§ 0.1 Canonical Transformations

Given a physical system that solves Hamilton's canonical equation, since the canonical variables (p_μ, q^ν) don't hold any intrinsic meaning, it's possible to find a new canonical coordinate set (P_μ, Q^ν) that represent the same state (we take t as a parameter during the coordinate transformation).
By definition, we must have a diffeomorphism between these two coordinate systems, therefore

$$\det\left|\frac{\partial(P_\mu, Q^\nu)}{\partial(p_\mu, q^\nu)}\right| \neq 0 \tag{1}$$

Where the reversibility condition must be satisfied. Note that in general, the system in the new coordinates (P_μ, Q^ν) is not Hamiltonian.

Definition 0.1.1 (Canonical Transformation)**.** A canonical transformation is a coordinate transformation such that

$$\mathcal{H}(p_\mu, q^\nu, t) \to \tilde{\mathcal{H}}(P_\mu, Q^\nu, t) \tag{2}$$

Where

$$\begin{cases} \dfrac{\partial \tilde{\mathcal{H}}}{\partial P_\mu} = \dot{Q}^\mu \\ \dfrac{\partial \tilde{\mathcal{H}}}{\partial Q^\mu} = -\dot{P}_\mu \end{cases} \tag{3}$$

In general $\mathcal{H} \neq \tilde{\mathcal{H}}$, but if $\mathcal{H} = \tilde{\mathcal{H}}$ the transformation is said to be «fully canonical»

§§ 0.1.1 Generating Functions of Canonical Transformation

Theorem 0.1 (Lie Condition)**.** *Given an invertible transformation $(p_\mu, q^\nu) \to (P_\mu, Q^\nu)$, the transformation is canonical if and only if, given $\lambda \in \mathbb{R},\ F, \psi : \Gamma^{2n} \to \mathbb{R}$, then it also maps $p_\mu \mathrm{d}q^\mu$ as follows*

$$p_\mu \mathrm{d}q^\mu \to \lambda P_\mu \mathrm{d}Q^\mu + \psi(P_\mu, Q^\nu, t)\mathrm{d}t + \mathrm{d}F \tag{4}$$

Or, in other words, it's necessary to verify that the following differential form is exact

$$p_\mu \mathrm{d}q^\mu - \lambda P_\mu \mathrm{d}Q^\mu = \psi \mathrm{d}t + \mathrm{d}F \tag{5}$$

Using Lie's condition, it's possible to define 4 different generating functions of canonical transformations, F_1, F_2, F_3, F_4, tied between themselves via Legendre transforms.

1. Generator of the 1st kind $F_1(q^\mu, Q^\nu, t)$. Lie's conditions becomes

$$p_\mu \mathrm{d}q^\mu - P_\mu \mathrm{d}Q^\mu = \psi \mathrm{d}t + \frac{\partial F_1}{\partial q^\mu}\mathrm{d}q^\mu + \frac{\partial F_1}{\partial Q^\mu}\mathrm{d}Q^\mu + \frac{\partial F_1}{\partial t}\mathrm{d}t \tag{6}$$

   In order to satisfy the theorem, it must hold that

$$\frac{\partial F_1}{\partial q^\mu} = p_\mu, \quad \frac{\partial F_1}{\partial Q^\mu} = -P_\mu, \quad \frac{\partial F_1}{\partial t} = -\psi \tag{7}$$

2. Generator of the 2nd kind $F_2(q^\mu, P_\nu, t)$

$$p_\mu \mathrm{d}q^\mu + Q^\mu \mathrm{d}P_\mu = \psi \mathrm{d}t + \frac{\partial F_2}{\partial q^\mu}\mathrm{d}q^\mu + \frac{\partial F_2}{\partial P_\mu}\mathrm{d}P_\mu + \frac{\partial F_2}{\partial t}\mathrm{d}t \tag{8}$$

I.e.

$$\frac{\partial F_2}{\partial q^\mu} = p_\mu, \quad \frac{\partial F_2}{\partial P_\mu} = Q^\mu, \quad \frac{\partial F_2}{\partial t} = -\psi \tag{9}$$

3. Generator of the 3rd kind $F_3(p_\mu, Q^\nu, t)$

$$q^\mu \mathrm{d}p_\mu + P_\mu \mathrm{d}Q^\mu = -\psi \mathrm{d}t - \frac{\partial F_3}{\partial p_\mu}\mathrm{d}p_\mu - \frac{\partial F_3}{\partial Q^\mu}\mathrm{d}Q^\mu - \frac{\partial F_3}{\partial t}\mathrm{d}t \tag{10}$$

Therefore

$$\frac{\partial F_3}{\partial p_\mu} = -q^\mu, \quad \frac{\partial F_3}{\partial Q^\mu} = -P_\mu, \quad \frac{\partial F_3}{\partial t} = -\psi \tag{11}$$

4. Generator of the 4th kind $F_4(p_\mu, P_\nu, t)$

$$q^\mu \mathrm{d}p_\mu - Q^\mu \mathrm{d}P_\mu = -\psi \mathrm{d}t - \frac{\partial F_4}{\partial p_\mu}\mathrm{d}p_\mu - \frac{\partial F_4}{\partial P_\mu}\mathrm{d}P_\mu - \frac{\partial F_4}{\partial t}\mathrm{d}t \tag{12}$$

Which means

$$\frac{\partial F_4}{\partial p_\mu} = -q^\mu, \quad \frac{\partial F_4}{\partial P_\mu} = -Q^\mu, \quad \frac{\partial F_4}{\partial t} = -\psi \tag{13}$$

Since $\tilde{\mathcal{H}} = \mathcal{H} - \psi$, it's obvious that if the generator function is stationary $\partial_t F_i = 0$, then the transformation is fully canonical.
A better way to see the previous list is as a series of Legendre transforms from (p_μ, q^ν) till (P_μ, Q^μ). In fact, we can write

$$\begin{aligned} F_2 &= F_1 + P_\mu Q^\mu \\ F_3 &= F_1 - p_\mu q^\mu \\ F_4 &= F_1 + P_\mu Q^\mu - p_\mu q^\mu \end{aligned} \tag{14}$$

§ 0.2 Poisson Brackets and Liouville's Theorem

Definition 0.2.1 (Poisson Brackets)**.** The space Γ^{2n} comes equipped with a bilinear transformation called the «Poisson brackets» .
Consider a function $f : \Gamma^{2n} \to \mathbb{R}$, then its total derivative with respect to time will be

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial p_\mu}\dot{p}_\mu + \frac{\partial f}{\partial q^\mu}\dot{q}^\mu \tag{15}$$

Substituting Hamilton's equations we have

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial q^\mu}\frac{\partial \mathcal{H}}{\partial p_\mu} - \frac{\partial f}{\partial q^\mu}\frac{\partial \mathcal{H}}{\partial p_\mu} = \frac{\partial f}{\partial t} + \{\mathcal{H}, f\} \tag{16}$$

Where we defined the poisson brackets as

$$\{\mathcal{H}, f\} = \frac{\partial \mathcal{H}}{\partial p_\mu}\frac{\partial f}{\partial q^\mu} - \frac{\partial \mathcal{H}}{\partial q^\mu}\frac{\partial f}{\partial p_\mu} \tag{17}$$

This operator is obviously bilinear and antisymmetric, in fact

$$\{f, \mathcal{H}\} = \frac{\partial f}{\partial p_\mu}\frac{\partial \mathcal{H}}{\partial q^\mu} - \frac{\partial f}{\partial q^\mu}\frac{\partial \mathcal{H}}{\partial p_\mu} = -\left(\frac{\partial \mathcal{H}}{\partial p_\mu}\frac{\partial f}{\partial q^\mu} - \frac{\partial \mathcal{H}}{\partial q^\mu}\frac{\partial f}{\partial p_\mu}\right) = -\{\mathcal{H}, f\}$$

Through this quick definition of this operator, one can immediately say, that if f is an integral of motion, one must have

$$\frac{\partial f}{\partial t} + \{\mathcal{H}, f\} = 0 \tag{18}$$

This operator can be directly generalized to two functions $g, h : \Gamma^{2n} \to \mathbb{R}$ as follows

$$\{g, h\} = \frac{\partial g}{\partial p_\mu}\frac{\partial h}{\partial q^\mu} - \frac{\partial g}{\partial q^\mu}\frac{\partial h}{\partial p_\mu} \tag{19}$$

Note that applying this operator to the canonical coordinates we obtain the two main properties of such

$$\begin{cases} \{q^\mu, q^\nu\} = \{p_\mu, p_\nu\} = 0 \\ \{p_\mu, q^\nu\} = \delta^\nu_\mu \end{cases} \tag{20}$$

It's also possible to derive the following identity through iteration, called the Jacobi identity

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0 \tag{21}$$

Applying canonical transformations to the definition of Poisson brackets it's possible to find more direct approaches for determining whether a transformation is canonical or not, using the following theorems

Theorem 0.2 (Invariance of Poisson Brackets)**.** ***Given two stationary functions*** $f, g : \Gamma^{2n} \to \mathbb{R}$ ***and a canonical transformation*** $(p_\mu, q^\nu) \to (P_\mu, Q^\nu)$***, such that***

$$\begin{aligned} \tilde{f}(P_\mu, Q^\nu) &= f\left(p_\mu(P, Q), q^\mu(P, Q)\right) \\ \tilde{g}(P_\mu, Q^\nu) &= q\left(p_\mu(P, Q), q^\mu(P, Q)\right) \end{aligned}$$

Then, if we define $\{\cdot, \cdot\}_{PQ}$ ***as the Poisson brackets in the new coordinate system, then***

$$\left\{\tilde{f}, \tilde{g}\right\}_{PQ} = \{f, g\} \tag{22}$$

I.e. Poisson brackets are invariant to canonical transformations.

Proof. Supposing that g is the Hamiltonian of some system, we can write

$$\{f, g\} = \frac{\mathrm{d}f}{\mathrm{d}t}$$

This implies that $\tilde{g}$ is the transformed Hamiltonian, therefore

$$\left\{\tilde{f}, \tilde{g}\right\}_{PQ} = \frac{\mathrm{d}\tilde{f}}{\mathrm{d}t}$$

Since canonical transformation preserve Hamilton's equations we must have

$$\frac{\mathrm{d}\tilde{f}}{\mathrm{d}t} = \frac{\mathrm{d}f}{\mathrm{d}t}$$

Which implies the statement of the theorem

$$\left\{\tilde{f}, \tilde{g}\right\}_{PQ} = \{f, g\}$$

This also proves that

$$\{Q^\mu, Q^\nu\}_{PQ} = \{P_\mu, P_\nu\}_{PQ} = 0$$

And

$$\{P_\nu, Q^\mu\}_{PQ} = \delta^\mu_\nu$$

□

Theorem 0.3 (Lie Condition on Poisson Brackets)**.** ***Given a transformation*** $(p_\mu, q^\nu) \to (P_\mu, Q^\nu)$***, it is canonical if and only if***

$$\begin{cases} \{Q^\mu, Q^\nu\} = \{P_\mu, P_\nu\} = 0 \\ \{P_\nu, Q^\mu\} = \delta^\mu_\nu \end{cases} \tag{23}$$

Another theorem that can be inferred is the so-called Liouville theorem, which states that an infinitesimal volume in the phase space is invariant to canonical transformations

Theorem 0.4 (Liouville)**.** ***Given an infinitesimal volume in*** Γ^{2n}***,*** $\mathrm{d}\Gamma = \mathrm{d}^n p\, \mathrm{d}^n q$***, then applying a canonical transformation we must have***

$$\mathrm{d}\tilde{\Gamma} = \mathrm{d}^n P\, \mathrm{d}^n Q = \mathrm{d}^n p\, \mathrm{d}^n q = \mathrm{d}\Gamma$$

Where if J ***is the determinant of the Jacobian, we must have*** $J = 1$

Proof. In order for the theorem to be demonstrated we must prove that

$$\int \mathrm{d}\tilde{\Gamma} = \int J \mathrm{d}\Gamma, \quad J = 1$$

By definition, we can write the determinant of the Jacobian as follows

$$J = \det\left|\frac{\partial(P_\mu, Q^\nu)}{\partial(p_\mu, q^\nu)}\right|$$

From here we write two intermediate canonical transformations and write the new Jacobian matrix as the product of the two matrices of the intermediate transformations.
We have, choosing the transformations $(p_\mu, q^\nu) \to (P_\mu, q^\nu) \to (P_\mu, Q^\nu)$, that our Jacobian can be written as follows

$$J = \det \left| \frac{\partial(P_\mu, Q^\nu)}{\partial(P_\mu, q^\nu)} \frac{\partial(P_\mu, q^\nu)}{\partial(p_\nu, q^\mu)} \right|$$

Simplifying the equal rows we have that

$$J = \det \left| \frac{\partial Q^\mu}{\partial q^\nu} \frac{\partial P_\nu}{\partial p_\mu} \right|$$

Imposing that the transformation is canonical we must have that it comes from a F_2 generating function, so that, using (9)

$$\frac{\partial Q^\mu}{\partial q^\nu} = \frac{\partial^2 F_2}{\partial q^\mu \partial P_\nu}, \quad \frac{\partial p_\mu}{\partial P_\nu} = \frac{\partial^2 F_2}{\partial P_\nu \partial q^\mu}$$

Imposing that the determinant of the Hessian of the generating function is some number d, we have

$$J = d/d = 1$$

Therefore

$$\int \mathrm{d}\tilde{\Gamma} = \int J \mathrm{d}\Gamma = \int \mathrm{d}\Gamma$$

□

§§ 0.2.1 Poincaré Recurrence

This theorem gives rise to a paradox known as ***Poincaré's recurrence theorem***. Basically this theorem states, against common sense, that an autonomous (time-independent) Hamiltonian system with some initial conditions (p^0_μ, q^ν_0) will evolve till returning to the initial conditions at some finite time t. Technically we have

Theorem 0.5 (Poincaré Recurrence Theorem)**.** *Given an autonomous Hamiltonian system confined in a subset $\Lambda \subset \Gamma^{2n}$ with some initial condition $x^\mu_0 \in \Lambda$, if we evolve $x^\mu_0 \to x^\mu(t)$ then*

$$\forall \tau \in \mathbb{R}\ \exists t^\star > \tau\ :\ \forall \epsilon > 0\ B_\epsilon(x^\mu(t^\star)) \cap B_\epsilon(x^\mu_0) \neq \{\}$$

I.e.

$$\forall \epsilon > 0\ x(t^\star) \in B_\epsilon(x_0)$$

Where $B_\epsilon(x^\mu)$ is the open ball centered in x^μ with radius ϵ

Proof. Begin by defining a sequence of times t_n such that $x^\mu(t_n) = x^\mu_n$, then

$$\exists n_1 \neq n_2 \in \mathbb{N}\ :\ B_\epsilon(x^\mu_{n_1}) \cap B_\epsilon(x^\mu_{n_2}) = \{\}$$

Defining a measure μ on the phase space we must have, that after n iterations, the total path measure will be

$$\mu\left(\bigsqcup_{i=1}^{n} B_\epsilon(x^\mu_i)\right) = \sum_{i=1}^{n} \mu\left(B_\epsilon(x^\mu_i)\right)$$

Considering time as a completely canonical transformation, we have for Liouville's theorem that

$$\forall i \neq j = 1, \cdots, n \quad \mu\left(B_\epsilon(x_i^\mu)\right) = \mu\left(B_\epsilon(x_j^\mu)\right)$$

Which implies that for $n \to \infty$

$$\mu\left(\bigsqcup_{i=1}^{\infty} B_\epsilon(x_i^\mu)\right) = \sum_{i=1}^{\infty} \mu\left(B_\epsilon(x_i^\mu)\right) \to \infty$$

This cannot be true, since by hypothesis we have that the motion of the system is confined in a set $\Lambda \subset \Gamma^{2n}$, therefore

$$\bigsqcup_{i=1}^{\infty} B_\epsilon(x_i^\mu) \subseteq \Lambda$$

In terms of measures this means

$$\mu\left(\bigsqcup_{i=1}^{\infty} B_\epsilon(x_i^\mu)\right) = \sum_{i=1}^{\infty} \mu\left(B_\epsilon(x_i^\mu)\right) \leq \mu\left(\Lambda\right) < \infty$$

Which implies that there must exists some set such that the intersection is not null, therefore there must exist, for some $n_1, n_2, k \in \mathbb{N}$

$$\mu\left(B_\epsilon(x_{n_1}^\mu) \cap B_\epsilon(x_{n_2}^\mu)\right) = \mu\left(B_\epsilon(x_{n_1-k}^\mu) \cap B_\epsilon(x_{n_2-k}^\mu)\right) \neq 0$$

Choosing $k = \min\{n_1, n_2\} = n_2$, where we supposed $n_2 < n_1$ we have

$$\mu\left(B_\epsilon(x_{n_1}^\mu) \cap B_\epsilon(x_{n_2}^\mu)\right) = \mu\left(B_\epsilon(x_{n_1-n_2}^\mu) \cap B_\epsilon(x_0^\mu)\right) \neq \{\}$$

Since $x_{n_1-n_2}^\mu = x^\mu(t_{n_1-n_2})$ and choosing $t^\star = t_{n_1-n_2}$ we have that for $t = t^\star$ the system will find itself in a ball of radius $\epsilon > 0$ from the initial value x_0^μ

$$B_\epsilon(x^\mu(t^\star)) \cap B_\epsilon(x_0^\mu) \neq \{\}$$

$\square$

§ 0.3 Hamilton-Jacobi Method

A really good use for the canonical transformation is to find a quick and trivial solution to Hamilton-Jacobi's equation.
The differential equation we intend to solve is the following

$$\frac{\partial \mathcal{S}}{\partial t} + \mathcal{H}\left(\frac{\partial \mathcal{S}}{\partial q^\mu}, q^\mu, t\right) = 0 \tag{24}$$

The complete solution of this equation can be inferred to be a function of the coordinates and $n+1$ parameters corresponding to the independent variables of the system, including time. Therefore we might write

$$\mathcal{S}(q^\mu, t) = f\left(q^\mu, t; \alpha_1, \cdots, \alpha_n\right) + A \tag{25}$$

Where $t, \alpha_1, \cdots, \alpha_n, A \in \mathbb{R}$ are our parameters.
We can choose now a canonical transformation to a new set of variables (α_μ, β^μ) that give the following relations

$$\frac{\partial f}{\partial q^\mu} = p_\mu, \quad \frac{\partial f}{\partial \alpha_\mu} = \beta^\mu, \quad \tilde{\mathcal{H}} = \mathcal{H} + \frac{\partial f}{\partial t} \tag{26}$$

Note that for our hypothesis f is a complete solution of Hamilton-Jacobi, therefore the last relation gives

$$\tilde{\mathcal{H}} = 0$$

Basically, with this canonical transformation, we mapped our Hamiltonian to a null Hamiltonian, for which the equations of motion are trivial, giving in the new variables $\alpha_\mu, \beta^\mu =$constant.
Using the definition of β^μ via the transformation, and using the reversibility of such, we can determine the q^μ and the analytical form of our complete solutions.