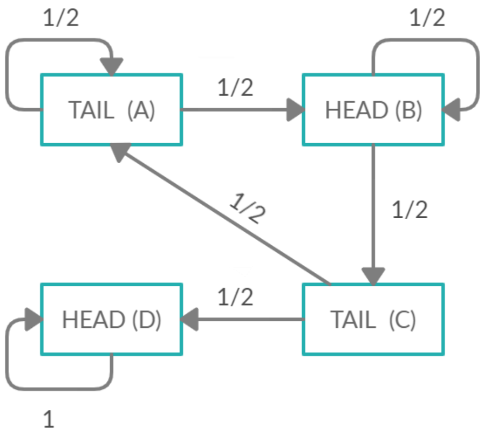
A fair coin is tossed repeatedly and independently. Use a Markov chain to find the expected number of tosses until the pattern HTH appears.



$$P = \begin{pmatrix} 1/2 & 1/2 & 0 & 0 \\ 0 & 1/2 & 1/2 & 0 \\ 1/2 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Fist method:

We have that:

$\pi_0 = (0.5 \quad 0.5 \quad 0 \quad 0)$

$\pi_1 = \pi_0 P = (0.25 \quad 0.5 \quad 0.25 \quad 0)$

$\pi_2 = \pi_1 P = (0.25 \quad 0.375 \quad 0 \quad 0.375)$

…

$\pi_8 = \pi_0 P^8 = (0.13 \quad 0.17 \quad 0.1 \quad 0.6)$

$\pi_9 = \pi_0 P^9 = (0.12 \quad 0.14 \quad 0.08 \quad 0.66)$

$\pi_{10} = \pi_0 P^{10} = (0.1 \quad 0.13 \quad 0.07 \quad 0.7)$

At this point the probability of be in the state (A), (B) or (C) continue decreasing and our expected value is (D), so we can assume that the expected number of tosses until the pattern appears is 10.

Second method:

We can see that this model has an absorbing state because we stop in the moment that the pattern HTH appears. Let us define the fundamental matrix of P:

$$N = (I - Q)^{-1} = \begin{pmatrix} 0.5 & -0.5 & 0 \\ 0 & 0.5 & -0.5 \\ -0.5 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 4 & 4 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 2 \end{pmatrix}$$

$$t = \begin{pmatrix} 4 & 4 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 10 \\ 8 \\ 6 \end{pmatrix}$$

According to this we have that the expected number of tosses if we start in (A) is 10 and if we start in (B) is 8, so we can assume that the expected number of tosses is 10.