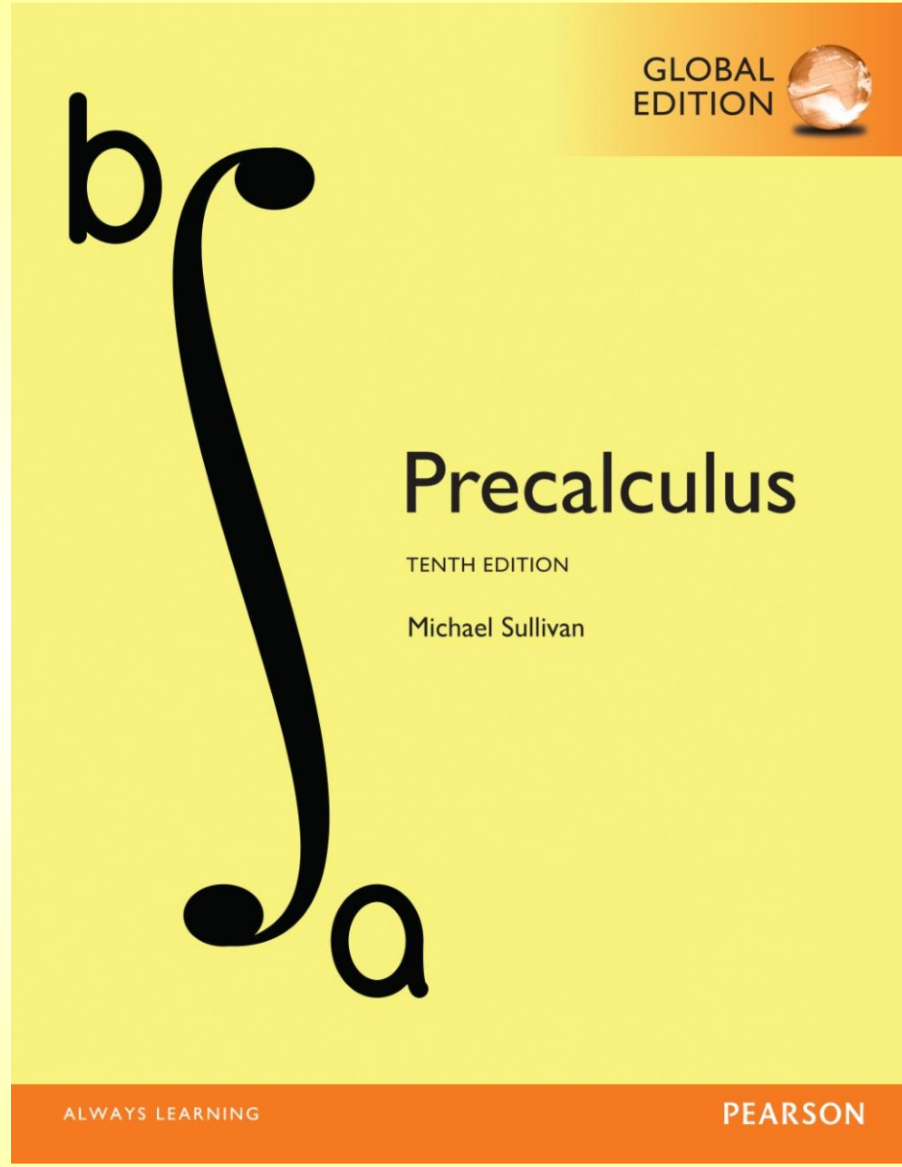
Appendix A

Section A3



A.3 Polynomials

OBJECTIVES

We have described algebra as a generalization of arithmetic in which letters are used to represent real numbers. From now on, we shall use the letters at the end of the alphabet, such as x, y, and z, to represent variables and use the letters at the beginning of the alphabet, such as a, b, and c, to represent constants. In the expressions $3x + 5$ and $ax + b$, it is understood that x is a variable and that a and b are constants, even though the constants a and b are unspecified. As you will find out, the context usually makes the intended meaning clear.

1 Recognize Monomials

DEFINITION

A **monomial** in one variable is the product of a constant and a variable raised to a nonnegative integer power. A monomial is of the form

$$ax^k$$

where a is a constant, x is a variable, and $k \geq 0$ is an integer. The constant a is called the **coefficient** of the monomial. If $a \neq 0$, then k is called the **degree** of the monomial.

NOTE The nonnegative integers are the integers 0, 1, 2, 3, ...

EXAMPLE 1 Examples of Monomials

Monomial	Coefficient	Degree	
(a) $6x^2$	6	2	
(b) $-\sqrt{2}x^3$	$-\sqrt{2}$	3	
(c) 3	3	0	Since $3 = 3 \cdot 1 = 3x^0, x \neq 0$
(d) $-5x$	-5	1	Since $-5x = -5x^1$
(e) x^4	1	4	Since $x^4 = 1 \cdot x^4$

EXAMPLE 2 Examples of Nonmonomial Expressions

(a) $3x^{1/2}$ is not a monomial, since the exponent of the variable x is $\frac{1}{2}$, and $\frac{1}{2}$ is not a nonnegative integer.

(b) $4x^{-3}$ is not a monomial, since the exponent of the variable x is -3, and -3 is not a nonnegative integer.

Now Work PROBLEM 15

2 Recognize Polynomials

Two monomials with the same variable raised to the same power are called **like terms**. For example, $2x^4$ and $-5x^4$ are like terms. In contrast, the monomials $2x^3$ and $2x^5$ are not like terms.

Example :

Which of the following is a monomial

(a) $3x^{\frac{1}{2}}$ **(b)** $4x^{-3}$ **(c)** $-5x^4$ **(d)** $\frac{2}{x}$

Solution :

Recall that x^n is a monomial if $n \in \mathbb{W}$

(a) $3x^{\frac{1}{2}}$

$\frac{1}{2} \notin \mathbb{W}$

$\Rightarrow$ Not a monomial

(b) $4x^{-3}$

$-3 \notin \mathbb{W}$

$\Rightarrow$ Not a monomial

(c) $-5x^4$

$4 \in \mathbb{W}$

$\Rightarrow$ Monomial

(d) $\frac{2}{x} = 2x^{-1}$

$-1 \notin \mathbb{W}$

$\Rightarrow$ Not a monomial

Polynomials

متعدد الحدود في متغير واحد هو تعبير جبري للنموذج

A **polynomial** in one variable is an algebraic expression of the form

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \qquad \textbf{(1)}$$

where $a_n, a_{n-1}, \ldots, a_1, a_0$ are constants,* called the **coefficients** of the polynomial, $n \geq 0$ is an integer, and x is a variable. If $a_n \neq 0$, it is called the **leading coefficient**, $a_n x^n$ is called the **leading term**, and n is the **degree** of the polynomial.

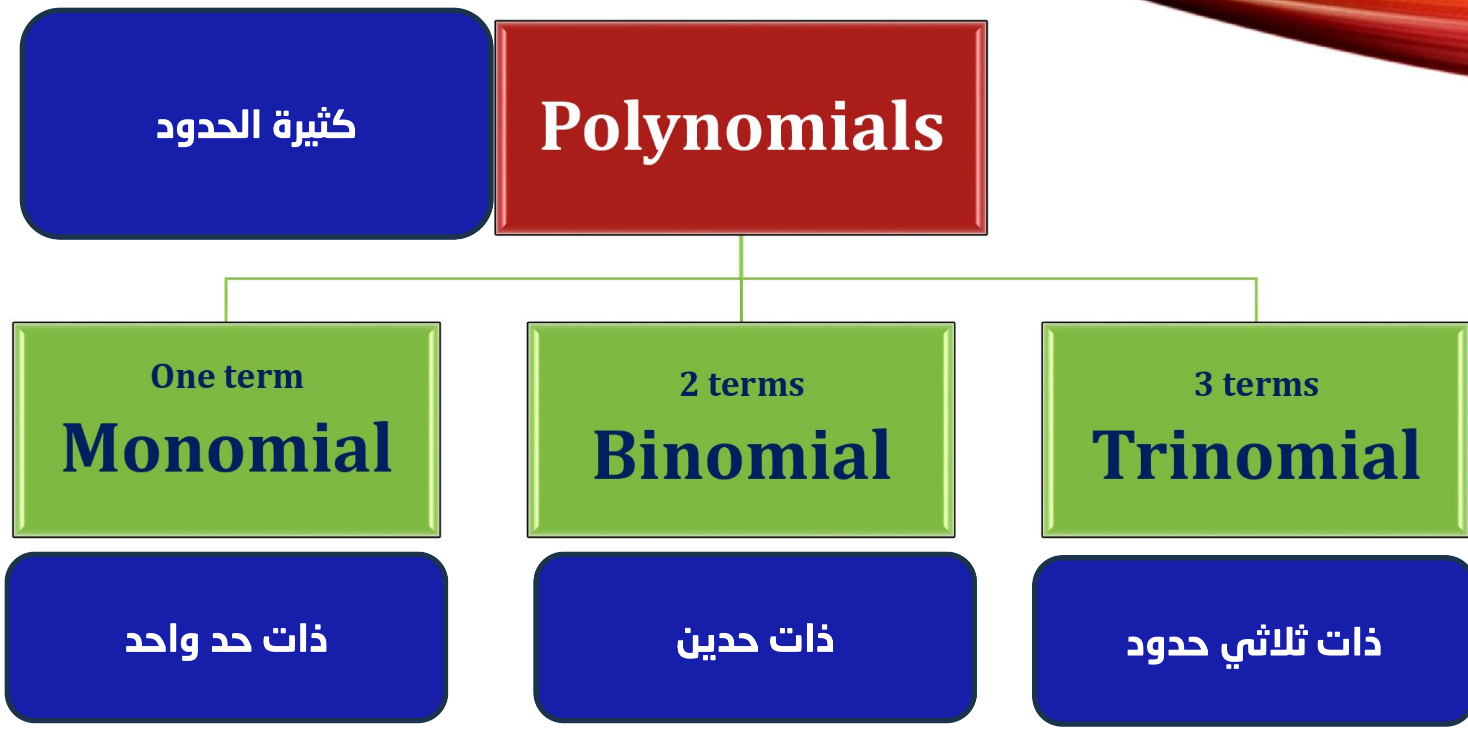
Degree:

- ❖ The degree of a polynomial is the highest exponent.
- ❖ Constant term has zero degree.
- ❖ Zero is a polynomial but has no degree.

❖ درجة كثير الحدود هي أعلى الأس.
❖ الحد الثابت له درجة صفر.
❖ الصفر هو متعدد الحدود ولكن ليس له درجة.

Example 3:

كثيرة الحدود	المعاملات	الدرجة
Polynomial	**Coefficients**	**Degree**
$-8x^3 + 4x^2 - 6x + 2$	$-8, 4, -6, 2$	3
$3x^2 - 5$	$3, -5$	2
$1x^2 - 2x + 8$	$1, -2, 8$	2
$5x^1 + \sqrt{2}$	$5, \sqrt{2}$	1
3	3	0 $(\because 3 \text{ is constant})$
0	0	No degree


كثيرة الحدود
Polynomials
One term
Monomial
2 terms
Binomial
3 terms
Trinomial
ذات حد واحد
ذات حدين
ذات ثلاثي حدود

Example : Classify بسط

(a) $3x^2 - 5x + 9$

(b) $9x + 2$

(c) $x^6 - 5x^4 + 8x^3 + 12$

(d) $-4x^6$

Solution :

(a) $3x^2 - 5x + 9$ Trinomial (3 terms) ٣ حدود

(b) $9x + 2$ Binomial (2 terms) ٢ حدود

(c) $x^6 - 5x^4 + 8x^3 + 12$ Polynomial (more than 3 terms) كثيرة حدود اكثر من ثلاثة

(d) $-4x^6$ Monomial (1 term) ذات حد واحد

Example :

Find the degree and coefficients of Monomials

أوجد درجات ومعاملات أحاديات الحد

(a) $6x^2$ (b) $-\sqrt{2}x^3$ (c) 3

Solution :

(a) $6x^2$

$\text{Coefficient} = 6$

$\text{Degree} = 2$

(b) $-\sqrt{2}x^3$

$\text{Coefficient} = -\sqrt{2}$

$\text{Degree} = 3$

(c) 3

$\text{Coefficient} = 3$

$\text{Degree} = 0$

Example :

أي من الاتية تمثل أحادية الحد

Which of the following is a monomial

(a) $3x^{\frac{1}{2}}$ (b) $4x^{-3}$ (c) $-5x^{4}$ (d) $\frac{2}{x}$

Solution :

$\boxed{\text{Recall that } x^n \text{ is a monomial if } n \in \mathbb{W}}$

(a) $3x^{\frac{1}{2}}$	(b) $4x^{-3}$	(c) $-5x^{4}$	(d) $\frac{2}{x} = 2x^{-1}$
$\frac{1}{2} \notin \mathbb{W}$	$-3 \notin \mathbb{W}$	$4 \in \mathbb{W}$	$-1 \notin \mathbb{W}$
$\Rightarrow$ Not a monomial	$\Rightarrow$ Not a monomial	$\Rightarrow$ Monomial	$\Rightarrow$ Not a monomial

Like terms:

- Note that $2x^3$ *and* $3x^3$ are like terms, but $4x^2$ *and* $4x^3$ are not like
- We can add or subtract like terms as:

$$2x^2 + 5x^2 = (2+5)x^2 = 7x^2$$
$$8x^3 - 5x^3 = (8-5)x^3 = 3x^3$$

لاحظ أن ٢ x3و٣ x3حدود متشابهة، لكن ٤ x2و٤ x3ليست متشابهة.
يمكننا جمع أو طرح الحدود المتشابهة على النحو التالي:

Multiplication of polynomials

FOIL Method

First, Outside, Inside, Last

طريقة ضرب كثيرة الحدود

$$(ax+b)(cx+d) = \underset{F}{acx^2} + \underset{O}{adx} + \underset{I}{bcx} + \underset{L}{bd}$$

(First: $ax \cdot cx$; Outside: $ax \cdot d$; Inside: $b \cdot cx$; Last: $b \cdot d$)

Example :

Multiply the following: (a) $(x+3)(x+1)$

(b) $(2x+1)(3x+4)$

Solution : Using the FOIL multiplication :

(a)

First $= x \cdot x = x^2$

Outside $= x \cdot 1 = x$

Inside $= 3 \cdot x = 3x$

Last $= 3 \cdot 1 = 3$

$$\Rightarrow (x+3)(x+1) = \underset{F}{x^2} + \underset{O}{x} + \underset{I}{3x} + \underset{L}{3}$$

$$= x^2 + 4x + 3$$

(b)

First $= 2x \cdot 3x = 6x^2$

Outside $= 2x \cdot 4 = 8x$

Inside $= 1 \cdot 3x = 3x$

Last $= 1 \cdot 4 = 4$

$$\Rightarrow (2x+1)(3x+4) = \underset{F}{6x^2} + \underset{O}{8x} + \underset{I}{3x} + \underset{L}{4}$$

$$= 6x^2 + 11x + 4$$

Special Product

الفرق بين مربعين

Difference of Two Squares

$$(x - a)(x + a) = x^2 - a^2 \qquad \textbf{(2)}$$

Example : Solve $(x-3)(x+3)$

Solution :

$$\begin{aligned}(x-3)(x+3) &= x^2 - 3^2 \\ &= x^2 - 9\end{aligned}$$

Special Product

مربعات ذات الحدين، أو المربعات الكاملة

Squares of Binomials, or Perfect Squares

$$(x + a)^2 = x^2 + 2ax + a^2 \qquad \textbf{(3a)}$$

$$(x - a)^2 = x^2 - 2ax + a^2 \qquad \textbf{(3b)}$$

Example : **Solve:**

(a) $(x + 2)^2$ **(b)** $(x - 3)^2$

Solution :

(a) $(x + 2)^2 = x^2 + 2(2)x + 2^2$
$= x^2 + 4x + 4$

(b) $(x - 3)^2 = x^2 - 2(3)x + 3^2$
$= x^2 - 6x + 9$

Special Product

مكعبات ذات الحدين، أو المكعبات الكاملة

Cubes of Binomials, or Perfect Cubes

$$(x+a)^3 = x^3 + 3ax^2 + 3a^2x + a^3 \qquad \textbf{(4a)}$$

$$(x-a)^3 = x^3 - 3ax^2 + 3a^2x - a^3 \qquad \textbf{(4b)}$$

Example : **Solve** $(2x+1)^3$

Solution :

$$(\underbrace{2x}_{x} + \underbrace{1}_{a})^3 \overset{(4a)}{=} (2x)^3 + 3(1)(2x)^2 + 3(1)^2(2x) + (1)^3$$

$$= 2^3x^3 + 3\left(4x^2\right) + 3(2x) + 1$$

$$= 8x^3 + 12x^2 + 6x + 1$$

3 Know Formulas for Special Products

Certain products, which we call **special products**, occur frequently in algebra. For example, we can find the product of two binomials using the **FOIL** (*F*irst, *O*uter, *I*nner, *L*ast) method.

$$(ax + b)(cx + d) = ax(cx + d) + b(cx + d)$$
$$= \overset{\text{First}}{ax \cdot cx} + \overset{\text{Outer}}{ax \cdot d} + \overset{\text{Inner}}{b \cdot cx} + \overset{\text{Last}}{b \cdot d}$$
$$= acx^2 + adx + bcx + bd$$
$$= acx^2 + (ad + bc)x + bd$$

EXAMPLE 4 **Using FOIL**

(a) $(x - 3)(x + 3) = \underset{F}{x^2} + \underset{O}{3x} - \underset{I}{3x} - \underset{L}{9} = x^2 - 9$

(b) $(x + 2)^2 = (x + 2)(x + 2) = x^2 + 2x + 2x + 4 = x^2 + 4x + 4$

(c) $(x - 3)^2 = (x - 3)(x - 3) = x^2 - 3x - 3x + 9 = x^2 - 6x + 9$

(d) $(x + 3)(x + 1) = x^2 + x + 3x + 3 = x^2 + 4x + 3$

(e) $(2x + 1)(3x + 4) = 6x^2 + 8x + 3x + 4 = 6x^2 + 11x + 4$

Now Work PROBLEM 45

Some products have been given special names because of their form. In the list that follows, x and a are real numbers.

الفرق بين مربعين

Difference of Two Squares

$$(x - a)(x + a) = x^2 - a^2 \qquad \textbf{(2)}$$

مربعات ذات الحدين، أو المربعات الكاملة

Squares of Binomials, or Perfect Squares

$$(x + a)^2 = x^2 + 2ax + a^2 \qquad \textbf{(3a)}$$
$$(x - a)^2 = x^2 - 2ax + a^2 \qquad \textbf{(3b)}$$

مكعبات ذات الحدين، أو المكعبات الكاملة

Cubes of Binomials, or Perfect Cubes

$$(x + a)^3 = x^3 + 3ax^2 + 3a^2x + a^3 \qquad \textbf{(4a)}$$
$$(x - a)^3 = x^3 - 3ax^2 + 3a^2x - a^3 \qquad \textbf{(4b)}$$

الفرق بين مكعبين

Difference of Two Cubes

$$(x - a)(x^2 + ax + a^2) = x^3 - a^3 \qquad \textbf{(5)}$$

مجموع مكعبين

Sum of Two Cubes

$$(x + a)(x^2 - ax + a^2) = x^3 + a^3 \qquad \textbf{(6)}$$

Now Work PROBLEMS 49, 53, AND 57

Long Division

Example: Divide 842 by 15.

Solution:

$$\begin{array}{r} \times \quad \boxed{56} \\ \boxed{15}\overline{)\boxed{842}} \\ -\ 75\downarrow \\ \hline 92 \\ -\ 90 \\ \hline \boxed{2} \end{array}$$

← Quotient (خارج القسمة)

(المقسوم عليه) Divisor →

← Dividend (المقسوم)

← Remainder (باقي القسمة)

$$\Rightarrow \; 842 = \overset{D}{15}\overset{Q}{(56)} + \overset{R}{2}$$

OR $$\frac{842}{15} = 56 + \frac{2}{15}.$$

Dividing Polynomials

degree $f(x) \geq$ degree $g(x)$

قسمة كثيرة الحدود

$$\frac{f(x)}{g(x)} = g(x)\overline{\big)\, f(x)}^{\,Q(x)}$$

$$\begin{array}{r} \vdots \\ \hline R(x) \end{array}$$

So, $\boxed{f(x) = g(x)Q(x) + R(x)}$ **Or** $\boxed{\frac{f(x)}{g(x)} = Q(x) + \frac{R(x)}{g(x)}}$

Dividing Polynomials

مثال : اوجد خارج قسمة كثيرة الحدود والباقي

Example: Find the quotient and the remainder when

$$\underbrace{3x^3 + 4x^2 + x + 7}_{degree=3} \text{ is divided by } \underbrace{x^2 + 1}_{degree=2}$$

Solution:

$$x^2 + 1 \overline{)3x^3 + 4x^2 + x + 7}$$

1. Divide
2. Multiply
3. Subtract

Dividing Polynomials

مثال : اوجد خارج قسمة كثيرة الحدود والباقي

Example: Find the quotient and the remainder when

$$\underbrace{3x^3 + 4x^2 + x + 7}_{degree = 3} \text{ is divided by } \underbrace{x^2 + 1}_{degree = 2}$$

Solution:

$$\begin{array}{r} 3x \\ \boxed{x^2} + 1\overline{)\boxed{3x^3} + 4x^2 + x + 7} \end{array}$$

1. Divide
2. Multiply
3. Subtract

$$\frac{3\cancel{x^3}}{\cancel{x^2}} = 3x$$

Dividing Polynomials

مثال : اوجد خارج قسمة كثيرة الحدود والباقي

Example: Find the quotient and the remainder when

$$\underbrace{3x^3 + 4x^2 + x + 7}_{degree = 3} \text{ is divided by } \underbrace{x^2 + 1}_{degree = 2}$$

Solution:

$$\begin{array}{r|l} & \times\ \boxed{3x} \\ \hline \boxed{x^2 + 1} & 3x^3 + 4x^2 + x + 7 \\ & 3x^3 + 3x \end{array}$$

1. Divide
2. Multiply
3. Subtract

$$\frac{3x^{\cancel{3}}}{\cancel{x^2}} = 3x$$

$$3x(x^2 + 1) = 3x^3 + 3x$$

Dividing Polynomials

مثال : اوجد خارج قسمة كثيرة الحدود والباقي

Example: Find the quotient and the remainder when

$$\underbrace{3x^3 + 4x^2 + x + 7}_{degree = 3} \text{ is divided by } \underbrace{x^2 + 1}_{degree = 2}$$

Solution:

$$\begin{array}{r l}
 & 3x \\
x^2 + 1 \,\big) & 3x^3 + 4x^2 + x + 7 \\
- & \cancel{3}x^3 + 3x \\
\hline
 & 4x^2 \boxed{-3x + x} + 7 \\
 & 4x^2 \boxed{-2x} + 7
\end{array}$$

1. Divide
2. Multiply
3. Subtract

$$\frac{3x^{\cancel{3}}}{\cancel{x^2}} = 3x$$

$$3x(x^2 + 1) = 3x^3 + 3x$$

Dividing Polynomials

مثال : اوجد خارج قسمة كثيرة الحدود والباقي

Example: Find the quotient and the remainder when

$$\underbrace{3x^3 + 4x^2 + x + 7}_{degree\,=\,3} \text{ is divided by } \underbrace{x^2 + 1}_{degree\,=2}$$

Solution:

$$\begin{array}{r l}
 & 3x + 4 \\
\boxed{x^2} + 1 \,\big) & 3\cancel{x}^3 + 4x^2 + x + 7 \\
- & \cancel{3}x^3 + 3x \qquad \downarrow \\
\hline
 & 4x^2 - 3x + x + 7 \\
\hline
 & \boxed{4x^2} - 2x + 7
\end{array}$$

1. Divide
2. Multiply
3. Subtract

$$\frac{3x^{\cancel{3}}}{\cancel{x}^2} = 3x$$

$$3x(x^2 + 1) = 3x^3 + 3x$$

$$\frac{4\cancel{x}^2}{\cancel{x}^2} = 4$$

Dividing Polynomials

مثال : اوجد خارج قسمة كثيرة الحدود والباقي

Example: Find the quotient and the remainder when

$$\underbrace{3x^3 + 4x^2 + x + 7}_{degree = 3} \text{ is divided by } \underbrace{x^2 + 1}_{degree = 2}$$

Solution:

$$\begin{array}{r l}
\times & 3x \boxed{+ 4} \\
\boxed{x^2 + 1} & \overline{)\,3x^3 + 4x^2 + x + 7} \\
- & \cancel{3}x^3 + 3x \quad \downarrow \\
\hline
 & 4x^2 - 3x + x + 7 \\
\hline
 & 4x^2 - 2x + 7 \\
 & 4x^2 + 4
\end{array}$$

1. Divide
2. Multiply
3. Subtract

$$\frac{3x^{\cancel{3}}}{\cancel{x^2}} = 3x$$

$$3x(x^2 + 1) = 3x^3 + 3x$$

$$\frac{4\cancel{x^2}}{\cancel{x^2}} = 4$$

$$4(x^2 + 1) = 4x^2 + 4$$

Dividing Polynomials

مثال : اوجد خارج قسمة كثيرة الحدود والباقي

Example: Find the quotient and the remainder when

$$\underbrace{3x^3 + 4x^2 + x + 7}_{degree = 3} \text{ is divided by } \underbrace{x^2 + 1}_{degree = 2}$$

Solution:

$$\boxed{3x + 4} \leftarrow \mathbf{Q}$$

$$
\begin{array}{r}
x^2 + 1 \overline{)\, 3x^3 + 4x^2 + x + 7} \\
- \quad \cancel{3x^3} + 3x \quad \downarrow \\
\hline
4x^2 - 3x + x + 7 \\
\hline
\cancel{4x^2} - 2x + 7 \\
- \quad \cancel{4x^2} + 4 \\
\hline
-2x + 7 - 4 \\
\hline
\end{array}
$$

$$\boxed{-2x + 3} \leftarrow \mathbf{R}$$

1. Divide
2. Multiply
3. Subtract

$$\frac{3\cancel{x^3}}{\cancel{x^2}} = 3x$$

$$3x(x^2 + 1) = 3x^3 + 3x$$

$$\frac{4\cancel{x^2}}{\cancel{x^2}} = 4$$

$$4(x^2 + 1) = 4x^2 + 4$$

$\Rightarrow$ **quotient** $= 3x + 4$

remainder $= -2x + 3$

Dividing Polynomials

مثال : اوجد خارج قسمة كثيرة الحدود والباقي

Example: Find the quotient and the remainder when

$$x^4 - 3x^3 + 2x - 5 \quad \text{is divided by} \quad x^2 - x + 1$$

Solution:

Dividing Polynomials

مثال : اوجد خارج قسمة كثيرة الحدود والباقي

Example: Find the quotient and the remainder when

$$\underbrace{x^4 - 3x^3 + 2x - 5}_{degree = 4} \text{ is divided by } \underbrace{x^2 - x + 1}_{degree = 2}$$

Solution:

$$\boxed{x^2 - 2x - 3} \leftarrow Q$$

$$
\begin{array}{r l}
x^2 - x + 1 \,\big)\, & x^4 - 3x^3 \qquad + 2x - 5 \\
- & \cancel{x^4} - x^3 + x^2 \qquad \downarrow \\
\hline
- & -2\cancel{x^3} - x^2 + 2x - 5 \\
 & -2x^3 + 2x^2 - 2x \\
\hline
- & -3\cancel{x^2} + 4x - 5 \\
 & -3x^2 + 3x - 3 \\
\hline
 & \boxed{x - 2} \leftarrow R
\end{array}
$$

1. Divide
2. Multiply
3. Subtract

$\Rightarrow$ **quotient** $= x^2 - 2x - 3$

remainder$= x - 2$

Factoring Polynomials

تحليل كثيرة الحدود

Example: **Identifying common factors**

مثال: تحديد العوامل المشتركة

كثيرة الحدود Polynomial	عامل مشترك *Common factor*	العامل المتبقي Remaining factor	الشكل العام Factored form
$2x + 4$			
$3x - 6$			
$2x^2 - 4x + 8$			
$8x - 12$			
$x^2 + x$			
$x^3 - 3x^2$			
$6x^2 + 9x$			

Factoring Polynomials

تحليل كثيرة الحدود

Example: Identifying common factors

مثال: تحديد العوامل المشتركة

كثيرة الحدود Polynomial	عامل مشترك *Common factor*	العامل المتبقي Remaining factor	الشكل العام Factored form
$2x + 4$	2	$x + 2$	$2(x + 2)$
$3x - 6$	3	$x - 2$	$3(x - 2)$
$2x^2 - 4x + 8$	2	$x^2 - 2x + 4$	$2(x^2 - 2x + 4)$
$8x - 12$	4	$2x - 3$	$4(2x - 3)$
$x^2 + x$	x	$x + 1$	$x(x + 1)$
$x^3 - 3x^2$	x^2	$x - 3$	$x^2(x - 3)$
$6x^2 + 9x$	$3x$	$2x + 3$	$3x(2x + 3)$