Randomized Algorithms

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Exercise 16

Let the random variable X be representable as a sum of random variables $X = \sum_{i=1}^{n} X_i$. Show that, if $\mathbf{E}[X_i X_j] = \mathbf{E}[X_i]\mathbf{E}[X_j]$ for every pair of i and j with $1 \le i < j \le n$, then $\mathbf{Var}[X] = \sum_{i=1}^{n} \mathbf{Var}[X_i]$.

$$
\begin{aligned}
\mathbf{Var}[X] &= \mathbf{E}[X^2] - \mathbf{E}^2[X] \\
&= \mathbf{E}\left[\left(\sum_{i=1}^{n} X_i\right)^2\right] - \left(\mathbf{E}\left[\sum_{i=1}^{n} X_i\right]\right)^2 \\
&= \mathbf{E}\left[\sum_{i=1}^{n}\sum_{j=1}^{n} X_i X_j\right] - \mathbf{E}\left[\sum_{i=1}^{n} X_i\right]\mathbf{E}\left[\sum_{i=1}^{n} X_i\right] \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} \mathbf{E}\left[X_i X_j\right] - \sum_{i=1}^{n}\sum_{j=1}^{n} \mathbf{E}\left[X_i\right]\mathbf{E}\left[X_j\right] \\
&= \sum_{i=1}^{n} \mathbf{E}\left[X_i^2\right] - \sum_{i=1}^{n} \mathbf{E}^2\left[X_i\right] \\
&= \sum_{i=1}^{n} \left(\mathbf{E}[X_i^2] - \mathbf{E}^2[X_i]\right) \\
&= \sum_{i=1}^{n} \mathbf{Var}\left[X_i\right]
\end{aligned}
$$