Exercise 25

Problem. Consider the knight's tour on a chess board: A knight selects one of the next positions at random independently of the past.

(a) Why is this process a Markov chain?

(b) What is the state space?

(c) Is it irreductible? Is it aperiodic?

(d) Find the stationary distribution. Give an interpretation of it: what does it mean, physically?

(e) Which are the most likely states in steady-state? Which are the least likely ones?

A solution.

(a) "knight selects one of the next positions at random independently of the past" means that the next position X_{n+1} is a function of the current position X_n and a random choice ξ_n of a neighbour.
We can define this function as $X_{n+1} = f(X_n, \xi_n)$.
Lets define the set P as the path from X_0 to X_{n-1}: $P = (X_0, X_1, \ldots, X_{n-1})$ and N as the next positions $N = (X_n, X_{n+1}, X_{n+2}, \ldots)$. Therefore, the values of P are $X_0, \xi_1, \xi_2, \ldots, \xi_{n-2}$ and the values of S are $X_n, \xi_{n+1}, \xi_{n+2}, \ldots$.
Since $X_0, \xi_1, \xi_2, \ldots, \xi_{n-2}$ are independent of $X_n, \xi_{n+1}, \xi_{n+2}, \ldots$, P is independent of S, hence, X_n is a Markov Chain.

(b) The state space is the set of squares of the chess board. There are 64 squares (8 rows $\times$ 8 columns). We can define the state space as the set of pairs of integers of $\{1, \ldots, 8\} \times \{1, \ldots, 8\}$:

$$S = \{(x, y) \mid 1 \leq x, y \leq 8\}$$

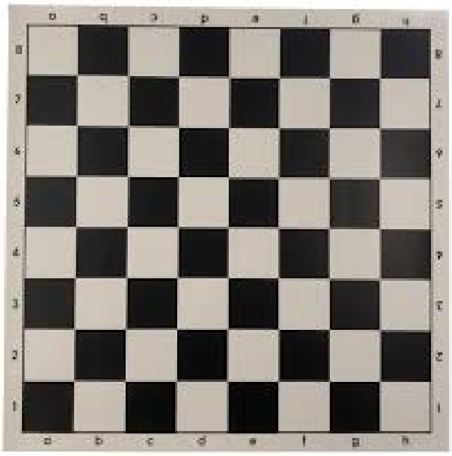
(c) Because of the L movement of the knight, given two positions X_i and X_j, there is always a path that takes the knight from position X_i to position X_j in a k number of steps, therefore, it's irreductible.

Given a position X_i, the knight only can return to the same position X_i in an even number of steps. Hence, the period is 2 so, it's not aperiodic.

(d) We have to find the stationary state S that satisfies

$$\pi = \pi P$$

Since we have 64 equations, we will try to guess S taking into accound the movement of the knight.



2	3	4	4	4	4	3	2
3	4	6	6	6	6	4	3
4	6	8	8	8	8	6	4
4	6	8	8	8	8	6	4
4	6	8	8	8	8	6	4
4	6	8	8	8	8	6	4
3	4	6	6	6	6	4	3
2	3	4	4	4	4	3	2

In the grid above we can see how many next positions have every position. Let us define the concept of neighbour: A position $y \in S$ is neighbour of $x \in S$ if the knight can move from x to y in one step, and vice versa. With that, let us guess if $x = (i, j)$, then $\pi(x)$ is proportional to the number $N(x)$ of neighbours of x:

$$\pi(x) = kN(x)$$

So,

$$\pi(x) = \sum_{y \in S} \pi(y) P_{y,x}$$

By taking away the constant k of each side

$$N(x) = \sum_{y \in S} N(y) P_{y,x}$$

But the sum of the right is zero unless x is a neighbour of y

$$N(x) = \sum_{y \in S:\ x \text{ neighbour of } y} N(y) P_{y,x}$$

By the statement, the knight chooses the next position randomly, therefore

$$P_{y,x} = \begin{cases} 1/N(y), & \text{if } x \text{ is neighbour of } y \\ 0, & \text{otherwise} \end{cases}$$

Which means

$$\begin{aligned}
N(x) &= \sum_{y \in S:\ x \text{ neighbour of } y} N(y)\frac{1}{N(y)} \\
&= \sum_{y \in S:\ x \text{ neighbour of } y} 1 \\
&= \sum_{y \in S:\ y \text{ neighbour of } x} 1 = N(x)
\end{aligned}$$

That means that our guess was correct.

Now, we have to count the neighbours of each square x which is 336 (see the grid above). We have that $k = 1/336$, and

$$\pi(1,1) = 2/336,\ \pi(1,2) = 3/336,\ \pi(1,3) = 4/336, \ldots, \pi(4,4) = 8/336, \ldots, \pi(8,8) = 2/336$$

That means,

$$P(X_0 = x) = \pi(x),\ \ x \in S$$

and, for $n \geq 1$

$$P(X_n = x) = \pi(x),\ \ x \in S$$

(e) The most likely are the 16 states of the square in the middle,

$$\{(i,j) \mid 3 \leq i,j \leq 6\}$$

The least likely are the four corners.