Tutorial on Signal
Introduction to Electrical and Electronic Engineering

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Overview

* **Learning Objectives:**
  - Signal Flow Graph
  - Difference Equations
* **Basics**
  - Building blocks of an LTI system
    - Three Building Blocks
    - Flow Graph Transformations
  - Difference Equations
    - Conventions
    - Two Special Discrete-time signals
    - Flow Graphs
* **Questions & Summary**

Building blocks(Three Building Blocks)

The three building blocks of an LTI system: **multiplication**, **addition**, and **delay**

- **Multiplication(gain)**

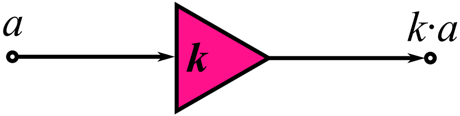


Figure : **Output equals to the input with a gain k**

- **Split/add(adder)**

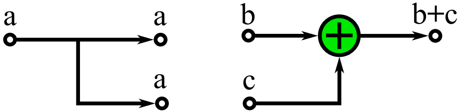


Figure : **On the left, a signal is split into two paths. On the right, two signals are added together**

Building blocks(Three Building Blocks)

The three building blocks of an LTI system: **multiplication**, **addition**, and **delay**

- **Delay**

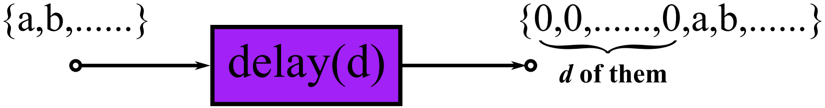


Figure : **Output equals to input with a delay of d time units**

Building blocks(Flow Graph Transformations)

Intuitively, some changes to the flow graphs are permitted:

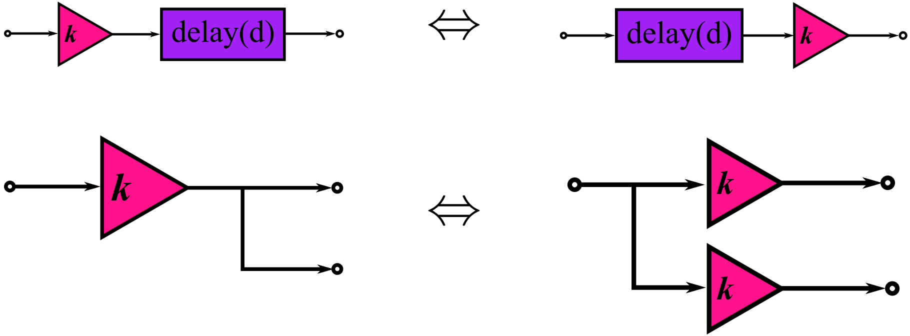


Figure : **Some examples of flow graph transformations**

Difference Equations(Expression & Conventions)

Expression:

- $y[n] = a_1 y[n-1] + a_2 y[n-2] + ... + b_0 x[n] + b_1 x[n-1] + ...$

Conventions:

- Signal: $x[n]$(square bracket)
- Use $x[n]$ for an input signal, $y[n]$ for an output signal
- Often $n = 0, 1, ..., N-1$(integer) for a length-N signal. We may also have an "infinite" length signal where n can be any nonnegative integers.
- Assume $x[n] = 0$ outside this range.

  $\Rightarrow$ No input, no output. System is "at rest"

Difference Equations(Two Special Discrete-time signals)

Impulse Signal(delta functions): $\delta[n]$

- $\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & otherwise(n \neq 0) \end{cases}$

  This is called an impulse because it is active only at the first time instance, and then it returns to zero and stays there forever

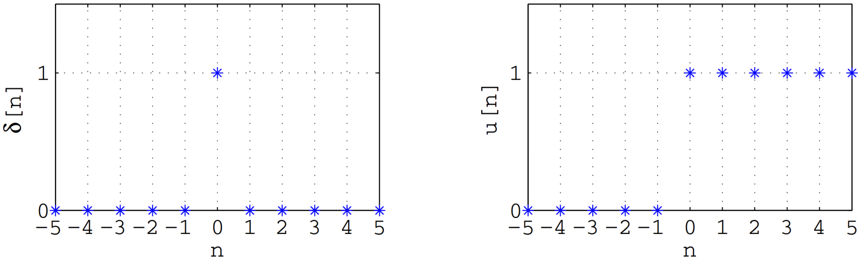
Unit Step Functions: $u[n]$

- $u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & otherwise(n < 0) \end{cases}$

  Notice that because we have assumed that all signals with negative indices are zero, the unit step appears to be equal to 1 all the time

Difference Equations(Two Special Discrete-time signals)

Relation of these two signals: $\delta[n]$ and $u[n]$

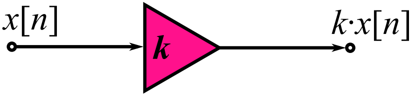


- $\delta[n] = u[n] - u[n-1]$
- $u[n] = \sum_{m=-\infty}^{n} \delta[m] = \sum_{k=0}^{\infty} \delta[n-k]$

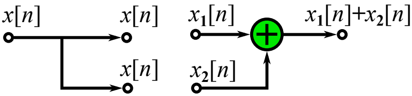
Difference Equations(Flow Graphs)

The three building blocks of an LTI system: **multiplication**, **addition**, and **delay**

- **Multiplication(gain)**
  (k can be integer, fraction, negative number...)



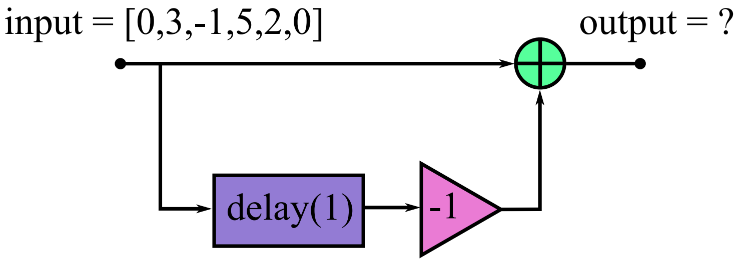
- **Split/add(adder)**
  (A signal becomes two **identical** copies)
  (Two signals added together)



- **Delay**
  (A signal is delayed by d **integer** units)

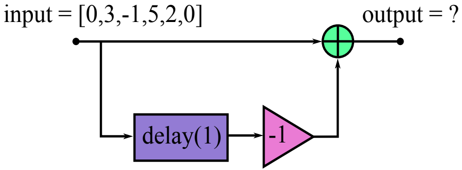
Question 1(a)

* Find the output of the system?



Solution(Q1(a))

* Find the output of the system?



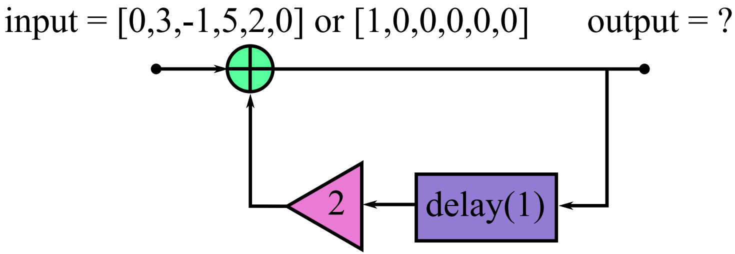
Assume: The system has no signal before the input.

$$\begin{aligned}
0 &\rightarrow \quad (0) && = 0 \\
3 &\rightarrow \quad (3) - 1(0) && = 3 \\
-1 &\rightarrow (-1) - 1(3) && = -4 \\
5 &\rightarrow \quad (5) - 1(-1) && = 6 \\
2 &\rightarrow \quad (2) - 1(5) && = -3 \\
0 &\rightarrow \quad (0) - 1(2) && = -2
\end{aligned}$$

Hence, output is $[0, 3, -4, 4, -3, -2]$

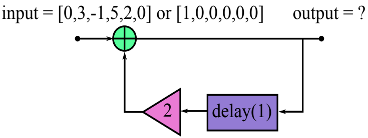
Question 1(b)

* Find the output of the system?



Solution(Q1(b))

* Find the output of the system?



Assume: The system has no signal before the input.

$$\begin{aligned}
0 &\rightarrow & (0) & = 0\\
3 &\rightarrow & (3)+2(0) & = 3\\
-1 &\rightarrow & (-1)+2(3) & = 5\\
5 &\rightarrow & (5)+2(5) & = 15\\
2 &\rightarrow & (2)+2(15) & = 32\\
0 &\rightarrow & (0)+2(32) & = 64
\end{aligned}$$

Hence, output is $[0, 3, 5, 15, 32, 64]$

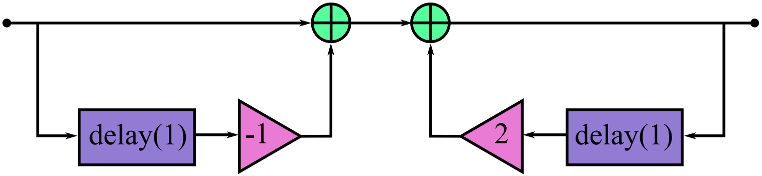
$$\begin{aligned}
1 &\rightarrow & (1) & = 1\\
0 &\rightarrow & (0)+2(1) & = 2\\
0 &\rightarrow & (0)+2(2) & = 4\\
0 &\rightarrow & (0)+2(4) & = 8\\
0 &\rightarrow & (0)+2(8) & = 16\\
0 &\rightarrow & (0)+2(16) & = 32
\end{aligned}$$

Output is $[1, 2, 4, 8, 16, 32]$

Question 1(c)

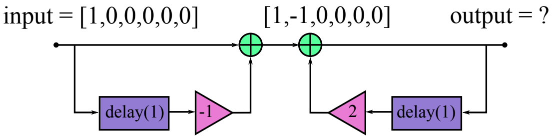
* Find the output of the system?

input = [1,0,0,0,0,0] output = ?



Solution(Q1(c))

* Find the output of the system?



Assume: The system has no signal before the input.

$$\begin{aligned} 1 &\rightarrow (1) &= 1 \\ 0 &\rightarrow (0) - 1(1) &= -1 \\ 0 &\rightarrow (0) - 1(0) &= 0 \\ 0 &\rightarrow (0) - 1(0) &= 0 \\ 0 &\rightarrow (0) - 1(0) &= 0 \\ 0 &\rightarrow (0) - 1(0) &= 0 \end{aligned}$$

$$\begin{aligned} 1 &\rightarrow (1) &= 1 \\ -1 &\rightarrow (-1) + 2(1) &= 1 \\ 0 &\rightarrow (0) + 2(1) &= 2 \\ 0 &\rightarrow (0) + 2(2) &= 4 \\ 0 &\rightarrow (0) + 2(4) &= 8 \\ 0 &\rightarrow (0) + 2(8) &= 16 \end{aligned}$$

Hence, output is $[1, 1, 2, 4, 8, 16]$

Question 2(a)

(a) Sketch each of the following input signals

i. $x[n] = \delta[n] + \delta[n-3]$

ii. $x[n] = u[n] - u[n-5]$

iii. $x[n] = \delta[n] + \frac{1}{2}\delta[n-1] + \frac{1}{2} \cdot 2\delta[n-2] + \frac{1}{2} \cdot 3\delta[n-3]$

where δ is unit impulse function and u is the unit step function.

Solution(Q2(a))

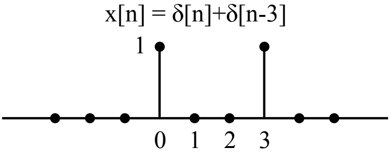
(a) Sketch each of the following input signals

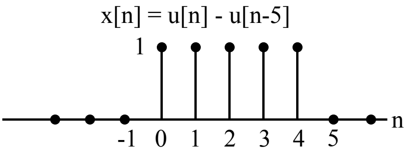
i. $x[n] = \delta[n] + \delta[n-3]$

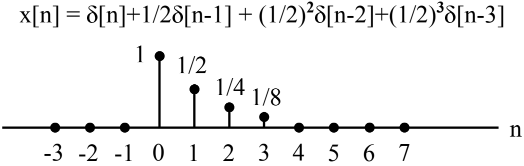
ii. $x[n] = u[n] - u[n-5]$

iii. $x[n] = \delta[n] + \frac{1}{2}\delta[n-1] + \frac{1}{2}\cdot 2\delta[n-2] + \frac{1}{2}\cdot 3\delta[n-3]$

where δ is unit impulse function and u is the unit step function.

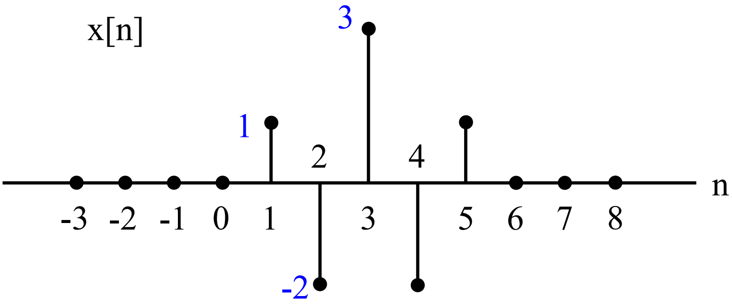






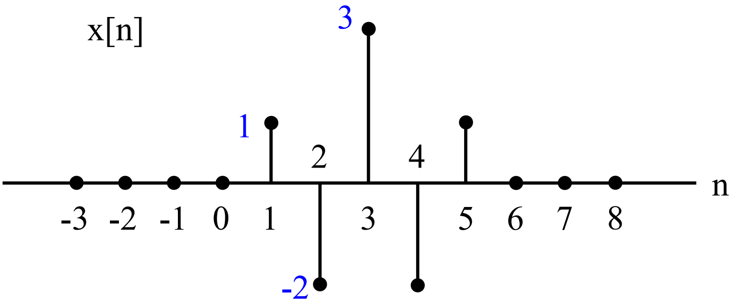
Question 2(b)

(b) Express the following as sums of weighted delayed impulses, i.e. in the form $x[n] = \sum_{k=-\infty}^{\infty} a_k \delta[n-k]$



Solution(Q2(b))

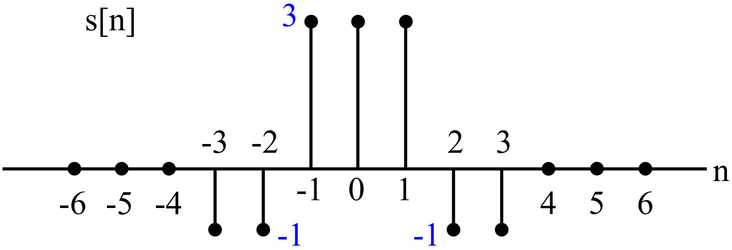
(b) Express the following as sums of weighted delayed impulses, i.e. in the form $x[n] = \sum_{k=-\infty}^{\infty} a_k \delta[n-k]$



Ans: $x[n] = \delta[n-1] - 2\delta[n-2] + 3\delta[n-3] - 2\delta[n-4] + \delta[n-5]$

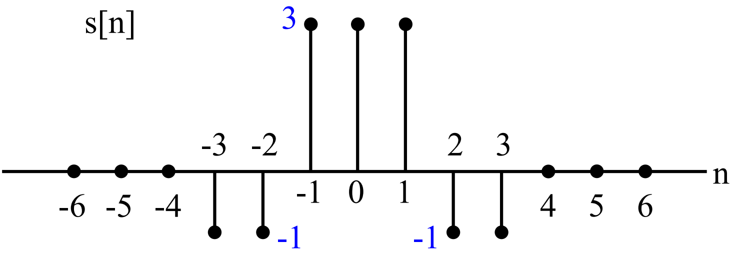
Question 2(c)

(c) Express the following sequence as sum of unit step function, i.e. in the form $s[n] = \sum_{k=-\infty}^{\infty} a_k u[n-k]$



Solution(Q2(c))

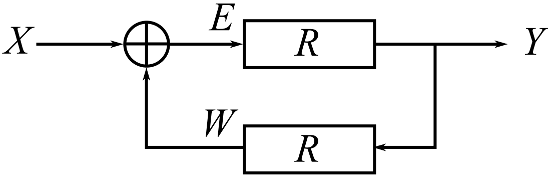
(c) Express the following sequence as sum of unit step function, i.e. in the form $s[n] = \sum_{k=-\infty}^{\infty} a_k u[n-k]$



Ans: $s[n] = -u[n+3] + 4u[n+1] - 4u[n-2] + u[n-4]$

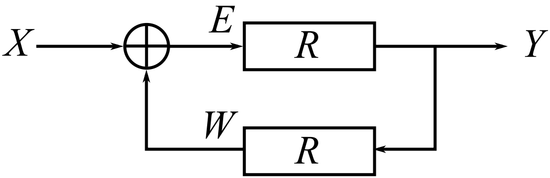
Question 3(a)

* Determine the difference equation that relates X and Y? **R: delay(1)**



Solution(Q3(a))

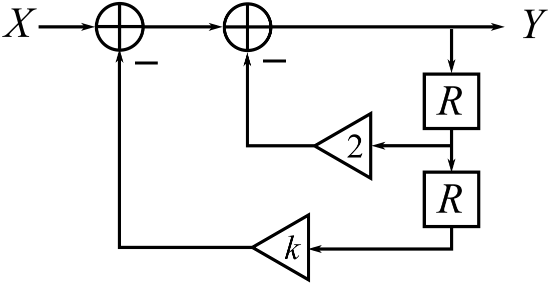
- Determine the difference equation that relates X and Y? **R: delay(1)**



- Express relations among signals algebraically
- $E = X + W$; $Y = RE$; $W = RY$
- Solve: $Y = RE = R(X + W) = R(X + RY)$
  $\rightarrow RX = Y - R^2 Y$
- Difference equation:$y[n] = x[n-1] + y[n-2]$

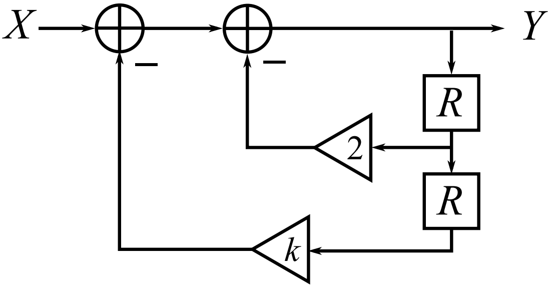
Question 3(b)

* Determine the difference equation that relates X and Y? **R: delay(1)**



Solution(Q3(b))

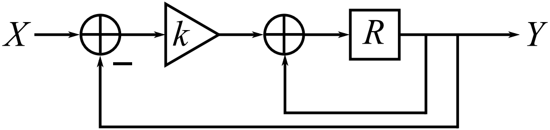
* Determine the difference equation that relates X and Y? **R: delay(1)**



* $Y = X - 2RY - kR^2Y$
* Difference equation: $y[n] = x[n] - 2y[n-1] - ky[n-2]$

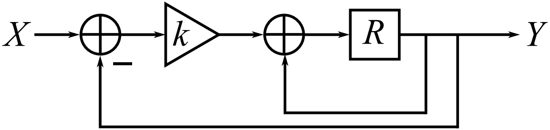
Question 3(c)

* Determine the difference equation that relates X and Y? **R: delay(1)**



Solution(Q3(c))

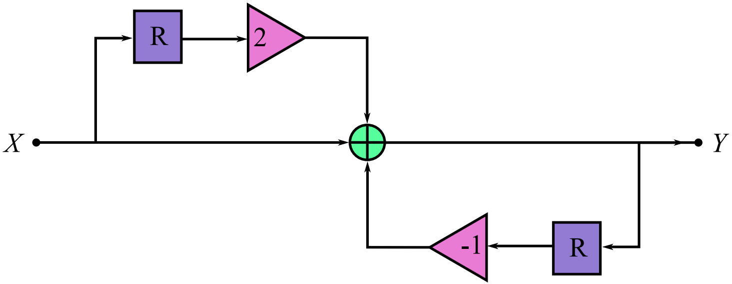
* Determine the difference equation that relates X and Y? **R: delay(1)**



* $Y = RY + kRX - kRY$
* Difference equation: $y[n] = y[n-1] + kx[n-1] - ky[n-1]$

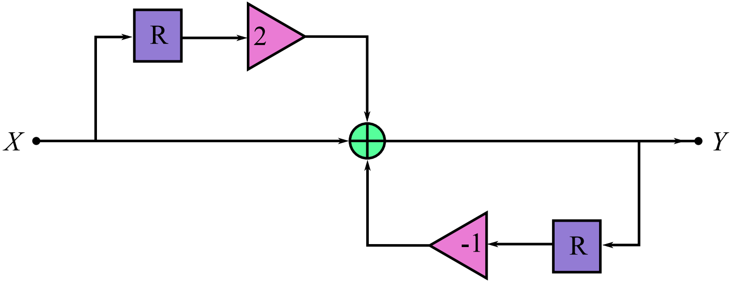
Question 3(d)

* Determine the difference equation that relates X and Y? **R: delay(1)**



Solution(Q3(d))

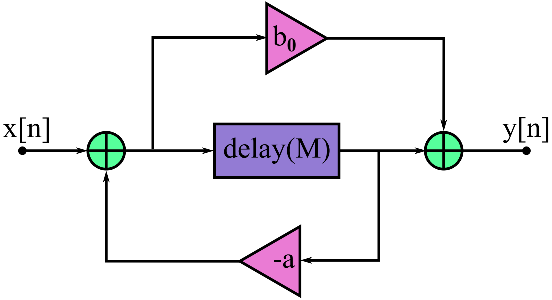
* Determine the difference equation that relates X and Y? **R: delay(1)**



* $Y = X + 2RY - RY$
* Difference equation:$y[n] = x[n] + 2x[n-1] - y[n-1]$

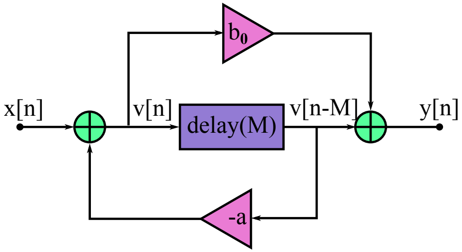
Question 3(e)

* Determine the difference equation that relates X and Y? **R: delay(1)**



Solution(Q3(e))

* Determine the difference equation that relates X and Y? **R: delay(1)**



* Difference equation:

$$v[n] = x[n] - av[n-M]; \;\; y[n] = b_0 v[n] + v[n-M]$$
$$\begin{aligned}\therefore y[n] &= b_0\{x[n] - av[n-M]\} + v[n-M] \\ &= b_0\{x[n] - av[n-M]\} + x[n-M] - av[n-2M] \\ &= b_0 x[n] + x[n-M] - a\{b_0 v[n-M] + v[n-2M]\} \\ &= b_0 x[n] + x[n-M] - ay[n-M]\end{aligned}$$

Question 4

[SP13 Final Exam] Consider the difference equation $y[n] = y[n-1] + k \cdot y[n-2] + x[n]$, where $x[n]$ is an impulse input. For what value(s) of k indicated below would the output converge to zero as n increases?

i $k = 0$

ii $k = -\frac{1}{2}$

iii $k = -1$

iv $k = -\frac{1}{2}$ and $k = 0$

v $k = -1$, $k = -\frac{1}{2}$, and $k = 0$

Solution(Q4)

Difference equation: $y[n] = y[n-1] + k \cdot y[n-2] + x[n]$

Impulse input $x[n]$: $x[n] = \delta[n] = \begin{cases} 1, & \text{if } n = 0 \\ 0, & \text{otherwise} \end{cases}$

For $k = 0$ $(y[n] = y[n-1] + x[n])$

n	$x[n]$	$y[n-1]$	$+$	$x[n]$	$=$	$y[n]$
0	1	0	$+$	1	$=$	1
1	0	1	$+$	0	$=$	1
2	0	1	$+$	0	$=$	1
3	0	1	$+$	0	$=$	1
4	0	1	$+$	0	$=$	1
5	0	1	$+$	0	$=$	1

Solution(Q4)

For $k = -\frac{1}{2}$ $(y[n] = y[n-1] - \frac{1}{2}y[n-2] + x[n])$

n	$x[n]$	$y[n-1]$	$-$	$\frac{1}{2}y[n-2]$	$+$	$x[n]$	$=$	$y[n]$
0	1	0	$-$	$\frac{1}{2}(0)$	$+$	1	$=$	1
1	0	1	$-$	$\frac{1}{2}(0)$	$+$	0	$=$	1
2	0	1	$-$	$\frac{1}{2}(1)$	$+$	0	$=$	$\frac{1}{2}$
3	0	$\frac{1}{2}$	$-$	$\frac{1}{2}(1)$	$+$	0	$=$	0
4	0	0	$-$	$\frac{1}{2}(\frac{1}{2})$	$+$	0	$=$	$\frac{-1}{4}$
5	0	$-\frac{1}{4}$	$-$	$\frac{1}{2}(0)$	$+$	0	$=$	$\frac{-1}{4}$
6	0	$-\frac{1}{4}$	$-$	$\frac{1}{2}(\frac{-1}{4})$	$+$	0	$=$	$\frac{-1}{8}$

Solution(Q4)

For $k = -\frac{1}{2}$ $(y[n] = y[n-1] - \frac{1}{2}y[n-2] + x[n])$

n	$x[n]$	$y[n-1]$	$-$	$\frac{1}{2}y[n-2]$	$+$	$x[n]$	$=$	$y[n]$
7	0	$\frac{-1}{8}$	$-$	$\frac{1}{2}(\frac{-1}{4})$	$+$	0	$=$	0
8	0	0	$-$	$\frac{1}{2}(\frac{-1}{8})$	$+$	0	$=$	$\frac{1}{16}$
9	0	$\frac{1}{16}$	$-$	$\frac{1}{2}(0)$	$+$	0	$=$	$\frac{1}{16}$
10	0	$\frac{1}{16}$	$-$	$\frac{1}{2}(\frac{1}{16})$	$+$	0	$=$	$\frac{1}{32}$
11	0	$\frac{1}{32}$	$-$	$\frac{1}{2}(\frac{1}{16})$	$+$	0	$=$	0
12	0	0	$-$	$\frac{1}{2}(\frac{1}{36})$	$+$	0	$=$	$\frac{-1}{72}$
13	0	$\frac{-1}{72}$	$-$	$\frac{1}{2}(0)$	$+$	0	$=$	$\frac{-1}{72}$

Solution(Q4)

For $k = -1$ $(y[n] = y[n-1] - y[n-2] + x[n])$

n	$x[n]$	$y[n-1]$	$-$	$\frac{1}{2}y[n-2]$	$+$	$x[n]$	$=$	$y[n]$
0	1	0	$-$	(0)	$+$	1	$=$	1
1	0	1	$-$	(0)	$+$	0	$=$	1
2	0	1	$-$	(1)	$+$	0	$=$	0
3	0	0	$-$	(1)	$+$	0	$=$	-1
4	0	-1	$-$	(0)	$+$	0	$=$	-1
5	0	-1	$-$	(-1)	$+$	0	$=$	0
6	0	0	$-$	(-1)	$+$	0	$=$	1

Solution(Q4)

For $k = -1$ $(y[n] = y[n-1] - y[n-2] + x[n])$

n	$x[n]$	$y[n-1]$	$-$	$\frac{1}{2}y[n-2]$	$+$	$x[n]$	$=$	$y[n]$
7	0	1	$-$	(0)	$+$	0	$=$	1
8	0	1	$-$	(1)	$+$	0	$=$	0
9	0	0	$-$	(1)	$+$	0	$=$	-1
10	0	-1	$-$	(0)	$+$	0	$=$	-1
11	0	-1	$-$	(-1)	$+$	0	$=$	0
12	0	0	$-$	(-1)	$+$	0	$=$	1
13	0	1	$-$	(0)	$+$	0	$=$	1

Question 5(a)

[FA12 Final Exam] Consider the difference equation
$y[n] = k \cdot y[n-1] + k \cdot y[n-2] + x[n]$.
Assume $x[n]$ is an impulse input, i.e. $x[0] = 1$ and $x[n] = 0$ for other values of n, and that $y[n] = 0$ for $n < 0$.

(a) Let $k = 1$. What is the value of $y[10]$?

(i) 2
(ii) 1
(iii) 0
(iv) -1
(v) -2

Solution(Q5(a))

$k = 1$, $y[10] = ?$

n	$x[n]$	$y[n]$	$=$	$y[n-1]$	$-$	$y[n-2]$	$+$	$x[n]$
0	1	1	$=$	0	$-$	0	$+$	1
1	0	1	$=$	1	$-$	0	$+$	0
2	0	0	$=$	1	$-$	1	$+$	0
3	0	-1	$=$	0	$-$	1	$+$	0
4	0	-1	$=$	-1	$-$	0	$+$	0
5	0	0	$=$	-1	$-$	-1	$+$	0

Solution(Q5(a))

$k = 1,\ y[10] = ?$

n	$x[n]$	$y[n]$	$=$	$y[n-1]$	$-$	$y[n-2]$	$+$	$x[n]$
6	0	1	$=$	0	$-$	-1	$+$	0
7	0	1	$=$	1	$-$	0	$+$	0
8	0	0	$=$	1	$-$	1	$+$	0
9	0	-1	$=$	0	$-$	1	$+$	0
10	0	-1	$=$	-1	$-$	0	$+$	0

Question 5(b)

[FA12 Final Exam] Consider the difference equation
$y[n] = k \cdot y[n-1] + k \cdot y[n-2] + x[n]$.
Assume $x[n]$ is an impulse input, i.e. $x[0] = 1$ and $x[n] = 0$ for other values of n, and that $y[n] = 0$ for $n < 0$.

(b) Let $k = -1$. What is the value of $y[10]$?

(i) 34
(ii) -34
(iii) 55
(iv) -55
(v) 89

Solution(Q5(b))

$k = -1$, $y[10] =?$

n	$x[n]$	$y[n]$	$=$	$-y[n-1]$	$+$	$y[n-2]$	$+$	$x[n]$
0	1	1	$=$	$-(0)$	$+$	0	$+$	1
1	0	-1	$=$	$-(1)$	$+$	0	$+$	0
2	0	2	$=$	$-(-1)$	$+$	1	$+$	0
3	0	-3	$=$	$-(2)$	$+$	-1	$+$	0
4	0	5	$=$	$-(-3)$	$+$	2	$+$	0
5	0	-8	$=$	$-(5)$	$+$	-3	$+$	0

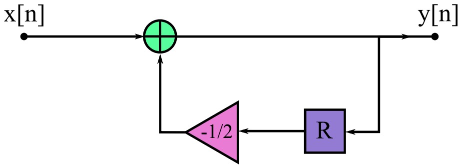
Solution(Q5(b))

$k = -1$, $y[10] =?$

n	$x[n]$	$y[n]$	$=$	$-y[n-1]$	$+$	$y[n-2]$	$+$	$x[n]$
6	0	13	$=$	$-(-8)$	$+$	5	$+$	0
7	0	-21	$=$	$-(13)$	$+$	-8	$+$	0
8	0	34	$=$	$-(-21)$	$+$	13	$+$	0
9	0	-55	$=$	$-(34)$	$+$	-21	$+$	0
10	0	89	$=$	$-(-55)$	$+$	34	$+$	0

Question 6

* Consider the block diagram relating the two signal $x[n]$ and $y[n]$ given in figure. **R: delay(1)**



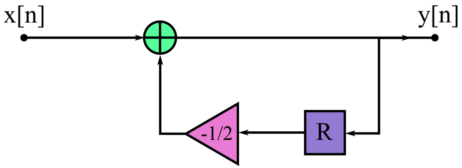
(a) Determine the difference equation relating $y[n]$ and $x[n]$.

(b) Assume that a solution to the difference equation in part (a) is given by $y[n] = k\alpha^n u[n]$, where $u[n]$ is unit step function and $x[n] = \delta[n]$. Find the appropriate value of k and α, and verify that $y[n]$ satisfies the difference equation.

(c) Verify your answer to part (b) by directly calculating $y[0]$, $y[1]$, and $y[2]$.

Solution(Q6(a))

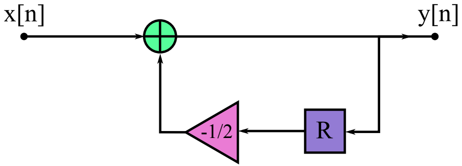
(a) Determine the difference equation relating $y[n]$ and $x[n]$.



* Thus, $y[n] = x[n] - \frac{1}{2}y[n-1]$
* or $y[n] + \frac{1}{2}y[n-1] = x[n]$

Solution(Q6(b))

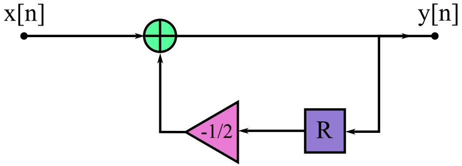
(b) Assume that a solution to the difference equation in part (a) is given by $y[n] = k\alpha^n u[n]$, where $u[n]$ is unit step function and $x[n] = \delta[n]$. Find the appropriate value of k and α, and verify that $y[n]$ satisfies the difference equation.



* For $n < 0$, $x[n] = \delta[n] = 0 \therefore y[n] = 0$
* For $n = 0$, $y[0] + \frac{1}{2}y[-1] = x[0]$
* Substituting $y[n] = k\alpha^n u[n]$
* $k\alpha^0 u[0] + \frac{1}{2}k\alpha^{-1}u[-1] = 1$
* $k(1)(1) + \frac{1}{2}k\alpha^{-1}(0) = 1 \Rightarrow k = 1$

Solution(Q6(b))

(b) Assume that a solution to the difference equation in part (a) is given by $y[n] = k\alpha^n u[n]$, where $u[n]$ is unit step function and $x[n] = \delta[n]$. Find the appropriate value of k and α, and verify that $y[n]$ satisfies the difference equation.



- For $n > 0$, $y[n] + \frac{1}{2}y[n-1] = x[n]$
- $k\alpha^n u[n] + \frac{1}{2}k\alpha^{n-1}u[n-1] = 0$
- $(1)(\alpha^n)(1) + \frac{1}{2}(1)\alpha^{n-1}(1) = 0$
- $\alpha^n + \frac{1}{2}\alpha^{n-1} = 0 \Rightarrow \alpha = -\frac{1}{2}$

Solution(Q6(b))

(b) The difference equation: $y[n] + \frac{1}{2}y[n-1] = x[n]$
For $k = 1$, $\alpha = -\frac{1}{2}$, $y[n] = (-\frac{1}{2})^n u[n]$
Substituting into the left side of the difference equation,
We have

$$y[n] + \frac{1}{2}y[n-1] = (-\frac{1}{2})^n u[n] + \frac{1}{2}(-\frac{1}{2})^{n-1} u[n-1]$$

$$= (-\frac{1}{2})^n u[n] - (-\frac{1}{2})^n u[n-1]$$

$$= \begin{cases} 1, & n = 0 \\ 0, & otherwise \end{cases} = \delta[n] = x[n] \qquad \text{Verified.}$$

Solution(Q6)

(c) We can successively calculate $y[n]$ by noting that $y[-1] = 0$ and that $y[n] = -\frac{1}{2}y[n-1] + x[n]$
so

$$
\begin{aligned}
n = 0,\ y[0] &= -\tfrac{1}{2} \cdot 0 + 1 = 1 \\
n = 1,\ y[1] &= -\tfrac{1}{2} \cdot 1 + 0 = -\tfrac{1}{2} \\
n = 2,\ y[2] &= -\tfrac{1}{2} \cdot (-\tfrac{1}{2}) + 0 = \tfrac{1}{4}
\end{aligned}
$$

Summary

- Building blocks
  - Three Building Blocks(**analyze signals one by one**)
  - Flow Graph Transformations
- Difference Equations
  - Conventions(**signals out of range is NULL or 0**)
  - Two Special Discrete-time signals($\delta[n]$ and $u[n]$)
  - Flow Graphs

The End