§ 0.1 Sequences of Functions

Definition 0.1.1 (Sequence of Functions)**.** Let S be a set and (X, d) a metric space, a *sequence of functions* is defined as follows

$$\begin{aligned} f_n :& S \longrightarrow (X, d) \\ & s \to f_n(s) \end{aligned} \tag{1}$$

Where, $\forall n \in \mathbb{N}$ a function $f_{(n)} : S \longrightarrow (X, d)$ is defined

Definition 0.1.2 (Pointwise Convergence)**.** A sequence of functions $(f_n)_{n\geq 0}$ is said to converge pointwise to a function $f : S \longrightarrow (X, d)$, and it's indicated as $f_n \to f$, if

$$\forall \epsilon > 0,\ \forall x \in S\ \exists N_\epsilon(x) \in \mathbb{N} \,:\, d(f_n(x), f(x)) < \epsilon\ \forall n \geq N_\epsilon(x) \tag{2}$$

It can be indicated also as follows

$$\lim_{n\to\infty} (f_n(x)) = f(x) \tag{3}$$

Definition 0.1.3 (Uniform Convergence)**.** Defining an $\|\cdot\|_\infty = \sup_{i\leq n} |\cdot|$ we have that the convergence of a sequence of functions is uniform, and it's indicated as $f_n \rightrightarrows f$, iff

$$\forall \epsilon > 0\ \exists N_\epsilon \in \mathbb{N} \,:\, d(f_n(x), f(x)) < \epsilon\ \forall n \geq N_\epsilon\ \forall x \in S \tag{4}$$

Or, using the norm $\|\cdot\|_\infty$

$$\forall \epsilon > 0\ \exists N_\epsilon \in \mathbb{N} \,:\, \|f_n - f\|_\infty < \epsilon \tag{5}$$

Theorem 0.1 (Continuity of Uniformly Convergent Sequences)**.** *Let $(f_n)_{n\geq 0} : (S, d_S) \longrightarrow (X, d)$ be a sequence of continuous functions. Then if $f_n \rightrightarrows f$, we have that $f \in C(S)$, where $C(S)$ is the space of continuous functions*

Proof.

$$\begin{aligned} &\forall x \in S,\ \exists \epsilon > 0 \,:\, f_n \rightrightarrows f,\ \therefore \forall n \geq N_\epsilon \in \mathbb{N} \,:\, d(f_n(x), f(x)) < \frac{\epsilon}{3} \\ &f_n \in C(S) \implies \exists \delta_\epsilon > 0 \,:\, d(f_n(x), f_n(y)) < \frac{\epsilon}{3},\ \forall x, y \in S \,:\, d_S(x, y) < \delta \\ &\therefore d(f(x), f(y)) \leq d(f(x), f_n(x)) + d(f_n(x), f_n(y)) + d(f_n(y), f(y)) < \epsilon \iff d_S(x, y) < \delta_\epsilon \end{aligned} \tag{6}$$

□

Theorem 0.2 (Integration of Sequences of Functions)**.** *Let $(f_n)_{n\geq 0}$ be a sequence of functions such that $f_n \rightrightarrows f$ Then we can define the following equality*

$$\lim_{n\to\infty} \int_a^b f_n(x)\,\mathrm{d}x = \int_a^b \lim_{n\to\infty} f_n(x)\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x \tag{7}$$

Proof. We already know that in the closed set $[a, b]$ we can say, since $f_n \rightrightarrows f$, that

$$\forall \epsilon > 0\ \exists N_\epsilon \in \mathbb{N} \,:\, \forall n \geq N_\epsilon\ \|f_n - f\|_\infty < \frac{\epsilon}{b - a} \tag{8}$$

Then, we have that

$$\forall n \geq N_\epsilon \left| \int_a^b f_n(x)\,\mathrm{d}x - \int_a^b f(x)\,\mathrm{d}x \right| \leq \|f_n - f\|_\infty (b-a) < \epsilon \tag{9}$$

$\square$

Theorem 0.3 (Differentiation of a Sequence of Functions)**.** *Define a sequence of functions as* $f_n : I \longrightarrow \mathbb{R}$, *with* $f_n(x) \in C^1(I)$. *If*

1. $\exists x_0 \in I \,:\, f_n(x_0) \to l$
2. $f_n' \rightrightarrows g \;\forall x \in I$

Then

$$f_n(x) \rightrightarrows f \implies \forall x \in I,\; f'(x) = \lim_{n\to\infty} f_n'(x) = g(x) \tag{10}$$

Proof. For the fundamental theorem of integral calculus, we can write, using the regularity of the $f_n(x)$ that

$$f_n(x) = f_n(x_0) + \int_{x_0}^x f_n(t)\,\mathrm{d}t$$

Taking the limit we have

$$\lim_{n\to\infty} f_n(x) = l + \int_{x_0}^x g(t)\,\mathrm{d}t = f(x)$$
$$\therefore f'(t) = g(t)$$

But, we also have that

$$\forall \epsilon > 0 \;\|f_n' - f'\|_\infty \leq |f_n(x_0) - l| + \|f_n' - g\|_\infty (b-a) < \epsilon$$
$$\therefore f_n \rightrightarrows f,\; f_n' \rightrightarrows f'$$

$\square$

§ 0.2 Series of Functions

Let now, for the rest of the section, $(X,d) = \mathbb{C}$.

Definition 0.2.1 (Series of Functions)**.** Let $(f_n)_{n\geq 0} \in \mathbb{C}$ be a sequence of functions, such that $f_n : S \to \mathbb{C}$. We can define the *series of functions* as follows

$$s_n(x) = \sum_{k=1}^n f_k(x) \tag{11}$$

Definition 0.2.2 (Convergent Series)**.** A series of functions $s_n(x) : S \to \mathbb{C}$ is said to be *convergent* or *pointwise convergent* if

$$s_n(x) = \sum_{k=0}^n f_k(x) \longrightarrow s(x) \tag{12}$$

Where $s(x) : S \to \mathbb{C}$ is the *sum* of the series.
This means that

$$\forall x \in S,\ \lim_{k\to\infty} s_k(x) = \sum_{k=0}^{\infty} f_k(x) = s(x) \tag{13}$$

Theorem 0.4. *Necessary Condition for the convergence of a series of functions:*
Let $(f_n) \in \mathbb{C}$ be a succession, then the series $s_n(x)$ defined as follows, converges to the function $s(x)$

$$s_n(x) = \sum_{k=0}^{n} f_k(x) = s(x) = \sum_{k=0}^{\infty} f_k(x)$$

Proof.

$$\forall x \in S\ \lim_{k\to\infty} f_k(x) = \lim_{n\to\infty} (s_n(x) - s_{n+1}(x)) = 0$$

□

Definition 0.2.3 (Uniform Convergence)**.** A series of functions is said to be *uniformly convergent* if and only if

$$\sum_{k=0}^{\infty} f_k(x) \rightrightarrows s(x) \iff s_n(x) = \sum_{k=0}^{n} f_k(x) \rightrightarrows s(x) \tag{14}$$

Definition 0.2.4 (Absolute Convergence)**.** A series of functions is said to be *absolutely convergent* if and only if

$$\sum_{k=0}^{\infty} f_k(x) \xrightarrow{\text{A}} s(x) \implies \sum_{k=0}^{\infty} |f_k(x)| \to s(x) \tag{15}$$

Theorem 0.5. *Let $\sum_{k=0}^{\infty} f_k(x) \xrightarrow{\text{A}} s(x)$, then*

$$\sum_{k=0}^{\infty} f_k(x) \xrightarrow{\text{A}} s(x) \implies \sum_{k=0}^{\infty} f_k(x) \to s(x) \tag{16}$$

Proof. Let

$$\begin{aligned} s_n(x) = \sum_{k=0}^{n} f_k(x)\ \therefore \exists g(x) : (S, d) \longrightarrow \mathbb{C},\ \exists N_\epsilon(x) \in \mathbb{N} :\ &\left| g(x) - \sum_{k=0}^{\infty} f_k(x) \right| = \\ = \sum_{k=n+1}^{\infty} |f_k(x)| < \epsilon\ \forall n \geq N_\epsilon(x) \\ \therefore \forall n, m \in \mathbb{N}, m > n \\ |s_m(x) - s_n(x)| = \left| \sum_{k=n+1}^{m} f_k(x) \right| \leq \sum_{k=n+1}^{\infty} |f_k(x)| < \epsilon\ \forall x \in S \\ \therefore (s_n(x)) \text{ is a Cauchy series in } \mathbb{C} \implies s_k(x) \to s(x) \end{aligned}$$

□

Definition 0.2.5 (Total Convergence)**.** A series of functions $s_k(x)$ is said to be *totally convergent* if

1. $\exists M_k \, : \, \sup_S |f_k(x)| \leq M_k \; \forall k \geq 1$
2. $\sum_{k=0}^{\infty} M_k \to M$

The total convergence is then indicated as $s_k(x) \xrightarrow{\text{T}} s(x)$

Proposition 1. Let

$$s_n(x) = \sum_{k=0}^{n} f_n(x)$$

Then

1. $f_n(x) \in C(S) \wedge s_k(x) \rightrightarrows s(x) \implies s(x) \in C(S)$
2. $f_n(x) \in C(S), \; s_k(x) \rightrightarrows s(x) \implies \int s(x) \, \mathrm{d}x = \lim_{k\to\infty} \int s_k(x) \, \mathrm{d}x$
3. $s_k(x) \xrightarrow{\text{A}} s(x) \implies s_k(x) \to s(x)$
4. $s_k(x) \rightrightarrows s(x) \implies s_k(x) \xrightarrow{\text{A}} s(x)$
5. $s_k(x) \xrightarrow{\text{T}} s(x) \implies s_k(x) \rightrightarrows s(x)$

§§ 0.2.1 Power Series and Convergence Tests

Theorem 0.6 (Weierstrass Test)**.** *Let* $(f_n) : (S, d) \to \mathbb{C}$ *a sequence of functions. If we have that*

$$\forall n > N_\epsilon \in \mathbb{N} \; \exists M_n > 0 \, : \, |f_n(x)| \leq M_n$$
$$\therefore \forall x \in S \; \sum_{k=0}^{n} f_k(x) \leq \sum_{k=1}^{\infty} M_k \to M \therefore \sum_{k=0} f_k(x)^n \rightrightarrows s(x)$$

Definition 0.2.6 (Power Series)**.** Let $z, z_0, (a_n) \in \mathbb{C}$. A *power series centered in* z_0 is defined as follows

$$\sum_{k=0}^{\infty} a_k (z - z_0)^k \tag{17}$$

Example 0.2.1. Take the *geometric series*. This is the best example of a power series centered in $z_0 = 0$, and it has the following form

$$\sum_{k=0}^{\infty} z^k \tag{18}$$

We can expand it as follows

$$\sum_{k=0}^{m} z^k = (1 - z) \left(1 + z + z^2 + \cdots + z^m\right) = 1 - z^{n+1} = \frac{1 + z^{n+1}}{1 - z} \; \forall |z| \neq 1 \tag{19}$$

Taking the limit, we have, therefore

$$\sum_{k=0}^{\infty} z^k = \frac{1}{1 - z} \; \forall |z| < 1 \tag{20}$$

Theorem 0.7 (Cauchy-Hadamard Criteria)**.** *Let $\sum_{k=0}^{\infty} a_k(z-z_0)^k$ be a power series, with $a_n, z, z_0 \in \mathbb{C}$. We define the Radius of convergence $R \in \mathbb{R}^\star = \mathbb{R} \cup \{\pm\infty\}$, with the Cauchy-Hadamard criteria*

$$\frac{1}{R} = \limsup_{n\to\infty} |a_n|^{\frac{1}{n}} = \begin{cases} +\infty & \frac{1}{R} = 0 \\ l & 0 < \frac{1}{R} = l < \infty \\ 0 & \frac{1}{R} = +\infty \end{cases} \tag{21}$$

Then $s_k(z) \rightrightarrows s(z)\ \forall |z| \in (-R, R)$

Theorem 0.8 (D'Alambert Criteria)**.** *From the power series we have defined before, we can write the D'Alambert criteria for convergence as follows*

$$\frac{1}{R} = \lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| \implies R = \lim_{k\to\infty} \left|\frac{a_k}{a_{k+1}}\right| \tag{22}$$

Where R is the previously defined radius of convergence

Theorem 0.9 (Abel)**.** *Let $R > 0$, then if a power series converges for $|z| = R$, it converges uniformly $\forall |z| \in [r, R] \subset (-R, R]$. It is valid analogously for $x = -R$*

Remark (Power Series Integration)*.* If the series has $R > 0$ and it converges in $|z| = R$, calling $s(x)$ the sum of the series, with $x = |z|$ we can say that

$$\int_0^R s(x)\,\mathrm{d}x = \sum_{k=0}^{\infty} \int_0^R a_k x^k\,\mathrm{d}x = \int_0^R \sum_{k=1}^{\infty} a_k x^k\,\mathrm{d}z = \sum_{k=0}^{\infty} a_k \frac{R^{k+1}}{k+1} \tag{23}$$

Remark (Power Series Derivation)*.* If Abel's theorem holds, we have also that, if we have $s(x)$ our power series sum, we can define the $n-$th derivative of this series as follows

$$\frac{\mathrm{d}^n s}{\mathrm{d}x^n} = \sum_{k=n}^{\infty} k(k-1)\cdots(k-n+1)a_k x^{k-n} \tag{24}$$

§ 0.3 Series Representation of Functions

§§ 0.3.1 Taylor Series

Theorem 0.10 (Taylor Series Expansion)**.** *Let $f : D \longrightarrow \mathbb{C}$ be a function such that $f \in H(B_R(z_0))$, with $B_r(z_0) \subseteq D$. Then*

$$f(z) = \sum_{n=0}^{n} \frac{1}{n!} \left.\frac{\mathrm{d}^n f}{\mathrm{d}z^n}\right|_{z_0} (z-z_0)^n \quad \|z - z_0\| < r \tag{25}$$

Proof. Taken $z \in B_r(z_0)$ and $\gamma(t) = z_0 + re^{it}$ $t \in [0, 2\pi]$ and $\|z - z_0\| < r < R$ we can write, using the integral representation of f

$$f(z) = \frac{1}{2\pi i} \oint_\gamma \frac{f(w)}{(w-z)}\,\mathrm{d}w = \frac{1}{2\pi i} \oint_\gamma \frac{f(w)}{(w-z_0)-(z-z_0)}\,\mathrm{d}w$$

From basic calculus we know already that if $z \neq w$

$$\frac{1}{w-z} = \frac{1}{w}\left(\frac{1-(z/w)^n}{1-z/w} + \frac{1}{1-z/w}\left(\frac{z}{w}\right)^n\right) =$$
$$= \frac{1}{w-z}\left(\frac{z}{w}\right)^n + \sum_{k=0}^{n-1}\frac{1}{w}\left(\frac{z}{w}\right)^n$$

Therefore, inserting it back into the integral representation, we have

$$f(z) = \sum_{k=0} \frac{(z-z_0)^k}{2\pi i}\oint_\gamma \frac{f(w)}{(w-z_0)^{k+1}}\,\mathrm{d}w + \frac{(z-z_0)^n}{2\pi i}\oint_\gamma \frac{f(w)}{[(w-z_0)-(z-z_0)]\,(w-z_0)^n}\,\mathrm{d}w$$

On the RHS as first term we have the $k-$th derivative of f and on the right there is the so called *remainder* $R_n(z)$. Therefore

$$f(z) = \sum_{k=0}^{n}\frac{1}{k!}\left.\frac{\mathrm{d}^k f}{\mathrm{d}z^k}\right|_{z_0}(z-z_0)^k + R_n(z)$$

It's easy to demonstrate that $R_n(z) \overset{n\to\infty}{\longrightarrow} 0$, and therefore

$$f(z) = \sum_{k=0}^{\infty}\frac{1}{k!}\left.\frac{\mathrm{d}^k f}{\mathrm{d}z^k}\right|_{z_0}(z-z_0)^k$$

$\square$

Definition 0.3.1 (Taylor Series for Scalar Fields)**.** Given a function $f : A \subset \mathbb{R}^n \longrightarrow \mathbb{R}$ $f \in C^m(A)$, given a multi-index α one can define the Taylor series of the scalar field as follows

$$f(x) = \sum_{|\alpha|\leq m}\frac{1}{\alpha!}\partial^\alpha f(x_0)(x-x_0)^\alpha + R_m(x)$$

Where, the remainder is defined in integral form as follows

$$R_m(x) = (m+1)\sum_{|\alpha|=m+1}\frac{(x-x_0)^\alpha}{\alpha!}\int_0^1 (1-t)^m \partial^\alpha f(x_0 + tx - tx_0)\,\mathrm{d}t$$

Definition 0.3.2 (MacLaurin Series)**.** Taken a Taylor series, such that $z_0 = 0$, we obtain a MacLaurin series.

$$f(z) = \sum_{k=0}^{\infty}\frac{1}{k!}\left.\frac{\mathrm{d}^k f}{\mathrm{d}z^k}\right|_{z=0} z^k \tag{26}$$

Definition 0.3.3 (Remainders)**.** We can have two kinds of remainder functions while calculating series:

1. Peano Remainders, $R_n(z) = \mathcal{O}(\|z-z_0\|^n)$
2. Lagrange Remainders, $R_n(x) = (n+1)!^{-1} f^{(n+1)}(\xi)(x-x_0)^{n+1},\ x, x_0 \in \mathbb{R}\ \xi \in (x, x_0)$

What we saw before as $R_n(z)$ is the remainder function for functions $f : D \subset \mathbb{C} \longrightarrow \mathbb{C}$.
A particularity of remainder function is that $R_n(z) \to 0$ always, if f is holomorphic

Theorem 0.11 (Integration of Power Series II). *Let $f, g : B_R(z_0) \longrightarrow \mathbb{C}$ and $\{\gamma\} \subset B_R(z_0)$ a piecewise smooth path. Taken*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \quad g \in C(\{\gamma\})$$

We have that

$$\sum_{n=0}^{\infty} a_n \int_\gamma g(z)(z - z_0)^n \, \mathrm{d}z = \int_\gamma g(z) f(z) \, \mathrm{d}z \tag{27}$$

Proof. Since $f, g \in C(\{\gamma\})$ by definition, and $f \in H(\overline{B_r}(z_0))$ with $r < R$, we have that $\exists \hat{K}_\gamma[fg]$.
Firstly we can write that $\forall z \in B_R(z_0)$

$$g(z)f(z) = \sum_{k=0}^{n-1} a_k g(z)(z - z_0)^k + g(z)R_n(z) = \sum_{k=0}^{n-1} a_k g(z)(z - z_0)^k + g(z) \sum_{k=n}^{\infty} a_k (z - z_0)^k$$

Then we can write

$$\int_\gamma g(z)f(z) \, \mathrm{d}z = \sum_{k=0}^{n-1} a_k \oint_\gamma g(z)(z - z_0)^k \, \mathrm{d}z + \int_\gamma g(z) R_n(z) \, \mathrm{d}z$$

Letting $M = \sup_{z \in \{\gamma\}} \|g(z)\|$, and noting that $\|R_n(z)\| < \epsilon$ for $\forall \epsilon > 0$ and for some $n \geq N_\epsilon \in \mathbb{N},\ z \in \{\gamma\}$ we have, using the Darboux inequality

$$\left\| \int_\gamma g(z) R_n(z) \, \mathrm{d}z \right\| \leq M L_\gamma \epsilon \to 0$$

□

Theorem 0.12 (Holomorphy of Power Series). *If a function $f(z)$ is expressable as a power series $f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k,\ \|z - z_0\| < R$ we have that $f \in H(B_R(z_0))$*

Proof. Take the previous theorem on the integration of power series, and choose $g(z) = 1$. Since $g(z) \in H(\mathbb{C})$ we also have that it'll be continuous on all paths $\{\gamma\} \subset \mathbb{C}$ piecewise smooth.
Take now a closed piecewise smooth path $\{\gamma\}$, then we can write

$$\oint_\gamma f(z)g(z) \, \mathrm{d}z = \oint_\gamma \sum_{k=0}^{\infty} a_k (z - z_0)^k = \sum_{k=0}^{\infty} a_k \oint_\gamma (z - z_0)^k \, \mathrm{d}z$$

Since the function $h(z) = (z - z_0)^k \in H(\mathbb{C})\ \forall k \neq 1$, we have, thanks to the Morera and Cauchy theorems

$$\oint_\gamma f(z) \, \mathrm{d}z = 0 \implies f(z) \in H(B_R(\overline{\{\gamma\}}))$$

□

Corollary 0.3.1 (Derivative of a Power Series II)**.** Take $f(z) = \sum_{k=0}^{\infty} a_k(z - z_0)^k \ \|z - z_0\| < R$. Then, $\forall z \in B_R(z_0)$ we have that

$$\frac{\mathrm{d}f}{\mathrm{d}z} = \sum_{k=1}^{\infty} a_k k(z - z_0)^{k-1} \tag{28}$$

Proof. Taken $z \in B_R(z_0)$ and a continuous function $g(w) \in C(\{\gamma\})$, with $\{\gamma\} \subset B_R(z_0)$ a closed simple piecewise smooth path. If $z \in \{\gamma\}^\circ$ and

$$g(w) = \frac{1}{2\pi i}\left(\frac{1}{(w - z)^2}\right)$$

We have, using the integral representation for holomorphic functions

$$\frac{1}{2\pi i}\oint_\gamma \frac{f(w)}{(w - z)^2}\,\mathrm{d}w = \sum_{k=0}^{\infty} \frac{a_k}{2\pi i}\oint_\gamma \frac{(w - z_0)^k}{(w - z_0)^2}\,\mathrm{d}w$$

Since $h(w) = (w - z_0)^k \in H(\mathbb{C})\ \forall k \neq 1$ we have, using again the integral representation for holomorphic functions

$$\begin{aligned}\frac{1}{2\pi i}\oint_\gamma \frac{f(w)}{(w - z)^2}\,\mathrm{d}w &= \frac{\mathrm{d}f}{\mathrm{d}z}\\ \frac{1}{2\pi i}\oint_\gamma \frac{(w - z_0)^k}{(w - z)^2}\,\mathrm{d}w &= k(z - z_0)^{k-1}\end{aligned}$$

Therefore

$$\frac{1}{2\pi i}\oint_\gamma \frac{f(w)}{(w - z)^2}\,\mathrm{d}w = \sum_{k=0}^{\infty} a_k k(z - z_0)^k = \frac{\mathrm{d}f}{\mathrm{d}z}$$

□

Corollary 0.3.2 (Uniqueness of the Taylor Series)**.** Taken an holomorphic function $f \in H(D)$ with $D \subset \mathbb{C}$ some connected open set, we have that

$$f(z) = \sum_{k=0}^{\infty} a_k(z - z_0)^k \quad a_k = \frac{1}{k!}\left.\frac{\mathrm{d}^k f}{\mathrm{d}z^k}\right|_{z_0} \quad \forall \|z - z_0\| < R$$

Proof. Taken $g(z)$ a continuous function over a closed piecewise simple path $\{\gamma\} \subset \mathbb{C}$, where

$$g(z) = \frac{1}{2\pi i}\left(\frac{1}{(z - z_0)^{k+1}}\right)$$

We have that

$$\frac{1}{2\pi i}\oint_\gamma \frac{f(z)}{(z - z_0)^{n+1}}\,\mathrm{d}z = \sum_{k=1}^{\infty} \frac{a_k}{2\pi i}\oint_\gamma (z - z_0)^{k-n-1}\,\mathrm{d}z$$

The integral on the RHS evaluates to δ_n^k, and thanks to the integral representation of $f(z)$ we can write

$$\frac{1}{2\pi i}\oint_\gamma \frac{f(z)}{(z - z_0)^{n+1}}\,\mathrm{d}z = \frac{1}{n!}\left.\frac{\mathrm{d}^n f}{\mathrm{d}z^n}\right|_{z_0} = n!a_n$$

□

§§ 0.3.2 Laurent Series

Definition 0.3.4 (Annulus Domain)**.** Let $0 \leq r < R \leq \infty$ and $z_0 \in \mathbb{C}$, we define the *annulus set* as follows

$$A_{rR}(z_0) := \{\, z \in \mathbb{C} |\, r < \|z - z_0\| < R\} \tag{29}$$

Special cases of this are the ones where $r = 0$, $R = \infty$ and $r = 0$, $R = \infty$

$$\begin{aligned} A_{0,R}(z_0) &= B_R(z_0) \setminus \{z_0\} \\ A_{r,\infty}(z_0) &= \mathbb{C} \setminus \overline{B}_r(z_0) \\ A_{0,\infty}(z_0) &= \mathbb{C} \setminus \{z_0\} \end{aligned}$$

Theorem 0.13 (Laurent Series Expansion)**.** *Let $f : A_{R_1R_2}(z_0) \longrightarrow \mathbb{C}$ be a function such that $f \in H\left(A_{R_1R_2}(z_0)\right)$, and $\{\gamma\} \subset A_{R_1R_2}(z_0)$ a closed simple piecewise smooth curve.*
Then f is expandable in a generalized power series or a Laurent series as follows

$$f(z) = \sum_{n=0}^{\infty} c_n^+ (z - z_0)^n + \sum_{n=1}^{\infty} \frac{c_n^-}{(z - z_0)^n} = \sum_{k=-\infty}^{\infty} c_k (z - z_0)^k \tag{30}$$

Where the coefficients are the following

$$\begin{aligned} c_n^- &= \frac{1}{2\pi i} \oint_\gamma f(z)(z - z_0)^{n-1} \,\mathrm{d}z \quad n \geq 0 \\ c_n^+ &= \frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{(z - z_0)^{n+1}} \,\mathrm{d}z \quad n > 0 \\ c_k &= \frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{(z - z_0)^{k+1}} \,\mathrm{d}z \quad k \in \mathbb{Z} \end{aligned} \tag{31}$$

Proof. Taken a random point $z \in A_{R_1R_2}(z_0)$, a closed simple piecewise smooth curve $\{\gamma\} \subset A_{R_1R_2}(z_0)$ and two circular smooth paths $\{\gamma_2\}, \{\gamma_3\} : \{\gamma_2\} \cup \{\gamma_3\} = \partial A_{r_1r_2}(z_0) \subset A_{R_1R_2}(z_0) \;\wedge\; \{\gamma\} \subset A_{r_1r_2}(z_0)$ and a third circular path $\{\gamma_3\} \subset A_{r_1r_2}(z_0)$, we can write immediately, using the omotopy between all the paths

$$\oint_{\gamma_2} \frac{f(w)}{w - z} \,\mathrm{d}w = \oint_{\gamma_1} \frac{f(w)}{w - z} \,\mathrm{d}w + \oint_{\gamma_3} \frac{f(w)}{w - z} \,\mathrm{d}w$$

Using the Cauchy integral representation we have that the integral on γ_3 yields immediately $2\pi i f(z)$, hence we can write

$$f(z) = \frac{1}{2\pi i} \oint_{\gamma_2} \frac{f(w)}{(w - z_0) - (z - z_0)} \,\mathrm{d}w + \frac{1}{2\pi i} \oint_{\gamma_1} \frac{f(w)}{(z_0 - z) - (w - z_0)} \,\mathrm{d}w$$

Using the two following identities for $z \neq w$

$$\begin{aligned} \frac{1}{w - z} &= \frac{1}{w - z} \left(\frac{z}{w}\right)^n + \sum_{k=0}^{n-1} \frac{1}{w} \left(\frac{z}{w}\right)^k \\ \frac{1}{z - w} &= \frac{1}{z - w} \left(\frac{w}{z}\right)^n + \sum_{k=1}^{n} \frac{1}{w} \left(\frac{w}{z}\right)^k \end{aligned}$$

We obtain that

$$f(z)=\sum_{k=0}^{n-1}\frac{(z-z_0)^k}{2\pi i}\oint_{\gamma_2}\frac{f(w)}{(w-z_0)^{k+1}}\,\mathrm{d}w+\rho_n(z)+\sum_{k=1}^{n}\frac{1}{2\pi i(z-z_0)^k}\oint_{\gamma_1}f(w)(w-z_0)^{k-1}\,\mathrm{d}w+\sigma_n(z)$$

Where, after choosing appropriate substitutions with some coefficients c_k^+, c_k^- we have

$$f(z)=\sum_{k=0}^{n-1}c_k^+(z-z_0)^k+\rho_n(z)+\sum_{k=1}^{n}\frac{c_k^-}{(z-z_0)^k}+\sigma_n(z)$$

Where ρ_n, σ_n are the two remainders of the series expansion, and are

$$\begin{aligned}\rho_n(z)&=\frac{(z-z_0)^n}{2\pi i}\oint_{\gamma_2}\frac{f(w)}{[(w-z_0)-(z-z_0)]\,(w-z_0)^n}\,\mathrm{d}w\\ \sigma_n(z)&=\frac{1}{2\pi i(z-z_0)^n}\oint_{\gamma_1}\frac{f(w)}{(w-z_0)-(z-z_0)}\,\mathrm{d}w\end{aligned}$$

In order to prove the theorem we now need to demonstrate that $\rho_n, \sigma_n \overset{n\to\infty}{\longrightarrow} 0$. Taken $M_1 = \sup_{z\in\{\gamma_1\}}\|f(z)\|,\ M_2=\sup_{z\in\{\gamma_2\}}\|f(z)\|$, we have, using the fact that both γ_1, γ_2 are circular

$$\begin{aligned}\|\rho_n(z)\|&\le\frac{M_2}{1-\frac{1}{r_2}\|z-z_0\|}\left(\frac{\|z-z_0\|}{r_2}\right)^n\overset{n\to\infty}{\longrightarrow}0\quad\|z-z_0\|<r_2\\ \|\sigma_n(z)\|&\le\frac{M_1}{\frac{1}{r_1}\|z-z_0\|-1}\left(\frac{r_1}{\|z-z_0\|}\right)^n\overset{n\to\infty}{\longrightarrow}0\quad r_1<\|z-z_0\|\end{aligned}$$

And the theorem is proved. □

Theorem 0.14 (Convergence of a Laurent Series)**.** *Being defined on an annulus set, the Laurent series of a function must have* two *radii of convergence. Given a function f holomorphic on a set $A_{R_1R_2}(z_0)$ we have*

$$\begin{aligned}\frac{1}{R_2}&=\limsup_{n\to\infty}\sqrt[n]{\|c_n\|}\\ R_1&=\limsup_{n\to\infty}\sqrt[n]{\|c_{-n}\|}\end{aligned}\tag{32}$$

It's equivalent of showing the convergence of the two series

$$f(z)=\sum_{k=0}^{\infty}c_k^+(z-z_0)^k+\sum_{k=1}^{\infty}\frac{c_k^-}{(z-z_0)^k}$$

Theorem 0.15 (Integral of a Laurent Series)**.** *Let $f(z)\in H\left(A_{R_1R_2}(z_0)\right)$ and take $\{\gamma\}\subset A_{R_1R_2}(z_0)$ a piecewise smooth curve, and $g\in C(\{\gamma\})$, then we have*

$$\sum_{n=-\infty}^{\infty}c_n\oint_{\gamma}g(z)(z-z_0)^n\,\mathrm{d}z=\oint_{\gamma}g(z)f(z)\,\mathrm{d}z$$

Proof. We begin by separating the sum in two parts, ending up with the following

$$\sum_{n=0}^{\infty} c_n^+ \oint_\gamma g(z)(z-z_0)^n \,\mathrm{d}z = \oint_\gamma g(z) f_+(z)\,\mathrm{d}z$$
$$\sum_{n=1}^{\infty} c_n^- \oint_\gamma \frac{g(z)}{(z-z_0)^n} \,\mathrm{d}z = \oint_\gamma g(z) f_-(z)\,\mathrm{d}z$$

Which is analogous to the integration of Taylor series. The same could be obtained keeping the bounds of the sum in all $\mathbb{Z}$ □

As for Taylor series, in a completely analogous fashion, a Laurent series is holomorphic and unique.
The derivative of a Laurent series, is then obviously the following

$$\frac{\mathrm{d}f}{\mathrm{d}z} = \sum_{n=-\infty}^{\infty} c_n n (z-z_0)^{n-1}$$

§§ 0.3.3 Multiplication and Division of Power Series

Theorem 0.16 (Product of Power Series)**.** *Take* $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n,\ z \in B_{R_1}(z_0)$ *and* $g(z) = \sum_{n=0}^{\infty} b_n (z-z_0)^n,\ z \in B_{R_2}(z_0)$*. Then*

$$f(z)g(z) = \sum_{n=0}^{\infty} c_n (z-z_0)^n, \quad c_n = \sum_{k=0}^{n} a_k b_{n-k} \quad \|z-z_0\| < \min(R_1, R_2) = R$$

Proof. Due to the holomorphy of both f and g, we have that the function fg has a Taylor series expansion

$$f(z)g(z) = \sum_{n=0}^{\infty} c_n (z-z_0) \quad \|z-z_0\| < R$$

We have then, using Leibniz's derivation rule

$$\begin{aligned} c_n &= \frac{1}{n!}\frac{\mathrm{d}^n}{\mathrm{d}z^n} f(z)g(z) = \frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k} \left.\frac{\mathrm{d}^k f}{\mathrm{d}z^k}\right|_{z_0} \left.\frac{\mathrm{d}^{n-k} g}{\mathrm{d}z^{n-k}}\right|_{z_0} = \\ &= \sum_{k=0}^{n} \frac{1}{k!}\left.\frac{\mathrm{d}^k f}{\mathrm{d}z^k}\right|_{z_0} \frac{1}{(n-k)!}\left.\frac{\mathrm{d}^{n-k} g}{\mathrm{d}z^{n-k}}\right|_{z_0} = \\ &= \sum_{k=0}^{n} a_k b_{n-k} \end{aligned}$$

□

Theorem 0.17 (Division of Power Series)**.** *Taken the two functions as before, with the added necessity that* $g(z) \neq 0$*, we have that*

$$\frac{f(z)}{g(z)} = \sum_{n=0}^{\infty} d_n (z-z_0)^n \quad d_n = \frac{1}{b_0}\left(a_n - \sum_{k=0}^{n-1} d_k b_{n-k}\right)$$

Proof. Everything hold as in the previous proof. Remembering that $(f/g)g = f$ and using the previous theorem, we obtain

$$a_n = \sum_{k=0}^{n} d_k b_{k-n}$$

And therefore, inverting

$$d_n = \frac{a_n}{b_0} - \frac{1}{b_0}\sum_{k=0}^{n-1} d_k b_{n-k}$$

□

§§ 0.3.4 Useful Expansions

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \quad \|z\| < \infty \tag{33}$$

$$\sin(z) = \frac{1}{2i}\left(e^{iz} - e^{-iz}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1} \quad \|z\| < \infty \tag{34}$$

$$\cos(z) = \frac{\mathrm{d}}{\mathrm{d}z}\sin(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n} \quad \|z\| < \infty \tag{35}$$

$$\cosh(z) = \cos(iz) = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} \quad \|z\| < \infty \tag{36}$$

$$\sinh(z) = \frac{\mathrm{d}}{\mathrm{d}z}\cosh(z) = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!} \quad \|z\| < \infty \tag{37}$$

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \quad \|z\| < 1 \tag{38}$$

$$\frac{1}{1+z} = \sum_{n=0}^{\infty} (-1)^n z^n \quad \|z\| < 1 \tag{39}$$

$$\frac{1}{z} = \sum_{n=0}^{\infty} (-1)^n (z-1)^n \quad \|z-1\| < 1 \tag{40}$$

$$(1+z)^s = \sum_{n=0}^{\infty} \binom{s}{n} z^n \quad s \in \mathbb{C},\ \|z\| < 1 \tag{41}$$

$$e^{\frac{1}{z}} = \sum_{n=0}^{\infty} \frac{1}{n! z^n} \quad 0 < \|z\| < \infty \tag{42}$$

§ 0.4 Residues

§§ 0.4.1 Singularities and Residues

Definition 0.4.1 (Singularity)**.** Given a function $f : G \longrightarrow \mathbb{C}$ we define a *singularity* a point $z_0 \in G$ such that

$$\forall \epsilon > 0 \; \exists z \in B_\epsilon(z_0) \; : \; f(z) \text{ is holomorphic} \tag{43}$$

Definition 0.4.2 (Isolated Singularity)**.** Given a function $f : G \longrightarrow \mathbb{C}$ we define an *isolated singularity* a point $z_0 \in G$ such that

$$\exists r > 0 \; : \; f \in H\left(A_{0r}(z_0)\right) \tag{44}$$

Definition 0.4.3 (Residue)**.** Let $z_0 \in G$ be an isolated singularity of $f : G \longrightarrow \mathbb{C}$, then $\exists r > 0$: $\forall z \in A_{0r}(z_0)$ the following Laurent series expansion holds

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n} = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n$$

The *residue* of the function f in z_0 is defined as follows

$$\operatorname*{Res}_{z=z_0} f(z) = b_1 = c_{-1} \tag{45}$$

A second definitiion is given by the following contour integral

$$\operatorname*{Res}_{z=z_0} f(z) = \frac{1}{2\pi i} \oint_\gamma f(z) \, \mathrm{d}z$$

Where γ is a simple closed path around z_0

Definition 0.4.4 (Winding Number)**.** Given a closed curve $\{\gamma\}$ we define the *winding number* or *index* of the curve around a point z_0 the following integral

$$n(\gamma, z_0) = \frac{1}{2\pi i} \oint_\gamma \frac{\mathrm{d}z}{z - z_0} \tag{46}$$

Theorem 0.18 (Residue Theorem)**.** *Given a function $f : G \longrightarrow \mathbb{C}$ such that $f \in H(D)$ where $D = G \setminus \{z_1, \cdots, z_n\}$ and z_k are isolated singularities, we have, taken a closed piecewise smooth curve $\{\gamma\}$, such that $\{z_1, \cdots, z_n\} \subset \{\gamma\}^\circ$*

$$\oint_\gamma f(z) \, \mathrm{d}z = 2\pi i \sum_{k=0}^{\infty} n(\gamma, z_k) \operatorname*{Res}_{z=z_k} f(z) \tag{47}$$

Proof. Firstly we can say that $\gamma \sim \sum_k \gamma_k$ where γ_k are simple curves around each z_k, then since the function is holomorphic in the regions $A_{0r}(z_k)$ with $k = 1, \cdots, n$ we can write

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_k)^n$$

Therefore, we have

$$\oint_\gamma f(z)\,\mathrm{d}z = \sum_{k=0}^{n} \oint_{\gamma_k} f(z)\,\mathrm{d}z = \sum_{k=0}^{n} \sum_{j=-\infty}^{\infty} c_j \oint_{\gamma_k} (z-z_k)^j\,\mathrm{d}z$$

We can then use the linearity of the integral operator and write

$$\oint_\gamma f(z)\,\mathrm{d}z = \sum_{k=0}^{n} \sum_{j=-\infty}^{-2} c_j \oint_{\gamma_k} (z-z_k)^j\,\mathrm{d}z + c_{-1} \oint_{\gamma_k} \frac{\mathrm{d}z}{z-z_k} + \sum_{j=0}^{\infty} c_j \oint_{\gamma_k} (z-z_k)^j\,\mathrm{d}z$$

Thanks to the Cauchy theorem we already know that the first and last integrals on the RHS must be null, therefore

$$\oint_\gamma f(z)\,\mathrm{d}z = \sum_{k=0}^{n} c_{-1} \oint_{\gamma_k} \frac{\mathrm{d}z}{z-z_k}$$

Recognizing the definition of residue and the winding number of the curve, we have the assert

$$\oint_\gamma f(z)\,\mathrm{d}z = 2\pi i \sum_{k=0} n(\gamma, z_k) \operatorname*{Res}_{z=z_k} f(z)$$

$\square$

Definition 0.4.5 (Residue at Infinity)**.** Given a function $f : G \longrightarrow \mathbb{C}$ and a piecewise smooth closed curve γ. If $f \in H(\{\gamma\} \cup \operatorname{extr}\{\gamma\})$ we have

$$\oint_\gamma f(z)\,\mathrm{d}z = -2\pi i \operatorname*{Res}_{z=\infty} f(z) = 2\pi i \operatorname*{Res}_{z=0} \frac{1}{z^2} f\left(\frac{1}{z}\right) \tag{48}$$

Theorem 0.19. *Given a function $f : G \longrightarrow \mathbb{C}$ as before, if the function has z_k singularities with $k = 1, \cdots, n$*

$$\operatorname*{Res}_{z=\infty} f(z) = \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z) \tag{49}$$

§§ 0.4.2 Classification of Singularities, Zeros and Poles

Definition 0.4.6 (Pole)**.** Given a function $f(z)$ with an isolated singular point in $z_0 \in \mathbb{C}$, we have that in $A_{0r}(z_0)$ the function can be expanded with a Laurent series

$$f(z) = \sum_{k=0}^{\infty} a_k (z-z_0)^k + \sum_{k=1}^{\infty} \frac{b_k}{(z-z_0)^k}$$

The point z_0 is called a *pole of order m* if $b_k = 0\ \forall k > m$

Definition 0.4.7 (Removable Singularity)**.** Given $f(z), z_0$ as before, we have that z_0 is a *removable singularity* if $b_k = 0\ \forall k \geq 1$

Definition 0.4.8 (Essential Singularity)**.** Given $f(z), z_0$ as before, we have that z_0 is an *essential singularity* if $b_k \neq 0$ for infinite values of k

Definition 0.4.9 (Meromorphic Function)**.** Let $f : G \subset \mathbb{C} \longrightarrow \mathbb{C}$ be a function. f is said to be *meromorphic* if $f \in H(\tilde{G})$ where $\tilde{G} = G \setminus \{z_1, \cdots, z_n\}$ where $z_k \in G$ are poles of the function

Theorem 0.20. *Let z_0 be an isolated singularity of a function $f(z)$. z_0 is a pole of order m if and only if*

$$f(z) = \frac{1}{(m-1)!} \left. \frac{\mathrm{d}^{m-1} g}{\mathrm{d}z^{m-1}} \right|_{z_0} \quad g \in H(B_\epsilon(z_0)) \ \epsilon > 0 \tag{50}$$

Proof. Let $f : G \longrightarrow \mathbb{C}$ be a meromorphic function and $g : G \longrightarrow \mathbb{C}$, $g \in H(G)$ where $f(z)$ has a pole in $z_0 \in G$ and $g(z_0) \neq 0$

$$f(z) = \frac{g(z)}{(z - z_0)^m}$$

Since $g(z)$ is holomorphic in z_0 we have that, for some r

$$g(z) = \sum_{k=0}^{\infty} \frac{1}{k!} \frac{\mathrm{d}^k g}{\mathrm{d}z_0^k} (z - z_0)^k \quad z \in B_r(z_0)$$

And therefore, $\forall z \in A_{0r}(z_0)$

$$f(z) = \frac{1}{(z - z_0)^m} g(z) = \sum_{k=0}^{\infty} \frac{1}{k!} \frac{\mathrm{d}^k g}{\mathrm{d}z_0^k} (z - z_0)^{k-m}$$

Since $g(z_0) \neq 0$ we have the assert.
Alternatively we start by hypothesizing that z_0 is already a pole of order m for f, and therefore we can write the following Laurent expansion for some $r > 0$

$$f(z) = \sum_{k=-m}^{\infty} c_k (z - z_0)^k \quad \forall z \in A_{0r}(z_0)$$

Where $c_{-m} \neq 0$. Therefore, we write

$$g(z) = \begin{cases} (z - z_0)^m f(z) & z \in A_{0r}(z_0) \\ c_{-m} & z = z_0 \end{cases}$$

And, expanding $g(z)$ for $z \in B_r(z_0)$ we obtain

$$g(z) = c_{-m} + c_{-m+1}(z - z_0) + \cdots + c_{-1}(z - z_0)^{m-1} + \sum_{k=0}^{\infty} c_k (z - z_0)^{k+m}$$

$g(z)$ is holomorphic in the previous domain of expansion, and therefore we have, since the Taylor expansion is unique

$$c_{-1} = \frac{1}{(m-1)!} \frac{\mathrm{d}^{m-1} g}{\mathrm{d}z_0^{m-1}} = \operatorname*{Res}_{z=z_0} f(z)$$

□

Definition 0.4.10 (Zero)**.** Let $f : G \longrightarrow \mathbb{C}$ be a holomorphic function. Taken $z_0 \in G$, it's said to be a *zero of order m* if

$$\begin{cases} \dfrac{\mathrm{d}^k f}{\mathrm{d}z_0^k} = 0 & k = 1, \cdots, m-1 \\ \dfrac{\mathrm{d}^m f}{\mathrm{d}z_0^m} \neq 0 \end{cases}$$

Theorem 0.21. *The point $z_0 \in G$ is a zero of order m for f if and only if*

$$f(z) = \frac{g(z)}{(z - z_0)^m} \quad g(z_0) \neq 0,\ g \in H(G)$$

Proof. Taken $f(z) = (z - z_0)^m g(z)$ such that $g(z_0) \neq 0$ we can expand $g(z)$ with Taylor and at the end obtain

$$f(z) = \sum_{k=0}^{\infty} \frac{1}{k!} \frac{\mathrm{d}^k g}{\mathrm{d}z_0^k} (z - z_0)^{k+m}$$

Since this is a Taylor expansion also for $f(z)$ we have that, for $j = 1, \cdots, m-1$

$$\frac{\mathrm{d}^j f}{\mathrm{d}z_0^j} = 0 \qquad \frac{\mathrm{d}^m f}{\mathrm{d}z_0^m} = m! g(z_0) \neq 0$$

The same is obtainable with the vice versa demonstrating the theorem ☐

Notation. Let f be a meromorphic function. We will define the following sets of points accordingly

1. Z_f^m as the set of zeros of order m
2. S_f as the set of isolated singularities of f
3. P_f^m as the set of poles of order m

We immediately see some special cases

1. P_f^∞ is the set of essential singularities of f
2. P_f^1 is the set of removable singularities of f

Theorem 0.22. *Let $f : D \longrightarrow \mathbb{C}$ be a function such that $f \in H(D)$, with D an open set, then*

1. $f(z) = 0\ \forall z \in D$
2. $\exists z_0 \,:\, f^{(k)}(z_0) = 0\ \forall k \geq 0$
3. *$Z_f \subset D$ has a limit point*

Proof. 3) $\implies$ 2)
Take $z_0 \in D$ as the limit point of Z_f. Since $f \in C(D)$ we have that $z_0 \in Z_f^m$. therefore

$$f(z) = (z - z_0)^m g(z) \quad g(z_0) \neq 0,\ g \in H(D) \implies \exists \delta > 0 \,:\, g(z) \neq 0 \quad \forall z \in B_\delta(z_0)$$

Therefore

$$f(z) \neq 0 \quad \forall z \in A_{0\delta}(z_0) \ \lightning$$

2) $\implies$ 1)
Suppose that $Z_{f^{(k)}} := \left\{ z \in D | \, f^{(k)}(z) = 0 \right\} \neq \{\}$. We have to demonstrate that this set is clopen in D.
Take $z \in \overline{Z_{f^{(k)}}}$ and a sequence $(z)_k \in Z_{f^{(k)}}$ such that $z_k \to z$. We have then

$$f^{(k)}(z) = \lim_{k\to\infty} f^{(k)}(z_k) = 0$$

Therefore $Z_{f^{(k)}} = \overline{Z_{f^{(k)}}}$ and the set is closed.
Take then $z \in Z_{f^{(k)}} \subset D$, since D is open we have that $\exists r > 0 \, : \, B_r(z) \subset D$, therefore

$$\forall w \in B_r(z), \;\; z \neq w \quad f(w) = \sum_{k=0}^{\infty} a_k (w-z)^k = 0 \implies \begin{cases} z = w \\ a_k = 0 \quad \forall k \geq 0 \end{cases}$$

Since $w \neq z$ we have that $B_r(z) \subset Z_{f^{(k)}}$ and the set is open. Taking both results we have that the set is clopen and $D = Z_{f^{(k)}}$ $\square$

Corollary 0.4.1. Let $f, g : D \longrightarrow \mathbb{C}$ and $f, g \in H(D)$. We have that $f = g$ iff the set $\{f(z) = g(z)\}$ has a limit point in D

Corollary 0.4.2 (Zeros of Holomorphic Functions)**.** Let $f : D \longrightarrow \mathbb{C}$ be a non-constant function $f \in H(D)$ with D an open connected set. Then

$$\forall z \in Z_f^m \quad m < \infty$$

Proof. Take $z_0 \in Z_f$, then since f is non-constant we have that Z_f has no limit points in D, therefore

$$\exists \delta > 0 \, : \, f(z) \neq 0 \quad \forall z \in A_{0\delta}(z_0) \;\wedge\; \exists m \geq 1 \, : \, \frac{\mathrm{d}^k f}{\mathrm{d}z_0^k} = 0 \;k \in [0, m), \;\frac{\mathrm{d}^m f}{\mathrm{d}z_0^m} \neq 0$$

Therefore $z_0 \in Z_f^m$ $\square$

Theorem 0.23. *Let $f : D \longrightarrow \mathbb{C}$ be a meromorphic function, such that*

$$f(z) = \frac{p(z)}{q(z)} \quad p, q \in H(D)$$

If $z_0 \in Z_q^m$ such that $p(z_0) \neq 0$, then $z_0 \in P_f^m$

Proof. $z_0 \in Z_q^m$ is an isolated singularity of q, therefore

$$\exists \delta > 0 \, : \, q(z) \neq 0 \quad \forall z \in A_{0\delta}(z_0) \;\therefore z_0 \in S_{p/q}$$

We therefore can take $q(z) = (z - z_0)^m g(z)$ and we have

$$f(z) = \frac{p(z)}{g(z)(z - z_0)^m} = \frac{h(z)}{(z - z_0)^m}$$

Where $h(z)$ is a holomorphic function such that $h(z_0) \neq 0$. By definition of pole we have $z_0 \in P_f^m$ $\square$

Theorem 0.24 (Quick Calculus of Residues for Rational Functions). *If $f(z) = p(z)/q(z)$ as before, there is a quick rule of thumb for calculating the residue in z_0. We can write*

$$\operatorname*{Res}_{z=z_0} f(z) = \frac{1}{(m-1)!} \frac{\mathrm{d}^{m-1} h}{\mathrm{d} z_0^{m-1}}$$

If the pole is a removable singularity, we have $z_0 \in P_f^1$ and

$$\operatorname*{Res}_{z=z_0} f(z) = \frac{p(z_0)}{q'(z_0)}$$

Theorem 0.25. *Let f be a meromorphic function. If $z_0 \in P_f^m$ we have*

$$\lim_{z \to z_0} f(z) = \infty$$

Proof.

$$z_0 \in P_f^m \implies f(z) = \frac{g(z)}{(z-z_0)^m}, \ z_0 \notin Z_g$$

Then

$$\lim_{z \to z_0} \frac{1}{f(z)} = \lim_{z \to z_0} \frac{(z-z_0)}{g(z)} = 0$$

□

Theorem 0.26. *If $z_0 \in P_f^1$, $\exists \epsilon > 0$ such that $f \in A_{0\epsilon}(z_0)$ and $\|f(z)\| \leq M$, $\forall z \in A_{0\epsilon}(z_0)$*

Proof. By definition we have that

$$\exists r > 0 \ : \ f \in H(A_{0\epsilon}(z_0))$$

And therefore the function is Laurent representable in this set as follows

$$f(z) = \sum_{k=0}^{\infty} c_k (z-z_0)^k \quad 0 < \|z - z_0\| < \epsilon$$

Taken the following holomorphic function

$$g(z) = \begin{cases} f(z) & z \in A_{0\epsilon}(z_0) \\ \sum_{z=0}^{\infty} c_k(z-z_0) & z = z_0 \end{cases}$$

We have that $g \in H\left(\overline{B_\epsilon}(z_0)\right)$ and therefore $\|f(z)\| \leq M \quad \forall z \in A_{0\epsilon}(z_0)$ □

Lemma 0.4.1 (Riemann). Take a function $f \in H\left(A_{0\epsilon}(z_0)\right)$ for some $\epsilon > 0$, then if $\|f(z)\| \leq M \ \forall z \in A_{0\epsilon}(z_0)$
The point z_0 is a removable singularity for f

Proof. In the set of holomorphy the function is representable with Laurent, therefore

$$f(z) = \sum_{k=0}^{\infty} c_k^+ (z - z_0)^k + \sum_{k=1}^{\infty} \frac{c_k^-}{(z - z_0)^k}$$

We have that the coefficients c_k^- are the following, where we integrate over a curve $\{\gamma\} := \{ z \in \mathbb{C} |\ \|z - z_0\| = \rho < \epsilon\}$

$$c_k^- = \frac{1}{2\pi i} \oint_\gamma f(z)(z - z_0)^{k-1} \, \mathrm{d}z$$

The function is limited, and therefore for Darboux

$$c_k^- \leq \rho^k M \to 0 \quad \forall k \geq 1$$

Therefore $z_0 \in P_f^1$ □

Theorem 0.27 (Quick Calculus Methods for Residues)**.** *Let f be a meromorphic function, then*

1. $z_0 \in P_f^n$ *then*

$$\operatorname*{Res}_{z=z_0} f(z) = \frac{1}{(n-1)!} \lim_{z \to z_0} \frac{\mathrm{d}^{n-1}}{\mathrm{d}z^{n-1}} (z - z_0)^n f(z) \tag{51}$$

2. $z_0 \in P_f^m$ *and* $f(z) = p(z)/(z - z_0)^m$, *where* $p \in \mathbb{C}_k[z]$ *with* $k \leq m - 2$ *and* $p(z_0) \neq 0$, *then*

$$\operatorname*{Res}_{z=z_0} f(z) = \operatorname*{Res}_{z=z_0} \frac{p(z)}{(z - z_0)^m} = 0$$

§ 0.5 Applications of Residue Calculus

§§ 0.5.1 Improper Integrals

Definition 0.5.1 (Improper Integral)**.** An *improper integral* is defined as the integral of a function in a domain where such function has a divergence, or where the interval is infinite. Some examples of such integrals, given a function $f(x)$ with divergences at $a, b \in \mathbb{R}$ are the following

$$\begin{aligned}
\int_c^\infty f(x) \, \mathrm{d}x &= \lim_{R \to \infty} \int_c^R f(x) \, \mathrm{d}x \\
\int_{-\infty}^d f(x) \, \mathrm{d}x &= \lim_{R \to \infty} \int_{-R}^d f(x) \, \mathrm{d}x \\
\int_{-\infty}^\infty f(x) \, \mathrm{d}x &= \int_{\mathbb{R}} f(x) \, \mathrm{d}x = \lim_{R \to \infty} \int_{-R}^R f(x) \, \mathrm{d}x \\
\int_a^b f(x) \, \mathrm{d}x &= \lim_{\epsilon \to 0^+} \int_{a+\epsilon}^{b-\epsilon} f(x) \, \mathrm{d}x \\
\int_e^h f(x) \, \mathrm{d}x &= \lim_{\epsilon \to 0^+} \left(\int_e^{a-\epsilon} f(x) \, \mathrm{d}x + \int_{a+\epsilon}^h f(x) \, \mathrm{d}x \right) \quad a \in (e, h)
\end{aligned}$$

Definition 0.5.2 (Cauchy Principal Value)**.** The previous definitions give rise to the following definition, the *Cauchy principal value.* Given an improper integral we define the Cauchy principal value as follows
Let $f(x)$ be a function with a singularity $c \in (a, b)$, and $g(x)$ another function then

$$\mathrm{PV}\int_{-\infty}^{\infty} g(x)\,\mathrm{d}x = \mathrm{PV}\int_{\mathbb{R}} g(x)\,\mathrm{d}x = \lim_{R\to\infty}\int_{-R}^{R} g(x)\,\mathrm{d}x$$
$$\mathrm{PV}\int_{a}^{b} f(x)\,\mathrm{d}x = \lim_{\epsilon\to 0^+}\left(\int_{a}^{c-\epsilon} f(x)\,\mathrm{d}x + \int_{c+\epsilon}^{b} f(x)\,\mathrm{d}x\right)$$

In the first case. PV is usually omitted.
For a complex integral, if $\gamma_R(t) = Re^{it}$ is a circumference, we have

$$\mathrm{PV}\int_{\gamma_R} f(z)\,\mathrm{d}z = \lim_{R\to\infty}\int_{\gamma_R} f(z)\,\mathrm{d}z$$

Notation (Circumferences and Parts of Circumference)**.** For a quick writing of the integrals in this section, we will use this notation for the following circumferences

$$\begin{aligned} C_R(t) &= Re^{it} \quad t \in [0, 2\pi] \\ C_{R\alpha\beta} &= Re^{it} \quad t \in [\alpha, \beta] \\ C_R^+(t) &= Re^{it} \quad t \in [0, \pi] \\ C_R^-(t) &= Re^{-it} \quad t \in [0, \pi] \\ \tilde{C^{\pm}}_R &= C_R^{\pm} \times [-R, R] \end{aligned}$$

Hypothesis 1. Let $R_0 > 0$ and $f \in C(D)$, where $D := \{\, z \in \mathbb{C} |\ \|z\| \geq R_0\} \cup \mathbb{R}$ and

$$\lim_{z\to\infty} zf(z) = 0$$

Hypothesis 2. Let $R_0 > 0$ and $f \in C(D)$, where $D := \{\, z \in \mathbb{C} |\ \|z\| \geq R_0\} \cup \mathbb{R}$ and

$$\lim_{z\to\infty} f(z) = 0$$

Theorem 0.28. *If* (1) *holds true, then*

$$\mathrm{PV}\int_{\gamma_R} f(z)\,\mathrm{d}z = 0 \quad \gamma_R = C_R, C_R^+, C_R^- \tag{52}$$

Also, if $f(x)$ is a real function

$$\int_{\mathbb{R}} f(x)\,\mathrm{d}x = \mathrm{PV}\int_{\tilde{C}_R^+} f(z)\,\mathrm{d}z = \mathrm{PV}\int_{\tilde{C}_R^-} f(z)\,\mathrm{d}z \tag{53}$$

Theorem 0.29. *Let $f(z)$ be an even function, if (1) holds we have*

$$\int_0^\infty f(x)\,\mathrm{d}x = \frac{1}{2}\,\mathrm{PV}\int_{\tilde{C}_R^+} f(z)\,\mathrm{d}z = \frac{1}{2}\,\mathrm{PV}\int_{\tilde{C}_R^-} f(z)\,\mathrm{d}z \tag{54}$$

Theorem 0.30. *Let $f(z) = g(z^k)$, $k \geq 2$. If (1) holds*

$$\int_0^\infty f(x)\,\mathrm{d}x = \frac{1}{1-e^{\frac{2i\pi}{k}}}\,\mathrm{PV}\int_{\tilde{C}_{R0,2\pi/k}} f(z)\,\mathrm{d}z \tag{55}$$

Theorem 0.31. *If (2) holds*

$$\begin{aligned} \int_{\mathbb{R}} f(x)e^{i\lambda x}\,\mathrm{d}x &= \mathrm{PV}\int_{\tilde{C}_R^+} f(z)e^{i\lambda z}\,\mathrm{d}z \quad \lambda > 0 \\ \int_{\mathbb{R}} f(x)e^{i\lambda x}\,\mathrm{d}x &= \mathrm{PV}\int_{\tilde{C}_R^-} f(z)e^{i\lambda z}\,\mathrm{d}z \quad \lambda > 0 \end{aligned} \tag{56}$$

From this, we can write then, for $\lambda > 0$

$$\begin{aligned} \int_{\mathbb{R}} f(x)\cos(i\lambda x)\,\mathrm{d}x &= \mathfrak{Re}\left(\mathrm{PV}\int_{\tilde{C}_R^+} f(z)e^{i\lambda z}\,\mathrm{d}z\right) \quad \lambda > 0 \\ \int_{\mathbb{R}} f(x)\sin(i\lambda x)\,\mathrm{d}x &= \mathfrak{Im}\left(\mathrm{PV}\int_{\tilde{C}_R^+} f(z)e^{i\lambda z}\,\mathrm{d}z\right) \quad \lambda > 0 \end{aligned} \tag{57}$$

Hypothesis 3. Let $f(z) = g(z)h(z)$ with $g(z)$ a meromorphic function such that $S_g \not\subset \mathbb{R}^+$ and

1. $h \in H(\mathbb{C} \setminus \mathbb{R}^+)$
2. $\lim_{z\to\infty} zf(z) = 0$
3. $\lim_{z\to 0} zf(z) = 0$

Definition 0.5.3 (Pacman Path)**.** Let $\Gamma_{Rr\epsilon}$ be what we will call as the *pacman path*, this path is formed by 4 different paths

$$\begin{aligned} \gamma_1(t) &= re^{it} \quad t \in [\epsilon, 2\pi - \epsilon] \\ \gamma_2 &= [-R, R] \\ \gamma_3(t) &= Re^{it} \quad t \in [\epsilon, 2\pi - \epsilon] \\ \gamma_4 &= [-R, R] \end{aligned} \tag{58}$$

We will abbreviate this as Γ

Theorem 0.32. *Given $f(x)$ a function such that (3) holds, we have that*

$$\int_0^\infty g(x)\Delta h(x)\,\mathrm{d}x = \mathrm{PV}\int_\Gamma g(z)h(z)\,\mathrm{d}z \tag{59}$$

Where

$$\Delta h(x) = \lim_{\epsilon \to 0^+} \left(h(x + i\epsilon) - h(x - i\epsilon)\right) \tag{60}$$

In general, we have the following conversion table

$h(z)$	$\Delta h(x)$
$-\frac{1}{2\pi i} \log_+(z)$	1
$\log_+(z)$	$-2\pi i$
$\log_+^2(z)$	$-2\pi i \log(x) + 4\pi^2$
$\log_+(z) - 2\pi i \log_+(z)$	$-4\pi i \log(x)$
$\frac{i}{4\pi} \log_+^2(z) + \frac{1}{2} \log_+(z)$	$\log(x)$
$[z^\alpha]^+$	$x^\alpha \left(1 - e^{2\pi i \alpha}\right)$

(61)

All the previous integrals are solved through a direct application of the residue theorem.

§§ 0.5.2 General Rules

Theorem 0.33 (Integrals of Trigonometric Functions)**.** *Let $f(\cos\theta, \sin\theta)$ be some rational function of cosines and sines. Then we have that*

$$\int_0^{2\pi} f(\cos\theta, \sin\theta)\, \mathrm{d}\theta = \int_{\|z\|=1} f\left(\frac{z + z^{-1}}{2}, \frac{z - z^{-1}}{2i}\right) \frac{\mathrm{d}z}{iz} \tag{62}$$

Theorem 0.34 (Integrals of Rational Functions)**.** *Let $f(x) = p_n(x)/q_m(x)$ with $m \geq n + 2$ and $q_m(x) \neq 0 \quad \forall x \in \mathbb{R}$, then*

$$\int_{\mathbb{R}} \frac{p_n(x)}{q_m(x)}\, \mathrm{d}x = 2\pi i \sum_k \operatorname*{Res}_{z=z_k} \frac{p_n(z)}{q_m(z)} \tag{63}$$

Lemma 0.5.1 (Jordan's Lemma)**.** Let $f(z)$ be a holomorphic function in $A := \{\, z \in \mathbb{C} |\ \|z\| > R_0,\ \mathfrak{Im}(z) \geq 0\}$.
Taken $\gamma(t) = Re^{it}$ $0 \leq t \leq \pi$ with $R > R_0$.
If $\exists M_R > 0 : \|f(z)\| \leq M_R\ \forall z \in \{\gamma\}$ and $M_R \to 0$, we have that

$$\mathrm{PV} \int_\gamma f(z) e^{iaz}\, \mathrm{d}z = 0 \quad a > 0 \tag{64}$$

Theorem 0.35. *Let $f(x) = p_n(x)/q_m(x)$ and $m \geq n + 1$ with $q_m(x) \neq 0$ $\forall x \in \mathbb{R}$, then $\forall a > 0$ we have that*

$$\int_{\mathbb{R}} \frac{p_n(x)}{q_m(x)} e^{iax}\, \mathrm{d}x = 2\pi i \sum_k \operatorname*{Res}_{z=z_k} \frac{p_n(z)}{q_m(z)} e^{iaz} \tag{65}$$

Lemma 0.5.2. Let $f(z)$ be a meromorphic function such that $z_0 \in P_f^1$ and $\gamma_r^\pm$ are semi circumferences parametrized as follows

$$\gamma_r^\pm(t) = z_0 + re^{\pm i\theta} \quad \theta \in [-\pi, 0]$$

Then

$$\mathrm{PV}\int_{\gamma_r^\pm} f(z)\,\mathrm{d}z = \pm\pi i \operatorname*{Res}_{z=z_0} f(z) \tag{66}$$

Theorem 0.36. *Let $f(x) = p_n(x)/q_m(x)$ with $m \geq n+2$ and $q_m(x)$ has $x_j \in Z_g^1\big|_{\mathbb{R}}$ then*

$$\int_{\mathbb{R}} \frac{p_n(x)}{q_m(x)}\,\mathrm{d}x = 2\pi i \sum_k \operatorname*{Res}_{z=z_k} \frac{p_n(z)}{q_m(z)} + \pi i \sum_j \operatorname*{Res}_{z=x_j} \frac{p_n(z)}{q_m(z)} \tag{67}$$

If $g(x) = r_\alpha(x)/s_\beta(x)e^{iax}$ and $\beta \geq \alpha + 1$ with $x_j \in Z_g^1\big|_{\mathbb{R}}$, then $\forall a > 0$

$$\int_{\mathbb{R}} \frac{r_\alpha(x)}{s_\beta(x)}e^{iax}\,\mathrm{d}x = 2\pi i \sum_k \operatorname*{Res}_{z=z_k} \frac{r_\alpha(z)}{s_\beta(z)}e^{iaz} + \pi i \sum_j \operatorname*{Res}_{z=x_j} \frac{r_\alpha(z)}{s_\beta(z)}e^{iaz} \tag{68}$$

z_k are all the zeros of q, s contained in the plane $\{\mathfrak{Im}(z) > 0\}$