Radomized Algorithms Exercises 3 Fall 2020.

20. Discuss the irreducibility and the periodicity of the following Markov chains:

$$(a)\ P = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix};\ (b)\ P = \begin{pmatrix} 1/2 & 1/2 \\ 1 & 0 \end{pmatrix};\ (c)\ P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix};$$

$$(d)\ P = \begin{pmatrix} 1/3 & 0 & 2/3 \\ 0 & 1 & 0 \\ 0 & 1/5 & 0 \end{pmatrix};\ (e)\ P = \begin{pmatrix} 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \\ 1/3 & 1/3 & 1/3 \end{pmatrix};$$

21. A Markov chain with state space $\{1, 2, 3\}$ has transition probability matrix

$$P = \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 0 & 1/2 & 1/2 \\ 0 & 0 & 1 \end{pmatrix}$$

Show that state 3 is absorbing and, starting from state 1, find the expected time until absorption occurs.

22. A fair coin is tossed repeatedly and independently. Use a Markov chain to find the expected number of tosses until the pattern HTH appears.

23. Consider a Markov chain with state space $\{0, 1, 2, 3\}$ and transition matrix:

$$P = \begin{bmatrix} 0 & 3/10 & 1/10 & 3/5 \\ 1/10 & 1/10 & 7/10 & 1/10 \\ 1/10 & 7/10 & 1/10 & 1/10 \\ 9/10 & 1/10 & 0 & 0 \end{bmatrix}$$

(a) Find the stationary distribution of the Markov Chain.

(b) Find the probability of being in state 3 after 32 steps if the chain begins at state 0.

(c) Find the probability of being in state 3 after 128 steps if the chain begins at a state chosen uniformly at random from the four states.

(d) Suppose that the chain begins in state 0. What is the smallest value of t for which $\max_i |P^t_{0,i} - \pi^*_i| \leq 0.01$? , where π^*_i is the stationary distribution.

24. I have 4 umbrellas, some at home, some in the office. I keep moving between home and office. I take an umbrella with me only if it rains. If it does not rain I leave the umbrella behind (at home or in the office). It may happen that all umbrellas are in one place, I am at the other, it starts raining and must leave, so I get wet.

(a) If the probability of rain is p, what is the probability that I get wet?

(b) If the current forecast shows a $p = 0.6$, how many umbrellas should I have so that, if I follow the strategy above, the probability I get wet is less than 0.1?

(*Hint: Use a MC*)

25. Consider the knight's tour on a chess board: A knight selects one of the next positions at random independently of the past.

    (a) Why is this process a Markov chain?

    (b) What is the state space?

    (c) Is it irreducible? Is it aperiodic?

    (d) Find the stationary distribution. Give an interpretation of it: what does it mean, physically?

    (e) Which are the most likely states in steady-state? Which are the least likely ones?

26. Consider a Markov chain with states $S = \{0, ..., N\}$ and transition probabilities $p_{i,i+1} = p$ and $p_{i,i-1} = q$, for $1 \leq i \leq N-1$, where $p + q = 1$, $0 < p < 1$; assume $p_{0,1} = 1$, and $p_{N,N-1} = 1$.

    (a) Draw the MC associated graph.

    (b) Is the Markov chain irreducible?

    (c) Is it aperiodic?

    (d) What is the period of the chain?

    (e) Find the stationary distribution.