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Question 1:

math 260 مراجعة لمادة

final exam.

Consider the differential equation:

$$y'' - 3y' - 4y = 2 + 8x\,, \qquad (E)$$

(1) Find the complementary solution " y_c " of equation (E):

$y'' - 3y' - 4y = 0$

$m^2 - 3m - 4 = 0$

$(m+1)(m-4) = 0$

$\boxed{m_1 = -1}$ $\boxed{m_2 = 4}$

$y_c = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

$y_c = c_1 e^{-1x} + c_2 e^{4x}$

(2) Find a particular solution " y_p " of equation (E):

$y_p = Ax + B$

$y_p' = A$

$y_p'' = 0$

$y_p'' - 3y_p' - 4y_p = 2 + 8x$

$0 - 3A - 4(Ax + B) = 2 + 8x$

$-3A - 4Ax - 4B = 2 + 8x$

$(-3A - 4B) + (-4Ax) = 2 + 8x$

$-3A - 4B = 2$ | $-4A = 8$

$-3(-2) - 4B = 2$ | $A = \frac{8}{-4}$

$6 - 4B = 2$ | $\boxed{A = -2}$

$-4B = 2 - 6$

$-4B = -4 \Rightarrow \boxed{B = 1}$

$y_p = Ax + B$

$y_p = -2x + 1$

(3) Find the general solution of (E):

$y = y_c + y_p$

$y = c_1 e^{-x} + c_2 e^{4x} - 2x + 1$

QUESTION 2:

Solve the system by using Cramer's rule

$$\begin{aligned} x\ +y\ +z &= 5 \\ 2x+3y-z &= -4 \\ x-2y-z &= 3 \end{aligned}$$

$$\begin{array}{cc} & \begin{array}{ccc} x & y & z \\ + & - & + \end{array} \\ A = & \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & -1 \\ 1 & -2 & -1 \end{vmatrix} \end{array}$$

$$\begin{aligned} &= 1(-3-2)-1(-2+1)+1(-4-3) \\ &= 1(-5)-1(-1)+1(-7) \\ &= -5+1-7 \\ &= -11 \\ &\neq 0 \end{aligned}$$

نواتج

$$x = \frac{\begin{vmatrix} 5 & 1 & 1 \\ -4 & 3 & -1 \\ 3 & -2 & -1 \end{vmatrix}}{-11} = \frac{5(-3-2)-1(4+3)+1(8-9)}{-11} = \frac{+33}{-11} = 3$$

نواتج

$$y = \frac{\begin{vmatrix} 1 & 5 & 1 \\ 2 & -4 & -1 \\ 1 & 3 & -1 \end{vmatrix}}{-11} = \frac{1(4+3)-5(-2+1)+1(6+4)}{-11} = \frac{22}{-11} = -2$$

نواتج

$$z = \frac{\begin{vmatrix} 1 & 1 & 5 \\ 2 & 3 & -4 \\ 1 & -2 & 3 \end{vmatrix}}{-11} = \frac{1(9-8)-1(6+4)+5(-4-3)}{-11} = \frac{-44}{-11} = 4$$

Question 3:

Solve the following linear system by using Gauss-Jordan Elimination:

$$\begin{aligned} x + y + z &= 3 \\ 2x + 3y - z &= -9 \\ -3x + y + 2z &= -1 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 2 & 3 & -1 & -9 \\ -3 & 1 & 2 & -1 \end{array}\right]$$

$$\begin{array}{c} \\ -2R_1+R_2 \\ 3R_1+R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & -3 & -15 \\ 0 & 4 & 5 & 8 \end{array}\right]$$

$$\begin{array}{c} -R_2+R_1 \\ \\ -4R_2+R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 4 & 18 \\ 0 & 1 & -3 & -15 \\ 0 & 0 & 17 & 68 \end{array}\right]$$

$-0+1=1$ $\quad -1+1=0$ $\quad 3+1=4$ $\quad 15+3 = 6$

$$\begin{array}{c} \\ \\ \frac{R_3}{17} \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 4 & 18 \\ 0 & 1 & -3 & -15 \\ 0 & 0 & 1 & 4 \end{array}\right]$$

$$\begin{array}{c} \ldots_3+R_1 \\ 3R_3+R_2 \\ \\ \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 4 \end{array}\right]$$

$x = 2$

$y = -3$

$z = 4$

(x, y, z)

$(2, -3, 4)$

QUESTION 4:

(i) If W is the set of all vectors (x_1, x_2) in R^2 such that $|x_1| = |x_2|$. Determine whether or not W is a subspace of R^2.

① $\vec{u} = (x_1, x_2)$; $|x_1| = |x_2| \in \vec{w}$

$\vec{v} = (x_1', x_2')$; $|x_1'| = |x_2'| \in \vec{w}$

$\vec{u} + \vec{v} = (x_1 + x_1', x_2 + x_2')$;

$|x_1 + x_1'| = |x_2 + x_2'| \in \vec{w}$

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$\vec{u} = (1, -1) \in \vec{w}$

$\vec{v} = (-2, 2) \in \vec{w}$

$\vec{u} + \vec{v} = (-1, 1) \in \vec{w}$

$2\vec{u} = (2, -2) \in \vec{w}$

② c scalar.

$c\vec{u} = c(x_1, x_2)$

$= (cx_1, cx_2)$; $|cx_1| = |cx_2| \in \vec{w}$

$\Rightarrow$ subspace.

(ii) Determine whether the given vectors $v_1 = (3, -1, -2)$, $v_2 = (-3, 1, -2)$ *and* $v_3 = (1, 0, 0)$ are linearly independent or linearly dependent.

$$\begin{array}{ccc} + & - & + \end{array}$$

$$\begin{vmatrix} 3 & -3 & 1 \\ -1 & 1 & 0 \\ -2 & -2 & 0 \end{vmatrix}$$

$= 3(0 - 0) - (-3)(0) + 1(2 - (-2))$

$= 0 - 0 + 4$

$= 4$

$\neq 0 \Rightarrow$ L. indp.

Question 5:

Find the eigenvalues and associated eigenvectors of the matrix

$$A = \begin{bmatrix} 5 & -9 \\ 2 & -1 \end{bmatrix}$$

$|A - \lambda I| = 0$

$$\begin{vmatrix} 5-\lambda & -9 \\ 2 & -1-\lambda \end{vmatrix}$$

$(5-\lambda)(-1-\lambda) - (-18) = 0$

$-5 - 5\lambda + \lambda + \lambda^2 + 18 = 0$

$1\lambda^2 - 4\lambda + 13 = 0$

$a = 1, \; b = -4, \; c = 13$

$$\lambda = \frac{4 \pm \sqrt{16 - 4(1)(13)}}{2(1)}$$

$$= \frac{4 \pm \sqrt{16-52}}{2}$$

$$= \frac{4 \pm \sqrt{-36}}{2}$$

$$= \frac{4 \pm 6i}{2}$$

$$= \frac{4}{2} \pm \frac{6}{2} i$$

$$= 2 \pm 3i$$

$\boxed{\lambda_1 = 2 + 3i}$

$(A - \lambda I)\vec{v} = \vec{0}$

$$\begin{bmatrix} 5-(2+3i) & -9 \\ 2 & -1-(2+3i) \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 5-2-3i & -9 \\ 2 & -1-2-3i \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3-3i & -9 \\ 2 & -3-3i \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$(3-3i)a - 9b = 0 \quad | \quad 2a + (-3-3i)b = 0$

$(3-3i)a = 9b$

let $a = 9 \Rightarrow b = (3-3i)$

$$\vec{v}_1 = \begin{bmatrix} 9 \\ 3-3i \end{bmatrix}$$

$$\vec{v}_2 = \begin{bmatrix} 9 \\ 3+3i \end{bmatrix}$$

مرافق

Question 6:

Find the general solution of

$$\frac{dx}{dt} = 2x + 3y$$
$$\frac{dy}{dt} = 2x + y$$

$$x' = \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix} x$$

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 2-\lambda & 3 \\ 2 & 1-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)(1-\lambda) - 6 = 0$$
$$2 - 2\lambda - \lambda + \lambda^2 - 6 = 0$$
$$\lambda^2 - 3\lambda - 4 = 0$$
$$(\lambda+1)(\lambda-4) = 0$$

$\boxed{\lambda_1 = -1}$. $\boxed{\lambda_2 = 4}$

$\boxed{\lambda_1 = -1}$ $\quad (A - \lambda I)\vec{v} = \vec{0}$

$$\begin{bmatrix} 3 & 3 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$3a + 3b = 0 \quad | \quad 2a + 2b = 0$

$\frac{3}{3}a = \frac{-3}{3}b$

$1a = -1b$

let $a = -1 \Rightarrow b = 1 \quad \vec{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\boxed{\lambda_2 = 4}$

$$\begin{bmatrix} -2 & 3 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$-2a + 3b = 0 \quad | \quad 2a - 3b = 0$

$\frac{-2a}{-1} = \frac{-3b}{-1}$

$2a = 3b$ let $a = 3 \Rightarrow b = 2$

$\vec{v}_2 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

$$\vec{X} = c_1 \vec{v}_1 e^{\lambda_1 t} + c_2 \vec{v}_2 e^{\lambda_2 t}$$
$$= c_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-1t} + c_2 \begin{bmatrix} 3 \\ 2 \end{bmatrix} e^{4t} \text{ (general sol)}$$

Question 7:

(a) Express the vectors $w = (-1,7)$ as a linear combination of the vectors $u = (1,2)$ and $v = (5,4)$.

$$W = c_1\vec{u} + c_2\vec{v}$$

$$\begin{array}{c} \begin{array}{ccc} u & v & w \end{array} \\ \left[\begin{array}{cc|c} 1 & 5 & -1 \\ 2 & 4 & 7 \end{array}\right] \end{array}$$

$$-2R_1+R_2 \left[\begin{array}{cc|c} 1 & 5 & -1 \\ 0 & -6 & 9 \end{array}\right]$$

$$\frac{R_2}{-6} \left[\begin{array}{cc|c} 1 & 5 & -1 \\ 0 & 1 & \frac{3}{2} \end{array}\right]$$

$$-5R_2+R_1 \left[\begin{array}{cc|c} 1 & 0 & \frac{13}{2} \\ 0 & 1 & \frac{-3}{2} \end{array}\right]$$

$$c_1 = \frac{13}{2}, \quad c_2 = \frac{-3}{2}$$

$$\vec{w} = c_1\vec{u} + c_2\vec{v}$$

$$\vec{w} = \frac{13}{2}\vec{u} - \frac{3}{2}\vec{v}$$

(b) Determine whether the given vectors $v_1 = (1,3,0)$, $v_2 = (1,-2,0)$ *and* $v_3 = (0,4,1)$ are linearly independent or linearly dependent.

$$\begin{array}{ccc} + & - & + \end{array}$$

$$\begin{vmatrix} 1 & 1 & 0 \\ 3 & -2 & 4 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= 1(-2-0) - 1(3-0) + 0$$

$$= 1(-2) - 1(3)$$

$$= -2 - 3$$

$$= -5$$

$\neq 0 \Rightarrow$ L. independent.

Question 8:

Solve the linear first-order equation

$$xdy + (2y - x^2)dx = 0$$

$$x\frac{dy}{dx} + (2y - x^2)\frac{dx}{dx} = 0$$

$$x\frac{dy}{dx} + 2y - x^2 = 0$$

$$x\frac{dy}{dx} + 2y = x^2$$

$$\frac{\cancel{x}}{\cancel{x}}\frac{dy}{dx} + \frac{2}{x}y = \frac{x^2}{x}$$

$$\frac{dy}{dx} + \underbrace{\frac{2}{x}}_{P(x)} y = \underbrace{x}_{Q(x)} \text{ linear.}$$

$$\rho = e^{\int \frac{2}{x}dx} = e^{2\ln x} = e^{\ln x^2} = x^2$$

$$\rho \cdot y = \int \rho \cdot Q(x)\, dx$$

$$x^2 \cdot y = \int x^2 \cdot x\, dx$$

$$x^2 y = \int x^3\, dx$$

$$x^2 y = \frac{x^4}{4} + c.$$

$$y = \frac{x^4}{4x^2} + \frac{c}{x^2} \Rightarrow y = \frac{x^2}{4} + cx^{-2}$$