

# Construction of Chebyshev nets with singularities

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## 1 Introduction

A global net on a surface  $M$  is two families of curves such that through each point  $p$  there passes exactly one curve of each family with tangent vectors forming a basis of the tangent plane at  $p$ . We will denote by  $\omega(p)$  the angle between these vector at  $p$  and call it angle function of the net. A global Chebyshev net (T-net in the following) is a net in which the tangent vectors to each family of line can be parallel displaced along the lines of the other family. A net is a Chebyshev net if around each point  $p$ , we can find a local chart  $\varphi : \Omega \subset \mathbb{R}^2 \mapsto M$ , such that the two family of line are the line of coordinate  $u \rightarrow \varphi(u, v_0)$  and  $v \rightarrow \varphi(u_0, v)$  with lines parametrized by arclength

$$|\partial_u \varphi| = |\partial_v \varphi| = 1.$$

When  $\omega(p) = 0[\pi]$ , lines are tangent and the net is no more a local chart of the surface. Singularities at these points are of cuspidal type or shallow tails. This is well describe in a paper of E. Ghys [?] introducing Chebyshev net on surfaces. It is not known when there exists global T-net on compact or non compact surfaces. The angle function  $\omega$  between the two family of lines appear in the metric of local coordinates

$$ds^2 = du^2 + 2 \cos \omega(u, v) du dv + dv^2$$

and the Gauss equation is related to a Sine-Gordon type equation of hyperbolic type:

$$\partial_{uv}^2 \omega = -K(u, v) \sin \omega(u, v).$$

Using Gauss Bonnet theorem which reformulate in Hazzidakis formula, it is known (see H. Hopf, [?], Lemma 2.3) that Chebyshev coordinates may not exist on a simply connected surface with  $|\int \int K dS| \geq 2\pi$ . P. L. Chebyshev proved the local existence of his coordinates under an analyticity assumption. In this paper, beside the hyperbolicity property of the Gauss equation, we will construct coordinates on the surface by finding the function  $\omega$  which satisfy boundary Cauchy data conditions on some boundary curves.

Piecewise smooth T-net is a weaker notion. We relax the regularity of lines of the net. Each family of lines are parallel displaced along the lines of the other family but it can happen that lines are only piecewise  $C^1$  with existing left and right tangent vector at singular point  $p$ . We denote by  $\eta_1$  the line of the other family passing through  $p$  and by  $\eta_2$  the line which is not  $C^1$  at  $p$ . The left and right vectors  $(\eta_2)_d(p)$  and  $(\eta_2)_\ell(p)$  form an angle  $\alpha \neq 0[\pi]$ . This angle is displaced along  $\eta_1$  by parallel transport, and any line of the second family is singular on  $\eta_1$ . We call this line of singularities "twin crystal line" as it can be observed in mineralogy. When twin crystal line meet at some point, this create bifurcation in the net. Bifurcation can appear when  $\eta_1$  and  $\eta_2$  are singular in the same point  $p$ . In this case, the bifurcation point is related to a "cross". However the number of twin crystal line which meet together at  $p$  can be greater than 4 but we consider only finite number case.

The first non local result has been proved by I. Ya. Bakelman [?]. He proved the existence of Chebyshev coordinates in any sector bounded by two orthogonal geodesics on a simply connected Aleksandrov surface under the condition that both the positive and negative integral curvatures,  $\int K^+ dS$  and  $\int K^- dS$  are less than  $\pi/2$ , (where  $K$  is the Gaussian curvature of  $M$ ,  $K^+ = \max(K, 0)$  and  $K^- = \max(-K, 0)$ ) but the net (And the surface) is not smooth. Much later, a similar result for smooth surfaces, was proved by S. L. Samelson and W. P. Dayawansa [?]: under the same restriction on the curvature, they proved that on each simply connected complete smooth surface there exists a global smooth Chebyshev net. Their proof is analytical. It is well known that the existence problem for the Chebyshev coordinates can be reduced to the existence problem for a solution to a hyperbolic system of second-order partial differential equations. This system is solved in [?] by the method of a paper of Ladyzhenskaya and Shubov [?]. Burago, Ivanov and Malev in [?] used Bakelman result in a sector decomposition of a surface described in a paper of Bonk and Lang [?]. They find four sectors satisfying hypothesis of Bakelman where they construct non smooth Chebyshev net on simply connected smooth surface  $M$  with  $\int K^\pm dS < 2\pi$ . Masson and Monasse in [?] improve the result of Samelson and Dayawansa by constructing smooth T-net for simply connected surfaces with  $\int K^\pm dS < 2\pi$  which is a sharp result by the lemma of H. Hopf.

Piecewise smooth T-net is strongly related to the construction of bi-Lipschitz map between surface. If  $M$  is a simply connected surface, a map  $f : \mathbb{R}^2 \mapsto M$  is bi-Lipschitz if there exists a constant  $L$  such that

$$\frac{1}{L}d(x, y) \leq \text{dist}_M(f(x), f(y)) \leq Ld(x, y).$$

For Chebyshev coordinates the Lipschitz constants can be estimated in terms of the net angles, i.e., via  $\min(\inf \omega, \inf(\pi - \omega))$ . M. Bonk and U. Lang [?] proved that if

$$\int K^+ dS < 2\pi \text{ and } \int K^- dS < C$$

then  $M$  is bi-Lipschitz equivalent to  $\mathbb{R}^2$  with Lipschitz constant  $\epsilon^{1/2}(2\pi + C)^{1/2}$ . The existence of piecewise smooth T-net immediately implies a weak (with a worse constant) version of the Bonk-Lang theorem.

In this paper, we consider complete and simply connected surface  $M$  and we construct by a new method piecewise smooth T-net by a decomposition of the surface in a finite number of sectors where we are able to produce smooth T-net. The boundary of the sector are the twin crystal lines on  $M$ . In the section 3, we describe the regularisation of piecewise smooth T-net outside a finite number of ball, arbitrarily small neighborhood of bifurcating points. More precisely the main theorem of this paper is

**Theorem 1** (Existence of a smooth Chebyshev net out of a finite number of balls). *Let  $M$  be a smooth complete immersion of the plane  $\mathbb{R}^2$  in  $\mathbb{R}^3$  satisfying*

$$\int_M K^+ < 2\pi \quad \text{and} \quad \int_M K^- < \infty, \quad (1)$$

where  $K$  is the Gaussian curvature of  $M$ ,  $K^+ = \max(K, 0)$  and  $K^- = \max(-K, 0)$ . Then there exists a piecewise smooth Chebyshev net with a finite number of bifurcating points  $\{p_i\}_{1 \leq i \leq N}$  on  $M$  with  $N \leq \frac{4}{\pi} \int_M K^- + 4$ . The surface  $M \setminus \bigcup_{1 \leq i \leq N} B_i$  admits a smooth Chebyshev net where  $B_i$  are disks centered at  $p_i$  with arbitrary small radius  $\varepsilon$ .

## 2 Preliminaries

We consider  $M$  complete and simply connected immersed surfaces. In a first part we will be interested in the construction of T-net in a sector of  $M$  bounded by two half line. More precisely, let  $L_u, L_v \in \mathbb{R}_*^+$  and  $\gamma_u : [0, L_u] \rightarrow M$  and  $\gamma_v : [0, L_v] \rightarrow M$  be two curves such that  $\gamma_u(0) = \gamma_v(0)$ , and forming an interior angle  $\angle(\gamma'_u(0), \gamma'_v(0)) \in (0, \pi)$  at their intersection and exterior angle  $\psi = \pi - \angle(\gamma'_u(0), \gamma'_v(0))$ .

We consider a slightly different notion of Chebyshev net which will refer to mappings (not necessarily homeomorphisms)  $\varphi : D = [0, L_u] \times [0, L_v] \rightarrow M$  satisfying

$$|\partial_u \varphi|_g(u, v) = 1, \quad (2a)$$

$$|\partial_v \varphi|_g(u, v) = 1, \quad (2b)$$

for all  $(u, v) \in D$ .

””Furthermore, note that the orientation of the boundary curve  $\gamma_v$  is reversed in this section to simplify the exposition.”” ???

We prove in the sequel the existence and the uniqueness of a Chebyshev net  $\varphi : D \rightarrow M$  verifying the Cauchy boundary conditions

$$\begin{aligned}\varphi(u, 0) &= \gamma_u(u), \quad \forall u \in [0, L_u], \\ \varphi(0, v) &= \gamma_v(v), \quad \forall v \in [0, L_v].\end{aligned}\tag{3}$$

In the following,  $M$  is a finite total curvature immersed plane with  $\int_M K^+ < 2\pi$  and  $\int_M K^- < \infty$ . Hence there exists a global bi-Lipschitz parametrization  $F : \mathbb{R}^2 \rightarrow M$  [?] and if  $g$  is the induced metric, for all  $X \in \mathbb{R}^2$  there exists  $C_{\text{surf}} > 1$  such that

$$\frac{1}{C_{\text{surf}}} |X| \leq |X|_g \leq C_{\text{surf}} |X|.\tag{4}$$

Following [?], we reformulate the problem of finding a Chebyshev net  $\varphi : D \rightarrow M$  as the problem of finding the angle distribution  $\omega : D \rightarrow (0, \pi)$  between the coordinate curves defined by  $\omega(u, v) = \angle(\partial_u \varphi, \partial_v \varphi)(u, v)$ . With this purpose in mind, we observe that  $\omega$  satisfies the integrability condition (in the form of a modified Sine-Gordon equation) [?]

$$\partial_{uv} \omega = -K(\varphi) \sin(\omega).\tag{5}$$

Equivalently,  $\omega$  satisfies the integrated form Gauss Bonnet formula of (5) called the Hazzidakis formula

**Signes de la formule et conventions de signes du parametrage**

$$\begin{aligned}\omega(u, v) &= \angle(\gamma'_u(0), \gamma'_v(0)) - \int_0^u \kappa_u(s) ds + \int_0^v \kappa_v(s) ds \\ &\quad - \int_0^u \int_0^v K(\varphi(s, t)) \sin[\omega(s, t)] ds dt,\end{aligned}\tag{6}$$

for all  $(u, v) \in D$ , with  $\kappa_u : [0, L_u] \rightarrow \mathbb{R}$  and  $\kappa_v : [0, L_v] \rightarrow \mathbb{R}$  the geodesic curvatures of  $\gamma_u$  and  $\gamma_v$  respectively (see Figure 1).

Moreover, remark that, since  $\varphi$  is a Chebyshev net, it satisfies the following property:

$$\kappa_u^{\text{map}}(u, v) = -\partial_u \omega(u, v),\tag{7a}$$

$$\kappa_v^{\text{map}}(u, v) = \partial_v \omega(u, v),\tag{7b}$$

for all  $(u, v) \in D$ , where  $\kappa_u^{\text{map}} : D \rightarrow \mathbb{R}$  and  $\kappa_v^{\text{map}} : D \rightarrow \mathbb{R}$  are respectively the geodesic curvatures of the  $u$ -coordinate curves and of the  $v$ -coordinate curves of  $\varphi$ . We obtain by combining (3) and (7) that the angle distribution  $\omega$  satisfies the boundary conditions

$$\omega(u, 0) = \angle(\gamma'_u(0), \gamma'_v(0)) - \int_0^u \kappa_u(s) ds, \quad \forall u \in [0, L_u],\tag{8a}$$

$$\omega(0, v) = \angle(\gamma'_u(0), \gamma'_v(0)) + \int_0^v \kappa_v(s) ds, \quad \forall v \in [0, L_v].\tag{8b}$$

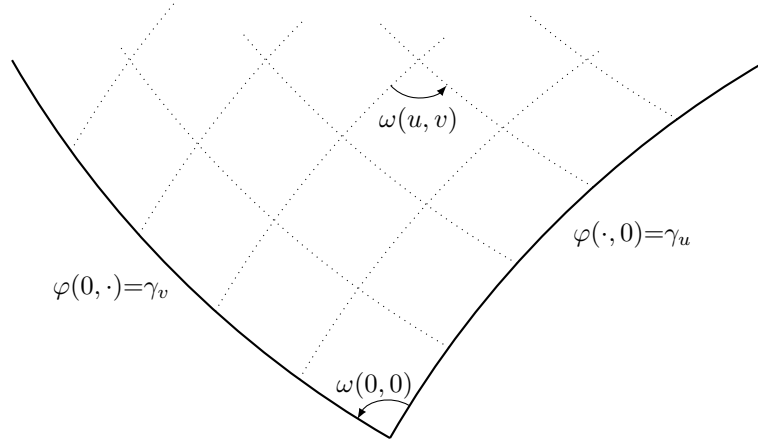


Figure 1: Illustration of the coordinate curves of a Chebyshev net  $\varphi$

We will show that we can associate to any angle distribution  $\omega : D \rightarrow (0, \pi)$  satisfying (8) a unique mapping  $\varphi := \mathcal{I}(\gamma, \omega) : D \rightarrow M$  satisfying (2b), (3), and (7b). We then show that this mapping also satisfies (2a) and (7a) whenever  $\omega$  satisfies the integrability condition (5). We prove the following result:

**Theorem 2** (Existence and uniqueness of smooth Chebyshev nets in a sector). *Let  $M$  be a smooth immersion of the plane  $\mathbb{R}^2$  in  $\mathbb{R}^3$ . Let  $\gamma_u : \mathbb{R}^+ \rightarrow M$  and  $\gamma_v : \mathbb{R}^+ \rightarrow M$  be two smooth curves with respective geodesic curvatures  $\kappa_u : \mathbb{R}^+ \rightarrow \mathbb{R}$  and  $\kappa_v : \mathbb{R}^+ \rightarrow \mathbb{R}$ , and such that  $\gamma_u(0) = \gamma_v(0)$ . Suppose that  $\psi = \pi - \angle(\gamma'_u(0), \gamma'_v(0)) \in (0, \pi)$  and that the sector  $\Omega$  delimited by  $\gamma_u$  and  $\gamma_v$  verifies*

$$\int_{\gamma_u} \kappa_u^+ + \int_{\gamma_v} \kappa_v^- + \int_{\Omega} K^+ < \pi - \psi \text{ and } \int_{\gamma_u} \kappa_u^- + \int_{\gamma_v} \kappa_v^+ + \int_{\Omega} K^- < \psi.$$

*Let  $L_u, L_v \in \mathbb{R}_*^+ \cup \{+\infty\}$  and  $D = [0, L_u] \times [0, L_v]$ . Then, there exists a unique angle distribution  $\omega : D \rightarrow (0, \pi)$  verifying the boundary conditions (8) and satisfying the Hazzidakis formula (6), with  $\varphi := \mathcal{I}(\gamma, \omega) : D \rightarrow M$  the unique mapping satisfying (2b), (3) and (7b). The mapping  $\varphi$  is a smooth Chebyshev net on  $\Omega$  with angles  $0 < \omega < \pi$ .*

*Proof of Proposition 11.* Using Theorem 12, we infer that there exists a unique angle distribution  $\omega : (\mathbb{R}^+)^2 \rightarrow \mathbb{R}/2\pi\mathbb{Z}$  satisfying the Hazzidakis formula (16), with  $\varphi : (\mathbb{R}^+)^2 \rightarrow M$  the unique mapping satisfying the boundary conditions (15) and the properties presented in the theorem. Then, using the continuity of the angle distribution  $\omega$  and  $\omega(0, 0) = \pi - \psi \in (0, \pi)$ , we infer that there exists  $\tilde{L}_1, \tilde{L}_2 > 0$  such that  $\omega(u, v) \in (0, \pi)$  for all  $(u, v) \in [0, \tilde{L}_1] \times [0, \tilde{L}_2]$ . Hence, by Theorem 12, the mapping  $\varphi$  satisfies (17) for all  $(u, v) \in [0, \tilde{L}_1] \times [0, \tilde{L}_2]$ . Then, in the same manner as in the proof of Lemma 10, we obtain that  $\varphi : (\mathbb{R}^+)^2 \rightarrow Q$  is a Chebyshev net. Suppose finally that  $\tilde{\varphi} : (\mathbb{R}^+)^2 \rightarrow M$  is a Chebyshev

net satisfying the boundary conditions (15). Then, using Property 9, we obtain that the angle distribution  $\tilde{\omega} : (\mathbb{R}^+)^2 \rightarrow (0, \pi)$  defined by  $\tilde{\omega} = \angle(\partial_u \tilde{\varphi}, \partial_v \tilde{\varphi})(u, v)$ , for all  $(u, v) \in (\mathbb{R}^+)^2$ , satisfies the Hazzidakis formula (16). We deduce from Theorem 12 that  $\varphi = \tilde{\varphi}$ . This concludes the proof.  $\square$

**Lemma 3** (Homeomorphism). *Let  $Q$  be a smooth sector delimited by the two smooth curves  $\eta_1 : \mathbb{R}^- \rightarrow M$  and  $\eta_2 : \mathbb{R}^+ \rightarrow M$  intersecting at  $p \in M$ , and satisfying (11). Assume that  $\varphi : (\mathbb{R}^+)^2 \rightarrow \varphi[(\mathbb{R}^+)^2] \subset M$  is a smooth mapping satisfying (??), and such that  $\varphi(u, 0) = \eta_2(u)$ ,  $\varphi(0, v) = \eta_1(-v)$  for all  $(u, v) \in (\mathbb{R}^+)^2$ . Then,  $\varphi : (\mathbb{R}^+)^2 \rightarrow Q$  is a homeomorphism.*

*Proof.* The proof is obtained in the same manner as in [?]. We just recall here the principal ideas. We denote  $\omega : (\mathbb{R}^+)^2 \rightarrow \mathbb{R}/2\pi\mathbb{Z}$  the angle distribution defined by  $\omega(u, v) = \angle(\partial_u \varphi, \partial_v \varphi)(u, v)$ , for all  $(u, v) \in (\mathbb{R}^+)^2$ . We denote  $\kappa_1 : \mathbb{R}^- \rightarrow \mathbb{R}$  and  $\kappa_2 : \mathbb{R}^+ \rightarrow \mathbb{R}$  the geodesic curvatures of  $\eta_1$  and  $\eta_2$  respectively. First, using (13), we obtain that

$$\omega(u, 0) = \omega(0, 0) - \int_0^u \kappa_2 = \pi - \psi - \int_0^u \kappa_2,$$

for all  $u \in \mathbb{R}^+$ . Then, using hypothesis (11), we deduce that  $\omega(u, 0) \in (0, \pi)$ , for all  $u \in \mathbb{R}^+$ . In the same manner, we obtain that  $\omega(0, v) = \pi - \psi - \int_{-v}^0 \kappa_1 \in (0, \pi)$ , for all  $v \in \mathbb{R}^+$ . Hence, using the continuity of  $\omega$ , we infer that there exists  $\tilde{D} = [0, l_1] \times [0, l_2] \subset (\mathbb{R}^+)^2$ , with  $l_1, l_2 \in \mathbb{R}_*^+$ , such that  $\omega(\tilde{D}) \subset (0, \pi)$ . Since (??) is satisfied, we infer that  $\varphi|_{\tilde{D}} : \tilde{D} \rightarrow \varphi(\tilde{D}) \subset M$  is a local homeomorphism, so that, up to reducing  $l_1$  and  $l_2$ ,  $\varphi$  is a homeomorphism. Moreover, since  $\omega(\tilde{D}) \subset (0, \pi)$ , we deduce that  $\angle(\eta_2'(u), \partial_v \varphi(u, 0)) \in (0, \pi)$ , for all  $u \in [0, l_1]$ , and  $\angle(\eta_1'(-v), \partial_u \varphi(0, v)) \in (0, \pi)$ , for all  $v \in [0, l_2]$ . We conclude that, up to reducing  $l_1$  and  $l_2$ , we have  $\varphi(\tilde{D}) \subset Q$ .

Reasoning by contradiction, we first suppose that  $\varphi$  is not a homeomorphism. Let  $U = [0, L_1] \times [0, L_2]$  and  $U_{cl} = [0, L_1] \times [0, L_2]$ , with  $L_1, L_2 > 0$ , be such that  $\varphi|_U : U \rightarrow \varphi(U) \subset M$  is a homeomorphism and such that  $\varphi|_{U_{cl}} : U_{cl} \rightarrow \varphi(U_{cl}) \subset M$  is not a homeomorphism. Using the Hazzidakis formula (14) and hypothesis (11), we easily obtain that  $\omega(U_{cl}) \subset (0, \pi)$ . Hence, the mapping  $\varphi|_{U_{cl}}$  is a local homeomorphism. Now, suppose that there exist  $(u_1, v_1), (u_2, v_2) \in (0, L_1] \times \{L_2\} \cup \{L_1\} \times (0, L_2]$  with  $\varphi(u_1, v_1) = \varphi(u_2, v_2)$ . Then, the two following cases are possible:

- case 1:  $u_1 = u_2 = L_1$  or  $v_1 = v_2 = L_2$ . We only consider the first subcase, since the reasoning for the second subcase is similar. Then, assuming that  $u_1 = u_2 = L_1$ , one can see that the Gauss–Bonnet formula applied to the curve  $\varphi(\{L_1\} \times [v_1, v_2])$  is in contradiction with (11).
- case 2:  $u_1 \neq u_2$  and  $v_1 \neq v_2$ . In this case, we can suppose, without loss of generality, that  $v_1 = L_2$  and  $u_2 = L_1$ . Then, the Gauss–Bonnet formula applied to the curve  $\varphi([u_1, L_1] \times \{L_2\}) \cup \varphi(\{L_1\} \times [L_2, v_2])$  yields a contradiction with (11).

We finally suppose that  $\varphi[(\mathbb{R}^+)^2] \not\subset Q$ . Then, let  $\tilde{U} = [0, \tilde{L}_1] \times [0, \tilde{L}_2]$ , with  $\tilde{L}_1, \tilde{L}_2 > 0$ , be such that  $\varphi(\tilde{U}) \subset Q$  and such that there exists  $(\tilde{u}, \tilde{v}) \in (0, \tilde{L}_1] \times \{\tilde{L}_2\} \cup \{\tilde{L}_1\} \times (0, \tilde{L}_2]$  with  $\varphi(\tilde{u}, \tilde{v}) \in \partial Q$ . Then,  $\varphi(\tilde{u}, \tilde{v}) \in \eta_1(\mathbb{R}^-)$  or  $\varphi(\tilde{u}, \tilde{v}) \in \eta_2(\mathbb{R}^+)$  and we obtain again a contradiction between the Gauss–Bonnet formula and (11). This concludes the proof.  $\square$

In a second part, we will apply the construction on sectors to consider the case where the boundary curve of the sector  $\eta : I \subset \mathbb{R} \rightarrow M$  we consider is arc-length parametrized, continuous on  $I$ , and piecewise smooth according to the following definition:

**Definition 4** (Piecewise smooth curves). *Let  $\eta : I \subset \mathbb{R} \rightarrow M$  be a continuous curve. We say that the curve  $\eta$  is piecewise smooth if there exists a partition of  $I$  in the form  $\bigcup_{i=1}^{N+1} [a_{i-1}, a_i] = I$ , with  $a_0 < \dots < a_{N+1}$ . Suppose moreover that  $\eta$  restricted to  $(a_{i-1}, a_i)$  is a smooth curve with all the derivatives having a finite limit from the right at  $a_{i-1}$  and from the left at  $a_i$ , for all  $i \in \{1, \dots, N+1\}$ . We say that the curve  $\eta$  is piecewise smooth.*

We define the total positive and negative turn angle of continuous piecewise smooth curves (see [?]):

**Definition 5** (Positive and negative turn angle  $\tau_{\pm}$ ). *Let  $\eta : I \subset \mathbb{R} \rightarrow M$  be a continuous piecewise smooth curve on the partition of  $I$  defined by  $a_0 < \dots < a_{N+1}$ . Then, for all  $i \in \{1, \dots, N+1\}$ , let  $\kappa_i : [a_{i-1}, a_i] \rightarrow \mathbb{R}$  be the geodesic curvature of  $\eta|_{[a_{i-1}, a_i]}$  defined by*

$$\kappa_i(s) = \langle \eta''(s), \eta'^{\perp}(s) \rangle_g,$$

with  $\eta'^{\perp}(s) \in T_{\eta(s)}M$  the vector such that  $\angle(\eta'(s), \eta'^{\perp}(s)) = \frac{\pi}{2}$ , for all  $s \in [a_{i-1}, a_i]$ . Let  $\psi_i = \angle(\eta'(a_i^-), \eta'(a_i^+))$ , for all  $i \in \{1, \dots, N\}$ . We suppose that  $-\pi < \psi_i < \pi$ , for all  $i \in \{1, \dots, N\}$ . We define the total positive and negative turn angles  $\tau(\eta)$  by

$$\tau_+(\eta) = \sum_{i=1}^{N+1} \int_{a_{i-1}}^{a_i} \kappa_i^+ + \sum_{i=1}^N \psi_i^+, \quad \tau_-(\eta) = \sum_{i=1}^{N+1} \int_{a_{i-1}}^{a_i} \kappa_i^- + \sum_{i=1}^N \psi_i^-, \quad (9)$$

and we define the total turn angle as  $\tau(\eta) = \tau_+(\eta) - \tau_-(\eta)$ . (Note that  $\tau_{\pm}(\eta)$  are different from  $(\tau(\eta))^{\pm}$ .) We denote  $\tau(\eta|_{[a,b]})$ , with  $a, b \in \mathbb{R}$  such that  $a < b$ , the restriction to  $[a, b]$  of the total turn angle. Note that, if any, the pointwise turns at  $a$  and  $b$  are included.

An illustration of the turn angle is presented in Figure 2. Finally, we denote  $\text{int}(D)$  the interior of any set  $D$ .

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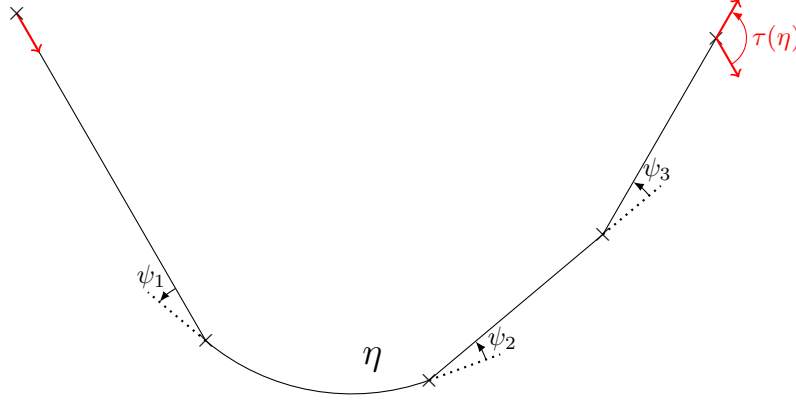


Figure 2: Illustration of the total turn angle  $\tau(\eta)$

### 3 Chebyshev nets on broken half-surfaces

In order to construct a Chebyshev net on broken half-surfaces with  $N \geq 1$  vertices, we first restrict ourselves in Section 3.1 to the case of a sector, which corresponds to the case  $N = 1$ . Then, in Section 3.2, we show that broken half-surfaces admit a Chebyshev parametrization as a particular piecewise smooth sector, under conditions on their total Gaussian curvature.

#### 3.1 Construction on a sector

We give in this section some existence results for Chebyshev nets on sectors.

**Definition 6** ((Smooth) sector). *A sector  $Q$  of the surface  $M$  is an unbounded connected domain of  $M$  delimited by the two curves  $\eta_1 : \mathbb{R}^- \rightarrow M$  and  $\eta_2 : \mathbb{R}^+ \rightarrow M$  intersecting only at  $p = \eta_1(0) = \eta_2(0)$ . Sectors are said to be smooth whenever the two curves  $\eta_1$  and  $\eta_2$  are smooth. The angle  $\psi = \angle(\eta_1'(0), \eta_2'(0))$  is supposed to be in  $(0, \pi)$  and is called the exterior angle of the sector  $Q$ .*

A sector with the notation introduced above is presented in Figure 3. Now, we recall a theorem obtained by Bakelman [?] and stated in the present form in [?]:

**Theorem 7** (I. Ya. Bakelman). *Let  $Q$  be a sector delimited by the two curves  $\eta_1 : \mathbb{R}^- \rightarrow M$  and  $\eta_2 : \mathbb{R}^+ \rightarrow M$  intersecting at  $p \in M$ . Suppose that  $Q$  satisfies the conditions*

$$\tau_+(\eta_1) + \tau_+(\eta_2) + \int_Q K^+ < \pi - \psi, \quad (10a)$$

$$\tau_-(\eta_1) + \tau_-(\eta_2) + \int_Q K^- < \psi, \quad (10b)$$



where  $\psi > 0$  is the exterior angle of  $Q$  at the vertex  $p$  and  $\tau_{\pm}(\eta_i)$ , with  $i \in \{1, 2\}$ , are the total positive and negative turn angles of  $\eta_i$  defined by (9). Then, there exist global Chebyshev coordinates in  $Q$  such that  $\eta_1$  and  $\eta_2$  are coordinate curves. Furthermore, the angle between the coordinate curves is bounded away from 0 and  $\pi$  by the positive real number

$$\min \left( \pi - \psi - \int_Q K^+ - \tau_+(\eta_1) - \tau_+(\eta_2), \psi - \int_Q K^- - \tau_-(\eta_1) - \tau_-(\eta_2) \right).$$

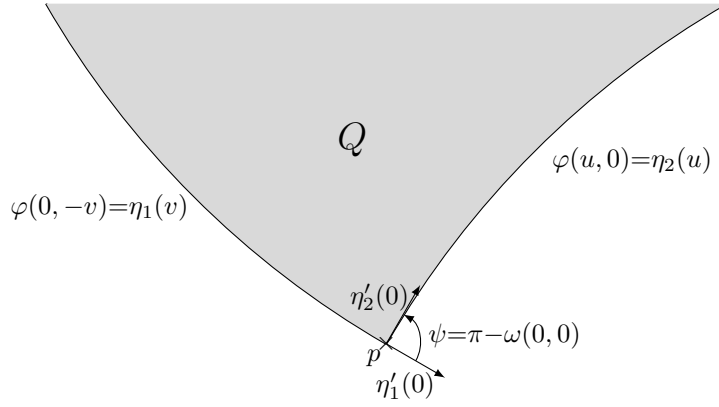


Figure 3: Illustration of a Chebyshev net  $\varphi$  on a sector  $Q$  of exterior angle  $\psi$

This theorem gives no information about the regularity of the Chebyshev net, even when the two delimiting curves of the sector are smooth curves. Our goal is now to sharpen Theorem 7 (see Proposition 11 below) to prove the existence of smooth Chebyshev nets on sectors delimited by smooth curves satisfying the counterpart of (10) for smooth curves, namely

$$\int_{\mathbb{R}^+} \kappa_2^+ + \int_{\mathbb{R}^-} \kappa_1^+ + \int_Q K^+ < \pi - \psi \text{ and } \int_{\mathbb{R}^+} \kappa_2^- + \int_{\mathbb{R}^-} \kappa_1^- + \int_Q K^- < \psi. \quad (11)$$

To this purpose, we state some preliminary results. First, we relate in Property 8 the geodesic curvatures of the coordinate curves of Chebyshev nets to the angle  $\omega$  between these coordinate curves. Then, we present the Hazzidakis formula in Property 9. See [?] for a proof of these properties.

**Property 8** (Geodesic curvature of coordinate curves). *Let  $\varphi : U \subset \mathbb{R}^2 \rightarrow \varphi(U) \subset M$  be a smooth mapping satisfying (??) and let  $(u_1, v_1) \in \mathbb{R}^2$  and  $(u_2, v_2) \in \mathbb{R}^2$ . We denote  $\omega : U \rightarrow \mathbb{R}/2\pi\mathbb{Z}$  the angle distribution defined by  $\omega(u, v) = \angle(\partial_u \varphi, \partial_v \varphi)(u, v)$ , for all  $(u, v) \in U$ . Then, supposing  $u_1, v_1$  and  $v_2$  are such that  $\{u_1\} \times [v_1, v_2] \subset U$ , we have*

$$\omega(u_1, v_2) = \omega(u_1, v_1) - \int_{-v_2}^{-v_1} \kappa_v, \quad (12)$$

where  $\kappa_v : [-v_2, -v_1] \rightarrow \mathbb{R}$  is the geodesic curvature of the curve  $\eta_1 : [-v_2, -v_1] \rightarrow M$  defined by  $\eta_1(v) = \varphi(u_1, -v)$ , for all  $v \in [-v_2, -v_1]$ . This property, illustrated in Figure 4, results from the parallel transport of the vector field  $\partial_u \varphi$  along  $\eta_1$ . Moreover, supposing  $u_1, u_2$  and  $v_1$  are such that  $[u_1, u_2] \times \{v_1\} \subset U$ , we have

$$\omega(u_2, v_1) = \omega(u_1, v_1) - \int_{u_1}^{u_2} \kappa_u, \quad (13)$$

where  $\kappa_u : [u_1, u_2] \rightarrow \mathbb{R}$  is the geodesic curvature of the curve  $\eta_2 : [u_1, u_2] \rightarrow M$  defined by  $\eta_2(u) = \varphi(u, v_1)$ , for all  $u \in [u_1, u_2]$ .

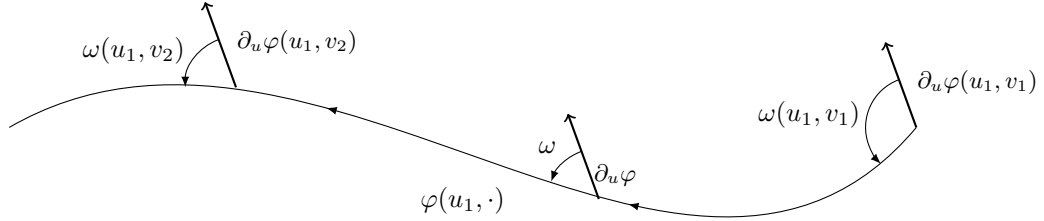


Figure 4: Illustration of the parallel transport of  $\partial_u \varphi$  along  $\varphi(u_1, \cdot)$

**Property 9** (Hazzidakis formula). *Let  $U = [u_1, u_2] \times [v_1, v_2]$ , with  $(u_1, v_1) \in \mathbb{R}^2$  and  $(u_2, v_2) \in \mathbb{R}^2$ . Let  $\varphi : U \rightarrow \Omega \subset M$ , with  $\Omega = \varphi(U)$ , be a smooth Chebyshev net. We denote  $\omega : U \rightarrow (0, \pi)$  the angle distribution defined by  $\omega = \angle(\partial_u \varphi, \partial_v \varphi)$  and we denote  $\eta_1 : [-v_2, -v_1] \rightarrow M$  and  $\eta_2 : [u_1, u_2] \rightarrow M$  the two curves respectively defined by  $\eta_1(v) = \varphi(u_1, -v)$ , for all  $v \in [-v_2, -v_1]$ , and  $\eta_2(u) = \varphi(u, v_1)$ , for all  $u \in [u_1, u_2]$ . We denote  $\kappa_u : [u_1, u_2] \rightarrow \mathbb{R}$  and  $\kappa_v : [-v_2, -v_1] \rightarrow \mathbb{R}$  their respective geodesic curvature. Then, the angle distribution  $\omega$  satisfies the following Hazzidakis formula*

$$\omega(u_2, v_2) = \omega(u_1, v_1) - \int_{u_1}^{u_2} \kappa_u - \int_{-v_2}^{-v_1} \kappa_v - \int_{\Omega} K. \quad (14)$$

**Lemma 10** (Homeomorphism). *Let  $Q$  be a smooth sector delimited by the two smooth curves  $\eta_1 : \mathbb{R}^- \rightarrow M$  and  $\eta_2 : \mathbb{R}^+ \rightarrow M$  intersecting at  $p \in M$ , and satisfying (11). Assume that  $\varphi : (\mathbb{R}^+)^2 \rightarrow \varphi[(\mathbb{R}^+)^2] \subset M$  is a smooth mapping satisfying (??), and such that  $\varphi(u, 0) = \eta_2(u)$ ,  $\varphi(0, v) = \eta_1(-v)$  for all  $(u, v) \in (\mathbb{R}^+)^2$ . Then,  $\varphi : (\mathbb{R}^+)^2 \rightarrow Q$  is a homeomorphism.*

*Proof.* The proof is obtained in the same manner as in [?]. We just recall here the principal ideas. We denote  $\omega : (\mathbb{R}^+)^2 \rightarrow \mathbb{R}/2\pi\mathbb{Z}$  the angle distribution defined by  $\omega(u, v) = \angle(\partial_u \varphi, \partial_v \varphi)(u, v)$ , for all  $(u, v) \in (\mathbb{R}^+)^2$ . We denote  $\kappa_1 : \mathbb{R}^- \rightarrow \mathbb{R}$

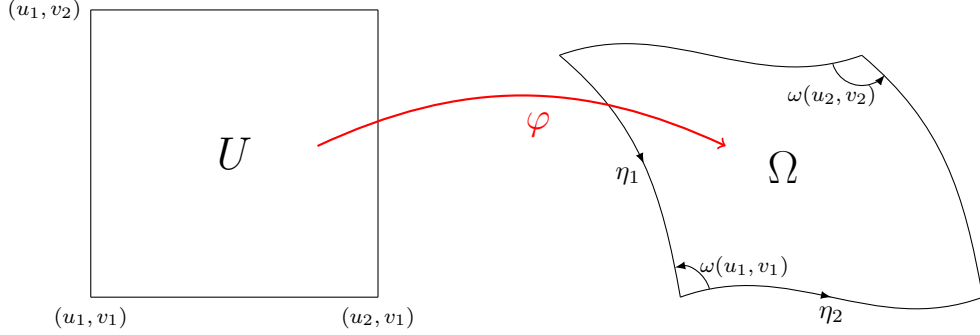


Figure 5: Illustration of the Hazzidakis formula

and  $\kappa_2 : \mathbb{R}^+ \rightarrow \mathbb{R}$  the geodesic curvatures of  $\eta_1$  and  $\eta_2$  respectively. First, using (13), we obtain that

$$\omega(u, 0) = \omega(0, 0) - \int_0^u \kappa_2 = \pi - \psi - \int_0^u \kappa_2,$$

for all  $u \in \mathbb{R}^+$ . Then, using hypothesis (11), we deduce that  $\omega(u, 0) \in (0, \pi)$ , for all  $u \in \mathbb{R}^+$ . In the same manner, we obtain that  $\omega(0, v) = \pi - \psi - \int_{-v}^0 \kappa_1 \in (0, \pi)$ , for all  $v \in \mathbb{R}^+$ . Hence, using the continuity of  $\omega$ , we infer that there exists  $\tilde{D} = [0, l_1] \times [0, l_2] \subset (\mathbb{R}^+)^2$ , with  $l_1, l_2 \in \mathbb{R}_*^+$ , such that  $\omega(\tilde{D}) \subset (0, \pi)$ . Since (??) is satisfied, we infer that  $\varphi|_{\tilde{D}} : \tilde{D} \rightarrow \varphi(\tilde{D}) \subset M$  is a local homeomorphism, so that, up to reducing  $l_1$  and  $l_2$ ,  $\varphi$  is a homeomorphism. Moreover, since  $\omega(\tilde{D}) \subset (0, \pi)$ , we deduce that  $\angle(\eta_2'(u), \partial_v \varphi(u, 0)) \in (0, \pi)$ , for all  $u \in [0, l_1]$ , and  $\angle(\eta_1'(-v), \partial_v \varphi(0, v)) \in (0, \pi)$ , for all  $v \in [0, l_2]$ . We conclude that, up to reducing  $l_1$  and  $l_2$ , we have  $\varphi(\tilde{D}) \subset Q$ .

Reasoning by contradiction, we first suppose that  $\varphi$  is not a homeomorphism. Let  $U = [0, L_1] \times [0, L_2]$  and  $U_{cl} = [0, L_1] \times [0, L_2]$ , with  $L_1, L_2 > 0$ , be such that  $\varphi|_U : U \rightarrow \varphi(U) \subset M$  is a homeomorphism and such that  $\varphi|_{U_{cl}} : U_{cl} \rightarrow \varphi(U_{cl}) \subset M$  is not a homeomorphism. Using the Hazzidakis formula (14) and hypothesis (11), we easily obtain that  $\omega(U_{cl}) \subset (0, \pi)$ . Hence, the mapping  $\varphi|_{U_{cl}}$  is a local homeomorphism. Now, suppose that there exist  $(u_1, v_1), (u_2, v_2) \in (0, L_1] \times \{L_2\} \cup \{L_1\} \times (0, L_2]$  with  $\varphi(u_1, v_1) = \varphi(u_2, v_2)$ . Then, the two following cases are possible:

- case 1:  $u_1 = u_2 = L_1$  or  $v_1 = v_2 = L_2$ . We only consider the first subcase, since the reasoning for the second subcase is similar. Then, assuming that  $u_1 = u_2 = L_1$ , one can see that the Gauss–Bonnet formula applied to the curve  $\varphi(\{L_1\} \times [v_1, v_2])$  is in contradiction with (11).
- case 2:  $u_1 \neq u_2$  and  $v_1 \neq v_2$ . In this case, we can suppose, without loss of generality, that  $v_1 = L_2$  and  $u_2 = L_1$ . Then, the Gauss–Bonnet

formula applied to the curve  $\varphi([u_1, L_1] \times \{L_2\}) \cup \varphi(\{L_1\} \times [L_2, v_2])$  yields a contradiction with (11).

We finally suppose that  $\varphi[(\mathbb{R}^+)^2] \not\subset Q$ . Then, let  $\tilde{U} = [0, \tilde{L}_1] \times [0, \tilde{L}_2]$ , with  $\tilde{L}_1, \tilde{L}_2 > 0$ , be such that  $\varphi(\tilde{U}) \subset Q$  and such that there exists  $(\tilde{u}, \tilde{v}) \in (0, \tilde{L}_1] \times \{\tilde{L}_2\} \cup \{\tilde{L}_1\} \times (0, \tilde{L}_2]$  with  $\varphi(\tilde{u}, \tilde{v}) \in \partial Q$ . Then,  $\varphi(\tilde{u}, \tilde{v}) \in \eta_1(\mathbb{R}^-)$  or  $\varphi(\tilde{u}, \tilde{v}) \in \eta_2(\mathbb{R}^+)$  and we obtain again a contradiction between the Gauss–Bonnet formula and (11). This concludes the proof.  $\square$

**Proposition 11** (Existence of smooth Chebyshev nets on sectors). *Let  $Q$  be a smooth sector delimited by the two smooth curves  $\eta_1 : \mathbb{R}^- \rightarrow M$  and  $\eta_2 : \mathbb{R}^+ \rightarrow M$ , and with exterior angle  $\psi \in (0, \pi)$ . Suppose that the geodesic curvatures  $\kappa_1 : \mathbb{R}^- \rightarrow \mathbb{R}$  and  $\kappa_2 : \mathbb{R}^+ \rightarrow \mathbb{R}$  of  $\eta_1$  and  $\eta_2$  respectively and the Gaussian curvature  $K$  of  $Q$  satisfy (11). Then, there exists a unique Chebyshev net  $\varphi : (\mathbb{R}^+)^2 \rightarrow Q$  such that*

$$\begin{aligned} \varphi(u, 0) &= \eta_2(u), & \forall u \in \mathbb{R}^+, \\ \varphi(0, v) &= \eta_1(-v), & \forall v \in \mathbb{R}^+. \end{aligned} \quad (15)$$

Moreover, the angle  $\omega = \angle(\partial_u \varphi, \partial_v \varphi) \in (0, \pi)$  of  $\varphi$  satisfies the Hazzidakis formula

$$\omega(u, v) = \pi - \psi - \int_0^u \kappa_2 - \int_{-v}^0 \kappa_1 - \int_{\varphi([0, u] \times [0, v])} K, \quad (16)$$

for all  $(u, v) \in (\mathbb{R}^+)^2$ .

The Hazzidakis formula in the sector  $Q$  is illustrated in Figure 6.

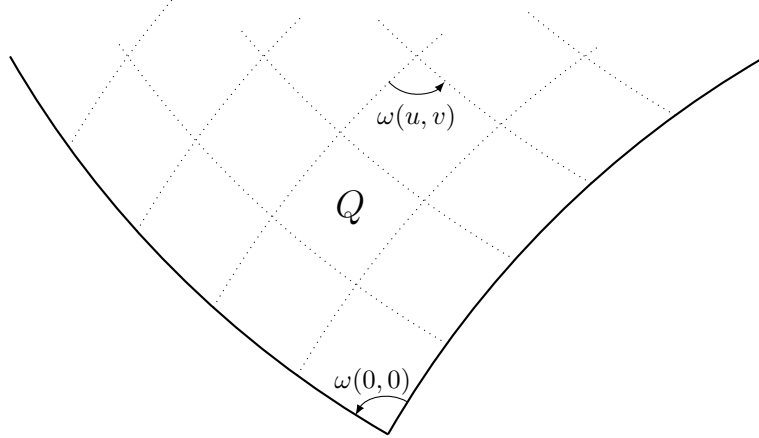


Figure 6: Illustration of the Hazzidakis formula in the sector  $Q$

Before proving this result, we recall a theorem proved in Section ??.

**Theorem 12** (Existence of a unique solution to integrability condition). *Let  $M$  be a smooth, open, complete, and simply connected surface. Let  $\eta_1 : \mathbb{R}^- \rightarrow M$*

and  $\eta_2 : \mathbb{R}^+ \rightarrow M$  be two smooth curves with respective geodesic curvatures  $\kappa_1 : \mathbb{R}^- \rightarrow \mathbb{R}$  and  $\kappa_2 : \mathbb{R}^+ \rightarrow \mathbb{R}$ , and such that  $\eta_1(0) = \eta_2(0)$ . Suppose that  $\psi := \angle(\eta'_1(0), \eta'_2(0)) \in (0, \pi)$ . Then, there exists a unique angle distribution  $\omega : (\mathbb{R}^+)^2 \rightarrow \mathbb{R}/2\pi\mathbb{Z}$  satisfying the Hazzidakis formula (16), with  $\varphi : (\mathbb{R}^+)^2 \rightarrow M$  the unique smooth mapping satisfying the boundary conditions (15), and such that its  $v$ -coordinate curves are arc-length parametrized curves with a geodesic curvature  $\kappa_2^{\text{map}} : D \rightarrow \mathbb{R}$  satisfying  $\kappa_2^{\text{map}}(u, v) = \partial_v \omega(u, v)$ , for all  $(u, v) \in (\mathbb{R}^+)^2$ .

Suppose moreover that there exists  $\tilde{D} = [0, \tilde{L}_2] \times [0, \tilde{L}_1]$ , with  $\tilde{L}_1, \tilde{L}_2 \in \mathbb{R}^+$ , such that  $0 < \omega(u, v) < \pi$ , for all  $(u, v) \in \tilde{D}$ . Then, the mapping  $\varphi$  satisfies

$$|\partial_u \varphi|_g(u, v) = |\partial_v \varphi|_g(u, v) = 1, \quad (17)$$

for all  $(u, v) \in \tilde{D}$ .

*Proof of Proposition 11.* Using Theorem 12, we infer that there exists a unique angle distribution  $\omega : (\mathbb{R}^+)^2 \rightarrow \mathbb{R}/2\pi\mathbb{Z}$  satisfying the Hazzidakis formula (16), with  $\varphi : (\mathbb{R}^+)^2 \rightarrow M$  the unique mapping satisfying the boundary conditions (15) and the properties presented in the theorem. Then, using the continuity of the angle distribution  $\omega$  and  $\omega(0, 0) = \pi - \psi \in (0, \pi)$ , we infer that there exists  $\tilde{L}_1, \tilde{L}_2 > 0$  such that  $\omega(u, v) \in (0, \pi)$  for all  $(u, v) \in [0, \tilde{L}_1] \times [0, \tilde{L}_2]$ . Hence, by Theorem 12, the mapping  $\varphi$  satisfies (17) for all  $(u, v) \in [0, \tilde{L}_1] \times [0, \tilde{L}_2]$ . Then, in the same manner as in the proof of Lemma 10, we obtain that  $\varphi : (\mathbb{R}^+)^2 \rightarrow Q$  is a Chebyshev net. Suppose finally that  $\tilde{\varphi} : (\mathbb{R}^+)^2 \rightarrow M$  is a Chebyshev net satisfying the boundary conditions (15). Then, using Property 9, we obtain that the angle distribution  $\tilde{\omega} : (\mathbb{R}^+)^2 \rightarrow (0, \pi)$  defined by  $\tilde{\omega} = \angle(\partial_u \tilde{\varphi}, \partial_v \tilde{\varphi})(u, v)$ , for all  $(u, v) \in (\mathbb{R}^+)^2$ , satisfies the Hazzidakis formula (16). We deduce from Theorem 12 that  $\varphi = \tilde{\varphi}$ . This concludes the proof.  $\square$

### 3.2 Construction on a broken half-surface

We now introduce broken half-surfaces which are defined to be half-surfaces with polygonal boundaries:

**Definition 13** ((Geodesic) broken half-surfaces). *Let  $N \geq 1$  be an integer. We say that  $B_c$  is a broken half-surface if  $B_c$  is a half-surface delimited by a piecewise smooth curve  $\gamma : \mathbb{R} \rightarrow M$  on the partition of  $\mathbb{R}$  defined by  $-\infty = a_0 < \dots < a_{N+1} = \infty$ . We denote  $p_i = \gamma(a_i)$ , for all  $i \in \{1, \dots, N\}$ , and we set  $\gamma_i := \gamma|_{[a_{i-1}, a_i]} : [a_{i-1}, a_i] \rightarrow M$ , for all  $i \in \{1, \dots, N+1\}$ . The points  $\{p_i\}_{1 \leq i \leq N}$  are called the vertices of  $B_c$ . We suppose moreover that the exterior angle  $\psi_i = \angle(\gamma'_i(a_i^-), \gamma'_{i+1}(a_i^+))$  at the vertex  $p_i$  satisfies  $\psi_i \in (0, \pi)$ , for all  $i \in \{1, \dots, N\}$ . Finally, we define*

$$|\psi|_{l^1} = \sum_{1 \leq i \leq N} \psi_i, \quad \text{and} \quad |\psi|_{l^\infty} = \max_{1 \leq i \leq N} \psi_i$$

and we suppose that  $|\psi|_{l^1} < \pi$ . Broken half-surfaces are called geodesic when the boundary curves are geodesic curves. Broken half-surfaces with  $N$  vertices are called  $N$ -half-surfaces.

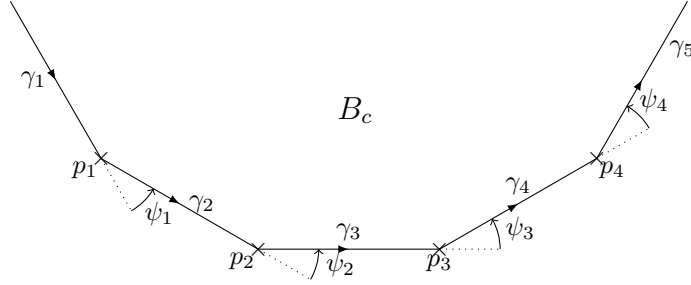


Figure 7: Illustration of a geodesic  $N$ -half-surface  $B_c$  with  $N = 4$

We depict the notation introduced in Definition 13 in Figure 7. Note that the edges composing  $\partial B_c$  are depicted as straight edges in this figure although they are more generally curved edges. We observe that 1-half-surfaces are smooth sectors. Now, in order to find Chebyshev nets on broken half-surfaces, we view them as sectors delimited by two piecewise smooth curves. This process, called sectorization, is described in the following definition.

**Definition 14** (Sectorization). *Let  $N \geq 1$  and let  $B_c$  be a  $N$ -half-surface delimited by the curves  $\{\gamma_i\}_{1 \leq i \leq N+1}$ . We denote  $\{p_i\}_{1 \leq i \leq N}$  the vertices of  $B_c$ . Let  $m \in \{1, \dots, N\}$ . We denote  $Q(B_c, p_m)$  the piecewise smooth sector delimited by the curves  $\eta_1^m : \mathbb{R}^- \rightarrow M$  and  $\eta_2^m : \mathbb{R}^+ \rightarrow M$  defined so that*

$$\eta_1^m(\mathbb{R}^-) = \bigcup_{i=1}^m \gamma_i([a_{i-1}, a_i]) \quad \text{and} \quad \eta_2^m(\mathbb{R}^+) = \bigcup_{i=m+1}^{N+1} \gamma_i([a_{i-1}, a_i]). \quad (18)$$

The sectorization of a broken half-surface is depicted in Figure 8. We

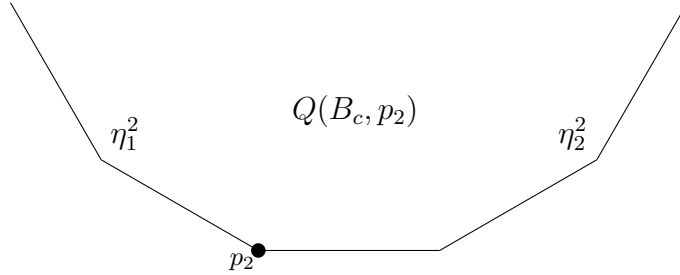


Figure 8: Illustration of the sectorization  $Q(B_c, p_2)$  of a  $N$ -half-surface  $B_c$  with  $N = 4$

give in the following proposition conditions on  $B_c$  for the existence of a sector  $Q(B_c, p_m)$ , for some  $m \in \{1, \dots, N\}$ , satisfying the conditions (10).

**Proposition 15** (From  $N$ -half-surfaces to sectors). *Let  $N \geq 1$  be an integer. Suppose that the  $N$ -half-surface  $B_c$  satisfies the conditions*

$$\tau_+(\partial B_c) + \int_{B_c} K^+ < \pi, \quad (19a)$$

$$\tau_-(\partial B_c) + \int_{B_c} K^- < |\psi|_{l^\infty}. \quad (19b)$$

*Then, there exists  $m \in \{1, \dots, N\}$  such that the piecewise smooth sector  $Q(B_c, p_m)$  satisfies the conditions (10).*

*Proof.* Let  $m = \operatorname{argmax}_{1 \leq i \leq N} \psi_i$  and denote  $Q_m = Q(B_c, p_m)$ . Then, a straightforward computation gives

$$\begin{aligned} \int_{Q_m} K^+ + \tau_+(\eta_1) + \tau_+(\eta_2) &= \int_{Q_m} K^+ + |\psi|_{l^1} - \psi_m + \sum_{i=1}^{N+1} \int_{a_{i-1}}^{a_i} \kappa_i^+ = \tau_+(\partial B_c) + \int_{B_c} K^+ - \psi_m, \\ \int_{Q_m} K^- + \tau_-(\eta_1) + \tau_-(\eta_2) &= \int_{Q_m} K^- + \sum_{i=1}^{N+1} \int_{a_{i-1}}^{a_i} \kappa_i^- < \psi_m. \end{aligned}$$

Then, conditions (10) follow from (19).  $\square$

**Corollary 16** (Existence of Chebyshev nets on  $N$ -half-surfaces). *Let  $N \geq 1$  be an integer. Let  $B_c$  be a  $N$ -half-surface delimited by the curves  $(\gamma_i)_{1 \leq i \leq N+1}$ . Suppose that  $B_c$  satisfies the conditions (19). Then, there exist Chebyshev coordinates on  $B_c$  such that  $(\gamma_i)_{1 \leq i \leq N+1}$  are coordinate curves. Moreover, the angle of the net is bounded away from 0 and  $\pi$  by*

$$\varepsilon = \min \left( \pi - \tau_+(\partial B_c) - \int_{B_c} K^+, \quad |\psi|_{l^\infty} - \tau_-(\partial B_c) - \int_{B_c} K^- \right). \quad (20)$$

*Proof.* The proof follows by combining Proposition 15 and Theorem 7.  $\square$

Finally, in the specific case of geodesic  $N$ -half-surfaces, we obtain the following theorem:

**Theorem 17** (Existence of Chebyshev nets on geodesic  $N$ -half-surfaces). *Let  $N \geq 1$ . Let  $B_c$  be a geodesic  $N$ -half-surface delimited by the geodesic curves  $\{\gamma_i\}_{1 \leq i \leq N+1}$ . Suppose  $B_c$  satisfies the conditions*

$$\int_{B_c} K^+ < \pi - |\psi|_{l^1}, \quad (21a)$$

$$\int_{B_c} K^- < |\psi|_{l^\infty}. \quad (21b)$$

*Then, there exist Chebyshev coordinates on  $B_c$  such that  $\{\gamma_i\}_{1 \leq i \leq N+1}$  are coordinate curves. Moreover, the angle of the net is bounded away from 0 and  $\pi$  by the positive real number*

$$\min \left( \pi - |\psi|_{l^1} - \int_{B_c} K^+, |\psi|_{l^\infty} - \int_{B_c} K^- \right). \quad (22)$$

## 4 Splitting of a surface into geodesic broken half-surfaces

In this section, we show how to split any surface  $M$  satisfying the curvature bound (1) into geodesic broken half-surfaces, each of them satisfying the conditions (21). This is the principal result of this section, stated in Theorem 22. This result is obtained in a similar manner to [?, Th. 4]: first, we split the surface into four sectors, all of them satisfying (21a) (see Theorem 20). Then, we split recursively each sector into broken half-surfaces, all of them satisfying (21a) (see Theorem 18). We finally prove that, after a finite number of splittings, all of the broken half-surfaces also satisfy the condition (21b).

### 4.1 Splitting of broken half-surfaces

We prove in this subsection the following theorem which extends the splitting of sectors, introduced in [?], to broken half-surfaces.

**Theorem 18** (Splitting of  $N$ -half-surfaces). *Let  $N \geq 1$  and let  $B_c$  be a  $N$ -half-surface with exterior angles  $\{\psi_i^0\}_{1 \leq i \leq N}$  satisfying*

$$\int_{B_c} K^+ < \pi - |\psi^0|_{l^1} - 2\varepsilon \text{ and } \int_{B_c} K^- < C, \quad (23)$$

*for positive  $C$  and  $\varepsilon$ . Then, there exist  $N_1, N_2 \geq 1$  such that*

$$N_1 + N_2 \in \{N + 1, N + 2\} \quad (24)$$

*and a geodesic curve  $\sigma^*$  splitting  $B_c$  into a  $N_1$ -half-surface  $B_c^1$  with exterior angles  $\{\psi_i^1\}_{1 \leq i \leq N_1}$  and a  $N_2$ -half-surface  $B_c^2$  with exterior angles  $\{\psi_i^2\}_{1 \leq i \leq N_2}$  satisfying*

$$\int_{B_c^1} K^+ < \pi - |\psi^1|_{l^1} - \varepsilon, \quad \int_{B_c^1} K^- < \frac{C}{2}, \quad (25a)$$

$$\int_{B_c^2} K^+ < \pi - |\psi^2|_{l^1} - \varepsilon, \quad \int_{B_c^2} K^- < \frac{C}{2}, \quad (25b)$$

*with  $|\psi^1|_{l^1} = \sum_{i=1}^{N_1} \psi_i^1$ , and  $|\psi^2|_{l^1} = \sum_{i=1}^{N_2} \psi_i^2$ .*

*Remark 19* ( $N_1$  and  $N_2$ ). Two different cases can happen for the splitting : either the geodesic curve  $\sigma^*$  intersects  $\partial B_c$  at some vertex and we have  $N_1 + N_2 = N + 1$ , or  $\sigma^*$  intersects  $\partial B_c$  in the interior of some edge and we have  $N_1 + N_2 = N + 2$ . See Figure 9 for an illustration of these two cases.

*Proof.* We adapt the proof of [?, Thm.3] to  $N$ -half-surfaces. Let us first recall the setting of this proof. We can assume the metric to be flat outside a compact set  $\tilde{D} \subset \text{int}(B_c)$  homeomorphic to the disk. Moreover, we can suppose that  $\tilde{D}$



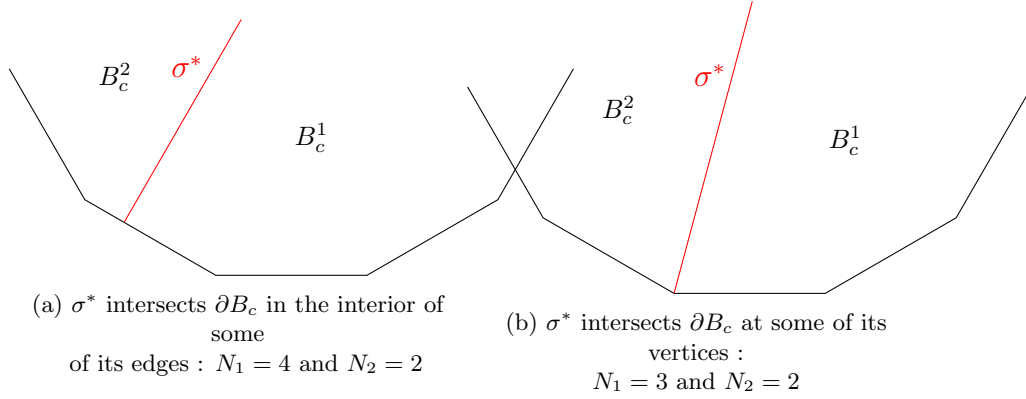


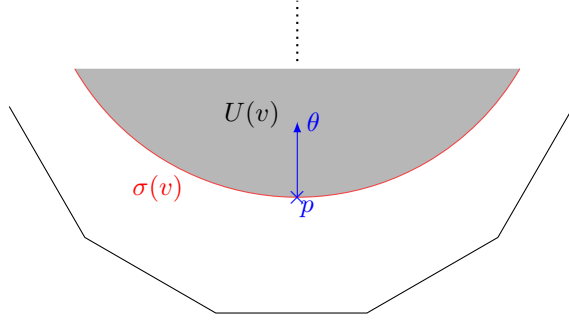
Figure 9: Illustration of the two possible cases for the splitting in Theorem 18 ( $N = 4$ )

is totally convex, i.e., all the geodesic curves joining two points  $p, q \in \tilde{D}$  are included in  $\tilde{D}$ . Let  $T_p \tilde{D}$  be the tangent plane at the point  $p \in \tilde{D}$  and let

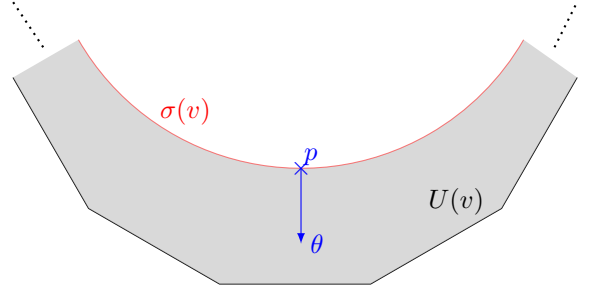
$$S\tilde{D} = \{(p, \theta), \text{ with } p \in \tilde{D}, \theta \in T_p \tilde{D}, |\theta|_g = 1\}$$

be the circle bundle over  $\tilde{D}$ . For any  $v = (p, \theta) \in S\tilde{D}$ , we denote  $-v = (p, -\theta)$  and  $\sigma(v) : \mathbb{R} \rightarrow M$  the unique geodesic curve passing through the point  $p$  orthogonally to  $v$  (the orientation of  $\sigma(v)$  has no importance in what follows). Since  $\tilde{D}$  is totally convex, the geodesic  $\sigma(v)$  splits  $\tilde{D}$  into two connected components. The vector  $v$  is directed inwards one of these components, which we denote by  $U(v)$ , and the other component is then denoted  $U(-v)$ . We now define a function  $\alpha$  which plays a similar role to the angular function  $\alpha$  in the original proof of [?, Thm.3]. The continuous function  $\alpha : S\tilde{D} \rightarrow [0, \pi]$  should satisfy  $\alpha(v) + \alpha(-v) = \pi - |\psi^0|_{l^1}$  for all  $v \in S\tilde{D}$ . For this definition, different cases, depicted in Figure 10, have to be considered:

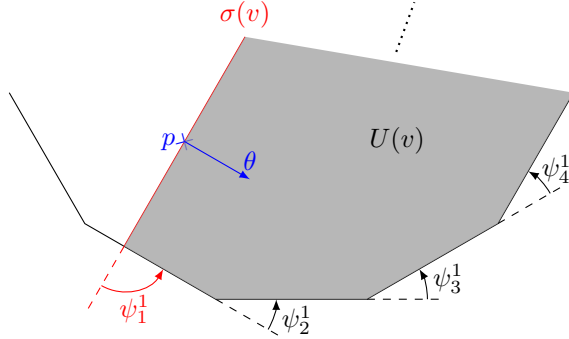
1.  $U(v)$  is a half-surface with boundary  $\sigma(v)$ ;
2.  $U(v)$  is a so-called polygonal strip;
3.  $U(v)$  and  $U(-v)$  are respectively a  $N_1$ -half-surface and a  $N_2$ -half-surface. We denote  $\{\psi_i^1(v)\}_{1 \leq i \leq N_1}$  the exterior angles of  $B_c^1$  and we set  $|\psi^1|_{l^1} := \sum_{i=1}^{N_1} \psi_i^1(v)$ ;
4.  $U(v)$  is a bounded polygonal domain with  $N_1$  vertices. We denote  $\{\psi_i^1\}_{1 \leq i \leq N_1}$  the exterior angles of  $U(v)$  and we set  $|\psi^1|_{l^1} := \sum_{i=1}^{N_1} \psi_i^1$ ;
5.  $U(v)$  is the complementary of a bounded polygonal domain (so that  $U(-v)$  is a bounded polygonal domain).



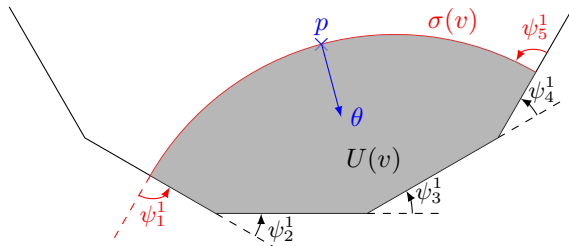
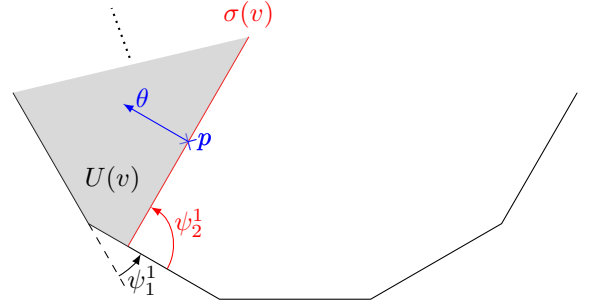
1)  $U(v)$  is a half-surface with boundary  $\sigma(v)$



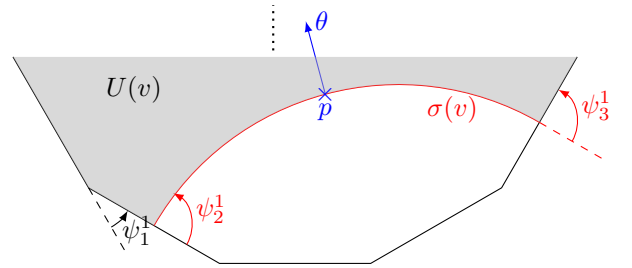
2)  $U(v)$  is a polygonal strip



3) Both  $U(v)$  and  $U(-v)$  are broken half-surfaces



4)  $U(v)$  is a bounded polygonal domain



5)  $U(v)$  is the complementary of a bounded polygonal domain

Figure 10: Illustration of the possible splittings of a  $N$ -half-surface with  $N = 4$  (proof of Theorem 18)

We emphasize that, for all  $v \in \tilde{D}$ ,  $U(v)$  belongs to one of the above cases. Then, we define the function  $\alpha$  in each of these cases as follows

$$\alpha(v) = \begin{cases} \pi - |\psi^0|_{l^1}, & \text{in case 1,} \\ 0, & \text{in case 2,} \\ \max \left[ \min \left( \pi - |\psi^1|_{l^1}, \pi - |\psi^0|_{l^1} \right), 0 \right], & \text{in case 3,} \\ \max \left[ \min \left( 2\pi - |\psi^1|_{l^1}, \pi - |\psi^0|_{l^1} \right), 0 \right], & \text{in case 4,} \\ \pi - |\psi^0|_{l^1} - \alpha(-v), & \text{in case 5.} \end{cases} \quad (26)$$

Using the continuity of  $|\psi^1|_{l^1}$  as  $\sigma(v)$  crosses the vertices of  $B_c$  and the continuity of all the case transitions, one can check that  $\alpha : S\tilde{D} \rightarrow [0, \pi]$  is a continuous function which satisfies  $\alpha(-v) + \alpha(v) = \pi - |\psi^0|_{l^1}$ . Now, we introduce the mapping  $\xi : S\tilde{D} \rightarrow \mathbb{R}^2$  defined by

$$\xi(v) = (\xi_1(v), \xi_2(v)) = \left( (\pi - |\psi^0|_{l^1} - 2\varepsilon) \frac{\int_{U(v)} K^+}{\int_{\tilde{D}} K^+} - \alpha(v) + \varepsilon, \quad \frac{\int_{U(v)} K^-}{\int_{\tilde{D}} K^-} - \frac{1}{2} \right).$$

Then,  $\xi_1$  satisfies

$$\begin{aligned} \xi_1(-v) &= (\pi - |\psi^0|_{l^1} - 2\varepsilon) \frac{\int_{\tilde{D}} K^+ - \int_{U(v)} K^+}{\int_{\tilde{D}} K^+} - \alpha(-v) + \varepsilon \\ &= \pi - |\psi^0|_{l^1} - 2\varepsilon - (\pi - |\psi^0|_{l^1} - 2\varepsilon) \frac{\int_{U(v)} K^+}{\int_{\tilde{D}} K^+} - (\pi - |\psi^0|_{l^1} - \alpha(v)) + \varepsilon \\ &= -\xi_1(v). \end{aligned}$$

In the same manner, we obtain that  $\xi_2(-v) = -\xi_2(v)$ , so that  $\xi(-v) = -\xi(v)$ . Therefore, using [?, Prop. 1], we can conclude that there exists  $v_0 \in S\tilde{D}$  such that  $\xi(v_0) = (0, 0)$ , which corresponds to

$$\alpha(v_0) = (\pi - |\psi^0|_{l^1} - 2\varepsilon) \frac{\int_{U(v_0)} K^+}{\int_{\tilde{D}} K^+} + \varepsilon, \quad \int_{U(v_0)} K^- = \frac{1}{2} \int_{\tilde{D}} K^-. \quad (27)$$

We now prove that  $U(v_0)$  necessarily matches case 3. First, by (27), we have

$$\alpha(v_0) \in [\varepsilon, \pi - |\psi^0|_{l^1} - \varepsilon], \quad (28)$$

which rules out cases 1 and 2. In order to rule out cases 4 and 5, suppose now that  $U(v)$  is a bounded polygonal domain. Applying the Gauss–Bonnet formula on  $U(v)$ , we infer that

$$|\psi^1|_{l^1} + \int_{U(v)} K = 2\pi,$$

which gives  $2\pi - |\psi^1|_{l^1} \leq \int_{U(v_0)} K^+$ . Then, using the hypotheses (23) and (27), we obtain that

$$\int_{U(v_0)} K^+ + \varepsilon < \alpha(v_0). \quad (29)$$

Combining these two results and the definition of  $\alpha$  in case 4, we obtain the following contradiction:

$$2\pi - |\psi^1|_{l^1} \leq \int_{U(v_0)} K^+ < \alpha(v_0) - \varepsilon \leq 2\pi - |\psi^1|_{l^1} - \varepsilon.$$

Finally, if  $U(v_0)$  matches case 5, then  $\xi(-v_0) = 0$ . Hence,  $U(-v_0)$  matches case 4 which leads, as above, to a contradiction. Therefore  $U(v_0)$  necessarily matches case 3, i.e., both  $U(v_0)$  and  $U(-v_0)$  are broken half-surfaces. Moreover, (26) and (28) show that  $\alpha(v_0) = \pi - |\psi^1|_{l^1}$ . Then, using (27) and (29), we infer that (25) is satisfied by  $U(v_0)$ . Since  $\xi(-v_0) = 0$ , we obtain, by symmetry, the same result for  $U(-v_0)$ . Finally, recalling Remark 19, we have  $N_1 + N_2 \in \{N + 1, N + 2\}$ . This concludes the proof.  $\square$

## 4.2 Recursive splitting

We first restate a result by Burago *et al* [?, Theorem 3] that allows one to split surfaces satisfying

$$\int_M K^+ < 2\pi - 4\varepsilon, \quad \text{and} \quad \int_M K^- < C - 4\varepsilon, \quad (30)$$

for positive  $C$  and  $\varepsilon$ , into four sectors delimited by geodesic curves, all of them satisfying the condition (21a). This result is stated in [?] with  $C = 2\pi$ , but the proof is valid in the general setting.

**Theorem 20** (Burago *et al.*). *Let  $M$  be a complete Riemannian 2-manifold homeomorphic to the plane and satisfying the conditions (30), for positive  $C$  and  $\varepsilon$ . Then, there exist four sectors  $\{Q_i\}_{1 \leq i \leq 4}$  with exterior angles  $\{\psi_i\}_{1 \leq i \leq 4}$  and delimited by geodesic curves such that  $\text{int}(Q_i) \cap \text{int}(Q_j) = \emptyset$  for all  $i \neq j$  and  $\cup_{1 \leq i \leq 4} Q_i = M$ , and such that, for all  $i \in \{1, \dots, 4\}$ , the sector  $Q_i$  satisfies the conditions*

$$\int_{Q_i} K^+ \leq \pi - \psi_i - \varepsilon, \quad \int_{Q_i} K^- \leq \frac{C}{2\pi} \psi_i - \varepsilon. \quad (31)$$

The four sectors obtained by this theorem are sketched in Figure 11. We also need the following lemma to bound from below the exterior angles of the broken half-surfaces resulting from splitting:

**Lemma 21** (Bound on exterior angles). *Let  $N \geq 1$  and let  $B_c$  be a  $N$ -half-surface satisfying the condition (21a). Let  $\sigma : \mathbb{R}^+ \rightarrow B_c$  be a geodesic curve with  $\sigma(0) \in \partial B_c$  and suppose that  $\sigma$  splits  $B_c$  into a  $N_1$ -half-surface  $B_c^1$  and a  $N_2$ -half-surface  $B_c^2$ , both of which satisfy the condition (21a). Then, denoting  $\{\psi_k^0\}_{1 \leq k \leq N}$ ,  $\{\psi_k^1\}_{1 \leq k \leq N_1}$ , and  $\{\psi_k^2\}_{1 \leq k \leq N_2}$ , the positive exterior angles of  $B_c$ ,  $B_c^1$ , and  $B_c^2$ , respectively, we have*

$$\forall i \in \{1, 2\}, \quad |\psi^i|_{l^\infty} \geq |\psi^0|_{l^\infty}. \quad (32)$$

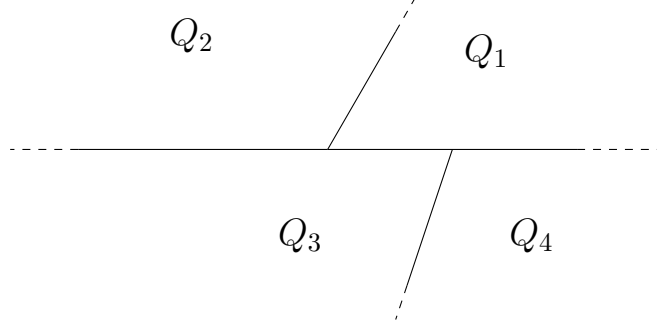


Figure 11: Illustration of the splitting of Theorem 20

*Proof.* We denote  $\psi_k^1$ , with  $k \in \{1, \dots, N_1\}$ , and  $\psi_l^2$ , with  $l \in \{1, \dots, N_2\}$ , the exterior angle of  $B_c^1$  and  $B_c^2$  respectively at the intersection of  $\sigma$  with  $\partial B_c$ . Two cases can occur: either the intersection point is not located at some vertex of  $\partial B_c$ , or  $\sigma$  intersects a vertex  $p_n \in \partial B_c$ , with  $n \in \{1, \dots, N\}$ , with exterior angle  $\psi_n^0$  (see Remark 19 and Figure 9). In the first case, we have  $\psi_k^1 + \psi_l^2 = \pi$ . In the second case, since  $\psi_n^0 \geq 0$ , we have  $\psi_k^1 + \psi_l^2 = \pi + \psi_n^0 \geq \pi$ . In both cases, we infer that

$$\psi_k^1 + \psi_l^2 \geq \pi. \quad (33)$$

Note that all the angles of both  $B_c^1$  and  $B_c^2$  are angles of  $B_c$ , except for the angle newly created by the intersection of  $\sigma$  with  $\partial B_c$ . Let  $\psi_m^0$ , with  $m \in \{1, \dots, N\}$ , be an exterior angle of  $B_c$  such that  $\psi_m^0 = |\psi^0|_{l^\infty}$ . Let  $p_m$  be the corresponding vertex of  $\partial B_c$ . We only prove that (32) is satisfied for  $i = 1$ , since the case  $i = 2$  is treated similarly. We remark that three cases can occur:

1. If  $p_m$  is contained in  $B_c^1$ , the result is straightforward;
2. If  $p_m$  is contained in  $B_c^2$ , applying (33) and the condition (21a) in  $B_c^1$ , we obtain that

$$\psi_k^1 \geq \pi - \psi_l^2 \geq |\psi^0|_{l^1} - \psi_l^2 + \int_{B_c^2} K^+ \geq \psi_m^0.$$

3. Finally, if  $\sigma$  intersects  $\partial B_c$  at  $p_m$ , we have  $\psi_k^1 + \psi_l^2 = \pi + \psi_m^0$ . Since  $\psi_l^2 \leq \pi$ , we infer that  $\psi_k^1 \geq \psi_m^0$ .

In all the cases, we obtain the expected result. This concludes the proof.  $\square$

We can now prove the main result of this section.

**Theorem 22** (Surface splitting into broken half-surfaces). *Let  $M$  be a smooth, complete, simply connected surface. Suppose that  $M$  satisfies the curvature bound (1), i.e.,*

$$\int_M K^+ < 2\pi \quad \text{and} \quad \int_M K^- < \infty,$$

with  $K$  the Gaussian curvature of  $M$ ,  $K^+ = \max(K, 0)$  and  $K^- = \max(-K, 0)$ . We set  $n_{\max} := \log_2 \left( \frac{1}{\pi} \int_M K^- + 1 \right) + 2$ . Then, there exist  $\mathcal{N}_{\text{pol}} \leq \frac{4}{\pi} \int_M K^- + 8$  geodesic  $N_\alpha$ -half-surfaces  $\{B_c^\alpha\}_{1 \leq \alpha \leq \mathcal{N}_{\text{pol}}}$ , with  $N_\alpha \leq n_{\max}$  for all  $\alpha \in \{1, \dots, \mathcal{N}_{\text{pol}}\}$ , satisfying the conditions (21) and such that  $\text{int}(B_c^\alpha) \cap \text{int}(B_c^\beta) = \emptyset$  for all  $\alpha \neq \beta$  and  $M = \bigcup_{\alpha=1}^{\mathcal{N}_{\text{pol}}} B_c^\alpha$ .

*Proof.* Let  $\bar{\varepsilon} = \frac{1}{5}(2\pi - \int_M K^+)$  and  $\bar{C} = \int_M K^- + 5\bar{\varepsilon}$ . Then, the hypotheses of Theorem 20 are satisfied by  $M$  with  $\varepsilon = \bar{\varepsilon}$  and  $C = \bar{C}$ . We denote  $\{S_{\alpha,0}\}_{1 \leq \alpha \leq 4}$  the four sectors satisfying (31) obtained by this theorem. As condition (21a) is satisfied by each 1-half-surface  $S_{\alpha,0}$  the proof consists in applying multiple times Theorem 18 to all of them so as to split these 1-half-surfaces recursively until condition (21b) on the total negative curvature is satisfied. Since each broken half-surface is treated similarly, we only enumerate one broken half-surface in each sector  $S_{\alpha,0}$  at each step of the subdivision. (Note that, otherwise, multi-indices should have been introduced.) See Figure 12 for an illustration of the resulting splitting. We denote, for all  $\alpha \in \{1, \dots, 4\}$ ,  $\psi_\alpha > 0$  the exterior angle of  $S_{\alpha,0}$  and we set  $C_\alpha = \frac{\bar{C}\psi_\alpha}{2\pi}$ . Since  $S_{\alpha,0}$  satisfies (31), the hypotheses of Theorem 18 are satisfied with  $B_c = S_{\alpha,0}$ ,  $\varepsilon = \frac{\bar{\varepsilon}}{3}$  and  $C = C_\alpha$ . Hence, there exists a splitting of  $S_{\alpha,0}$  into two broken half-surfaces satisfying the conditions (25). By symmetry, we consider only one of them, denoted  $S_{\alpha,1}$ . In the same manner, we apply recursively Theorem 18 with  $B_c = S_{\alpha,n-1}$ ,  $\varepsilon = \frac{\bar{\varepsilon}}{3 \cdot 2^{n-1}}$  and  $C = \frac{C_\alpha}{2^{n-1}}$ . This yields a splitting of  $S_{\alpha,n-1}$  into two broken half-surfaces satisfying the conditions (25). By symmetry, we consider only one of them, the  $N$ -half-surface denoted  $S_{\alpha,n}$ , whose exterior angles are denoted  $\{\psi_k^n\}_{1 \leq k \leq N}$ . Note that  $N \leq n+1$  by (24). Then, Lemma 21 ensures that  $|\psi^n|_{l^\infty} \geq \psi_\alpha$  (with  $|\psi^n|_{l^\infty} = \max_{1 \leq k \leq N} \psi_k^n$ ). Therefore, condition (21b) is satisfied by  $S_{\alpha,n}$  whenever

$$\int_{S_{\alpha,n}} K^- < |\psi^n|_{l^\infty} \leq \psi_\alpha. \quad (34)$$

Since  $\int_{S_{\alpha,n}} K^- \leq \frac{C_\alpha}{2^n}$ , we infer that (34) is satisfied whenever  $n = n_\alpha^{\max}$ , where

$$n_\alpha^{\max} = \left\lceil \log_2 \left( \frac{C_\alpha}{\psi_\alpha} \right) \right\rceil = \left\lceil \log_2 \left( \frac{\bar{C}}{2\pi} \right) \right\rceil$$

and  $\lceil \cdot \rceil$  is the ceiling function. Therefore, we have proved that there exist  $\mathcal{N}_\alpha$   $N_i$ -half-surfaces  $\{B_c^{\alpha,i}\}_{1 \leq i \leq \mathcal{N}_\alpha}$ , with  $N_i \leq n_\alpha^{\max} + 1$ , such that  $\text{int}(B_c^{\alpha,i}) \cap \text{int}(B_c^{\alpha,j}) = \emptyset$  for all  $i \neq j$ . Moreover, we have  $\mathcal{N}_\alpha \leq \mathcal{N}_\alpha^{\max}$ , with:

$$\mathcal{N}_\alpha^{\max} = 2^{n_\alpha^{\max}} \leq \frac{\bar{C}}{\pi} \leq \frac{1}{\pi} \int_M K^- + 2. \quad (35)$$

We finally define  $\mathcal{N}_{\text{pol}} = \sum_{\alpha=1}^4 \mathcal{N}_\alpha$  and the set of  $N_i$ -half-surfaces  $\{B_c^i\}_{1 \leq i \leq \mathcal{N}_{\text{pol}}}$  as the union of the sets  $\{B_c^{\alpha,i}\}_{1 \leq i \leq \mathcal{N}_\alpha}$ , for  $\alpha \in \{1, \dots, 4\}$ . For all  $i \in \{1, \dots, \mathcal{N}_{\text{pol}}\}$ ,

the number  $N_i$  of vertices of  $B_c^i$  satisfies

$$\begin{aligned} N_i &\leq \max_{\alpha \in \{1, \dots, 4\}} n_\alpha^{\max} + 1 = \left\lceil \log_2 \left( \frac{\bar{C}}{2\pi} \right) \right\rceil + 1 \\ &\leq \log_2 \left( \frac{1}{2\pi} \int_M K^- + 1 \right) + 2 = n_{\max}. \end{aligned}$$

Moreover, using (35), we obtain that the number  $\mathcal{N}_{\text{pol}}$  of polygons satisfies

$$\mathcal{N}_{\text{pol}} = \sum_{\alpha=1}^4 \mathcal{N}_\alpha \leq \frac{4}{\pi} \int_M K^- + 8.$$

The claim follows.  $\square$

*Remark 23* (Tree representation). The construction can be seen as a binary tree of broken half-surfaces, each splitting being an edge,  $n_{\max}$  being the maximal depth of the tree, and  $\mathcal{N}_{\text{pol}}$  being the maximal number of leaves of the tree. Once the splitting is achieved, we renumber the broken half-surfaces to obtain the set  $\{B_c^\alpha\}_{1 \leq \alpha \leq \mathcal{N}_{\text{pol}}}$ .

**Definition 24** (Skeleton). *The graph in the surface  $M$  defined by the vertices of the boundaries of the broken half-surfaces  $\{B_c^\alpha\}_{1 \leq \alpha \leq \mathcal{N}_{\text{pol}}}$  obtained using Theorem 22 and the edges (geodesic curves) joining the vertices is called the skeleton. The vertices and the edges in the skeleton are respectively enumerated as  $\{p_c^i\}_{1 \leq i \leq \mathcal{N}_{\text{ver}}}$  and  $\{\gamma_c^i\}_{1 \leq i \leq \mathcal{N}_{\text{ed}}}$ .*

An example of skeleton is then presented in Figure 13.

## 5 Proof of the main theorem

We prove in this section Theorem 1 on the existence of piecewise smooth Chebyshev nets with singularities on surfaces  $M$  satisfying the curvature bound (1). We first gather the results of Theorem 17 and Theorem 22 to construct a Chebyshev net with singularities on  $M$  (Theorem 25). Then, we show that the Chebyshev parametrization obtained on each broken half-surface by Theorem 17 is piecewise smooth (Theorem 30). The proof of Theorem 1 then follows from Theorem 25 and Theorem 30.

### 5.1 Existence of a Chebyshev net with conical singularities

**Theorem 25** (Existence of Chebyshev nets with singularities). *Let  $M$  be a smooth, complete, simply connected surface satisfying the curvature bound (1). Then, there exists a Chebyshev net with conical singularities  $\mathcal{C} = (\{B_e^i\}_{1 \leq i \leq \mathcal{N}_{\text{pol}}}, \{\varphi_e^i\}_{1 \leq i \leq \mathcal{N}_{\text{pol}}}, T)$ , with  $\mathcal{N}_{\text{pol}} \leq \frac{4}{\pi} \int_M K^- + 8$ , on  $M$ .*

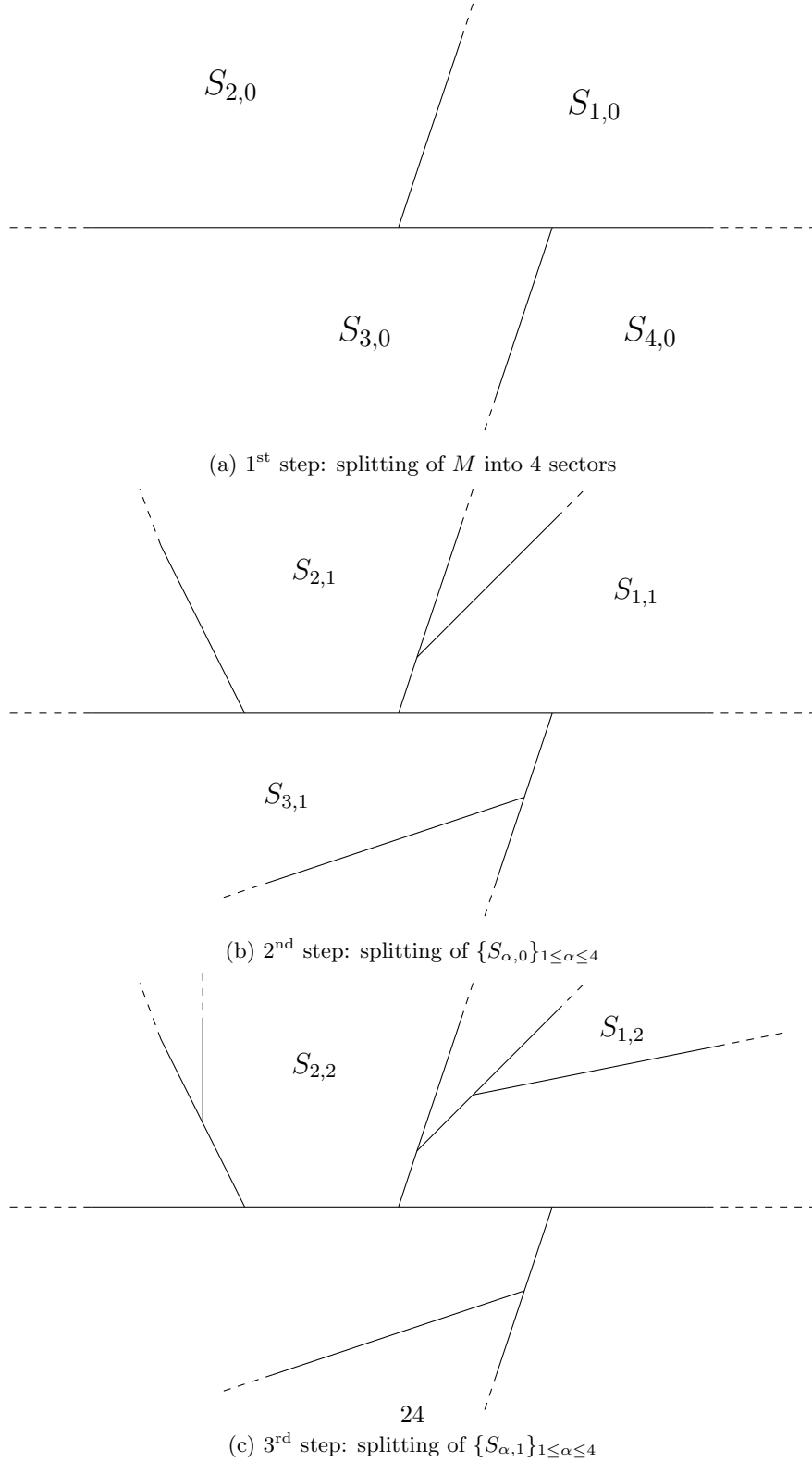


Figure 12: Illustration of the recursive splitting used for the proof of Theorem 22



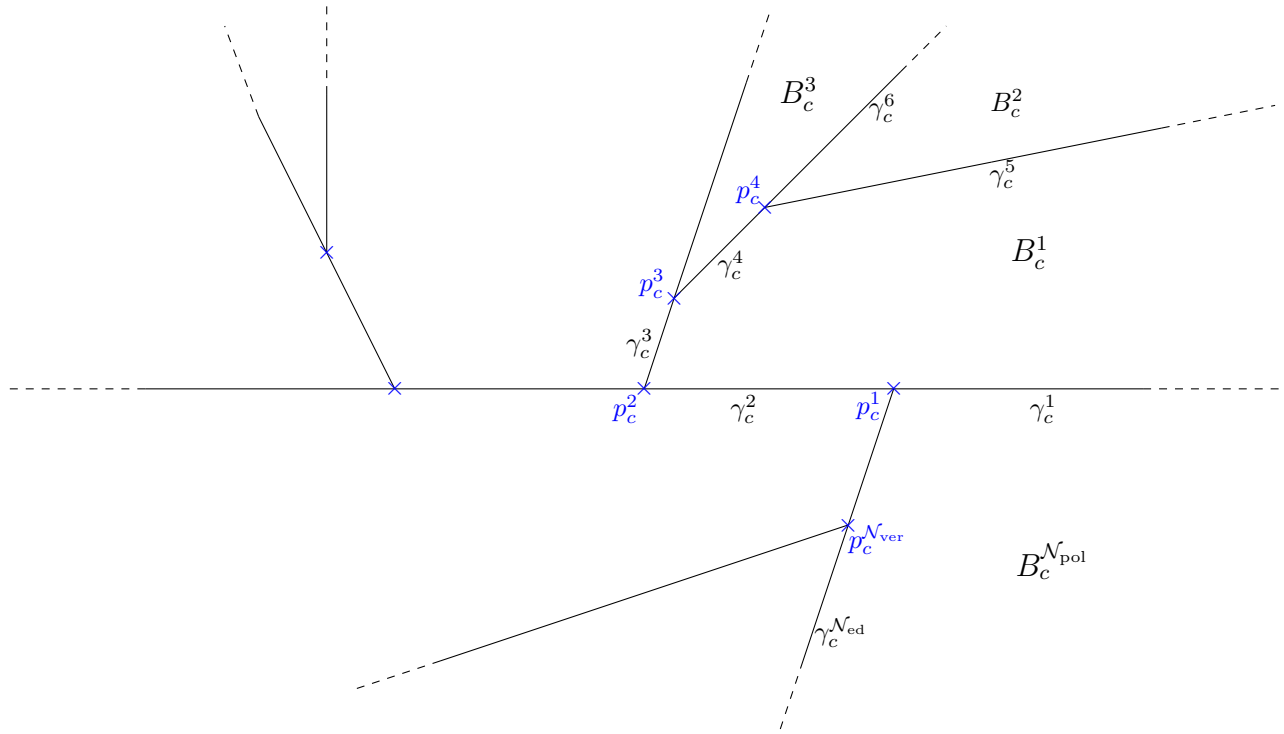


Figure 13: An example of skeleton

*Proof.* First, we apply Theorem 22 to obtain a splitting of  $M$  into broken half-surfaces  $\{B_c^i\}_{1 \leq i \leq \mathcal{N}_{\text{pol}}}$ , with  $\mathcal{N}_{\text{pol}} \leq \frac{4}{\pi} \int_M K^- + 8$ , all of them satisfying the conditions

$$\int_{B_c^i} K^+ < \pi - |\psi^i|_{l^1}, \quad \int_{B_c^i} K^- < |\psi^i|_{l^\infty}.$$

Then, owing to Theorem 17, we infer that there exists a Chebyshev parametrization  $\varphi_i : (\mathbb{R}^+)^2 \rightarrow B_c^i$ , for all  $i \in \{1, \dots, \mathcal{N}_{\text{pol}}\}$ . We set  $B_e^i = (\mathbb{R}^+)^2$ , for all  $i \in \{1, \dots, \mathcal{N}_{\text{pol}}\}$ . We construct the equivalence table  $T : \{1, \dots, \mathcal{N}_{\text{pol}}\} \rightarrow \mathbb{N}$  as follows. For all  $i \in \{1, \dots, \mathcal{N}_{\text{pol}}\}$ , we set  $T(i, i) = 0$ . For all  $i, j \in \{1, \dots, \mathcal{N}_{\text{pol}}\}$  such that  $i \neq j$ , we set  $T(i, j) = T(j, i) = 0$  if  $B_c^i \cap B_c^j = \emptyset$ . We now suppose that  $B_c^i \cap B_c^j \neq \emptyset$ . Let  $\{\gamma_c^{i, \alpha}\}_{1 \leq \alpha \leq N_i}$  and  $\{\gamma_c^{j, \beta}\}_{1 \leq \beta \leq N_j}$  be the edges of the skeleton that are included in  $B_c^i$  and  $B_c^j$  respectively. Then, by construction (see Theorem 18),  $B_c^i \cap B_c^j = \gamma_c^{i, \alpha_0} = \gamma_c^{j, \beta_0}$  for some  $\alpha_0 \in \{1, \dots, N_i\}$  and  $\beta_0 \in \{1, \dots, N_j\}$ , and, we set  $T(i, j) = \alpha_0$  and  $T(j, i) = \beta_0$  (see Figure 14). We conclude that  $\mathcal{C} = (\{B_e^i\}_{1 \leq i \leq \mathcal{N}_{\text{pol}}}, \{\varphi_e^i\}_{1 \leq i \leq \mathcal{N}_{\text{pol}}}, T)$  is a Chebyshev net with conical singularities on  $M$ .  $\square$

## 5.2 Piecewise smooth Chebyshev nets on broken half-surfaces

Let us notice that whenever all the broken half-surfaces  $\{B_c^i\}_{1 \leq i \leq \mathcal{N}_{\text{pol}}}$  obtained in Theorem 22 are smooth sectors (only one vertex), Theorem 1 follows from Propositions 11 and 25. In the general case, we need to prove that, for all  $i \in \{1, \dots, \mathcal{N}_{\text{pol}}\}$ , the Chebyshev parametrization  $\varphi_i : B_e^i = (\mathbb{R}^+)^2 \rightarrow B_c^i$  obtained from Theorem 17 on the geodesic broken half-surface  $B_c^i$  satisfying the conditions

$$\int_{B_c^i} K^+ < \pi - |\psi^i|_{l^1}, \quad \int_{B_c^i} K^- < |\psi^i|_{l^\infty}, \quad (36)$$

is piecewise smooth. With this purpose in mind, we proceed as in Section 3: we first consider the case of a piecewise smooth sector  $Q$  of exterior angle  $\psi \in (0, \pi)$  satisfying the conditions

$$\tau_+(\eta_1) + \tau_+(\eta_2) + \int_Q K^+ < \pi - \psi, \quad (37a)$$

$$\tau_-(\eta_1) + \tau_-(\eta_2) + \int_Q K^- < \psi. \quad (37b)$$

Before this, we state the following lemma on the geodesic curvature of parameter curves of a smooth Chebyshev net. The notations used in this lemma are depicted in Figure 15.

**Lemma 26.** *Let  $a, b \in \mathbb{R}_*^+$  and let  $\varphi : [0, a] \times [0, b] \rightarrow M$  be a smooth Chebyshev net on  $M$  such that  $0 < \omega(u, v) < \pi$  for all  $u, v \in [0, a] \times [0, b]$ . We define  $\Omega = \varphi([0, a] \times [0, b])$ ,  $\eta_1(v) = \varphi(0, -v)$ ,  $\sigma_1(v) = \varphi(a, -v)$  for all  $v \in [-b, 0]$ , and*

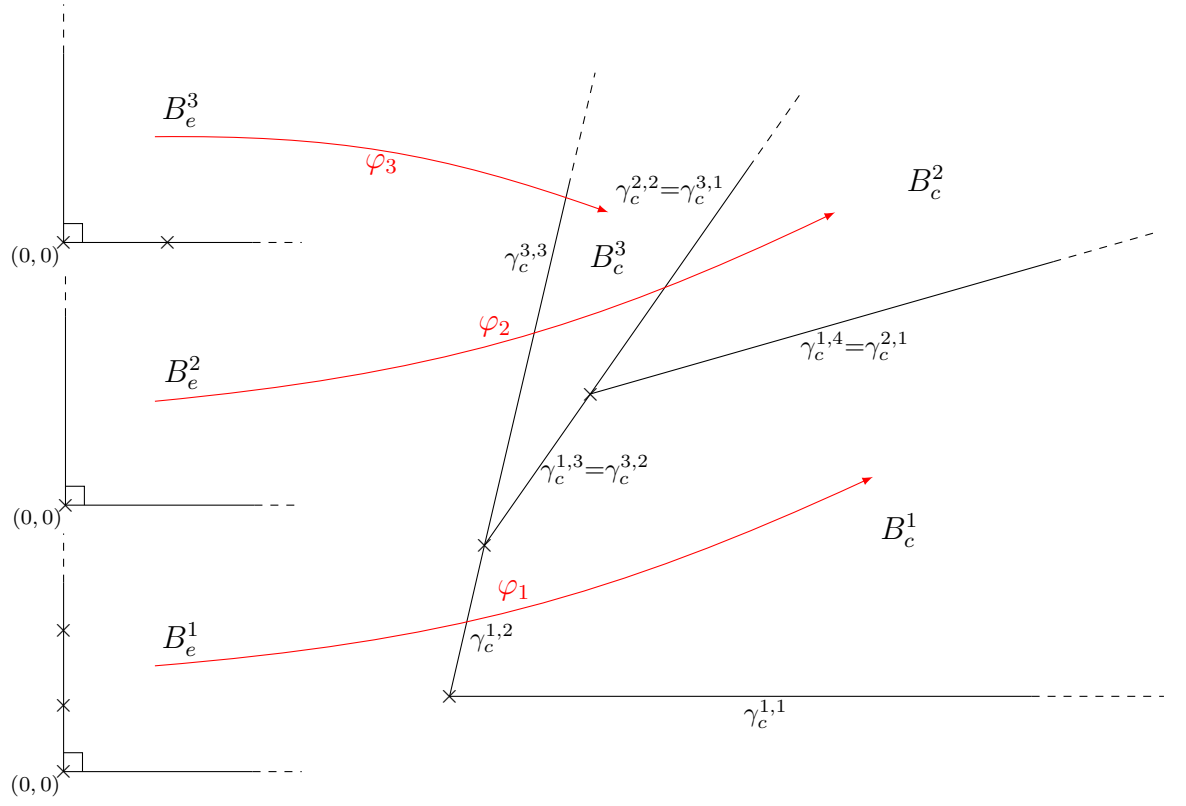


Figure 14: Illustration of the construction of the Chebyshev net with conical singularities (the crosses are the vertices)

$\eta_2(u) = \varphi(u, 0)$ ,  $\sigma_2(u) = \varphi(u, b)$  for all  $u \in [0, a]$ . Then, the geodesic curvatures  $\kappa_{\eta_1}$ ,  $\kappa_{\eta_2}$ ,  $\kappa_{\sigma_1}$ , and  $\kappa_{\sigma_2}$  of  $\eta_1$ ,  $\eta_2$ ,  $\sigma_1$ , and  $\sigma_2$ , respectively, are related by

$$\int_0^a \kappa_{\sigma_2} = \int_0^a \kappa_{\eta_2} + \int_{\Omega} K, \quad \int_{-b}^0 \kappa_{\sigma_1} = \int_{-b}^0 \kappa_{\eta_1} + \int_{\Omega} K, \quad (38)$$

and satisfy

$$\int_0^a \kappa_{\eta_2}^+ - \int_0^a \kappa_{\sigma_2}^+ + \int_{\Omega} K^+ \geq 0, \quad \int_{-b}^0 \kappa_{\eta_1}^+ - \int_{-b}^0 \kappa_{\sigma_1}^+ + \int_{\Omega} K^+ \geq 0, \quad (39a)$$

$$\int_0^a \kappa_{\eta_2}^- - \int_0^a \kappa_{\sigma_2}^- + \int_{\Omega} K^- \geq 0, \quad \int_{-b}^0 \kappa_{\eta_1}^- - \int_{-b}^0 \kappa_{\sigma_1}^- + \int_{\Omega} K^- \geq 0. \quad (39b)$$

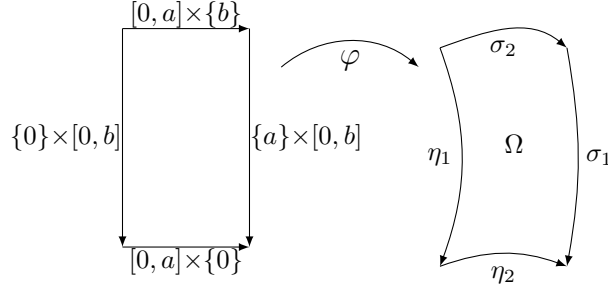


Figure 15: Illustration of the notation of Lemma 26

*Proof.* We only prove the lemma for the curves  $\eta_1$  and  $\sigma_1$ . The formulas for  $\eta_2$  and  $\sigma_2$  are obtained in a similar way. Using (13) and the Hazzidakis formula (14) with  $u = a$  and  $v = b$ , we obtain

$$\omega(a, b) = \omega(0, 0) - \int_0^a \kappa_{\eta_2} - \int_{-b}^0 \kappa_{\eta_1} - \int_{\Omega} K = \omega(a, 0) - \int_{-b}^0 \kappa_{\eta_1} - \int_{\Omega} K,$$

so that

$$\int_{-b}^0 \kappa_{\sigma_1} = - \int_0^b \partial_v \omega(a, v) dv = \int_{-b}^0 \kappa_{\eta_1} + \int_{\Omega} K.$$

To prove the inequalities (39), we first note that (38) implies

$$\int_0^a \kappa_{\eta_1}^+ - \int_0^a \kappa_{\sigma_1}^+ + \int_{\Omega} K^+ = \int_0^a \kappa_{\eta_1}^- - \int_0^a \kappa_{\sigma_1}^- + \int_{\Omega} K^-.$$

Subdividing the curves  $\eta_1$  and  $\sigma_1$  according to the sign changes of  $\kappa_{\eta_1}$  and  $\kappa_{\sigma_1}$ , it is possible to assume that the sign of  $\kappa_{\eta_1}$  and  $\kappa_{\sigma_1}$  is constant on  $[0, a]$ . The discussion is then simplified to the two following cases:

- if  $\kappa_{\eta_1}$  and  $\kappa_{\sigma_1}$  have the same sign (say, nonnegative), then

$$\int_0^a \kappa_{\eta_1}^- - \int_0^a \kappa_{\sigma_1}^- + \int_{\Omega} K^- = \int_{\Omega} K^- \geq 0;$$

- if  $\kappa_{\eta_1}$  and  $\kappa_{\sigma_1}$  have different signs (say,  $\kappa_{\eta_1} \geq 0$  and  $\kappa_{\sigma_1} \leq 0$ ), then

$$\int_0^a \kappa_{\eta_1}^+ - \int_0^a \kappa_{\sigma_1}^+ + \int_{\Omega} K^+ = \int_0^a \kappa_{\eta_1}^+ + \int_{\Omega} K^+ \geq 0.$$

□

**Theorem 27** (Existence of piecewise smooth Chebyshev nets on sectors). *Let  $Q$  be a sector delimited by the two piecewise smooth curves  $\eta_1 : \mathbb{R}^- \rightarrow M$  and  $\eta_2 : \mathbb{R}^+ \rightarrow M$ . We denote  $\pi - \theta_1 \in (0, \pi)$  the exterior angle of this sector and we suppose that  $Q$  satisfies the conditions (37). Then, there exist  $\mathcal{N}_{\text{piece}} \geq 1$  polygons  $\{B_e^i\}_{1 \leq i \leq \mathcal{N}_{\text{piece}}}$  such that  $(\mathbb{R}^+)^2 = \cup_{i=1}^{\mathcal{N}_{\text{piece}}} B_e^i$  and  $\text{int}(B_e^i) \cap \text{int}(B_e^j) = \emptyset$  for all  $i \neq j$ , and Chebyshev coordinates  $\varphi$  on  $Q$  such that  $\eta_1$  and  $\eta_2$  are coordinate curves. Moreover, the angle  $\omega = \angle(\partial_u \varphi, \partial_v \varphi)$  of the net satisfies the nonsmooth Hazzidakis formula*

$$\omega(u, v) = \theta_1 - \tau(\eta_2|_{[0, u]}) - \tau(\eta_1|_{[-v, 0]}) - \int_{\varphi([0, u] \times [0, v])} K, \quad (40)$$

for all  $u, v \in \mathbb{R}^+$ , and is bounded away from 0 and  $\pi$  by the positive real number

$$\min \left( \pi - \psi - \int_Q K^+ - \tau_+(\eta_1) - \tau_+(\eta_2), \psi - \int_Q K^- - \tau_-(\eta_1) - \tau_-(\eta_2) \right). \quad (41)$$

Finally,  $\varphi|_{B_e^i} : B_e^i \rightarrow \varphi(B_e^i) \subset Q$  is a diffeomorphism, for all  $1 \leq i \leq \mathcal{N}_{\text{piece}}$ .

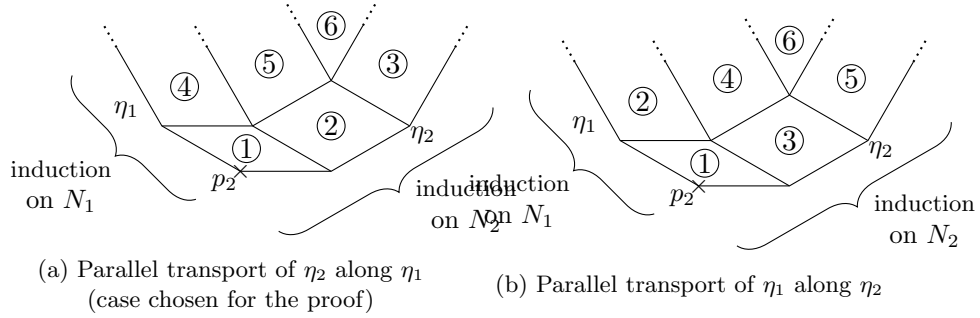


Figure 16: Parallel transport of  $\eta_1$  and  $\eta_2$  along each other ( $N_1 = 1$ ,  $N_2 = 2$ ):  
numbering of the double induction process

*Proof.* We first split the two curves  $\eta_1$  and  $\eta_2$  into smooth pieces. We denote  $N_1 + 1 \geq 1$  and  $N_2 + 1 \geq 1$  the number of smooth pieces of the curves  $\eta_1$  and  $\eta_2$ , respectively. Then, we parallel transport the curve  $\eta_2$  along each smooth piece of  $\eta_1$ . This is done recursively on  $N_1$ . The parallel transport of  $\eta_2$  along a piece of  $\eta_1$  is obtained by induction on  $N_2$ . Hence, we have two nested induction arguments (see Figure 16). We observe that, by symmetry, the role of the two curves can be switched, as can be seen in the same figure. Hence, we can always suppose that  $N_1 \geq N_2$ . Once the construction is over, we prove the nonsmooth Hazzidakis formula (40).

**Step 1** (*Formulation of the first induction process (on  $N_1 \geq 0$ )*). We suppose that  $N_2 \in \{0, \dots, N_1\}$  is a given fixed integer and we denote  $(\mathcal{H}_{N_1+1})$  the following induction hypothesis:

for any sector  $Q$  of exterior angle  $\pi - \theta_1 \in (0, \pi)$ , delimited by the two curves  $\eta_1$  and  $\eta_2$  having respectively  $N_1 + 1$  and  $N_2 + 1$  smooth pieces and satisfying (37), there exist polygons  $\{B_e^i\}_{1 \leq i \leq \mathcal{N}_{\text{piece}}}$  and a Chebyshev net  $\varphi : (\mathbb{R}^+)^2 \rightarrow Q$  such that:

- $(\mathbb{R}^+)^2 = \cup_{i=1}^{\mathcal{N}_{\text{piece}}} B_e^i$  and  $\text{int}(B_e^i) \cap \text{int}(B_e^j) = \emptyset$  for all  $i \neq j$ ;
- $\eta_1$  and  $\eta_2$  are coordinate curves;
- the angle  $\omega = \angle(\partial_u \varphi, \partial_v \varphi)$  of  $\varphi$  satisfies the nonsmooth Hazzidakis formula (40);
- $\varphi|_{B_e^i} : B_e^i \rightarrow \varphi(B_e^i) \subset Q$  is a diffeomorphism, for all  $i \in \{1, \dots, \mathcal{N}_{\text{piece}}\}$ .

**Step 2** (*Proof of the first induction process (1<sup>st</sup> part of the construction)*). We firstly check that  $(\mathcal{H}_1)$  holds. Since  $N_2 \leq N_1$ , we have  $N_2 = 0$ . Hence, the sector  $Q$  is delimited by the two smooth curves  $\eta_1$  and  $\eta_2$  and  $(\mathcal{H}_1)$  holds, with  $\mathcal{N}_{\text{piece}} = 1$ , by Proposition 11. Now, for  $N_1 \geq 1$ , we suppose that  $(\mathcal{H}_{N_1})$  holds and we prove that  $(\mathcal{H}_{N_1+1})$  also holds. Thus, we suppose that  $Q$  is delimited by two curves  $\eta_1$  and  $\eta_2$  having respectively  $N_1 + 1 \geq 2$  and  $N_2 + 1 \geq 1$  smooth pieces. Let

$$-\infty = a_{1,N_1+1} < \dots < a_{1,0} = 0 = a_{2,0} < \dots < a_{2,N_2+1} = \infty$$

be such that, for all  $l = 1, 2$ ,  $\eta_l$  restricted to  $[a_{l,i-1}, a_{l,i}]$  is a smooth curve, for all  $i \in \{1, \dots, N_l + 1\}$ . We denote  $\eta_{l,i} : [a_{l,i-1}, a_{l,i}] \rightarrow M$  this piece of the curve  $\eta_l$  and  $\kappa_{l,i} : [a_{l,i-1}, a_{l,i}] \rightarrow \mathbb{R}$  its geodesic curvature. We denote  $\psi_{l,i} = (-1)^l \angle(\eta'_{l,i}(a_{l,i}), \eta'_{l,i+1}(a_{l,i}))$  for all  $i \in \{1, \dots, N_l\}$  and  $l \in \{1, 2\}$  (see Figure 17). To abbreviate the notation, we set  $\tilde{\eta}_{1,1} = \eta_{1,1}$ .

**Step 3** (*Formulation of the second induction process (on  $n \in \{1, \dots, N_2 + 1\}$ )*). We parallel transport in what follows the curve  $\eta_2$  along  $\eta_{1,1}$ . See Figure 17 for the notation and Figure 18 for an illustration of the construction. For all  $n \in \{1, \dots, N_2 + 1\}$ , we denote  $(\tilde{\mathcal{H}}_n)$  the following induction hypothesis:

for all  $j \in \{1, \dots, n\}$ , let  $\tilde{\eta}_{1,j} : [a_{1,1}, 0] \rightarrow Q$  be a smooth curve intersecting  $\eta_{2,j}$  at  $\eta_{2,j}(a_{2,j-1}) = \tilde{\eta}_{1,j}(0)$ . If  $n > 1$ , for all  $j \in \{1, \dots, n-1\}$ , let  $B_e^j =$

$[a_{2,j-1}, a_{2,j}] \times [0, -a_{1,1}] \subset (\mathbb{R}^+)^2$  and assume that  $\varphi_j : B_e^j \rightarrow B_c^j \subset Q$ , with  $B_c^j = \varphi_j(B_e^j)$ , are Chebyshev nets such that

$$\begin{aligned} \varphi_j(a_{2,j-1}, v) &= \tilde{\eta}_{1,j}(-v), & \varphi_j(a_{2,j}, v) &= \tilde{\eta}_{1,j+1}(-v), & \forall v \in [0, -a_{1,1}], \\ \varphi_j(u, 0) &= \eta_{2,j}(u), & & & \forall u \in [a_{2,j-1}, a_{2,j}]. \end{aligned}$$

Then, there exists a Chebyshev net  $\varphi_n : B_e^n \subset (\mathbb{R}^+)^2 \rightarrow B_c^n \subset Q$ , with  $B_e^n = [a_{2,n-1}, a_{2,n}] \times [0, -a_{1,1}]$  and  $B_c^n = \varphi_n(B_e^n)$ . Moreover, the set  $B_c^n$  satisfies:

1. if  $n > 1$ , then  $B_c^{n-1} \cap B_c^n = \tilde{\eta}_{1,n}$ ;
2. if  $n > 2$ , then  $B_c^j \cap B_c^n = \emptyset$ , for all  $j \in \{1, \dots, n-2\}$ .

Finally, suppose that  $n < N_2 + 1$  and denote  $\tilde{\eta}_{1,n+1} : [a_{1,1}, 0] \rightarrow Q$  the curve defined by  $\tilde{\eta}_{1,n+1}(v) = \varphi_n(a_{2,n}, -v)$  for all  $v \in [a_{1,1}, 0]$ . This curve intersects  $\eta_{2,n+1}$  at  $\eta_{2,n+1}(a_{2,n}) = \tilde{\eta}_{1,n+1}(0)$ , forming an interior angle denoted  $\theta_{n+1} \in [0, \pi]$ . Then,  $\theta_{n+1}$  and the geodesic curvature  $\tilde{\kappa}_{1,n+1} : [a_{1,1}, 0] \rightarrow \mathbb{R}$  of  $\tilde{\eta}_{1,n+1}$  satisfy respectively

$$\theta_{n+1} = \theta_1 - \tau(\eta_2|_{[0, a_{2,n}]}) \in (0, \pi), \quad (42a)$$

$$\int_{-v}^0 \tilde{\kappa}_{1,n+1} = \int_{-v}^0 \kappa_{1,1} + \sum_{k=1}^n \int_{\varphi_k([a_{2,k-1}, a_{2,k}] \times [0, v])} K, \quad \forall v \in [0, -a_{1,1}]. \quad (42b)$$

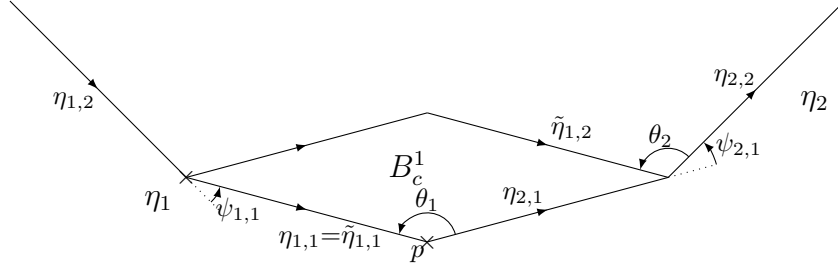


Figure 17: Illustration of the parallel transport of  $\eta_{2,1}$  along  $\eta_{1,1}$  ( $N_1 = 1$ ,  $N_2 = 1$ )

**Step 4 (Proof of the second induction process).** To prove that  $(\tilde{\mathcal{H}}_1)$  holds, we apply Theorem 12 with the curves  $\eta_{1,1}$  and  $\eta_{2,1}$  of respective length  $-a_{1,1}$  and  $a_{2,1}$  and forming an interior angle  $\theta_1 \in (0, \pi)$  by hypothesis. We obtain a mapping  $\varphi_1 : B_e^1 \subset (\mathbb{R}^+)^2 \rightarrow B_c^1 \subset M$ , with  $B_e^1 = [a_{2,0}, a_{2,1}] \times [0, -a_{1,1}]$  and  $B_c^1 = \varphi_1(B_e^1)$ . Then, using the same argument as the proof of Lemma 10, we obtain that the mapping  $\varphi_1$  is a Chebyshev net and that  $B_c^1 \subset Q$ . Using Lemma 26, we infer that  $\tilde{\eta}_{1,2}$  has a geodesic curvature  $\tilde{\kappa}_{1,2}$  satisfying

$$\int_{-v}^0 \tilde{\kappa}_{1,1} = \int_{-v}^0 \kappa_{1,1} + \int_{\varphi_1([0, a_{2,1}] \times [0, v])} K,$$

for all  $v \in [0, -a_{1,1}]$ . Moreover, we deduce from (13) that the interior angle  $\theta_2 = \angle(\eta'_{2,2}(0), -\tilde{\eta}'_{1,2}(0))$  satisfies

$$\theta_2 = \theta_1 - \int_0^{a_{2,1}} \kappa_{2,1} - \psi_{2,2} = \theta_1 - \tau(\eta_2|_{[0, a_{2,1}]}).$$

Since hypothesis (37) (with  $\psi = \pi - \theta_1$ ) is satisfied by  $Q$ , we obtain that  $\theta_2 \in (0, \pi)$ . Hence,  $(\tilde{\mathcal{H}}_1)$  holds. We now suppose that  $(\tilde{\mathcal{H}}_{n-1})$  holds for  $n \in \{0, \dots, N_2 + 1\}$ . Let  $\{\tilde{\eta}_{1,j}\}_{1 \leq j \leq n}$  and  $\{\varphi_j\}_{1 \leq j \leq n-1}$  be respectively curves and Chebyshev nets as in the hypothesis of  $(\mathcal{H}_n)$ . Since  $\theta_n \in (0, \pi)$ , we apply Theorem 12 to the curves  $\tilde{\eta}_{1,n}$  and  $\eta_{2,n}$  to obtain a mapping  $\varphi_n : B_e^n \subset (\mathbb{R}^+)^2 \rightarrow B_c^n \subset M$ , with  $B_e^n = [a_{2,n-1}, a_{2,n}] \times [0, -a_{1,1}]$  and  $B_c^n = \varphi_n(B_e^n)$ . Using the same argument as in the proof of Lemma 10, we obtain that  $\varphi$  is a Chebyshev net, that  $B_c^n \subset Q$  and that  $B_c^n$  satisfies the statements 1 and 2 of  $(\tilde{\mathcal{H}}_n)$ .

We now suppose that  $n < N_2 + 1$ . Then, from Lemma 26 and from the assertion (42b) of  $(\tilde{\mathcal{H}}_{n-1})$ , we infer that the geodesic curvature  $\tilde{\kappa}_{1,n+1}$  of  $\tilde{\eta}_{1,n+1}$  satisfies

$$\int_{-v}^0 \tilde{\kappa}_{1,n+1} = \int_{-v}^0 \tilde{\kappa}_{1,n} + \int_{\varphi_n([a_{2,n-1}, a_{2,n}] \times [0, v])} K = \int_{-v}^0 \kappa_{1,0} + \sum_{k=1}^n \int_{\varphi_k([a_{2,k-1}, a_{2,k}] \times [0, v])} K,$$

for all  $v \in [0, -a_{1,1}]$ . Finally, from (13) and from the assertion (42a) of  $(\tilde{\mathcal{H}}_{n-1})$ , we infer that the interior angle  $\theta_{n+1} = \angle(\eta'_{2,n+1}(a_{2,n}), -\tilde{\eta}'_{1,n+1}(0))$  satisfy

$$\theta_{n+1} = \theta_n - \int_{a_{2,n-1}}^{a_{2,n}} \kappa_{2,n} - \psi_{2,n} = \theta_1 - \tau(\eta_2|_{[0, a_{2,n}]}).$$

Then, using hypotheses (37), we obtain that  $\theta_{n+1} \in (0, \pi)$ . This concludes the proof of the statement  $(\tilde{\mathcal{H}}_n)$ .

**Step 5** (*Proof of the first induction process (2<sup>nd</sup> part of the construction)*).

By application of  $(\tilde{\mathcal{H}}_n)$ , for all  $n \in \{1, \dots, N_2 + 1\}$ , we obtain the existence of polygons  $\{B_e^i\}_{1 \leq i \leq N_2+1}$  and, for all  $i \in \{1, \dots, N_2 + 1\}$ , of Chebyshev nets  $\varphi_i : B_e^i \rightarrow B_c^i \subset Q$ , with  $B_c^i = \varphi_i(B_e^i)$ . Then, for all  $i \in \{1, \dots, N_2 + 1\}$ , we denote  $\tilde{\eta}_{2,i} : [a_{2,i-1}, a_{2,i}] \rightarrow Q$  the curve defined by  $\tilde{\eta}_{2,i}(u) = \varphi_i(u, -a_{1,1})$ , for all  $u \in [a_{2,i-1}, a_{2,i}]$ . We denote  $\tilde{\eta}_2 : \mathbb{R}^+ \rightarrow Q$  the junction of the curves  $\tilde{\eta}_{2,i}$ , with  $i \in \{1, \dots, N_2 + 1\}$ , defined so that

$$\tilde{\eta}_2(\mathbb{R}^+) = \bigcup_{i=1}^{N_2+1} \tilde{\eta}_{2,i}([a_{2,i-1}, a_{2,i}]).$$

We denote  $B_e^{\text{band}} = \mathbb{R}^+ \times [0, -a_{1,1}]$ ,  $B_c^{\text{band}} = \bigcup_{i=1}^{N_2+1} B_c^i$  and  $\tilde{Q} = Q \setminus B_c^{\text{band}}$ . Let us construct the Chebyshev net on the half-band  $B_e^{\text{band}}$ . This mapping, denoted  $\varphi_{\text{band}} : B_e^{\text{band}} \rightarrow B_c^{\text{band}}$ , is defined by

$$\varphi_{\text{band}}(u, v) = \varphi_i(u, v), \quad \text{whenever } (u, v) \in P_e^i \text{ for } i \in \{1, \dots, N_2 + 1\}. \quad (43)$$



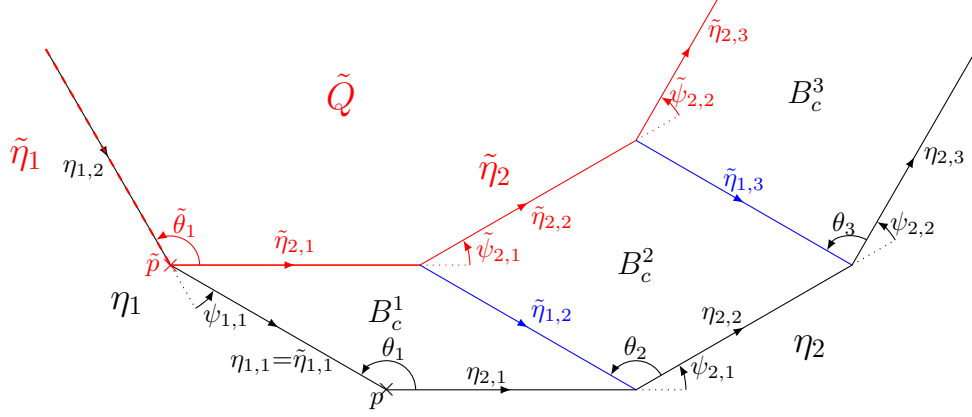


Figure 18: Illustration of the recursive parallel transport of  $\eta_2$  along  $\eta_{1,1}$  ( $N_1 = 1$ ,  $N_2 = 2$ )

Then, using that  $\varphi_{\text{band}}|_{B_c^i}$  is a diffeomorphism for all  $i \in \{1, \dots, N_2 + 1\}$  and using the statements 1 and 2 of  $(\tilde{\mathcal{H}}_n)$ , for all  $n \in \{1, \dots, N_2 + 1\}$ , we obtain that the mapping  $\varphi_{\text{band}}$  is a homeomorphism. We denote  $\tilde{\eta}_1 : \mathbb{R}^- \rightarrow \eta_1[(-\infty, a_{1,1}]]$  the curve defined by  $\tilde{\eta}_1(t) = \eta_1(t + a_{1,1})$ . Then,  $\tilde{Q}$  is a sector delimited by the curves  $\tilde{\eta}_1$  and  $\tilde{\eta}_2$  with respectively  $N_1$  and  $N_2 + 1$  smooth pieces. We now prove that  $\tilde{Q}$  satisfies the hypotheses of  $(\mathcal{H}_{N_1})$  (on the interior angle and on the total curvature). First note that, using (12), we obtain that the interior angle  $\tilde{\theta}_1$  of  $\tilde{Q}$  satisfies

$$\tilde{\theta}_1 = \theta_1 - \int_{a_{1,1}}^0 \kappa_{1,1} - \psi_{1,1} = \theta_1 - \tau(\eta_1|_{[a_{1,1}, 0]}). \quad (44)$$

Then, using the hypotheses (37) on  $Q$ , we obtain that  $\tilde{\theta}_1 \in (0, \pi)$ .

To simplify the notation, we use in what follows the convention that  $\sum_{k=1}^{i-1}(\dots) = 0$  when  $i = 1$ . As parallel transport preserves the angles, we have  $\tilde{\psi}_{2,i} = \psi_{2,i}$ , for all  $i \in \{1, \dots, N_2\}$ . Therefore, from (38) and the definition of  $\varphi_{\text{band}}$ , we infer that, for all  $i \in \{1, \dots, N_2 + 1\}$  and  $u \in [a_{2,i-1}, a_{2,i})$ ,

$$\begin{aligned} \tau(\tilde{\eta}_2|_{[0,u]}) &= \sum_{k=1}^{i-1} \int_{a_{2,k-1}}^{a_{2,k}} \tilde{\kappa}_{2,k} + \int_{a_{2,i-1}}^u \tilde{\kappa}_{2,i} + \sum_{k=1}^{i-1} \tilde{\psi}_{2,k} \\ &= \sum_{k=1}^{i-1} \int_{a_{2,k-1}}^{a_{2,k}} \kappa_{2,k} + \sum_{k=1}^{i-1} \int_{B_c^k} K + \int_{a_{2,i-1}}^u \kappa_{2,i} + \int_{\varphi_i([a_{2,i-1}, u] \times [0, -a_{1,1}])} K + \sum_{k=1}^{i-1} \psi_{2,k} \\ &= \tau(\eta_2|_{[0,u]}) + \int_{\varphi_{\text{band}}([0,u] \times [0, -a_{1,1}])} K. \end{aligned} \quad (45)$$

Then, in the same manner as in Lemma 26, we deduce from (45) that

$$\tau_+(\tilde{\eta}_2) \leq \tau_+(\eta_2) + \int_{B_c^{\text{band}}} K^+ \quad \text{and} \quad \tau_-(\tilde{\eta}_2) \leq \tau_-(\eta_2) + \int_{B_c^{\text{band}}} K^-. \quad (46)$$

Moreover, we have

$$\tau_{\pm}(\tilde{\eta}_1) = \tau_{\pm}(\eta_1) - \tau_{\pm}(\eta_1|_{[a_{1,1},0]}). \quad (47)$$

We now prove that  $\tilde{Q}$  satisfies the condition (37a). First, we have

$$\begin{aligned} \tau_+(\tilde{\eta}_1) + \tau_+(\tilde{\eta}_2) + \int_{\tilde{Q}} K^+ + \pi - \tilde{\theta}_1 &\leq \tau_+(\eta_1) - \tau_+(\eta_1|_{[a_{1,1},0]}) + \tau_+(\eta_2) + \int_{B_c^{\text{band}}} K^+ \\ &\quad + \int_{\tilde{Q}} K^+ + \pi - \theta_1 + \tau(\eta_1|_{[a_{1,1},0]}) \\ &\leq \tau_+(\eta_2) + \tau_+(\eta_1) + \pi - \theta_1 + \int_Q K^+, \quad (48) \end{aligned}$$

using (44), (46) and (47) for the first inequality. Since hypothesis (37a) (with  $\psi = \pi - \theta_1$ ) is satisfied by  $Q$ , we infer from (48) that

$$\tau_+(\tilde{\eta}_1) + \tau_+(\tilde{\eta}_2) + \int_{\tilde{Q}} K^+ + \pi - \tilde{\theta}_1 < \pi.$$

Finally, we prove that  $\tilde{Q}$  satisfies the condition (37b). From (46) and (47), we deduce that

$$\tau_-(\tilde{\eta}_1) + \tau_-(\tilde{\eta}_2) + \int_{\tilde{Q}} K^- \leq \tau_-(\eta_1) - \tau_-(\eta_1|_{[a_{1,1},0]}) + \tau_-(\eta_2) + \int_Q K^-. \quad (49)$$

Then, using the hypothesis (37b) on  $Q$ , we infer from (49) that

$$\tau_-(\tilde{\eta}_1) + \tau_-(\tilde{\eta}_2) + \int_{\tilde{Q}} K^- < \pi - \theta_1 - \tau_-(\eta_1|_{[a_{1,1},0]}).$$

Finally, using (44), we conclude that

$$\tau_-(\tilde{\eta}_1) + \tau_-(\tilde{\eta}_2) + \int_{\tilde{Q}} K^- \leq \pi - \tilde{\theta}_1 - \tau(\eta_1|_{[a_{1,1},0]}) - \tau_-(\eta_1|_{[a_{1,1},0]}) < \pi - \tilde{\theta}_1.$$

Hence, the hypotheses of  $(\mathcal{H}_{N_1})$  are satisfied by  $\tilde{Q}$ . We obtain the existence of polygons  $\{\tilde{B}_e^i\}_{1 \leq i \leq \tilde{N}_{\text{piece}}}$  such that  $\text{int}(\tilde{B}_e^i) \cap \text{int}(\tilde{B}_e^j) = \emptyset$  for all  $i \neq j$  and  $\bigcup_{i=1}^{\tilde{N}_{\text{piece}}} \tilde{B}_e^i = (\mathbb{R}^+)^2$ , and a Chebyshev parametrization of  $\tilde{Q}$  denoted  $\tilde{\varphi} : (\mathbb{R}^+)^2 \rightarrow \tilde{Q}$ . The translation of each polygon  $\tilde{B}_e^i$  by the vector  $(0, -a_{1,1})$ , i.e., the set  $\{\tilde{B}_e^i + (0, -a_{1,1})\}_{1 \leq i \leq \tilde{N}_{\text{piece}}}$  is then joined to the set  $\{B_e^i\}_{1 \leq i \leq N_2+1}$  to obtain  $\{B_e^i\}_{1 \leq i \leq \mathcal{N}_{\text{piece}}}$ . Moreover, we define the mapping  $\varphi : (\mathbb{R}^+)^2 \rightarrow Q$  by

$$\varphi(u, v) = \begin{cases} \varphi_{\text{band}}(u, v), & \text{if } (u, v) \in B_e^{\text{band}}, \\ \tilde{\varphi}(u, v + a_{1,1}), & \text{otherwise.} \end{cases}$$

We can show, in the same manner as for  $\varphi_{\text{band}}$ , that  $\varphi$  is a homeomorphism.

**Step 6** (*Proof of the first induction process (nonsmooth Hazzidakis formula)*). Let  $(u, v) \in (\mathbb{R}^+)^2$ . First, we suppose that  $(u, v) \in B_e^{\text{band}}$ , so that  $(u, v) \in B_e^j$  for some  $j \in \{1, \dots, N_2 + 1\}$ . By the Hazzidakis formula in the smooth setting (14), the angle  $\omega_j$  between the parameter curves of  $\varphi_j$  satisfies

$$\omega_j(u, v) = \theta_j - \int_{a_{2,j-1}}^u \kappa_{2,j} - \int_{-v}^0 \tilde{\kappa}_{1,j} - \int_{\varphi_j([a_{2,j-1}, u] \times [0, v])} K. \quad (50)$$

Then, from (42) and (50), we infer that

$$\begin{aligned} \omega(u, v) &= \omega_j(u, v) = \theta_1 - \tau(\eta_2|_{[0, u]}) - \tau(\eta_1|_{[-v, 0]}) - \sum_{k=1}^{j-1} \int_{\varphi_k([a_{2,k-1}, a_{2,k}] \times [0, v])} K \\ &\quad - \int_{\varphi_j([a_{2,j-1}, u] \times [0, v])} K \\ &= \theta_1 - \tau(\eta_1|_{[-v, 0]}) - \tau(\eta_2|_{[0, u]}) - \int_{\varphi([0, u] \times [0, v])} K. \end{aligned}$$

We now suppose that  $(u, v) \notin B_e^{\text{band}}$  and we denote  $\bar{u} = u$  and  $\bar{v} = v + a_{1,1}$ . Then,  $\bar{u}, \bar{v} \in \mathbb{R}^+$  and, by  $(\mathcal{H}_{N_1})$ , the nonsmooth Hazzidakis formula (40) is satisfied by the angle  $\tilde{\omega}$  between the coordinate curves of  $\tilde{\varphi}$ . Hence, using (40) on  $\tilde{\omega}$  and (45), we obtain that

$$\begin{aligned} \omega(u, v) &= \tilde{\omega}(\bar{u}, \bar{v}) = \tilde{\theta}_1 - \tau(\tilde{\eta}_1|_{[-\bar{v}, 0]}) - \tau(\tilde{\eta}_2|_{[0, \bar{u}]}) - \int_{\tilde{\varphi}([0, \bar{u}] \times [0, \bar{v}])} K, \\ &= \theta_1 - \tau(\eta_1|_{[a_{1,1}, 0]}) - \tau(\eta_1|_{[-v, a_{1,1}]}) - \tau(\eta_2|_{[0, u]}) \\ &\quad - \int_{\varphi([0, u] \times [0, -a_{1,1}])} K - \int_{\varphi([0, u] \times [-a_{1,1}, v])} K \\ &= \theta_1 - \tau(\eta_1|_{[-v, 0]}) - \tau(\eta_2|_{[0, u]}) - \int_{\varphi([0, u] \times [0, v])} K. \end{aligned}$$

Hence,  $(\mathcal{H}_{N_1+1})$  holds. This concludes the proof.  $\square$

*Remark 28* (Explicit value of  $\mathcal{N}_{\text{piece}}$ ). We easily see from the proof that  $\mathcal{N}_{\text{piece}} = (N_1 + 1)(N_2 + 1)$ , where  $N_1 + 1$  and  $N_2 + 1$  are respectively the number of smooth pieces of  $\eta_1$  and  $\eta_2$ .

**Corollary 29** (Existence of piecewise smooth Chebyshev nets on  $N$ -half-surfaces). *Let  $N \geq 1$  and let  $B_c$  be a  $N$ -half-surface delimited by the curves  $\{\gamma_c^i\}_{1 \leq i \leq N+1}$ . We suppose that  $B_c$  satisfies the conditions*

$$\begin{aligned} \tau_+(\partial B_c) + \int_{B_c} K^+ &< \pi, \\ \tau_-(\partial B_c) + \int_{B_c} K^- &< |\psi|_{l^\infty}. \end{aligned}$$

Then, there exist  $\mathcal{N}_{\text{piece}} \geq 1$  polygons  $\{B_e^i\}_{1 \leq i \leq \mathcal{N}_{\text{piece}}}$  such that  $(\mathbb{R}^+)^2 = \cup_{i=1}^{\mathcal{N}_{\text{piece}}} B_e^i$  and  $\text{int}(B_e^i) \cap \text{int}(B_e^j) = \emptyset$  for all  $i \neq j$ , and Chebyshev coordinates  $\varphi$  on  $B_c$  such that  $\{\gamma_c^i\}_{1 \leq i \leq N+1}$  are coordinate curves. Moreover, the angle  $\omega = \angle(\partial_u \varphi, \partial_v \varphi)$  of the net is bounded away from 0 and  $\pi$  by the positive real number

$$\varepsilon = \min \left( \pi - \tau_+(\partial B_c) - \int_{B_c} K^+, \quad |\psi|_{l^\infty} - \tau_-(\partial B_c) - \int_{B_c} K^- \right).$$

Finally, the mapping  $\varphi|_{B_e^i} : B_e^i \rightarrow B_c^i \subset B_c$ , with  $B_c^i = \varphi(B_e^i)$ , is a diffeomorphism, for all  $i \in \{1, \dots, \mathcal{N}_{\text{piece}}\}$ .

*Proof.* The proof follows by combining Theorem 27 and Proposition 15.  $\square$

**Theorem 30** (Existence of piecewise smooth Chebyshev nets on geodesic  $N$ -half-surfaces).  
Let  $N \geq 1$  and let  $B_c$  be a geodesic  $N$ -half-surface delimited by the geodesic curves  $\{\gamma_c^i\}_{1 \leq i \leq N+1}$ . We suppose that  $B_c$  satisfies the conditions

$$\begin{aligned} \int_{B_c} K^+ &< \pi - |\psi|_{l^1}, \\ \int_{B_c} K^- &< |\psi|_{l^\infty}. \end{aligned}$$

Then, there exist  $\mathcal{N}_{\text{piece}} \geq 1$  polygons  $\{B_e^i\}_{1 \leq i \leq \mathcal{N}_{\text{piece}}}$  such that  $(\mathbb{R}^+)^2 = \cup_{i=1}^{\mathcal{N}_{\text{piece}}} B_e^i$  and  $\text{int}(B_e^i) \cap \text{int}(B_e^j) = \emptyset$  for all  $i \neq j$ , and Chebyshev coordinates  $\varphi$  on  $B_c$  such that  $\{\gamma_c^i\}_{1 \leq i \leq N+1}$  are coordinate curves. Moreover, the angle  $\omega = \angle(\partial_u \varphi, \partial_v \varphi)$  of the net is bounded away from 0 and  $\pi$  by the positive real number

$$\min \left( \pi - |\psi|_{l^1} - \int_{B_c} K^+, |\psi|_{l^\infty} - \int_{B_c} K^- \right).$$

Finally, the mapping  $\varphi|_{B_e^i} : B_e^i \rightarrow B_c^i \subset B_c$ , with  $B_c^i = \varphi(B_e^i)$ , is a diffeomorphism, for all  $i \in \{1, \dots, \mathcal{N}_{\text{piece}}\}$ .

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## 6 Construction of a Chebyshev net from its angle distribution

We now prove that a Chebyshev net can be constructed uniquely from its angle distribution. We start by showing in Subsection 6.1 that the construction of curves from their geodesic curvature, initial point and initial tangent vector is a well-posed problem. We then define, following [?], the mapping  $\mathcal{I}(\gamma, \omega)$  which, for given boundary curves  $\gamma_u$  and  $\gamma_v$ , associates with any angle distribution  $\omega$  satisfying the boundary conditions (8) the candidate Chebyshev net  $\varphi$ ,

with angle distribution  $\omega$ , satisfying the boundary conditions (3) (Subsection 6.2). The parametrization  $\varphi$  is constructed in such a way that the  $v$ -coordinate curves are arc-length parametrized curves with geodesic curvatures satisfying (7b). We moreover show the continuity of the mapping  $\mathcal{I}$  with respect to the angle distribution and to the delimiting curves  $\gamma_u$  and  $\gamma_v$ . In Subsection 6.3, we show that the candidate Chebyshev net  $\varphi$  has improved regularity in the  $u$ -coordinate whenever  $\omega$  satisfies the integrability condition (5). Finally, in Subsection 6.4, we prove that  $\varphi$  is indeed a Chebyshev net if  $\omega$  satisfies the integrability condition and has a sufficient regularity.

## 6.1 Construction of curves from their geodesic curvature

**Proposition 31** (Construction of curves from their geodesic curvature). *Let  $M$  be a smooth, open, complete, and simply connected surface. Let  $L_{\max} \in \mathbb{R}^+$ ,  $L \in (0, L_{\max})$  and  $k, r \in \mathbb{N}$ . Let  $x \in M$ , let  $V \in T_x M$  be a unit vector, i.e., a vector such that  $|V|_g = 1$ , and let  $\kappa \in C^k([0, L], \mathbb{R})$ . Then, there exists a unique (arc-length parametrized) curve  $\sigma(x, V, \kappa) := \sigma \in \Gamma^{k+2}([0, L])$  such that  $\sigma(0) = x$ ,  $\sigma'(0) = V$ , and with geodesic curvature  $\kappa$ .*

*Moreover, let  $L_1, L_2 \in (0, L_{\max})$ , let  $x_1, x_2 \in C^r([0, L_1], M)$  be initial position distributions and let  $V_1, V_2 \in C^r([0, L_1], TM)$ , with  $|V_1|_g = |V_2|_g = 1$ , be initial derivatives distribution. Let  $D_{1,2} = [0, L_1] \times [0, L_2]$  and let  $\kappa_1, \kappa_2 \in C^r([0, L_1], C^k([0, L_2], \mathbb{R}))$  be geodesic curvatures. We denote  $\sigma_1, \sigma_2 : [0, L_1] \rightarrow \Gamma^{k+2}([0, L_2])$  the two families of curves defined by  $\sigma_m(\eta, \cdot) := \sigma(x_m(\eta), V_m(\eta), \kappa_m(\eta, \cdot))$ , for all  $\eta \in [0, L_1]$  and  $m \in \{1, 2\}$ . Then, we have*

$$\sigma_1, \sigma_2 \in \Phi^{r,k+2}(D_{1,2}) = C^r([0, L_1], \Gamma^{k+2}([0, L_2])),$$

and, for all  $m \in \{1, 2\}$ ,

$$\|\sigma_m\|_{\Phi^{r,k+2}(D_{1,2})} \leq C, \quad (51)$$

where the constant  $C$  depends on  $L_{\max}$ ,  $\|x_m\|_{C^r([0, L_1])}$ ,  $\|V_m\|_{C^r([0, L_1])}$ , and  $\|\kappa_m\|_{C^r([0, L_1], C^k([0, L_2]))}$ . Finally, we have

$$\begin{aligned} \|\sigma_1 - \sigma_2\|_{\Phi^0(D_{1,2})} &\leq C \left( L_2 \|\kappa_1 - \kappa_2\|_{C^0(D_{1,2})} \right. \\ &\quad \left. + \|x_1 - x_2\|_{C^0([0, L_1])} + \|V_1 - V_2\|_{C^0([0, L_1])} \right), \end{aligned} \quad \text{if } k = 0 \text{ and } r = 0, \quad (52a)$$

$$\begin{aligned} \|\sigma_1 - \sigma_2\|_{\Phi^{r,k+2}(D_{1,2})} &\leq \tilde{C} \left( \|\kappa_1 - \kappa_2\|_{C^r([0, L_1], C^k([0, L_2]))} \right. \\ &\quad \left. + \|x_1 - x_2\|_{C^r([0, L_1])} + \|V_1 - V_2\|_{C^r([0, L_1])} \right), \end{aligned} \quad (52b)$$

where the constants  $C$  and  $\tilde{C}$  depend on  $L_{\max}$ ,  $\|x_m\|_{C^r([0, L_1])}$ ,  $\|V_m\|_{C^r([0, L_1])}$ , and  $\|\kappa_m\|_{C^r([0, L_1], C^k([0, L_2]))}$ , with  $m \in \{1, 2\}$ .

We illustrate the family of curves  $\sigma_1$  introduced in Proposition 31 in Figure 19.

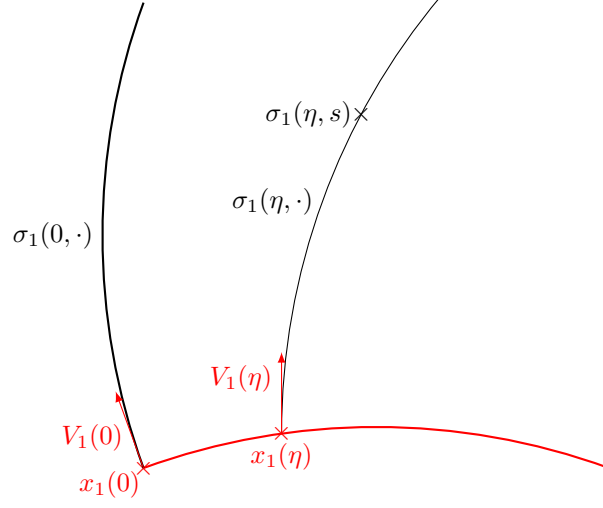


Figure 19: Illustration of the construction of the family of curves  $\sigma_1$

*Proof.* To prove the claim, we proceed as follows. We first introduce in Step 1 the geodesic curvature equation that permits to define uniquely a curve from its geodesic curvature. The existence and the uniqueness of the curve is proved in Step 2 using Cauchy–Lipschitz theorem. Then, in order to apply an induction argument, we prove (51) and (52) in the case  $k = 0$  and  $r = 0$  using Grönwall’s inequality. The equation satisfied by the derivatives of the solution is presented in a generic form to facilitate the end of the proof in Step 3. Finally, we prove (51) and (52b) in Steps 4 and 5 using induction arguments. Whenever there is no ambiguity, the domain of the variables will be omitted.

**Step 1** (*Formulation of the geodesic curvature equation*). Let  $(\sigma^1, \sigma^2)$ , with  $\sigma^1, \sigma^2 : [0, L] \rightarrow \mathbb{R}$ , be the coordinates of the candidate curve  $\sigma : [0, L] \rightarrow M$  with geodesic curvature  $\kappa$ . We denote  $\frac{D}{dt}$  the covariant derivative along the curve  $\sigma$ . The geodesic curvature equation for arc-length parametrized curves gives

$$\frac{D}{dt}\sigma' = \kappa\sigma'^{\perp}, \quad (53)$$

which can be written, in coordinates,

$$\dot{X} = G(X) + \kappa f(X), \quad (54)$$

with  $X(0) = (x^1, x^2, V^1, V^2)$ . Here, we have denoted

$$X = \begin{pmatrix} \sigma^1 \\ \sigma^2 \\ \sigma'^1 \\ \sigma'^2 \end{pmatrix}, \quad f(X) = \begin{pmatrix} 0 \\ 0 \\ N^1 \\ N^2 \end{pmatrix}, \quad G(X) = \begin{pmatrix} X^3 \\ X^4 \\ -\sum_{1 \leq i, j \leq 2} \Gamma_{i,j}^1(X^1, X^2)X^{2+i}X^{2+j} \\ -\sum_{1 \leq i, j \leq 2} \Gamma_{i,j}^2(X^1, X^2)X^{2+i}X^{2+j} \end{pmatrix},$$

$\Gamma_{ij}^k$  being the smooth Christoffel symbols,  $G$  the smooth geodesic function and  $N$  the normal vector defined by

$$N = \begin{pmatrix} N^1 \\ N^2 \end{pmatrix} = \sqrt{\det g(X^1, X^2)} g^{-1}(X^1, X^2) \begin{pmatrix} -X^4 \\ X^3 \end{pmatrix}.$$

**Step 2** (*Proof for  $k = 0$  and  $r = 0$* ). As  $G + \kappa f$  is continuous and locally Lipschitz continuous with respect to  $X$ , owing to Cauchy–Lipschitz theorem, there exists a local solution to (54). Moreover, since  $\sigma$  is by construction arc-length parametrized, we have  $|\sigma'|_g = 1$  and we infer from (4) that  $|\sigma'| \leq C_{\text{surf}}$ . Hence, the image of  $X$  is included in a ball  $\mathcal{B}$  whose radius depends only on  $L_{\text{max}}$  and  $C_{\text{surf}}$ . We deduce that the solution is defined on  $[0, L]$ . Moreover,  $G$  and  $f$  are Lipschitz continuous with respect to  $X$  on  $\mathcal{B}$ . Hence,  $G + \kappa f$  is Lipschitz continuous with respect to  $X$  on  $\mathcal{B}$ , so that the uniqueness of the solution follows. Moreover, we infer from (54) that  $\sigma \in \Gamma^2([0, L])$ .

Let  $m \in \{1, 2\}$  and let  $X_m = (\sigma_m^1, \sigma_m^2, \sigma_m'^1, \sigma_m'^2)^t : D_{1,2} \rightarrow \mathbb{R}^4$  be such that  $X_m(\eta, \cdot)$  is the unique solution to (54) with  $\kappa = \kappa_m(\eta, \cdot)$ , for all  $\eta \in [0, L_1]$ . First, owing to [?, Chap. V, Th. 2.1], we have  $X_m \in C^0([0, L_1], C^1([0, L_2]))$ , so that  $\sigma_m \in \Phi^{0,2}(D_{1,2})$ . Then, since  $|\sigma'| \leq C_{\text{surf}}$ , we infer that  $\|\sigma_m\|_{\Phi^{0,2}(D_{1,2})} \leq C$ , where the constant  $C$  depends on  $\|\sigma_m(\cdot, 0)\|_{C^0([0, L_1])}$  and  $\|X_m'\|_{C^0(D_{1,2})}$ . Moreover, due to  $|f(X_m)|_g = |\sigma_m'|_g = 1$ , we have

$$|f(X_m)| \leq C_{\text{surf}}. \quad (55)$$

Furthermore, since  $X_m(\{\eta\} \times [0, L_2]) \subset \mathcal{B}$ , we infer that  $G(X_m)$  is bounded and that this bound only depends on  $\mathcal{B}$ , so that it only depends on  $X_m(\eta, 0)$ ,  $L_{\text{max}}$  and  $C_{\text{surf}}$ , for all  $\eta \in [0, L_1]$ . We conclude that

$$\|\sigma_m\|_{\Phi^{0,2}(D_{1,2})} \leq C,$$

where the constant  $C$  depends on  $\|x_m\|_{C^0([0, L_1])}$  and  $\|\kappa_m\|_{C^0(D_{1,2})}$ , and (51) holds with  $k = r = 0$ . Then, since the restriction of  $G$  and  $f$  to  $\mathcal{B}$  are Lipschitz continuous in the variable  $X$  with coefficients denoted respectively  $C_G$  and  $C_f$ , we have

$$|\dot{X}_1 - \dot{X}_2| \leq (C_G + |\kappa_1| C_f) |X_2 - X_1| + |\kappa_2 - \kappa_1| \|f(X_2)\|_{C^0(D_{1,2})}.$$

Therefore, using Grönwall's inequality and (55), we infer that

$$|X_1(t) - X_2(t)| \leq \exp \left[ t(C_G + C_f \|\kappa_1\|_{C^0(D_{1,2})}) \right] \left( |X_2(0) - X_1(0)| + C_{\text{surf}} \int_0^t |\kappa_2 - \kappa_1| \right), \quad (56)$$

for all  $t \in [0, L]$ . We deduce that (52a) and (52b) are satisfied for  $k = r = 0$ .

**Step 3** (*Differential equation on the derivatives of  $X$* ). Let  $m \in \{1, 2\}$ . Owing to [?, Chap. V, Th. 4.1], we have that  $\sigma \in \Gamma^{k+2}([0, L])$  and  $\sigma_m \in \Phi^{r, k+2}(D_{1,2})$ . We use the following notation: for  $I = (i_1, i_2) \in \mathbb{N}^2$ , we set  $\partial^I f(\eta, t) = \partial_{\eta}^{i_1} \partial_t^{i_2} f(\eta, t)$ . We denote  $(\mathcal{H}_{r,k})$ , with  $r, k \in \mathbb{N}^*$ , the following induction hypothesis:

for all  $I = (i_1, i_2) \in \{0, \dots, r\} \times \{0, \dots, k\}$  such that  $I \neq (0, 0)$ , we have

$$\begin{aligned} \partial_t \partial^I X_m &= (\nabla G(X_m) + \kappa_m \nabla f(X_m)) \partial^I X_m \\ &\quad + \sum_{0 \leq \alpha \leq i_1} \sum_{0 \leq \beta \leq i_2} F_{\alpha, \beta}^I \left[ (\partial^{(p, q)} X_m)_{0 \leq p + \alpha \leq i_1, 0 \leq q + \beta \leq i_2, p + q < i_1 + i_2} \right] \partial^{(\alpha, \beta)} \kappa_m, \end{aligned} \quad (57)$$

where  $F_{\alpha, \beta}^I : \mathbb{R}^{4n_0} \rightarrow \mathbb{R}^4$ , with

$$n_0 = \begin{cases} (i_1 - \alpha + 1)(i_2 - \beta + 1), & \text{if } \alpha + \beta \neq 0, \\ (i_1 + 1)(i_2 + 1) - 1, & \text{otherwise,} \end{cases}$$

are  $C^\infty$  functions, for all  $(\alpha, \beta) \in \{0, \dots, i_1\} \times \{0, \dots, i_2\}$ .

We denote  $\partial_1$  and  $\partial_2$  the derivatives with respect to  $\eta$  and  $t$  respectively. We first obtain from (54) that, for all  $m \in \{1, 2\}$  and  $i \in \{1, 2\}$ ,

$$\partial_t \partial_i X_m = \nabla G(X_m) \partial_i X_m + \partial_i \kappa_m f(X_m) + \kappa_m \nabla f(X_m) \partial_i X_m.$$

Hence, (57) is satisfied for  $I = (0, 1)$  and  $I = (1, 0)$ , so that  $(\mathcal{H}_{1,1})$  holds. We now suppose that  $(\mathcal{H}_{r,k})$  holds for  $r, k \in \mathbb{N}^*$ . Then, for  $I = (i_1, i_2)$  with  $i_1 = r$  and  $i_2 = k$ , we have

$$\begin{aligned} \partial_t \partial_i \partial^I X_m &= (\nabla G(X_m) + \kappa_m \nabla f(X_m)) \partial_i \partial^I X_m + \partial_i [\nabla G(X_m) + \kappa_m \nabla f(X_m)] \partial^I X_m \\ &\quad + \sum_{0 \leq \alpha \leq i_1} \sum_{0 \leq \beta \leq i_2} \left[ \partial_i \left( F_{\alpha, \beta}^I \left[ (\partial^{(p, q)} X_m)_{0 \leq p + \alpha \leq i_1, 0 \leq q + \beta \leq i_2, p + q < i_1 + i_2} \right] \right) \partial^{(\alpha, \beta)} \kappa_m \right. \\ &\quad \left. + F_{\alpha, \beta}^I \partial_i \partial^{(\alpha, \beta)} \kappa_m \right]. \end{aligned}$$

The first term is in the form of the first term on the right-hand side of (57) and the last two terms can be put in the form of the second term on the right-hand side of (57). Equation (57) is then satisfied for  $I = (r + 1, k)$  and  $I = (r, k + 1)$ , so that  $(\mathcal{H}_{r+1,k})$  and  $(\mathcal{H}_{r,k+1})$  hold. The claim follows.

**Step 4 (Proof of (51)).** Let  $m \in \{1, 2\}$ . First note that we have by definition

$$\|\sigma_m\|_{\Phi^{r,k+2}(D_{1,2})} \leq \sum_{i_1=0}^r \sum_{i_2=0}^{k+1} \|\partial_\eta^{i_1} \partial_t^{i_2} X_m\|_{C^0(D_{1,2})}.$$

Then, to obtain (51), we only need to prove that

$$\|\partial^I X_m\|_{C^0(D_{1,2})} \leq C, \quad (58)$$

where the constant  $C$  depends on  $L_{\max}$ ,  $\|x_m\|_{C^{i_1}([0, L_1])}$ ,  $\|V_m\|_{C^{i_1}([0, L_1])}$ , and  $\|\kappa_m\|_{C^{i_1}([0, L_1], C^{i_2}([0, L_2])})$ , for all  $I = (i_1, i_2) \in \{0, \dots, r\} \times \{0, \dots, k+1\}$ . We prove



(58) by induction on  $I \in \{0, \dots, r\} \times \{0, \dots, k+1\}$ . Hence, we first consider the case  $i_2 = 0$ , in which case we have, using (57) and Grönwall's inequality,

$$|\partial_\eta^{i_1} X_m(t)| \leq \exp \left[ \|\nabla G(X_m) + \kappa_{\sigma_m} \nabla f(X_m)\|_{C^0(D_{1,2})} t \right] \\ \times \left( |\partial_\eta^{i_1} X_m(0)| + \sum_{\alpha=0}^{i_1} \int_0^t \left| F_{\alpha,0}^{(i_1,0)} [X_m, \dots, \partial_\eta^{i_1-1} X_m] \partial^{(\alpha,0)} \kappa_{\sigma_m} \right| \right),$$

for all  $i_1 \in \{1, \dots, r\}$ . Moreover, from Step 2, we have that (58) holds in the case where  $I = (0, 0)$ . Then, since the functions  $F_{\alpha,0}^{(i_1,0)}$  have  $C^\infty$ -regularity with respect to  $(X_m, \dots, \partial_\eta^{i_1-1} X_m)$ , for all  $\alpha \in \{0, \dots, i_1\}$  and  $i_1 \in \{1, \dots, r\}$ , an induction argument on  $i_1 \in \{0, \dots, r\}$  permits to prove that (58) holds for all  $I \in \{0, \dots, r\} \times \{0\}$ .

Now, noting that  $\partial_t \partial^I X_m = \partial_\eta^{i_1} \partial_t^{i_2+1} X_m$ , we infer from (57) that

$$\|\partial_t \partial^I X_m\|_{C^0(D_{1,2})} \leq \|\nabla G(X_m) + \kappa \nabla f(X_m)\|_{C^0(D_{1,2})} \|\partial^I X_m\|_{C^0(D_{1,2})} \\ + \sum_{0 \leq \alpha \leq i_1} \sum_{0 \leq \beta \leq i_2} \left\| F_{\alpha,\beta}^I \left[ (\partial^{(p,q)} X_m)_{0 \leq p+\alpha \leq i_1, 0 \leq q+\beta \leq i_2, p+q < i_1+i_2} \right] \right\|_{C^0(D_{1,2})} \|\partial^{(\alpha,\beta)} \kappa_m\|_{C^0(D_{1,2})},$$

for all  $I = (i_1, i_2) \in \{0, \dots, r\} \times \{0, \dots, k\}$  such that  $I \neq (0, 0)$ . Finally, since (58) holds for all  $I = (i_1, 0) \in \{0, \dots, r\} \times \{0\}$  and for  $I = (0, 1)$  by step 2, and since all the functions  $F_{\alpha,\beta}^I$  have  $C^\infty$ -regularity, the induction argument on  $I$  to prove that (58) holds for all  $I \in \{0, \dots, r\} \times \{0, \dots, k+1\}$  is straightforward. We conclude that (51) holds.

**Step 5** (*Proof of (52b)*). Let  $I = (i_1, i_2) \in \{0, \dots, r\} \times \{0, \dots, k\}$  be such that  $I \neq (0, 0)$  and let  $m \in \{1, 2\}$ . Owing to (51),  $\partial^I X_m$  is bounded by a constant depending only on  $\|\kappa_m\|_{C^r([0,L_1], C^k([0,L_2])}$ ,  $\|x_m\|_{C^r([0,L_1])}$ , and  $\|V_m\|_{C^r([0,L_1])}$ , for all  $\tilde{I} \in \{0, \dots, r\} \times \{0, \dots, k\}$ . Hence, the smooth functions  $F_{\alpha,\beta}^I$  are Lipschitz continuous on the compact set defined by the image of the derivatives of  $X_m$ , for all  $(\alpha, \beta) \in \{0, \dots, i_1\} \times \{0, \dots, i_2\}$ . We denote  $C_{F_{\alpha,\beta}^I}$  their respective Lipschitz coefficients in this compact set and we set  $W_m = \partial^I X_m$ . Using (57), we easily obtain that

$$|\partial_t W_2 - \partial_t W_1| \leq (\|\nabla G(X_1)\|_{C^0} + |\kappa_1| \|\nabla f(X_1)\|_{C^0}) |W_2 - W_1| \\ + \left( |\kappa_1 \nabla f(X_1) - \kappa_2 \nabla f(X_2)| + |\nabla G(X_1) - \nabla G(X_2)| \right) \|W_2\|_{C^0} \\ + \sum_{\alpha=0}^{i_1} \sum_{\beta=0}^{i_2} C_{F_{\alpha,\beta}^I} \sum_{\substack{p \in \{0, \dots, i_1\}, q \in \{0, \dots, i_2\} \\ p+q < i_1+i_2}} |\partial^{(p,q)} X_2 - \partial^{(p,q)} X_1| \|\partial^{(\alpha,\beta)} \kappa_1\|_{C^0} \\ + \sum_{\alpha=0}^{i_1} \sum_{\beta=0}^{i_2} \left\| F_{\alpha,\beta}^I \left[ (\partial^{(p,q)} X_2)_{0 \leq p+\alpha \leq i_1, 0 \leq q+\beta \leq i_2, p+q < i_1+i_2} \right] \right\|_{C^0} |\partial^{(\alpha,\beta)} \kappa_1 - \partial^{(\alpha,\beta)} \kappa_2|,$$

where  $C^0$  refers to the norm  $C^0(D_{1,2})$ . Using Grönwall's inequality, we infer

that

$$\begin{aligned}
|W_1 - W_2|(\eta, t) &\leq \exp \left[ \|\nabla G(X_1) + \kappa_1 \nabla f(X_1)\|_{C^0} t \right] \times \\
&\left( |W_1 - W_2|(\eta, 0) + \|W_2\|_{C^0} \max_{i \in \{1, 2\}} \max (\|\kappa_i\|_{C^0}, \|\nabla f(X_i)\|_{C^0}, 1) \int_0^t [|\kappa_2 - \kappa_1| + (C_{\nabla f} + C_{\nabla G})|X_2 - X_1|] \right. \\
&\quad + \sum_{\alpha=0}^{i_1} \sum_{\beta=0}^{i_2} C_{F_{\alpha, \beta}}^I \|\partial^{(\alpha, \beta)} \kappa_1\|_{C^0} \sum_{\substack{p \in \{0, \dots, i_1\}, \\ p+q < i_1+i_2}} \int_0^t |\partial^{(p, q)} X_2 - \partial^{(p, q)} X_1| \\
&\quad \left. + \sum_{\alpha=0}^{i_1} \sum_{\beta=0}^{i_2} \left\| F_{\alpha, \beta}^I \left[ (\partial^{(p, q)} X_2)_{0 \leq p+\alpha \leq i_1, 0 \leq q+\beta \leq i_2, p+q < i_1+i_2} \right] \right\|_{C^0} \int_0^t |\partial^{(\alpha, \beta)} \kappa_1 - \partial^{(\alpha, \beta)} \kappa_2| \right),
\end{aligned}$$

where  $C^0$  refers to the norm  $C^0(D_{1,2})$ , and  $C_{\nabla f}$  and  $C_{\nabla G}$  are the Lipschitz constants of  $\nabla f$  and  $\nabla G$ , respectively. Then, we obtain using (51) that

$$\begin{aligned}
\|\sigma_1 - \sigma_2\|_{\Phi^{i_1, i_2+1}(D_{1,2})} &\leq C \left( \|\kappa_1 - \kappa_2\|_{C^{i_1}([0, L_1], C^{i_2}([0, L_2]))} + \|x_1 - x_2\|_{C^{i_1}([0, L_1])} \right. \\
&\quad \left. + \|V_1 - V_2\|_{C^{i_1}([0, L_1])} + \Sigma_1 + \Sigma_2 \right),
\end{aligned}$$

where the constant  $C$  depends on  $L_{\max}$ ,  $\|x_l\|_{C^{i_1}([0, L_1])}$ ,  $\|V_l\|_{C^{i_1}([0, L_1])}$ , and  $\|\kappa_l\|_{C^{i_1}([0, L_1], C^{i_2}([0, L_2]))}$ , with  $l \in \{1, 2\}$ , and

$$\Sigma_1 = \begin{cases} \|\sigma_1 - \sigma_2\|_{\Phi^{i_1-1, i_2+1}(D_{1,2})}, & \text{if } i_1 > 0, \\ 0, & \text{otherwise,} \end{cases} \quad \Sigma_2 = \begin{cases} \|\sigma_1 - \sigma_2\|_{\Phi^{i_1, i_2}(D_{1,2})}, & \text{if } i_2 > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Hence, using (57), we obtain in the same manner that

$$\begin{aligned}
\|\sigma_1 - \sigma_2\|_{\Phi^{i_1, i_2+2}(D_{1,2})} &\leq C \left( \|\kappa_1 - \kappa_2\|_{C^{i_1}([0, L_1], C^{i_2}([0, L_2]))} + \|x_1 - x_2\|_{C^{i_1}([0, L_1])} \right. \\
&\quad \left. + \|V_1 - V_2\|_{C^{i_1}([0, L_1])} + \Sigma_1 + \Sigma_2 \right),
\end{aligned}$$

where the constant  $C$  depends on  $L_{\max}$ ,  $\|x_l\|_{C^r([0, L_1])}$ ,  $\|V_l\|_{C^r([0, L_1])}$ , and  $\|\kappa_l\|_{C^r([0, L_1], C^k([0, L_2]))}$ , with  $l \in \{1, 2\}$ . We then obtain (52) by a straightforward induction argument on  $I = (i_1, i_2) \in \{0, \dots, r\} \times \{0, \dots, k\}$ , recalling that the case  $I = (0, 0)$  was proved in Step 2. This concludes the proof of the proposition.  $\square$

## 6.2 Construction of the parametrization

Let  $\mathcal{R}_x(\theta)$  be the rotation of angle  $\theta$  in the tangent plane  $T_x M$  at  $x \in M$  and let  $D = [0, L_u] \times [0, L_v]$ , with  $L_u, L_v \in \mathbb{R}_*^+$ . Let  $r, k \in \mathbb{N}$ . Using the notation of Proposition 31, given an angle distribution  $\omega \in \Theta_{\gamma}^{r+1, k+1}(D)$  satisfying the boundary conditions (8) given by the two curves  $\gamma = (\gamma_u, \gamma_v) \in \Gamma^{r+2}([0, L_u]) \times \Gamma^{k+2}([0, L_v])$ , we set

$$x(\eta) = \gamma_u(\eta) \in M, \quad V(\eta) = \mathcal{R}_{\gamma_u(\eta)}(\omega(\eta, 0))\gamma'_u(\eta) \in T_{\gamma_u(\eta)} M, \quad (59)$$

and  $\kappa(\eta, s) = \partial_v \omega(\eta, s)$ , for all  $(\eta, s) \in D$ . Let  $\varphi_\omega : D \rightarrow M$  be the family of curves such that the curve  $\varphi_\omega(\eta, \cdot) \in \Gamma^{k+2}([0, L_v])$  has initial position  $\varphi_\omega(\eta, 0) = x(\eta)$ , initial tangent vector  $\partial_v \varphi_\omega(\eta, 0) = V(\eta)$ , and geodesic curvature  $\kappa(\eta, \cdot)$ , for all  $\eta \in [0, L_u]$  (see Figure 20). Note that the mapping  $\varphi_\omega$  also depends on the curves  $\gamma = (\gamma_u, \gamma_v)$  but since these curves are kept fixed in what follows, we do not mention them explicitly. We denote

$$\begin{aligned} \mathcal{I}(\gamma, \cdot) : \Theta_\gamma^{r+1, k+1}(D) &\rightarrow \Phi^{r+1, k+2}(D), \\ \omega &\mapsto \varphi_\omega, \end{aligned}$$

the mapping that associates with each angle distribution  $\omega$  with  $\Theta^{r+1, k+1}$ -regularity and satisfying the boundary conditions (8), the mapping  $\varphi_\omega : D \rightarrow M$ .

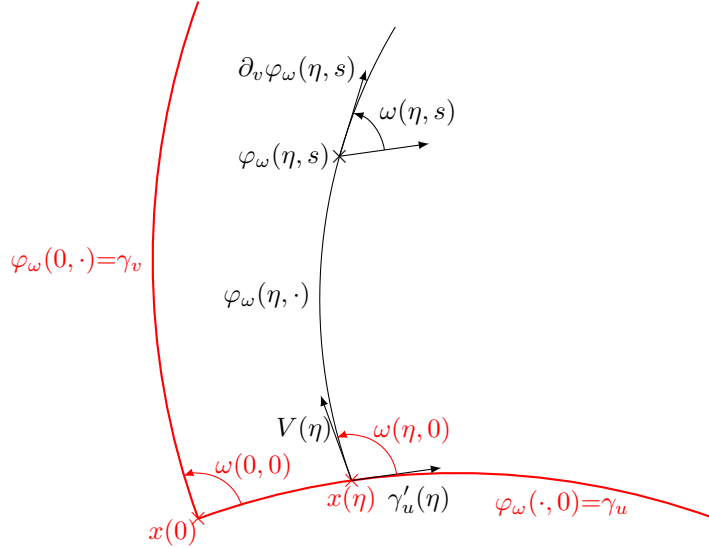


Figure 20: Illustration of the construction of the parametrization  $\varphi_\omega$

**Proposition 32** (Continuity of the construction). *Let  $M$  be a smooth, open, complete, and simply connected surface, let  $k, r \in \mathbb{N}$ , and let  $D = [0, L_u] \times [0, L_v]$ , with  $L_u, L_v \in \mathbb{R}_*^+$ . For all  $\gamma = (\gamma_u, \gamma_v) \in \Gamma^{r+2}([0, L_u]) \times \Gamma^{k+2}([0, L_v])$ , the mapping  $\mathcal{I}(\gamma, \cdot)$  is well defined. Moreover, let  $\gamma_1, \gamma_2 \in \Gamma^{r+2}([0, L_u]) \times \Gamma^{k+2}([0, L_v])$ , with  $\gamma_1 = (\gamma_{u,1}, \gamma_{v,1})$  and  $\gamma_2 = (\gamma_{u,2}, \gamma_{v,2})$ , be such that  $\gamma_{u,1}(0) = \gamma_{u,2}(0) = \gamma_{v,1}(0) = \gamma_{v,2}(0)$  and such that  $\gamma'_{u,1}(0) = \gamma'_{u,2}(0)$  and  $\gamma'_{v,1}(0) = \gamma'_{v,2}(0)$ . Consider the two angle distributions  $\omega_m \in \Theta_{\gamma_m}^{r+1, k+1}(D)$ , for  $m \in \{1, 2\}$ . Then, we have*

$$\|\mathcal{I}(\gamma_m, \omega_m)\|_{\Phi^{r+1, k+2}(D)} \leq C \quad (60)$$

where the constant  $C$  depends on  $L_u, L_v, \|\gamma_{u,m}\|_{\Gamma^s([0, L_u])}$ , and  $\|\omega_m\|_{\Theta^{r+1, k+1}(D)}$ , with  $s = \max(r+1, 2)$ . Moreover, for all  $L \in (0, L_v]$ , setting  $D_L = [0, L_u] \times [0, L]$ ,

we have

$$\begin{aligned} \|\mathcal{I}(\gamma_1, \omega_1) - \mathcal{I}(\gamma_2, \omega_2)\|_{\Phi^0(D_L)} \leq & C \left( L \|\omega_1 - \omega_2\|_{\Theta^1(D_L)} \right. \\ & \left. + \|\gamma_{u,2} - \gamma_{u,1}\|_{\Gamma^2([0, L_u])} \right), \end{aligned} \quad \text{if } r = 0 \text{ and } k = 0, \quad (61a)$$

$$\begin{aligned} \|\mathcal{I}(\gamma_1, \omega_1) - \mathcal{I}(\gamma_2, \omega_2)\|_{\Phi^{1,k+2}(D_L)} \leq & C \left( \|\omega_1 - \omega_2\|_{\Theta^{1,k+1}(D_L)} \right. \\ & \left. + \|\gamma_{u,2} - \gamma_{u,1}\|_{\Gamma^2([0, L_u])} \right), \end{aligned} \quad \text{if } r = 0, \quad (61b)$$

$$\begin{aligned} \|\mathcal{I}(\gamma_1, \omega_1) - \mathcal{I}(\gamma_2, \omega_2)\|_{\Phi^{r+1,k+2}(D_L)} \leq & C \left( \|\omega_1 - \omega_2\|_{\Theta^{r+1,k+1}(D_L)} \right. \\ & \left. + \|\gamma_{u,2} - \gamma_{u,1}\|_{\Gamma^{r+1}([0, L_u])} \right), \end{aligned} \quad \text{if } r > 0, \quad (61c)$$

where the constant  $C$  depends on  $L_u$ ,  $L_v$ ,  $\|\gamma_{u,i}\|_{\Gamma^s([0, L_u])}$ , and  $\|\omega_i\|_{\Theta^{r+1,k+1}(D)}$ , with  $i \in \{1, 2\}$  and  $s = \max(r+1, 2)$ .

*Proof.* Since the construction of the mapping  $\varphi_\omega$  is the same as that in the construction of Proposition 31, we only need to prove that the boundary conditions used in the construction of  $\varphi_\omega$  are smooth enough. We denote  $\kappa_{u,1} : [0, L_u] \rightarrow \mathbb{R}$  and  $\kappa_{u,2} : [0, L_v] \rightarrow \mathbb{R}$  the geodesic curvatures of the curves  $\gamma_{u,1}$  and  $\gamma_{u,2}$  respectively. Using the notation of Proposition 31, for all  $m \in \{1, 2\}$  and  $u \in [0, L_u]$ , we set  $x_m(u) = \gamma_{u,m}(u)$  and

$$V_m(u) = \mathcal{R}_{\gamma_{u,m}(u)}(\omega_m(u, 0))\gamma'_{u,m}(u) = \cos(\omega_m(u, 0))\gamma'_{u,m}(u) + \sin(\omega_m(u, 0))\gamma'^{\perp}_{u,m}(u), \quad (62)$$

where  $\gamma'^{\perp}_{u,m}$  is the direct  $\frac{\pi}{2}$ -rotation of  $\gamma'_{u,m}$ . Since  $|V_m|_g = 1$ , we infer from (4) that  $\|V_m\|_{C^0([0, L_u])} \leq C_{\text{surf}}$ , for all  $m \in \{1, 2\}$ . Furthermore, using that  $\omega_1 \in \Theta_{\gamma_1}^{r+1,k+1}(D)$  and  $\omega_2 \in \Theta_{\gamma_2}^{r+1,k+1}(D)$  both satisfy the boundary conditions (8), we obtain

$$|\omega_2 - \omega_1|(u, 0) = \left| \int_0^u \kappa_{u,2} - \int_0^u \kappa_{u,1} \right| \leq \int_0^u |\kappa_{u,2} - \kappa_{u,1}|.$$

Hence, using (62), we obtain by a straightforward computation that

$$\|V_2 - V_1\|_{C^0([0, L_u])} \leq C \left( \|\kappa_{u,2} - \kappa_{u,1}\|_{C^0([0, L_u])} + \|\gamma'_2 - \gamma'_1\|_{C^0([0, L_u])} \right). \quad (63)$$

Let  $m \in \{1, 2\}$ . Since  $\omega_m$  satisfies the boundary conditions (8) given by the arc-length parametrized curve  $\gamma_{u,m}$ , we infer from the definition of geodesic curvature (??) that

$$\begin{aligned} \frac{D}{du} \gamma'_{u,m} &= \kappa_{u,m} \gamma'^{\perp}_{u,m} = -\partial_u \omega_m(\cdot, 0) \gamma'^{\perp}_{u,m}, \\ \frac{D}{du} \gamma'^{\perp}_{u,m} &= -\kappa_{u,m} \gamma'_{u,m} = \partial_u \omega_m(\cdot, 0) \gamma'_{u,m}, \end{aligned}$$

where  $\frac{D}{du}$  is the covariant derivative along the curve  $\gamma_{u,m}$ . Hence, we deduce from (62) that

$$\begin{aligned} \frac{D}{du} V_m(u) = & \partial_u \omega_m(u, 0) (\cos(\omega_m(u, 0)) \gamma'_{u,m}{}^\perp - \sin(\omega_m(u, 0)) \gamma'_{u,m}) \\ & + \cos(\omega_m(u, 0)) \frac{D}{du} \gamma'_{u,m} + \sin(\omega_m(u, 0)) \frac{D}{du} \gamma'_{u,m}{}^\perp = 0, \end{aligned} \quad (64)$$

for all  $u \in [0, L_u]$ . Therefore, in the same manner as in Proposition 31, using that  $V_m$  is bounded, (63) and (64), we obtain

$$\|V_m\|_{C^{r+1}([0, L_u])} \leq C, \quad \|V_1 - V_2\|_{C^{r+1}([0, L_u])} \leq \tilde{C} \|\gamma_{u,1} - \gamma_{u,2}\|_{\Gamma^s([0, L_u])}, \quad (65)$$

where  $s = \max(r+1, 2)$ , the constant  $C$  depends on  $L_u, L_v$ , and  $\|\gamma_{u,m}\|_{\Gamma^s([0, L_u])}$ , and the constant  $\tilde{C}$  depends on  $L_u, L_v$ , and  $\|\gamma_{u,l}\|_{\Gamma^s([0, L_u])}$ , with  $l \in \{1, 2\}$ . Moreover, since  $x_m = \gamma_{u,m}$ , we have

$$\|x_m\|_{C^{r+1}([0, L_u])} = \|\gamma_{u,m}\|_{\Gamma^{r+1}([0, L_u])}, \quad \|x_1 - x_2\|_{C^{r+1}([0, L_u])} = \|\gamma_{u,1} - \gamma_{u,2}\|_{\Gamma^{r+1}([0, L_u])}. \quad (66)$$

We set  $\kappa_{v,m}^{\text{map}}(u, \cdot) := \partial_v \omega_m(u, \cdot) : [0, L_v] \rightarrow \mathbb{R}$ , for all  $u \in [0, L_u]$ . We conclude the proof using (65), (66) and Proposition 31 with regularity  $(r+1, k)$ , i.e., with  $x_1, x_2 \in C^{r+1}([0, L_u])$ ,  $V_1, V_2 \in C^{r+1}([0, L_u])$  and  $\kappa_{v,1}^{\text{map}}, \kappa_{v,2}^{\text{map}} \in C^{r+1}([0, L_u], C^k([0, L_v]))$ .  $\square$

### 6.3 Regularity of the candidate Chebyshev nets

Let us now consider the case where we have the same regularity in both coordinates in the data permitting the construction of  $\varphi_\omega$ , i.e.,  $r = k$ . Then, taking  $r = k$  in Proposition 32 gives an optimal estimate in the second coordinate regularity but a suboptimal estimate in the first coordinate, since we expect  $C^{k+2}$ -regularity in both directions. We show in this section that, whenever the mapping constructed satisfies the integrability equation (5), it has indeed the expected regularity in the first variable as well.

**Proposition 33** (Regularity of mappings satisfying the integrability condition). *Keeping the assumptions of Proposition 32 with  $k = r \in \mathbb{N}$ , we moreover suppose that  $\omega_m \in \Theta^{k+1}(D)$  satisfies the Hazzidakis formula (6) with  $\varphi_m := \mathcal{I}(\gamma_m, \omega_m)$ , i.e.,*

$$\begin{aligned} \omega_m(u, v) = & \angle(\gamma'_{u,m}(0), \gamma'_{v,m}(0)) - \int_0^u \kappa_{u,m} + \int_0^v \kappa_{v,m} \\ & - \int_0^u \int_0^v K[\mathcal{I}(\gamma_m, \omega_m)(t, s)] \sin(\omega_m(t, s)) dt ds, \end{aligned} \quad (67)$$

for all  $m \in \{1, 2\}$ . Then, we have  $\varphi_m \in \Phi^{k+2}(D)$  and

$$\|\mathcal{I}(\gamma_m, \omega_m)\|_{\Phi^{k+2}(D)} \leq C, \quad (68)$$

where the constant  $C$  depends on  $L_u, L_v, \|\gamma_{u,m}\|_{\Gamma^{k+2}([0,L_u])}$ , and  $\|\omega_m\|_{\Theta^{k+1}(D)}$ , for all  $m \in \{1, 2\}$ . Moreover, we have

$$\begin{aligned} \|\mathcal{I}(\gamma_1, \omega_1) - \mathcal{I}(\gamma_2, \omega_2)\|_{\Phi^{k+2}(D)} &\leq C \left( \|\omega_1 - \omega_2\|_{\Theta^{k+1}(D)} \right. \\ &\quad \left. + \|\gamma_{2,u} - \gamma_{1,u}\|_{\Gamma^{k+2}([0,L_u])} \right) \end{aligned} \quad (69)$$

where the constant  $C$  depends on  $L_u, L_v, \|\gamma_{u,i}\|_{\Gamma^{k+2}([0,L_u])}$  and  $\|\omega_i\|_{\Theta^{k+1}(D)}$ , for  $i \in \{1, 2\}$ .

*Proof.* Let  $m \in \{1, 2\}$ . First, owing to Proposition 32, we have  $\varphi_m = \mathcal{I}(\gamma_m, \omega_m) \in \Phi^{k+1,k+2}(D)$ . We denote  $\kappa_{v,m}^{\text{map}} = \partial_v \omega_m \in C^{k+1}([0, L_u], C^k([0, L_v]))$  the geodesic curvatures of the  $v$ -coordinate curves of  $\varphi_m$ . Let us remark that, since  $\omega_m \in \Theta^1(D)$  satisfies the Hazzidakis formula (67),  $\omega_m$  satisfies the integrability condition (5), i.e.,

$$\partial_{uv} \omega_m = -K(\varphi_m) \sin(\omega_m). \quad (70)$$

Then, the claim is obtained by remarking that (70) implies that the geodesic curvatures  $\kappa_{v,m}^{\text{map}}$  of the  $v$ -coordinate curves satisfy

$$\kappa_{v,m}^{\text{map}} \in C^{k+2}([0, L_u], C^k([0, L_v])). \quad (71)$$

Indeed, for all  $i, j \in \{0, \dots, k\}$ , we have

$$\partial_u^{i+1} \partial_v^j \kappa_{v,m}^{\text{map}} = \partial_u^i \partial_v^j (\partial_u \partial_v \omega_m) = \partial_u^i \partial_v^j (-K(\varphi_m) \sin(\omega_m)). \quad (72)$$

Hence, since  $\omega_m \in \Theta_{\gamma_m}^{k+1}(D)$  and  $\varphi_m \in \Phi^{k+1,k+2}(D)$  and since  $K$  is smooth, we infer that (71) holds and that (72) is satisfied for all  $i \in \{0, \dots, k+1\}$  and  $j \in \{0, \dots, k\}$ . Furthermore, we have

$$\partial_u^2 \kappa_{v,m}^{\text{map}} = \partial_u (-K(\varphi_m) \sin(\omega_m)) = -\sin(\omega_m) \nabla K(\varphi_m) \partial_u \varphi_m - K(\varphi_m) \cos(\omega_m) \partial_u \omega_m, \quad (73)$$

so that we easily obtain  $\|\partial_u^2 \kappa_{v,m}^{\text{map}}\|_{C^k([0,L_u], C^k([0,L_v]))} \leq C$ , where the constant  $C$  depends on  $L_u, L_v, \|\omega_m\|_{\Theta^{k+1}(D)}$ , and  $\|\varphi_m\|_{\Phi^{k+1}(D)}$ . Using moreover (60), we conclude that the constant  $C$  only depends on  $L_u, L_v, \|\omega_m\|_{\Theta^{k+1}(D)}$ , and  $\|\gamma_{u,m}\|_{\Gamma^s([0,L_u])}$ , with  $s = \max(k+1, 2)$ . We infer that

$$\|\kappa_{v,m}^{\text{map}}\|_{C^{k+2}([0,L_u], C^k([0,L_v]))} \leq C, \quad (74)$$

where the constant  $C$  depends on  $L_u, L_v, \|\omega_m\|_{\Theta^{k+1}(D)}$ , and  $\|\gamma_{u,m}\|_{\Gamma^s([0,L_u])}$ . Recalling that  $s = \max(k+1, 2)$ , we deduce from (73) and (61) that

$$\begin{aligned} \|\partial_u^2 \kappa_{v,2}^{\text{map}} - \partial_u^2 \kappa_{v,1}^{\text{map}}\|_{C^k([0,L_u], C^k([0,L_v]))} &\leq \tilde{C} \left( \|\omega_2 - \omega_1\|_{\Theta^{k+1}(D)} + \|\varphi_2 - \varphi_1\|_{\Phi^{k+1}(D)} \right) \\ &\leq C \left( \|\omega_2 - \omega_1\|_{\Theta^{k+1}(D)} + \|\gamma_{u,2} - \gamma_{u,1}\|_{\Gamma^s([0,L_u])} \right), \end{aligned}$$

where the constants  $C, \tilde{C}$  depend on  $L_u, L_v, \|\gamma_{u,i}\|_{\Gamma^s([0,L_u])}$ , and  $\|\omega_i\|_{\Theta^{k+1}(D)}$ , with  $i \in \{1, 2\}$ . We conclude that

$$\|\kappa_{v,2}^{\text{map}} - \kappa_{v,1}^{\text{map}}\|_{C^{k+2}([0,L_u], C^k([0,L_v]))} \leq C \left( \|\omega_2 - \omega_1\|_{\Theta^{k+1}(D)} + \|\gamma_{u,2} - \gamma_{u,1}\|_{\Gamma^s([0,L_u])} \right), \quad (75)$$

where the constant  $C$  depends on  $L_u, L_v, \|\gamma_{u,i}\|_{\Gamma^s([0,L_u])}$ , and  $\|\omega_i\|_{\Theta^{k+1}(D)}$ , with  $i \in \{1, 2\}$ . As in the proof of Proposition 32, for all  $u \in [0, L_u]$ , we set  $x_m(u) = \gamma_{u,m}(u)$  and

$$V_m(u) = \mathcal{R}_{\gamma_{u,m}(u)}(\omega_m(u, 0))\gamma'_{u,m}(u).$$

Before we apply Proposition 31 with regularity  $(k+2, k)$ , let us first note that, as in the proof of Proposition 32, the estimate on  $x_1, x_2 \in C^{k+2}([0, L_u])$ ,  $V_1, V_2 \in C^{k+2}([0, L_u])$  is given by (65), (66). Hence, the claim follows from the estimate on  $\kappa_{v,1}^{\text{map}}, \kappa_{v,2}^{\text{map}} \in C^{k+2}([0, L_u], C^k([0, L_v]))$  given by (74) and (75), and Proposition 31 with regularity  $(k+2, k)$ .  $\square$

## 6.4 From integrability conditions to Chebyshev nets

**Proposition 34** (From integrability conditions to Chebyshev nets). *Let  $M$  be a smooth, open, complete, and simply connected surface, let  $D = [0, L_u] \times [0, L_v]$ , with  $L_u, L_v \in \mathbb{R}_*^+$ , and let  $k \geq 1$ . Let  $\gamma = (\gamma_u, \gamma_v) \in \Gamma^{k+2}([0, L_u]) \times \Gamma^{k+2}([0, L_v])$ . Assume that  $\omega \in \Theta_\gamma^{k+1}(D)$  is an angle distribution satisfying the integrability condition (5), with  $\varphi := \mathcal{I}(\gamma, \omega) \in \Phi^{k+2}(D)$ . Suppose moreover that  $0 < \omega(u, v) < \pi$ , for all  $(u, v) \in D$ . Then, the mapping  $\varphi$  is a Chebyshev net in the sense that it satisfies (2).*

*Proof.* First, owing to Proposition 33, we have that  $\varphi \in \Phi^{k+2}(D)$ , so that  $\varphi$  has  $C^3$ -regularity. Since the  $v$ -coordinate curves are arc-length parametrized curves, we have by construction  $|\partial_v \varphi|_g(u, v) = 1$ , and we set  $R(u, v) = |\partial_u \varphi|_g(u, v)$ , for all  $(u, v) \in D$ . Then, since  $\gamma_u$  is an arc-length parametrized curve, we have  $R(u, 0) = 1$ , for all  $u \in [0, L_u]$ . The proof amounts to showing that

$$\exists L \in (0, L_v], \quad \partial_v R(u, v) = 0, \quad \forall (u, v) \in [0, L_u] \times [0, L]. \quad (76)$$

Indeed, suppose that (76) is satisfied and denote  $I \subset [0, L_v]$  the maximal interval on which we have  $R(u, v) = 1$ , for all  $(u, v) \in [0, L_u] \times I$ . Owing to (76), we first have that  $[0, L] \subset I$ , so that  $I$  is nonempty. Moreover, suppose that  $[0, \tilde{L}_0] \subset I$ , for some  $\tilde{L}_0 \in (0, L_v]$ . Then, since the angle distribution  $\omega|_{[0, L_u] \times [\tilde{L}_0, L_v]}$  and the mapping  $\varphi|_{[0, L_u] \times [\tilde{L}_0, L_v]}$  satisfy the hypotheses of the proposition, we infer from (76) that there exists  $\tilde{L}_1 \in (\tilde{L}_0, L_v]$  such that  $[0, \tilde{L}_1] \subset I$ . Hence,  $I$  is open in  $[0, L_v]$ , and we deduce from the continuity of  $R$  that  $I$  is closed. Therefore, (76) implies the claim.

We now suppose that  $L \in (0, L_v]$  is small enough so that  $R(u, v) > 0$ , for all  $(u, v) \in [0, L_u] \times [0, L]$ , we set  $D_L = [0, L_u] \times [0, L]$ , and we set  $X_1(u, v) := \langle \frac{\partial_u \varphi}{R}(u, v), \partial_v \varphi(u, v) \rangle_g$  and  $X_2(u, v) := \langle \frac{\partial_u \varphi^\perp}{R}(u, v), \partial_v \varphi(u, v) \rangle_g$ , for all  $(u, v) \in$

$D_L$ . Note that by definition  $X_1^2 + X_2^2 = 1$  in  $D_L$ . Recalling that  $R(\cdot, 0) = 1$  by construction of  $\varphi$ , we have

$$X_1(u, 0) = \langle \partial_u \varphi, \partial_v \varphi \rangle_g(u, 0) = \cos(\omega(u, 0)), \quad (77a)$$

$$X_2(u, 0) = \langle \partial_u \varphi^\perp, \partial_v \varphi \rangle_g(u, 0) = \sin(\omega(u, 0)), \quad (77b)$$

for all  $u \in [0, L_u]$ . Hence, since  $X_2$  is continuous, up to reducing  $L$ , we have  $X_2 > 0$  due to the assumption that  $0 < \omega < \pi$ .

We prove (76) as follows. We show that an identification of the Gaussian curvature computed using the local coordinates  $\varphi$  with (5) leads to the following integro-differential equation on  $R$  in the  $v$ -coordinates:

$$\partial_{vv}R = \partial_v \omega \partial_v R T + \frac{(\partial_v R)^2}{R} T^2 - K(\varphi) X_2 \left( \sin \omega \int_0^v \frac{\partial_v R}{X_2} \sin \omega + \cos \omega \int_0^v \frac{\partial_v R}{X_2} \cos \omega \right), \quad (78)$$

with  $T = \frac{X_1}{X_2}$ . We show that  $\partial_v R = 0$  is the unique solution to (78) to prove the claim (76). First, we compute in Step 1 the initial conditions satisfied by  $\partial_v R$  necessary to obtain the uniqueness of the solution of (78), i.e., we prove that  $\partial_v R(\cdot, 0) = 0$ . Then, we compute in Steps 2-4 the Gaussian curvature  $K$  in terms of  $R$ ,  $X_1$  and  $X_2$ . Using (5), we reduce the Gaussian curvature to (78) in Step 5 and we conclude in Step 6.

**Step 1 (Initial conditions).** We prove that  $\partial_v R(u, 0) = 0$ , for all  $u \in [0, L_u]$ .

We denote in what follows  $D_{\partial_u} Y$  and  $D_{\partial_v} Y$  the covariant derivative of the vector field  $Y$  in the directions  $\partial_u \varphi$  and  $\partial_v \varphi$ , respectively. First, since  $R(\cdot, 0) = 1$ , we have that  $\partial_u R(u, 0) = 0$ , for all  $u \in [0, L_u]$ . Moreover, since  $\omega$  satisfies the boundary conditions (8), we have that  $D_{\partial_u} \partial_u \varphi(u, 0) = -[\partial_u \omega \partial_u \varphi^\perp](u, 0)$ . Combining these results with (77), we obtain

$$\begin{aligned} \frac{1}{2} \partial_v (R^2)(u, 0) &= \langle D_{\partial_v} \partial_u \varphi, \partial_u \varphi \rangle_g(u, 0) = \partial_u (R X_1)(u, 0) - \langle D_{\partial_u} \partial_u \varphi, \partial_v \varphi \rangle_g(u, 0) \\ &= R(u, 0) \partial_u X_1(u, 0) + \langle \partial_u \omega \partial_u \varphi^\perp, \partial_v \varphi \rangle_g(u, 0) \\ &= \frac{d}{du} [\cos(\omega(u, 0))] + \partial_u \omega(u, 0) \sin(\omega(u, 0)) = 0, \end{aligned}$$

for all  $u \in [0, L_u]$ . Therefore, we have  $\partial_v R(u, 0) = 0$ , for all  $u \in [0, L_u]$ .

**Step 2 (Computation of the Gaussian curvature (1<sup>st</sup> part)).** We prove the following relations on the parallel transport of vectors:

$$D_{\partial_u} \partial_u \varphi = \left( \frac{\partial_u R}{R} + \frac{X_1}{X_2^2} (\partial_v R - \partial_u X_1) \right) \partial_u \varphi + \left( \frac{R}{X_2^2} (\partial_u X_1 - \partial_v R) \right) \partial_v \varphi, \quad (79a)$$

$$D_{\partial_v} \partial_u \varphi = D_{\partial_u} \partial_v \varphi = \frac{\partial_v R}{R X_2^2} \partial_u \varphi - \frac{\partial_v R X_1}{X_2^2} \partial_v \varphi, \quad (79b)$$

$$D_{\partial_v} \partial_v \varphi = \partial_v \omega \partial_v \varphi^\perp. \quad (79c)$$



The results (79a) and (79b) easily follow from the identities

$$\begin{aligned}
\langle D_{\partial_u} \partial_u \varphi, \partial_u \varphi \rangle_g &= R \partial_u R, \\
\langle D_{\partial_u} \partial_u \varphi, \partial_v \varphi \rangle_g &= \partial_u (R X_1) - \langle \partial_u \varphi, D_{\partial_v} \partial_u \varphi \rangle_g = \partial_u R X_1 + R \partial_u X_1 - R \partial_v R, \\
\langle D_{\partial_u} \partial_v \varphi, \partial_u \varphi \rangle_g &= R \partial_v R, \\
\langle D_{\partial_u} \partial_v \varphi, \partial_v \varphi \rangle_g &= 0,
\end{aligned}$$

where we have used in the last equality the fact that  $|\partial_v \varphi|_g = 1$ . Equation (79c) is obtained using that the  $v$ -coordinate curves of  $\varphi$  are arc-length parametrized and have by construction a geodesic curvature given by  $\partial_v \omega$ .

**Step 3** (*Computation of the Gaussian curvature (2<sup>nd</sup> part)*). We prove that  $X_1$  and  $X_2$  satisfy

$$\partial_v X_1 = -\partial_v \omega X_2 - \frac{\partial_v R}{R} X_1, \quad (80a)$$

$$\partial_v X_2 = \partial_v \omega X_1 + \frac{X_1}{X_2} \frac{\partial_v R}{R} X_1. \quad (80b)$$

First, using that  $\langle D_{\partial_v} \partial_u \varphi, \partial_v \varphi \rangle = 0$  and (79c), we obtain

$$\begin{aligned}
\partial_v X_1 &= \langle D_{\partial_v} \partial_v \varphi, \frac{\partial_u \varphi}{R} \rangle_g + \langle D_{\partial_v} \frac{\partial_u \varphi}{R}, \partial_v \varphi \rangle_g = \partial_v \omega \langle \partial_v \varphi^\perp, \frac{\partial_u \varphi}{R} \rangle_g + \langle -\frac{\partial_v R}{R^2} \partial_u \varphi + \frac{1}{R} D_{\partial_v} \partial_u \varphi, \partial_v \varphi \rangle_g \\
&= -\partial_v \omega X_2 - \frac{\partial_v R}{R} X_1,
\end{aligned}$$

which proves (80a). Then, using that the geodesic curvatures of the  $v$ -coordinate curves of  $\varphi$  is  $\partial_v \omega$ , we obtain  $D_{\partial_v} \partial_v \varphi^\perp = -\partial_v \omega \partial_v \varphi$ . Moreover, a straightforward computation gives

$$\partial_v \varphi^\perp = -\frac{1}{R X_2} \partial_u \varphi + T \partial_v \varphi, \quad \langle \partial_u \varphi^\perp, \partial_v \varphi \rangle_g = -\langle \partial_u \varphi, \partial_v \varphi^\perp \rangle_g.$$

Combining these results, we infer that

$$\begin{aligned}
\partial_v X_2 &= -\langle D_{\partial_v} \frac{\partial_u \varphi}{R}, \partial_v \varphi^\perp \rangle_g - \langle D_{\partial_v} \partial_v \varphi^\perp, \frac{\partial_u \varphi}{R} \rangle_g \\
&= \frac{\partial_v R}{R^2} \langle \partial_u \varphi, \partial_v \varphi^\perp \rangle_g - \frac{1}{R} \langle D_{\partial_u} \partial_v \varphi, \partial_v \varphi^\perp \rangle_g + \partial_v \omega \langle \partial_v \varphi, \frac{\partial_u \varphi}{R} \rangle_g \\
&= -\frac{\partial_v R}{R} X_2 - \frac{1}{R} \langle D_{\partial_u} \partial_v \varphi, -\frac{1}{R X_2} \partial_u \varphi + T \partial_v \varphi \rangle_g + \partial_v \omega X_1 \\
&= -\frac{\partial_v R}{R} X_2 + \frac{\partial_v (R^2)}{2 R^2 X_2} + \partial_v \omega X_1 = \frac{\partial_v R}{R} \left( \frac{1}{X_2} - X_2 \right) + \partial_v \omega X_1 \\
&= \partial_v \omega X_1 + \frac{X_1}{X_2} \frac{\partial_v R}{R} X_1.
\end{aligned}$$

**Step 4** (*Computation of the Gaussian curvature (3<sup>rd</sup> part)*). We now compute the expression of the Gaussian curvature  $K$  in the local parametrization  $\varphi$ .

Note that the metric induced by  $\varphi$  is  $\tilde{g} = R^2 du^2 + 2RX_1 dudv + dv^2$  giving  $\det \tilde{g} = R^2 X_2^2$ . Then, recall that, by definition, the Gaussian curvature  $K$  satisfies

$$K \det \tilde{g} = \langle D_{\partial_v} D_{\partial_u} \partial_u \varphi - D_{\partial_u} D_{\partial_v} \partial_u \varphi, \partial_v \varphi \rangle_g. \quad (81)$$

Using (79a), we first obtain that

$$\begin{aligned} \langle D_{\partial_v} D_{\partial_u} \partial_u \varphi, \partial_v \varphi \rangle_g &= \langle D_{\partial_v} (A \partial_u \varphi + B \partial_v \varphi), \partial_v \varphi \rangle_g \\ &= A \langle D_{\partial_v} \partial_u \varphi, \partial_v \varphi \rangle_g + B \langle D_{\partial_v} \partial_v \varphi, \partial_v \varphi \rangle_g + RX_1 \partial_v A + \partial_v B, \end{aligned}$$

with  $A = \frac{\partial_u R}{R} + \frac{X_1}{X_2^2}(\partial_v R - \partial_u X_1)$  and  $B = \frac{R}{X_2^2}(\partial_u X_1 - \partial_v R)$ . Then, we infer that

$$\begin{aligned} \langle D_{\partial_v} D_{\partial_u} \partial_u \varphi, \partial_v \varphi \rangle_g &= RX_1 \partial_v A + \partial_v B \\ &= \partial_v \left( \frac{\partial_u R}{R} \right) RX_1 + RX_1^2 \partial_v \left( \frac{\partial_v R - \partial_u X_1}{X_2^2} \right) + RX_1 \partial_v X_1 \frac{\partial_v R - \partial_u X_1}{X_2^2} \\ &\quad - \partial_v R \frac{\partial_v R - \partial_u X_1}{X_2^2} - R \partial_v \left( \frac{\partial_v R - \partial_u X_1}{X_2^2} \right) \\ &= \partial_v \left( \frac{\partial_u R}{R} \right) RX_1 - RX_2^2 \partial_v \left( \frac{\partial_v R - \partial_u X_1}{X_2^2} \right) \\ &\quad + RX_1 \partial_v X_1 \frac{\partial_v R - \partial_u X_1}{X_2^2} - \partial_v R \frac{\partial_v R - \partial_u X_1}{X_2^2}. \end{aligned} \quad (82)$$

Secondly, using (79b), we obtain that

$$\begin{aligned} \langle D_{\partial_u} D_{\partial_v} \partial_u \varphi, \partial_v \varphi \rangle_g &= \langle D_{\partial_u} \left[ \frac{\partial_v R}{RX_2^2} \partial_u \varphi - \frac{\partial_v RX_1}{X_2^2} \partial_v \varphi \right], \partial_v \varphi \rangle_g \\ &= \frac{\partial_v R}{RX_2^2} \langle D_{\partial_u} \partial_u \varphi, \partial_v \varphi \rangle_g - \frac{\partial_v RX_1}{X_2^2} \langle D_{\partial_u} \partial_v \varphi, \partial_v \varphi \rangle_g + RX_1 \partial_u \left( \frac{\partial_v R}{RX_2^2} \right) - \partial_u \left( \frac{\partial_v RX_1}{X_2^2} \right) \\ &= \frac{\partial_v R}{RX_2^2} \langle D_{\partial_u} \partial_u \varphi, \partial_v \varphi \rangle_g - \frac{\partial_u R}{R^2} \frac{\partial_v R}{X_2^2} RX_1 + \partial_u \left( \frac{\partial_v R}{X_2^2} \right) X_1 - \partial_u \left( \frac{\partial_v R}{X_2^2} \right) X_1 - \frac{\partial_u X_1 \partial_v R}{X_2^2} \\ &= \frac{\partial_v R}{X_2^2} \left( \frac{1}{R} \langle D_{\partial_u} \partial_u \varphi, \partial_v \varphi \rangle_g - \frac{X_1 \partial_u R}{R} - \partial_u X_1 \right). \end{aligned}$$

Since  $\langle D_{\partial_u} \partial_u \varphi, \partial_v \varphi \rangle_g = \partial_u RX_1 + R \partial_u X_1 - R \partial_v R$ , we infer that

$$\langle D_{\partial_v} D_{\partial_u} \partial_u \varphi, \partial_v \varphi \rangle_g = -\frac{(\partial_v R)^2}{X_2^2}. \quad (83)$$

Combining (81), (82) and (83) gives

$$\begin{aligned}
K \det \tilde{g} &= -RX_2^2 \partial_v \left( \frac{\partial_v R - \partial_u X_1}{X_2^2} \right) + RX_1 \partial_v X_1 \frac{\partial_v R - \partial_u X_1}{X_2^2} + \frac{\partial_v R \partial_u X_1}{X_2^2} + \partial_v \left( \frac{\partial_u R}{R} \right) RX_1 \\
&= \frac{1}{X_2^2} [RX_1 \partial_v X_1 (\partial_v R - \partial_u X_1) + \partial_v R \partial_u X_1] - R \partial_{vv} R + R \partial_{uv} X_1 \\
&\quad + 2 \partial_v X_2 \frac{R(\partial_v R - \partial_u X_1)}{X_2} + \partial_v \left( \frac{\partial_u R}{R} \right) RX_1 \\
&= \frac{1}{X_2^2} \left[ -RX_1 \left( \partial_v \omega X_2 + \frac{\partial_v R}{R} X_1 \right) (\partial_v R - \partial_u X_1) + \partial_v R \partial_u X_1 \right] - R \partial_{vv} R - R \partial_u \left[ \partial_v \omega X_2 + \frac{\partial_v R}{R} X_1 \right] \\
&\quad + 2 \left( \partial_v \omega X_1 + \frac{X_1^2 \partial_v R}{RX_2} \right) \frac{R(\partial_v R - \partial_u X_1)}{X_2} + \partial_v \left( \frac{\partial_u R}{R} \right) RX_1,
\end{aligned}$$

using (80) for the last equality. Then, we split the computation in two parts.

First, we have

$$\begin{aligned}
C &:= \frac{1}{X_2^2} \left[ -RX_1 \left( \partial_v \omega X_2 + \frac{\partial_v R}{R} X_1 \right) (\partial_v R - \partial_u X_1) + \partial_v R \partial_u X_1 \right] \\
&= \frac{1}{X_2^2} \left[ -R \partial_v \omega X_1 X_2 \partial_v R + RX_1 X_2 \partial_v \omega \partial_u X_1 - X_1^2 (\partial_v R)^2 + X_1^2 \partial_u X_1 \partial_v R + \partial_v R \partial_u X_1 \right] \\
&= -R \partial_v R \partial_v \omega T + RT \partial_v \omega \partial_u X_1 - T^2 (\partial_v R)^2 + T^2 \partial_u X_1 \partial_v R + \frac{\partial_v R \partial_u X_1}{X_2^2}.
\end{aligned} \tag{84}$$

Then, we obtain

$$\begin{aligned}
E &:= -R \partial_{vv} R - R \partial_u \left[ \partial_v \omega X_2 + \frac{\partial_v R}{R} X_1 \right] \\
&\quad + 2 \left( \partial_v \omega X_1 + \frac{X_1^2 \partial_v R}{RX_2} \right) \frac{R(\partial_v R - \partial_u X_1)}{X_2} + \partial_v \left( \frac{\partial_u R}{R} \right) RX_1 \\
&= -R \partial_{vv} R - R \left[ \partial_{uv} \omega X_2 + \partial_v \omega \partial_u X_2 + \partial_v \left( \frac{\partial_u R}{R} \right) X_1 + \frac{\partial_v R}{R} \partial_u X_1 \right] \\
&\quad + 2 \partial_v \omega T R (\partial_v R - \partial_u X_1) + 2T^2 \partial_v R (\partial_v R - \partial_u X_1) + \partial_v \left( \frac{\partial_u R}{R} \right) RX_1 \\
&= -R \partial_{vv} R - R \partial_{uv} \omega X_2 - R \partial_v \omega (\partial_u X_2 + T \partial_u X_1) - \partial_v R \partial_u X_1 (1 + T^2) \\
&\quad - T \partial_u X_1 R \partial_v \omega - T^2 \partial_v R \partial_u X_1 + 2T \partial_v R \partial_v \omega R + 2T^2 (\partial_v R)^2.
\end{aligned}$$

Moreover, since  $X_1^2 + X_2^2 = 1$ , we have  $1 + T^2 = \frac{1}{X_2^2}$  and  $T \partial_u X_1 + \partial_u X_2 = 0$ .

We infer that

$$\begin{aligned}
E &= -R \partial_{vv} R - R \partial_{uv} \omega X_2 - \frac{\partial_v R \partial_u X_1}{X_2^2} - T \partial_u X_1 R \partial_v \omega \\
&\quad - T^2 \partial_v R \partial_u X_1 + 2T \partial_v R \partial_v \omega R + 2T^2 (\partial_v R)^2.
\end{aligned} \tag{85}$$

We obtain by combining (84) and (85) that

$$K \det \tilde{g} = C + E = T^2 (\partial_v R)^2 - R \partial_{uv} \omega X_2 - R \partial_{vv} R + T R \partial_v R \partial_v \omega. \tag{86}$$

Using that  $\det \tilde{g} = X_2^2 R^2$  and dividing by  $R$ , we finally obtain

$$-\partial_v \omega \partial_v R T - \frac{(\partial_v R)^2}{R} T^2 + \partial_{vv} R = -X_2(\partial_{uv} \omega + K X_2 R). \quad (87)$$

**Step 5** (*Bound on the right-hand side of the equation (87)*). By (87) and the integrability condition (5), we have

$$\partial_{vv} R = \partial_v \omega \partial_v R T + \frac{(\partial_v R)^2}{R} T^2 + K(\varphi) X_2 [\sin(\omega) - X_2 R]. \quad (88)$$

The proof is now reduced to showing that  $\partial_v R = 0$  is the unique solution to (88) such that  $\partial_v R(\cdot, 0) = 0$ . To this end, we bound the right-hand side of this equation. Let  $F_1 = \sin(\omega) - R X_2$  and  $F_2 = \cos(\omega) - R X_1$ . We infer from (80b) that

$$\begin{aligned} \partial_v F_1 &= \partial_v \omega \cos(\omega) - \partial_v R X_2 - R \partial_v X_2 = \partial_v \omega (\cos(\omega) - R X_1) - \partial_v R (X_2 + T X_1) \\ &= \partial_v \omega F_2 - \frac{\partial_v R}{X_2}, \end{aligned}$$

using that  $X_2 + T X_1 = X_2 + \frac{1-X_2^2}{X_2} = \frac{1}{X_2}$  in the last equality. In the same manner, we deduce from (80a) that  $\partial_v F_2 = -\partial_v \omega F_1$ , so that the couple  $(F_1, F_2)$  satisfies the system of differential equations

$$\begin{cases} \partial_v F_1 = \partial_v \omega F_2 - \frac{\partial_v R}{X_2}, \\ \partial_v F_2 = -\partial_v \omega F_1, \end{cases}$$

with  $F_1(0) = F_2(0) = 0$ , since  $R(u, 0) = 1$ ,  $X_1(u, 0) = \cos \omega(u, 0)$ , and  $X_2(u, 0) = \sin \omega(u, 0)$ , for all  $u \in [0, L_1]$ . A straightforward computation shows that the unique solution to this linear ordinary differential equation is

$$\begin{aligned} (\sin \omega - X_2 R)(u, v) &= F_1(u, v) = -\sin \omega(u, v) \int_0^v \frac{\partial_v R}{X_2}(u, s) \sin \omega(u, s) ds \\ &\quad - \cos \omega(u, v) \int_0^v \frac{\partial_v R}{X_2}(u, s) \cos \omega(u, s) ds, \end{aligned} \quad (89a)$$

$$\begin{aligned} (\cos \omega - X_1 R)(u, v) &= F_2(u, v) = -\cos \omega(u, v) \int_0^v \frac{\partial_v R}{X_2}(u, s) \sin \omega(u, s) ds \\ &\quad + \sin \omega(u, v) \int_0^v \frac{\partial_v R}{X_2}(u, s) \cos \omega(u, s) ds, \end{aligned} \quad (89b)$$

since  $\partial_v R(u, 0) = 0$ , for all  $u \in [0, L_u]$ , by Step 1.

**Step 6** (*Conclusion*). Finally, we infer from (88) and (89a) that

$$\partial_{vv} R = \partial_v \omega \partial_v R T + \frac{(\partial_v R)^2}{R} T^2 - K(\varphi) X_2 \left( \sin \omega \int_0^v \frac{\partial_v R}{X_2} \sin \omega + \cos \omega \int_0^v \frac{\partial_v R}{X_2} \cos \omega \right). \quad (90)$$

Then, since  $0 < \omega(u, v) < \pi$ , for all  $(u, v) \in D_L$ , we have that  $T$  and  $\frac{1}{X_2}$  are bounded. Using moreover that  $\frac{1}{R}$ ,  $\partial_v \omega$  and  $K \circ \varphi$  are bounded, and using  $\partial_v R(\cdot, 0) = 0$ , we infer from (90) that

$$\begin{aligned} |\partial_v R(t)| &\leq \tilde{C} \left( \int_0^t |\partial_v R(s)| ds + \int_0^t |(\partial_v R)^2(s)| ds + \int_0^t \int_0^s |\partial_v R(l)| dl ds \right) \\ &\leq C \int_0^t |\partial_v R(s)| ds, \end{aligned}$$

for all  $u \in [0, L_u]$  and  $t \in [0, L]$ . Using Grönwall's inequality, we conclude that  $\partial_v R(u, v) = 0$ , for all  $(u, v) \in D_L$ . The claim follows.  $\square$

## 7 Existence and uniqueness of angle distribution

Let  $k \in \mathbb{N}$ , let  $D = [0, L_u] \times [0, L_v]$ , with  $L_u, L_v \in \mathbb{R}_*^+$ , and let  $\gamma = (\gamma_u, \gamma_v) \in \Gamma^{k+2}([0, L_u]) \times \Gamma^{k+2}([0, L_v])$  be two curves of geodesic curvatures  $\kappa_u \in C^k([0, L_u], \mathbb{R})$  and  $\kappa_v \in C^k([0, L_v], \mathbb{R})$ , respectively. In this section, we consider the Hazzidakis formula (6) as an equation on  $\omega \in \Theta_\gamma^{k+1}(D)$ , i.e., on angle distributions satisfying the boundary conditions (8). Hence, we define the mapping  $F : \Theta_\gamma^{k+1}(D) \rightarrow \Theta_\gamma^{k+1}(D)$  by

$$F[\omega](u, v) = \angle(\gamma'_u(0), \gamma'_v(0)) - \int_0^u \kappa_u + \int_0^v \kappa_v - \int_0^u \int_0^v K[\mathcal{I}(\gamma, \omega)(t, s)] \sin(\omega(t, s)) dt ds,$$

and we prove in what follows that there exists a unique solution to

$$\omega(u, v) = F[\omega](u, v), \tag{91}$$

for all  $(u, v) \in D$ . We first show in Subsection 7.1 that there exists a unique  $\omega^* \in \Theta_\gamma^{k+1}([0, L_u] \times [0, L_0])$ , for  $L_0 \in (0, L_v]$  small enough, satisfying (91). We also prove that this solution depends continuously on the curves  $\gamma_u$  and  $\gamma_v$ . Then, we extend this result to finite rectangles  $D = [0, L_u] \times [0, L_v]$ , with  $L_u, L_v \in \mathbb{R}_*^+$ , in Subsection 7.2. Finally, we prove by a density argument on the regularity of  $\gamma_u$  and  $\gamma_v$  that the associated parametrization  $\mathcal{I}(\gamma, \omega^*)$  is indeed a Chebyshev net.

### 7.1 Local existence of a solution

We first suppose that  $k = 0$  and we state the local existence of the angle distribution  $\omega$  in the following proposition.

**Proposition 35** (Local existence of a solution). *Let  $M$  be a smooth, open, complete, and simply connected surface, let  $L_u, L_v \in \mathbb{R}_*^+$ , and let  $\gamma = (\gamma_u, \gamma_v) \in$*

$\Gamma^2([0, L_u]) \times \Gamma^2([0, L_v])$  be such that  $\gamma_u(0) = \gamma_v(0)$  and  $\angle(\gamma'_u(0), \gamma'_v(0)) \in (0, \pi)$ . Then, there exists  $L_0 \in (0, L_v]$ , depending only on  $\|\gamma_u\|_{\Gamma^2([0, L_u])}$  and  $\|\gamma_v\|_{\Gamma^2([0, L_v])}$ , such that there exists a unique solution  $\omega_\gamma^* \in \Theta_\gamma^1([0, L_u] \times [0, L_0])$  to (91). Moreover, we have

$$\|\omega_\gamma^*\|_{\Theta^1([0, L_u] \times [0, L_0])} \leq C, \quad (92)$$

where the constant  $C$  depends on  $\|\gamma_u\|_{\Gamma^2([0, L_u])}$  and  $\|\gamma_v\|_{\Gamma^2([0, L_v])}$ .

*Proof.* Let  $D = [0, L_u] \times [0, L_v]$  and let  $D_L = [0, L_u] \times [0, L]$ , with  $L \in (0, L_v]$ . We prove the claim by application of the Banach fixed-point theorem to the functional  $F : \Theta_\gamma^1(D_L) \rightarrow \Theta_\gamma^1(D_L)$ , supposing that  $L$  is small enough. Hence, we first prove that  $F$  is stable in some bounded closed subset of  $\Theta_\gamma^1(D_L)$  (Step 1) and we then show that  $F^2$  is a contraction mapping in this space (Step 2). We conclude using the Banach fixed-point theorem in Step 3.

**Step 1 (Stability in a closed subset).** We denote  $\kappa_{u,1} \in C^0([0, L_u])$  and  $\kappa_{v,1} \in C^0([0, L_v])$  the geodesic curvatures of  $\gamma_u$  and  $\gamma_v$  respectively. We set  $\varphi_\gamma := \mathcal{I}(\gamma, \omega_\gamma) \in \Phi^{1,2}(D)$ . Since  $\varphi_\gamma$  is bounded by (60), we have that  $K \circ \varphi_\gamma$  is bounded. Moreover, a straightforward computation gives

$$\begin{aligned} \|F(\omega_\gamma)\|_{C^0(D)} &\leq \pi + L_u \|\kappa_{u,1}\|_{C^0([0, L_u])} + L_v \|\kappa_{v,1}\|_{C^0([0, L_v])} + L_u L_v \|K \circ \varphi_\gamma\|_{C^0(D)}, \\ \|\partial_u F(\omega_\gamma)\|_{C^0(D)} &\leq \|\kappa_{u,1}\|_{C^0([0, L_u])} + L_v \|K \circ \varphi_\gamma\|_{C^0(D)}, \\ \|\partial_v F(\omega_\gamma)\|_{C^0(D)} &\leq \|\kappa_{v,1}\|_{C^0([0, L_v])} + L_u \|K \circ \varphi_\gamma\|_{C^0(D)}, \\ \|\partial_{uv} F(\omega_\gamma)\|_{C^0(D)} &\leq \|K \circ \varphi_\gamma\|_{C^0(D)}, \end{aligned}$$

for all  $\omega_\gamma \in \Theta_\gamma^1(D)$ . Then, there exists  $\mathcal{R}(\gamma) > 0$  such that  $F : \mathcal{B}_{\Theta^1(D_L)}(\mathcal{R}(\gamma)) \rightarrow \mathcal{B}_{\Theta^1(D_L)}(\mathcal{R}(\gamma))$ , where  $\mathcal{B}_{\Theta^1(D_L)}(\mathcal{R}(\gamma))$  is the closed ball of radius  $\mathcal{R}(\gamma)$  centered at the origin in  $\Theta_\gamma^1(D_L)$ . In what follows, we restrict  $F$  to this ball.

**Step 2 (Contraction mapping).** We now prove the following result:

Let  $\sigma = (\sigma_u, \sigma_v) \in \Gamma^2([0, L_u]) \times \Gamma^2([0, L_v])$  be such that  $\gamma_u(0) = \gamma_v(0) = \sigma_u(0) = \sigma_v(0)$ ,  $\gamma'_u(0) = \sigma'_u(0)$  and  $\gamma'_v(0) = \sigma'_v(0)$ . Let  $\omega_\gamma \in \mathcal{B}_{\Theta^1(D_L)}(\mathcal{R}(\gamma))$  and  $\omega_\sigma \in \mathcal{B}_{\Theta^1(D_L)}(\mathcal{R}(\sigma))$ . Then, we have

$$\begin{aligned} \|F^2(\omega_\gamma) - F^2(\omega_\sigma)\|_{\Theta^1(D_L)} &\leq C \left( \|\gamma_u - \sigma_u\|_{\Gamma^2([0, L_u])} + \|\gamma_v - \sigma_v\|_{\Gamma^2([0, L_v])} \right. \\ &\quad \left. + L \|\omega_\gamma - \omega_\sigma\|_{\Theta^1(D_L)} \right), \end{aligned} \quad (93)$$

where the constant  $C$  depends on  $\mathcal{R}(\gamma)$  and  $\mathcal{R}(\sigma)$ . Note that (93) holds for all  $L \in (0, L_v]$  and  $\gamma, \sigma \in \Gamma^2([0, L_u]) \times \Gamma^2([0, L_v])$ .

In this step, the domain of definition of the two-dimensional variables for the different norms is always  $D_L = [0, L_u] \times [0, L]$  and will not be specified. In all the subsequent estimates, unless explicitly mentioned, the constants only depend on  $\mathcal{R}(\gamma)$  and  $\mathcal{R}(\sigma)$ . We set  $\varphi_\sigma := \mathcal{I}(\sigma, \omega_\sigma) \in \Phi^{1,2}$  and we denote  $\kappa_{u,2} \in C^0([0, L_u])$  and  $\kappa_{v,2} \in C^0([0, L_v])$  the geodesic curvatures of  $\sigma_u$  and  $\sigma_v$ , respectively. First,

we prove that

$$\|F(\omega_\gamma) - F(\omega_\sigma)\|_{\Theta^0} \leq \tilde{C} \left( \|\gamma_u - \sigma_u\|_{\Gamma^2([0, L_u])} + \|\gamma_v - \sigma_v\|_{\Gamma^2([0, L_v])} + L\|\omega_\gamma - \omega_\sigma\|_{\Theta^1} \right), \quad (94a)$$

$$\begin{aligned} \|F(\omega_\gamma) - F(\omega_\sigma)\|_{\Theta^1} &\leq C \left( \|\gamma_u - \sigma_u\|_{\Gamma^2([0, L_u])} + \|\gamma_v - \sigma_v\|_{\Gamma^2([0, L_v])} + L\|\omega_\gamma - \omega_\sigma\|_{\Theta^1} \right. \\ &\quad \left. + \|\omega_\gamma - \omega_\sigma\|_{\Theta^0} \right). \end{aligned} \quad (94b)$$

To this end, first note that we have

$$\begin{aligned} |F(\omega_\gamma) - F(\omega_\sigma)|(u, v) &\leq \int_0^u |\kappa_{u,1} - \kappa_{u,2}| + \int_0^v |\kappa_{v,1} - \kappa_{v,2}| + |\angle(\gamma'_u(0), \gamma'_v(0)) - \angle(\sigma'_u(0), \sigma'_v(0))| \\ &\quad + \int_0^u \int_0^v |K(\varphi_\sigma) \sin(\omega_\sigma) - K(\varphi_\gamma) \sin(\omega_\gamma)| \\ &\leq C \left( \|\gamma_u - \sigma_u\|_{\Gamma^2([0, L_u])} + \|\gamma_v - \sigma_v\|_{\Gamma^2([0, L_v])} \right. \\ &\quad \left. + L\|K(\varphi_\sigma) \sin(\omega_\sigma) - K(\varphi_\gamma) \sin(\omega_\gamma)\|_{C^0} \right), \end{aligned}$$

for all  $(u, v) \in D_L$ . Doing the same for  $\partial_u F(\omega)$ ,  $\partial_v F(\omega)$  and  $\partial_{uv} F(\omega) = -K(\varphi) \sin(\omega)$ , we obtain

$$\begin{aligned} \|F(\omega_\gamma) - F(\omega_\sigma)\|_{\Theta^0} &\leq \tilde{C} \left( \|\gamma_u - \sigma_u\|_{\Gamma^2([0, L_u])} + \|\gamma_v - \sigma_v\|_{\Gamma^2([0, L_v])} \right. \\ &\quad \left. + L\|K(\varphi_\gamma) \sin(\omega_\gamma) - K(\varphi_\sigma) \sin(\omega_\sigma)\|_{C^0} \right), \\ \|F(\omega_\gamma) - F(\omega_\sigma)\|_{\Theta^1} &\leq C \left( \|\gamma_u - \sigma_u\|_{\Gamma^2([0, L_u])} + \|\gamma_v - \sigma_v\|_{\Gamma^2([0, L_v])} \right. \\ &\quad \left. + \|K(\varphi_\gamma) \sin(\omega_\gamma) - K(\varphi_\sigma) \sin(\omega_\sigma)\|_{C^0} \right). \end{aligned}$$

Moreover, since  $K$  is smooth and since  $\varphi_\gamma$  and  $\varphi_\sigma$  are bounded, we have

$$\|K(\varphi_\gamma) \sin(\omega_\gamma) - K(\varphi_\sigma) \sin(\omega_\sigma)\|_{C^0} \leq C \left( \|\omega_\gamma - \omega_\sigma\|_{\Theta^0} + \|\varphi_\gamma - \varphi_\sigma\|_{\Phi^0} \right),$$

and we conclude the proof of (94) with

$$\|\varphi_\gamma - \varphi_\sigma\|_{\Phi^0} \leq C \left( L\|\omega_\gamma - \omega_\sigma\|_{\Theta^1} + \|\gamma_u - \sigma_u\|_{\Gamma^2([0, L_u])} \right), \quad (95)$$

obtained using (61a) of Proposition 32. Finally, using (94), we obtain

$$\begin{aligned} \|F^2(\omega_\gamma) - F^2(\omega_\sigma)\|_{\Theta^1} &\leq \tilde{C} \left( \|\gamma_u - \sigma_u\|_{\Gamma^2([0, L_u])} + \|\gamma_v - \sigma_v\|_{\Gamma^2([0, L_v])} \right. \\ &\quad \left. + L\|F(\omega_\gamma) - F(\omega_\sigma)\|_{\Theta^1} + \|F(\omega_\gamma) - F(\omega_\sigma)\|_{\Theta^0} \right) \\ &\leq C \left( \|\gamma_u - \sigma_u\|_{\Gamma^2([0, L_u])} + \|\gamma_v - \sigma_v\|_{\Gamma^2([0, L_v])} + L\|\omega_\gamma - \omega_\sigma\|_{\Theta^1} \right). \end{aligned}$$

The inequality (93) follows.

**Step 3 (Conclusion).** We infer from (93) that there exists  $L_0 \in (0, L_v]$  such that  $F^2$  is a contraction mapping in  $\mathcal{B}_{\Theta^1(D_{L_0})}(\mathcal{R}(\gamma))$  for the norm in  $\Theta^1(D_{L_0})$ . Hence, the claim follows from the Banach fixed-point theorem on the closed subset  $\mathcal{B}_{\Theta^1(D_{L_0})}(\mathcal{R}(\gamma))$  of the Banach space  $C^1([0, L_u], C^1([0, L_0]))$ .

□

Let  $R_0 \in \mathbb{R}_*^+$  and let  $\mathcal{B}_{\Gamma^2 \times \Gamma^2}(R_0)$  be the closed ball of radius  $R_0$  centered at the origin in  $\Gamma^2([0, L_u]) \times \Gamma^2([0, L_v])$  (supposing  $R_0$  is large enough for this set not to be empty). As the length of integration  $L_0 \in (0, L_v]$  of the  $v$ -coordinate curves of  $\varphi$  defined in Proposition 35 only depends on the norm in  $\Gamma^2([0, L_u]) \times \Gamma^2([0, L_v])$  of the initial conditions  $\gamma = (\gamma_u, \gamma_v)$ , we can define the mapping

$$\begin{aligned} \mathcal{J}_{0, R_0} : \mathcal{B}_{\Gamma^2 \times \Gamma^2}(R_0) &\rightarrow \Theta^1([0, L_u] \times [0, L_0(R_0)]), \\ \gamma = (\gamma_u, \gamma_v) &\mapsto \omega_\gamma = F(\omega_\gamma), \end{aligned} \quad (96)$$

which maps the boundary conditions  $\gamma$  to the solution  $\omega_\gamma$  to (91). We then state the following proposition which asserts the continuity of the mapping  $\mathcal{J}_{0, R_0}$  with respect to these boundary conditions.

**Proposition 36** (Continuity with respect to the boundary conditions). *Let  $M$  be a smooth, open, complete, and simply connected surface, let  $R_0 \in \mathbb{R}_*^+$ , and let  $D = [0, L_u] \times [0, L_v]$ , with  $L_u, L_v \in \mathbb{R}_*^+$ . We equip  $\mathcal{B}_{\Gamma^2 \times \Gamma^2}(R_0)$  and  $\Theta^1([0, L_u] \times [0, L_0(R_0)])$  with the norms in  $\Gamma^2([0, L_u]) \times \Gamma^2([0, L_v])$  and  $\Theta^1([0, L_u] \times [0, L_0(R_0)])$ , respectively. Then, the mappings  $\mathcal{J}_{0, R_0}$  and*

$$\begin{aligned} \mathcal{I} \circ (\text{Id}, \mathcal{J}_{0, R_0}) : \mathcal{B}_{\Gamma^2 \times \Gamma^2}(R_0) &\rightarrow \Phi^2([0, L_u] \times [0, L_0(R_0)]) \\ \gamma = (\gamma_u, \gamma_v) &\mapsto \varphi_\omega := \mathcal{I}(\gamma, \mathcal{J}_{0, R_0}(\gamma)) \end{aligned}$$

are Lipschitz continuous, with  $\text{Id}$  the identity operator in  $\mathcal{B}_{\Gamma^2 \times \Gamma^2}(R_0)$ .

*Proof.* Let  $\gamma, \sigma \in \mathcal{B}_{\Gamma^2 \times \Gamma^2}(R_0)$ , with  $\gamma = (\gamma_u, \gamma_v)$  and  $\sigma = (\sigma_u, \sigma_v)$ , be such that  $\gamma_u(0) = \gamma_v(0) = \sigma_u(0) = \sigma_v(0)$ ,  $\gamma'_u(0) = \sigma'_u(0)$  and  $\gamma'_v(0) = \sigma'_v(0)$ . Suppose moreover that  $\angle(\gamma'_u(0), \gamma'_v(0)) = \angle(\sigma'_u(0), \sigma'_v(0)) \in (0, \pi)$ . We set  $D_{L_0} = [0, L_u] \times [0, L_0(R_0)]$ ,  $\omega_\gamma := \mathcal{J}_{0, R_0}(\gamma) \in \Theta^1_\gamma(D_{L_0})$ , and  $\omega_\sigma := \mathcal{J}_{0, R_0}(\sigma) \in \Theta^1_\sigma(D_{L_0})$ . Let us recall from the proof of Proposition 35 that  $F^2 : \mathcal{B}_{\Theta^1(D_{L_0})}(\mathcal{R}(\gamma)) \rightarrow \mathcal{B}_{\Theta^1(D_{L_0})}(\mathcal{R}(\gamma))$  is a contraction mapping, with  $\mathcal{R}(\gamma) \in \mathbb{R}_*^+$  a constant depending only on  $R_0$ , and  $\mathcal{B}_{\Theta^1(D_{L_0})}(\mathcal{R}(\gamma))$  the ball centered at the origin with radius  $\mathcal{R}(\gamma)$  in  $\Theta^1(D_{L_0})$ . Since  $\omega_\gamma$  and  $\omega_\sigma$  are both contained in this ball, we deduce from (93) that

$$\begin{aligned} \|F^2(\omega_\gamma) - F^2(\omega_\sigma)\|_{\Theta^1(D_{L_0})} &\leq C_1 \left[ \|\gamma_u - \sigma_u\|_{\Gamma^2([0, L_u])} + \|\gamma_v - \sigma_v\|_{\Gamma^2([0, L_v])} \right] \\ &\quad + C_2 \|\omega_\gamma - \omega_\sigma\|_{\Theta^1(D_{L_0})}, \end{aligned}$$

with  $C_1 \in \mathbb{R}_*^+$  and  $C_2 \in (0, 1)$  two constants. As  $F(\omega) = \omega$ , we infer that

$$\|\omega_\gamma - \omega_\sigma\|_{\Theta^1(D_{L_0})} \leq \frac{C_1}{1 - C_2} \left[ \|\gamma_u - \sigma_u\|_{\Gamma^2([0, L_u])} + \|\gamma_v - \sigma_v\|_{\Gamma^2([0, L_v])} \right],$$



which proves that  $\mathcal{J}_{0,R_0}$  is Lipschitz continuous. Finally, using additionally Proposition 33, we infer that  $\mathcal{I} \circ (\text{Id}, \mathcal{J}_{0,R_0})$  is Lipschitz continuous.  $\square$

We now prove that  $C^{k+2}$ -regularity for the boundary conditions  $\gamma$  implies  $\Theta^{k+1}$ -regularity for the solution  $\omega$ , for all  $k \in \mathbb{N}$ .

**Proposition 37** (Regularity of the solution). *Let  $M$  be a smooth, open, complete, and simply connected surface, let  $k \in \mathbb{N}$ , and let  $D = [0, L_u] \times [0, L_v]$ , with  $L_u, L_v \in \mathbb{R}_*^+$ . Let  $\gamma = (\gamma_u, \gamma_v) \in \Gamma^{k+2}([0, L_u]) \times \Gamma^{k+2}([0, L_v])$  be such that  $\gamma_u(0) = \gamma_v(0)$  and  $\angle(\gamma'_u(0), \gamma'_v(0)) \in (0, \pi)$ . Then, there exists  $L_0 \in (0, L_v]$ , depending only on  $\|\gamma_u\|_{\Gamma^2([0, L_u])}$  and  $\|\gamma_v\|_{\Gamma^2([0, L_v])}$ , such that there exists a unique solution  $\omega_\gamma \in \Theta_\gamma^{k+1}([0, L_u] \times [0, L_0])$  to (91). Moreover, we have*

$$\|\omega_\gamma\|_{\Theta^{k+1}([0, L_u] \times [0, L_0])} \leq C, \quad (97)$$

where the constant  $C$  depends on  $\|\gamma_u\|_{\Gamma^{k+2}([0, L_u])}$  and  $\|\gamma_v\|_{\Gamma^{k+2}([0, L_v])}$ .

*Proof.* Owing to Proposition 35, there exists  $L_0 \in (0, L_v]$  such that there exists a unique solution  $\omega_\gamma \in \Theta_\gamma^1(D_{L_0})$  to (91). We prove in what follows that  $\omega_\gamma \in \Theta_\gamma^{k+1}(D_{L_0})$ . In this proof, the domain of definition of the two-dimensional variables for the different spaces is always  $D_{L_0} = [0, L_u] \times [0, L_0]$  and it will not be specified. Owing to Proposition 33, we have that  $\varphi_\omega = \mathcal{I}(\gamma, \omega) \in \Phi^{r+2}$  whenever  $\omega_\gamma \in \Theta_\gamma^{r+1}$ , for all  $r \in \{0, \dots, k\}$ . Therefore, using that  $\omega_\gamma = F(\omega_\gamma)$ , we obtain by an induction argument on  $r \in \{0, \dots, k\}$  that  $\omega_\gamma \in \Theta_\gamma^{k+1}$  (the only limiting factor being the regularity of the boundary curves  $\gamma$ ). Hence, we have  $\varphi_\omega \in \Phi^{k+2}$ . Now, to prove (97), we note that

$$\partial^{(i_1, 0)} F(\omega_\gamma)(u, v) = \partial_u^{i_1-1} \kappa_u + \int_0^v \partial_u^{i_1-1} [K(\varphi_\omega) \sin(\omega_\gamma)](u, t) dt, \quad (98a)$$

$$\partial^{(0, i_2)} F(\omega_\gamma)(u, v) = \partial_v^{i_2-1} \kappa_v + \int_0^u \partial_v^{i_2-1} [K(\varphi_\omega) \sin(\omega_\gamma)](s, v) ds, \quad (98b)$$

$$\partial^I F(\omega_\gamma)(u, v) = \partial^{(i_1-1, i_2-1)} [K(\varphi_\omega) \sin(\omega_\gamma)](u, v), \quad (98c)$$

for all  $I = (i_1, i_2) \in \{1, \dots, k+1\}^2$ . Furthermore, a straightforward computation gives

$$\|K(\varphi_\omega) \sin(\omega_\gamma)\|_{C^k([0, L_u], C^k([0, L_0]))} \leq C, \quad (99)$$

where the constant  $C$  depends on  $\|\varphi_\omega\|_{\Phi^k}$  and  $\|\omega_\gamma\|_{\Theta^k}$ . Using moreover Proposition 33, we infer that the constant  $C$  only depends on  $\|\gamma_u\|_{\Gamma^s([0, L_u])}$ , with  $s = \max(k, 2)$ , and  $\|\omega_\gamma\|_{\Theta^l}$ , with  $l = \max(k, 1)$ . Then, if  $k \geq 1$ , we deduce from (98) and (99) that  $\|\omega_\gamma\|_{\Theta^{k+1}} \leq C$ , where the constant  $C$  depends on  $\|\gamma_u\|_{\Gamma^{k+2}([0, L_u])}$ ,  $\|\gamma_v\|_{\Gamma^{k+2}([0, L_v])}$ , and  $\|\omega_\gamma\|_{\Theta^k}$ . Finally, since the case  $k = 0$  follows from (92), we obtain (97) by a straightforward induction argument on  $k \geq 0$ . The claim follows.  $\square$

Let  $\mathcal{B}_{\Gamma^{k+2} \times \Gamma^{k+2}}(R_k)$  be the closed ball of radius  $R_k \in \mathbb{R}_*^+$  centered at the origin in  $\Gamma^{k+2}([0, L_u]) \times \Gamma^{k+2}([0, L_v])$ . We denote  $\mathcal{J}_{k, R_k}$  the restriction of the mapping (96) to this ball, i.e.,

$$\mathcal{J}_{k, R_k} : \mathcal{B}_{\Gamma^{k+2} \times \Gamma^{k+2}}(R_k) \rightarrow \Theta^{k+1}([0, L_u] \times [0, L_0(R_k)]).$$

We can now state the equivalent of Proposition 36 in  $\Gamma^{k+2}([0, L_u]) \times \Gamma^{k+2}([0, L_v])$ .

**Proposition 38** (Continuity with respect to boundary conditions). *Let  $M$  be a smooth, open, complete, and simply connected surface and let  $k \in \mathbb{N}$ . Let  $R_k \in \mathbb{R}_*^+$  and let  $D = [0, L_u] \times [0, L_v]$ , with  $L_u, L_v \in \mathbb{R}_*^+$ . We equip  $\mathcal{B}_{\Gamma^{k+2} \times \Gamma^{k+2}}(R_k)$  and  $\Theta^{k+1}([0, L_u] \times [0, L_0(R_k)])$  with the norms in  $\Gamma^{k+2}([0, L_u]) \times \Gamma^{k+2}([0, L_v])$  and  $\Theta^{k+1}(D)$ , respectively. Then, the mappings  $\mathcal{J}_{k, R_k}$  and*

$$\begin{aligned} \mathcal{I} \circ (\text{Id}, \mathcal{J}_{k, R_k}) : \mathcal{B}_{\Gamma^{k+2} \times \Gamma^{k+2}}(R_k) &\rightarrow \Phi^{k+2}([0, L_u] \times [0, L_0(R_k)]), \\ \gamma = (\gamma_u, \gamma_v) &\mapsto \varphi_\omega := \mathcal{I}(\gamma, \mathcal{J}_{k, R_k}(\gamma)), \end{aligned}$$

are Lipschitz continuous, with  $\text{Id}$  the identity operator in  $\mathcal{B}_{\Gamma^{k+2} \times \Gamma^{k+2}}(R_k)$ .

*Proof.* Similarly to the previous proofs, the domain of definition of the two-dimensional variables for the different norms is always  $D_{L_0} = [0, L_u] \times [0, L_0(R_k)]$  and it will not be specified. Let  $\gamma = (\gamma_u, \gamma_v) \in \mathcal{B}_{\Gamma^{k+2} \times \Gamma^{k+2}}(R_k)$  be such that  $\gamma_u(0) = \gamma_v(0)$  and  $\angle(\gamma'_u(0), \gamma'_v(0)) \in (0, \pi)$ . Then, let  $\sigma = (\sigma_u, \sigma_v) \in \mathcal{B}_{\Gamma^{k+2} \times \Gamma^{k+2}}(R_k)$  be such that  $\gamma_u(0) = \sigma_u(0) = \sigma_v(0)$ ,  $\gamma'_u(0) = \sigma'_u(0)$  and  $\gamma'_v(0) = \sigma'_v(0)$ . We set  $\omega_\gamma := \mathcal{J}_{k, R_k}(\gamma) \in \Theta_\gamma^{k+1}$ ,  $\omega_\sigma := \mathcal{J}_{k, R_k}(\sigma) \in \Theta_\sigma^{k+1}$  and we set  $\varphi_\gamma := \mathcal{I}(\gamma, \omega_\gamma) \in \Phi^{k+1}$  and  $\varphi_\sigma := \mathcal{I}(\sigma, \omega_\sigma) \in \Phi^{k+1}$ . Using (98), a straightforward computation gives

$$\begin{aligned} \|F(\omega_\gamma) - F(\omega_\sigma)\|_{\Theta^{k+1}} &\leq C \left( \|\gamma_u - \sigma_u\|_{\Gamma^{k+2}([0, L_u])} + \|\gamma_v - \sigma_v\|_{\Gamma^{k+2}([0, L_v])} \right. \\ &\quad \left. + \|K(\varphi_\gamma) \sin(\omega_\gamma) - K(\varphi_\sigma) \sin(\omega_\sigma)\|_{C^k([0, L_u], C^k([0, L_0]))} \right). \end{aligned}$$

Moreover, we deduce from Proposition 33 that

$$\begin{aligned} \|K(\varphi_\gamma) \sin(\omega_\gamma) - K(\varphi_\sigma) \sin(\omega_\sigma)\|_{C^k([0, L_u], C^k([0, L_0]))} &\leq \tilde{C} \left( \|\varphi_\gamma - \varphi_\sigma\|_{\Phi^k} + \|\omega_\gamma - \omega_\sigma\|_{\Theta^k} \right) \\ &\leq C \left( \|\gamma_u - \sigma_u\|_{\Gamma^s([0, L_u])} + \|\omega_\gamma - \omega_\sigma\|_{\Theta^l} \right), \end{aligned}$$

with  $s = \max(k, 2)$  and  $l = \max(k, 1)$ . Hence, if  $k \geq 1$ , we have

$$\|\omega_\gamma - \omega_\sigma\|_{\Theta^{k+1}} \leq C \left( \|\gamma_u - \sigma_u\|_{\Gamma^{k+2}([0, L_u])} + \|\gamma_v - \sigma_v\|_{\Gamma^{k+2}([0, L_v])} + \|\omega_\gamma - \omega_\sigma\|_{\Theta^k} \right).$$

Since the Lipschitz continuity of  $\mathcal{J}_{k, R_k}$  in the case where  $k = 0$  follows from Proposition 36, we then obtain the Lipschitz continuity of  $\mathcal{J}_{k, R_k}$  in the general case by a straightforward induction argument on  $k \geq 0$ . Finally, the Lipschitz continuity of  $\mathcal{I} \circ (\text{Id}, \mathcal{J}_{k, R_k})$  follows from Proposition 33. This concludes the proof.  $\square$

In what follows, we will not make explicit the dependency of the mapping  $\mathcal{J}_{k, R_k}$  on  $R_k$ , so that it will be denoted  $\mathcal{J}_k$ .

## 7.2 Extension to rectangles

We now extend Propositions 35 and 37 on the existence and uniqueness of a solution to the fixed-point equation (91) and Propositions 36 and 38 on the continuity with respect to boundary conditions to the whole domain  $D = [0, L_u] \times [0, L_v]$ , with  $L_u, L_v \in \mathbb{R}_*^+$ . In the same manner as above, we start with the case where  $k = 0$ .

**Proposition 39** (Global existence of a solution). *Let  $M$  be a smooth, open, complete, and simply connected surface, let  $D = [0, L_u] \times [0, L_v]$ , with  $L_u, L_v \in \mathbb{R}_*^+$ , and let  $\gamma = (\gamma_u, \gamma_v) \in \Gamma^2([0, L_u]) \times \Gamma^2([0, L_v])$  be such that  $\gamma_u(0) = \gamma_v(0)$  and  $\angle(\gamma'_u(0), \gamma'_v(0)) \in (0, \pi)$ . Then, there exists a unique solution  $\omega \in \Theta_\gamma^1(D)$  to (91).*

*Proof.*

**Step 1** (*Restriction of  $F$  to angle distributions coinciding with the local solution*). First, owing to Proposition 35, there exists  $L_0 \in (0, L_v]$  such that there exists a unique solution  $\omega_\gamma = \mathcal{J}_0(\gamma) \in \Theta_\gamma^1(D_{L_0})$ , with  $D_{L_0} = [0, L_u] \times [0, L_0]$ , to (91). Suppose that  $L_0 < L_v$ . Otherwise, we have the expected result. We set  $\varphi_\gamma := \mathcal{I}(\gamma, \omega_\gamma) \in \Phi^1(D_{L_0})$ . Since we cannot expect in the general setting that the  $u$ -coordinate curves of the mapping  $\varphi_\gamma$  are arc-length parametrized, we cannot construct an extension of  $\varphi_\gamma$  using the curves  $\varphi_\gamma(\cdot, L_0)$  and  $\gamma_v$  as new boundary conditions. Therefore, we prove the claim using a fixed-point argument on the angle distributions  $\tilde{\omega}_\gamma$  defined to be extensions of  $\omega_\gamma : D_{L_0} \rightarrow \mathbb{R}$ . Let  $L_1 \in (L_0, L_v]$  be such that  $L_1 \leq 2L_0$  and let  $D_{L_1} = [0, L_u] \times [0, L_1]$ . We define the set  $S_{1,\gamma}(D_{L_1})$  composed of extensions of  $\omega_\gamma$  as follows:

$$S_{1,\gamma}(D_{L_1}) = \left\{ \tilde{\omega}_\gamma \in \Theta_\gamma^1(D_{L_1}) \text{ s.t. } \tilde{\omega}_\gamma|_{D_{L_0}} = \omega_\gamma \right\}.$$

Note that  $S_{1,\gamma}(D_{L_1})$  is clearly not empty and, to abbreviate the notation, we also denote  $\omega_\gamma$  the generic elements of  $S_{1,\gamma}$ . We now adapt the proof of Proposition 35 to show that  $F^2$  is a contraction mapping in some bounded subset of  $S_{1,\gamma}(D_{L_1})$  that is stable by  $F$ . Recall from this proof that there exists  $\mathcal{R}(\gamma) > 0$ , depending only on  $\|\gamma_u\|_{\Gamma^2([0, L_u])}$  and  $\|\gamma_v\|_{\Gamma^2([0, L_v])}$ , such that  $F : \mathcal{B}_{\Theta^1(D_{L_1})}(\mathcal{R}(\gamma)) \rightarrow \mathcal{B}_{\Theta^1(D_{L_1})}(\mathcal{R}(\gamma))$ , where  $\mathcal{B}_{\Theta^1(D_{L_1})}(\mathcal{R}(\gamma))$  is the closed ball of radius  $\mathcal{R}(\gamma)$  centered at the origin in  $\Theta_\gamma^1(D_{L_1})$ .

**Step 2** (*Contraction mapping*). Let  $\sigma = (\sigma_u, \sigma_v) \in \Gamma^2([0, L_u]) \times \Gamma^2([0, L_v])$  be such that  $\sigma_u(0) = \sigma_v(0) = \gamma_u(0)$ ,  $\gamma'_u(0) = \sigma'_u(0)$ , and  $\gamma'_v(0) = \sigma'_v(0)$ . Let  $\omega_\sigma \in S_{1,\sigma}(D_{L_1})$ . We now prove the following counterpart of (93):

$$\begin{aligned} \|F^2(\omega_\gamma) - F^2(\omega_\sigma)\|_{\Theta^1(D_{L_1})} &\leq C \left( \|\gamma_u - \sigma_u\|_{\Gamma^2([0, L_u])} + \|\gamma_v - \sigma_v\|_{\Gamma^2([0, L_v])} + L_0 \|\omega_\gamma - \omega_\sigma\|_{\Theta^1(D_{L_0})} \right. \\ &\quad \left. + (L_1 - L_0) \|\omega_\gamma - \omega_\sigma\|_{\Theta^1([0, L_u] \times [L_0, L_1])} \right), \end{aligned} \tag{100}$$

where the constant  $C$  is independent of  $L_0$  and  $L_1$ . A simple modification of the proof of (52a) implying (61a) gives the following counterpart of (95):

$$\begin{aligned} \|\mathcal{I}(\gamma, \omega_\gamma) - \mathcal{I}(\sigma, \omega_\sigma)\|_{\Phi^0(D_{L_1})} &\leq C \left( (L_1 - L_0) \|\omega_\gamma - \omega_\sigma\|_{\Theta^1([0, L_u] \times [L_0, L_1])} \right. \\ &\quad \left. + L_0 \|\omega_\gamma - \omega_\sigma\|_{\Theta^1(D_{L_0})} + \|\gamma_u - \sigma_u\|_{\Gamma^2([0, L_u])} \right), \end{aligned}$$

where the constant  $C$  is independent of  $L_0$  and  $L_1$ . Hence, we obtain the following counterpart of (94):

$$\begin{aligned} \|F(\omega_\gamma) - F(\omega_\sigma)\|_{\Theta^0(D_{L_1})} &\leq C \left( \|\gamma_u - \sigma_u\|_{\Gamma^2([0, L_u])} + \|\gamma_v - \sigma_v\|_{\Gamma^2([0, L_v])} \right. \\ &\quad \left. + L_0 \|\omega_\gamma - \omega_\sigma\|_{\Theta^1(D_{L_0})} + (L_1 - L_0) \|\omega_\gamma - \omega_\sigma\|_{\Theta^1([0, L_u] \times [L_0, L_1])} \right), \end{aligned} \quad (101a)$$

$$\begin{aligned} \|F(\omega_\gamma) - F(\omega_\sigma)\|_{\Theta^1(D_{L_1})} &\leq C \left( \|\gamma_u - \sigma_u\|_{\Gamma^2([0, L_u])} + \|\gamma_v - \sigma_v\|_{\Gamma^2([0, L_v])} \right. \\ &\quad \left. + L_0 \|\omega_\gamma - \omega_\sigma\|_{\Theta^1(D_{L_0})} + (L_1 - L_0) \|\omega_\gamma - \omega_\sigma\|_{\Theta^1([0, L_u] \times [L_0, L_1])} \right. \\ &\quad \left. + \|\omega_\gamma - \omega_\sigma\|_{\Theta^0(D_{L_0})} + \|\omega_\gamma - \omega_\sigma\|_{\Theta^0([0, L_u] \times [L_0, L_1])} \right), \end{aligned} \quad (101b)$$

where the constant  $C$  is independent of  $L_0$  and  $L_1$ . Then, (100) follows from (101) in the same manner as in the proof of Proposition 35.

**Step 3 (Conclusion).** We set  $\mathcal{B}_{S_{1,\gamma}(D_{L_1})}(\mathcal{R}(\gamma)) := \mathcal{B}_{\Theta^1(D_{L_1})}(\mathcal{R}(\gamma)) \cap S_{1,\gamma}(D_{L_1})$ . We clearly have that  $F$  maps  $\mathcal{B}_{S_{1,\gamma}(D_{L_1})}(\mathcal{R}(\gamma))$  onto itself, so that we now restrict  $F$  to this ball. We infer from (100) that there exists  $L_1^* \in (L_0, L_v]$ , with  $L_1^* \leq 2L_0$ , such that  $F^2$  is a contraction mapping in  $\mathcal{B}_{S_{1,\gamma}(D_{L_1^*})}(\mathcal{R}(\gamma))$ . Hence, using the Banach fixed-point theorem, we obtain that there exists a unique solution  $\omega_\gamma^* \in S_{1,\gamma}(D_{L_1^*})$ . Moreover, since the constants in (100) are independent of  $L_0$ , we infer that  $L_1^*$  is independent of  $L_0$ , so that we can repeat (a finite number of times) the argument until we reach  $L_v$ . The claim follows.  $\square$

**Proposition 40** (Regularity of the global solution). *Let  $M$  be a smooth, open, complete, and simply connected surface, let  $k \in \mathbb{N}$ , and let  $D = [0, L_u] \times [0, L_v]$ , with  $L_u, L_v \in \mathbb{R}_*^+$ . Let  $\gamma = (\gamma_u, \gamma_v) \in \Gamma^{k+2}([0, L_u]) \times \Gamma^{k+2}([0, L_v])$  be such that  $\gamma_u(0) = \gamma_v(0)$  and  $\angle(\gamma_u'(0), \gamma_v'(0)) \in (0, \pi)$ . Then, there exists a unique solution  $\omega \in \Theta_\gamma^{k+1}(D)$  to (91).*

*Proof.* The claim is obtained in the same manner as in Proposition 37.  $\square$

For all  $k \in \mathbb{N}$  and  $R_k \in \mathbb{R}_*^+$ , we now extend  $\mathcal{J}_{k,R_k}$  to the whole rectangle  $D = [0, L_u] \times [0, L_v]$  as follows:

$$\mathcal{J}_{k,R_k} : \mathcal{B}_{\Gamma^{k+2} \times \Gamma^{k+2}}(R_k) \rightarrow \Theta^{k+1}(D).$$

We prove in the following proposition that  $\mathcal{J}_{k,R_k}$  is Lipschitz continuous.

**Proposition 41** (Continuity with respect to boundary conditions). *Let  $M$  be a smooth, open, complete, and simply connected surface, let  $k \in \mathbb{N}$ , let  $R_k \in \mathbb{R}_*^+$ , and let  $D = [0, L_u] \times [0, L_v]$ , with  $L_u, L_v \in \mathbb{R}_*^+$ . We equip  $\mathcal{B}_{\Gamma^{k+2} \times \Gamma^{k+2}}(R_k)$  and  $\Theta^{k+1}(D)$  with the norms in  $\Gamma^{k+2}([0, L_u]) \times \Gamma^{k+2}([0, L_v])$  and  $\Theta^{k+1}(D)$ , respectively. Then, the mappings  $\mathcal{J}_{k, R_k}$  and*

$$\begin{aligned} \mathcal{I} \circ (\text{Id}, \mathcal{J}_{k, R_k}) : \mathcal{B}_{\Gamma^{k+2} \times \Gamma^{k+2}}(R_k) &\rightarrow \Phi^{k+2}(D) \\ \gamma = (\gamma_u, \gamma_v) &\mapsto \varphi_\omega := \mathcal{I}(\gamma, \mathcal{J}_{k, R_k}(\gamma)) \end{aligned}$$

are Lipschitz continuous, with  $\text{Id}$  the identity operator in  $\mathcal{B}_{\Gamma^{k+2} \times \Gamma^{k+2}}(R_k)$ .

*Proof.* We first prove the claim in the case where  $k = 0$ . Let  $\gamma, \sigma \in \mathcal{B}_{\Gamma^{k+2} \times \Gamma^{k+2}}(R_k)$ , with  $(\gamma_u, \gamma_v)$  and  $\sigma = (\sigma_u, \sigma_v)$ , be such that  $\gamma_u(0) = \gamma_v(0) = \sigma_u(0) = \sigma_v(0)$ ,  $\gamma'_u(0) = \sigma'_u(0)$  and  $\gamma'_v(0) = \sigma'_v(0)$ . We moreover suppose that  $\angle(\gamma'_u(0), \gamma'_v(0)) = \angle(\sigma'_u(0), \sigma'_v(0)) \in (0, \pi)$ . We denote  $\omega_\gamma := \mathcal{J}_{0, R_0}(\gamma) \in \Theta_\gamma^1(D)$  and  $\omega_\sigma := \mathcal{J}_{0, R_0}(\sigma) \in \Theta_\sigma^1(D)$ . In the same manner as in the proof of Proposition 36, we infer from (100) that

$$\begin{aligned} \|\omega_\gamma - \omega_\sigma\|_{\Theta^1(D_{L_1})} &\leq \tilde{C} \left[ \|\gamma_u - \sigma_u\|_{\Gamma^2([0, L_u])} + \|\gamma_v - \sigma_v\|_{\Gamma^2([0, L_v])} + L_0 \|\omega_\gamma - \omega_\sigma\|_{\Theta^1(D_{L_0})} \right] \\ &\leq C \left[ \|\gamma_u - \sigma_u\|_{\Gamma^2([0, L_u])} + \|\gamma_v - \sigma_v\|_{\Gamma^2([0, L_v])} \right], \end{aligned}$$

using that the mapping (96) is Lipschitz continuous (Proposition 36) for the second inequality. Since the extension process is only operated a finite number of times, we easily obtain by repeating the above argument that the mapping  $\mathcal{J}_{0, R_0}$  is Lipschitz continuous. Then, the proof for the case  $k > 0$  is obtained in the same manner as Proposition 38. Finally, the Lipschitz continuity of  $\mathcal{I} \circ (\text{Id}, \mathcal{J}_{k, R_k})$  follows from Proposition 33.  $\square$

### 7.3 Proof of the main result

*Proof of Theorem ??.* First, owing to Proposition 40, we obtain the existence of an angle distribution  $\omega_\gamma := \mathcal{J}_k(\gamma) \in \Theta_\gamma^{k+1}(D)$  satisfying (91). Then, the continuity of  $\mathcal{J}_k$  and  $\mathcal{I}$  is a direct consequence of Proposition 41. Hence, to prove the claim, we suppose that  $0 < \omega(u, v) < \pi$ , for all  $(u, v) \in D$ , and we show that  $\varphi := \mathcal{I}(\gamma, \omega_\gamma) \in \Theta_\gamma^{k+2}(D)$  satisfies (2). We suppose that  $k = 0$ . Otherwise, this is a direct consequence of Proposition 34. Using the density of  $\Gamma^3([0, L_u])$  and  $\Gamma^3([0, L_v])$  in  $\Gamma^2([0, L_u])$  and  $\Gamma^2([0, L_v])$  respectively, let  $(\gamma_{u,n})_{n \in \mathbb{N}} \subset \Gamma^3([0, L_u])$  and  $(\gamma_{v,n})_{n \in \mathbb{N}} \subset \Gamma^3([0, L_v])$  be sequences satisfying  $\gamma_{u,n}(0) = \gamma_u(0)$ ,  $\gamma_{v,n}(0) = \gamma_v(0)$ ,  $\gamma'_{u,n}(0) = \gamma'_u(0)$ ,  $\gamma'_{v,n}(0) = \gamma'_v(0)$ , for all  $n \in \mathbb{N}$ , and

$$\begin{aligned} \gamma_{u,n} &\xrightarrow[n \rightarrow \infty]{\Gamma^2} \gamma_u, \quad \text{and} \quad \|\gamma_{u,n}\|_{\Gamma^2([0, L_u])} \leq \|\gamma_u\|_{\Gamma^2([0, L_u])}, \quad \forall n \in \mathbb{N}, \\ \gamma_{v,n} &\xrightarrow[n \rightarrow \infty]{\Gamma^2} \gamma_v, \quad \text{and} \quad \|\gamma_{v,n}\|_{\Gamma^2([0, L_v])} \leq \|\gamma_v\|_{\Gamma^2([0, L_v])}, \quad \forall n \in \mathbb{N}. \end{aligned}$$

We set  $\gamma_n = (\gamma_{u,n}, \gamma_{v,n}) \in \Gamma^3([0, L_u]) \times \Gamma^3([0, L_v])$ ,  $\omega_n = \mathcal{J}_1(\gamma_n) \in \Theta_\gamma^2(D)$ , and  $\varphi_n = \mathcal{I}(\gamma_n, \omega_n) \in \Phi^3(D)$ , for all  $n \in \mathbb{N}$ . Owing to Proposition 38, we infer that

the sequence  $(\omega_n)_{n \in \mathbb{N}}$  converges to  $\omega$  in the  $C^0(D)$ -norm. Hence, there exists  $n_0 \in \mathbb{N}$  such that  $0 < \omega_n(u, v) < \pi$ , for all  $(u, v) \in D$  and all  $n \geq n_0$ . Since  $\omega_n \in \Theta_\gamma^2(D)$  and  $\gamma_n \in \Gamma^3([0, L_u]) \times \Gamma^3([0, L_v])$ , we obtain from Proposition 34 that  $\varphi_n$  is a Chebyshev net, for all  $n \geq n_0$ . Moreover, owing to Proposition 38, we have that  $\varphi_n \xrightarrow[n \rightarrow \infty]{\Phi^2(D)} \varphi$ . Finally, since  $\varphi_n$  satisfies (2) for all  $(u, v) \in D$  and all  $n \geq n_0$ , we conclude that  $\varphi$  satisfies (2) for all  $(u, v) \in D$ . The claim follows.  $\square$