17. Suppose that we flip a fair coin, ***n*** times to obtain **n** random bits. Consider all $m = \binom{n}{2}$ pairs of these bits in some order. Let Y_i be the **exclusive-or** $\oplus$ of the *i-th* pair of bits, and let $Y = \sum_{i=1}^{m} Y_i$ be the number of Y_i that equal 1.

(a) Show that each $Y_i = 0$ with prob = $\frac{1}{2}$ (therefore, $Y_i = 1$ with probability also $\frac{1}{2}$)

(b) Show that Y_i are not mutually independent.

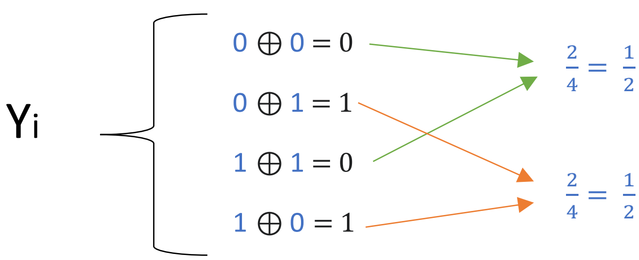
(c) Show that $E[Y_iY_j] = E[Y_i]E[Y_j]$.

(d) Find Var[Y].

(e) Use Chebyshev to bound $\Pr[|Y - E[Y]| \geq n]$.

(a)

Symmetric difference



$$P(Y_i = 1) = \frac{2 \times 2^{n-2}}{2^n} = \frac{1}{2}$$

$$P(Y_i = 0) = 1 - P(Y_i = 1) = \frac{1}{2}$$

(b)

To investigate their independence, let's consider $P(Y_i = 1; Y_j = 1; Y_k = 1)$.

$Y_i = 1$	=>	The first and the second variable
$Y_j = 1$	=>	The second and the third variable
$Y_k = 1$	=>	The first and the third variable

If they are mutually independent, we should have:

$P(Y_i = 1) = \frac{2 \times 2^{n-2}}{2^n} = \frac{1}{2}$ The same for Y_j and Y_k

$P(Y_i = 1; Y_j = 1; Y_k = 1) = P(Y_i = 1) \times P(Y_j = 1) \times P(Y_k = 1) = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \mathbf{\frac{1}{8}}$

Nevertheless, these events cannot be happened at the same time. Therefore $P(Y_i = 1; Y_j = 1; Y_k = 1)$ = 0

$P(Y_i = 1; Y_j = 1; Y_k = 1) = 0$ => $0 \neq \mathbf{\frac{1}{8}}$

Let's assume:

000
001
011
010
110
111
101
100

(c)

First, we will show, that Y_i are pairwise independent. Consider 2 possibilities.

1.) Y_i and Y_j do not share a bit position. They are obviously independent.

2.) Y_i and Y_j share a bit position. Consider all 3-bit strings to confirm the independence.

Y_i	=>	The Sth and the Rth variable
Y_j	=>	The Kth and the Rth variable

...	R	...	K	...	S	...
...	0	...	0	...	0	...

Or

...	R	...	K	...	S	...
...	1	...	1	...	1	...

$$P(Y_i = 0, Y_j = 0) = \frac{2 \times 2^{n-3}}{2^n} = \frac{1}{4}$$

$$P(Y_i = 0) = \frac{2 \times 2^{n-2}}{2^n} = \frac{1}{2}$$

$$P(Y_i = 0; Y_j = 0) = P(Y_i = 0) \times P(Y_j = 0) = \frac{1}{2} \times \frac{1}{2} = \mathbf{\frac{1}{4}}$$

(d)

When our variables are independent:

$$Var(Y) = \sum_{i=1}^{m} Var(Y_i)$$

$$\sum_{i=1}^{m} E(Y_i^2) - E(Y_i)^2 = \sum_{i=1}^{m} \left(0^2 \frac{1}{2} + 1^2 \frac{1}{2}\right) - \left(0 \frac{1}{2} + 1 \frac{1}{2}\right)^2 = \sum_{i=1}^{m} \frac{1}{4} = m \times \frac{1}{4} = \frac{m}{4}$$

(e)

From the previous part we have $\mathrm{Var(Y)} = \frac{\mathrm{m}}{4}$

$$P[|Y - E(Y)| \geq n] \leq \frac{Var(Y)}{n^2} = \frac{1}{8}\frac{n(n-1)}{n^2} \leq \frac{1}{8}$$