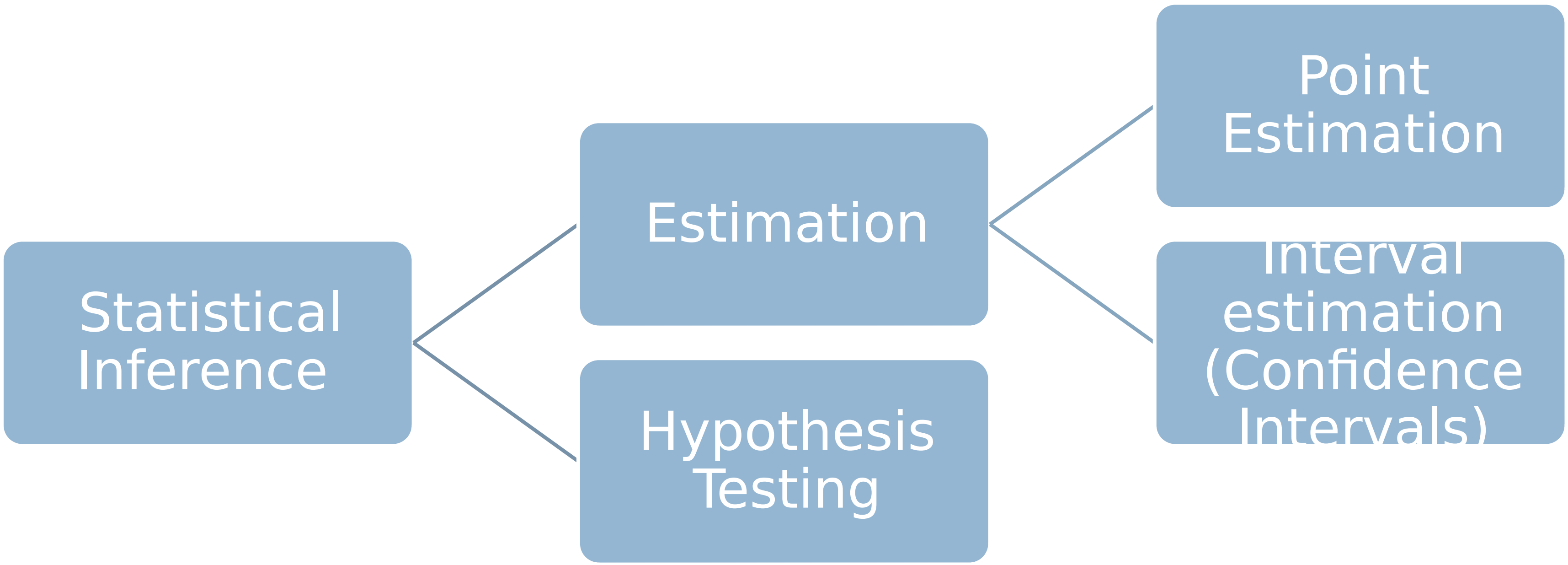
# HYPOTHESIS TESTING

## STATISTICAL INFERENCE

Pau Fonseca, Phd

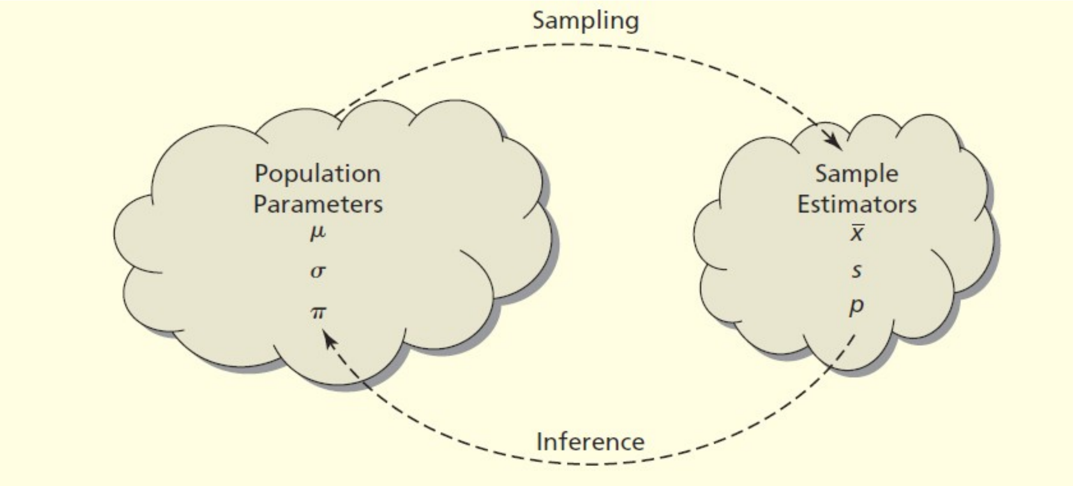
Nihan Acar-Denizli, Phd

# Statistical Inference



# Statistical Inference

- An estimator is statistic derived from a sample to infer the  value of a population parameter.
- An estimate is the value of the estimatr in a particular sample.



# Sampling Error

- The sampling error is the difference between an estimate and the corresponding population parameter and it is defined by,

- The sampling error is the difference between an estimate and the corresponding population parameter and it is defined by,

$$\text{Sampling Error} = \bar{X} - \mu$$

Sampling Error

- Usually the parameter is unknown so the sampling error can not be computed.

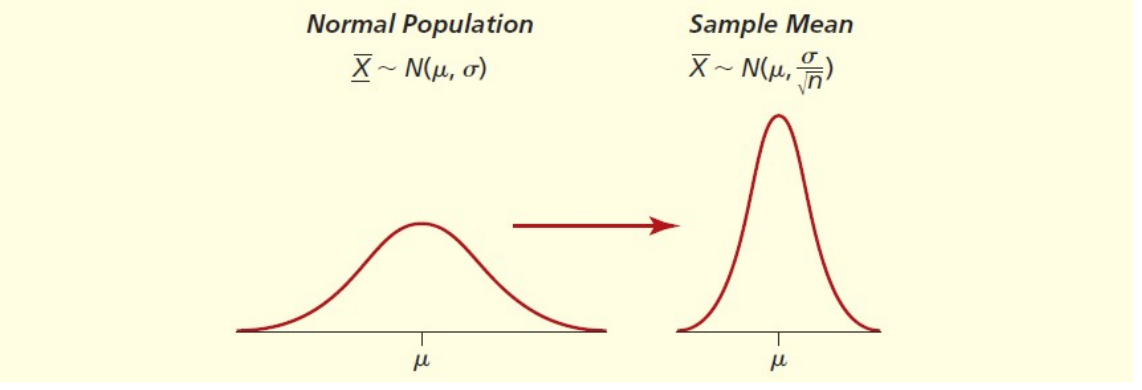
- Usually the parameter is unknown so the sampling error can not be computed.

# Central Limit Theorem

- If the population is exactly normal, the sample mean has exactly a normal distribution centered at with the standard deviation,

$$\sigma_x = \frac{\sigma}{\sqrt{n}}$$

If the population is exactly normal, the sample mean has exactly a normal distribution centered at $\mu$ with the standard deviation,



# Confidence Intervals

- Confidence interval is an interval estimate by specifying the probability that a given interval will contain $\mu$.

- If the sample is large enough that $\bar{X}$ approximately is normal by the Central Limit Theorem then the probability that the interval contains the true mean is $1-\alpha$.

# Confidence Level

- The probability $1-\alpha$ is usually expressed as a percentage called the confidence level.

- The confidence level gives the probability that an interval constructed in this manner will contain $\mu$.

# Confidence Intervals for the Mean

- Confidence Interval of a Mean when the Variance is Known:

Confidence Interval of a Mean when the Variance is Known:

$$P\left(\bar{X} - z_{\alpha/2}\,\frac{\sigma}{\sqrt{n}} < \mu < \bar{X} + z_{\alpha/2}\,\frac{\sigma}{\sqrt{n}}\right) = 1 - \alpha$$

$$\bar{X} \pm z_{\alpha/2}\,\frac{\sigma}{\sqrt{n}}$$

# Confidence Intervals for the Mean

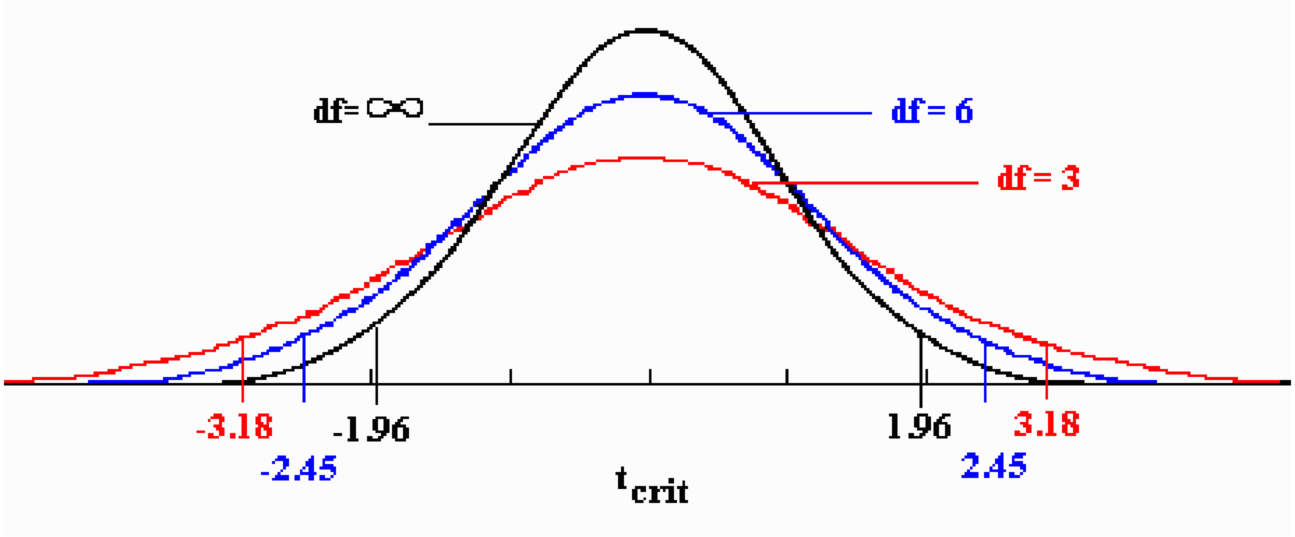
- Confidence Interval of a Mean when the Variance is Unknown:

Confidence Interval of a Mean when the Variance is Unknown:

$$P\left(\bar{X} - t_{\alpha/2}\,\frac{s}{\sqrt{n}} < \mu < \bar{X} + t_{\alpha/2}\,\frac{s}{\sqrt{n}}\right) = 1 - \alpha$$

$$\bar{X} \pm t_{\alpha/2}\,\frac{s}{\sqrt{n}}$$

# The Standard Nomal (z) and t distributions



# Degrees of Freedom

- Degrees of freedom indicates the number of sample values that are free to vary.
- If a sample contains n number of observations, then the degrees of freedom is $df = n-1$

# Hypotheses

- Null Hypothesis ($H_0$): The statements based on similarity or equality.
  - represents a research claim that is liked to be proved.
  - always contains an equal sign ($=, \leq, \geq$).
  - expresses the belief or assumption about the value of the population parameter.

# Hypotheses

- Alternative Hypothesis ($H_a$): The statements based on differences.
  - represents the assumption that the test is attempting to demostrate.
  - never contains an equal sign regarding the specified value of the population parameter ($\neq, <, >$).
  - expresses the opposite of the belief or assumption in $H_0$.

# Hypotheses

- The average connection speed is 54 Mbps, as claimed by the internet service provider.
- The average number of users increased by 2000 this year.
- The service times have Normal distribution.

# Hypotheses

- The average connection speed is 54 Mbps, as claimed by the internet service provider.
- The average number of users increased by 2000 this year.
- The service times have Normal distribution.

The service times follow a Normal distribution.

- The average connection speed is 54 Mbps, as claimed by the internet service provider.

$H_0: \mu = 54$ $\quad\quad$ $H_a: \mu \neq 54$

- The average number of users increased by 2000 this year.

$H_0: \mu \leq 2000$ $\quad\quad$ $H_a: \mu > 2000$

- The service times have Normal distribution.

$H_0$: The service times follow a Normal distribution.

$H_a$: The service times <u>does not</u> follow a Normal distribution.

# Some Important Definitions

- Level of Significance ($\alpha$) : The probability of rejecting the null hypothesis when it is true. It is also known as Type I error.

  $\alpha$ = P(reject H0|H0 is true)

- Test statistic : A statistic value which helps to determine whether a null hypothesis should be rejected.
- Critical value : The critical value is the largest value of the test statistic that result in the rejection of the null hypothesis.

# Some Important Definitions

- p value (p) : It is a probability that provides a measure of evidence against $H_0$.
- It answers the question : "If the null hypothesis were true, then what is the probability of observing a test statistic at least as extreme as the one we observed?"
- The smaller p-values indicate more evidence against $H_0$.

# Type of errors

An error in the analysis!

More work is needed.

| | $H_0$ is True | $H_0$ is False |
|---|---|---|
| Reject $H_0$ | Type I error ($\alpha$) (False Positive) | Correct inference (True Positive) ($1-\beta$) |
| Fail to reject $H_0$ | Correct inference (True Negative) ($1-\alpha$) | Type II error ($\beta$) (False Negative) |

$$P(H_1 | H_0) = \alpha$$

$$P(H_0 | H_1) = \beta$$

$$P(H_1 | H_1) = 1 - \beta \quad \text{(Power of Test)}$$

$$P(H_0 | H_0) = 1 - \alpha \quad \text{(Confidence level)}$$

# Example: Type I vs. Type II Error

- Consider the following hypothesis of a person's judgement:

$H_0$ : The defendant is innocent.

$H_a$ : The defendant is guilty.

| Reality/Decision | $H_0$ rejected (Guilty) | $H_0$ accepted (Innocent) |
|---|---|---|
| $H_0$ True (Innocent) | Type I Error ($\alpha$) | Right Decision (1-$\alpha$) |
| $H_0$ False (Guilty) | Right Decision (1-$\beta$) | Type II Error ($\beta$) |

- In a ... defendant is innocent unless the evidence gathered by the prosecutor is suficient to reject this assumption

# Type of errors

- Type I error (False Positive): "rejecting the null hypothesis when it is true".
- Type II error (False negative): "accepting the null hypothesis when it is false".
- Type III error: “solving the wrong problem [representation]“.
- Type IV error: "the incorrect interpretation of a correctly rejected hypothesis".

Dirty Rotten Strategies: How We Trick Ourselves and Others into Solving the Wrong Problems Precisely (High Reliability and Crisis Management), Ian I. Mitroff, Abraham Silvers. ISBN-13: 978-0804759960

# Type of errors

- Type 1 errors are more important than Type 2 errors ✉ evidence.
  - *p*-values are only correlated with evidence.
  - Evidence in science is necessarily *relative*. When data is more likely assuming one model is true (e.g., a null model) compared to another model (e.g., the alternative model), we can say the model provides **evidence** for the null compared to the alternative hypothesis. *p*-values only give you the probability of the data under one model – what you need for evidence is the relative likelihood of two models

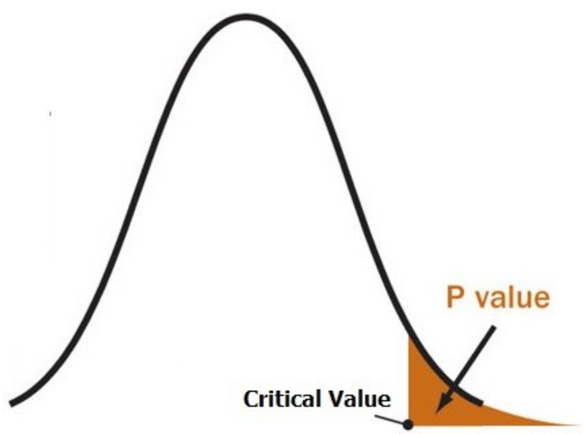
# Steps of Hypothesis Testing

1. Determine the null and alternative hypothesis
2. Specify the level of significance and the decision rules
3. Calculate the related test statistic to test the hypothesis
4. Make a decision (by using p value or critical value approach)

# Hypothesis Testing

- p-value approach:
  - Use the test statistic to compute the p-value.
  - Reject $H_0$ if the p-value $\le \alpha$.

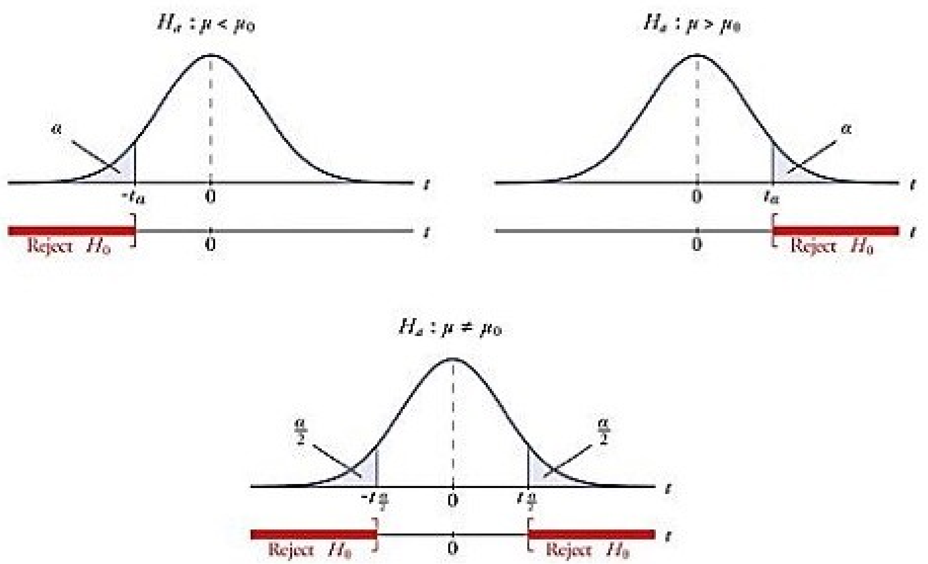
p-value approach:
Use the test statistic to compute the p value
Reject  if the p-value .



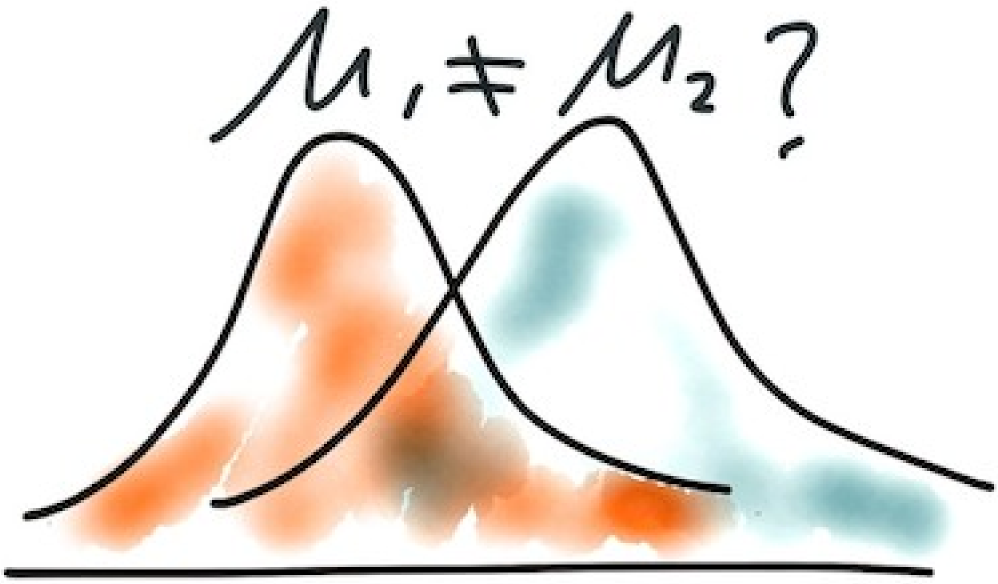
# Hypothesis Testing

- Critical value approach:
  - Use the level of significance to determine the critical value and the rejection rule.
  - Use the value of the test statistic and the rejection rule to determine whether to reject $H_0$.


- Critical value approach:
  - Use the level of significance to determine the critical value and the rejection rule.
  - Use the value of the test statistic and the rejection rule to determine whether to r

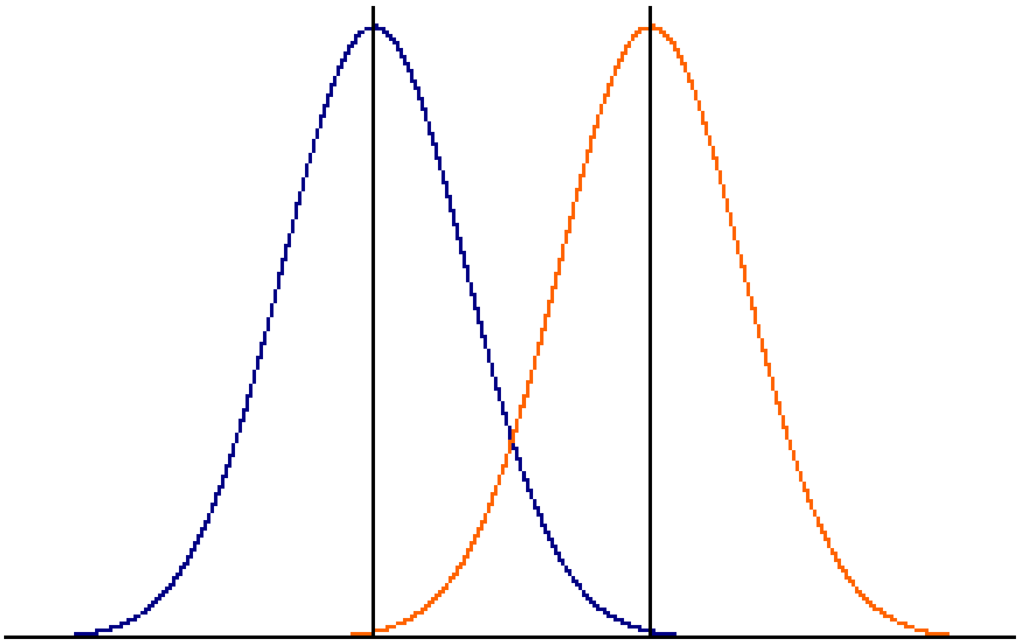



# Comparison of Means



# Comparison of Two Distributions with Equal Variances

# Comparison of two configurations with equal variances.



# Comparison of two configurations with equal variances.

- We define the hypothesis test:
- We define the hypothesis test:
  - $H_0: \mu_A \leq \mu_B$
  - $H_1: \mu_A > \mu_B$
- Thanks the central limit theorem we obtain that:
- Thanks the central limit theorem we obtain that:

$$\bar{y}_A \approx N(\mu_A, \frac{\sigma_A}{\sqrt{n_A}})$$

$$\bar{y}_B \approx N(\mu_B, \frac{\sigma_B}{\sqrt{n_B}})$$

# Comparison of two configurations with equal variances.

- We can deduce that:

$$\bar{y}_A - \bar{y}_B \approx N(\mu_A - \mu_B, \sqrt{\frac{\sigma_A^2}{n_A} + \frac{\sigma_B^2}{n_B}})$$

$$\frac{(\bar{y}_A - \bar{y}_B) - (\mu_A - \mu_B)}{\sqrt{\frac{\sigma_A^2}{n_A} + \frac{\sigma_B^2}{n_B}}} \approx N(0,1)$$

# Comparison of two configurations with equal variances.

- We define the test, and calculate s, the common sample variance:

$$\frac{(\bar{y}_A - \bar{y}_B) - (\mu_A - \mu_B)}{s\sqrt{\frac{1}{n_A} + \frac{1}{n_B}}} \approx t_n$$

- Where $n = n_A + n_B - 2$

# Confidence Interval for the Difference of Means (Equal Variances)

- For the case of equal variances:

$$\bar{y}_A - \bar{y}_B \pm t_{\alpha/2} s_p \sqrt{\frac{1}{n_A} + \frac{1}{n_B}}$$

where $s_p$ is a pooled variance,

$$s_p = \frac{(n_A - 1)s_A^2 + (n_B - 1)s_B^2}{n_A + n_B - 2}$$

# Comparison of two configurations with equal variances.

- The test is defined as is shown:

$$\frac{\bar{y}_A - \bar{y}_B}{s\sqrt{\frac{1}{n_A} + \frac{1}{n_B}}} > t_{\alpha,n}$$

- We reject $H_0$ if this is true.

# Example

Equal variances

# Example

| Observation | Values for pop A | Values for pop B |
|---|---|---|
| 1 | 24.3 | 24.4 |
| 2 | 25.6 | 21.5 |
| 3 | 26.7 | 25.1 |
| 4 | 22.7 | 22.8 |
| 5 | 24.8 | 25.2 |
| 6 | 23.8 | 23.5 |
| 7 | 25.9 | 22.2 |
| 8 | 26.4 | 23.5 |
| 9 | 25.8 | 23.3 |
| 10 | 25.4 | 24.7 |

# Example

- Mean of the sample.
  - A=25.14, B=23.62


- Mean of the sample.
  - A=25.14, B=23.62

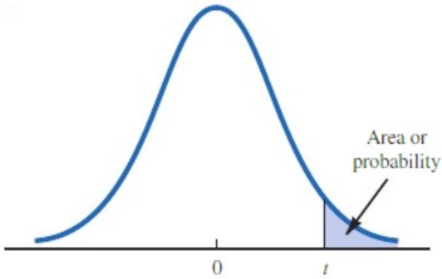

- $H_0: \mu_A \leq \mu_B$
- $H_1: \mu_A > \mu_B$

# Example

- Equal variances:

$$\frac{\bar{y}_A - \bar{y}_B}{s\sqrt{\frac{1}{n_A} + \frac{1}{n_B}}} > t_{\alpha,n}$$

# t-table




Entries in the table give $t$ values for an area or probability in the upper tail of the $t$ distribution. For example, with 10 degrees of freedom and a .05 area in the upper tail, $t_{.05} = 1.812$.

| Degrees of Freedom | Area in Upper Tail | | | | | |
|---|---|---|---|---|---|---|
| | .20 | .10 | .05 | .025 | .01 | .005 |
| 1 | 1.376 | 3.078 | 6.314 | 12.706 | 31.821 | 63.656 |
| 2 | 1.061 | 1.886 | 2.920 | 4.303 | 6.965 | 9.925 |
| 3 | .978 | 1.638 | 2.353 | 3.182 | 4.541 | 5.841 |
| 4 | .941 | 1.533 | 2.132 | 2.776 | 3.747 | 4.604 |
| 5 | .920 | 1.476 | 2.015 | 2.571 | 3.365 | 4.032 |
| 6 | .906 | 1.440 | 1.943 | 2.447 | 3.143 | 3.707 |
| 7 | .896 | 1.415 | 1.895 | 2.365 | 2.998 | 3.499 |
| 8 | .889 | 1.397 | 1.860 | 2.306 | 2.896 | 3.355 |
| 9 | .883 | 1.383 | 1.833 | 2.262 | 2.821 | 3.250 |
| 10 | .879 | 1.372 | 1.812 | 2.228 | 2.764 | 3.169 |
| 11 | .876 | 1.363 | 1.796 | 2.201 | 2.718 | 3.106 |
| 12 | .873 | 1.356 | 1.782 | 2.179 | 2.681 | 3.055 |
| 13 | .870 | 1.350 | 1.771 | 2.160 | 2.650 | 3.012 |
| 14 | .868 | 1.345 | 1.761 | 2.145 | 2.624 | 2.977 |
| 15 | .866 | 1.341 | 1.753 | 2.131 | 2.602 | 2.947 |
| 16 | .865 | 1.337 | 1.746 | 2.120 | 2.583 | 2.921 |
| 17 | .863 | 1.333 | 1.740 | 2.110 | 2.567 | 2.898 |
| 18 | .862 | 1.330 | 1.734 | 2.101 | 2.552 | 2.878 |
| 19 | .861 | 1.328 | 1.729 | 2.093 | 2.539 | 2.861 |

# Example

- The standard deviation is:
  - $s = 1.242;\ 1.237$
  - $S_A = 1.242;\ S_B = 1.237$

$$\frac{25.14 - 23.62}{1.24\sqrt{\frac{1}{10} + \frac{1}{10}}} = 2.74 > t_{0.05,18} = 1.734$$

- Reject $H_0$


- Reject $H_0$

# Comparison of Two Distributions with Unequal Variances

# Comparison of two configurations with unequal variances

- If we cannot assume equal variances.

$$t' = \frac{(\overline{y}_A - \overline{y}_B) - (\mu_A - \mu_B)}{\sqrt{\dfrac{s_A^2}{n_A} + \dfrac{s_B^2}{n_B}}}$$

# Two configurations comparison.

- If $n_A = n_B = n$, the signification level is determined using as a reference distribution a **t of Student** with n-1 degrees of freedom.
- If $n_A \neq n_B$, with the value calculated of $t'$ we can find different signification values $p_A$ and $p_B$ in the student distributions, with $n_A$ -1 and $n_B$ -1 degrees of freedom respectively.

# Two configurations comparison

- The signification level of the test:

$$\alpha = \frac{\omega_A p_A + \omega_B p_B}{\omega_A + \omega_B}$$

- with:

$$\omega_A = \frac{S_A^2}{n_A} \qquad \omega_B = \frac{S_B^2}{n_B}$$

# Confidence Interval for the Difference of Means (Unequal Variances)

- For the case of unequal variances:

$$\bar{y}_A - \bar{y}_B \pm t_{\alpha/2}\sqrt{\frac{s_A^{\ 2}}{n_A} + \frac{s_B^{\ 2}}{n_B}}$$

# Equal variance test

- Hypothesis test:
  - H0: $\sigma_A^2 = \sigma_B^2$
  - H1: $\sigma_A^2 \neq \sigma_B^2$

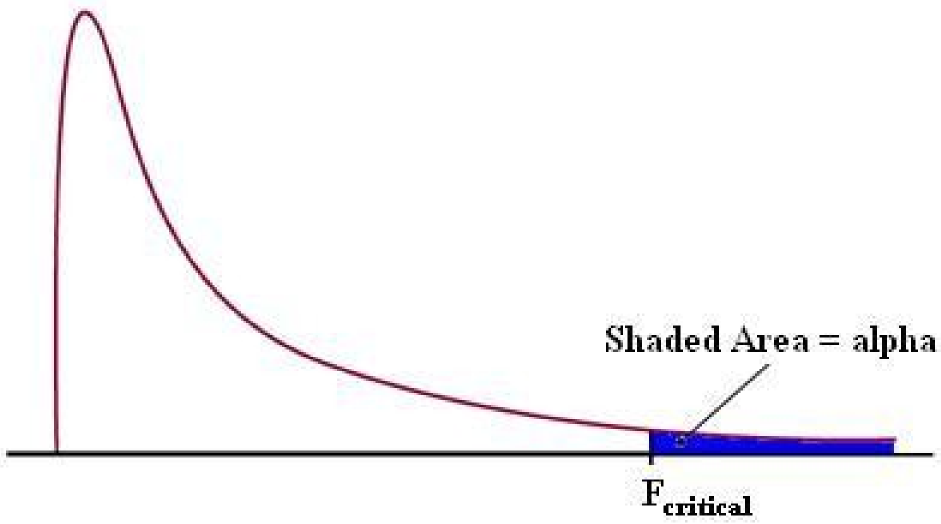
$$F = \frac{variance\ between\ treatments}{variance\ within\ treatments}$$

$$\frac{S_A^2}{S_B^2} \approx F_{n,m}$$

- F of Snedecor
  - $n = n_A - 1$
  - $m = n_B - 1.$

# F distribution and F table

- F table: http://www.z-table.com/f-distribution-table.html



# Example

- $S_A^2$ =1.54
- $S_B^2$ = 2.18

$$\frac{S_B^2}{S_A^2} = \frac{2.18}{1.54} = 1.42 < F_{0.05,9,9} = 3.18$$

- Accept $H_0$

# Example

- $S_A^2 = 1.54$
- $S_B^2 = 16.3$

$$\frac{S_B^2}{S_A^2} = \frac{16.3}{1.54} = 10.58 > F_{0.05,9,9} = 3.18$$

- Reject $H_0$

# To know more

- Part III: Statistics and Chapter 9.4: Statistical Inference I of **Probability and Statistics for Computer Scientists** (2014 Ed.)