Construction of Chebyshev nets with singularities

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1 Introduction

A global net on a surface M is two families of curves such that through each point p there passes exactly one curve of each family with tangent vectors forming a basis of the tangent plane at p. We will denote by $\omega(p)$ the angle between these vector at p and call it angle function of the net. A global Chebyshev net (T-net in the following) is a net in which the tangent vectors to each family of line can be parallel displaced along the lines of the other family. A net is a Chebyshev net if around each point p, we can find a local chart $\varphi : \Omega \subset \mathbb{R}^2 \mapsto M$, such that the two family of line are the line of coordinate $u \to \varphi(u, v_0)$ and $v \to \varphi(u_0, v)$ with lines parametrized by arclength

$$|\partial_u \varphi| = |\partial_v \varphi| = 1.$$

When $\omega(p) = 0[\pi]$, lines are tangent and the net is no more a local chart of the surface. Singularities at these points are of cuspidal type or shallow tails. This is well describe in a paper of E. Ghys [?] introducing Chebyshev net on surfaces. It is not known when there exists global T-net on compact or non compact surfaces. The angle function ω between the two family of lines appear in the metric of local coordinates

$$\mathrm{d}s^2 = \mathrm{d}u^2 + 2\cos\omega(u,v)\mathrm{d}u\mathrm{d}v + \mathrm{d}v^2$$

and the Gauss equation is related to a Sine-Gordon type equation of hyperbolic type:

$$\partial^2_{uv}\omega = -K(u,v)\sin\omega(u,v).$$

Using Gauss Bonnet theorem which reformulate in Hazzidakis formula, it is known (see H. Hopf, [?], Lemma 2.3) that Chebyschev coordinates may not exist on a simply connected surface with $|\int\int KdS| \geq 2\pi$. P. L. Chebyshev proved the local existence of his coordinates under an analyticity assumption. In this paper, beside the hyperbolicity property of the Gauss equation, we will construct coordinates on the surface by finding the function ω which satisfy boundary Cauchy data conditions on some boundary curves.

Piecewise smooth T-net is a weaker notion. We relax the regularity of lines of the net. Each family of lines are parallel displaced along the lines of the other family but it can happen that lines are only piecewise C^1 with existing left and right tangent vector at singular point p. We denote by η_1 the line of the other family passing through p and by η_2 the line which is not C^1 at p. The left and right vectors $(\eta_2')_d(p)$ and $(\eta_2')_\ell(p)$ form an angle $\alpha \neq 0[\pi]$. This angle is displaced along η_1 by parallel transport, and any line of the second family is singular on η_1. We call this line of singularities "twin crystal line" as it can be observed in mineralogy. When twin crystal line meet at some point, this create bifurcation in the net. Bifurcation can appear when η_1 and η_2 are singular in the same point p. In this case, the bifurcation point is related to a "cross". However the number of twin crystal line which meet together at p can be greater than 4 but we consider only finite number case.

The first non local result has been proved by I. Ya. Bakelman [?]. He proved the existence of Chebyshev coordinates in any sector bounded by two orthogonal geodesics on a simply connected Aleksandrov surface under the condition that both the positive and negative integral curvatures, $\int K^+dS$ and $\int K^-dS$ are less than $\pi/2$, (where K is the Gaussian curvature of M, $K^+ = \max(K, 0)$ and $K^- = \max(-K, 0)$) but the net (And the surface) is not smooth. Much later, a similar result for smooth surfaces, was proved by S. L. Samelson and W. P. Dayawansa [?]: under the same restriction on the curvature, they proved that on each simply connected complete smooth surface there exists a global smooth Chebyshev net. Their proof is analytical. It is well known that the existence problem for the Chebyshev coordinates can be reduced to the existence problem for a solution to a hyperbolic system of second-order partial differential equations. This system is solved in [?] by the method of a paper of Ladyzhenskaya and Shubov [?]. Burago, Ivanov and Malev in [?] used Bakelman result in a sector decomposition of a surface described in a paper of Bonk and Lang [?]. They find four sectors satisfying hypothesis of Bakelman where they construct non smooth Chebyshev net on simply connected smooth surface M with $\int K^\pm dS < 2\pi$. Masson and Monasse in [?] improve the result of Samelson and Dayawansa by constructing smooth T-net for simply connected surfaces with $\int K^\pm dS < 2\pi$ which is a sharp result by the lemma of H. Hopf.

Piecewise smooth T-net is strongly related to the construction of bi-Lipschitz map between surface. If M is a simply connected surface, a map $f : \mathbb{R}^2 \mapsto M$ is bi-Lipschitz if there exists a constant L such that

$$\frac{1}{L}d(x, y,) \leq \mathrm{dist}_M(f(x), f(y)) \leq Ld(x, y).$$

For Chebyshev coordinates the Lipschitz constants can be estimated in terms of the net angles, i.e., via $\min(\inf\omega, \inf(\pi-\omega))$. M. Bonk and U. Lang [**?**] proved that if

$$\int K^+ dS < 2\pi \text{ and } \int K^- dS < C$$

then M is bi-Lipschitz equivalent to $\mathbb{R}^2$ with Lipschitz constant $\epsilon^{1/2}(2\pi+C)^{1/2}$. The existence of piecewise smooth T-net immediately implies a weak (with a worse constant) version of the Bonk-Lang theorem.

In this paper, we consider complete and simply connected surface M and we construct by a new method piecewise smooth T-net by a decomposition of the surface in a finite number of sectors where we are able to produce smooth T-net. The boundary of the sector are the twin crystal lines on M. In a the section 3, we describe the regularisation of piecewise smooth T-net outside a finite number of ball, arbitrarly small neighborhood of bifurcating points. More precisely the main theorem of this paper is

Theorem 1 (Existence of a smooth Chebyshev net out of a finite number of balls)**.** *Let M be a smooth complete immersion of the plane $\mathbb{R}^2$ in $\mathbb{R}^3$ satisfying*

$$\int_M K^+ < 2\pi \quad \textit{and} \quad \int_M K^- < \infty, \tag{1}$$

where K is the Gaussian curvature of M, $K^+ = \max(K,0)$ and $K^- = \max(-K,0)$. Then there exists a piecewise smooth Chebyshev net with a finite number of bifurcating points $\{p_i\}_{1\le i\le\mathcal{N}}$ on M with $\mathcal{N} \le \frac{4}{\pi}\int_M K^- + 4$. The surface $M \setminus \bigcup_{1\le i\le\mathcal{N}} \mathcal{B}_i$ admits a smooth Chebyshev net where B_i are disks centered at p_i with arbitrary small radius ε.

2 Preliminaries

We consider a complete and simply connected immersed surface. Let $D = [0,L_u]\times[0,L_v]$, with $L_u, L_v \in \mathbb{R}_*^+$, and let $\gamma_u : [0,L_u] \to M$ and $\gamma_v : [0,L_v] \to M$ be two curves such that $\gamma_u(0) = \gamma_v(0)$, and forming an interior angle $\angle(\gamma_u'(0), \gamma_v'(0)) \in (0,\pi)$ at their intersection and exterior angle $\psi = \pi - \angle(\gamma_u'(0), \gamma_v'(0))$. For clarity of exposition, throughout this section, we consider a slightly different notion of Chebyshev net which will refer to mappings (not necessarily homeomorphisms) $\varphi : D \to M$ satisfying

$$|\partial_u\varphi|_g(u,v) = 1, \tag{2a}$$
$$|\partial_v\varphi|_g(u,v) = 1, \tag{2b}$$

for all $(u,v) \in D$. Furthermore, note that the orientation of the boundary curve γ_v is reversed in this section to simplify the exposition. We prove in the sequel the existence and the uniqueness of a Chebyshev net $\varphi : D \to M$ verifying the Cauchy boundary conditions

$$\begin{aligned} \varphi(u,0) &= \gamma_u(u), \quad \forall u \in [0,L_u], \\ \varphi(0,v) &= \gamma_v(v), \quad \forall v \in [0,L_v]. \end{aligned} \tag{3}$$

Since M is a plane immersion, and since $\int_M K^+ < 2\pi$ and $\int_M K^- < \infty$, there exists a global bi-Lipschitz parametrization of M [**?**] and we identify $M = (\mathbb{R}^2, g)$ such that there exists $C_{\text{surf}} > 1$ such that for all $X \in \mathbb{R}^2$,

$$\frac{1}{C_{\text{surf}}}|X| \leq |X|_g \leq C_{\text{surf}}\,|X|. \tag{4}$$

Following [**?**], we reformulate the problem of finding a Chebyshev net $\varphi : D \to M$ as the problem of finding the angle distribution $\omega : D \to (0,\pi)$ between the coordinate curves defined by $\omega(u,v) = \angle(\partial_u\varphi, \partial_v\varphi)(u,v)$. With this purpose in mind, we observe that ω satisfies the integrability condition (in the form of a modified Sine-Gordon equation) [**?**]

$$\partial_{uv}\omega = -K(\varphi)\sin(\omega). \tag{5}$$

Equivalently, ω satisfies the integrated form Gauss Bonnet formula of (5) called the Hazzidakis formula

$$\begin{aligned}\omega(u,v) = \angle\big(\gamma_u'(0), \gamma_v'(0)\big) - \int_0^u \kappa_u(s)ds + \int_0^v \kappa_v(s)ds&\\ - \int_0^u\int_0^v K(\varphi(s,t))\sin\big[\omega(s,t)\big]ds\ dt,&\end{aligned} \tag{6}$$

for all $(u,v) \in D$, with $\kappa_u : [0, L_u] \to \mathbb{R}$ and $\kappa_v : [0, L_v] \to \mathbb{R}$ the geodesic curvatures of γ_u and γ_v respectively (see Figure 1).

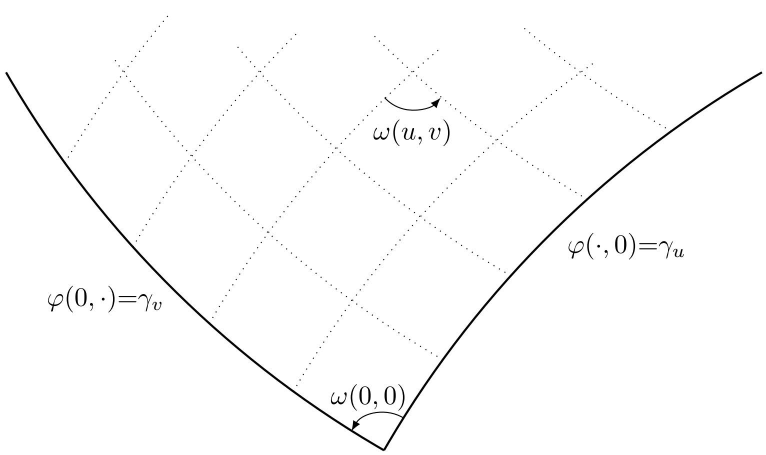

Figure 1: Illustration of the coordinate curves of a Chebyshev net φ

Moreover, remark that, since φ is a Chebyshev net, it satisfies the following property:

$$\kappa_u^{\text{map}}(u,v) = -\partial_u\omega(u,v), \tag{7a}$$
$$\kappa_v^{\text{map}}(u,v) = \partial_v\omega(u,v), \tag{7b}$$

for all $(u, v) \in D$, where $\kappa_u^{\mathrm{map}} : D \to \mathbb{R}$ and $\kappa_v^{\mathrm{map}} : D \to \mathbb{R}$ are respectively the geodesic curvatures of the u-coordinate curves and of the v-coordinate curves of φ. We obtain by combining (3) and (7) that the angle distribution ω satisfies the boundary conditions

$$\omega(u, 0) = \angle(\gamma_u'(0), \gamma_v'(0)) - \int_0^u \kappa_u(s)\mathrm{d}s, \quad \forall u \in [0, L_u], \tag{8a}$$

$$\omega(0, v) = \angle(\gamma_u'(0), \gamma_v'(0)) + \int_0^v \kappa_v(s)\mathrm{d}s, \quad \forall v \in [0, L_v]. \tag{8b}$$

We will show that we can associate to any angle distribution $\omega : D \to (0, \pi)$ satisfying (8) a unique mapping $\varphi := \mathcal{I}(\gamma, \omega) : D \to M$ satisfying (2b), (3), and (7b). We then show that this mapping also satisfies (2a) and (7a) whenever ω satisfies the integrability condition (5). We prove the following result:

Theorem 2 (Existence and uniqueness of smooth Chebyshev nets in a sector)**.** *Let M be a smooth immersion of the plane $\mathbb{R}^2$ in $\mathbb{R}^3$. Let $\gamma_u : \mathbb{R}^+ \to M$ and $\gamma_v : \mathbb{R}^+ \to M$ be two smooth curves with respective geodesic curvatures $\kappa_u : \mathbb{R}^+ \to \mathbb{R}$ and $\kappa_v : \mathbb{R}^+ \to \mathbb{R}$, and such that $\gamma_u(0) = \gamma_v(0)$. Suppose that $\psi = \pi - \angle(\gamma_u'(0), \gamma_v'(0)) \in (0, \pi)$ and that the sector Ω delimited by γ_u and γ_v verifies*

$$\int_{\gamma_u} \kappa_u^+ + \int_{\gamma_v} \kappa_v^- + \int_\Omega K^+ < \pi - \psi \;\; and \;\; \int_{\gamma_u} \kappa_u^- + \int_{\gamma_v} \kappa_v^+ + \int_\Omega K^- < \psi.$$

Let $L_u, L_v \in \mathbb{R}_*^+ \cup \{+\infty\}$ and $D = [0, L_u]\times[0, L_v]$. Then, there exists a unique angle distribution $\omega : D \to (0, \pi)$ verifying the boundary conditions (8) *and satisfying the Hazzidakis formula* (6)*, with $\varphi := \mathcal{I}(\gamma, \omega) : D \to M$ the unique mapping satisfying* (2b), (3) *and* (7b)*. The mapping φ is a smooth Chebyshev net on Ω with angles $0 < \omega < \pi$.*

Proof of Proposition 25. Using Theorem 26, we infer that there exists a unique angle distribution $\omega : (\mathbb{R}^+)^2 \to \mathbb{R}/2\pi\mathbb{Z}$ satisfying the Hazzidakis formula (70), with $\varphi : (\mathbb{R}^+)^2 \to M$ the unique mapping satisfying the boundary conditions (69) and the properties presented in the theorem. Then, using the continuity of the angle distribution ω and $\omega(0, 0) = \pi - \psi \in (0, \pi)$, we infer that there exists $\tilde{L}_1, \tilde{L}_2 > 0$ such that $\omega(u, v) \in (0, \pi)$ for all $(u, v) \in [0, \tilde{L}_1]\times[0, \tilde{L}_2]$. Hence, by Theorem 26, the mapping φ satisfies (71) for all $(u, v) \in [0, \tilde{L}_1]\times[0, \tilde{L}_2]$. Then, in the same manner as in the proof of Lemma 24, we obtain that $\varphi : (\mathbb{R}^+)^2 \to Q$ is a Chebyshev net. Suppose finally that $\tilde{\varphi} : (\mathbb{R}^+)^2 \to M$ is a Chebyshev net satisfying the boundary conditions (69). Then, using Property 23, we obtain that the angle distribution $\tilde{\omega} : (\mathbb{R}^+)^2 \to (0, \pi)$ defined by $\tilde{\omega} = \angle(\partial_u\tilde{\varphi}, \partial_v\tilde{\varphi})(u, v)$, for all $(u, v) \in (\mathbb{R}^+)^2$, satisfies the Hazzidakis formula (70). We deduce from Theorem 26 that $\varphi = \tilde{\varphi}$. This concludes the proof. □

Lemma 3 (Homeomorphism)**.** *Let Q be a smooth sector delimited by the two smooth curves $\eta_1 : \mathbb{R}^- \to M$ and $\eta_2 : \mathbb{R}^+ \to M$ intersecting at $p \in M$, and satisfying* (65)*. Assume that $\varphi : (\mathbb{R}^+)^2 \to \varphi[(\mathbb{R}^+)^2] \subset M$ is a smooth mapping* *satisfying* (61)*, and such that* $\varphi(u,0) = \eta_2(u)$*,* $\varphi(0,v) = \eta_1(-v)$ *for all* $(u,v) \in (\mathbb{R}^+)^2$*. Then,* $\varphi : (\mathbb{R}^+)^2 \to Q$ *is a homeomorphism.*

Proof. The proof is obtained in the same manner as in [?]. We just recall here the principal ideas. We denote $\omega : (\mathbb{R}^+)^2 \to \mathbb{R}/2\pi\mathbb{Z}$ the angle distribution defined by $\omega(u,v) = \angle(\partial_u\varphi, \partial_v\varphi)(u,v)$, for all $(u,v) \in (\mathbb{R}^+)^2$. We denote $\kappa_1 : \mathbb{R}^- \to \mathbb{R}$ and $\kappa_2 : \mathbb{R}^+ \to \mathbb{R}$ the geodesic curvatures of η_1 and η_2 respectively. First, using (67), we obtain that

$$\omega(u,0) = \omega(0,0) - \int_0^u \kappa_2 = \pi - \psi - \int_0^u \kappa_2,$$

for all $u \in \mathbb{R}^+$. Then, using hypothesis (65), we deduce that $\omega(u,0) \in (0,\pi)$, for all $u \in \mathbb{R}^+$. In the same manner, we obtain that $\omega(0,v) = \pi - \psi - \int_{-v}^0 \kappa_1 \in (0,\pi)$, for all $v \in \mathbb{R}^+$. Hence, using the continuity of ω, we infer that there exists $\tilde{D} = [0,l_1]\times[0,l_2] \subset (\mathbb{R}^+)^2$, with $l_1, l_2 \in \mathbb{R}^+_*$, such that $\omega(\tilde{D}) \subset (0,\pi)$. Since (61) is satisfied, we infer that $\varphi\big|_{\tilde{D}} : \tilde{D} \to \varphi(\tilde{D}) \subset M$ is a local homeomorphism, so that, up to reducing l_1 and l_2, φ is a homeomorphism. Moreover, since $\omega(\tilde{D}) \subset (0,\pi)$, we deduce that $\angle(\eta_2'(u), \partial_v\varphi(u,0)) \in (0,\pi)$, for all $u \in [0,l_1]$, and $\angle(\eta_1'(-v), \partial_v\varphi(0,v)) \in (0,\pi)$, for all $v \in [0,l_2]$. We conclude that, up to reducing l_1 and l_2, we have $\varphi(\tilde{D}) \subset Q$.

Reasoning by contradiction, we first suppose that φ is not a homeomorphism. Let $U = [0,L_1)\times[0,L_2)$ and $U_{\mathrm{cl}} = [0,L_1]\times[0,L_2]$, with $L_1, L_2 > 0$, be such that $\varphi\big|_U : U \to \varphi(U) \subset M$ is a homeomorphism and such that $\varphi\big|_{U_{\mathrm{cl}}} : U_{\mathrm{cl}} \to \varphi(U_{\mathrm{cl}}) \subset M$ is not a homeomorphism. Using the Hazzidakis formula (68) and hypothesis (65), we easily obtain that $\omega(U_{\mathrm{cl}}) \subset (0,\pi)$. Hence, the mapping $\varphi\big|_{U_{\mathrm{cl}}}$ is a local homeomorphism. Now, suppose that there exist $(u_1,v_1),(u_2,v_2) \in (0,L_1]\times\{L_2\}\cup\{L_1\}\times(0,L_2]$ with $\varphi(u_1,v_1) = \varphi(u_2,v_2)$. Then, the two following cases are possible:

- case 1: $u_1 = u_2 = L_1$ or $v_1 = v_2 = L_2$. We only consider the first subcase, since the reasoning for the second subcase is similar. Then, assuming that $u_1 = u_2 = L_1$, one can see that the Gauss–Bonnet formula applied to the curve $\varphi(\{L_1\}\times[v_1,v_2])$ is in contradiction with (65).

- case 2: $u_1 \neq u_2$ and $v_1 \neq v_2$. In this case, we can suppose, without loss of generality, that $v_1 = L_2$ and $u_2 = L_1$. Then, the Gauss–Bonnet formula applied to the curve $\varphi([u_1,L_1]\times\{L_2\}) \cup \varphi(\{L_1\}\times[L_2,v_2])$ yields a contradiction with (65).

We finally suppose that $\varphi[(\mathbb{R}^+)^2] \not\subset Q$. Then, let $\tilde{U} = [0,\tilde{L}_1)\times[0,\tilde{L}_2)$, with $\tilde{L}_1, \tilde{L}_2 > 0$, be such that $\varphi(\tilde{U}) \subset Q$ and such that there exists $(\tilde{u},\tilde{v}) \in (0,\tilde{L}_1]\times\{\tilde{L}_2\}\cup\{\tilde{L}_1\}\times(0,\tilde{L}_2]$ with $\varphi(\tilde{u},\tilde{v}) \in \partial Q$. Then, $\varphi(\tilde{u},\tilde{v}) \in \eta_1(\mathbb{R}^-)$ or $\varphi(\tilde{u},\tilde{v}) \in \eta_2(\mathbb{R}^+)$ and we obtain again a contradiction between the Gauss–Bonnet formula and (65). This concludes the proof. $\square$

First, unless explicitly mentioned, any curve $\eta : I \subset \mathbb{R} \to M$ we consider in what follows is arc-length parametrized, continuous on I, and piecewise smooth according to the following definition:

Definition 4 (Piecewise smooth curves)**.** *Let $\eta : I \subset \mathbb{R} \to M$ be a continuous curve. We say that the curve η is* piecewise smooth *if there exists a partition of I in the form $\bigcup_{i=1}^{N+1}[a_{i-1}, a_i] = I$, with $a_0 < ... < a_{N+1}$. Suppose moreover that η restricted to (a_{i-1}, a_i) is a smooth curve with all the derivatives having a finite limit from the right at a_{i-1} and from the left at a_i, for all $i \in \{1, ..., N+1\}$. We say that the curve η is piecewise smooth.*

We denote $\angle(X, Y) \in (-\pi, \pi]$ the oriented angle (using the orientation of M) between the vectors X and Y in the tangent plane T_pM at any point $p \in M$. We define the total positive and negative turn angle of continuous piecewise smooth curves (see [**?**]):

Definition 5 (Positive and negative turn angle $\tau_\pm$)**.** *Let $\eta : I \subset \mathbb{R} \to M$ be a continuous piecewise smooth curve on the partition of I defined by $a_0 < ... < a_{N+1}$. Then, for all $i \in \{1, ..., N+1\}$, let $\kappa_i : [a_{i-1}, a_i] \to \mathbb{R}$ be the geodesic curvature of $\eta\big|_{[a_{i-1},a_i]}$ defined by*

$$\kappa_i(s) = \langle \eta''(s), \eta'^{\perp}(s) \rangle_g,$$

with $\eta'^{\perp}(s) \in T_{\eta(s)}M$ the vector such that $\angle(\eta'(s), \eta'^{\perp}(s)) = \frac{\pi}{2}$, for all $s \in [a_{i-1}, a_i]$. Let $\psi_i = \angle(\eta'(a_i^-), \eta'(a_i^+))$, for all $i \in \{1, ..., N\}$. We suppose that $-\pi < \psi_i < \pi$, for all $i \in \{1, ..., N\}$. We define the total positive and negative turn angles *$\tau(\eta)$ by*

$$\tau_+(\eta) = \sum_{i=1}^{N+1} \int_{a_{i-1}}^{a_i} \kappa_i^+ + \sum_{i=1}^{N} \psi_i^+, \quad \tau_-(\eta) = \sum_{i=1}^{N+1} \int_{a_{i-1}}^{a_i} \kappa_i^- + \sum_{i=1}^{N} \psi_i^-, \tag{9}$$

and we define the total turn angle *as $\tau(\eta) = \tau_+(\eta) - \tau_-(\eta)$. (Note that $\tau_\pm(\eta)$ are different from $(\tau(\eta))^\pm$.) We denote $\tau\big(\eta\big|_{[a,b]}\big)$, with $a, b \in \mathbb{R}$ such that $a < b$, the restriction to $[a, b]$ of the total turn angle. Note that, if any, the pointwise turns at a and b are included.*

An illustration of the turn angle is presented in Figure 5. Finally, we denote $\mathrm{int}(D)$ the interior of any set D.

3 Construction of a Chebyshev net from its angle distribution

We now prove that a Chebyshev net can be constructed uniquely from its angle distribution. We start by showing in Subsection 3.1 that the construction of curves from their geodesic curvature, initial point and initial tangent vector is a well-posed problem. We then define, following [**?**], the mapping $\mathcal{I}(\gamma, \omega)$ which, for given boundary curves γ_u and γ_v, associates with any angle distribution ω satisfying the boundary conditions (8) the candidate Chebyshev net φ, with angle distribution ω, satisfying the boundary conditions (3) (Subsection

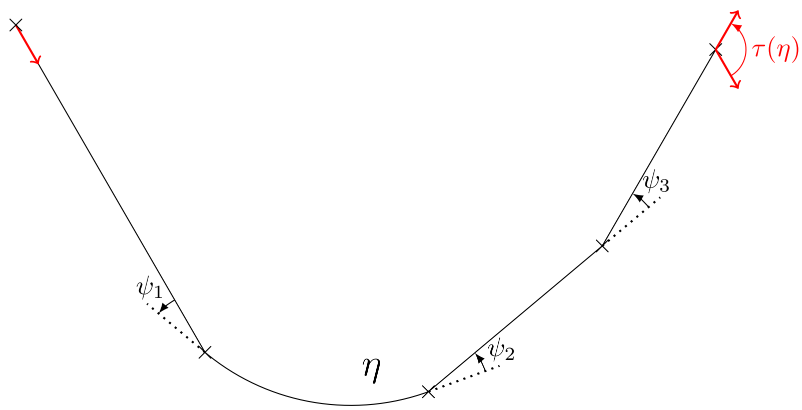

Figure 2: Illustration of the total turn angle $\tau(\eta)$

3.2). The parametrization φ is constructed in such a way that the v-coordinate curves are arc-length parametrized curves with geodesic curvatures satisfying (7b). We moreover show the continuity of the mapping $\mathcal{I}$ with respect to the angle distribution and to the delimiting curves γ_u and γ_v. In Subsection 3.3, we show that the candidate Chebyshev net φ has improved regularity in the u-coordinate whenever ω satisfies the integrability condition (5). Finally, in Subsection 3.4, we prove that φ is indeed a Chebyshev net if ω satisfies the integrability condition and has a sufficient regularity.

3.1 Construction of curves from their geodesic curvature

Proposition 6 (Construction of curves from their geodesic curvature)**.** *Let M be a smooth, open, complete, and simply connected surface. Let $L_{\max} \in \mathbb{R}_*^+$, $L \in (0, L_{\max})$ and $k, r \in \mathbb{N}$. Let $x \in M$, let $V \in T_xM$ be a unit vector, i.e., a vector such that $|V|_g = 1$, and let $\kappa \in C^k([0,L],\mathbb{R})$. Then, there exists a unique (arc-length parametrized) curve $\sigma(x,V,\kappa) := \sigma \in \Gamma^{k+2}([0,L])$ such that $\sigma(0) = x$, $\sigma'(0) = V$, and with geodesic curvature κ.*

Moreover, let $L_1, L_2 \in (0, L_{\max})$, let $x_1, x_2 \in C^r([0,L_1], M)$ be initial position distributions and let $V_1, V_2 \in C^r([0,L_1], TM)$, with $|V_1|_g = |V_2|_g = 1$, be initial derivatives distribution. Let $D_{1,2} = [0,L_1]\times[0,L_2]$ and let $\kappa_1, \kappa_2 \in C^r([0,L_1], C^k([0,L_2],\mathbb{R}))$ be geodesic curvatures. We denote $\sigma_1, \sigma_2 : [0,L_1] \to \Gamma^{k+2}([0,L_2])$ the two families of curves defined by $\sigma_m(\eta,\cdot) := \sigma(x_m(\eta), V_m(\eta), \kappa_m(\eta,\cdot))$, for all $\eta \in [0,L_1]$ and $m \in \{1,2\}$. Then, we have

$$\sigma_1, \sigma_2 \in \Phi^{r,k+2}(D_{1,2}) = C^r([0,L_1], \Gamma^{k+2}([0,L_2])),$$

and, for all $m \in \{1,2\}$,

$$\|\sigma_m\|_{\Phi^{r,k+2}(D_{1,2})} \leq C, \tag{10}$$

where the constant C *depends on* $L_{\max}$*,* $\|x_m\|_{C^r([0,L_1])}$*,* $\|V_m\|_{C^r([0,L_1])}$*, and* $\|\kappa_m\|_{C^r([0,L_1],C^k([0,L_2]))}$*. Finally, we have*

$$\begin{aligned}\|\sigma_1 - \sigma_2\|_{\Phi^0(D_{1,2})} \leq C\Big(&L_2\|\kappa_1 - \kappa_2\|_{C^0(D_{1,2})} \\ &+ \|x_1 - x_2\|_{C^0([0,L_1])} + \|V_1 - V_2\|_{C^0([0,L_1])}\Big),\end{aligned} \quad \textit{if } k = 0 \textit{ and } r = 0, \tag{11a}$$

$$\begin{aligned}\|\sigma_1 - \sigma_2\|_{\Phi^{r,k+2}(D_{1,2})} \leq \tilde{C}\Big(&\|\kappa_1 - \kappa_2\|_{C^r([0,L_1],C^k([0,L_2]))} \\ &+ \|x_1 - x_2\|_{C^r([0,L_1])} + \|V_1 - V_2\|_{C^r([0,L_1])}\Big),\end{aligned} \tag{11b}$$

where the constants C *and* $\tilde{C}$ *depend on* $L_{\max}$*,* $\|x_m\|_{C^r([0,L_1])}$*,* $\|V_m\|_{C^r([0,L_1])}$*, and* $\|\kappa_m\|_{C^r([0,L_1],C^k([0,L_2]))}$*, with* $m \in \{1,2\}$*.*

We illustrate the family of curves σ_1 introduced in Proposition 6 in Figure 3.

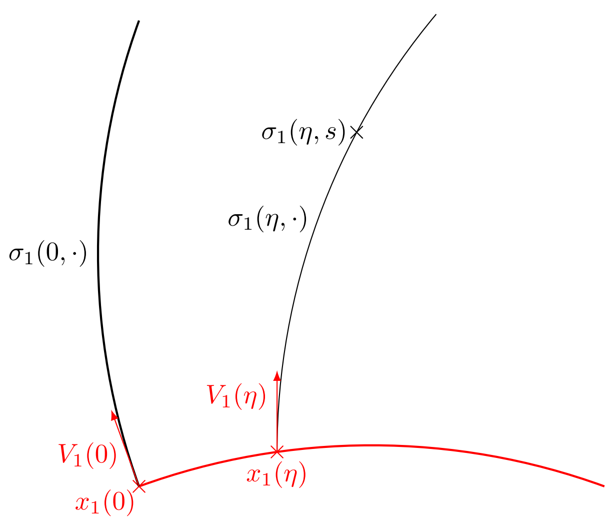

Figure 3: Illustration of the construction of the family of curves σ_1

Proof. To prove the claim, we proceed as follows. We first introduce in Step 1 the geodesic curvature equation that permits to define uniquely a curve from its geodesic curvature. The existence and the uniqueness of the curve is proved in Step 2 using Cauchy–Lipschitz theorem. Then, in order to apply an induction argument, we prove (10) and (11) in the case $k = 0$ and $r = 0$ using GrÃ¶nwall's inequality. The equation satisfied by the derivatives of the solution is presented in a generic form to facilitate the end of the proof in Step 3. Finally, we prove (10) and (11b) in Steps 4 and 5 using induction arguments. Whenever there is no ambiguity, the domain of the variables will be omitted.

Step 1 *(Formulation of the geodesic curvature equation).* Let (σ^1, σ^2), with $\sigma^1, \sigma^2 : [0, L] \to \mathbb{R}$, be the coordinates of the candidate curve $\sigma : [0, L] \to M$ with geodesic curvature κ. We denote $\frac{D}{dt}$ the covariant derivative along the curve σ. The geodesic curvature equation for arc-length parametrized curves gives

$$\frac{D}{dt}\sigma' = \kappa \sigma'^{\perp}, \tag{12}$$

which can be written, in coordinates,

$$\dot{X} = G(X) + \kappa f(X), \tag{13}$$

with $X(0) = (x^1, x^2, V^1, V^2)$. Here, we have denoted

$$X = \begin{pmatrix} \sigma^1 \\ \sigma^2 \\ \sigma'^1 \\ \sigma'^2 \end{pmatrix}, \quad f(X) = \begin{pmatrix} 0 \\ 0 \\ N^1 \\ N^2 \end{pmatrix}, \quad G(X) = \begin{pmatrix} X^3 \\ X^4 \\ -\displaystyle\sum_{1\le i,j\le 2} \Gamma^1_{i,j}(X^1, X^2) X^{2+i} X^{2+j} \\ -\displaystyle\sum_{1\le i,j\le 2} \Gamma^2_{i,j}(X^1, X^2) X^{2+i} X^{2+j} \end{pmatrix},$$

Γ^k_{ij} being the smooth Christoffel symbols, G the smooth geodesic function and N the normal vector defined by

$$N = \begin{pmatrix} N^1 \\ N^2 \end{pmatrix} = \sqrt{\det g(X^1, X^2)}\, g^{-1}(X^1, X^2) \begin{pmatrix} -X^4 \\ X^3 \end{pmatrix}.$$

Step 2 *(Proof for $k = 0$ and $r = 0$).* As $G + \kappa f$ is continuous and locally Lipschitz continuous with respect to X, owing to Cauchy–Lipschitz theorem, there exists a local solution to (13). Moreover, since σ is by construction arc-length parametrized, we have $|\sigma'|_g = 1$ and we infer from (4) that $|\sigma'| \le C_{\mathrm{surf}}$. Hence, the image of X is included in a ball $\mathcal{B}$ whose radius depends only on $L_{\max}$ and C_{surf}. We deduce that the solution is defined on $[0, L]$. Moreover, G and f are Lipschitz continuous with respect to X on $\mathcal{B}$. Hence, $G + \kappa f$ is Lipschitz continuous with respect to X on $\mathcal{B}$, so that the uniqueness of the solution follows. Moreover, we infer from (13) that $\sigma \in \Gamma^2([0, L])$.

Let $m \in \{1, 2\}$ and let $X_m = (\sigma^1_m, \sigma^2_m, \sigma'^1_m, \sigma'^2_m)^t : D_{1,2} \to \mathbb{R}^4$ be such that $X_m(\eta, \cdot)$ is the unique solution to (13) with $\kappa = \kappa_m(\eta, \cdot)$, for all $\eta \in [0, L_1]$. First, owing to [**?**, Chap. V, Th. 2.1], we have $X_m \in C^0([0, L_1], C^1([0, L_2]))$, so that $\sigma_m \in \Phi^{0,2}(D_{1,2})$. Then, since $|\sigma'| \le C_{\mathrm{surf}}$, we infer that $\|\sigma_m\|_{\Phi^{0,2}(D_{1,2})} \le C$, where the constant C depends on $\|\sigma_m(\cdot, 0)\|_{C^0([0,L_1])}$ and $\|X'_m\|_{C^0(D_{1,2})}$. Moreover, due to $|f(X_m)|_g = |\sigma'_m|_g = 1$, we have

$$|f(X_m)| \le C_{\mathrm{surf}}\,. \tag{14}$$

Furthermore, since $X_m(\{\eta\} \times [0, L_2]) \subset \mathcal{B}$, we infer that $G(X_m)$ is bounded and that this bound only depends on $\mathcal{B}$, so that it only depends on $X_m(\eta, 0)$, $L_{\max}$ and C_{surf}, for all $\eta \in [0, L_1]$. We conclude that

$$\|\sigma_m\|_{\Phi^{0,2}(D_{1,2})} \le C,$$

where the constant C depends on $\|x_m\|_{C^0([0,L_1])}$ and $\|\kappa_m\|_{C^0(D_{1,2})}$, and (10) holds with $k = r = 0$. Then, since the restrition of G and f to $\mathcal{B}$ are Lipschitz continuous in the variable X with coefficients denoted respectively C_G and C_f, we have

$$|\dot{X}_1 - \dot{X}_2| \leq (C_G + |\kappa_1| C_f)|X_2 - X_1| + |\kappa_2 - \kappa_1| \ \|f(X_2)\|_{C^0(D_{1,2}))}.$$

Therefore, using GrÃ¶nwall's inequality and (14), we infer that

$$|X_1(t) - X_2(t)| \leq \exp\left[t\big(C_G + C_f \|\kappa_1\|_{C^0(D_{1,2})}\big)\right] \left(|X_2(0) - X_1(0)| + C_{\mathrm{surf}} \int_0^t |\kappa_2 - \kappa_1|\right), \tag{15}$$

for all $t \in [0, L]$. We deduce that (11a) and (11b) are satisfied for $k = r = 0$.

Step 3 *(Differential equation on the derivatives of X).* Let $m \in \{1, 2\}$. Owing to [**?**, Chap. V, Th. 4.1], we have that $\sigma \in \Gamma^{k+2}([0, L])$ and $\sigma_m \in \Phi^{r,k+2}(D_{1,2})$. We use the following notation: for $I = (i_1, i_2) \in \mathbb{N}^2$, we set $\partial^I f(\eta, t) = \partial_\eta^{i_1} \partial_t^{i_2} f(\eta, t)$. We denote $(\mathcal{H}_{r,k})$, with $r, k \in \mathbb{N}^*$, the following induction hypothesis:

for all $I = (i_1, i_2) \in \{0, ..., r\} \times \{0, ..., k\}$ such that $I \neq (0, 0)$, we have

$$\begin{aligned} \partial_t \partial^I X_m = \ & (\nabla G(X_m) + \kappa_m \nabla f(X_m)) \partial^I X_m \\ & + \sum_{0 \leq \alpha \leq i_1} \sum_{0 \leq \beta \leq i_2} F^I_{\alpha,\beta} \left[(\partial^{(p,q)} X_m)_{0 \leq p+\alpha \leq i_1, \ 0 \leq q+\beta \leq i_2, \ p+q < i_1+i_2} \right] \partial^{(\alpha,\beta)} \kappa_m, \end{aligned} \tag{16}$$

where $F^I_{\alpha,\beta} : \mathbb{R}^{4n_0} \to \mathbb{R}^4$, with

$$n_0 = \begin{cases} (i_1 - \alpha + 1)(i_2 - \beta + 1), & \text{if } \alpha + \beta \neq 0, \\ (i_1 + 1)(i_2 + 1) - 1, & \text{otherwise}, \end{cases}$$

are C^∞ functions, for all $(\alpha, \beta) \in \{0, ..., i_1\} \times \{0, ..., i_2\}$.

We denote ∂_1 and ∂_2 the derivatives with respect to η and t respectively. We first obtain from (13) that, for all $m \in \{1, 2\}$ and $i \in \{1, 2\}$,

$$\partial_t \partial_i X_m = \nabla G(X_m) \partial_i X_m + \partial_i \kappa_m f(X_m) + \kappa_m \nabla f(X_m) \partial_i X_m.$$

Hence, (16) is satisfied for $I = (0, 1)$ and $I = (1, 0)$, so that $(\mathcal{H}_{1,1})$ holds. We now suppose that $(\mathcal{H}_{r,k})$ holds for $r, k \in \mathbb{N}^*$. Then, for $I = (i_1, i_2)$ with $i_1 = r$ and $i_2 = k$, we have

$$\begin{aligned} \partial_t \partial_i \partial^I X_m = & (\nabla G(X_m) + \kappa_m \nabla f(X_m)) \partial_i \partial^I X_m + \partial_i \big[\nabla G(X_m) + \kappa_m \nabla f(X_m) \big] \partial^I X_m \\ & + \sum_{0 \leq \alpha \leq i_1} \sum_{0 \leq \beta \leq i_2} \Bigg[\partial_i \left(F^I_{\alpha,\beta} \left[(\partial^{(p,q)} X_m)_{0 \leq p+\alpha \leq i_1, \ 0 \leq q+\beta \leq i_2, \ p+q < i_1+i_2} \right] \right) \partial^{(\alpha,\beta)} \kappa_m \\ & \qquad + F^I_{\alpha,\beta} \partial_i \partial^{(\alpha,\beta)} \kappa_m \Bigg]. \end{aligned}$$

The first term is in the form of the first term on the right-hand side of (16) and the last two terms can be put in the form of the second term on the right-hand side of (16). Equation (16) is then satisfied for $I=(r+1,k)$ and $I=(r,k+1)$, so that $(\mathcal{H}_{r+1,k})$ and $(\mathcal{H}_{r,k+1})$ hold. The claim follows.

Step 4 *(Proof of* (10)*).* Let $m\in\{1,2\}$. First note that we have by definition

$$\|\sigma_m\|_{\Phi^{r,k+2}(D_{1,2})}\leq \sum_{i_1=0}^{r}\sum_{i_2=0}^{k+1}\|\partial_\eta^{i_1}\partial_t^{i_2}X_m\|_{C^0(D_{1,2})}.$$

Then, to obtain (10), we only need to prove that

$$\|\partial^I X_m\|_{C^0(D_{1,2})}\leq C, \tag{17}$$

where the constant C depends on $L_{\max}$, $\|x_m\|_{C^{i_1}([0,L_1])}$, $\|V_m\|_{C^{i_1}([0,L_1])}$, and $\|\kappa_m\|_{C^{i_1}([0,L_1],C^{i_2}([0,L_2]))}$, for all $I=(i_1,i_2)\in\{0,...,r\}\times\{0,...,k+1\}$. We prove (17) by induction on $I\in\{0,...,r\}\times\{0,...,k+1\}$. Hence, we first consider the case $i_2=0$, in which case we have, using (16) and GrÃ¶nwall's inequality,

$$\begin{aligned}|\partial_\eta^{i_1}X_m(t)|\leq& \exp\left[\|\nabla G(X_m)+\kappa_{\sigma_m}\nabla f(X_m)\|_{C^0(D_{1,2})}t\right]\\ &\times\Big(|\partial_\eta^{i_1}X_m(0)|+\sum_{\alpha=0}^{i_1}\int_0^t\left|F_{\alpha,0}^{(i_1,0)}\left[X_m,..,\partial_\eta^{i_1-1}X_m\right]\partial^{(\alpha,0)}\kappa_{\sigma_m}\right|\Big),\end{aligned}$$

for all $i_1\in\{1,...,r\}$. Moreover, from Step 2, we have that (17) holds in the case where $I=(0,0)$. Then, since the functions $F_{\alpha,0}^{(i_1,0)}$ have C^∞-regularity with respect to $(X_m,...,\partial_\eta^{i_1-1}X_m)$, for all $\alpha\in\{0,...,i_1\}$ and $i_1\in\{1,...r\}$, an induction argument on $i_1\in\{0,...,r\}$ permits to prove that (17) holds for all $I\in\{0,...,r\}\times\{0\}$.

Now, noting that $\partial_t\partial^I X_m=\partial_\eta^{i_1}\partial_t^{i_2+1}X_m$, we infer from (16) that

$$\begin{aligned}\|\partial_t\partial^I X_m\|_{C^0(D_{1,2})}\leq&\ \|\nabla G(X_m)+\kappa\nabla f(X_m)\|_{C^0(D_{1,2})}\|\partial^I X_m\|_{C^0(D_{1,2})}\\ +\sum_{0\leq\alpha\leq i_1}\sum_{0\leq\beta\leq i_2}&\left\|F_{\alpha,\beta}^I\left[(\partial^{(p,q)}X_m)_{0\leq p+\alpha\leq i_1,\ 0\leq q+\beta\leq i_2,\ p+q<i_1+i_2}\right]\right\|_{C^0(D_{1,2})}\|\partial^{(\alpha,\beta)}\kappa_m\|_{C^0(D_{1,2})},\end{aligned}$$

for all $I=(i_1,i_2)\in\{0,...,r\}\times\{0,...,k\}$ such that $I\neq(0,0)$. Finally, since (17) holds for all $I=(i_1,0)\in\{0,...,r\}\times\{0\}$ and for $I=(0,1)$ by step 2, and since all the functions $F_{\alpha,\beta}^I$ have C^∞-regularity, the induction argument on I to prove that (17) holds for all $I\in\{0,...,r\}\times\{0,...,k+1\}$ is straightforward. We conclude that (10) holds.

Step 5 *(Proof of* (11b)*).* Let $I=(i_1,i_2)\in\{0,...,r\}\times\{0,...,k\}$ be such that $I\neq(0,0)$ and let $m\in\{1,2\}$. Owing to (10), $\partial^{\tilde{I}}X_m$ is bounded by a constant depending only on $\|\kappa_m\|_{C^r([0,L_1],C^k([0,L_2])}$, $\|x_m\|_{C^r([0,L_1])}$, and $\|V_m\|_{C^r([0,L_1])}$, for all $\tilde{I}\in\{0,...,r\}\times\{0,...,k\}$. Hence, the smooth functions $F_{\alpha,\beta}^I$ are Lipschitz continuous on the compact set defined by the image of the derivatives of X_m, for all $(\alpha,\beta)\in\{0,...,i_1\}\times\{0,...,i_2\}$. We denote $C_{F_{\alpha,\beta}^I}$ their respective Lipschitz coefficients in this compact set and we set $W_m = \partial^I X_m$. Using (16), we easily obtain that

$$
\begin{aligned}
|\partial_t W_2 - \partial_t W_1| \leq & \big(\|\nabla G(X_1)\|_{C^0} + |\kappa_1| \|\nabla f(X_1)\|_{C^0}\big)|W_2 - W_1| \\
& + \Big(\big|\kappa_1 \nabla f(X_1) - \kappa_2 \nabla f(X_2)\big| + \big|\nabla G(X_1) - \nabla G(X_2)\big|\Big)\|W_2\|_{C^0} \\
& + \sum_{\alpha=0}^{i_1}\sum_{\beta=0}^{i_2} C_{F^I_{\alpha,\beta}} \sum_{\substack{p\in\{0,\dots,i_1\},\ q\in\{0,\dots,i_2\} \\ p+q<i_1+i_2}} \big|\partial^{(p,q)}X_2 - \partial^{(p,q)}X_1\big| \|\partial^{(\alpha,\beta)}\kappa_1\|_{C^0} \\
& + \sum_{\alpha=0}^{i_1}\sum_{\beta=0}^{i_2} \Big\|F^I_{\alpha,\beta}\Big[(\partial^{(p,q)}X_2)_{0\leq p+\alpha\leq i_1,\ 0\leq q+\beta\leq i_2,\ p+q<i_1+i_2}\Big]\Big\|_{C^0} |\partial^{(\alpha,\beta)}\kappa_1 - \partial^{(\alpha,\beta)}\kappa_2|,
\end{aligned}
$$

where C^0 refers to the norm $C^0(D_{1,2})$. Using GrÃ¶nwall's inequality, we infer that

$$
\begin{aligned}
|W_1 - W_2|(\eta,t) \leq & \exp\big[\|\nabla G(X_1) + \kappa_1 \nabla f(X_1)\|_{C^0} t\big] \times \\
& \Bigg(|W_1 - W_2|(\eta,0) + \|W_2\|_{C^0} \max_{i\in\{1,2\}} \max\big(\|\kappa_i\|_{C^0}, \|\nabla f(X_i)\|_{C^0}, 1\big) \int_0^t \big[|\kappa_2 - \kappa_1| + \big(C_{\nabla f} + C_{\nabla G}\big)|X_2 - X_1|\big] \\
& + \sum_{\alpha=0}^{i_1}\sum_{\beta=0}^{i_2} C_{F^I_{\alpha,\beta}} \|\partial^{(\alpha,\beta)}\kappa_1\|_{C^0} \sum_{\substack{p\in\{0,\dots,i_1\},\ q\in\{0,\dots,i_2\} \\ p+q<i_1+i_2}} \int_0^t |\partial^{(p,q)}X_2 - \partial^{(p,q)}X_1| \\
& + \sum_{\alpha=0}^{i_1}\sum_{\beta=0}^{i_2} \Big\|F^I_{\alpha,\beta}\Big[(\partial^{(p,q)}X_2)_{0\leq p+\alpha\leq i_1,\ 0\leq q+\beta\leq i_2,\ p+q<i_1+i_2}\Big]\Big\|_{C^0} \int_0^t \big|\partial^{(\alpha,\beta)}\kappa_1 - \partial^{(\alpha,\beta)}\kappa_2\big|\Bigg),
\end{aligned}
$$

where C^0 refers to the norm $C^0(D_{1,2})$, and $C_{\nabla f}$ and $C_{\nabla G}$ are the Lipschitz constants of ∇f and ∇G, respectively. Then, we obtain using (10) that

$$
\begin{aligned}
\|\sigma_1 - \sigma_2\|_{\Phi^{i_1,i_2+1}(D_{1,2})} \leq C\Big(& \|\kappa_1 - \kappa_2\|_{C^{i_1}([0,L_1],C^{i_2}([0,L_2]))} + \|x_1 - x_2\|_{C^{i_1}([0,L_1])} \\
& + \|V_1 - V_2\|_{C^{i_1}([0,L_1])} + \Sigma_1 + \Sigma_2\Big),
\end{aligned}
$$

where the constant C depends on $L_{\max}$, $\|x_l\|_{C^{i_1}([0,L_1])}$, $\|V_l\|_{C^{i_1}([0,L_1])}$, and $\|\kappa_l\|_{C^{i_1}([0,L_1],C^{i_2}([0,L_2]))}$, with $l \in \{1,2\}$, and

$$
\Sigma_1 = \begin{cases} \|\sigma_1 - \sigma_2\|_{\Phi^{i_1-1,i_2+1}(D_{1,2})}, & \text{if } i_1 > 0, \\ 0, & \text{otherwise,} \end{cases} \qquad \Sigma_2 = \begin{cases} \|\sigma_1 - \sigma_2\|_{\Phi^{i_1,i_2}(D_{1,2})}, & \text{if } i_2 > 0, \\ 0, & \text{otherwise.} \end{cases}
$$

Hence, using (16), we obtain in the same manner that

$$
\begin{aligned}
\|\sigma_1 - \sigma_2\|_{\Phi^{i_1,i_2+2}(D_{1,2})} \leq C\Big(& \|\kappa_1 - \kappa_2\|_{C^{i_1}([0,L_1],C^{i_2}([0,L_2]))} + \|x_1 - x_2\|_{C^{i_1}([0,L_1])} \\
& + \|V_1 - V_2\|_{C^{i_1}([0,L_1])} + \Sigma_1 + \Sigma_2\Big),
\end{aligned}
$$

where the constant C depends on $L_{\max}$, $\|x_l\|_{C^r([0,L_1])}$, $\|V_l\|_{C^r([0,L_1])}$, and $\|\kappa_l\|_{C^r([0,L_1],C^k([0,L_2]))}$, with $l \in \{1,2\}$. We then obtain (11) by a straightforward induction argument on $I = (i_1, i_2) \in \{0,...,r\}\times\{0,...,k\}$, recalling that the case $I = (0,0)$ was proved in Step 2. This concludes the proof of the proposition.

□

3.2 Construction of the parametrization

Let $\mathcal{R}_x(\theta)$ be the rotation of angle θ in the tangent plane T_xM at $x \in M$ and let $D = [0, L_u] \times [0, L_v]$, with $L_u, L_v \in \mathbb{R}_*^+$. Let $r, k \in \mathbb{N}$. Using the notation of Proposition 6, given an angle distribution $\omega \in \Theta_\gamma^{r+1,k+1}(D)$ satisfying the boundary conditions (8) given by the two curves $\gamma = (\gamma_u, \gamma_v) \in \Gamma^{r+2}([0, L_u]) \times \Gamma^{k+2}([0, L_v])$, we set

$$x(\eta) = \gamma_u(\eta) \in M, \quad V(\eta) = \mathcal{R}_{\gamma_u(\eta)}(\omega(\eta, 0))\gamma_u'(\eta) \in T_{\gamma_u(\eta)}M, \tag{18}$$

and $\kappa(\eta, s) = \partial_v \omega(\eta, s)$, for all $(\eta, s) \in D$. Let $\varphi_\omega : D \to M$ be the family of curves such that the curve $\varphi_\omega(\eta, \cdot) \in \Gamma^{k+2}([0, L_v])$ has initial position $\varphi_\omega(\eta, 0) = x(\eta)$, initial tangent vector $\partial_v \varphi_\omega(\eta, 0) = V(\eta)$, and geodesic curvature $\kappa(\eta, \cdot)$, for all $\eta \in [0, L_u]$ (see Figure 4). Note that the mapping φ_ω also depends on the curves $\gamma = (\gamma_u, \gamma_v)$ but since these curves are kept fixed in what follows, we do not mention them explicitly. We denote

$$\begin{aligned} \mathcal{I}(\gamma, \cdot) : \Theta_\gamma^{r+1,k+1}(D) &\to \Phi^{r+1,k+2}(D), \\ \omega &\mapsto \varphi_\omega, \end{aligned}$$

the mapping that associates with each angle distribution ω with $\Theta^{r+1,k+1}$-regularity and satisfying the boundary conditions (8), the mapping $\varphi_\omega : D \to M$.

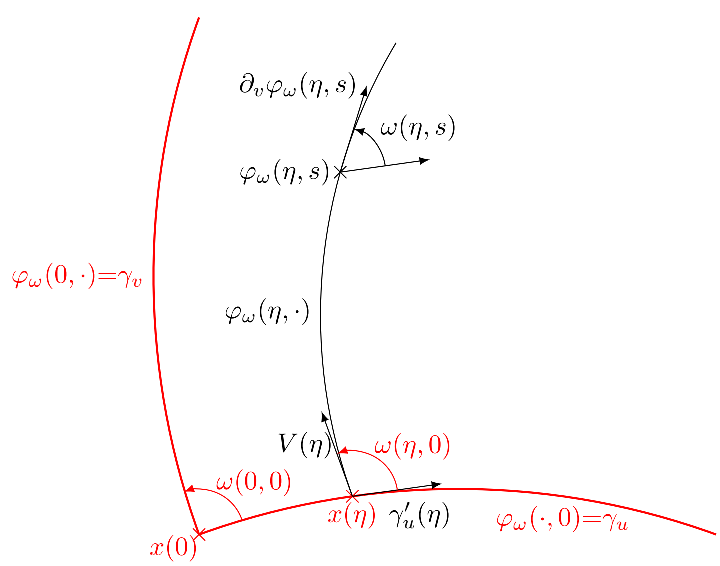

Figure 4: Illustration of the construction of the parametrization φ_ω

Proposition 7 (Continuity of the construction)**.** *Let M be a smooth, open, complete, and simply connected surface, let $k, r \in \mathbb{N}$, and let $D = [0, L_u] \times [0, L_v]$,* *with $L_u, L_v \in \mathbb{R}_*^+$. For all $\gamma = (\gamma_u, \gamma_v) \in \Gamma^{r+2}([0, L_u]) \times \Gamma^{k+2}([0, L_v])$, the mapping $\mathcal{I}(\gamma, \cdot)$ is well defined. Moreover, let $\gamma_1, \gamma_2 \in \Gamma^{r+2}([0, L_u]) \times \Gamma^{k+2}([0, L_v])$, with $\gamma_1 = (\gamma_{u,1}, \gamma_{v,1})$ and $\gamma_2 = (\gamma_{u,2}, \gamma_{v,2})$, be such that $\gamma_{u,1}(0) = \gamma_{u,2}(0) = \gamma_{v,1}(0) = \gamma_{v,2}(0)$ and such that $\gamma'_{u,1}(0) = \gamma'_{u,2}(0)$ and $\gamma'_{v,1}(0) = \gamma'_{v,2}(0)$. Consider the two angle distributions $\omega_m \in \Theta^{r+1,k+1}_{\gamma_m}(D)$, for $m \in \{1, 2\}$. Then, we have*

$$\|\mathcal{I}(\gamma_m, \omega_m)\|_{\Phi^{r+1,k+2}(D)} \leq C \tag{19}$$

where the constant C depends on L_u, L_v, $\|\gamma_{u,m}\|_{\Gamma^s([0,L_u])}$, and $\|\omega_m\|_{\Theta^{r+1,k+1}(D)}$, with $s = \max(r+1, 2)$. Moreover, for all $L \in (0, L_v]$, setting $D_L = [0, L_u] \times [0, L]$, we have

$$\begin{aligned}
\|\mathcal{I}(\gamma_1, \omega_1) - \mathcal{I}(\gamma_2, \omega_2)\|_{\Phi^0(D_L)} \leq\ & C\Big(L\|\omega_1 - \omega_2\|_{\Theta^1(D_L)} \\
& + \|\gamma_{u,2} - \gamma_{u,1}\|_{\Gamma^2([0,L_u])}\Big), \quad \textit{if } r = 0 \textit{ and } k = 0,
\end{aligned} \tag{20a}$$

$$\begin{aligned}
\|\mathcal{I}(\gamma_1, \omega_1) - \mathcal{I}(\gamma_2, \omega_2)\|_{\Phi^{1,k+2}(D_L)} \leq C\Big(& \|\omega_1 - \omega_2\|_{\Theta^{1,k+1}(D_L)} \\
& + \|\gamma_{u,2} - \gamma_{u,1}\|_{\Gamma^2([0,L_u])}\Big), \quad \textit{if } r = 0,
\end{aligned} \tag{20b}$$

$$\begin{aligned}
\|\mathcal{I}(\gamma_1, \omega_1) - \mathcal{I}(\gamma_2, \omega_2)\|_{\Phi^{r+1,k+2}(D_L)} \leq C\Big(& \|\omega_1 - \omega_2\|_{\Theta^{r+1,k+1}(D_L)} \\
& + \|\gamma_{u,2} - \gamma_{u,1}\|_{\Gamma^{r+1}([0,L_u])}\Big), \quad \textit{if } r > 0,
\end{aligned} \tag{20c}$$

where the constant C depends on L_u, L_v, $\|\gamma_{u,i}\|_{\Gamma^s([0,L_u])}$, and $\|\omega_i\|_{\Theta^{r+1,k+1}(D)}$, with $i \in \{1, 2\}$ and $s = \max(r + 1, 2)$.

Proof. Since the construction of the mapping φ_ω is the same as that in the construction of Proposition 6, we only need to prove that the boundary conditions used in the construction of φ_ω are smooth enough. We denote $\kappa_{u,1} : [0, L_u] \to \mathbb{R}$ and $\kappa_{u,2} : [0, L_v] \to \mathbb{R}$ the geodesic curvatures of the curves $\gamma_{u,1}$ and $\gamma_{u,2}$ respectively. Using the notation of Proposition 6, for all $m \in \{1, 2\}$ and $u \in [0, L_u]$, we set $x_m(u) = \gamma_{u,m}(u)$ and

$$V_m(u) = \mathcal{R}_{\gamma_{u,m}(u)}(\omega_m(u, 0))\gamma'_{u,m}(u) = \cos(\omega_m(u, 0))\gamma'_{u,m}(u) + \sin(\omega_m(u, 0))\gamma'^{\perp}_{u,m}(u), \tag{21}$$

where $\gamma'^{\perp}_{u,m}$ is the direct $\frac{\pi}{2}$-rotation of $\gamma'_{u,m}$. Since $|V_m|_g = 1$, we infer from (4) that $\|V_m\|_{C^0([0,L_u])} \leq C_{\text{surf}}$, for all $m \in \{1, 2\}$. Furthermore, using that $\omega_1 \in \Theta^{r+1,k+1}_{\gamma_1}(D)$ and $\omega_2 \in \Theta^{r+1,k+1}_{\gamma_2}(D)$ both satisfy the boundary conditions (8), we obtain

$$|\omega_2 - \omega_1|(u, 0) = \Big|\int_0^u \kappa_{u,2} - \int_0^u \kappa_{u,1}\Big| \leq \int_0^u |\kappa_{u,2} - \kappa_{u,1}|.$$

Hence, using (21), we obtain by a straightforward computation that

$$\|V_2 - V_1\|_{C^0([0,L_u])} \leq C\Big(\|\kappa_{u,2} - \kappa_{u,1}\|_{C^0([0,L_u])} + \|\gamma'_2 - \gamma'_1\|_{C^0([0,L_u])}\Big). \tag{22}$$

Let $m \in \{1,2\}$. Since ω_m satisfies the boundary conditions (8) given by the arc-length parametrized curve $\gamma_{u,m}$, we infer from the definition of geodesic curvature (**??**) that

$$\frac{D}{du}\gamma'_{u,m} = \kappa_{u,m}\gamma'^{\perp}_{u,m} = -\partial_u\omega_m(\cdot,0)\gamma'^{\perp}_{u,m},$$
$$\frac{D}{du}\gamma'^{\perp}_{u,m} = -\kappa_{u,m}\gamma'_{u,m} = \partial_u\omega_m(\cdot,0)\gamma'_{u,m},$$

where $\frac{D}{du}$ is the covariant derivative along the curve $\gamma_{u,m}$. Hence, we deduce from (21) that

$$\begin{aligned}\frac{D}{du}V_m(u) = \quad & \partial_u\omega_m(u,0)\left(\cos(\omega_m(u,0))\gamma'^{\perp}_{u,m} - \sin(\omega_m(u,0))\gamma'_{u,m}\right) \\ & + \cos(\omega_m(u,0))\frac{D}{du}\gamma'_{u,m} + \sin(\omega_m(u,0))\frac{D}{du}\gamma'^{\perp}_{u,m} = 0, \end{aligned} \tag{23}$$

for all $u \in [0,L_u]$. Therefore, in the same manner as in Proposition 6, using that V_m is bounded, (22) and (23), we obtain

$$\|V_m\|_{C^{r+1}([0,L_u])} \leq C, \quad \|V_1 - V_2\|_{C^{r+1}([0,L_u])} \leq \tilde{C}\|\gamma_{u,1} - \gamma_{u,2}\|_{\Gamma^s([0,L_u])}, \tag{24}$$

where $s = \max(r+1,2)$, the constant C depends on L_u, L_v, and $\|\gamma_{u,m}\|_{\Gamma^s([0,L_u])}$, and the constant $\tilde{C}$ depends on L_u, L_v, and $\|\gamma_{u,l}\|_{\Gamma^s([0,L_u])}$, with $l \in \{1,2\}$. Moreover, since $x_m = \gamma_{u,m}$, we have

$$\|x_m\|_{C^{r+1}([0,L_u])} = \|\gamma_{u,m}\|_{\Gamma^{r+1}([0,L_u])}, \quad \|x_1 - x_2\|_{C^{r+1}([0,L_u])} = \|\gamma_{u,1} - \gamma_{u,2}\|_{\Gamma^{r+1}([0,L_u])}. \tag{25}$$

We set $\kappa^{\mathrm{map}}_{v,m}(u,\cdot) := \partial_v\omega_m(u,\cdot) : [0,L_v] \to \mathbb{R}$, for all $u \in [0,L_u]$. We conclude the proof using (24), (25) and Proposition 6 with regularity $(r+1,k)$, i.e., with $x_1, x_2 \in C^{r+1}([0,L_u])$, $V_1, V_2 \in C^{r+1}([0,L_u])$ and $\kappa^{\mathrm{map}}_{v,1}, \kappa^{\mathrm{map}}_{v,2} \in C^{r+1}([0,L_u], C^k([0,L_v]))$. □

3.3 Regularity of the candidate Chebyshev nets

Let us now consider the case where we have the same regularity in both coordinates in the data permitting the construction of φ_ω, i.e., $r = k$. Then, taking $r = k$ in Proposition 7 gives an optimal estimate in the second coordinate regularity but a suboptimal estimate in the first coordinate, since we expect C^{k+2}-regularity in both directions. We show in this section that, whenever the mapping constructed satisfies the integrability equation (5), it has indeed the expected regularity in the first variable as well.

Proposition 8 (Regularity of mappings satisfying the integrability condition)**.** *Keeping the assumptions of Proposition 7 with $k = r \in \mathbb{N}$, we moreover suppose that $\omega_m \in \Theta^{k+1}(D)$ satisfies the Hazzidakis formula* (6) *with $\varphi_m := \mathcal{I}(\gamma_m, \omega_m)$,* *i.e.*,

$$\begin{aligned}\omega_m(u,v) = \angle\big(\gamma'_{u,m}(0),\gamma'_{v,m}(0)\big) - \int_0^u \kappa_{u,m} + \int_0^v \kappa_{v,m} \\ - \int_0^u\int_0^v K\big[\mathcal{I}(\gamma_m,\omega_m)(t,s)\big]\sin(\omega_m(t,s))\mathrm{d}t\mathrm{d}s,\end{aligned} \tag{26}$$

for all $m\in\{1,2\}$. *Then, we have* $\varphi_m\in\Phi^{k+2}(D)$ *and*

$$\|\mathcal{I}(\gamma_m,\omega_m)\|_{\Phi^{k+2}(D)} \le C, \tag{27}$$

where the constant C *depends on* L_u, L_v, $\|\gamma_{u,m}\|_{\Gamma^{k+2}([0,L_u])}$, *and* $\|\omega_m\|_{\Theta^{k+1}(D)}$, *for all* $m\in\{1,2\}$. *Moreover, we have*

$$\begin{aligned}\|\mathcal{I}(\gamma_1,\omega_1)-\mathcal{I}(\gamma_2,\omega_2)\|_{\Phi^{k+2}(D)} \le C\Big(\|\omega_1-\omega_2\|_{\Theta^{k+1}(D)} \\ + \|\gamma_{2,u}-\gamma_{1,u}\|_{\Gamma^{k+2}([0,L_u])}\Big)\end{aligned} \tag{28}$$

where the constant C *depends on* L_u, L_v, $\|\gamma_{u,i}\|_{\Gamma^{k+2}([0,L_u])}$ *and* $\|\omega_i\|_{\Theta^{k+1}(D)}$, *for* $i\in\{1,2\}$.

Proof. Let $m\in\{1,2\}$. First, owing to Proposition 7, we have $\varphi_m=\mathcal{I}(\gamma_m,\omega_m)\in\Phi^{k+1,k+2}(D)$. We denote $\kappa^{\mathrm{map}}_{v,m}=\partial_v\omega_m\in C^{k+1}([0,L_u],C^k([0,L_v]))$ the geodesic curvatures of the v-coordinate curves of φ_m. Let us remark that, since $\omega_m\in\Theta^1(D)$ satisfies the Hazzidakis formula (26), ω_m satisfies the integrability condition (5), i.e.,

$$\partial_{uv}\omega_m = -K(\varphi_m)\sin(\omega_m). \tag{29}$$

Then, the claim is obtained by remarking that (29) implies that the geodesic curvatures $\kappa^{\mathrm{map}}_{v,m}$ of the v-coordinate curves satisfy

$$\kappa^{\mathrm{map}}_{v,m}\in C^{k+2}([0,L_u],C^k([0,L_v])). \tag{30}$$

Indeed, for all $i,j\in\{0,...,k\}$, we have

$$\partial_u^{i+1}\partial_v^j\kappa^{\mathrm{map}}_{v,m} = \partial_u^i\partial_v^j\big(\partial_u\partial_v\omega_m\big) = \partial_u^i\partial_v^j\Big(-K(\varphi_m)\sin(\omega_m)\Big). \tag{31}$$

Hence, since $\omega_m\in\Theta^{k+1}_{\gamma_m}(D)$ and $\varphi_m\in\Phi^{k+1,k+2}(D)$ and since K is smooth, we infer that (30) holds and that (31) is satisfied for all $i\in\{0,...,k+1\}$ and $j\in\{0,...,k\}$. Furthermore, we have

$$\partial_u^2\kappa^{\mathrm{map}}_{v,m} = \partial_u\big(-K(\varphi_m)\sin(\omega_m)\big) = -\sin(\omega_m)\nabla K(\varphi_m)\partial_u\varphi_m - K(\varphi_m)\cos(\omega_m)\partial_u\omega_m, \tag{32}$$

so that we easily obtain $\big\|\partial_u^2\kappa^{\mathrm{map}}_{v,m}\big\|_{C^k([0,L_u],C^k([0,L_v]))}\le C$, where the constant C depends on L_u, L_v, $\|\omega_m\|_{\Theta^{k+1}(D)}$, and $\|\varphi_m\|_{\Phi^{k+1}(D)}$. Using moreover (19), we conclude that the constant C only depends on L_u, L_v, $\|\omega_m\|_{\Theta^{k+1}(D)}$, and $\|\gamma_{u,m}\|_{\Gamma^s([0,L_u])}$, with $s=\max(k+1,2)$. We infer that

$$\big\|\kappa^{\mathrm{map}}_{v,m}\big\|_{C^{k+2}([0,L_u],C^k([0,L_v]))}\le C, \tag{33}$$

where the constant C depends on L_u, L_v, $\|\omega_m\|_{\Theta^{k+1}(D)}$, and $\|\gamma_{u,m}\|_{\Gamma^s([0,L_u])}$. Recalling that $s = \max(k+1, 2)$, we deduce from (32) and (20) that

$$\begin{aligned}\left\|\partial_u^2\kappa_{v,2}^{\mathrm{map}} - \partial_u^2\kappa_{v,1}^{\mathrm{map}}\right\|_{C^k([0,L_u],C^k([0,L_v]))} &\leq \tilde{C}\Big(\|\omega_2-\omega_1\|_{\Theta^{k+1}(D)} + \|\varphi_2-\varphi_1\|_{\Phi^{k+1}(D)}\Big)\\ &\leq C\Big(\|\omega_2-\omega_1\|_{\Theta^{k+1}(D)} + \|\gamma_{u,2}-\gamma_{u,1}\|_{\Gamma^s([0,L_u])}\Big),\end{aligned}$$

where the constants $C, \tilde{C}$ depend on L_u, L_v, $\|\gamma_{u,i}\|_{\Gamma^s([0,L_u])}$, and $\|\omega_i\|_{\Theta^{k+1}(D)}$, with $i \in \{1,2\}$. We conclude that

$$\left\|\kappa_{v,2}^{\mathrm{map}} - \kappa_{v,1}^{\mathrm{map}}\right\|_{C^{k+2}([0,L_u],C^k([0,L_v]))} \leq C\Big(\|\omega_2-\omega_1\|_{\Theta^{k+1}(D)} + \|\gamma_{u,2}-\gamma_{u,1}\|_{\Gamma^s([0,L_u])}\Big), \tag{34}$$

where the constant C depends on L_u, L_v, $\|\gamma_{u,i}\|_{\Gamma^s([0,L_u])}$, and $\|\omega_i\|_{\Theta^{k+1}(D)}$, with $i \in \{1,2\}$. As in the proof of Proposition 7, for all $u \in [0, L_u]$, we set $x_m(u) = \gamma_{u,m}(u)$ and

$$V_m(u) = \mathcal{R}_{\gamma_{u,m}(u)}(\omega_m(u,0))\gamma'_{u,m}(u).$$

Before we apply Proposition 6 with regularity $(k+2, k)$, let us first note that, as in the proof of Proposition 7, the estimate on $x_1, x_2 \in C^{k+2}([0, L_u])$, $V_1, V_2 \in C^{k+2}([0, L_u])$ is given by (24), (25). Hence, the claim follows from the estimate on $\kappa_{v,1}^{\mathrm{map}}, \kappa_{v,2}^{\mathrm{map}} \in C^{k+2}([0, L_u], C^k([0, L_v]))$ given by (33) and (34), and Proposition 6 with regularity $(k+2, k)$. $\square$

3.4 From integrability conditions to Chebyshev nets

Proposition 9 (From integrability conditions to Chebyshev nets)**.** *Let M be a smooth, open, complete, and simply connected surface, let $D = [0, L_u] \times [0, L_v]$, with $L_u, L_v \in \mathbb{R}_*^+$, and let $k \geq 1$. Let $\gamma = (\gamma_u, \gamma_v) \in \Gamma^{k+2}([0, L_u]) \times \Gamma^{k+2}([0, L_v])$. Assume that $\omega \in \Theta_\gamma^{k+1}(D)$ is an angle distribution satisfying the integrability condition* (5)*, with $\varphi := \mathcal{I}(\gamma, \omega) \in \Phi^{k+2}(D)$. Suppose moreover that $0 < \omega(u,v) < \pi$, for all $(u,v) \in D$. Then, the mapping φ is a Chebyshev net in the sense that it satisfies* (2)*.*

Proof. First, owing to Proposition 8, we have that $\varphi \in \Phi^{k+2}(D)$, so that φ has C^3-regularity. Since the v-coordinate curves are arc-length parametrized curves, we have by construction $|\partial_v\varphi|_g(u,v) = 1$, and we set $R(u,v) = |\partial_u\varphi|_g(u,v)$, for all $(u,v) \in D$. Then, since γ_u is an arc-length parametrized curve, we have $R(u,0) = 1$, for all $u \in [0, L_u]$. The proof amounts to showing that

$$\exists L \in (0, L_v], \quad \partial_v R(u,v) = 0, \quad \forall (u,v) \in [0, L_u] \times [0, L]. \tag{35}$$

Indeed, suppose that (35) is satisfied and denote $I \subset [0, L_v]$ the maximal interval on which we have $R(u,v) = 1$, for all $(u,v) \in [0, L_u] \times I$. Owing to (35), we first have that $[0, L] \subset I$, so that I is nonempty. Moreover, suppose that $[0, \tilde{L}_0] \subset I$, for some $\tilde{L}_0 \in (0, L_v]$. Then, since the angle distribution $\omega\big|_{[0,L_u]\times[\tilde{L}_0,L_v]}$ and the mapping $\varphi\big|_{[0,L_u]\times[\tilde{L}_0,L_v]}$ satisfy the hypotheses of the proposition, we infer from

(35) that there exists $\tilde{L}_1 \in (\tilde{L}_0, L_v]$ such that $[0, \tilde{L}_1] \subset I$. Hence, I is open in $[0, L_v]$, and we deduce from the continuity of R that I is closed. Therefore, (35) implies the claim.

We now suppose that $L \in (0, L_v]$ is small enough so that $R(u,v) > 0$, for all $(u,v) \in [0,L_u]\times[0,L]$, we set $D_L = [0,L_u]\times[0,L]$, and we set $X_1(u,v) := \left\langle \frac{\partial_u \varphi}{R}(u,v), \partial_v\varphi(u,v)\right\rangle_g$ and $X_2(u,v) := \left\langle \frac{\partial_u \varphi^\perp}{R}(u,v), \partial_v\varphi(u,v)\right\rangle_g$, for all $(u,v) \in D_L$. Note that by definition $X_1^2 + X_2^2 = 1$ in D_L. Recalling that $R(\cdot,0) = 1$ by construction of φ, we have

$$X_1(u,0) = \left\langle \partial_u\varphi, \partial_v\varphi\right\rangle_g(u,0) = \cos\big(\omega(u,0)\big), \tag{36a}$$

$$X_2(u,0) = \left\langle \partial_u\varphi^\perp, \partial_v\varphi\right\rangle_g(u,0) = \sin\big(\omega(u,0)\big), \tag{36b}$$

for all $u \in [0, L_u]$. Hence, since X_2 is continuous, up to reducing L, we have $X_2 > 0$ due to the assumption that $0 < \omega < \pi$.

We prove (35) as follows. We show that an identification of the Gaussian curvature computed using the local coordinates φ with (5) leads to the following integro-differential equation on R in the v-coordinates:

$$\partial_{vv}R = \partial_v\omega\partial_v RT + \frac{(\partial_v R)^2}{R}T^2 - K(\varphi)X_2\Big(\sin\omega\int_0^v \frac{\partial_v R}{X_2}\sin\omega + \cos\omega\int_0^v\frac{\partial_v R}{X_2}\cos\omega\Big), \tag{37}$$

with $T = \dfrac{X_1}{X_2}$. We show that $\partial_v R = 0$ is the unique solution to (37) to prove the claim (35). First, we compute in Step 1 the initial conditions satisfied by $\partial_v R$ necessary to obtain the uniqueness of the solution of (37), i.e., we prove that $\partial_v R(\cdot, 0) = 0$. Then, we compute in Steps 2-4 the Gaussian curvature K in terms of R, X_1 and X_2. Using (5), we reduce the Gaussian curvature to (37) in Step 5 and we conclude in Step 6.

Step 1 *(Initial conditions)*. We prove that $\partial_v R(u,0) = 0$, for all $u \in [0, L_u]$.

We denote in what follows $D_{\partial_u}Y$ and $D_{\partial_v}Y$ the covariant derivative of the vector field Y in the directions $\partial_u\varphi$ and $\partial_v\varphi$, respectively. First, since $R(\cdot,0) = 1$, we have that $\partial_u R(u,0) = 0$, for all $u \in [0, L_u]$. Moreover, since ω satisfies the boundary conditions (8), we have that $D_{\partial_u}\partial_u\varphi(u,0) = -[\partial_u\omega\partial_u\varphi^\perp](u,0)$. Combining these results with (36), we obtain

$$\begin{aligned}
\frac{1}{2}\partial_v(R^2)(u,0) &= \left\langle D_{\partial_v}\partial_u\varphi, \partial_u\varphi\right\rangle_g(u,0) = \partial_u(RX_1)(u,0) - \left\langle D_{\partial_u}\partial_u\varphi, \partial_v\varphi\right\rangle_g(u,0)\\
&= R(u,0)\partial_u X_1(u,0) + \left\langle \partial_u\omega\partial_u\varphi^\perp, \partial_v\varphi\right\rangle_g(u,0)\\
&= \frac{d}{du}[\cos(\omega(u,0))] + \partial_u\omega(u,0)\sin(\omega(u,0)) = 0,
\end{aligned}$$

for all $u \in [0, L_u]$. Therefore, we have $\partial_v R(u,0) = 0$, for all $u \in [0, L_u]$.

Step 2 *(Computation of the Gaussian curvature (1^{st} part))*. We prove the following relations on the parallel transport of vectors:

$$D_{\partial_u}\partial_u\varphi = \left(\frac{\partial_u R}{R} + \frac{X_1}{X_2^2}(\partial_v R - \partial_u X_1)\right)\partial_u\varphi + \left(\frac{R}{X_2^2}(\partial_u X_1 - \partial_v R)\right)\partial_v\varphi, \tag{38a}$$

$$D_{\partial_v}\partial_u\varphi = D_{\partial_u}\partial_v\varphi = \frac{\partial_v R}{RX_2^2}\partial_u\varphi - \frac{\partial_v R X_1}{X_2^2}\partial_v\varphi, \tag{38b}$$

$$D_{\partial_v}\partial_v\varphi = \partial_v\omega\partial_v\varphi^\perp. \tag{38c}$$

The results (38a) and (38b) easily follow from the identities

$$\begin{aligned}
&\langle D_{\partial_u}\partial_u\varphi, \partial_u\varphi\rangle_g = R\partial_u R,\\
&\langle D_{\partial_u}\partial_u\varphi, \partial_v\varphi\rangle_g = \partial_u(RX_1) - \langle \partial_u\varphi, D_{\partial_v}\partial_u\varphi\rangle_g = \partial_u R X_1 + R\partial_u X_1 - R\partial_v R,\\
&\langle D_{\partial_u}\partial_v\varphi, \partial_u\varphi\rangle_g = R\partial_v R,\\
&\langle D_{\partial_u}\partial_v\varphi, \partial_v\varphi\rangle_g = 0,
\end{aligned}$$

where we have used in the last equality the fact that $|\partial_v\varphi|_g = 1$. Equation (38c) is obtained using that the v-coordinate curves of φ are arc-length parametrized and have by construction a geodesic curvature given by $\partial_v\omega$.

Step 3 *(Computation of the Gaussian curvature (2^{nd} part)).* We prove that X_1 and X_2 satisfy

$$\partial_v X_1 = -\partial_v\omega X_2 - \frac{\partial_v R}{R}X_1, \tag{39a}$$

$$\partial_v X_2 = \partial_v\omega X_1 + \frac{X_1}{X_2}\frac{\partial_v R}{R}X_1. \tag{39b}$$

First, using that $\langle D_{\partial_v}\partial_u\varphi, \partial_v\varphi\rangle = 0$ and (38c), we obtain

$$\begin{aligned}
\partial_v X_1 &= \langle D_{\partial_v}\partial_v\varphi, \tfrac{\partial_u\varphi}{R}\rangle_g + \langle D_{\partial_v}\tfrac{\partial_u\varphi}{R}, \partial_v\varphi\rangle_g = \partial_v\omega\langle\partial_v\varphi^\perp, \tfrac{\partial_u\varphi}{R}\rangle_g + \langle -\frac{\partial_v R}{R^2}\partial_u\varphi + \frac{1}{R}D_{\partial_v}\partial_u\varphi, \partial_v\varphi\rangle_g\\
&= -\partial_v\omega X_2 - \frac{\partial_v R}{R}X_1,
\end{aligned}$$

which proves (39a). Then, using that the geodesic curvatures of the v-coordinate curves of φ is $\partial_v\omega$, we obtain $D_{\partial_v}\partial_v\varphi^\perp = -\partial_v\omega\partial_v\varphi$. Moreover, a straightforward comptation gives

$$\partial_v\varphi^\perp = -\frac{1}{RX_2}\partial_u\varphi + T\partial_v\varphi, \quad \langle\partial_u\varphi^\perp, \partial_v\varphi\rangle_g = -\langle\partial_u\varphi, \partial_v\varphi^\perp\rangle_g.$$

Combining these results, we infer that

$$\begin{aligned}
\partial_v X_2 &= -\big\langle D_{\partial_v} \tfrac{\partial_u \varphi}{R}, \partial_v \varphi^\perp \big\rangle_g - \big\langle D_{\partial_v} \partial_v \varphi^\perp, \tfrac{\partial_u \varphi}{R} \big\rangle_g \\
&= \frac{\partial_v R}{R^2} \big\langle \partial_u \varphi, \partial_v \varphi^\perp \big\rangle_g - \frac{1}{R} \big\langle D_{\partial_u} \partial_v \varphi, \partial_v \varphi^\perp \big\rangle_g + \partial_v \omega \big\langle \partial_v \varphi, \tfrac{\partial_u \varphi}{R} \big\rangle_g \\
&= -\frac{\partial_v R}{R} X_2 - \frac{1}{R} \big\langle D_{\partial_u} \partial_v \varphi, -\frac{1}{R X_2} \partial_u \varphi + T \partial_v \varphi \big\rangle_g + \partial_v \omega X_1 \\
&= -\frac{\partial_v R}{R} X_2 + \frac{\partial_v (R^2)}{2 R^2 X_2} + \partial_v \omega X_1 = \frac{\partial_v R}{R} \big(\frac{1}{X_2} - X_2 \big) + \partial_v \omega X_1 \\
&= \partial_v \omega X_1 + \frac{X_1}{X_2} \frac{\partial_v R}{R} X_1.
\end{aligned}$$

Step 4 *(Computation of the Gaussian curvature (3^{rd} part)).* We now compute the expression of the Gaussian curvature K in the local parametrization φ. Note that the metric induced by φ is $\tilde{g} = R^2 du^2 + 2 R X_1 du dv + dv^2$ giving $\det \tilde{g} = R^2 X_2^2$. Then, recall that, by definition, the Gaussian curvature K satisfies

$$K \det \tilde{g} = \big\langle D_{\partial_v} D_{\partial_u} \partial_u \varphi - D_{\partial_u} D_{\partial_v} \partial_u \varphi, \partial_v \varphi \big\rangle_g. \tag{40}$$

Using (38a), we first obtain that

$$\begin{aligned}
\big\langle D_{\partial_v} D_{\partial_u} \partial_u \varphi, \partial_v \varphi \big\rangle_g &= \big\langle D_{\partial_v} \big(A \partial_u \varphi + B \partial_v \varphi \big), \partial_v \varphi \big\rangle_g \\
&= A \big\langle D_{\partial_v} \partial_u \varphi, \partial_v \varphi \big\rangle_g + B \langle D_{\partial_v} \partial_v \varphi, \partial_v \varphi \big\rangle_g + R X_1 \partial_v A + \partial_v B,
\end{aligned}$$

with $A = \frac{\partial_u R}{R} + \frac{X_1}{X_2^2} (\partial_v R - \partial_u X_1)$ and $B = \frac{R}{X_2^2} (\partial_u X_1 - \partial_v R)$. Then, we infer that

$$\begin{aligned}
\big\langle D_{\partial_v} D_{\partial_u} \partial_u \varphi, \partial_v \varphi \big\rangle_g &= R X_1 \partial_v A + \partial_v B \\
&= \partial_v \left(\frac{\partial_u R}{R} \right) R X_1 + R X_1^2 \partial_v \left(\frac{\partial_v R - \partial_u X_1}{X_2^2} \right) + R X_1 \partial_v X_1 \frac{\partial_v R - \partial_u X_1}{X_2^2} \\
&\quad - \partial_v R \frac{\partial_v R - \partial_u X_1}{X_2^2} - R \partial_v \left(\frac{\partial_v R - \partial_u X_1}{X_2^2} \right) \\
&= \partial_v \left(\frac{\partial_u R}{R} \right) R X_1 - R X_2^2 \partial_v \left(\frac{\partial_v R - \partial_u X_1}{X_2^2} \right) \\
&\quad + R X_1 \partial_v X_1 \frac{\partial_v R - \partial_u X_1}{X_2^2} - \partial_v R \frac{\partial_v R - \partial_u X_1}{X_2^2}.
\end{aligned} \tag{41}$$

Secondly, using (38b), we obtain that

$$\begin{aligned}
\big\langle D_{\partial_u} D_{\partial_v} \partial_u \varphi, \partial_v \varphi \big\rangle_g &= \big\langle D_{\partial_u} \left[\frac{\partial_v R}{R X_2^2} \partial_u \varphi - \frac{\partial_v R X_1}{X_2^2} \partial_v \varphi \right], \partial_v \varphi \big\rangle_g \\
&= \frac{\partial_v R}{R X_2^2} \big\langle D_{\partial_u} \partial_u \varphi, \partial_v \varphi \big\rangle_g - \frac{\partial_v R X_1}{X_2^2} \big\langle D_{\partial_u} \partial_v \varphi, \partial_v \varphi \big\rangle_g + R X_1 \partial_u \left(\frac{\partial_v R}{R X_2^2} \right) - \partial_u \left(\frac{\partial_v R X_1}{X_2^2} \right) \\
&= \frac{\partial_v R}{R X_2^2} \big\langle D_{\partial_u} \partial_u \varphi, \partial_v \varphi \big\rangle_g - \frac{\partial_u R}{R^2} \frac{\partial_v R}{X_2^2} R X_1 + \partial_u \left(\frac{\partial_v R}{X_2^2} \right) X_1 - \partial_u \left(\frac{\partial_v R}{X_2^2} \right) X_1 - \frac{\partial_u X_1 \partial_v R}{X_2^2} \\
&= \frac{\partial_v R}{X_2^2} \left(\frac{1}{R} \big\langle D_{\partial_u} \partial_u \varphi, \partial_v \varphi \big\rangle_g - \frac{X_1 \partial_u R}{R} - \partial_u X_1 \right).
\end{aligned}$$

Since $\langle D_{\partial_u}\partial_u\varphi, \partial_v\varphi\rangle_g = \partial_u R X_1 + R\partial_u X_1 - R\partial_v R$, we infer that

$$\left\langle D_{\partial_v} D_{\partial_u}\partial_u\varphi, \partial_v\varphi\right\rangle_g = -\frac{(\partial_v R)^2}{X_2^2}. \tag{42}$$

Combining (40), (41) and (42) gives

$$\begin{aligned}
K\det\tilde g &= -RX_2^2\partial_v\left(\frac{\partial_v R - \partial_u X_1}{X_2^2}\right) + RX_1\partial_v X_1\frac{\partial_v R - \partial_u X_1}{X_2^2} + \frac{\partial_v R\partial_u X_1}{X_2^2} + \partial_v\left(\frac{\partial_u R}{R}\right)RX_1 \\
&= \frac{1}{X_2^2}\left[RX_1\partial_v X_1(\partial_v R - \partial_u X_1) + \partial_v R\partial_u X_1\right] - R\partial_{vv}R + R\partial_{uv}X_1 \\
&\quad + 2\partial_v X_2\frac{R(\partial_v R - \partial_u X_1)}{X_2} + \partial_v\left(\frac{\partial_u R}{R}\right)RX_1 \\
&= \frac{1}{X_2^2}\left[-RX_1\Big(\partial_v\omega X_2 + \frac{\partial_v R}{R}X_1\Big)(\partial_v R - \partial_u X_1) + \partial_v R\partial_u X_1\right] - R\partial_{vv}R - R\partial_u\left[\partial_v\omega X_2 + \frac{\partial_v R}{R}X_1\right] \\
&\quad + 2\left(\partial_v\omega X_1 + \frac{X_1^2\partial_v R}{RX_2}\right)\frac{R(\partial_v R - \partial_u X_1)}{X_2} + \partial_v\left(\frac{\partial_u R}{R}\right)RX_1,
\end{aligned}$$

using (39) for the last equality. Then, we split the computation in two parts. First, we have

$$\begin{aligned}
C &:= \frac{1}{X_2^2}\left[-RX_1\Big(\partial_v\omega X_2 + \frac{\partial_v R}{R}X_1\Big)(\partial_v R - \partial_u X_1) + \partial_v R\partial_u X_1\right] \\
&= \frac{1}{X_2^2}\Big[-R\partial_v\omega X_1X_2\partial_v R + RX_1X_2\partial_v\omega\partial_u X_1 - X_1^2(\partial_v R)^2 + X_1^2\partial_u X_1\partial_v R + \partial_v R\partial_u X_1\Big] \\
&= -R\partial_v R\partial_v\omega T + RT\partial_v\omega\partial_u X_1 - T^2(\partial_v R)^2 + T^2\partial_u X_1\partial_v R + \frac{\partial_v R\partial_u X_1}{X_2^2}.
\end{aligned} \tag{43}$$

Then, we obtain

$$\begin{aligned}
E &:= -R\partial_{vv}R - R\partial_u\left[\partial_v\omega X_2 + \frac{\partial_v R}{R}X_1\right] \\
&\quad + 2\left(\partial_v\omega X_1 + \frac{X_1^2\partial_v R}{RX_2}\right)\frac{R(\partial_v R - \partial_u X_1)}{X_2} + \partial_v\left(\frac{\partial_u R}{R}\right)RX_1 \\
&= -R\partial_{vv}R - R\Big[\partial_{uv}\omega X_2 + \partial_v\omega\partial_u X_2 + \partial_v\left(\frac{\partial_u R}{R}\right)X_1 + \frac{\partial_v R}{R}\partial_u X_1\Big] \\
&\quad + 2\partial_v\omega TR\big(\partial_v R - \partial_u X_1\big) + 2T^2\partial_v R\big(\partial_v R - \partial_u X_1\big) + \partial_v\left(\frac{\partial_u R}{R}\right)RX_1 \\
&= -R\partial_{vv}R - R\partial_{uv}\omega X_2 - R\partial_v\omega(\partial_u X_2 + T\partial_u X_1) - \partial_v R\partial_u X_1(1+T^2) \\
&\quad - T\partial_u X_1 R\partial_v\omega - T^2\partial_v R\partial_u X_1 + 2T\partial_v R\partial_v\omega R + 2T^2(\partial_v R)^2.
\end{aligned}$$

Moreover, since $X_1^2 + X_2^2 = 1$, we have $1 + T^2 = \frac{1}{X_2^2}$ and $T\partial_u X_1 + \partial_u X_2 = 0$. We infer that

$$\begin{aligned}
E =& -R\partial_{vv}R - R\partial_{uv}\omega X_2 - \frac{\partial_v R\partial_u X_1}{X_2^2} - T\partial_u X_1 R\partial_v\omega \\
& - T^2\partial_v R\partial_u X_1 + 2T\partial_v R\partial_v\omega R + 2T^2(\partial_v R)^2.
\end{aligned} \tag{44}$$

We obtain by combining (43) and (44) that

$$K \det \tilde{g} = C + E = T^2(\partial_v R)^2 - R\partial_{uv}\omega X_2 - R\partial_{vv}R + TR\partial_v R\partial_v\omega. \tag{45}$$

Using that $\det \tilde{g} = X_2^2 R^2$ and dividing by R, we finally obtain

$$-\partial_v\omega\partial_v RT - \frac{(\partial_v R)^2}{R}T^2 + \partial_{vv}R = -X_2(\partial_{uv}\omega + KX_2 R). \tag{46}$$

Step 5 *(Bound on the right-hand side of the equation* (46)*)*. By (46) and the integrability condition (5), we have

$$\partial_{vv}R = \partial_v\omega\partial_v RT + \frac{(\partial_v R)^2}{R}T^2 + K(\varphi)X_2\big[\sin(\omega) - X_2 R\big]. \tag{47}$$

The proof is now reduced to showing that $\partial_v R = 0$ is the unique solution to (47) such that $\partial_v R(\cdot, 0) = 0$. To this end, we bound the right-hand side of this equation. Let $F_1 = \sin(\omega) - RX_2$ and $F_2 = \cos(\omega) - RX_1$. We infer from (39b) that

$$\begin{aligned}\partial_v F_1 &= \partial_v\omega\cos(\omega) - \partial_v RX_2 - R\partial_v X_2 = \partial_v\omega\big(\cos(\omega) - RX_1\big) - \partial_v R\big(X_2 + TX_1\big)\\ &= \partial_v\omega F_2 - \frac{\partial_v R}{X_2},\end{aligned}$$

using that $X_2 + TX_1 = X_2 + \frac{1-X_2^2}{X_2} = \frac{1}{X_2}$ in the last equality. In the same manner, we deduce from (39a) that $\partial_v F_2 = -\partial_v\omega F_1$, so that the couple (F_1, F_2) satisfies the system of differential equations

$$\begin{cases}\partial_v F_1 = \partial_v\omega F_2 - \dfrac{\partial_v R}{X_2},\\ \partial_v F_2 = -\partial_v\omega F_1,\end{cases}$$

with $F_1(0) = F_2(0) = 0$, since $R(u,0) = 1$, $X_1(u,0) = \cos\omega(u,0)$, and $X_2(u,0) = \sin\omega(u,0)$, for all $u \in [0, L_1]$. A straightforward computation shows that the unique solution to this linear ordinary differential equation is

$$\begin{aligned}(\sin\omega - X_2 R)(u,v) = F_1(u,v) = &- \sin\omega(u,v)\int_0^v \frac{\partial_v R}{X_2}(u,s)\sin\omega(u,s)\mathrm{d}s\\ &- \cos\omega(u,v)\int_0^v \frac{\partial_v R}{X_2}(u,s)\cos\omega(u,s)\mathrm{d}s,\end{aligned} \tag{48a}$$

$$\begin{aligned}(\cos\omega - X_1 R)(u,v) = F_2(u,v) = &- \cos\omega(u,v)\int_0^v \frac{\partial_v R}{X_2}(u,s)\sin\omega(u,s)\mathrm{d}s\\ &+ \sin\omega(u,v)\int_0^v \frac{\partial_v R}{X_2}(u,s)\cos\omega(u,s)\mathrm{d}s,\end{aligned} \tag{48b}$$

since $\partial_v R(u,0) = 0$, for all $u \in [0, L_u]$, by Step 1.

Step 6 *(Conclusion).* Finally, we infer from (47) and (48a) that

$$\partial_{vv}R = \partial_v\omega\partial_v RT + \frac{(\partial_v R)^2}{R}T^2 - K(\varphi)X_2\Big(\sin\omega\int_0^v \frac{\partial_v R}{X_2}\sin\omega + \cos\omega\int_0^v \frac{\partial_v R}{X_2}\cos\omega\Big). \tag{49}$$

Then, since $0 < \omega(u,v) < \pi$, for all $(u,v) \in D_L$, we have that T and $\frac{1}{X_2}$ are bounded. Using moreover that $\frac{1}{R}$, $\partial_v\omega$ and $K\circ\varphi$ are bounded, and using $\partial_v R(\cdot,0) = 0$, we infer from (49) that

$$\begin{aligned} |\partial_v R(t)| &\leq \tilde{C}\Big(\int_0^t |\partial_v R(s)|\mathrm{d}s + \int_0^t |(\partial_v R)^2(s)|\mathrm{d}s + \int_0^t\int_0^s |\partial_v R(l)|\mathrm{d}l\mathrm{d}s\Big) \\ &\leq C\int_0^t |\partial_v R(s)|\mathrm{d}s, \end{aligned}$$

for all $u \in [0,L_u]$ and $t \in [0,L]$. Using GrÃ¶nwall's inequality, we conclude that $\partial_v R(u,v) = 0$, for all $(u,v) \in D_L$. The claim follows.

□

4 Existence and uniqueness of angle distribution

Let $k \in \mathbb{N}$, let $D = [0,L_u]\times[0,L_v]$, with $L_u, L_v \in \mathbb{R}_*^+$, and let $\gamma = (\gamma_u,\gamma_v) \in \Gamma^{k+2}([0,L_u])\times\Gamma^{k+2}([0,L_v])$ be two curves of geodesic curvatures $\kappa_u \in C^k([0,L_u],\mathbb{R})$ and $\kappa_v \in C^k([0,L_v],\mathbb{R})$, respectively. In this section, we consider the Hazzidakis formula (6) as an equation on $\omega \in \Theta_\gamma^{k+1}(D)$, i.e., on angle distributions satisfying the boundary conditions (8). Hence, we define the mapping $F : \Theta_\gamma^{k+1}(D) \to \Theta_\gamma^{k+1}(D)$ by

$$F[\omega](u,v) = \angle\big(\gamma_u'(0),\gamma_v'(0)\big) - \int_0^u \kappa_u + \int_0^v \kappa_v - \int_0^u\int_0^v K\big[\mathcal{I}(\gamma,\omega)(t,s)\big]\sin(\omega(t,s))\mathrm{d}t\mathrm{d}s,$$

and we prove in what follows that there exists a unique solution to

$$\omega(u,v) = F[\omega](u,v), \tag{50}$$

for all $(u,v) \in D$. We first show in Subsection 4.1 that there exists a unique $\omega^* \in \Theta_\gamma^{k+1}([0,L_u]\times[0,L_0])$, for $L_0 \in (0,L_v]$ small enough, satisfying (50). We also prove that this solution depends continuously on the curves γ_u and γ_v. Then, we extend this result to finite rectangles $D = [0,L_u]\times[0,L_v]$, with $L_u, L_v \in \mathbb{R}_*^+$, in Subsection 4.2. Finally, we prove by a density argument on the regularity of γ_u and γ_v that the associated parametrization $\mathcal{I}(\gamma,\omega^*)$ is indeed a Chebyshev net.

4.1 Local existence of a solution

We first suppose that $k = 0$ and we state the local existence of the angle distribution ω in the following proposition.

Proposition 10 (Local existence of a solution)**.** *Let M be a smooth, open, complete, and simply connected surface, let $L_u, L_v \in \mathbb{R}_*^+$, and let $\gamma = (\gamma_u, \gamma_v) \in \Gamma^2([0, L_u])\times\Gamma^2([0, L_v])$ be such that $\gamma_u(0) = \gamma_v(0)$ and $\angle(\gamma_u'(0), \gamma_v'(0)) \in (0, \pi)$. Then, there exists $L_0 \in (0, L_v]$, depending only on $\|\gamma_u\|_{\Gamma^2([0,L_u])}$ and $\|\gamma_v\|_{\Gamma^2([0,L_v])}$, such that there exists a unique solution $\omega_\gamma^* \in \Theta_\gamma^1([0, L_u]\times[0, L_0])$ to* (50)*. Moreover, we have*

$$\|\omega_\gamma^*\|_{\Theta^1([0,L_u]\times[0,L_0])} \leq C, \tag{51}$$

where the constant C depends on $\|\gamma_u\|_{\Gamma^2([0,L_u])}$ and $\|\gamma_v\|_{\Gamma^2([0,L_v])}$.

Proof. Let $D = [0, L_u]\times[0, L_v]$ and let $D_L = [0, L_u]\times[0, L]$, with $L \in (0, L_v]$. We prove the claim by application of the Banach fixed-point theorem to the functional $F : \Theta_\gamma^1(D_L) \to \Theta_\gamma^1(D_L)$, supposing that L is small enough. Hence, we first prove that F is stable in some bounded closed subset of $\Theta_\gamma^1(D_L)$ (Step 1) and we then show that F^2 is a contraction mapping in this space (Step 2). We conclude using the Banach fixed-point theorem in Step 3.

Step 1 *(Stability in a closed subset)*. We denote $\kappa_{u,1} \in C^0([0, L_u])$ and $\kappa_{v,1} \in C^0([0, L_v])$ the geodesic curvatures of γ_u and γ_v respectively. We set $\varphi_\gamma := \mathcal{I}(\gamma, \omega_\gamma) \in \Phi^{1,2}(D)$. Since φ_γ is bounded by (19), we have that $K \circ \varphi_\gamma$ is bounded. Moreover, a straightforward computation gives

$$\begin{aligned}
\|F(\omega_\gamma)\|_{C^0(D)} &\leq \pi + L_u\|\kappa_{u,1}\|_{C^0([0,L_u])} + L_v\|\kappa_{v,1}\|_{C^0([0,L_v])} + L_uL_v\|K \circ \varphi_\gamma\|_{C^0(D)},\\
\|\partial_u F(\omega_\gamma)\|_{C^0(D)} &\leq \|\kappa_{u,1}\|_{C^0([0,L_u])} + L_v\|K \circ \varphi_\gamma\|_{C^0(D)},\\
\|\partial_v F(\omega_\gamma)\|_{C^0(D)} &\leq \|\kappa_{v,1}\|_{C^0([0,L_v])} + L_u\|K \circ \varphi_\gamma\|_{C^0(D)},\\
\|\partial_{uv} F(\omega_\gamma)\|_{C^0(D)} &\leq \|K \circ \varphi\|_{C^0(D)},
\end{aligned}$$

for all $\omega_\gamma \in \Theta_\gamma^1(D)$. Then, there exists $\mathcal{R}(\gamma) > 0$ such that $F : \mathcal{B}_{\Theta^1(D_L)}(\mathcal{R}(\gamma)) \to \mathcal{B}_{\Theta^1(D_L)}(\mathcal{R}(\gamma))$, where $\mathcal{B}_{\Theta^1(D_L)}(\mathcal{R}(\gamma))$ is the closed ball of radius $\mathcal{R}(\gamma)$ centered at the origin in $\Theta_\gamma^1(D_L)$. In what follows, we restrict F to this ball.

Step 2 *(Contraction mapping)*. We now prove the following result:

Let $\sigma = (\sigma_u, \sigma_v) \in \Gamma^2([0, L_u])\times\Gamma^2([0, L_v])$ be such that $\gamma_u(0) = \gamma_v(0) = \sigma_u(0) = \sigma_v(0)$, $\gamma_u'(0) = \sigma_u'(0)$ and $\gamma_v'(0) = \sigma_v'(0)$. Let $\omega_\gamma \in \mathcal{B}_{\Theta^1(D_L)}(\mathcal{R}(\gamma))$ and $\omega_\sigma \in \mathcal{B}_{\Theta^1(D_L)}(\mathcal{R}(\sigma))$. Then, we have

$$\begin{aligned}
\|F^2(\omega_\gamma) - F^2(\omega_\sigma)\|_{\Theta^1(D_L)} \leq C\Big(&\|\gamma_u - \sigma_u\|_{\Gamma^2([0,L_u])} + \|\gamma_v - \sigma_v\|_{\Gamma^2([0,L_v])}\\
&+ L\|\omega_\gamma - \omega_\sigma\|_{\Theta^1(D_L)}\Big),
\end{aligned} \tag{52}$$

where the constant C depends on $\mathcal{R}(\gamma)$ and $\mathcal{R}(\sigma)$. Note that (52) *holds for all $L \in (0, L_v]$ and $\gamma, \sigma \in \Gamma^2([0, L_u])\times\Gamma^2([0, L_v])$.*

In this step, the domain of definition of the two-dimensional variables for the different norms is always $D_L = [0, L_u] \times [0, L]$ and will not be specified. In all the subsequent estimates, unless explicitly mentioned, the constants only depend on $\mathcal{R}(\gamma)$ and $\mathcal{R}(\sigma)$. We set $\varphi_\sigma := \mathcal{I}(\sigma, \omega_\sigma) \in \Phi^{1,2}$ and we denote $\kappa_{u,2} \in C^0([0, L_u])$ and $\kappa_{v,2} \in C^0([0, L_v])$ the geodesic curvatures of σ_u and σ_v, respectively. First, we prove that

$$\|F(\omega_\gamma) - F(\omega_\sigma)\|_{\Theta^0} \leq \tilde{C}\Big(\|\gamma_u - \sigma_u\|_{\Gamma^2([0,L_u])} + \|\gamma_v - \sigma_v\|_{\Gamma^2([0,L_v])} + L\|\omega_\gamma - \omega_\sigma\|_{\Theta^1}\Big), \tag{53a}$$

$$\begin{aligned}\|F(\omega_\gamma) - F(\omega_\sigma)\|_{\Theta^1} \leq C\Big(&\|\gamma_u - \sigma_u\|_{\Gamma^2([0,L_u])} + \|\gamma_v - \sigma_v\|_{\Gamma^2([0,L_v])} + L\|\omega_\gamma - \omega_\sigma\|_{\Theta^1} \\ &+ \|\omega_\gamma - \omega_\sigma\|_{\Theta^0}\Big).\end{aligned} \tag{53b}$$

To this end, first note that we have

$$\begin{aligned}|F(\omega_\gamma) - F(\omega_\sigma)|(u,v) &\leq \int_0^u |\kappa_{u,1} - \kappa_{u,2}| + \int_0^v |\kappa_{v,1} - \kappa_{v,2}| + \big|\angle\big(\gamma_u'(0), \gamma_v'(0)\big) - \angle\big(\sigma_u'(0), \sigma_v'(0)\big)\big| \\ &\quad + \int_0^u \int_0^v |K(\varphi_\sigma)\sin(\omega_\sigma) - K(\varphi_\gamma)\sin(\omega_\gamma)| \\ &\leq C\Big(\|\gamma_u - \sigma_u\|_{\Gamma^2([0,L_u])} + \|\gamma_v - \sigma_v\|_{\Gamma^2([0,L_v])} \\ &\quad + L\|K(\varphi_\sigma)\sin(\omega_\sigma) - K(\varphi_\gamma)\sin(\omega_\gamma)\|_{C^0}\Big),\end{aligned}$$

for all $(u, v) \in D_L$. Doing the same for $\partial_u F(\omega)$, $\partial_v F(\omega)$ and $\partial_{uv} F(\omega) = -K(\varphi)\sin(\omega)$, we obtain

$$\begin{aligned}\|F(\omega_\gamma) - F(\omega_\sigma)\|_{\Theta^0} &\leq \tilde{C}\Big(\|\gamma_u - \sigma_u\|_{\Gamma^2([0,L_u])} + \|\gamma_v - \sigma_v\|_{\Gamma^2([0,L_v])} \\ &\qquad + L\|K(\varphi_\gamma)\sin(\omega_\gamma) - K(\varphi_\sigma)\sin(\omega_\sigma)\|_{C^0}\Big), \\ \|F(\omega_\gamma) - F(\omega_\sigma)\|_{\Theta^1} &\leq C\Big(\|\gamma_u - \sigma_u\|_{\Gamma^2([0,L_u])} + \|\gamma_v - \sigma_v\|_{\Gamma^2([0,L_v])} \\ &\qquad + \|K(\varphi_\gamma)\sin(\omega_\gamma) - K(\varphi_\sigma)\sin(\omega_\sigma)\|_{C^0}\Big).\end{aligned}$$

Moreover, since K is smooth and since φ_γ and φ_σ are bounded, we have

$$\|K(\varphi_\gamma)\sin(\omega_\gamma) - K(\varphi_\sigma)\sin(\omega_\sigma)\|_{C^0} \leq C\Big(\|\omega_\gamma - \omega_\sigma\|_{\Theta^0} + \|\varphi_\gamma - \varphi_\sigma\|_{\Phi^0}\Big),$$

and we conclude the proof of (53) with

$$\|\varphi_\gamma - \varphi_\sigma\|_{\Phi^0} \leq C\Big(L\|\omega_\gamma - \omega_\sigma\|_{\Theta^1} + \|\gamma_u - \sigma_u\|_{\Gamma^2([0,L_u])}\Big), \tag{54}$$

obtained using (20a) of Proposition 7. Finally, using (53), we obtain

$$\begin{aligned}\|F^2(\omega_\gamma) - F^2(\omega_\sigma)\|_{\Theta^1} &\leq \tilde{C}\Big(\|\gamma_u - \sigma_u\|_{\Gamma^2([0,L_u])} + \|\gamma_v - \sigma_v\|_{\Gamma^2([0,L_v])} \\ &\qquad + L\|F(\omega_\gamma) - F(\omega_\sigma)\|_{\Theta^1} + \|F(\omega_\gamma) - F(\omega_\sigma)\|_{\Theta^0}\Big) \\ &\leq C\Big(\|\gamma_u - \sigma_u\|_{\Gamma^2([0,L_u])} + \|\gamma_v - \sigma_v\|_{\Gamma^2([0,L_u])} + L\|\omega_\gamma - \omega_\sigma\|_{\Theta^1}\Big).\end{aligned}$$

The inequality (52) follows.

Step 3 *(Conclusion).* We infer from (52) that there exists $L_0 \in (0, L_v]$ such that F^2 is a contraction mapping in $\mathcal{B}_{\Theta^1(D_{L_0})}(\mathcal{R}(\gamma))$ for the norm in $\Theta^1(D_{L_0})$. Hence, the claim follows from the Banach fixed-point theorem on the closed subset $\mathcal{B}_{\Theta^1(D_{L_0})}(\mathcal{R}(\gamma))$ of the Banach space $C^1([0, L_u], C^1([0, L_0]))$.

□

Let $R_0 \in \mathbb{R}_*^+$ and let $\mathcal{B}_{\Gamma^2\times\Gamma^2}(R_0)$ be the closed ball of radius R_0 centered at the origin in $\Gamma^2([0, L_u])\times\Gamma^2([0, L_v])$ (supposing R_0 is large enough for this set not to be empty). As the length of integration $L_0 \in (0, L_v]$ of the v-coordinate curves of φ defined in Proposition 10 only depends on the norm in $\Gamma^2([0, L_u])\times\Gamma^2([0, L_v])$ of the initial conditions $\gamma = (\gamma_u, \gamma_v)$, we can define the mapping

$$\begin{array}{rccl} \mathcal{J}_{0,R_0}: & \mathcal{B}_{\Gamma^2\times\Gamma^2}(R_0) & \to & \Theta^1([0, L_u]\times[0, L_0(R_0)]), \\ & \gamma = (\gamma_u, \gamma_v) & \mapsto & \omega_\gamma = F(\omega_\gamma), \end{array} \tag{55}$$

which maps the boundary conditions γ to the solution ω_γ to (50). We then state the following proposition which asserts the continuity of the mapping $\mathcal{J}_{0,R_0}$ with respect to these boundary conditions.

Proposition 11 (Continuity with respect to the boundary conditions)**.** *Let M be a smooth, open, complete, and simply connected surface, let $R_0 \in \mathbb{R}_*^+$, and let $D = [0, L_u]\times[0, L_v]$, with $L_u, L_v \in \mathbb{R}_*^+$. We equip $\mathcal{B}_{\Gamma^2\times\Gamma^2}(R_0)$ and $\Theta^1([0, L_u]\times[0, L_0(R_0)])$ with the norms in $\Gamma^2([0, L_u])\times\Gamma^2([0, L_v])$ and $\Theta^1([0, L_u]\times[0, L_0(R_0)])$, respectively. Then, the mappings $\mathcal{J}_{0,R_0}$ and*

$$\begin{array}{rccl} \mathcal{I} \circ \big(\mathrm{Id}, \mathcal{J}_{0,R_0}\big): & \mathcal{B}_{\Gamma^2\times\Gamma^2}(R_0) & \to & \Phi^2([0, L_u]\times[0, L_0(R_0)]) \\ & \gamma = (\gamma_u, \gamma_v) & \mapsto & \varphi_\omega := \mathcal{I}(\gamma, \mathcal{J}_{0,R_0}(\gamma)) \end{array}$$

are Lipschitz continuous, with Id *the identity operator in $\mathcal{B}_{\Gamma^2\times\Gamma^2}(R_0)$.*

Proof. Let $\gamma, \sigma \in \mathcal{B}_{\Gamma^2\times\Gamma^2}(R_0)$, with $\gamma = (\gamma_u, \gamma_v)$ and $\sigma = (\sigma_u, \sigma_v)$, be such that $\gamma_u(0) = \gamma_v(0) = \sigma_u(0) = \sigma_v(0)$, $\gamma_u'(0) = \sigma_u'(0)$ and $\gamma_v'(0) = \sigma_v'(0)$. Suppose moreover that $\angle\big(\gamma_u'(0), \gamma_v'(0)\big) = \angle\big(\sigma_u'(0), \sigma_v'(0)\big) \in (0, \pi)$. We set $D_{L_0} = [0, L_u]\times[0, L_0(R_0)]$, $\omega_\gamma := \mathcal{J}_{0,R_0}(\gamma) \in \Theta^1_\gamma(D_{L_0})$, and $\omega_\sigma := \mathcal{J}_{0,R_0}(\sigma) \in \Theta^1_\sigma(D_{L_0})$. Let us recall from the proof of Proposition 10 that $F^2 : \mathcal{B}_{\Theta^1(D_{L_0})}(\mathcal{R}(\gamma)) \to \mathcal{B}_{\Theta^1(D_{L_0})}(\mathcal{R}(\gamma))$ is a contraction mapping, with $\mathcal{R}(\gamma) \in \mathbb{R}_*^+$ a constant depending only on R_0, and $\mathcal{B}_{\Theta^1(D_{L_0})}(\mathcal{R}(\gamma))$ the ball centered at the origin with radius

$\mathcal{R}(\gamma)$ in $\Theta^1(D_{L_0})$. Since ω_γ and ω_σ are both contained in this ball, we deduce from (52) that

$$\|F^2(\omega_\gamma) - F^2(\omega_\sigma)\|_{\Theta^1(D_{L_0})} \leq C_1\Big[\|\gamma_u - \sigma_u\|_{\Gamma^2([0,L_u])} + \|\gamma_v - \sigma_v\|_{\Gamma^2([0,L_v])}\Big] + C_2\|\omega_\gamma - \omega_\gamma\|_{\Theta^1(D_{L_0})},$$

with $C_1 \in \mathbb{R}_*^+$ and $C_2 \in (0,1)$ two constants. As $F(\omega) = \omega$, we infer that

$$\|\omega_\gamma - \omega_\sigma\|_{\Theta^1(D_{L_0})} \leq \frac{C_1}{1-C_2}\Big[\|\gamma_u - \sigma_u\|_{\Gamma^2([0,L_u])} + \|\gamma_v - \sigma_v\|_{\Gamma^2([0,L_v])}\Big],$$

which proves that $\mathcal{J}_{0,R_0}$ is Lipschitz continuous. Finally, using additionally Proposition 8, we infer that $\mathcal{I} \circ \big(\mathrm{Id}, \mathcal{J}_{0,R_0}\big)$ is Lipschitz continuous. $\square$

We now prove that C^{k+2}-regularity for the boundary conditions γ implies Θ^{k+1}-regularity for the solution ω, for all $k \in \mathbb{N}$.

Proposition 12 (Regularity of the solution)**.** *Let M be a smooth, open, complete, and simply connected surface, let $k \in \mathbb{N}$, and let $D = [0,L_u]\times[0,L_v]$, with $L_u, L_v \in \mathbb{R}_*^+$. Let $\gamma = (\gamma_u, \gamma_v) \in \Gamma^{k+2}([0,L_u])\times\Gamma^{k+2}([0,L_v])$ be such that $\gamma_u(0) = \gamma_v(0)$ and $\angle(\gamma_u'(0), \gamma_v'(0)) \in (0,\pi)$. Then, there exists $L_0 \in (0, L_v]$, depending only on $\|\gamma_u\|_{\Gamma^2([0,L_u])}$ and $\|\gamma_v\|_{\Gamma^2([0,L_v])}$, such that there exists a unique solution $\omega_\gamma \in \Theta_\gamma^{k+1}([0,L_u]\times[0,L_0])$ to* (50)*. Moreover, we have*

$$\|\omega_\gamma\|_{\Theta^{k+1}([0,L_u]\times[0,L_0])} \leq C, \tag{56}$$

where the constant C depends on $\|\gamma_u\|_{\Gamma^{k+2}([0,L_u])}$ and $\|\gamma_v\|_{\Gamma^{k+2}([0,L_v])}$.

Proof. Owing to Proposition 10, there exists $L_0 \in (0, L_v]$ such that there exists a unique solution $\omega_\gamma \in \Theta_\gamma^1(D_{L_0})$ to (50). We prove in what follows that $\omega_\gamma \in \Theta_\gamma^{k+1}(D_{L_0})$. In this proof, the domain of definition of the two-dimensional variables for the different spaces is always $D_{L_0} = [0,L_u]\times[0,L_0]$ and it will not be specified. Owing to Proposition 8, we have that $\varphi_\omega = \mathcal{I}(\gamma,\omega) \in \Phi^{r+2}$ whenever $\omega_\gamma \in \Theta_\gamma^{r+1}$, for all $r \in \{0,...,k\}$. Therefore, using that $\omega_\gamma = F(\omega_\gamma)$, we obtain by an induction argument on $r \in \{0,...,k\}$ that $\omega_\gamma \in \Theta_\gamma^{k+1}$ (the only limiting factor being the regularity of the boundary curves γ). Hence, we have $\varphi_\omega \in \Phi^{k+2}$. Now, to prove (56), we note that

$$\partial^{(i_1,0)}F(\omega_\gamma)(u,v) = \partial_u^{i_1-1}\kappa_u + \int_0^v \partial_u^{i_1-1}\big[K(\varphi_\omega)\sin(\omega_\gamma)\big](u,t)\mathrm{d}t, \tag{57a}$$

$$\partial^{(0,i_2)}F(\omega_\gamma)(u,v) = \partial_v^{i_2-1}\kappa_v + \int_0^u \partial_v^{i_2-1}\big[K(\varphi_\omega)\sin(\omega_\gamma)\big](s,v)\mathrm{d}s, \tag{57b}$$

$$\partial^I F(\omega_\gamma)(u,v) = \partial^{(i_1-1,i_2-1)}\big[K(\varphi_\omega)\sin(\omega_\gamma)\big](u,v), \tag{57c}$$

for all $I = (i_1, i_2) \in \{1,...,k+1\}^2$. Furthermore, a straightforward computation gives

$$\|K(\varphi_\omega)\sin(\omega_\gamma)\|_{C^k([0,L_u],C^k([0,L_0]))} \leq C, \tag{58}$$

where the constant C depends on $\|\varphi_\omega\|_{\Phi^k}$ and $\|\omega_\gamma\|_{\Theta^k}$. Using moreover Proposition 8, we infer that the constant C only depends on $\|\gamma_u\|_{\Gamma^s([0,L_u])}$, with $s = \max(k, 2)$, and $\|\omega_\gamma\|_{\Theta^l}$, with $l = \max(k, 1)$. Then, if $k \geq 1$, we deduce from (57) and (58) that $\|\omega_\gamma\|_{\Theta^{k+1}} \leq C$, where the constant C depends on $\|\gamma_u\|_{\Gamma^{k+2}([0,L_u])}$, $\|\gamma_v\|_{\Gamma^{k+2}([0,L_v])}$, and $\|\omega_\gamma\|_{\Theta^k}$. Finally, since the case $k = 0$ follows from (51), we obtain (56) by a straightforward induction argument on $k \geq 0$. The claim follows. □

Let $\mathcal{B}_{\Gamma^{k+2}\times\Gamma^{k+2}}(R_k)$ be the closed ball of radius $R_k \in \mathbb{R}_*^+$ centered at the origin in $\Gamma^{k+2}([0, L_u])\times\Gamma^{k+2}([0, L_v])$. We denote $\mathcal{J}_{k,R_k}$ the restriction of the mapping (55) to this ball, i.e.,

$$\mathcal{J}_{k,R_k} : \mathcal{B}_{\Gamma^{k+2}\times\Gamma^{k+2}}(R_k) \to \Theta^{k+1}([0, L_u]\times[0, L_0(R_k)]).$$

We can now state the equivalent of Proposition 11 in $\Gamma^{k+2}([0, L_u])\times\Gamma^{k+2}([0, L_v])$.

Proposition 13 (Continuity with respect to boundary conditions)**.** *Let M be a smooth, open, complete, and simply connected surface and let $k \in \mathbb{N}$. Let $R_k \in \mathbb{R}_*^+$ and let $D = [0, L_u]\times[0, L_v]$, with $L_u, L_v \in \mathbb{R}_*^+$. We equip $\mathcal{B}_{\Gamma^{k+2}\times\Gamma^{k+2}}(R_k)$ and $\Theta^{k+1}([0, L_u]\times[0, L_0(R_k)])$ with the norms in $\Gamma^{k+2}([0, L_u])\times\Gamma^{k+2}([0, L_v])$ and $\Theta^{k+1}(D)$, respectively. Then, the mappings $\mathcal{J}_{k,R_k}$ and*

$$\begin{array}{rccl} \mathcal{I} \circ \big(\mathrm{Id}, \mathcal{J}_{k,R_k}\big) : & \mathcal{B}_{\Gamma^{k+2}\times\Gamma^{k+2}}(R_k) & \to & \Phi^{k+2}([0, L_u]\times[0, L_0(R_k)]), \\ & \gamma = (\gamma_u, \gamma_v) & \mapsto & \varphi_\omega := \mathcal{I}(\gamma, \mathcal{J}_{k,R_k}(\gamma)), \end{array}$$

are Lipschitz continuous, with Id *the identity operator in $\mathcal{B}_{\Gamma^{k+2}\times\Gamma^{k+2}}(R_k)$.*

Proof. Similarly to the previous proofs, the domain of definition of the two-dimensional variables for the different norms is always $D_{L_0} = [0, L_u]\times[0, L_0(R_k)]$ and it will not be specified. Let $\gamma = (\gamma_u, \gamma_v) \in \mathcal{B}_{\Gamma^{k+2}\times\Gamma^{k+2}}(R_k)$ be such that $\gamma_u(0) = \gamma_v(0)$ and $\angle(\gamma_u'(0), \gamma_v'(0) \in (0, \pi)$. Then, let $\sigma = (\sigma_u, \sigma_v) \in \mathcal{B}_{\Gamma^{k+2}\times\Gamma^{k+2}}(R_k)$ be such that $\gamma_u(0) = \sigma_u(0) = \sigma_v(0)$, $\gamma_u'(0) = \sigma_u'(0)$ and $\gamma_v'(0) = \sigma_v'(0)$. We set $\omega_\gamma := \mathcal{J}_{k,R_k}(\gamma) \in \Theta_\gamma^{k+1}$, $\omega_\sigma := \mathcal{J}_{k,R_k}(\sigma) \in \Theta_\sigma^{k+1}$ and we set $\varphi_\gamma := \mathcal{I}(\gamma, \omega_\gamma) \in \Phi^{k+1}$ and $\varphi_\sigma := \mathcal{I}(\sigma, \omega_\sigma) \in \Phi^{k+1}$. Using (57), a straightforward computation gives

$$\begin{aligned} \|F(\omega_\gamma) - F(\omega_\sigma)\|_{\Theta^{k+1}} \leq C\Big(&\|\gamma_u - \sigma_u\|_{\Gamma^{k+2}([0,L_u])} + \|\gamma_v - \sigma_v\|_{\Gamma^{k+2}([0,L_v])} \\ &+ \|K(\varphi_\gamma)\sin(\omega_\gamma) - K(\varphi_\sigma)\sin(\omega_\sigma)\|_{C^k([0,L_u],C^k([0,L_0]))}\Big). \end{aligned}$$

Moreover, we deduce from Proposition 8 that

$$\begin{aligned} \|K(\varphi_\gamma)\sin(\omega_\gamma) - K(\varphi_\sigma)\sin(\omega_\sigma)\|_{C^k([0,L_u],C^k([0,L_0]))} &\leq \tilde{C}\Big(\|\varphi_\gamma - \varphi_\sigma\|_{\Phi^k} + \|\omega_\gamma - \omega_\sigma\|_{\Theta^k}\Big) \\ &\leq C\Big(\|\gamma_u - \sigma_u\|_{\Gamma^s([0,L_u])} + \|\omega_\gamma - \omega_\sigma\|_{\Theta^l}\Big), \end{aligned}$$

with $s = \max(k, 2)$ and $l = \max(k, 1)$. Hence, if $k \geq 1$, we have

$$\|\omega_\gamma - \omega_\sigma\|_{\Theta^{k+1}} \leq C\Big(\|\gamma_u - \sigma_u\|_{\Gamma^{k+2}([0,L_u])} + \|\gamma_v - \sigma_v\|_{\Gamma^{k+2}([0,L_v])} + \|\omega_\gamma - \omega_\sigma\|_{\Theta^k}\Big).$$

Since the Lipschitz continuity of $\mathcal{J}_{k,R_k}$ in the case where $k = 0$ follows from Proposition 11, we then obtain the Lipschitz continuity of $\mathcal{J}_{k,R_k}$ in the general case by a straightforward induction argument on $k \geq 0$. Finally, the Lipschitz continuity of $\mathcal{I} \circ \big(\mathrm{Id}, \mathcal{J}_{k,R_k}\big)$ follows from Proposition 8. This concludes the proof. $\square$

In what follows, we will not make explicit the dependency of the mapping $\mathcal{J}_{k,R_k}$ on R_k, so that it will be denoted $\mathcal{J}_k$.

4.2 Extension to rectangles

We now extend Propositions 10 and 12 on the existence and uniqueness of a solution to the fixed-point equation (50) and Propositions 11 and 13 on the continuity with respect to boundary conditions to the whole domain $D = [0, L_u]\times[0, L_v]$, with $L_u, L_v \in \mathbb{R}_*^+$. In the same manner as above, we start with the case where $k = 0$.

Proposition 14 (Global existence of a solution)**.** *Let M be a smooth, open, complete, and simply connected surface, let $D = [0, L_u]\times[0, L_v]$, with $L_u, L_v \in \mathbb{R}_*^+$, and let $\gamma = (\gamma_u, \gamma_v) \in \Gamma^2([0, L_u])\times\Gamma^2([0, L_v])$ be such that $\gamma_u(0) = \gamma_v(0)$ and $\angle\big(\gamma_u'(0), \gamma_v'(0)\big) \in (0, \pi)$. Then, there exists a unique solution $\omega \in \Theta^1_\gamma(D)$ to* (50)*.*

Proof.

Step 1 *(Restriction of F to angle distributions coinciding with the local solution).* First, owing to Proposition 10, there exists $L_0 \in (0, L_v]$ such that there exists a unique solution $\omega_\gamma = \mathcal{J}_0(\gamma) \in \Theta^1_\gamma(D_{L_0})$, with $D_{L_0} = [0, L_u]\times[0, L_0]$, to (50). Suppose that $L_0 < L_v$. Otherwise, we have the expected result. We set $\varphi_\gamma := \mathcal{I}(\gamma, \omega_\gamma) \in \Phi^1(D_{L_0})$. Since we cannot expect in the general setting that the u-coordinate curves of the mapping φ_γ are arc-length parametrized, we cannot construct an extension of φ_γ using the curves $\varphi_\gamma(\cdot, L_0)$ and γ_v as new boundary conditions. Therefore, we prove the claim using a fixed-point argument on the angle distributions $\tilde{\omega}_\gamma$ defined to be extensions of $\omega_\gamma : D_{L_0} \to \mathbb{R}$. Let $L_1 \in (L_0, L_v]$ be such that $L_1 \leq 2L_0$ and let $D_{L_1} = [0, L_u]\times[0, L_1]$. We define the set $S_{1,\gamma}(D_{L_1})$ composed of extensions of ω_γ as follows:

$$S_{1,\gamma}(D_{L_1}) = \Big\{\tilde{\omega}_\gamma \in \Theta^1_\gamma(D_{L_1}) \text{ s.t. } \tilde{\omega}_\gamma\big|_{D_{L_0}} = \omega_\gamma\Big\}.$$

Note that $S_{1,\gamma}(D_{L_1})$ is clearly not empty and, to abbreviate the notation, we also denote ω_γ the generic elements of $S_{1,\gamma}$. We now adapt the proof of Proposition 10 to show that F^2 is a contraction mapping in some bounded subset of $S_{1,\gamma}(D_{L_1})$ that is stable by F. Recall from this proof that there exists $\mathcal{R}(\gamma) > 0$, depending only on $\|\gamma_u\|_{\Gamma^2([0,L_u])}$ and $\|\gamma_v\|_{\Gamma^2([0,L_v])}$, such that $F : \mathcal{B}_{\Theta^1(D_{L_1})}(\mathcal{R}(\gamma)) \to \mathcal{B}_{\Theta^1(D_{L_1})}(\mathcal{R}(\gamma))$, where $\mathcal{B}_{\Theta^1(D_{L_1})}(\mathcal{R}(\gamma))$ is the closed ball of radius $\mathcal{R}(\gamma)$ centered at the origin in $\Theta^1_\gamma(D_{L_1})$.

Step 2 *(Contraction mapping).* Let $\sigma = (\sigma_u, \sigma_v) \in \Gamma^2([0, L_u]) \times \Gamma^2([0, L_v])$ be such that $\sigma_u(0) = \sigma_v(0) = \gamma_u(0)$, $\gamma_u'(0) = \sigma_u'(0)$, and $\gamma_v'(0) = \sigma_v'(0)$. Let $\omega_\sigma \in S_{1,\sigma}(D_{L_1})$. We now prove the following counterpart of (52):

$$\|F^2(\omega_\gamma) - F^2(\omega_\sigma)\|_{\Theta^1(D_{L_1})} \leq C\Big(\|\gamma_u - \sigma_u\|_{\Gamma^2([0,L_u])} + \|\gamma_v - \sigma_v\|_{\Gamma^2([0,L_v])} + L_0\|\omega_\gamma - \omega_\sigma\|_{\Theta^1(D_{L_0})} + (L_1 - L_0)\|\omega_\gamma - \omega_\sigma\|_{\Theta^1([0,L_u]\times[L_0,L_1])}\Big), \tag{59}$$

where the constant C is independent of L_0 and L_1. A simple modification of the proof of (11a) implying (20a) gives the following counterpart of (54):

$$\|\mathcal{I}(\gamma, \omega_\gamma) - \mathcal{I}(\sigma, \omega_\sigma)\|_{\Phi^0(D_{L_1})} \leq C\Big((L_1 - L_0)\|\omega_\gamma - \omega_\sigma\|_{\Theta^1([0,L_u]\times[L_0,L_1])} + L_0\|\omega_\gamma - \omega_\sigma\|_{\Theta^1(D_{L_0})} + \|\gamma_u - \sigma_u\|_{\Gamma^2([0,L_u])}\Big),$$

where the constant C is independent of L_0 and L_1. Hence, we obtain the following counterpart of (53):

$$\|F(\omega_\gamma) - F(\omega_\sigma)\|_{\Theta^0(D_{L_1})} \leq C\Big(\|\gamma_u - \sigma_u\|_{\Gamma^2([0,L_u])} + \|\gamma_v - \sigma_v\|_{\Gamma^2([0,L_v])} + L_0\|\omega_\gamma - \omega_\sigma\|_{\Theta^1(D_{L_0})} + (L_1 - L_0)\|\omega_\gamma - \omega_\sigma\|_{\Theta^1([0,L_u]\times[L_0,L_1])}\Big), \tag{60a}$$

$$\|F(\omega_\gamma) - F(\omega_\sigma)\|_{\Theta^1(D_{L_1})} \leq C\Big(\|\gamma_u - \sigma_u\|_{\Gamma^2([0,L_u])} + \|\gamma_v - \sigma_v\|_{\Gamma^2([0,L_v])} + L_0\|\omega_\gamma - \omega_\sigma\|_{\Theta^1(D_{L_0})} + (L_1 - L_0)\|\omega_\gamma - \omega_\sigma\|_{\Theta^1([0,L_u]\times[L_0,L_1])} + \|\omega_\gamma - \omega_\sigma\|_{\Theta^0(D_{L_0})} + \|\omega_\gamma - \omega_\sigma\|_{\Theta^0([0,L_u]\times[L_0,L_1])}\Big), \tag{60b}$$

where the constant C is independent of L_0 and L_1. Then, (59) follows from (60) in the same manner as in the proof of Proposition 10.

Step 3 *(Conclusion).* We set $\mathcal{B}_{S_{1,\gamma}(D_{L_1})}(\mathcal{R}(\gamma)) := \mathcal{B}_{\Theta^1(D_{L_1})}(\mathcal{R}(\gamma)) \cap S_{1,\gamma}(D_{L_1})$. We clearly have that F maps $\mathcal{B}_{S_{1,\gamma}(D_{L_1})}(\mathcal{R}(\gamma))$ onto itself, so that we now restrict F to this ball. We infer from (59) that there exists $L_1^* \in (L_0, L_v]$, with $L_1^* \leq 2L_0$, such that F^2 is a contraction mapping in $\mathcal{B}_{S_{1,\gamma}(D_{L_1^*})}(\mathcal{R}(\gamma))$. Hence, using the Banach fixed-point theorem, we obtain that there exists a unique solution $\omega_\gamma^* \in S_{1,\gamma}(D_{L_1^*})$. Moreover, since the constants in (59) are independent of L_0, we infer that L_1^* is independent of L_0, so that we can repeat (a finite number of times) the argument until we reach L_v. The claim follows.

□

Proposition 15 (Regularity of the global solution)**.** *Let M be a smooth, open, complete, and simply connected surface, let $k \in \mathbb{N}$, and let $D = [0, L_u] \times [0, L_v]$, with $L_u, L_v \in \mathbb{R}_*^+$. Let $\gamma = (\gamma_u, \gamma_v) \in \Gamma^{k+2}([0, L_u]) \times \Gamma^{k+2}([0, L_v])$ be such that $\gamma_u(0) = \gamma_v(0)$ and $\angle\big(\gamma_u'(0), \gamma_v'(0)\big) \in (0, \pi)$. Then, there exists a unique solution $\omega \in \Theta_\gamma^{k+1}(D)$ to (50).*

Proof. The claim is obtained in the same manner as in Proposition 12. $\square$

For all $k \in \mathbb{N}$ and $R_k \in \mathbb{R}_*^+$, we now extend $\mathcal{J}_{k,R_k}$ to the whole rectangle $D = [0, L_u]\times[0, L_v]$ as follows:

$$\mathcal{J}_{k,R_k} : \mathcal{B}_{\Gamma^{k+2}\times\Gamma^{k+2}}(R_k) \to \Theta^{k+1}(D).$$

We prove in the following proposition that $\mathcal{J}_{k,R_k}$ is Lipschitz continuous.

Proposition 16 (Continuity with respect to boundary conditions)**.** *Let M be a smooth, open, complete, and simply connected surface, let $k \in \mathbb{N}$, let $R_k \in \mathbb{R}_*^+$, and let $D = [0, L_u]\times[0, L_v]$, with $L_u, L_v \in \mathbb{R}_*^+$. We equip $\mathcal{B}_{\Gamma^{k+2}\times\Gamma^{k+2}}(R_k)$ and $\Theta^{k+1}(D)$ with the norms in $\Gamma^{k+2}([0, L_u])\times\Gamma^{k+2}([0, L_v])$ and $\Theta^{k+1}(D)$, respectively. Then, the mappings $\mathcal{J}_{k,R_k}$ and*

$$\begin{array}{rccl} \mathcal{I} \circ \big(\mathrm{Id}, \mathcal{J}_{k,R_k}\big) : & \mathcal{B}_{\Gamma^{k+2}\times\Gamma^{k+2}}(R_k) & \to & \Phi^{k+2}(D) \\ & \gamma = (\gamma_u, \gamma_v) & \mapsto & \varphi_\omega := \mathcal{I}(\gamma, \mathcal{J}_{k,R_k}(\gamma)) \end{array}$$

are Lipschitz continuous, with Id *the identity operator in $\mathcal{B}_{\Gamma^{k+2}\times\Gamma^{k+2}}(R_k)$.*

Proof. We first prove the claim in the case where $k = 0$. Let $\gamma, \sigma \in \mathcal{B}_{\Gamma^{k+2}\times\Gamma^{k+2}}(R_k)$, with (γ_u, γ_v) and $\sigma = (\sigma_u, \sigma_v)$, be such that $\gamma_u(0) = \gamma_v(0) = \sigma_u(0) = \sigma_v(0)$, $\gamma_u'(0) = \sigma_u'(0)$ and $\gamma_v'(0) = \sigma_v'(0)$. We moreover suppose that $\angle\big(\gamma_u'(0), \gamma_v'(0)\big) = \angle\big(\sigma_u'(0), \sigma_v'(0)\big) \in (0, \pi)$. We denote $\omega_\gamma := \mathcal{J}_{0,R_0}(\gamma) \in \Theta^1_\gamma(D)$ and $\omega_\sigma := \mathcal{J}_{0,R_0}(\sigma) \in \Theta^1_\sigma(D)$. In the same manner as in the proof of Proposition 11, we infer from (59) that

$$\begin{aligned} \|\omega_\gamma - \omega_\sigma\|_{\Theta^1(D_{L_1})} &\le \tilde{C}\Big[\|\gamma_u - \sigma_u\|_{\Gamma^2([0,L_u])} + \|\gamma_v - \sigma_v\|_{\Gamma^2([0,L_v])} + L_0\|\omega_\gamma - \omega_\sigma\|_{\Theta^1(D_{L_0})}\Big] \\ &\le C\Big[\|\gamma_u - \sigma_u\|_{\Gamma^2([0,L_u])} + \|\gamma_v - \sigma_v\|_{\Gamma^2([0,L_v])}\Big], \end{aligned}$$

using that the mapping (55) is Lipschitz continuous (Proposition 11) for the second inequality. Since the extension process is only operated a finite number of times, we easily obtain by repeating the above argument that the mapping $\mathcal{J}_{0,R_0}$ is Lipschitz continuous. Then, the proof for the case $k > 0$ is obtained in the same manner as Proposition 13. Finally, the Lipschitz continuity of $\mathcal{I} \circ \big(\mathrm{Id}, \mathcal{J}_{k,R_k}\big)$ follows from Proposition 8. $\square$

4.3 Proof of the main result

Proof of Theorem **??**. First, owing to Proposition 15, we obtain the existence of an angle distribution $\omega_\gamma := \mathcal{J}_k(\gamma) \in \Theta^{k+1}_\gamma(D)$ satisfying (50). Then, the continuity of $\mathcal{J}_k$ and $\mathcal{I}$ is a direct consequence of Proposition 16. Hence, to prove the claim, we suppose that $0 < \omega(u, v) < \pi$, for all $(u, v) \in D$, and we show that $\varphi := \mathcal{I}(\gamma, \omega_\gamma) \in \Theta^{k+2}_\gamma(D)$ satisfies (2). We suppose that $k = 0$. Otherwise, this is a direct consequence of Proposition 9. Using the density of $\Gamma^3([0, L_u])$ and $\Gamma^3([0, L_v])$ in $\Gamma^2([0, L_u])$ and $\Gamma^2([0, L_v])$ respectively, let $(\gamma_{u,n})_{n\in\mathbb{N}} \subset \Gamma^3([0, L_u])$

and $(\gamma_{v,n})_{n\in\mathbb{N}} \subset \Gamma^3([0,L_v])$ be sequences satisfying $\gamma_{u,n}(0) = \gamma_u(0)$, $\gamma_{v,n}(0) = \gamma_v(0)$, $\gamma'_{u,n}(0) = \gamma'_u(0)$, $\gamma'_{v,n}(0) = \gamma'_v(0)$, for all $n \in \mathbb{N}$, and

$$\gamma_{u,n} \xrightarrow[n\to\infty]{\Gamma^2} \gamma_u, \quad \text{and} \quad \|\gamma_{u,n}\|_{\Gamma^2([0,L_u])} \leq \|\gamma_u\|_{\Gamma^2([0,L_u])}, \quad \forall n \in \mathbb{N},$$
$$\gamma_{v,n} \xrightarrow[n\to\infty]{\Gamma^2} \gamma_v, \quad \text{and} \quad \|\gamma_{v,n}\|_{\Gamma^2([0,L_v])} \leq \|\gamma_v\|_{\Gamma^2([0,L_v])}, \quad \forall n \in \mathbb{N}.$$

We set $\gamma_n = (\gamma_{u,n}, \gamma_{v,n}) \in \Gamma^3([0,L_u])\times\Gamma^3([0,L_v])$, $\omega_n = \mathcal{J}_1(\gamma_n) \in \Theta^2_\gamma(D)$, and $\varphi_n = \mathcal{I}(\gamma_n, \omega_n) \in \Phi^3(D)$, for all $n \in \mathbb{N}$. Owing to Proposition 13, we infer that the sequence $(\omega_n)_{n\in\mathbb{N}}$ converges to ω in the $C^0(D)$-norm. Hence, there exists $n_0 \in \mathbb{N}$ such that $0 < \omega_n(u,v) < \pi$, for all $(u,v) \in D$ and all $n \geq n_0$. Since $\omega_n \in \Theta^2_\gamma(D)$ and $\gamma_n \in \Gamma^3([0,L_u])\times\Gamma^3([0,L_v])$, we obtain from Proposition 9 that φ_n is a Chebyshev net, for all $n \geq n_0$. Moreover, owing to Proposition 13, we have that $\varphi_n \xrightarrow[n\to\infty]{\Phi^2(D)} \varphi$. Finally, since φ_n satisfies (2) for all $(u,v) \in D$ and all $n \geq n_0$, we conclude that φ satisfies (2) for all $(u,v) \in D$. The claim follows. $\square$

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5 Introduction

We call *surface* a Riemannian 2-manifold, whose metric will be denoted g, and we consider complete, simply connected surfaces. A Chebyshev net $\varphi : U \subset \mathbb{R}^2 \to \varphi(U) \subset M$, with U an open set, is a coordinate system satisfying

$$|\partial_u \varphi|_g(u,v) = |\partial_v \varphi|_g(u,v) = 1, \tag{61}$$

for all $(u,v) \in U$. We construct in what follows Chebyshev nets with a finite set of conical singularities on surfaces with finite total negative curvature and total positive curvature lower than 2π. The main result of this section is the following theorem.

Theorem 17 (Existence of piecewise smooth Chebyshev nets with singularities)**.** *Let M be a smooth, complete, simply connected surface satisfying*

$$\int_M K^+ < 2\pi \quad \textit{and} \quad \int_M K^- < \infty, \tag{62}$$

where K is the Gaussian curvature of M, $K^+ = \max(K,0)$ and $K^- = \max(-K,0)$. Then, there exists a piecewise smooth Chebyshev net with conical singularities $\mathcal{C} = \left(\{B^i_e\}_{1\leq i\leq \mathcal{N}_{\mathrm{pol}}}, \{\varphi^i_e\}_{1\leq i\leq \mathcal{N}_{\mathrm{pol}}}, T\right)$, with $\mathcal{N}_{\mathrm{pol}} \leq \frac{4}{\pi}\int_M K^- + 8$, on M.

We highlight that the proof of the theorem is constructive. The section is organized as follows. In Section 6, we first consider the construction of Chebyshev nets on broken half-surfaces, defined to be half-surfaces with polygonal boundaries. We prove the existence of Chebyshev nets on broken half-surfaces under some condition on their total curvature in Theorem 31. Then, we show in Section 7 that we can split surfaces satisfying (1) into broken half-surfaces (Theorem 36), each of them satisfying the conditions of Theorem 31. Finally, we combine these two results to construct the Chebyshev net with conical singularities in Section 8 and we show that this Chebyshev net is piecewise smooth.

Before this, let us introduce some notation. First, unless explicitly mentioned, any curve $\eta : I \subset \mathbb{R} \to M$ we consider in what follows is arc-length parametrized, continuous on I, and piecewise smooth according to the following definition:

Definition 18 (Piecewise smooth curves)**.** *Let $\eta : I \subset \mathbb{R} \to M$ be a continuous curve. We say that the curve η is* piecewise smooth *if there exists a partition of I in the form $\bigcup_{i=1}^{N+1}[a_{i-1}, a_i] = I$, with $a_0 < ... < a_{N+1}$. Suppose moreover that η restricted to (a_{i-1}, a_i) is a smooth curve with all the derivatives having a finite limit from the right at a_{i-1} and from the left at a_i, for all $i \in \{1, ..., N+1\}$. We say that the curve η is piecewise smooth.*

We denote $\angle(X, Y) \in (-\pi, \pi]$ the oriented angle (using the orientation of M) between the vectors X and Y in the tangent plane T_pM at any point $p \in M$. We define the total positive and negative turn angle of continuous piecewise smooth curves (see [**?**]):

Definition 19 (Positive and negative turn angle $\tau_\pm$)**.** *Let $\eta : I \subset \mathbb{R} \to M$ be a continuous piecewise smooth curve on the partition of I defined by $a_0 < ... < a_{N+1}$. Then, for all $i \in \{1, ..., N+1\}$, let $\kappa_i : [a_{i-1}, a_i] \to \mathbb{R}$ be the geodesic curvature of $\eta\big|_{[a_{i-1},a_i]}$ defined by*

$$\kappa_i(s) = \langle \eta''(s), \eta'^{\perp}(s) \rangle_g,$$

with $\eta'^{\perp}(s) \in T_{\eta(s)}M$ the vector such that $\angle(\eta'(s), \eta'^{\perp}(s)) = \frac{\pi}{2}$, for all $s \in [a_{i-1}, a_i]$. Let $\psi_i = \angle(\eta'(a_i^-), \eta'(a_i^+))$, for all $i \in \{1, ..., N\}$. We suppose that $-\pi < \psi_i < \pi$, for all $i \in \{1, ..., N\}$. We define the total positive and negative turn angles *$\tau(\eta)$ by*

$$\tau_+(\eta) = \sum_{i=1}^{N+1} \int_{a_{i-1}}^{a_i} \kappa_i^+ + \sum_{i=1}^{N} \psi_i^+, \quad \tau_-(\eta) = \sum_{i=1}^{N+1} \int_{a_{i-1}}^{a_i} \kappa_i^- + \sum_{i=1}^{N} \psi_i^-, \tag{63}$$

and we define the total turn angle *as $\tau(\eta) = \tau_+(\eta) - \tau_-(\eta)$. (Note that $\tau_\pm(\eta)$ are different from $(\tau(\eta))^\pm$.) We denote $\tau\big(\eta\big|_{[a,b]}\big)$, with $a, b \in \mathbb{R}$ such that $a < b$, the restriction to $[a, b]$ of the total turn angle. Note that, if any, the pointwise turns at a and b are included.*

An illustration of the turn angle is presented in Figure 5. Finally, we denote $\mathrm{int}(D)$ the interior of any set D.

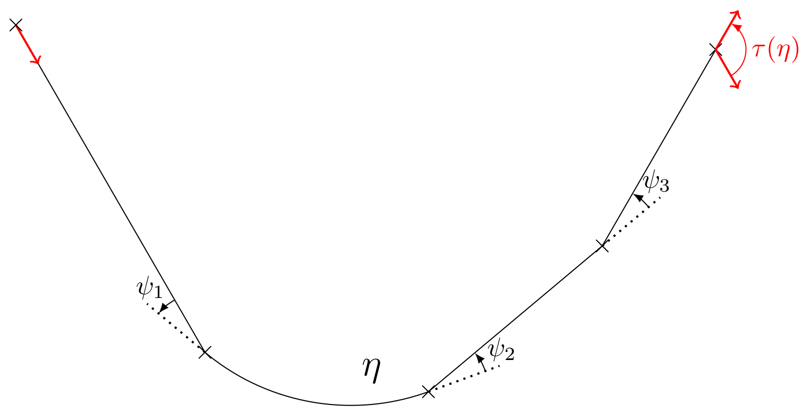


Figure 5: Illustration of the total turn angle $\tau(\eta)$

6 Chebyshev nets on broken half-surfaces

In order to construct a Chebyshev net on broken half-surfaces with $N \geq 1$ vertices, we first restrict ourselves in Section 6.1 to the case of a sector, which corresponds to the case $N = 1$. Then, in Section 6.2, we show that broken half-surfaces admit a Chebyshev parametrization as a particular piecewise smooth sector, under conditions on their total Gaussian curvature.

6.1 Construction on a sector

We give in this section some existence results for Chebyshev nets on sectors.

Definition 20 ((Smooth) sector)**.** *A sector Q of the surface M is an unbounded connected domain of M delimited by the two curves $\eta_1 : \mathbb{R}^- \to M$ and $\eta_2 : \mathbb{R}^+ \to M$ intersecting only at $p = \eta_1(0) = \eta_2(0)$.* Sectors *are said to be smooth whenever the two curves η_1 and η_2 are smooth. The angle $\psi = \angle(\eta_1'(0), \eta_2'(0))$ is supposed to be in $(0, \pi)$ and is called the exterior angle of the* sector *Q.*

A sector with the notation introduced above is presented in Figure 6. Now, we recall a theorem obtained by Bakelman [**?**] and stated in the present form in [**?**]:

Theorem 21 (I. Ya. Bakelman)**.** *Let Q be a sector delimited by the two curves $\eta_1 : \mathbb{R}^- \to M$ and $\eta_2 : \mathbb{R}^+ \to M$ intersecting at $p \in M$. Suppose that Q satisfies the conditions*

$$\tau_+(\eta_1) + \tau_+(\eta_2) + \int_Q K^+ < \pi - \psi, \tag{64a}$$

$$\tau_-(\eta_1) + \tau_-(\eta_2) + \int_Q K^- < \psi, \tag{64b}$$

where $\psi > 0$ is the exterior angle of Q at the vertex p and $\tau_\pm(\eta_i)$, with $i \in \{1,2\}$, are the total positive and negative turn angles of η_i defined by (63)*. Then, there exist global Chebyshev coordinates in Q such that η_1 and η_2 are coordinate curves. Furthermore, the angle between the coordinate curves is bounded away from 0 and π by the positive real number*

$$\min\left(\pi - \psi - \int_Q K^+ - \tau_+(\eta_1) - \tau_+(\eta_2),\ \psi - \int_Q K^- - \tau_-(\eta_1) - \tau_-(\eta_2)\right).$$

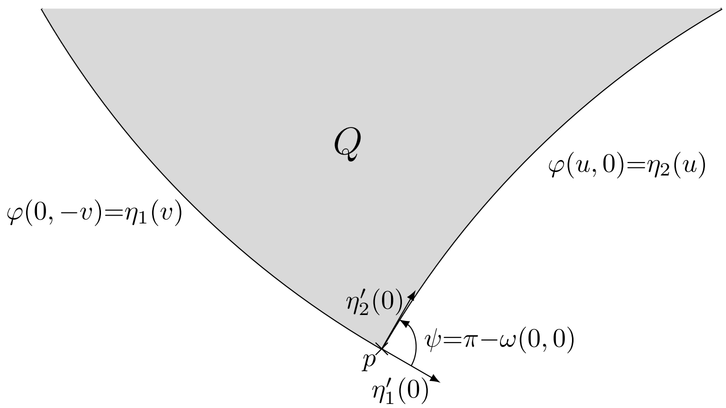


Figure 6: Illustration of a Chebyshev net φ on a sector Q of exterior angle ψ

This theorem gives no information about the regularity of the Chebyshev net, even when the two delimiting curves of the sector are smooth curves. Our goal is now to sharpen Theorem 21 (see Proposition 25 below) to prove the existence of smooth Chebyshev nets on sectors delimited by smooth curves satisfying the counterpart of (64) for smooth curves, namely

$$\int_{\mathbb{R}^+} \kappa_2^+ + \int_{\mathbb{R}^-} \kappa_1^+ + \int_Q K^+ < \pi - \psi \text{ and } \int_{\mathbb{R}^+} \kappa_2^- + \int_{\mathbb{R}^-} \kappa_1^- + \int_Q K^- < \psi. \quad (65)$$

To this purpose, we state some preliminary results. First, we relate in Property 22 the geodesic curvatures of the coordinate curves of Chebyshev nets to the angle ω between these coordinate curves. Then, we present the Hazzidakis formula in Property 23. See [**?**] for a proof of these properties.

Property 22 (Geodesic curvature of coordinate curves)**.** *Let $\varphi : U \subset \mathbb{R}^2 \to \varphi(U) \subset M$ be a smooth mapping satisfying* (61) *and let $(u_1, v_1) \in \mathbb{R}^2$ and $(u_2, v_2) \in \mathbb{R}^2$. We denote $\omega : U \to \mathbb{R}/2\pi\mathbb{Z}$ the angle distribution defined by $\omega(u,v) = \angle(\partial_u\varphi, \partial_v\varphi)(u,v)$, for all $(u,v) \in U$. Then, supposing u_1, v_1 and v_2 are such that $\{u_1\}\times[v_1,v_2] \subset U$, we have*

$$\omega(u_1, v_2) = \omega(u_1, v_1) - \int_{-v_2}^{-v_1} \kappa_v, \quad (66)$$

where $\kappa_v : [-v_2, -v_1] \to \mathbb{R}$ is the geodesic curvature of the curve $\eta_1 : [-v_2, -v_1] \to M$ defined by $\eta_1(v) = \varphi(u_1, -v)$, for all $v \in [-v_2, -v_1]$. This property, illustrated in Figure 7, results from the parallel transport of the vector field $\partial_u\varphi$ along η_1. Moreover, supposing u_1, u_2 and v_1 are such that $[u_1, u_2]\times\{v_1\} \subset U$, we have

$$\omega(u_2, v_1) = \omega(u_1, v_1) - \int_{u_1}^{u_2} \kappa_u, \tag{67}$$

where $\kappa_u : [u_1, u_2] \to \mathbb{R}$ is the geodesic curvature of the curve $\eta_2 : [u_1, u_2] \to M$ defined by $\eta_2(u) = \varphi(u, v_1)$, for all $u \in [u_1, u_2]$.

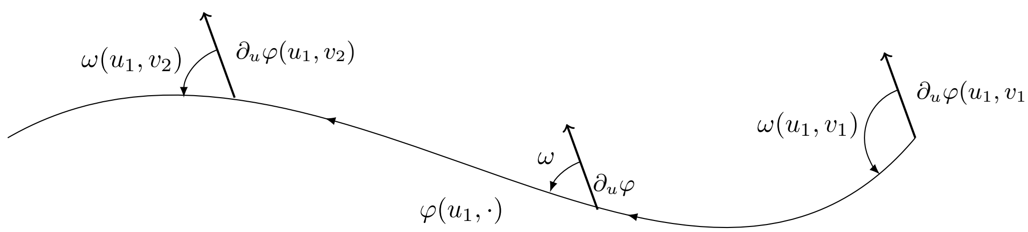


Figure 7: Illustration of the parallel transport of $\partial_u\varphi$ along $\varphi(u_1, \cdot)$

Property 23 (Hazzidakis formula)**.** *Let $U = [u_1, u_2]\times[v_1, v_2]$, with $(u_1, v_1) \in \mathbb{R}^2$ and $(u_2, v_2) \in \mathbb{R}^2$. Let $\varphi : U \to \Omega \subset M$, with $\Omega = \varphi(U)$, be a smooth Chebyshev net. We denote $\omega : U \to (0, \pi)$ the angle distribution defined by $\omega = \angle(\partial_u\varphi, \partial_v\varphi)$ and we denote $\eta_1 : [-v_2, -v_1] \to M$ and $\eta_2 : [u_1, u_2] \to M$ the two curves respectively defined by $\eta_1(v) = \varphi(u_1, -v)$, for all $v \in [-v_2, -v_1]$, and $\eta_2(u) = \varphi(u, v_1)$, for all $u \in [u_1, u_2]$. We denote $\kappa_u : [u_1, u_2] \to \mathbb{R}$ and $\kappa_v : [-v_2, -v_1] \to \mathbb{R}$ their respective geodesic curvature. Then, the angle distribution ω satisfies the following Hazzidakis formula*

$$\omega(u_2, v_2) = \omega(u_1, v_1) - \int_{u_1}^{u_2} \kappa_u - \int_{-v_2}^{-v_1} \kappa_u - \int_{\Omega} K. \tag{68}$$

Lemma 24 (Homeomorphism)**.** *Let Q be a smooth sector delimited by the two smooth curves $\eta_1 : \mathbb{R}^- \to M$ and $\eta_2 : \mathbb{R}^+ \to M$ intersecting at $p \in M$, and satisfying* (65)*. Assume that $\varphi : (\mathbb{R}^+)^2 \to \varphi[(\mathbb{R}^+)^2] \subset M$ is a smooth mapping satisfying* (61)*, and such that $\varphi(u, 0) = \eta_2(u)$, $\varphi(0, v) = \eta_1(-v)$ for all $(u, v) \in (\mathbb{R}^+)^2$. Then, $\varphi : (\mathbb{R}^+)^2 \to Q$ is a homeomorphism.*

Proof. The proof is obtained in the same manner as in [**?**]. We just recall here the principal ideas. We denote $\omega : (\mathbb{R}^+)^2 \to \mathbb{R}/2\pi\mathbb{Z}$ the angle distribution defined

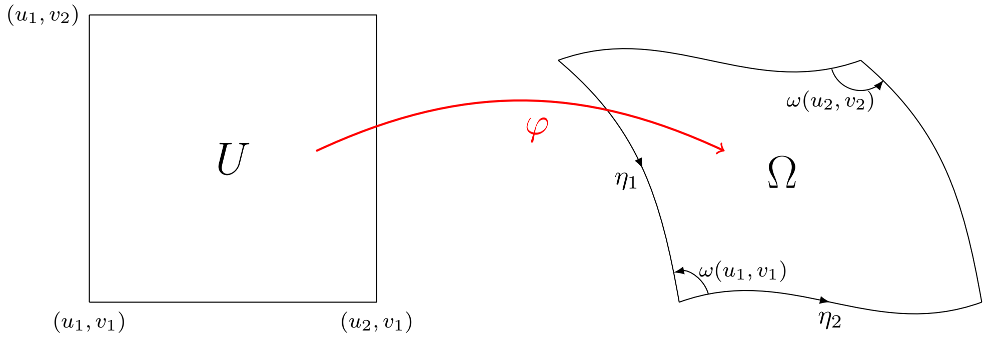


Figure 8: Illustration of the Hazzidakis formula

by $\omega(u,v) = \angle(\partial_u\varphi, \partial_v\varphi)(u,v)$, for all $(u,v) \in (\mathbb{R}^+)^2$. We denote $\kappa_1 : \mathbb{R}^- \to \mathbb{R}$ and $\kappa_2 : \mathbb{R}^+ \to \mathbb{R}$ the geodesic curvatures of η_1 and η_2 respectively. First, using (67), we obtain that

$$\omega(u,0) = \omega(0,0) - \int_0^u \kappa_2 = \pi - \psi - \int_0^u \kappa_2,$$

for all $u \in \mathbb{R}^+$. Then, using hypothesis (65), we deduce that $\omega(u,0) \in (0,\pi)$, for all $u \in \mathbb{R}^+$. In the same manner, we obtain that $\omega(0,v) = \pi - \psi - \int_{-v}^0 \kappa_1 \in (0,\pi)$, for all $v \in \mathbb{R}^+$. Hence, using the continuity of ω, we infer that there exists $\tilde{D} = [0,l_1]\times[0,l_2] \subset (\mathbb{R}^+)^2$, with $l_1, l_2 \in \mathbb{R}^+_*$, such that $\omega(\tilde{D}) \subset (0,\pi)$. Since (61) is satisfied, we infer that $\varphi\big|_{\tilde{D}} : \tilde{D} \to \varphi(\tilde{D}) \subset M$ is a local homeomorphism, so that, up to reducing l_1 and l_2, φ is a homeomorphism. Moreover, since $\omega(\tilde{D}) \subset (0,\pi)$, we deduce that $\angle(\eta_2'(u), \partial_v\varphi(u,0)) \in (0,\pi)$, for all $u \in [0,l_1]$, and $\angle(\eta_1'(-v), \partial_v\varphi(0,v)) \in (0,\pi)$, for all $v \in [0,l_2]$. We conclude that, up to reducing l_1 and l_2, we have $\varphi(\tilde{D}) \subset Q$.

Reasoning by contradiction, we first suppose that φ is not a homeomorphism. Let $U = [0,L_1)\times[0,L_2)$ and $U_{\mathrm{cl}} = [0,L_1]\times[0,L_2]$, with $L_1, L_2 > 0$, be such that $\varphi\big|_U : U \to \varphi(U) \subset M$ is a homeomorphism and such that $\varphi\big|_{U_{\mathrm{cl}}} : U_{\mathrm{cl}} \to \varphi(U_{\mathrm{cl}}) \subset M$ is not a homeomorphism. Using the Hazzidakis formula (68) and hypothesis (65), we easily obtain that $\omega(U_{\mathrm{cl}}) \subset (0,\pi)$. Hence, the mapping $\varphi\big|_{U_{\mathrm{cl}}}$ is a local homeomorphism. Now, suppose that there exist $(u_1,v_1), (u_2,v_2) \in (0,L_1]\times\{L_2\}\cup\{L_1\}\times(0,L_2]$ with $\varphi(u_1,v_1) = \varphi(u_2,v_2)$. Then, the two following cases are possible:

- case 1: $u_1 = u_2 = L_1$ or $v_1 = v_2 = L_2$. We only consider the first subcase, since the reasoning for the second subcase is similar. Then, assuming that $u_1 = u_2 = L_1$, one can see that the Gauss–Bonnet formula applied to the curve $\varphi(\{L_1\}\times[v_1,v_2])$ is in contradiction with (65).
- case 2: $u_1 \neq u_2$ and $v_1 \neq v_2$. In this case, we can suppose, without loss of generality, that $v_1 = L_2$ and $u_2 = L_1$. Then, the Gauss–Bonnet

formula applied to the curve $\varphi([u_1, L_1]\times\{L_2\}) \cup \varphi(\{L_1\}\times[L_2, v_2])$ yields a contradiction with (65).

We finally suppose that $\varphi[(\mathbb{R}^+)^2] \not\subset Q$. Then, let $\tilde{U} = [0, \tilde{L}_1)\times[0, \tilde{L}_2)$, with $\tilde{L}_1, \tilde{L}_2 > 0$, be such that $\varphi(\tilde{U}) \subset Q$ and such that there exists $(\tilde{u}, \tilde{v}) \in (0, \tilde{L}_1]\times\{\tilde{L}_2\} \cup \{\tilde{L}_1\}\times(0, \tilde{L}_2]$ with $\varphi(\tilde{u}, \tilde{v}) \in \partial Q$. Then, $\varphi(\tilde{u}, \tilde{v}) \in \eta_1(\mathbb{R}^-)$ or $\varphi(\tilde{u}, \tilde{v}) \in \eta_2(\mathbb{R}^+)$ and we obtain again a contradiction between the Gauss–Bonnet formula and (65). This concludes the proof. $\square$

Proposition 25 (Existence of smooth Chebyshev nets on sectors)**.** *Let Q be a smooth sector delimited by the two smooth curves $\eta_1 : \mathbb{R}^- \to M$ and $\eta_2 : \mathbb{R}^+ \to M$, and with exterior angle $\psi \in (0, \pi)$. Suppose that the geodesic curvatures $\kappa_1 : \mathbb{R}^- \to \mathbb{R}$ and $\kappa_2 : \mathbb{R}^+ \to \mathbb{R}$ of η_1 and η_2 respectively and the Gaussian curvature K of Q satisfy* (65)*. Then, there exists a unique Chebyshev net $\varphi : (\mathbb{R}^+)^2 \to Q$ such that*

$$\begin{aligned} \varphi(u, 0) &= \eta_2(u), \qquad \forall u \in \mathbb{R}^+, \\ \varphi(0, v) &= \eta_1(-v), \quad \forall v \in \mathbb{R}^+. \end{aligned} \tag{69}$$

Moreover, the angle $\omega = \angle(\partial_u\varphi, \partial_v\varphi) \in (0, \pi)$ of φ satisfies the Hazzidakis formula

$$\omega(u, v) = \pi - \psi - \int_0^u \kappa_2 - \int_{-v}^0 \kappa_1 - \int_{\varphi([0,u]\times[0,v])} K, \tag{70}$$

for all $(u, v) \in (\mathbb{R}^+)^2$.

The Hazzidakis formula in the sector Q is illustrated in Figure 9.

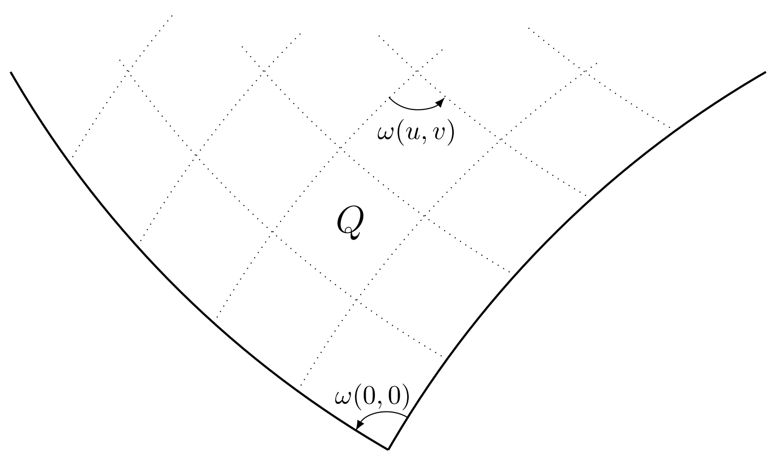


Figure 9: Illustration of the Hazzidakis formula in the sector Q

Before proving this result, we recall a theorem proved in Section **??**.

Theorem 26 (Existence of a unique solution to integrability condition)**.** *Let M be a smooth, open, complete, and simply connected surface. Let $\eta_1 : \mathbb{R}^- \to M$*

and $\eta_2 : \mathbb{R}^+ \to M$ *be two smooth curves with respective geodesic curvatures* $\kappa_1 : \mathbb{R}^- \to \mathbb{R}$ *and* $\kappa_2 : \mathbb{R}^+ \to \mathbb{R}$*, and such that* $\eta_1(0) = \eta_2(0)$*. Suppose that* $\psi := \angle(\eta_1'(0), \eta_2'(0)) \in (0, \pi)$*. Then, there exists a unique angle distribution* $\omega : (\mathbb{R}^+)^2 \to \mathbb{R}/2\pi\mathbb{Z}$ *satisfying the Hazzidakis formula* (70)*, with* $\varphi : (\mathbb{R}^+)^2 \to M$ *the unique smooth mapping satisfying the boundary conditions* (69)*, and such that its* v*-coordinate curves are arc-length parametrized curves with a geodesic curvature* $\kappa_2^{\mathrm{map}} : D \to \mathbb{R}$ *satisfying* $\kappa_2^{\mathrm{map}}(u,v) = \partial_v \omega(u,v)$*, for all* $(u,v) \in (\mathbb{R}^+)^2$*.*

Suppose moreover that there exists $\tilde{D} = [0, \tilde{L}_2] \times [0, \tilde{L}_1]$*, with* $\tilde{L}_1, \tilde{L}_2 \in \mathbb{R}_*^+$*, such that* $0 < \omega(u,v) < \pi$*, for all* $(u,v) \in \tilde{D}$*. Then, the mapping* φ *satisfies*

$$|\partial_u \varphi|_g(u,v) = |\partial_v \varphi|_g(u,v) = 1, \tag{71}$$

for all $(u,v) \in \tilde{D}$*.*

Proof of Proposition 25. Using Theorem 26, we infer that there exists a unique angle distribution $\omega : (\mathbb{R}^+)^2 \to \mathbb{R}/2\pi\mathbb{Z}$ satisfying the Hazzidakis formula (70), with $\varphi : (\mathbb{R}^+)^2 \to M$ the unique mapping satisfying the boundary conditions (69) and the properties presented in the theorem. Then, using the continuity of the angle distribution ω and $\omega(0,0) = \pi - \psi \in (0,\pi)$, we infer that there exists $\tilde{L}_1, \tilde{L}_2 > 0$ such that $\omega(u,v) \in (0,\pi)$ for all $(u,v) \in [0, \tilde{L}_1] \times [0, \tilde{L}_2]$. Hence, by Theorem 26, the mapping φ satisfies (71) for all $(u,v) \in [0, \tilde{L}_1] \times [0, \tilde{L}_2]$. Then, in the same manner as in the proof of Lemma 24, we obtain that $\varphi : (\mathbb{R}^+)^2 \to Q$ is a Chebyshev net. Suppose finally that $\tilde{\varphi} : (\mathbb{R}^+)^2 \to M$ is a Chebyshev net satisfying the boundary conditions (69). Then, using Property 23, we obtain that the angle distribution $\tilde{\omega} : (\mathbb{R}^+)^2 \to (0,\pi)$ defined by $\tilde{\omega} = \angle(\partial_u \tilde{\varphi}, \partial_v \tilde{\varphi})(u,v)$, for all $(u,v) \in (\mathbb{R}^+)^2$, satisfies the Hazzidakis formula (70). We deduce from Theorem 26 that $\varphi = \tilde{\varphi}$. This concludes the proof. □

6.2 Construction on a broken half-surface

We now introduce broken half-surfaces which are defined to be half-surfaces with polygonal boundaries:

Definition 27 ((Geodesic) broken half-surfaces)**.** *Let* $N \geq 1$ *be an integer. We say that* B_c *is a* broken half-surface *if* B_c *is a half-surface delimited by a piecewise smooth curve* $\gamma : \mathbb{R} \to M$ *on the partition of* $\mathbb{R}$ *defined by* $-\infty = a_0 < ... < a_{N+1} = \infty$*. We denote* $p_i = \gamma(a_i)$*, for all* $i \in \{1, ..., N\}$*, and we set* $\gamma_i := \gamma\big|_{[a_{i-1}, a_i]} : [a_{i-1}, a_i] \to M$*, for all* $i \in \{1, ..., N+1\}$*. The points* $\{p_i\}_{1 \leq i \leq N}$ *are called the vertices of* B_c*. We suppose moreover that the exterior angle* $\psi_i = \angle(\gamma_i'(a_i^-), \gamma_{i+1}'(a_i^+))$ *at the vertex* p_i *satisfies* $\psi_i \in (0,\pi)$*, for all* $i \in \{1, ..., N\}$*. Finally, we define*

$$|\psi|_{l^1} = \sum_{1 \leq i \leq N} \psi_i, \quad \textit{and} \quad |\psi|_{l^\infty} = \max_{1 \leq i \leq N} \psi_i$$

and we suppose that $|\psi|_{l^1} < \pi$*.* Broken half-surfaces *are called geodesic when the boundary curves are geodesic curves. Broken half-surfaces with* N *vertices are called* N-half-surfaces*.*

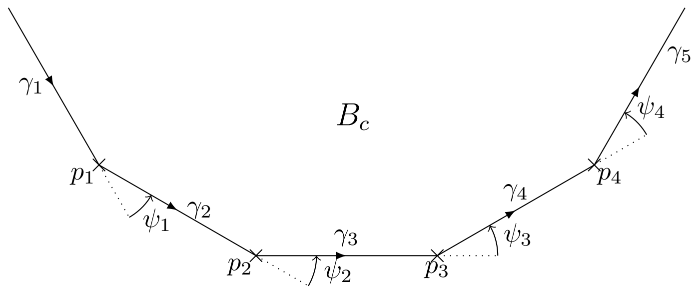


Figure 10: Illustration of a geodesic N-half-surface B_c with $N = 4$

We depict the notation introduced in Definition 27 in Figure 10. Note that the edges composing ∂B_c are depicted as straight edges in this figure although they are more generally curved edges. We observe that 1-half-surfaces are smooth sectors. Now, in order to find Chebyshev nets on broken half-surfaces, we view them as sectors delimited by two piecewise smooth curves. This process, called sectorization, is described in the following definition.

Definition 28 (Sectorization)**.** *Let $N \geq 1$ and let B_c be a N-half-surface delimited by the curves $\{\gamma_i\}_{1 \leq i \leq N+1}$. We denote $\{p_i\}_{1 \leq i \leq N}$ the vertices of B_c. Let $m \in \{1, ..., N\}$. We denote $Q(B_c, p_m)$ the piecewise smooth sector delimited by the curves $\eta_1^m : \mathbb{R}^- \to M$ and $\eta_2^m : \mathbb{R}^+ \to M$ defined so that*

$$\eta_1^m(\mathbb{R}^-) = \bigcup_{i=1}^{m} \gamma_i([a_{i-1}, a_i]) \quad \textit{and} \quad \eta_2^m(\mathbb{R}^+) = \bigcup_{i=m+1}^{N+1} \gamma_i([a_{i-1}, a_i]). \tag{72}$$

The sectorization of a broken half-surface is depicted in Figure 11. We

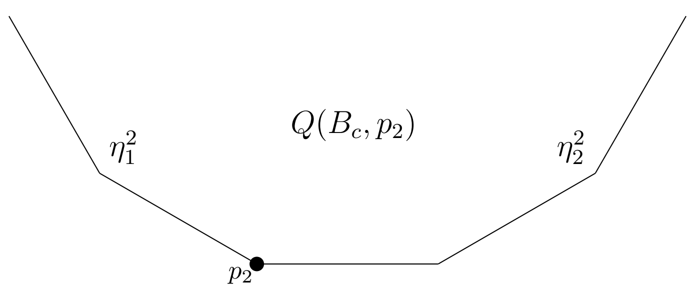


Figure 11: Illustration of the sectorization $Q(B_c, p_2)$ of a N-half-surface B_c with $N = 4$

give in the following proposition conditions on B_c for the existence of a sector $Q(B_c, p_m)$, for some $m \in \{1, ..., N\}$, satisfying the conditions (64).

Proposition 29 (From N-half-surfaces to sectors)**.** *Let $N \geq 1$ be an integer. Suppose that the N-half-surface B_c satisfies the conditions*

$$\tau_+(\partial B_c) + \int_{B_c} K^+ < \pi, \tag{73a}$$

$$\tau_-(\partial B_c) + \int_{B_c} K^- < |\psi|_{l^\infty}. \tag{73b}$$

Then, there exists $m \in \{1, ..., N\}$ such that the piecewise smooth sector $Q(B_c, p_m)$ satisfies the conditions (64)*.*

Proof. Let $m = \operatorname{argmax}_{1\leq i\leq N} \psi_i$ and denote $Q_m = Q(B_c, p_m)$. Then, a straightforward computation gives

$$\int_{Q_m} K^+ + \tau_+(\eta_1) + \tau_+(\eta_2) = \int_{Q_m} K^+ + |\psi|_{l^1} - \psi_m + \sum_{i=1}^{N+1} \int_{a_{i-1}}^{a_i} \kappa_i^+ = \tau_+(\partial B_c) + \int_{B_c} K^+ - \psi_m,$$

$$\int_{Q_m} K^- + \tau_-(\eta_1) + \tau_-(\eta_2) = \int_{Q_m} K^- + \sum_{i=1}^{N+1} \int_{a_{i-1}}^{a_i} \kappa_i^- < \psi_m.$$

Then, conditions (64) follow from (73). □

Corollary 30 (Existence of Chebyshev nets on N-half-surfaces)**.** *Let $N \geq 1$ be an integer. Let B_c be a N-half-surface delimited by the curves $(\gamma_i)_{1\leq i\leq N+1}$. Suppose that B_c satisfies the conditions* (73)*. Then, there exist Chebyshev coordinates on B_c such that $(\gamma_i)_{1\leq i\leq N+1}$ are coordinate curves. Moreover, the angle of the net is bounded away from 0 and π by*

$$\varepsilon = \min\left(\pi - \tau_+(\partial B_c) - \int_{B_c} K^+, \quad |\psi|_{l^\infty} - \tau_-(\partial B_c) - \int_{B_c} K^-\right). \tag{74}$$

Proof. The proof follows by combining Proposition 29 and Theorem 21. □

Finally, in the specific case of geodesic N-half-surfaces, we obtain the following theorem:

Theorem 31 (Existence of Chebyshev nets on geodesic N-half-surfaces)**.** *Let $N \geq 1$. Let B_c be a geodesic N-half-surface delimited by the geodesic curves $\{\gamma_i\}_{1\leq i\leq N+1}$. Suppose B_c satisfies the conditions*

$$\int_{B_c} K^+ < \pi - |\psi|_{l^1}, \tag{75a}$$

$$\int_{B_c} K^- < |\psi|_{l^\infty}. \tag{75b}$$

Then, there exist Chebyshev coordinates on B_c such that $\{\gamma_i\}_{1\leq i\leq N+1}$ are coordinate curves. Moreover, the angle of the net is bounded away from 0 and π by the positive real number

$$\min\left(\pi - |\psi|_{l^1} - \int_{B_c} K^+, |\psi|_{l^\infty} - \int_{B_c} K^-\right). \tag{76}$$

7 Splitting of a surface into geodesic broken half-surfaces

In this section, we show how to split any surface M satisfying the curvature bound (1) into geodesic broken half-surfaces, each of them satisfying the conditions (75). This is the principal result of this section, stated in Theorem 36. This result is obtained in a similar manner to [**?**, Th. 4]: first, we split the surface into four sectors, all of them satisfying (75a) (see Theorem 34). Then, we split recursively each sector into broken half-surfaces, all of them satisfying (75a) (see Theorem 32). We finally prove that, after a finite number of splittings, all of the broken half-surfaces also satisfy the condition (75b).

7.1 Splitting of broken half-surfaces

We prove in this subsection the following theorem which extends the splitting of sectors, introduced in [**?**], to broken half-surfaces.

Theorem 32 (Splitting of N-half-surfaces)**.** *Let $N \geq 1$ and let B_c be a N-half-surface with exterior angles $\{\psi_i^0\}_{1\leq i\leq N}$ satisfying*

$$\int_{B_c} K^+ < \pi - |\psi^0|_{l^1} - 2\varepsilon \text{ and } \int_{B_c} K^- < C, \tag{77}$$

for positive C and ε. Then, there exist $N_1, N_2 \geq 1$ such that

$$N_1 + N_2 \in \{N+1, N+2\} \tag{78}$$

and a geodesic curve σ^* splitting B_c into a N_1-half-surface B_c^1 with exterior angles $\{\psi_i^1\}_{1\leq i\leq N_1}$ and a N_2-half-surface B_c^2 with exterior angles $\{\psi_i^2\}_{1\leq i\leq N_2}$ satisfying

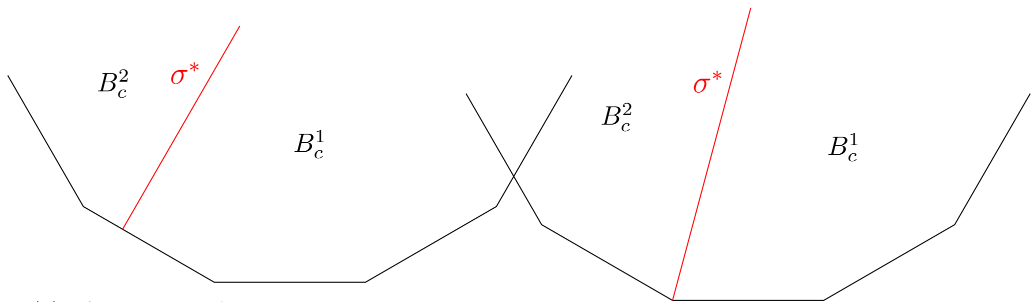
$$\int_{B_c^1} K^+ < \pi - |\psi^1|_{l^1} - \varepsilon, \quad \int_{B_c^1} K^- < \frac{C}{2}, \tag{79a}$$

$$\int_{B_c^2} K^+ < \pi - |\psi^2|_{l^1} - \varepsilon, \quad \int_{B_c^2} K^- < \frac{C}{2}, \tag{79b}$$

with $|\psi^1|_{l^1} = \sum_{i=1}^{N_1} \psi_i^1$, and $|\psi^2|_{l^1} = \sum_{i=1}^{N_2} \psi_i^2$.

Remark 33 (N_1 and N_2). Two different cases can happen for the splitting : either the geodesic curve σ^* intersects ∂B_c at some vertex and we have $N_1 + N_2 = N+1$, or σ^* intersects ∂B_c in the interior of some edge and we have $N_1 + N_2 = N + 2$. See Figure 12 for an illustration of these two cases.

Proof. We adapt the proof of [**?**, Thm.3] to N-half-surfaces. Let us first recall the setting of this proof. We can assume the metric to be flat outside a compact set $\tilde{D} \subset \mathrm{int}(B_c)$ homeomorphic to the disk. Moreover, we can suppose that $\tilde{D}$



(a) σ^* intersects ∂B_c in the interior of some of its edges : $N_1 = 4$ and $N_2 = 2$

(b) σ^* intersects ∂B_c at some of its vertices : $N_1 = 3$ and $N_2 = 2$

Figure 12: Illustration of the two possible cases for the splitting in Theorem 32 ($N = 4$)

is totally convex, i.e., all the geodesic curves joining two points $p, q \in \tilde{D}$ are included in $\tilde{D}$. Let $T_p\tilde{D}$ be the tangent plane at the point $p \in \tilde{D}$ and let

$$S\tilde{D} = \left\{(p, \theta), \text{ with } p \in \tilde{D},\ \theta \in T_p\tilde{D},\ |\theta|_g = 1\right\}$$

be the circle bundle over $\tilde{D}$. For any $v = (p, \theta) \in S\tilde{D}$, we denote $-v = (p, -\theta)$ and $\sigma(v) : \mathbb{R} \to M$ the unique geodesic curve passing through the point p orthogonally to v (the orientation of $\sigma(v)$ has no importance in what follows). Since $\tilde{D}$ is totally convex, the geodesic $\sigma(v)$ splits $\tilde{D}$ into two connected components. The vector v is directed inwards one of these components, which we denote by $U(v)$, and the other component is then denoted $U(-v)$. We now define a function α which plays a similar role to the angular function α in the original proof of [**?**, Thm.3]. The continuous function $\alpha : S\tilde{D} \to [0, \pi]$ should satisfy $\alpha(v) + \alpha(-v) = \pi - |\psi^0|_{l^1}$ for all $v \in S\tilde{D}$. For this definition, different cases, depicted in Figure 13, have to be considered:

1. $U(v)$ is a half-surface with boundary $\sigma(v)$;
2. $U(v)$ is a so-called polygonal strip;
3. $U(v)$ and $U(-v)$ are respectively a N_1-half-surface and a N_2-half-surface. We denote $\{\psi_i^1(v)\}_{1 \leq i \leq N_1}$ the exterior angles of B_c^1 and we set $|\psi^1|_{l^1} := \sum_{i=1}^{N_1} \psi_i^1(v)$;
4. $U(v)$ is a bounded polygonal domain with N_1 vertices. We denote $\{\psi_i^1\}_{1 \leq i \leq N_1}$ the exterior angles of $U(v)$ and we set $|\psi^1|_{l^1} := \sum_{i=1}^{N_1} \psi_i^1$;
5. $U(v)$ is the complementary of a bounded polygonal domain (so that $U(-v)$ is a bounded polygonal domain).

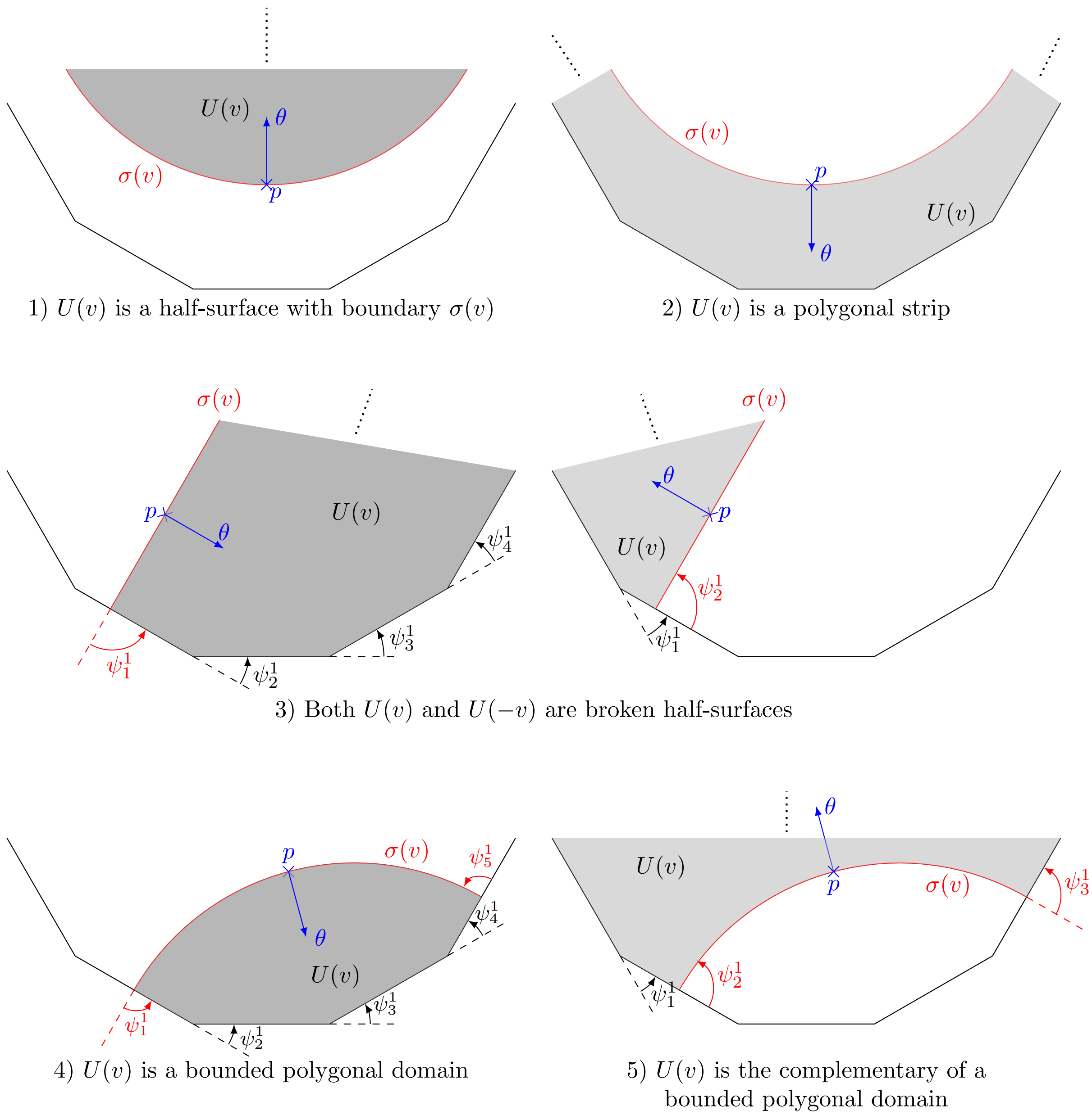


Figure 13: Illustration of the possible splittings of a N-half-surface with $N = 4$ (proof of Theorem 32)

We emphasize that, for all $v \in \tilde{D}$, $U(v)$ belongs to one of the above cases. Then, we define the function α in each of these cases as follows

$$\alpha(v) = \begin{cases} \pi - |\psi^0|_{l^1}, & \text{in case 1,} \\ 0, & \text{in case 2,} \\ \max\left[\min\left(\pi - |\psi^1|_{l^1}, \pi - |\psi^0|_{l^1}\right), 0\right], & \text{in case 3,} \\ \max\left[\min\left(2\pi - |\psi^1|_{l^1}, \pi - |\psi^0|_{l^1}\right), 0\right], & \text{in case 4,} \\ \pi - |\psi^0|_{l^1} - \alpha(-v), & \text{in case 5.} \end{cases} \tag{80}$$

Using the continuity of $|\psi^1|_{l^1}$ as $\sigma(v)$ crosses the vertices of B_c and the continuity of all the case transitions, one can check that $\alpha : S\tilde{D} \to [0, \pi]$ is a continuous function which satisfies $\alpha(-v)+\alpha(v) = \pi - |\psi^0|_{l^1}$. Now, we introduce the mapping $\xi : S\tilde{D} \to \mathbb{R}^2$ defined by

$$\xi(v) = (\xi_1(v), \xi_2(v)) = \left((\pi - |\psi^0|_{l^1} - 2\varepsilon) \frac{\int_{U(v)} K^+}{\int_{\tilde{D}} K^+} - \alpha(v) + \varepsilon, \quad \frac{\int_{U(v)} K^-}{\int_{\tilde{D}} K^-} - \frac{1}{2} \right).$$

Then, ξ_1 satisfies

$$\begin{aligned} \xi_1(-v) &= (\pi - |\psi^0|_{l^1} - 2\varepsilon) \frac{\int_{\tilde{D}} K^+ - \int_{U(v)} K^+}{\int_{\tilde{D}} K^+} - \alpha(-v) + \varepsilon \\ &= \pi - |\psi^0|_{l^1} - 2\varepsilon - (\pi - |\psi^0|_{l^1} - 2\varepsilon) \frac{\int_{U(v)} K^+}{\int_{\tilde{D}} K^+} - (\pi - |\psi^0|_{l^1} - \alpha(v)) + \varepsilon \\ &= -\xi_1(v). \end{aligned}$$

In the same manner, we obtain that $\xi_2(-v) = -\xi_2(v)$, so that $\xi(-v) = -\xi(v)$. Therefore, using [**?**, Prop. 1], we can conclude that there exists $v_0 \in S\tilde{D}$ such that $\xi(v_0) = (0, 0)$, which corresponds to

$$\alpha(v_0) = (\pi - |\psi^0|_{l^1} - 2\varepsilon) \frac{\int_{U(v_0)} K^+}{\int_{\tilde{D}} K^+} + \varepsilon, \quad \int_{U(v_0)} K^- = \frac{1}{2} \int_{\tilde{D}} K^-. \tag{81}$$

We now prove that $U(v_0)$ necessarily matches case 3. First, by (81), we have

$$\alpha(v_0) \in [\varepsilon, \pi - |\psi^0|_{l^1} - \varepsilon], \tag{82}$$

which rules out cases 1 and 2. In order to rule out cases 4 and 5, suppose now that $U(v)$ is a bounded polygonal domain. Applying the Gauss–Bonnet formula on $U(v)$, we infer that

$$|\psi^1|_{l^1} + \int_{U(v)} K = 2\pi,$$

which gives $2\pi - |\psi^1|_{l^1} \le \int_{U(v_0)} K^+$. Then, using the hypotheses (77) and (81), we obtain that

$$\int_{U(v_0)} K^+ + \varepsilon < \alpha(v_0). \tag{83}$$

Combining these two results and the definition of α in case 4, we obtain the following contradiction:

$$2\pi - |\psi^1|_{l^1} \leq \int_{U(v_0)} K^+ < \alpha(v_0) - \varepsilon \leq 2\pi - |\psi^1|_{l^1} - \varepsilon.$$

Finally, if $U(v_0)$ matches case 5, then $\xi(-v_0) = 0$. Hence, $U(-v_0)$ matches case 4 which leads, as above, to a contradiction. Therefore $U(v_0)$ necessarily matches case 3, i.e., both $U(v_0)$ and $U(-v_0)$ are broken half-surfaces. Moreover, (80) and (82) show that $\alpha(v_0) = \pi - |\psi^1|_{l^1}$. Then, using (81) and (83), we infer that (79) is satisfied by $U(v_0)$. Since $\xi(-v_0) = 0$, we obtain, by symmetry, the same result for $U(-v_0)$. Finally, recalling Remark 33, we have $N_1 + N_2 \in \{N+1, N+2\}$. This concludes the proof. $\square$

7.2 Recursive splitting

We first restate a result by Burago *et al* [**?**, Theorem 3] that allows one to split surfaces satisfying

$$\int_M K^+ < 2\pi - 4\varepsilon, \quad \text{and} \quad \int_M K^- < C - 4\varepsilon, \tag{84}$$

for positive C and ε, into four sectors delimited by geodesic curves, all of them satisfying the condition (75a). This result is stated in [**?**] with $C = 2\pi$, but the proof is valid in the general setting.

Theorem 34 (Burago *et al.*)**.** *Let M be a complete Riemannian 2-manifold homeomorphic to the plane and satisfying the conditions* (84)*, for positive C and ε. Then, there exist four sectors $\{Q_i\}_{1\leq i\leq 4}$ with exterior angles $\{\psi_i\}_{1\leq i\leq 4}$ and delimited by geodesic curves such that $int(Q_i) \cap int(Q_j) = \varnothing$ for all $i \neq j$ and $\cup_{1\leq i\leq 4} Q_i = M$, and such that, for all $i \in \{1, ..., 4\}$, the sector Q_i satisfies the conditions*

$$\int_{Q_i} K^+ \leq \pi - \psi_i - \varepsilon, \quad \int_{Q_i} K^- \leq \frac{C}{2\pi}\psi_i - \varepsilon. \tag{85}$$

The four sectors obtained by this theorem are sketched in Figure 14. We also need the following lemma to bound from below the exterior angles of the broken half-surfaces resulting from splitting:

Lemma 35 (Bound on exterior angles)**.** *Let $N \geq 1$ and let B_c be a N-half-surface satisfying the condition* (75a)*. Let $\sigma : \mathbb{R}^+ \to B_c$ be a geodesic curve with $\sigma(0) \in \partial B_c$ and suppose that σ splits B_c into a N_1-half-surface B_c^1 and a N_2-half-surface B_c^2, both of which satisfy the condition* (75a)*. Then, denoting $\{\psi_k^0\}_{1\leq k\leq N}$, $\{\psi_k^1\}_{1\leq k\leq N_1}$, and $\{\psi_k^2\}_{1\leq k\leq N_2}$, the positive exterior angles of B_c, B_c^1, and B_c^2, respectively, we have*

$$\forall i \in \{1, 2\}, \quad |\psi^i|_{l^\infty} \geq |\psi^0|_{l^\infty}. \tag{86}$$

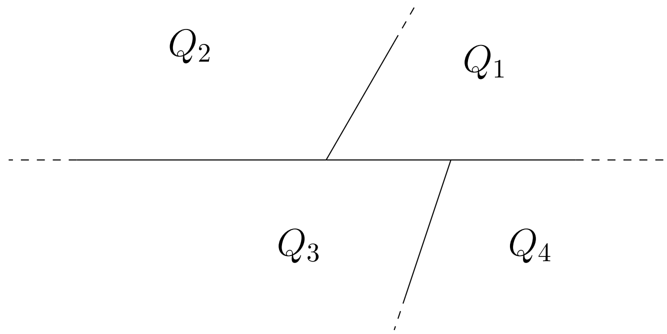


Figure 14: Illustration of the splitting of Theorem 34

Proof. We denote ψ_k^1, with $k \in \{1, ..., N_1\}$, and ψ_l^2, with $l \in \{1, ..., N_2\}$, the exterior angle of B_c^1 and B_c^2 respectively at the intersection of σ with ∂B_c. Two cases can occur: either the intersection point is not located at some vertex of ∂B_c, or σ intersects a vertex $p_n \in \partial B_c$, with $n \in \{1, ..., N\}$, with exterior angle ψ_n^0 (see Remark 33 and Figure 12). In the first case, we have $\psi_k^1 + \psi_l^2 = \pi$. In the second case, since $\psi_n^0 \geq 0$, we have $\psi_k^1 + \psi_l^2 = \pi + \psi_n^0 \geq \pi$. In both cases, we infer that

$$\psi_k^1 + \psi_l^2 \geq \pi. \tag{87}$$

Note that all the angles of both B_c^1 and B_c^2 are angles of B_c, except for the angle newly created by the intersection of σ with ∂B_c. Let ψ_m^0, with $m \in \{1, ..., N\}$, be an exterior angle of B_c such that $\psi_m^0 = |\psi^0|_{l^\infty}$. Let p_m be the corresponding vertex of ∂B_c. We only prove that (86) is satisfied for $i = 1$, since the case $i = 2$ is treated similarly. We remark that three cases can occur:

1. If p_m is contained in B_c^1, the result is straightforward;
2. If p_m is contained in B_c^2, applying (87) and the condition (75a) in B_c^1, we obtain that
$$\psi_k^1 \geq \pi - \psi_l^2 \geq |\psi^0|_{l^1} - \psi_l^2 + \int_{B_c^2} K^+ \geq \psi_m^0.$$
3. Finally, if σ intersects ∂B_c at p_m, we have $\psi_k^1 + \psi_l^2 = \pi + \psi_m^0$. Since $\psi_l^2 \leq \pi$, we infer that $\psi_k^1 \geq \psi_m^0$.

In all the cases, we obtain the expected result. This concludes the proof. □

We can now prove the main result of this section.

Theorem 36 (Surface splitting into broken half-surfaces)**.** *Let M be a smooth, complete, simply connected surface. Suppose that M satisfies the curvature bound* (1)*, i.e.,*

$$\int_M K^+ < 2\pi \quad \textit{and} \quad \int_M K^- < \infty,$$

with K the Gaussian curvature of M, $K^+ = \max(K, 0)$ and $K^- = \max(-K, 0)$. We set $n_{\max} := \log_2\left(\frac{1}{\pi}\int_M K^- + 1\right) + 2$. Then, there exist $\mathcal{N}_{\rm pol} \leq \frac{4}{\pi}\int_M K^- + 8$ geodesic N_α-half-surfaces $\{B_c^\alpha\}_{1\leq\alpha\leq\mathcal{N}_{\rm pol}}$, with $N_\alpha \leq n_{\max}$ for all $\alpha \in \{1, ..., \mathcal{N}_{\rm pol}\}$, satisfying the conditions (75) *and such that* $\mathrm{int}(B_c^\alpha)\cap\mathrm{int}(B_c^\beta) = \varnothing$ *for all $\alpha \neq \beta$ and $M = \cup_{\alpha=1}^{\mathcal{N}_{\rm pol}} B_c^\alpha$.*

Proof. Let $\bar{\varepsilon} = \frac{1}{5}\left(2\pi - \int_M K^+\right)$ and $\bar{C} = \int_M K^- + 5\bar{\varepsilon}$. Then, the hypotheses of Theorem 34 are satisfied by M with $\varepsilon = \bar{\varepsilon}$ and $C = \bar{C}$. We denote $\{S_{\alpha,0}\}_{1\leq\alpha\leq 4}$ the four sectors satisfying (85) obtained by this theorem. As condition (75a) is satisfied by each 1-half-surface $S_{\alpha,0}$ the proof consists in applying multiple times Theorem 32 to all of them so as to split these 1-half-surfaces recursively until condition (75b) on the total negative curvature is satisfied. Since each broken half-surface is treated similarly, we only enumerate one broken half-surface in each sector $S_{\alpha,0}$ at each step of the subdividision. (Note that, otherwise, multi-indices should have been introduced.) See Figure 15 for an illustration of the resulting splitting. We denote, for all $\alpha \in \{1, ..., 4\}$, $\psi_\alpha > 0$ the exterior angle of $S_{\alpha,0}$ and we set $C_\alpha = \frac{\bar{C}\psi_\alpha}{2\pi}$. Since $S_{\alpha,0}$ satisfies (85), the hypotheses of Theorem 32 are satisfied with $B_c = S_{\alpha,0}$, $\varepsilon = \frac{\bar{\varepsilon}}{3}$ and $C = C_\alpha$. Hence, there exists a splitting of $S_{\alpha,0}$ into two broken half-surfaces satisfying the conditions (79). By symmetry, we consider only one of them, denoted $S_{\alpha,1}$. In the same manner, we apply recursively Theorem 32 with $B_c = S_{\alpha,n-1}$, $\varepsilon = \frac{\bar{\varepsilon}}{3.2^{n-1}}$ and $C = \frac{C_\alpha}{2^{n-1}}$. This yields a splitting of $S_{\alpha,n-1}$ into two broken half-surfaces satisfying the conditions (79). By symmetry, we consider only one of them, the N-half-surface denoted $S_{\alpha,n}$, whose exterior angles are denoted $\{\psi_k^n\}_{1\leq k\leq N}$. Note that $N \leq n+1$ by (78). Then, Lemma 35 ensures that $|\psi^n|_{l^\infty} \geq \psi_\alpha$ (with $|\psi^n|_{l^\infty} = \max_{1\leq k\leq N}\psi_k^n$). Therefore, condition (75b) is satisfied by $S_{\alpha,n}$ whenever

$$\int_{S_{\alpha,n}} K^- < |\psi^n|_{l^\infty} \leq \psi_\alpha. \tag{88}$$

Since $\int_{S_{\alpha,n}} K^- \leq \frac{C_\alpha}{2^n}$, we infer that (88) is satisfied whenever $n = n_\alpha^{\max}$, where

$$n_\alpha^{\max} = \left\lceil \log_2\left(\frac{C_\alpha}{\psi_\alpha}\right)\right\rceil = \left\lceil \log_2\left(\frac{\bar{C}}{2\pi}\right)\right\rceil$$

and $\lceil\cdot\rceil$ is the ceiling function. Therefore, we have proved that there exist $\mathcal{N}_\alpha$ N_i-half-surfaces $\{B_c^{\alpha,i}\}_{1\leq i\leq\mathcal{N}_\alpha}$, with $N_i \leq n_\alpha^{\max} + 1$, such that $\mathrm{int}(B_c^{\alpha,i}) \cap \mathrm{int}(B_c^{\alpha,j}) = \varnothing$ for all $i \neq j$. Moreover, we have $\mathcal{N}_\alpha \leq \mathcal{N}_\alpha^{\max}$, with:

$$\mathcal{N}_\alpha^{\max} = 2^{n_\alpha^{\max}} \leq \frac{\bar{C}}{\pi} \leq \frac{1}{\pi}\int_M K^- + 2. \tag{89}$$

We finally define $\mathcal{N}_{\rm pol} = \sum_{\alpha=1}^4 \mathcal{N}_\alpha$ and the set of N_i-half-surfaces $\{B_c^i\}_{1\leq i\leq\mathcal{N}_{\rm pol}}$ as the union of the sets $\{B_c^{\alpha,i}\}_{1\leq i\leq\mathcal{N}_\alpha}$, for $\alpha \in \{1, ..., 4\}$. For all $i \in \{1, ..., \mathcal{N}_{\rm pol}\}$, the number N_i of vertices of B_c^i satisfies

$$N_i \leq \max_{\alpha\in\{1,\dots,4\}} n_\alpha^{\max} + 1 = \left\lceil \log_2\left(\frac{\bar{C}}{2\pi}\right)\right\rceil + 1$$
$$\leq \log_2\left(\frac{1}{2\pi}\int_M K^- + 1\right) + 2 = n_{\max}.$$

Moreover, using (89), we obtain that the number $\mathcal{N}_{\mathrm{pol}}$ of polygons satisfies

$$\mathcal{N}_{\mathrm{pol}} = \sum_{\alpha=1}^{4} \mathcal{N}_\alpha \leq \frac{4}{\pi}\int_M K^- + 8.$$

The claim follows. □

Remark 37 (Tree representation). The construction can be seen as a binary tree of broken half-surfaces, each splitting being an edge, $n_{\max}$ being the maximal depth of the tree, and $\mathcal{N}_{\mathrm{pol}}$ being the maximal number of leaves of the tree. Once the splitting is achieved, we renumber the broken half-surfaces to obtain the set $\{B_c^\alpha\}_{1\leq\alpha\leq\mathcal{N}_{\mathrm{pol}}}$.

Definition 38 (Skeleton). *The graph in the surface M defined by the vertices of the boundaries of the broken half-surfaces $\{B_c^\alpha\}_{1\leq\alpha\leq\mathcal{N}_{\mathrm{pol}}}$ obtained using Theorem 36 and the edges (geodesic curves) joining the vertices is called the* skeleton. *The vertices and the edges in the* skeleton *are respectively enumerated as $\{p_c^i\}_{1\leq i\leq\mathcal{N}_{\mathrm{ver}}}$ and $\{\gamma_c^i\}_{1\leq i\leq\mathcal{N}_{\mathrm{ed}}}$.*

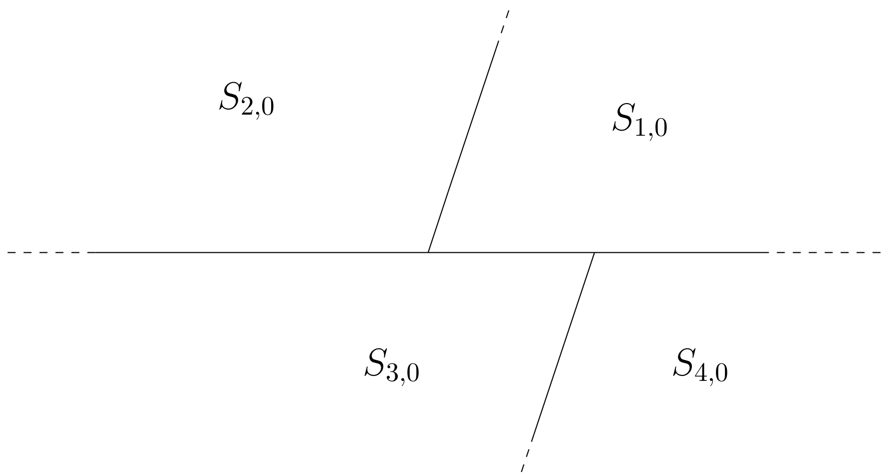
An example of skeleton is then presented in Figure 16.

8 Proof of the main theorem

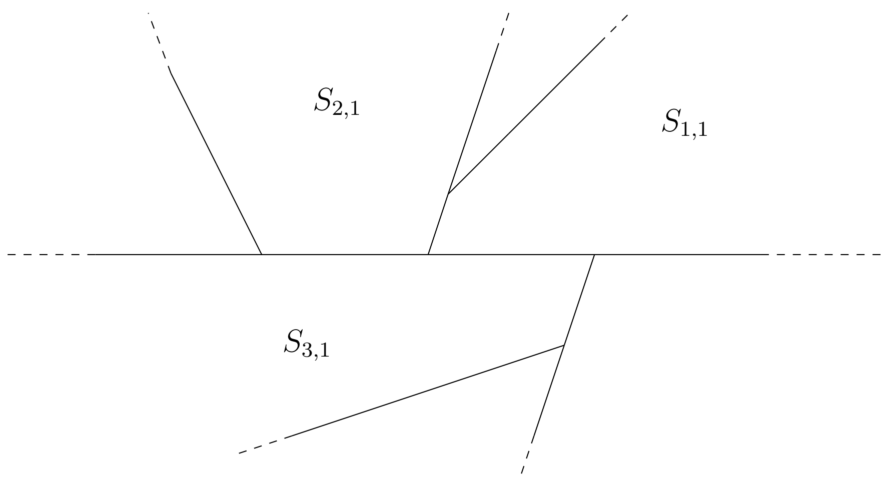
We prove in this section Theorem 1 on the existence of piecewise smooth Chebyshev nets with singularities on surfaces M satisfying the curvature bound (1). We first gather the results of Theorem 31 and Theorem 36 to construct a Chebyshev net with singularities on M (Theorem 39). Then, we show that the Chebyshev parametrization obtained on each broken half-surface by Theorem 31 is piecewise smooth (Theorem 44). The proof of Theorem 1 then follows from Theorem 39 and Theorem 44.

8.1 Existence of a Chebyshev net with conical singularities

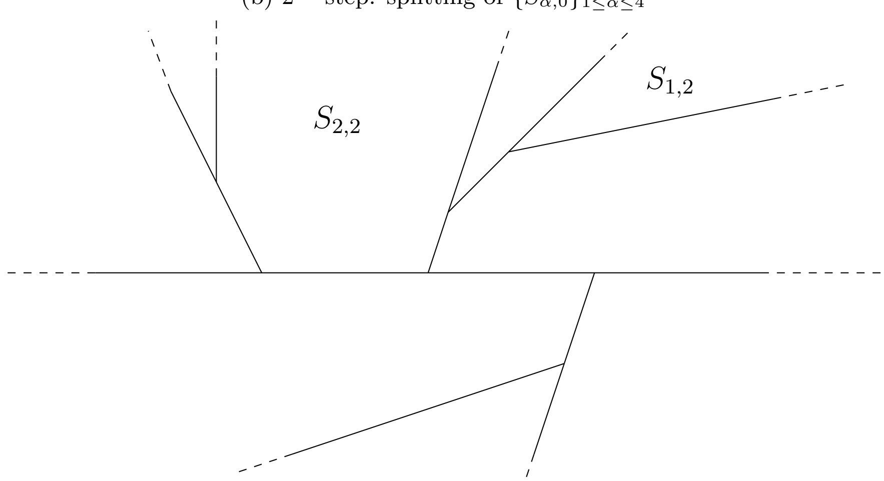
Theorem 39 (Existence of Chebyshev nets with singularities). *Let M be a smooth, complete, simply connected surface satisfying the curvature bound (1). Then, there exists a Chebyshev net with conical singularities $\mathcal{C} = \left(\{B_e^i\}_{1\leq i\leq\mathcal{N}_{\mathrm{pol}}}, \{\varphi_e^i\}_{1\leq i\leq\mathcal{N}_{\mathrm{pol}}}, T\right)$, with $\mathcal{N}_{\mathrm{pol}} \leq \frac{4}{\pi}\int_M K^- + 8$, on M.*



(a) 1^{st} step: splitting of M into 4 sectors



(b) 2^{nd} step: splitting of $\{S_{\alpha,0}\}_{1\leq\alpha\leq 4}$



(c) 3^{rd} step: splitting of $\{S_{\alpha,1}\}_{1\leq\alpha\leq 4}$

Figure 15: Illustration of the recursive splitting used for the proof of Theorem 36

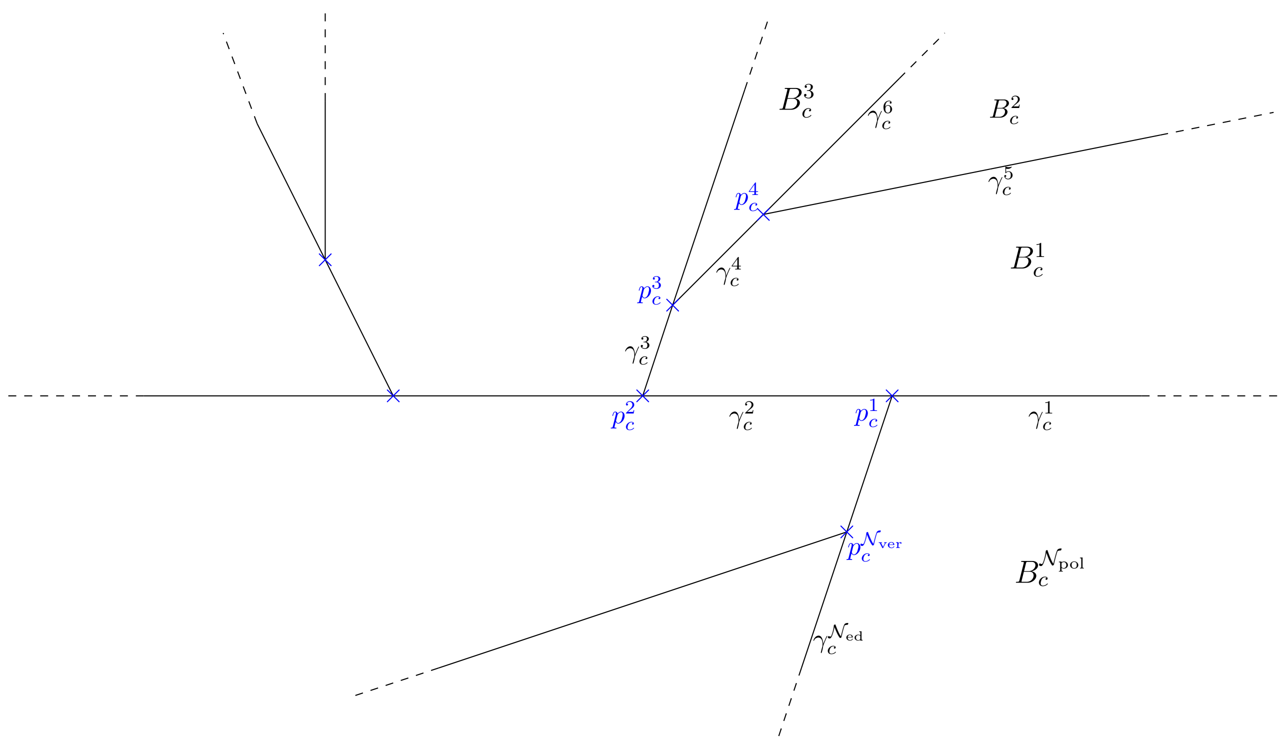


Figure 16: An example of skeleton

Proof. First, we apply Theorem 36 to obtain a splitting of M into broken half-surfaces $\{B_c^i\}_{1\le i\le \mathcal{N}_{\mathrm{pol}}}$, with $\mathcal{N}_{\mathrm{pol}} \le \frac{4}{\pi}\int_M K^- + 8$, all of them satisfying the conditions

$$\int_{B_c^i} K^+ < \pi - |\psi^i|_{l^1}, \quad \int_{B_c^i} K^- < |\psi^i|_{l^\infty}.$$

Then, owing to Theorem 31, we infer that there exists a Chebyshev parametrization $\varphi_i : (\mathbb{R}^+)^2 \to B_c^i$, for all $i \in \{1, ..., \mathcal{N}_{\mathrm{pol}}\}$. We set $B_e^i = (\mathbb{R}^+)^2$, for all $i \in \{1, ..., \mathcal{N}_{\mathrm{pol}}\}$. We construct the equivalence table $T : \{1, ..., \mathcal{N}_{\mathrm{pol}}\} \to \mathbb{N}$ as follows. For all $i \in \{1, ..., \mathcal{N}_{\mathrm{pol}}\}$, we set $T(i, i) = 0$. For all $i, j \in \{1, ..., \mathcal{N}_{\mathrm{pol}}\}$ such that $i \ne j$, we set $T(i, j) = T(j, i) = 0$ if $B_c^i \cap B_c^j = \varnothing$. We now suppose that $B_c^i \cap B_c^j \ne \varnothing$. Let $\{\gamma_c^{i,\alpha}\}_{1\le\alpha\le N_i}$ and $\{\gamma_c^{j,\beta}\}_{1\le\beta\le N_j}$ be the edges of the skeleton that are included in B_c^i and B_c^j respectively. Then, by construction (see Theorem 32), $B_c^i \cap B_c^j = \gamma_c^{i,\alpha_0} = \gamma_c^{j,\beta_0}$ for some $\alpha_0 \in \{1, ..., N_i\}$ and $\beta_0 \in \{1, ..., N_j\}$, and, we set $T(i, j) = \alpha_0$ and $T(j, i) = \beta_0$ (see Figure 17). We conclude that $\mathcal{C} = \left(\{B_e^i\}_{1\le i\le\mathcal{N}_{\mathrm{pol}}}, \{\varphi_e^i\}_{1\le i\le\mathcal{N}_{\mathrm{pol}}}, T\right)$ is a Chebyshev net with conical singularities on M. $\square$

8.2 Piecewise smooth Chebyshev nets on broken half-surfaces

Let us notice that whenever all the broken half-surfaces $\{B_c^i\}_{1\le i\le\mathcal{N}_{\mathrm{pol}}}$ obtained in Theorem 36 are smooth sectors (only one vertex), Theorem 1 follows from Propositions 25 and 39. In the general case, we need to prove that, for all $i \in \{1, ..., \mathcal{N}_{\mathrm{pol}}\}$, the Chebyshev parametrization $\varphi_i : B_e^i = (\mathbb{R}^+)^2 \to B_c^i$ obtained from Theorem 31 on the geodesic broken half-surface B_c^i satisfying the conditions

$$\int_{B_c^i} K^+ < \pi - |\psi^i|_{l^1}, \quad \int_{B_c^i} K^- < |\psi^i|_{l^\infty}, \tag{90}$$

is piecewise smooth. With this purpose in mind, we proceed as in Section 6: we first consider the case of a piecewise smooth sector Q of exterior angle $\psi \in (0, \pi)$ satisfying the conditions

$$\tau_+(\eta_1) + \tau_+(\eta_2) + \int_Q K^+ < \pi - \psi, \tag{91a}$$

$$\tau_-(\eta_1) + \tau_-(\eta_2) + \int_Q K^- < \psi. \tag{91b}$$

Before this, we state the following lemma on the geodesic curvature of parameter curves of a smooth Chebyshev net. The notations used in this lemma are depicted in Figure 18.

Lemma 40. *Let $a, b \in \mathbb{R}_*^+$ and let $\varphi : [0, a]\times[0, b] \to M$ be a smooth Chebyshev net on M such that $0 < \omega(u, v) < \pi$ for all $u, v \in [0, a]\times[0, b]$. We define $\Omega = \varphi([0, a]\times[0, b])$, $\eta_1(v) = \varphi(0, -v)$, $\sigma_1(v) = \varphi(a, -v)$ for all $v \in [-b, 0]$, and*

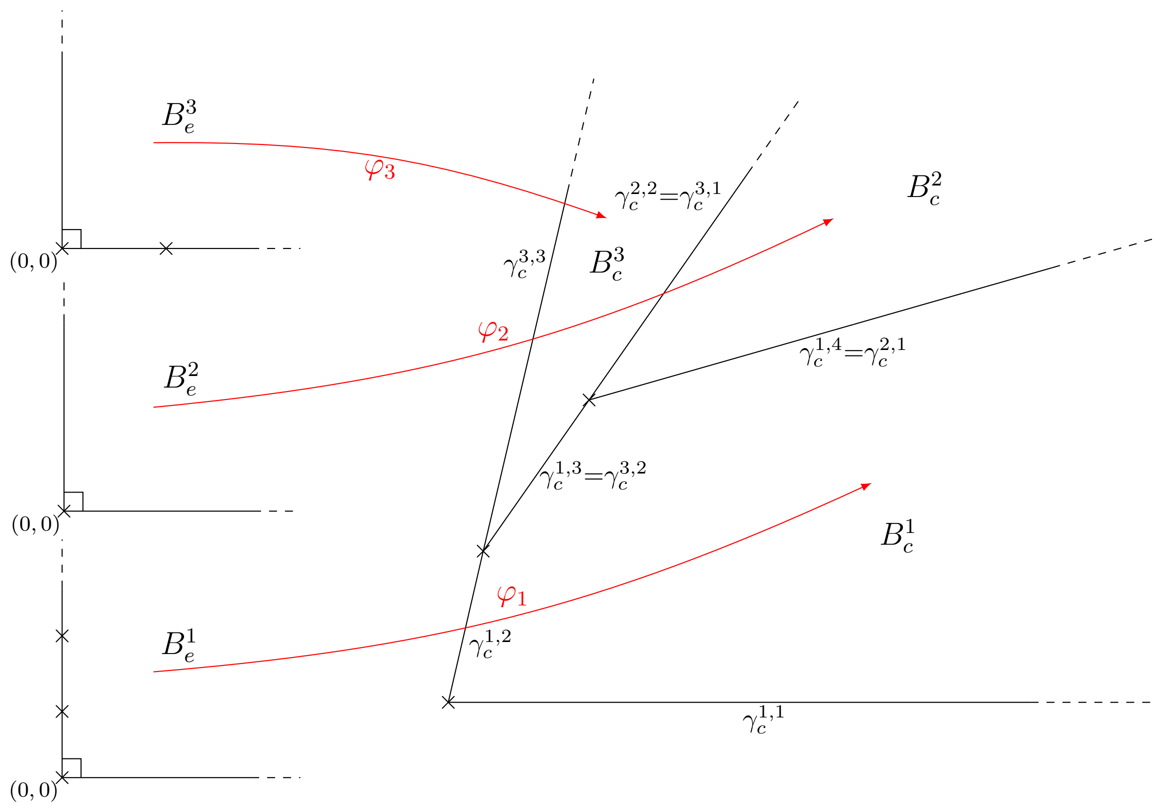


Figure 17: Illustration of the construction of the Chebyshev net with conical singularities (the crosses are the vertices)

$\eta_2(u) = \varphi(u,0)$, $\sigma_2(u) = \varphi(u,b)$ for all $u \in [0,a]$. Then, the geodesic curvatures κ_{η_1}, κ_{η_2}, κ_{σ_1}, and κ_{σ_2} of η_1, η_2, σ_1, and σ_2, respectively, are related by

$$\int_0^a \kappa_{\sigma_2} = \int_0^a \kappa_{\eta_2} + \int_\Omega K, \quad \int_{-b}^0 \kappa_{\sigma_1} = \int_{-b}^0 \kappa_{\eta_1} + \int_\Omega K, \tag{92}$$

and satisfy

$$\int_0^a \kappa^+_{\eta_2} - \int_0^a \kappa^+_{\sigma_2} + \int_\Omega K^+ \geq 0, \quad \int_{-b}^0 \kappa^+_{\eta_1} - \int_{-b}^0 \kappa^+_{\sigma_1} + \int_\Omega K^+ \geq 0, \tag{93a}$$

$$\int_0^a \kappa^-_{\eta_2} - \int_0^a \kappa^-_{\sigma_2} + \int_\Omega K^- \geq 0, \quad \int_{-b}^0 \kappa^-_{\eta_1} - \int_{-b}^0 \kappa^-_{\sigma_1} + \int_\Omega K^- \geq 0. \tag{93b}$$

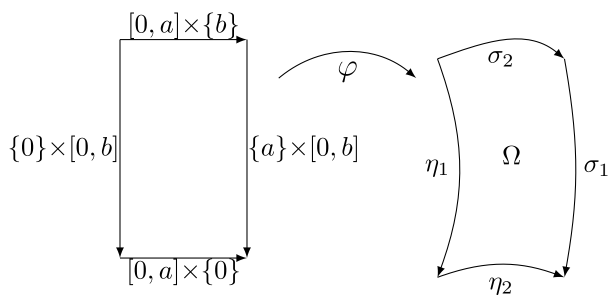


Figure 18: Illustration of the notation of Lemma 40

Proof. We only prove the lemma for the curves η_1 and σ_1. The formulas for η_2 and σ_2 are obtained in a similar way. Using (67) and the Hazzidakis formula (68) with $u = a$ and $v = b$, we obtain

$$\omega(a,b) = \omega(0,0) - \int_0^a \kappa_{\eta_2} - \int_{-b}^0 \kappa_{\eta_1} - \int_\Omega K = \omega(a,0) - \int_{-b}^0 \kappa_{\eta_1} - \int_\Omega K,$$

so that

$$\int_{-b}^0 \kappa_{\sigma_1} = -\int_0^b \partial_v \omega(a,v)\mathrm{d}v = \int_{-b}^0 \kappa_{\eta_1} + \int_\Omega K.$$

To prove the inequalities (93), we first note that (92) implies

$$\int_0^a \kappa^+_{\eta_1} - \int_0^a \kappa^+_{\sigma_1} + \int_\Omega K^+ = \int_0^a \kappa^-_{\eta_1} - \int_0^a \kappa^-_{\sigma_1} + \int_\Omega K^-.$$

Subdividing the curves η_1 and σ_1 according to the sign changes of κ_{η_1} and κ_{σ_1}, it is possible to assume that the sign of κ_{η_1} and κ_{σ_1} is constant on $[0,a]$. The discussion is then simplified to the two following cases:

- if κ_{η_1} and κ_{σ_1} have the same sign (say, nonnegative), then

$$\int_0^a \kappa_{\eta_1}^- - \int_0^a \kappa_{\sigma_1}^- + \int_\Omega K^- = \int_\Omega K^- \geq 0;$$

- if κ_{η_1} and κ_{σ_1} have different signs (say, $\kappa_{\eta_1} \geq 0$ and $\kappa_{\sigma_1} \leq 0$), then

$$\int_0^a \kappa_{\eta_1}^+ - \int_0^a \kappa_{\sigma_1}^+ + \int_\Omega K^+ = \int_0^a \kappa_{\eta_1}^+ + \int_\Omega K^+ \geq 0.$$

$\square$

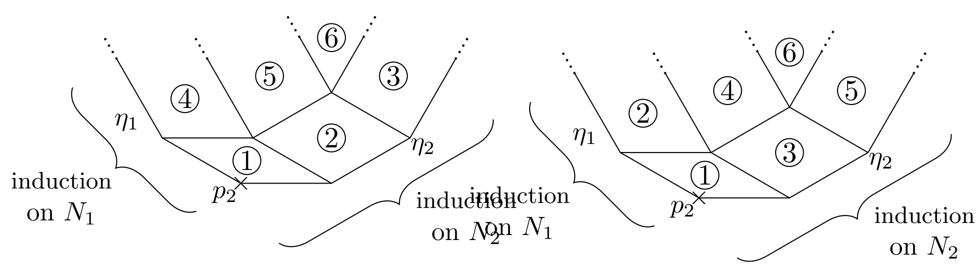
Theorem 41 (Existence of piecewise smooth Chebyshev nets on sectors)**.** *Let Q be a sector delimited by the two piecewise smooth curves $\eta_1 : \mathbb{R}^- \to M$ and $\eta_2 : \mathbb{R}^+ \to M$. We denote $\pi - \theta_1 \in (0, \pi)$ the exterior angle of this sector and we suppose that Q satisfies the conditions* (91)*. Then, there exist $\mathcal{N}_{\text{piece}} \geq 1$ polygons $\{B_e^i\}_{1 \leq i \leq \mathcal{N}_{\text{piece}}}$ such that $(\mathbb{R}^+)^2 = \cup_{i=1}^{\mathcal{N}_{\text{piece}}} B_e^i$ and $int(B_e^i) \cap int(B_e^j) = \varnothing$ for all $i \neq j$, and Chebyshev coordinates φ on Q such that η_1 and η_2 are coordinate curves. Moreover, the angle $\omega = \angle(\partial_u \varphi, \partial_v \varphi)$ of the net satisfies the nonsmooth Hazzidakis formula*

$$\omega(u, v) = \theta_1 - \tau\big(\eta_2\big|_{[0,u]}\big) - \tau\big(\eta_1\big|_{[-v,0]}\big) - \int_{\varphi([0,u]\times[0,v])} K, \tag{94}$$

for all $u, v \in \mathbb{R}^+$, and is bounded away from 0 and π by the positive real number

$$\min\left(\pi - \psi - \int_Q K^+ - \tau_+(\eta_1) - \tau_+(\eta_2),\ \psi - \int_Q K^- - \tau_-(\eta_1) - \tau_-(\eta_2)\right). \tag{95}$$

Finally, $\varphi\big|_{B_e^i} : B_e^i \to \varphi(B_e^i) \subset Q$ is a diffeomorphism, for all $1 \leq i \leq \mathcal{N}_{\text{piece}}$.



(a) Parallel transport of η_2 along η_1 (case chosen for the proof)

(b) Parallel transport of η_1 along η_2

Figure 19: Parallel transport of η_1 and η_2 along each other ($N_1 = 1$, $N_2 = 2$): numbering of the double induction process

Proof. We first split the two curves η_1 and η_2 into smooth pieces. We denote $N_1 + 1 \geq 1$ and $N_2 + 1 \geq 1$ the number of smooth pieces of the curves η_1 and η_2, respectively. Then, we parallel transport the curve η_2 along each smooth piece of η_1. This is done recursively on N_1. The parallel transport of η_2 along a piece of η_1 is obtained by induction on N_2. Hence, we have two nested induction arguments (see Figure 19). We observe that, by symmetry, the role of the two curves can be switched, as can be seen in the same figure. Hence, we can always suppose that $N_1 \geq N_2$. Once the construction is over, we prove the nonsmooth Hazzidakis formula (94).

Step 1 *(Formulation of the first induction process (on $N_1 \geq 0$)).* We suppose that $N_2 \in \{0, ..., N_1\}$ is a given fixed integer and we denote $(\mathcal{H}_{N_1+1})$ the following induction hypothesis:

for any sector Q of exterior angle $\pi - \theta_1 \in (0, \pi)$, delimited by the two curves η_1 and η_2 having respectively $N_1 + 1$ and $N_2 + 1$ smooth pieces and satisfying (91)*, there exist polygons $\{B_c^i\}_{1 \leq i \leq \mathcal{N}_{\text{piece}}}$ and a Chebyshev net $\varphi : (\mathbb{R}^+)^2 \to Q$ such that:*

- $(\mathbb{R}^+)^2 = \cup_{i=1}^{\mathcal{N}_{\text{piece}}} B_e^i$ *and* $int(B_e^i) \cap int(B_e^j) = \varnothing$ *for all* $i \neq j$*;*
- η_1 *and* η_2 *are coordinate curves;*
- *the angle* $\omega = \angle(\partial_u \varphi, \partial_v \varphi)$ *of* φ *satisfies the nonsmooth Hazzidakis formula* (94)*;*
- $\varphi\big|_{B_e^i} : B_e^i \to \varphi(B_e^i) \subset Q$ *is a diffeomorphism, for all* $i \in \{1, ..., \mathcal{N}_{\text{piece}}\}$*.*

Step 2 *(Proof of the first induction process (1^{st} part of the construction)).* We firstly check that $(\mathcal{H}_1)$ holds. Since $N_2 \leq N_1$, we have $N_2 = 0$. Hence, the sector Q is delimited by the two smooth curves η_1 and η_2 and $(\mathcal{H}_1)$ holds, with $\mathcal{N}_{\text{piece}} = 1$, by Proposition 25. Now, for $N_1 \geq 1$, we suppose that $(\mathcal{H}_{N_1})$ holds and we prove that $(\mathcal{H}_{N_1+1})$ also holds. Thus, we suppose that Q is delimited by two curves η_1 and η_2 having respectively $N_1 + 1 \geq 2$ and $N_2 + 1 \geq 1$ smooth pieces. Let

$$-\infty = a_{1,N_1+1} < ... < a_{1,0} = 0 = a_{2,0} < ... < a_{2,N_2+1} = \infty$$

be such that, for all $l = 1, 2$, η_l restricted to $[a_{l,i-1}, a_{l,i}]$ is a smooth curve, for all $i \in \{1, ..., N_l + 1\}$. We denote $\eta_{l,i} : [a_{l,i-1}, a_{l,i}] \to M$ this piece of the curve η_l and $\kappa_{l,i} : [a_{l,i-1}, a_{l,i}] \to \mathbb{R}$ its geodesic curvature. We denote $\psi_{l,i} = (-1)^l \angle\big(\eta'_{l,i}(a_{l,i}), \eta'_{l,i+1}(a_{l,i})\big)$ for all $i \in \{1, ..., N_l\}$ and $l \in \{1, 2\}$ (see Figure 20). To abbreviate the notation, we set $\tilde{\eta}_{1,1} = \eta_{1,1}$.

Step 3 *(Formulation of the second induction process (on $n \in \{1, ..., N_2 + 1\}$)).* We parallel transport in what follows the curve η_2 along $\eta_{1,1}$. See Figure 20 for the notation and Figure 21 for an illustration of the construction. For all $n \in \{1, ..., N_2 + 1\}$, we denote $(\tilde{\mathcal{H}}_n)$ the following induction hypothesis:

for all $j \in \{1, ..., n\}$, let $\tilde{\eta}_{1,j} : [a_{1,1}, 0] \to Q$ be a smooth curve intersecting $\eta_{2,j}$ at $\eta_{2,j}(a_{2,j-1}) = \tilde{\eta}_{1,j}(0)$. If $n > 1$, for all $j \in \{1, ..., n-1\}$, let $B_e^j =$

$[a_{2,j-1}, a_{2,j}]\times[0, -a_{1,1}] \subset (\mathbb{R}^+)^2$ *and assume that* $\varphi_j : B_e^j \to B_c^j \subset Q$, *with* $B_c^j = \varphi_j(B_e^j)$, *are Chebyshev nets such that*

$$\begin{aligned} \varphi_j(a_{2,j-1}, v) = \tilde{\eta}_{1,j}(-v), \quad \varphi_j(a_{2,j}, v) = \tilde{\eta}_{1,j+1}(-v), \quad & \forall v \in [0, -a_{1,1}], \\ \varphi_j(u, 0) = \eta_{2,j}(u), \quad & \forall u \in [a_{2,j-1}, a_{2,j}]. \end{aligned}$$

Then, there exists a Chebyshev net $\varphi_n : B_e^n \subset (\mathbb{R}^+)^2 \to B_c^n \subset Q$, *with* $B_e^n = [a_{2,n-1}, a_{2,n}]\times[0, -a_{1,1}]$ *and* $B_c^n = \varphi_n(B_e^n)$. *Moreover, the set* B_c^n *satisfies:*

1. *if* $n > 1$, *then* $B_c^{n-1} \cap B_c^n = \tilde{\eta}_{1,n}$*;*
2. *if* $n > 2$, *then* $B_c^j \cap B_c^n = \varnothing$, *for all* $j \in \{1, ..., n-2\}$.

Finally, suppose that $n < N_2 + 1$ *and denote* $\tilde{\eta}_{1,n+1} : [a_{1,1}, 0] \to Q$ *the curve defined by* $\tilde{\eta}_{1,n+1}(v) = \varphi_n(a_{2,n}, -v)$ *for all* $v \in [a_{1,1}, 0]$. *This curve intersects* $\eta_{2,n+1}$ *at* $\eta_{2,n+1}(a_{2,n}) = \tilde{\eta}_{1,n+1}(0)$, *forming an interior angle denoted* $\theta_{n+1} \in [0, \pi]$. *Then,* θ_{n+1} *and the geodesic curvature* $\tilde{\kappa}_{1,n+1} : [a_{1,1}, 0] \to \mathbb{R}$ *of* $\tilde{\eta}_{1,n+1}$ *satisfy respectively*

$$\theta_{n+1} = \theta_1 - \tau(\eta_2\big|_{[0,a_{2,n}]}) \in (0, \pi), \tag{96a}$$

$$\int_{-v}^{0} \tilde{\kappa}_{1,n+1} = \int_{-v}^{0} \kappa_{1,1} + \sum_{k=1}^{n} \int_{\varphi_k([a_{2,k-1}, a_{2,k}]\times[0,v])} K, \quad \forall v \in [0, -a_{1,1}]. \tag{96b}$$

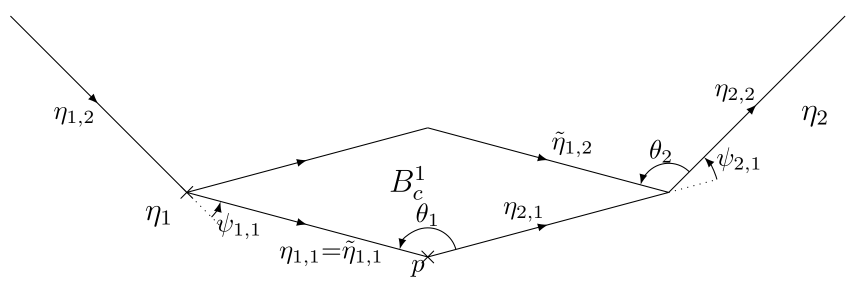


Figure 20: Illustration of the parallel transport of $\eta_{2,1}$ along $\eta_{1,1}$ ($N_1 = 1$, $N_2 = 1$)

Step 4 *(Proof of the second induction process).* To prove that $(\tilde{\mathcal{H}}_1)$ holds, we apply Theorem 26 with the curves $\eta_{1,1}$ and $\eta_{2,1}$ of respective length $-a_{1,1}$ and $a_{2,1}$ and forming an interior angle $\theta_1 \in (0, \pi)$ by hypothesis. We obtain a mapping $\varphi_1 : B_e^1 \subset (\mathbb{R}^+)^2 \to B_c^1 \subset M$, with $B_e^1 = [a_{2,0}, a_{2,1}]\times[0, -a_{1,1}]$ and $B_c^1 = \varphi_1(B_e^1)$. Then, using the same argument as the proof of Lemma 24, we obtain that the mapping φ_1 is a Chebyshev net and that $B_c^1 \subset Q$. Using Lemma 40, we infer that $\tilde{\eta}_{1,2}$ has a geodesic curvature $\tilde{\kappa}_{1,2}$ satisfying

$$\int_{-v}^{0} \tilde{\kappa}_{1,1} = \int_{-v}^{0} \kappa_{1,1} + \int_{\varphi_1([0,a_{2,1}]\times[0,v])} K,$$

for all $v \in [0, -a_{1,1}]$. Moreover, we deduce from (67) that the interior angle $\theta_2 = \angle(\eta'_{2,2}(0), -\tilde{\eta}'_{1,2}(0))$ satisfies

$$\theta_2 = \theta_1 - \int_0^{a_{2,1}} \kappa_{2,1} - \psi_{2,2} = \theta_1 - \tau(\eta_2\big|_{[0,a_{2,1}]}).$$

Since hypothesis (91) (with $\psi = \pi - \theta_1$) is satisfied by Q, we obtain that $\theta_2 \in (0, \pi)$. Hence, $(\tilde{\mathcal{H}}_1)$ holds. We now suppose that $(\tilde{\mathcal{H}}_{n-1})$ holds for $n \in \{0, ..., N_2 + 1\}$. Let $\{\tilde{\eta}_{1,j}\}_{1\leq j\leq n}$ and $\{\varphi_j\}_{1\leq j\leq n-1}$ be respectively curves and Chebyshev nets as in the hypothesis of $(\tilde{\mathcal{H}}_n)$. Since $\theta_n \in (0, \pi)$, we apply Theorem 26 to the curves $\tilde{\eta}_{1,n}$ and $\eta_{2,n}$ to obtain a mapping $\varphi_n : B_e^n \subset (\mathbb{R}^+)^2 \to B_c^n \subset M$, with $B_e^n = [a_{2,n-1}, a_{2,n}]\times[0, -a_{1,1}]$ and $B_c^n = \varphi_n(B_e^n)$. Using the same argument as in the proof of Lemma 24, we obtain that φ is a Chebyshev net, that $B_c^n \subset Q$ and that B_c^n satisfies the statements 1 and 2 of $(\tilde{\mathcal{H}}_n)$.

We now suppose that $n < N_2 + 1$. Then, from Lemma 40 and from the assertion (96b) of $(\tilde{\mathcal{H}}_{n-1})$, we infer that the geodesic curvature $\tilde{\kappa}_{1,n+1}$ of $\tilde{\eta}_{1,n+1}$ satisfies

$$\int_{-v}^0 \tilde{\kappa}_{1,n+1} = \int_{-v}^0 \tilde{\kappa}_{1,n} + \int_{\varphi_n([a_{2,n-1},a_{2,n}]\times[0,v])} K = \int_{-v}^0 \kappa_{1,0} + \sum_{k=1}^n \int_{\varphi_k([a_{2,k-1},a_{2,k}]\times[0,v])} K,$$

for all $v \in [0, -a_{1,1}]$. Finally, from (67) and from the assertion (96a) of $(\tilde{\mathcal{H}}_{n-1})$, we infer that the interior angle $\theta_{n+1} = \angle(\eta'_{2,n+1}(a_{2,n}), -\tilde{\eta}'_{1,n+1}(0))$ satisfy

$$\theta_{n+1} = \theta_n - \int_{a_{2,n-1}}^{a_{2,n}} \kappa_{2,n} - \psi_{2,n} = \theta_1 - \tau(\eta_2\big|_{[0,a_{2,n}]}).$$

Then, using hypotheses (91), we obtain that $\theta_{n+1} \in (0, \pi)$. This concludes the proof of the statement $(\tilde{\mathcal{H}}_n)$.

Step 5 *(Proof of the first induction process (2^{nd} part of the construction)).* By application of $(\tilde{\mathcal{H}}_n)$, for all $n \in \{1, ..., N_2 + 1\}$, we obtain the existence of polygons $\{B_e^i\}_{1\leq i\leq N_2+1}$ and, for all $i \in \{1, ..., N_2 + 1\}$, of Chebyshev nets $\varphi_i : B_e^i \to B_c^i \subset Q$, with $B_c^i = \varphi_i(B_e^i)$. Then, for all $i \in \{1, ..., N_2 + 1\}$, we denote $\tilde{\eta}_{2,i} : [a_{2,i-1}, a_{2,i}] \to Q$ the curve defined by $\tilde{\eta}_{2,i}(u) = \varphi_i(u, -a_{1,1})$, for all $u \in [a_{2,i-1}, a_{2,i}]$. We denote $\tilde{\eta}_2 : \mathbb{R}^+ \to Q$ the junction of the curves $\tilde{\eta}_{2,i}$, with $i \in \{1, ..., N_2 + 1\}$, defined so that

$$\tilde{\eta}_2(\mathbb{R}^+) = \bigcup_{i=1}^{N_2+1} \tilde{\eta}_{2,i}([a_{2,i-1}, a_{2,i}]).$$

We denote $B_e^{\text{band}} = \mathbb{R}^+\times[0, -a_{1,1}]$, $B_c^{\text{band}} = \cup_{i=1}^{N_2+1} B_c^i$ and $\tilde{Q} = Q \setminus B_c^{\text{band}}$. Let us construct the Chebyshev net on the half-band B_e^{band}. This mapping, denoted $\varphi_{\text{band}} : B_e^{\text{band}} \to B_c^{\text{band}}$, is defined by

$$\varphi_{\text{band}}(u, v) = \varphi_i(u, v), \quad \text{whenever } (u, v) \in P_e^i \text{ for } i \in \{1, ..., N_2 + 1\}. \tag{97}$$

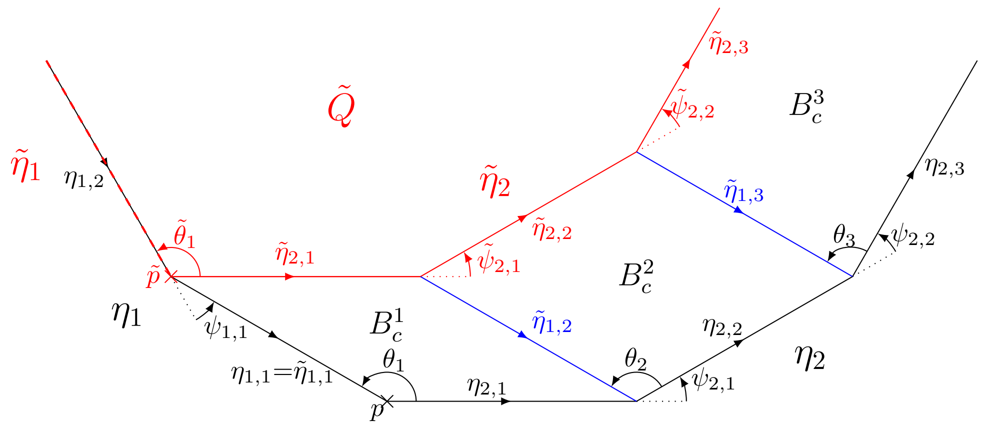


Figure 21: Illustration of the recursive parallel transport of η_2 along $\eta_{1,1}$ ($N_1 = 1$, $N_2 = 2$)

Then, using that $\varphi_{\text{band}}\big|_{B_e^i}$ is a diffeomorphism for all $i \in \{1, ..., N_2 + 1\}$ and using the statements 1 and 2 of $(\tilde{\mathcal{H}}_n)$, for all $n \in \{1, ..., N_2 + 1\}$, we obtain that the mapping φ_{band} is a homeomorphism. We denote $\tilde{\eta}_1 : \mathbb{R}^- \to \eta_1\big[(-\infty, a_{1,1}]\big]$ the curve defined by $\tilde{\eta}_1(t) = \eta_1(t + a_{1,1})$. Then, $\tilde{Q}$ is a sector delimited by the curves $\tilde{\eta}_1$ and $\tilde{\eta}_2$ with respectively N_1 and $N_2 + 1$ smooth pieces. We now prove that $\tilde{Q}$ satisfies the hypotheses of $(\mathcal{H}_{N_1})$ (on the interior angle and on the total curvature). First note that, using (66), we obtain that the interior angle $\tilde{\theta}_1$ of $\tilde{Q}$ satisfies

$$\tilde{\theta}_1 = \theta_1 - \int_{a_{1,1}}^{0} \kappa_{1,1} - \psi_{1,1} = \theta_1 - \tau(\eta_1\big|_{[a_{1,1},0]}). \tag{98}$$

Then, using the hypotheses (91) on Q, we obtain that $\tilde{\theta}_1 \in (0, \pi)$.

To simplify the notation, we use in what follows the convention that $\sum_{k=1}^{i-1}(...) = 0$ when $i = 1$. As parallel transport preserves the angles, we have $\tilde{\psi}_{2,i} = \psi_{2,i}$, for all $i \in \{1, ..., N_2\}$. Therefore, from (92) and the definition of φ_{band}, we infer that, for all $i \in \{1, ..., N_2 + 1\}$ and $u \in [a_{2,i-1}, a_{2,i})$,

$$\begin{aligned}
\tau(\tilde{\eta}_2\big|_{[0,u]}) &= \sum_{k=1}^{i-1} \int_{a_{2,k-1}}^{a_{2,k}} \tilde{\kappa}_{2,k} + \int_{a_{2,i-1}}^{u} \tilde{\kappa}_{2,i} + \sum_{k=1}^{i-1} \tilde{\psi}_{2,k} \\
&= \sum_{k=1}^{i-1} \int_{a_{2,k-1}}^{a_{2,k}} \kappa_{2,k} + \sum_{k=1}^{i-1} \int_{B_c^k} K + \int_{a_{2,i-1}}^{u} \kappa_{2,i} + \int_{\varphi_i([a_{2,i-1},u]\times[0,-a_{1,1}])} K + \sum_{k=1}^{i-1} \psi_{2,k} \\
&= \tau(\eta_2\big|_{[0,u]}) + \int_{\varphi_{\text{band}}([0,u]\times[0,-a_{1,1}])} K.
\end{aligned} \tag{99}$$

Then, in the same manner as in Lemma 40, we deduce from (99) that

$$\tau_+(\tilde{\eta}_2) \le \tau_+(\eta_2) + \int_{B_c^{\text{band}}} K^+ \quad \text{and} \quad \tau_-(\tilde{\eta}_2) \le \tau_-(\eta_2) + \int_{B_c^{\text{band}}} K^-. \tag{100}$$

Moreover, we have

$$\tau_\pm(\tilde\eta_1) = \tau_\pm(\eta_1) - \tau_\pm(\eta_1\big|_{[a_{1,1},0]}). \tag{101}$$

We now prove that $\tilde Q$ satisfies the condition (91a). First, we have

$$\begin{aligned}\tau_+(\tilde\eta_1) + \tau_+(\tilde\eta_2) + \int_{\tilde Q} K^+ + \pi - \tilde\theta_1 &\le \tau_+(\eta_1) - \tau_+(\eta_1\big|_{[a_{1,1},0]}) + \tau_+(\eta_2) + \int_{B_c^{\text{band}}} K^+ \\ &\quad + \int_{\tilde Q} K^+ + \pi - \theta_1 + \tau(\eta_1\big|_{[a_{1,1},0]}) \\ &\le \tau_+(\eta_2) + \tau_+(\eta_1) + \pi - \theta_1 + \int_Q K^+,\end{aligned} \tag{102}$$

using (98), (100) and (101) for the first inequality. Since hypothesis (91a) (with $\psi = \pi - \theta_1$) is satisfied by Q, we infer from (102) that

$$\tau_+(\tilde\eta_1) + \tau_+(\tilde\eta_2) + \int_{\tilde Q} K^+ + \pi - \tilde\theta_1 < \pi.$$

Finally, we prove that $\tilde Q$ satisfies the condition (91b). From (100) and (101), we deduce that

$$\tau_-(\tilde\eta_1) + \tau_-(\tilde\eta_2) + \int_{\tilde Q} K^- \le \tau_-(\eta_1) - \tau_-(\eta_1\big|_{[a_{1,1},0]}) + \tau_-(\eta_2) + \int_Q K^-. \tag{103}$$

Then, using the hypothesis (91b) on Q, we infer from (103) that

$$\tau_-(\tilde\eta_1) + \tau_-(\tilde\eta_2) + \int_{\tilde Q} K^- < \pi - \theta_1 - \tau_-(\eta_1\big|_{[a_{1,1},0]}).$$

Finally, using (98), we conclude that

$$\tau_-(\tilde\eta_1) + \tau_-(\tilde\eta_2) + \int_{\tilde Q} K^- \le \pi - \tilde\theta_1 - \tau(\eta_1\big|_{[a_{1,1},0]}) - \tau_-(\eta_1\big|_{[a_{1,1},0]}) < \pi - \tilde\theta_1.$$

Hence, the hypotheses of $(\mathcal{H}_{N_1})$ are satisfied by $\tilde Q$. We obtain the existence of polygons $\{\tilde B_e^i\}_{1\le i\le \tilde{\mathcal{N}}_{\text{piece}}}$ such that $\operatorname{int}(\tilde B_e^i) \cap \operatorname{int}(\tilde B_e^j) = \varnothing$ for all $i \ne j$ and $\cup_{i=1}^{\tilde{\mathcal{N}}_{\text{piece}}} \tilde B_e^i = (\mathbb{R}^+)^2$, and a Chebyshev parametrization of $\tilde Q$ denoted $\tilde\varphi : (\mathbb{R}^+)^2 \to \tilde Q$. The translation of each polygon $\tilde B_e^i$ by the vector $(0, -a_{1,1})$, i.e., the set $\{\tilde B_e^i + (0, -a_{1,1})\}_{1\le i\le \tilde{\mathcal{N}}_{\text{piece}}}$ is then joined to the set $\{B_e^i\}_{1\le i\le N_2+1}$ to obtain $\{B_e^i\}_{1\le i\le \mathcal{N}_{\text{piece}}}$. Moreover, we define the mapping $\varphi : (\mathbb{R}^+)^2 \to Q$ by

$$\varphi(u,v) = \begin{cases} \varphi_{\text{band}}(u,v), & \text{if } (u,v) \in B_e^{\text{band}}, \\ \tilde\varphi(u, v + a_{1,1}), & \text{otherwise.} \end{cases}$$

We can show, in the same manner as for φ_{band}, that φ is a homeomorphism.

Step 6 *(Proof of the first induction process (nonsmooth Hazzidakis formula))*. Let $(u,v)\in(\mathbb{R}^+)^2$. First, we suppose that $(u,v)\in B_e^{\text{band}}$, so that $(u,v)\in B_e^j$ for some $j\in\{1,...,N_2+1\}$. By the Hazzidakis formula in the smooth setting (68), the angle ω_j between the parameter curves of φ_j satisfies

$$\omega_j(u,v)=\theta_j-\int_{a_{2,j-1}}^{u}\kappa_{2,j}-\int_{-v}^{0}\tilde{\kappa}_{1,j}-\int_{\varphi_j([a_{2,j-1},u]\times[0,v])}K. \tag{104}$$

Then, from (96) and (104), we infer that

$$\begin{aligned}\omega(u,v)=\omega_j(u,v)&=\theta_1-\tau(\eta_2\big|_{[0,u]})-\tau(\eta_1\big|_{[-v,0]})-\sum_{k=1}^{j-1}\int_{\varphi_k([a_{2,k-1},a_{2,k}]\times[0,v])}K\\&\quad-\int_{\varphi_j([a_{2,j-1},u]\times[0,v])}K\\&=\theta_1-\tau(\eta_1\big|_{[-v,0]})-\tau(\eta_2\big|_{[0,u]})-\int_{\varphi([0,u]\times[0,v])}K.\end{aligned}$$

We now suppose that $(u,v)\notin B_e^{\text{band}}$ and we denote $\bar{u}=u$ and $\bar{v}=v+a_{1,1}$. Then, $\bar{u},\bar{v}\in\mathbb{R}^+$ and, by $(\mathcal{H}_{N_1})$, the nonsmooth Hazzidakis formula (94) is satisfied by the angle $\tilde{\omega}$ between the coordinate curves of $\tilde{\varphi}$. Hence, using (94) on $\tilde{\omega}$ and (99), we obtain that

$$\begin{aligned}\omega(u,v)=\tilde{\omega}(\bar{u},\bar{v})&=\tilde{\theta}_1-\tau(\tilde{\eta}_1\big|_{[-\bar{v},0]})-\tau(\tilde{\eta}_2\big|_{[0,\bar{u}]})-\int_{\tilde{\varphi}([0,\bar{u}]\times[0,\bar{v}])}K,\\&=\theta_1-\tau(\eta_1\big|_{[a_{1,1},0]})-\tau(\eta_1\big|_{[-v,a_{1,1}]})-\tau(\eta_2\big|_{[0,u]})\\&\quad-\int_{\varphi([0,u]\times[0,-a_{1,1}])}K-\int_{\varphi([0,u]\times[-a_{1,1},v])}K\\&=\theta_1-\tau(\eta_1\big|_{[-v,0]})-\tau(\eta_2\big|_{[0,u]})-\int_{\varphi([0,u]\times[0,v])}K.\end{aligned}$$

Hence, $(\mathcal{H}_{N_1+1})$ holds. This concludes the proof.

□

Remark 42 (Explicit value of $\mathcal{N}_{\text{piece}}$). We easily see from the proof that $\mathcal{N}_{\text{piece}}=(N_1+1)(N_2+1)$, where N_1+1 and N_2+1 are respectively the number of smooth pieces of η_1 and η_2.

Corollary 43 (Existence of piecewise smooth Chebyshev nets on N-half-surfaces)**.** *Let $N\geq 1$ and let B_c be a N-half-surface delimited by the curves $\{\gamma_c^i\}_{1\leq i\leq N+1}$. We suppose that B_c satisfies the conditions*

$$\begin{aligned}\tau_+(\partial B_c)+\int_{B_c}K^+&<\pi,\\ \tau_-(\partial B_c)+\int_{B_c}K^-&<|\psi|_{l^\infty}.\end{aligned}$$

Then, there exist $\mathcal{N}_{\text{piece}} \geq 1$ *polygons* $\{B_e^i\}_{1\leq i\leq \mathcal{N}_{\text{piece}}}$ *such that* $(\mathbb{R}^+)^2 = \cup_{i=1}^{\mathcal{N}_{\text{piece}}} B_e^i$ *and* $int(B_e^i) \cap int(B_e^j) = \varnothing$ *for all* $i \neq j$*, and Chebyshev coordinates* φ *on* B_c *such that* $\{\gamma_c^i\}_{1\leq i\leq N+1}$ *are coordinate curves. Moreover, the angle* $\omega = \angle(\partial_u\varphi, \partial_v\varphi)$ *of the net is bounded away from* 0 *and* π *by the positive real number*

$$\varepsilon = \min\left(\pi - \tau_+(\partial B_c) - \int_{B_c} K^+, \quad |\psi|_{l^\infty} - \tau_-(\partial B_c) - \int_{B_c} K^-\right).$$

Finally, the mapping $\varphi\big|_{B_e^i} : B_e^i \to B_c^i \subset B_c$*, with* $B_c^i = \varphi(B_e^i)$*, is a diffeomorphism, for all* $i \in \{1, ..., \mathcal{N}_{\text{piece}}\}$*.*

Proof. The proof follows by combining Theorem 41 and Proposition 29. □

Theorem 44 (Existence of piecewise smooth Chebyshev nets on geodesic N-half-surfaces)**.** *Let* $N \geq 1$ *and let* B_c *be a geodesic* N*-half-surface delimited by the geodesic curves* $\{\gamma_c^i\}_{1\leq i\leq N+1}$*. We suppose that* B_c *satisfies the conditions*

$$\int_{B_c} K^+ < \pi - |\psi|_{l^1},$$
$$\int_{B_c} K^- < |\psi|_{l^\infty}.$$

Then, there exist $\mathcal{N}_{\text{piece}} \geq 1$ *polygons* $\{B_e^i\}_{1\leq i\leq \mathcal{N}_{\text{piece}}}$ *such that* $(\mathbb{R}^+)^2 = \cup_{i=1}^{\mathcal{N}_{\text{piece}}} B_e^i$ *and* $int(B_e^i) \cap int(B_e^j) = \varnothing$ *for all* $i \neq j$*, and Chebyshev coordinates* φ *on* B_c *such that* $\{\gamma_c^i\}_{1\leq i\leq N+1}$ *are coordinate curves. Moreover, the angle* $\omega = \angle(\partial_u\varphi, \partial_v\varphi)$ *of the net is bounded away from* 0 *and* π *by the positive real number*

$$\min\left(\pi - |\psi|_{l^1} - \int_{B_c} K^+, |\psi|_{l^\infty} - \int_{B_c} K^-\right).$$

Finally, the mapping $\varphi\big|_{B_e^i} : B_e^i \to B_c^i \subset B_c$*, with* $B_c^i = \varphi(B_e^i)$*, is a diffeomorphism, for all* $i \in \{1, ..., \mathcal{N}_{\text{piece}}\}$*.*