13. Let X and Y be independent geometric random variables, where X has parameter p and Y has parameter q.

- What is the probability that $X = Y$?

$$\mathbf{Pr[X = Y]} = \sum_x (1-p)^{x-1} p (1-q)^{x-1} q = \sum_x [(1-p)(1-q)]^{x-1} pq$$

We have the property for random variables were:

$$\mathbf{Pr[X} \geq i] = \sum_{x=i}^{\infty} (1-p)^{x-1} p = (1-p)^{i-1}$$

We obtain that:

$$\mathbf{Pr[X = Y]} = \frac{pq}{p+q-pq}$$

- What is $\mathbf{E}[\ \max(X,\ Y)\ ]$?

In the previous class we saw that :

$$\mathbf{E}[max(X,Y)] = \mathbf{E}[X] + \mathbf{E}[Y] - \mathbf{E}[min(X,Y)]$$

Taking the result of the 13.c, we know that *min(X,Y)* is a geometric variable with mean (p+q-pq), therefore its expectation is $\frac{1}{p+q-pq}$

$$\mathbf{E}[max(X,Y)] = 1/p + 1/q - \frac{1}{p+q-pq}$$

- What is $\mathbf{Pr}\ [\min(X,\ Y) = k]$?

We have that this expression can be split in two disjoint events:

$$\mathbf{Pr}[min(X,Y) = k\,] = \mathbf{Pr}[X = k,\, Y \geq k] + \mathbf{Pr}[X > k,\, Y = k] =$$

$$= \mathbf{Pr}[X = k]\mathbf{Pr}[Y \geq k] + \mathbf{Pr}[X > k]\mathbf{Pr}[Y = k]$$

Recalling from exercise 13.a:

$$\mathbf{Pr}[X > k] = \mathbf{Pr}[X \geq k] - \mathbf{Pr[X = k]} = (1-p)^{k-1}(1-p)$$

Taking this into account we have that:

$$\mathbf{Pr}[min(X,Y) = k\,] = (1-p)^{k-1} p (1-q)^{k-1} + (1-p)^{k-1}(1-p)(1-q)^{k-1} q$$

$$= [(1-p)(1-q)]^{k-1}(p + (1-p)q) = [(1-p)(1-q)]^{k-1}(p+q-pq)$$

- What is $\mathbf{E}[X \mid X \leq Y\ ]$?

$$\mathbf{E}[X|X \leq Y] = \sum_x x Pr[X = x | x \leq Y]$$

$$= \sum_x x \frac{Pr[X = x \cap x \leq Y]}{Pr[X \leq Y]}$$

The bottom part is solved as follows:

$$Pr[X \leq Y] = \sum_z Pr[X = z \cap z \leq Y] = \sum_z Pr[X = z] Pr[z \leq Y] = \sum_z (1-p)^{z-1} p (1-q)^{z-1}$$

$$p \sum_z [(1-p)(1-q)]^{z-1} = \frac{p}{p+q-pq}$$

The top part is solved as follows:

$$\mathbf{E}[X|X \leq Y] = \frac{p+q-pq}{p} \sum_x x Pr[X = x] Pr[x \leq Y]$$

$$\frac{p+q-pq}{p} \sum_x x(1-p)^{x-1} p(1-q)^{x-1} = (p+q-pq) \sum_x x(1-p-q-pq)^{x-1}$$

This is equal to the expectation of a geometric rando variable with mean (p+q-pq), so:

$$\mathbf{E}[X|X \leq Y] = \frac{1}{p+q-pq}$$