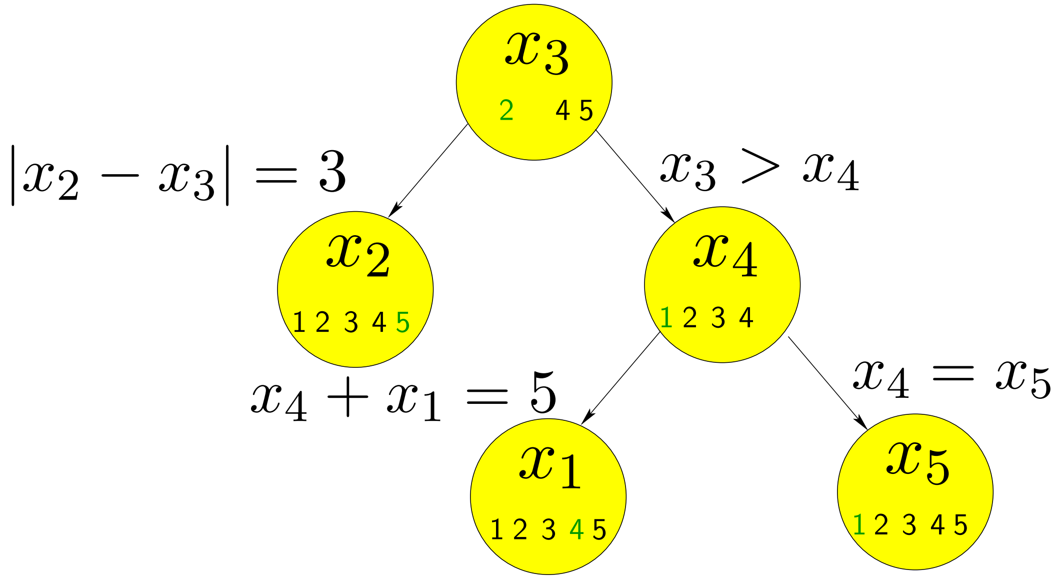
DAC and Acyclic CSP's

- Consider a CSP with 5 integer vars with domain $[1, 5]$, and constraints $|x_2 - x_3| = 3,\ x_3 > x_4,\ x_4 + x_1 = 5,\ x_4 = x_5$
- Let us take the topological ordering $(x_3, x_4, x_5, x_1, x_2)$
- Second we build the assignment



DAC and Acyclic CSP's

- Consider a CSP with 5 integer vars with domain $[1, 5]$, and constraints $|x_2 - x_3| = 3,\ x_3 > x_4,\ x_4 + x_1 = 5,\ x_4 = x_5$
- Let us take the topological ordering $(x_3, x_4, x_5, x_1, x_2)$
- Second we build the assignment



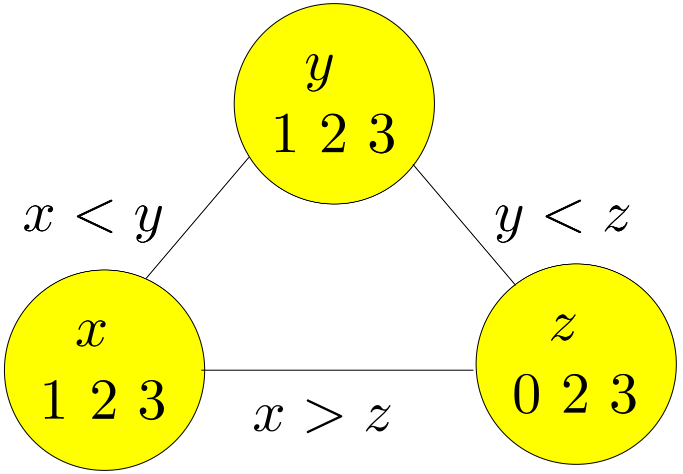
Bounds Consistency

- AC may be too costly when domains are very large
- Idea: Establish an order on domain values and require supports only for the extreme values (i.e., $\min\{d_i\}$ and $\max\{d_i\}$)
- Consider a CSP (X, D, C) with ordered domains
  - Variable $x_i \in X$ is bounds-consistent wrt. x_j iff $\min\{d_i\}$ and $\max\{d_i\}$ have a support in x_j
  - Constraint $c_{ij} \in C$ is bounds-consistent iff x_i is bounds-consistent wrt. x_j, and x_j wrt. x_i
  - The CSP is bounds-consistent iff all its constraints are bounds-consistent
- Notation: BC means bounds-consistent
- BC weaker than AC, but can be enforced more efficiently in practice

Bounds Consistency

- Let $x, y \in X$ be integer variables with domains $[1, 10]$.
  Constraint $|x - y| > 5$ is BC (but not AC nor DAC)

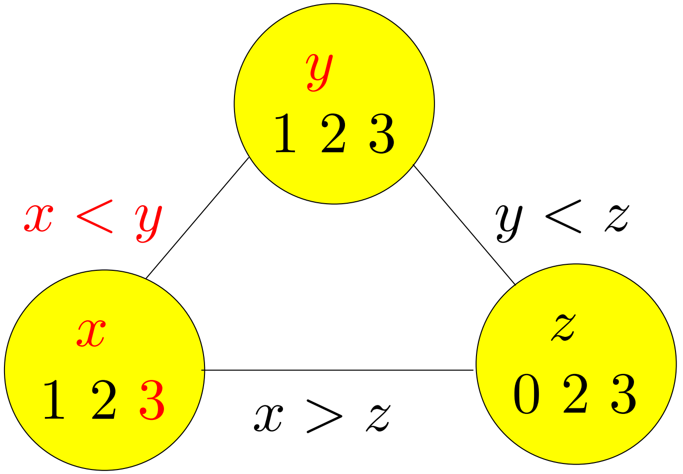
- Consider CSP with vars x, y, z, domains $d_x = d_y = \{1, 2, 3\}$ and $d_z = \{0, 2, 3\}$, and constraints $x < y,\ \ y < z,\ \ x > z$



Bounds Consistency

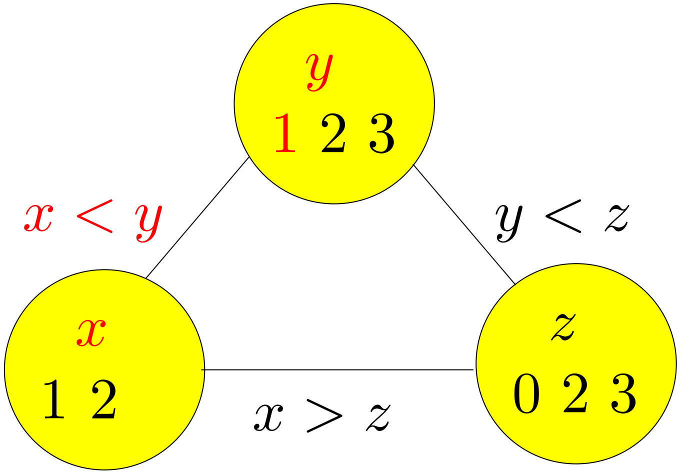
- Let $x, y \in X$ be integer variables with domains $[1, 10]$.
  Constraint $|x - y| > 5$ is BC (but not AC nor DAC)

- Consider CSP with vars x, y, z, domains $d_x = d_y = \{1, 2, 3\}$ and $d_z = \{0, 2, 3\}$, and constraints $x < y, \;\; y < z, \;\; x > z$



Bounds Consistency

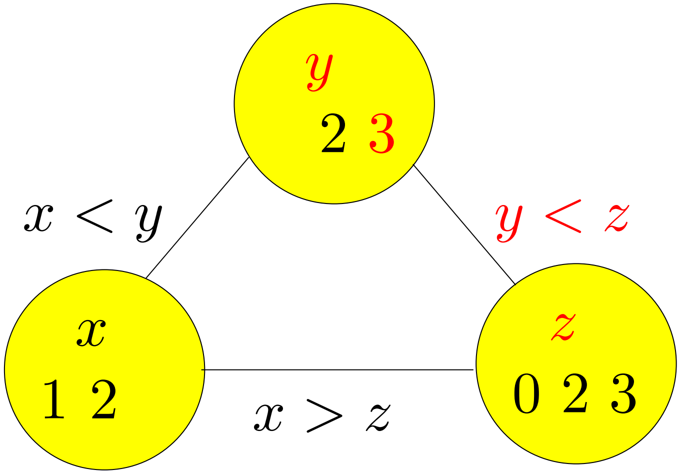
- Let $x, y \in X$ be integer variables with domains $[1, 10]$.
  Constraint $|x - y| > 5$ is BC (but not AC nor DAC)
- Consider CSP with vars x, y, z, domains $d_x = d_y = \{1, 2, 3\}$ and $d_z = \{0, 2, 3\}$, and constraints $x < y, \;\; y < z, \;\; x > z$



Bounds Consistency

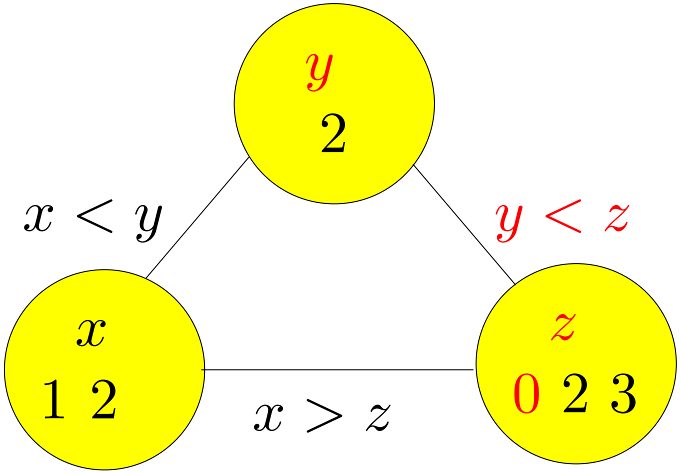
- Let $x, y \in X$ be integer variables with domains $[1, 10]$.
  Constraint $|x - y| > 5$ is BC (but not AC nor DAC)

- Consider CSP with vars x, y, z, domains $d_x = d_y = \{1, 2, 3\}$ and $d_z = \{0, 2, 3\}$, and constraints $x < y,\ \ y < z,\ \ x > z$



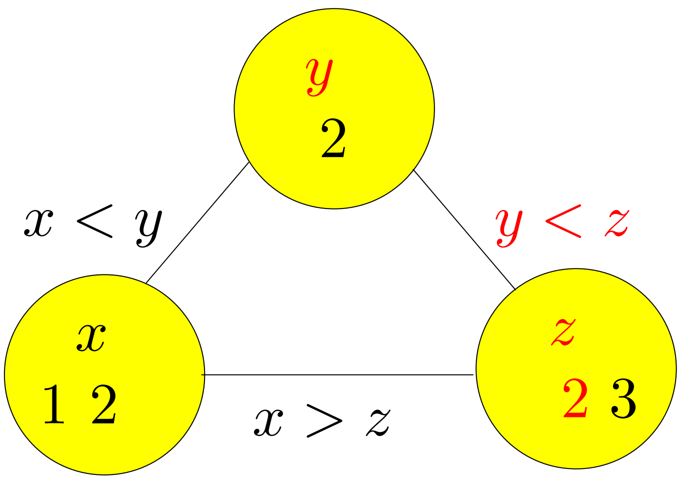
Bounds Consistency

- Let $x, y \in X$ be integer variables with domains $[1, 10]$.
  Constraint $|x - y| > 5$ is BC (but not AC nor DAC)
- Consider CSP with vars x, y, z, domains $d_x = d_y = \{1, 2, 3\}$ and $d_z = \{0, 2, 3\}$, and constraints $x < y, \;\; y < z, \;\; x > z$



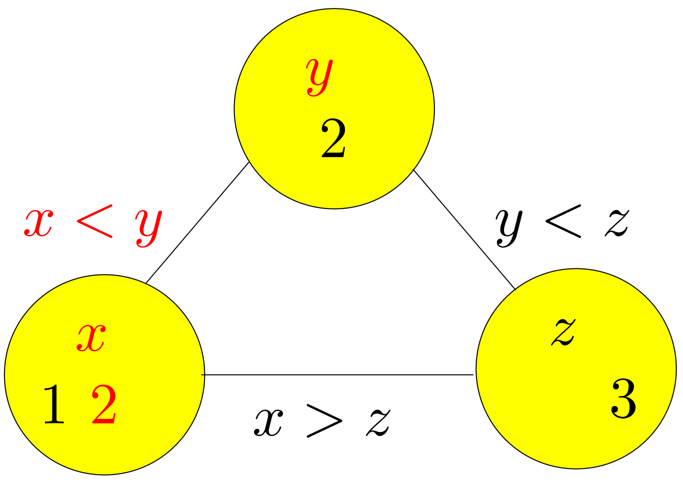
Bounds Consistency

- Let $x, y \in X$ be integer variables with domains $[1, 10]$.
  Constraint $|x - y| > 5$ is BC (but not AC nor DAC)
- Consider CSP with vars x, y, z, domains $d_x = d_y = \{1, 2, 3\}$ and $d_z = \{0, 2, 3\}$, and constraints $x < y,\ \ y < z,\ \ x > z$



Bounds Consistency

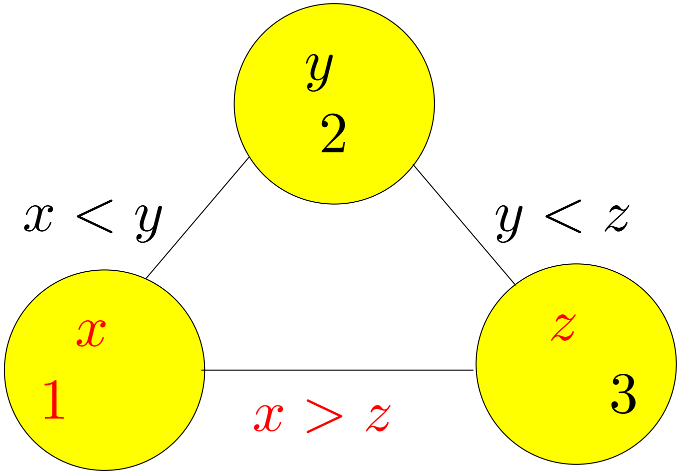
- Let $x, y \in X$ be integer variables with domains $[1, 10]$.
  Constraint $|x - y| > 5$ is BC (but not AC nor DAC)
- Consider CSP with vars x, y, z, domains $d_x = d_y = \{1, 2, 3\}$ and $d_z = \{0, 2, 3\}$, and constraints $x < y, \;\; y < z, \;\; x > z$



Bounds Consistency

- Let $x, y \in X$ be integer variables with domains $[1, 10]$.
  Constraint $|x - y| > 5$ is BC (but not AC nor DAC)

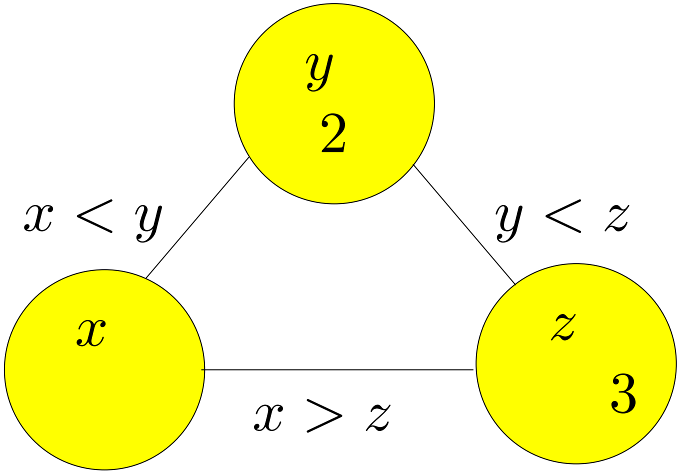
- Consider CSP with vars x, y, z, domains $d_x = d_y = \{1, 2, 3\}$ and $d_z = \{0, 2, 3\}$, and constraints $x < y, \;\; y < z, \;\; x > z$



Bounds Consistency

- Let $x, y \in X$ be integer variables with domains $[1, 10]$.
  Constraint $|x - y| > 5$ is BC (but not AC nor DAC)

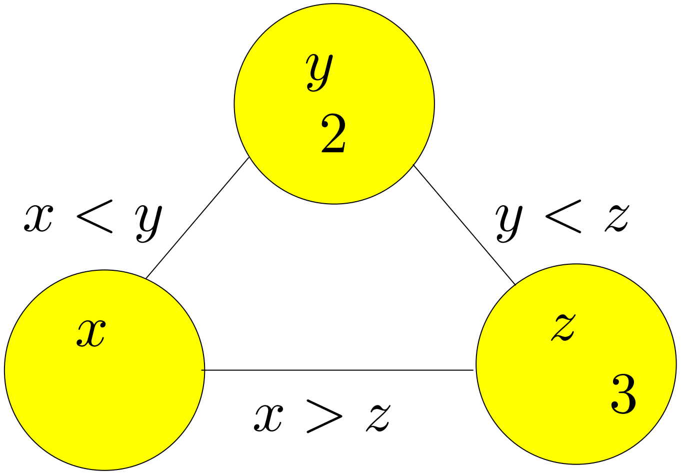
- Consider CSP with vars x, y, z, domains $d_x = d_y = \{1, 2, 3\}$ and $d_z = \{0, 2, 3\}$, and constraints $x < y, \;\; y < z, \;\; x > z$



Bounds Consistency

- Let $x, y \in X$ be integer variables with domains $[1, 10]$.
  Constraint $|x - y| > 5$ is BC (but not AC nor DAC)

- Consider CSP with vars x, y, z, domains $d_x = d_y = \{1, 2, 3\}$ and $d_z = \{0, 2, 3\}$, and constraints $x < y,\ \ y < z,\ \ x > z$



- The examples show that DAC and BC are incomparable
  (and both weaker than AC)

BC-3: ReviseBounds(i, j)

- Natural adaptation of AC-3 to bounds consistency: BC-3
- Based on function `ReviseBounds`(i, j),
  which removes values from the extremes of d_i without support in d_j

  Returns true if some value is removed

function `ReviseBounds`(i, j)
 change $:=$ false
 while $|d_i| \neq \emptyset \wedge (\forall_{b \in d_j} \neg c_{ij}(\min\{d_i\}, b))$ **do**
  change $:=$ true
  remove $\min\{d_i\}$ **from** d_i
 while $|d_i| \neq \emptyset \wedge \forall_{b \in d_j} \neg c_{ij}(\max\{d_i\}, b)$ **do**
  change $:=$ true
  remove $\max\{d_i\}$ **from** d_i
 return change

- The time complexity of `ReviseBounds`(i, j) is $O(|d_i| \cdot |d_j|)$

BC-3

$// \ (i,j) \in Q$ means
"cannot guarantee $\min\{d_i\}$, $\max\{d_i\}$ have support in d_j"

procedure $\texttt{BC3}(X, D, C)$
  $Q := \{(i,j),(j,i) | \ c_{ij} \in C\}$ // each constraint is added twice
  while $Q \neq \emptyset$ **do**
    $(i,j) := \texttt{Fetch}(Q)$
    if $\texttt{ReviseBounds}(i,j)$ **then**
      $Q := Q \cup \{(k,i) | \ c_{ki} \in C, k \neq j\}$

- Space complexity: $O(e)$
- Time complexity: $O(e \cdot m^3)$
- Same asymptotic costs of AC-3, but BC-3 is more efficient in practice

Stronger than AC

- Singleton AC (SAC)
- Neighborhood Inverse Consistency (NIC)

Singleton AC

- Let $AC(P)$ denote the CSP resulting from enforcing AC on P

- If $AC(P[x_i \to a])$ has an empty domain,
  then $P[x_i \to a]$ does not have any solution, and
  $a \in d_i$ is unfeasible in P

- A CSP $P = (X, D, C)$ is singleton arc-consistent (SAC) iff
  $\forall d_i \in D, \forall a \in d_i$, problem $AC(P[x_i \to a])$ has no empty domains

Singleton AC

- Let us enforce SAC in the 4-queens problem

Singleton AC

- Let us enforce SAC in the 4-queens problem

Q	X	X	X
X	X		
X		X	
X			X

Singleton AC

- Let us enforce SAC in the 4-queens problem

Q	X	X	X
X	X		
X	Q	X	
X	X	X	X

Singleton AC

- Let us enforce SAC in the 4-queens problem

Q	X	X	X
X	X	X	X
X		X	Q
X			X

Singleton AC

- Let us enforce SAC in the 4-queens problem

X	X	X	
X	Q	X	X
X	X	X	
	X		X

Singleton AC

- Let us enforce SAC in the 4-queens problem

X	X	X	Q
X	Q	X	X
X	X	X	X
	X		X

Singleton AC

- Let us enforce SAC in the 4-queens problem

X			X
	X	X	
	X	X	
X			X

by symmetry

Enforcing SAC

procedure $\texttt{SAC}(P)$
    $P := AC(P)$
    repeat
        change $:= \mathrm{false}$
        for each $d_i \in D$, $a \in d_i$ **do**
            if $AC(P[x_i \rightarrow a])$ has an empty domain **then**
                change $:= \mathrm{true}$
                remove a **from** d_i
        if change **then**
            $P := AC(P)$
    until $\neg$ change

- Complexity: $O(e \cdot n^2 \cdot m^4)$

Neighborhood Inverse Consistency

- Let $P = (X, D, C)$ be a CSP.
- The neighborhood of $x_i \in X$, noted N_i, is
  the set of vars containing x_i and all x_j such that $c_{ij} \in C$
- The projection of P on N_i, noted $P[N_i]$, is
  the problem obtained from P by
  taking all variables in N_i and all constraints c such that $\mathrm{scope}(c) \subseteq N_i$
- If $a \in d_i$ is unfeasible in $P[N_i]$, then so is in P
- A CSP $P = (X, D, C)$ is neighborhood inverse consistent (NIC) iff for
  every $x_i \in X$, $a \in d_i$, a is feasible in $P[N_i]$

Enforcing NIC

procedure $\texttt{NIC}(P)$
  $P := AC(P)$
  repeat
    change $:= \text{false}$
    for each $d_i \in D$, $a \in d_i$ **do**
      if $SOL(P[N_i][x_i \to a]) = \emptyset$ **then**
        change $:= \text{true}$
        remove a **from** d_i
    endfunction
  until $\neg$ change

- Complexity: $O(g^2 \cdot m^g \cdot n^2 \cdot m^2)$,
  where g is the degree of the interaction graph
  (i.e, the max number of neighbors that some vertex has)