Particle Physics

Università degli studi di Roma "La Sapienza"
Physics and Astrophysics BSc

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Notes on Particle Physics

November 3, 2023

Version 1.0

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Written by

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Contents

Part I

Nuclear and Particle Physics

1 Scattering and Cross Sections

§ 1.1 A Small Intro to Some Wacky Units

Since in particle physics usually we deal with particular calculations, it's preferable to avoid using the SI system of units, and instead pass to what I like to call, the system of «*God given units*», where the most common fundamental constants are taken as unitary, i.e. $\hbar = c = 1$.
With this choice is also common to write energy in terms of eV, i.e., using $c = 1$ and $E = \gamma mc^2$, we have that

$$[E] = [m] = \mathrm{eV}$$

Using the dispersion relation we also get

$$E = p^2 + m^2 \Longrightarrow [p] = \mathrm{eV}$$

Therefore mass, energy and momentum are all expressed in eV. Using that $1\ \mathrm{J} = (1.602)^{-1} \times 10^{19}\ \mathrm{eV}$ we get the conversion value

$$1\ \mathrm{kg} = 5.6 \times 10^{35}\ \mathrm{eV} \tag{1.1}$$

In these units we have that the mass of the electron m_e and the mass of the proton m_p are

$$\begin{aligned} m_e &= 9.109 \times 10^{-31}\ \mathrm{kg} = 0.511\ \mathrm{MeV} \\ m_p &= 1.673 \times 10^{-27}\ \mathrm{kg} = 938.3\ \mathrm{MeV} \end{aligned} \tag{1.2}$$

The second consequence of taking $\hbar = c = 1$ is that time can also be expressed in terms of eV. In fact since $[\hbar] = Js$ in the SI system, and $\hbar = 1$ in the GGS[1], we have

$$\hbar = 1.055 \times 10^{-34}\ \mathrm{Js} = 6.583 \times 10^{-22}\ \mathrm{MeVs}$$

Therefore

$$1\ \mathrm{s} = \frac{1}{\hbar}\ \mathrm{MeV}^{-1} = 1.519 \times 10^{21}\ \mathrm{MeV}^{-1} \tag{1.3}$$

Combining both $\hbar c$ we have $[\hbar c] = MeVm$, therefore we can think of expressing distances with this unit. Multiplying the constants we get

$$\hbar c = 197.35 \times 10^{-15}\ \mathrm{MeVm} = 197.35\ \mathrm{MeVfm}$$

[1] God Given System

This implies that

$$1 \text{ fm} = 5.608 \text{ GeV}^{-1} \tag{1.4}$$

In these units is also quick to see that

$$\alpha = \frac{e^2}{4\pi\epsilon_0\hbar c}\frac{1}{137} \tag{1.5}$$

§ 1.2 Cross Section

Consider a beam of particles colliding with a target which is long d. The N_p particles of the beam will react with the N_t particles of the target N_R times in some time T.

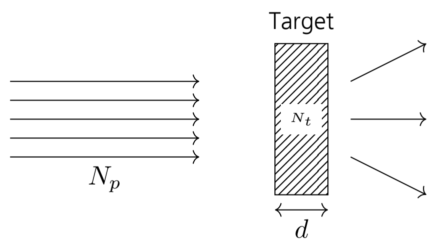


Figure 1.1: Stylization of a beam of particles colliding with a target thick d

Considering the previous data, we can immediately say that the probabilty of interaction between a beam particle and a target particle in a time T, if the particle density of the target is n_t, is

$$\sigma = \frac{1}{n_t d}\frac{N_R/T}{N_p/T} \approx \frac{1}{n_t d}\frac{\dot{N}_R}{\dot{N}_p} \tag{1.6}$$

This probability is commonly known as the «*cross-section*» for the reaction.

Consider now a beam with cross sectional surface S

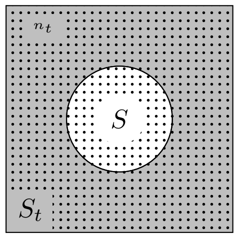


Figure 1.2: Cross section of the beam with surface S hitting a target with surface S_t and particle density n_t

Since we have $S < S_t$ as in figure (1.2) we have that the target is going to get hit in a fraction of his surface. Considering that the target is long d, we have that the amount of particles that might get hit by the beam will be

$$N_t = n_t S d \Longrightarrow n_t d = \frac{N_t}{S} \tag{1.7}$$

Inserting it back into the formula of the cross section we have

$$\sigma = \frac{\dot{N}_R}{N_t}\frac{1}{S\dot{N}_p} \tag{1.8}$$

Since $S\dot{N}_p$ gives the amount of particles passing through a cross-sectional surface of the beam per unit time (i.e., a flux), we can define the flux of particles ϕ_p as follows

$$\phi_p = S\frac{\mathrm{d}N_p}{\mathrm{d}t} \tag{1.9}$$

I.e.

$$\sigma = \frac{1}{N_t \phi_p}\frac{\mathrm{d}N_R}{\mathrm{d}t} \tag{1.10}$$

Noting also that if the particle beam is long L, if its particle density is n_p we have, if N_p is the total number of beam particles moving with average velocity v_p

$$N_p = n_p v_p S \mathrm{d}t$$

And therefore, since N_p is fixed

$$\phi_p = \frac{1}{S}\frac{\mathrm{d}N_p}{\mathrm{d}t} = n_p v_p$$

Which gives

$$\sigma = \frac{1}{n_p v_p N_t}\frac{\mathrm{d}N_R}{\mathrm{d}t}$$

And solving for $\dot{N}_R$ we have

$$\frac{\mathrm{d}N_R}{\mathrm{d}t} = \sigma n_t d \frac{\mathrm{d}N_p}{\mathrm{d}t} \tag{1.11}$$

Which is the differential equation that gives the number of reactions per unit time. Note how $[\sigma] = L^2$, therefore it has units of m^2 in the S.I., in particle physics these units are quite uncomfortable since the cross sections evaluated are infinitesimally small, and another unit is used, the «*barn*». As follows there are the conversions

$$1 \text{ b} = 10^{-28} \text{ m}^2 \tag{1.12}$$

For common reactions we have

$$\begin{aligned}
&\mathrm{p} + \mathrm{p} \to \mathrm{X} &\qquad& \sigma_{pp} \approx 10\ \mathrm{mb} = 10^{-30}\ \mathrm{m}^2 \\
&\nu_\mathrm{e} + \mathrm{p} \to \mathrm{X} && \sigma_{\nu p} \approx 10\ \mathrm{fb} = 10^{-14}\ \mathrm{b} \\
&\chi + \mathrm{p} \to \mathrm{X} && \sigma_{\chi p} \approx 10^{-2}\ \mathrm{fb} = 10^{-17}\ \mathrm{b} = 10^{-45}\ \mathrm{m}^2
\end{aligned} \tag{1.13}$$

§§ 1.2.1 Crossed Beam Interaction and Luminosity

Consider two beams colliding frontally like they'd do in the LHC, a 27km particle accelerator ring.

Figure 1.3: Schematization of the two beams colliding at some point inside a particle collider

The target we're now considering is a second beam with cross sectional surface $S_2 = S_1 = S$ and N_2 particles. Rewriting the formulas using $N_p = N_1$ and $N_t = N_2$ we get

$$\frac{\mathrm{d}N_R}{\mathrm{d}t} = \sigma n_2 d \frac{\mathrm{d}N_1}{\mathrm{d}t} = \frac{\sigma}{S} N_2 \frac{\mathrm{d}N_1}{\mathrm{d}t}$$

Using $\phi_1 = S^{-1}\dot{N}_1$ we simply get

$$\frac{\mathrm{d}N_R}{\mathrm{d}t} = \sigma \phi_1 N_2 \tag{1.14}$$

We can define a new quantity, f_{int}, i.e. the «*interaction frequency*» of the beam, we have that the flux can be redefined as follows

$$\phi_1 = \frac{N_1 f_{int}}{S} \tag{1.15}$$

And this gives

$$\frac{\mathrm{d}N_R}{\mathrm{d}t} = \sigma \frac{N_1 N_2 f_{int}}{S} = \sigma \mathcal{L} \tag{1.16}$$

The new quantity $\mathcal{L}$ is called the «*instantaneous luminosity*» of the collider and it has units of $\mathrm{L}^{-2}\mathrm{T}^{-1}$ and is commonly expressed therefore in $\mathrm{Hz\ b}^{-1}$ or $\mathrm{Hz\ cm}^{-2}$.
Integrating the previous formula we get the «*integrated luminosity*» L, measured in b^{-1}

$$L = \int_{t_0}^{t_1} \mathcal{L} \mathrm{d}t$$

§§ 1.2.2 Higgs Bosons, Exclusive and Inclusive Cross Sections

Going back to what we said for the luminosity of a detector, for the well known Large Hadron Collider we have

$$\mathcal{L}_{LHC} \approx 10^{34} \ \frac{\text{Hz}}{\text{cm}^2} = 100 \ \frac{\text{Hz}}{\text{b}}$$

More specifically, during the years and the various runs it varied. The highest value was reached with run 3 of the LHC in 2018, where

$$L_{LHC} = 163 \ \text{fb}^{-1}$$

This value can be used to estimate the number of events happened for a certain reaction. Take as an example the creation of Higgs boson

$$p + p \rightarrow H + X$$

The number of Higgs produced will be given using the integrated luminosity of LHC in run 3 and the cross section for the creation of $H + X$ particles

$$N_H = L\sigma_{HX}$$

Here $\sigma_{HX} \approx 10^{-10}$ b and $\sqrt{s} = 13$ TeV.

$$N_H = 163 \cdot 10^{-10} \text{bfb}^{-1} = 163 \cdot 10^5$$

Therefore, in run 3, between 2015 and 2018 around 16 million Higgs bosons were produced in the LHC.
It's important tho to note that there are various ways a Higgs boson can decay, which will be what is going to be measured at the detector. These decays will be

$$\begin{aligned} &H \rightarrow bb \\ &H \rightarrow \gamma\gamma \\ &H \rightarrow ZZ \\ &H \rightarrow WW \end{aligned}$$

The probability for having one of these decays is known as the «*branching factor*» BF.
Suppose that now we want to see how many reactions of the kind $H \rightarrow \gamma\gamma$ happened in run 3, then we will have to weigh the total number of Higgs bosons with its branching factor

$$N_{\gamma\gamma} = N_H BF_{H\rightarrow\gamma\gamma} \tag{1.17}$$

Suppose instead that we want to analyze a more complex decay of the Higgs boson

$$\begin{aligned} &H \rightarrow ZZ \\ &ZZ \rightarrow \begin{cases} e^+ + e^- \\ \mu^+ + \mu^- \\ \tau^+ + \tau^- \end{cases} \end{aligned} \tag{1.18}$$

The branching factor for the $H \rightarrow ZZ$ is $BF_{H\rightarrow ZZ} = 3\%$, which means that

$$N_{ZZ} = 163 \cdot 10^5 \cdot 3 \cdot 10^{-2} = 489 \cdot 10^3$$

This is not enough to check how many Higgs have been produced, since only the $Z + Z \to e^+ + e^-$ reaction is measured, and the total chain reaction we gotta consider becomes

$$\begin{aligned}
p + p &\longrightarrow H + X \\
&\qquad \hookrightarrow Z^\pm + Z^\pm \\
&\qquad\qquad\qquad \hookrightarrow e^+ + e^- \\
&\qquad\qquad \hookrightarrow e^+ + e^-
\end{aligned}$$

Therefore, the real number of measured Higgs bosons $N_H^\star$ will be

$$N_H^\star = \sigma_H L B F_{ZZ} B F_{Zee}^2 = 163 \cdot 10^5 \cdot 9 \cdot 10^{-4} \approx 432$$

I.e., if the LHC's CMS detector was a perfect detector we would have measured *at most* 432 Higgs bosons in the 4 full years of the third run.
A major problem now comes into play. What if I don't know the cross section for the reaction σ_{HX}?. In general we have that the number of particles is given by the number of counts of the detector, the integrated luminosity is measured directly, and the branching factors can be either measured or determined theoretically. Therefore, simply inverting and adding a factor ε determining the efficiency of the detector, which accounts for imperfections in its surface, we have

$$\sigma_{HX} = \frac{N_H}{L B F_{HZZ} B F_{Zee}^2} \frac{1}{\varepsilon}$$

This is not enough for a proper determination of values, since in the LHC there are various reactions that get measured. We have that for each reaction with cross section σ_i, the total cross section of the $p + p \to X$ reaction is

$$\sigma_{ppX} = \sum_{i=1}^{n} \sigma_i \tag{1.19}$$

The cross sections on the right are known as «*exclusive cross sections*» and the one on the left is the «*total cross section*» for the measured event.
Note that in searching for the Higgs boson, we actually want to measure the photons from pair annihilation of the two electrons of the previous reaction, since their energies will peak around $m_H c^2$, giving the complete reaction

$$\begin{aligned}
p + p &\longrightarrow H + X \\
&\qquad \hookrightarrow Z^\pm + Z^\pm \\
&\qquad\qquad\qquad \hookrightarrow e^+ + e^- \longrightarrow \gamma + \gamma \\
&\qquad\qquad \hookrightarrow e^+ + e^- \longrightarrow \gamma + \gamma
\end{aligned}$$

Generally we're seeing the decay of a state $|pp\rangle$ into a state $|e^+e^-e^+e^-\rangle$, passing through a $Z^\pm$ state. Using SR we have

$$\begin{aligned}
P_{Z_1}^\mu &= p_{e_1^+}^\mu + p_{e_1^-}^\mu \\
P_{Z_2}^\mu &= p_{e_2^+}^\mu + p_{e_2^-}^\mu
\end{aligned} \tag{1.20}$$

The 4 moment of the 4 electrons is measured, and using the well known formula for the invariant mass we have that

$$m_{ZZ} = \sqrt{P^{\mu}_{(1)} P^{(2)}_{\mu}}$$

Which, also gives the energies of the emitted photons in the center of mass of the decay.
Measuring various combinations of such at LHC in 2012, finally a peak in counts of reactions at around $E^{\star}_{\gamma\gamma} \approx 125$ GeV was measured, which corresponds to the measured mass of this famous boson.

§§ 1.2.3 Mean Free Path

Starting again for the usual and now well known relation for giving the number of reactions per unit time for some beam of particles hitting a target long l, we can imagine to generalize everything to differential lengths $l \to \mathrm{d}x$, giving us

$$\frac{\mathrm{d}N_R}{\mathrm{d}t} = \sigma n_t \frac{\mathrm{d}N_p}{\mathrm{d}t} \mathrm{d}x$$

The interaction probability will be

$$P_{int} = \frac{\dot{N}_R}{\dot{N}_p} = \sigma n_t \mathrm{d}x = \sigma \frac{N_t}{S}$$

Define now an «*absorption coefficient*» μ. This coefficient indicates the amount of particles of the beam that get "absorbed" by the target per unit length. Using some reasoning with the previous formulae we must have that it must depend linearly to the linear density of particles of the target n_t, and therefore

$$\mu = \sigma n_t \tag{1.21}$$

We have now that the pdf for the interaction energy is

$$P_{int} = \mu \mathrm{d}x \tag{1.22}$$

This, must also be proportional to the derivative of the flux, since the flux must lower by some quantity while passing through the target. Finally we have

$$\frac{\mathrm{d}\phi_p}{\mathrm{d}x} = -P_{int}\phi_p = -\mu\phi_p \mathrm{d}x \tag{1.23}$$

Solving the ODE we get

$$\phi_p(x) = \phi_0 e^{-\mu x} \tag{1.24}$$

Noting that μ has units of L^{-1} we can imagine to define a length λ as the inverse of this attenuation factor, known as the «*mean free path*» of the particle.
The main utility of this length is that it can be used to determine the interaction cross section experimentally. In fact, imagine sending a flux of particles through N targets with increasing lengths d_i as in the next figure

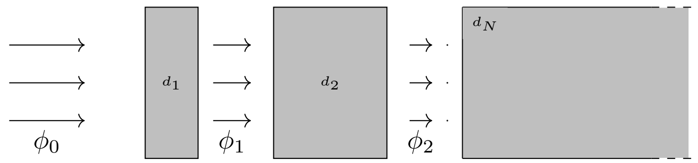


Figure 1.4: Example sketch of the experiment

Since $\phi \propto e^{-\mu x}$ we could use an exponential fit to find μ. Using $\mu = \sigma n_t$ it's possible to estimate σ if the properties of the target element are known

§ 1.3 Differential Cross Section

Consider now a realistic approach for the collision of a beam of particles with a target. In this realistic approach the detectors will occupy part of the space around the target, and therefore there will be some preferred angles in order to detect properly the particle.
This final angle depends on the flight angle of the scattered particle.

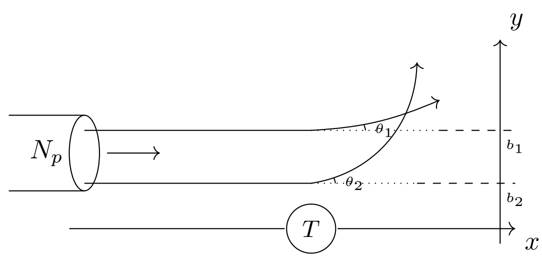


Figure 1.5: Sketch of the scattering process in study

In the previous figure it's possible to see the sketch of this kind of scattering. The two parameters b_1, b_2 are known as the «*impact parameters*» of the two particles, while the two angles θ_1, θ_2 are the «*flight angles*» of the two particles.
In general, it's not hard to believe that this flight angle will depend on the impact parameter b which is the vertical distance between the target and the unperturbed particle path's y height.
From this supposition we can immediately say

$$\begin{cases} g(b) = \theta \\ f(\theta) = b \end{cases} \Longrightarrow \mathrm{d}b \propto \mathrm{d}\theta$$

Imagining a toroidal detector around the target we can transform this reasoning into 3D, where $\mathrm{d}\theta \to \mathrm{d}\Omega = \sin\theta \mathrm{d}\theta \mathrm{d}\varphi$. In this case, noting that $b\mathrm{d}b = \sin\theta \mathrm{d}\theta$ we get a new «*differential cross-section*» $\mathrm{d}\sigma$

$$\mathrm{d}\sigma = b\mathrm{d}b\mathrm{d}\theta \tag{1.25}$$

Manipulating this a bit we get

$$\mathrm{d}\sigma = \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\mathrm{d}\Omega = \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}|\sin\theta \mathrm{d}\theta \mathrm{d}\varphi| = b\mathrm{d}b\mathrm{d}\varphi$$

Or, rearranging everything

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{b}{|\sin\theta|}\left|\frac{\mathrm{d}b}{\mathrm{d}\theta}\right| \tag{1.26}$$

Where the absolute value comes from $\sigma > 0$.

Example 1.3.1 (An Easy Example of Differential Cross-Section). As an example it's really exemplar finding the differential cross section for a particle hitting a rigid sphere with radius R.

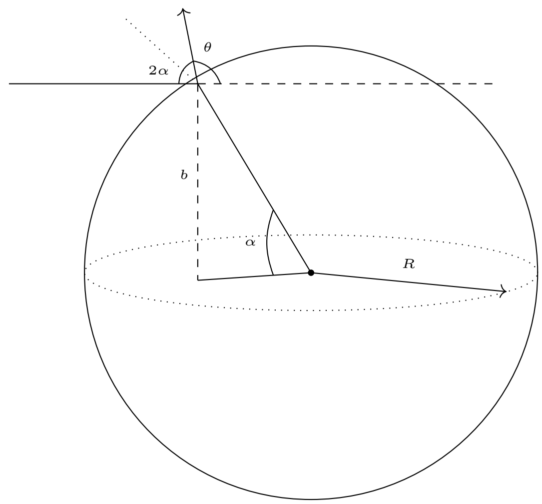


Figure 1.6: Quick sketch of the particle beam colliding with the cited sphere

This rigid sphere can be imagined as a potential wall in 3D spherical coordinates, where, simply

$$\mathcal{U}(R) = \begin{cases} 0 & r > R \\ \infty & r < R \end{cases}$$

It's obvious from the picture also that:

$$\begin{cases} b = R\sin\alpha \\ 2\alpha + \theta = \pi \end{cases}$$

Solving the second equations for $\alpha(\theta)$ we get that

$$\alpha = \frac{\pi - \theta}{2}$$

And therefore

$$\sin(\alpha) = \cos\left(\frac{\theta}{2}\right) \Longrightarrow b = R\cos\left(\frac{\theta}{2}\right)$$

Deriving b with respect to θ and getting its absolute value we get

$$\left|\frac{\mathrm{d}b}{\mathrm{d}\theta}\right| = \frac{R}{2}\left|\sin\left(\frac{\theta}{2}\right)\right|$$

And, simply substituting everything into (1.26) we get

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{R^2}{2|\sin\theta|}\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right) = \frac{R^2}{4}$$

Having now a simple differential equation for σ we have finally

$$\sigma = \frac{R^2}{4}\int_{4\pi}\mathrm{d}\Omega = \pi R^2$$

Which means that, if the beam has cross sectional surface S, the probability of interaction of the beam with the sphere is

$$P_{int} = \frac{\pi R^2}{S}$$

This is in complete accord with the impulsive idea that the interaction probability in this case will be given from the exposed surface of the sphere divided by the surface of the beam, giving without problems a function of reactions in terms of cross-sectional surface of the beam

$$N_R(S) = \frac{\sigma}{S} = \frac{\pi R^2}{S}$$

Having said all of this one might rightfully ask how would someone measure the differential cross section.
Start with hitting the target in question with a beam of particles with known flux, then the number of counts per unit time will be the usual well known formula

$$\frac{\mathrm{d}N_R}{\mathrm{d}t} = \sigma n_t d\frac{\mathrm{d}N_p}{\mathrm{d}t}$$

Substituting inside this the differential quantities we have

$$\frac{\mathrm{d}N_R}{\mathrm{d}t} = n_t d\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\frac{\mathrm{d}N_p}{\mathrm{d}t}\mathrm{d}\Omega$$

Expressing $n_t d$ in terms of differential surface we can say immediately that

$$\frac{n_t}{S}\frac{\mathrm{d}N_p}{\mathrm{d}t}\mathrm{d}S = \phi_p(x)N_t$$

Which gives

$$\frac{\mathrm{d}N_R}{\mathrm{d}t} = \phi_p(x)N_t\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\mathrm{d}\Omega$$

Since every term on the right is known and the number of counts per unit time is measured by the detector we can solve for the differential cross section, getting

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{\phi_p(x)N_t}\frac{\mathrm{d}N_R}{\mathrm{d}t}\frac{1}{\mathrm{d}\Omega} \tag{1.27}$$

2 Nuclear Physics

§ 2.1 First Discoveries

At the end of the '800s we managed to discover radioactivity, and it was divided as follows

- X rays, discovered by Röntgen in 1895
- Natural radioactivity observed in phosphorescent materials as ^{238}U salts, of which
  - α rays
  - β rays
  - γ rays

The firsts, X-rays, were known for passing easily through matter and leave traces in photographic tables, they're now known as high energy photons
The last, natural radiation in the shape of α, β and γ radiation are now today known as emission of particular particles by atomic nuclei

1. γ rays, today known as photons for which $E > E_X$
2. β rays, today known as electrons and their antimatter counterpart, the positron
3. α rays, now known as Helium nuclei, which are emitted only from heavy nuclei

§§ 2.1.1 Thompson and the Discovery of the Electron

One of the first experiments was conducted by Thompson, Milliken et. al which studied the nature of β particles using a CRT as in picture

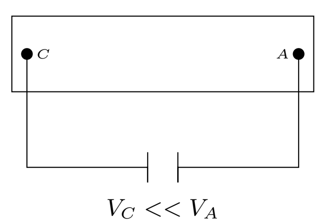


Figure 2.1: An example of a CRT's functioning, the cathode C and the anode A are enclosed in a glass tube

The Thompson-Milliken experiment goes as follows: Filling the CRT tube with different gases it's possible to measure events in function of the kind of gas, its pressure, the potential difference $V_A - V_C$ and obviously in function of the material used to produce the cathode.
The observations that the scientists reported were

- A green luminescence close to the anodes
- Electric shocks present also with small ΔV
- The electrical screening between the anode and cathode grows with ΔV
- All these effects are independent from the presence of a magnetic field B

They ended up with an hypothesis: CRT rays must be charged particles.
Thompson went forward proving this setting a new experiment, set up as in the following picture

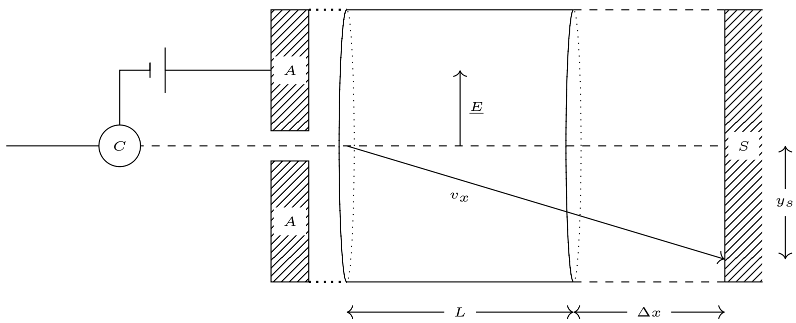


Figure 2.2: Scheme of the Thompson experiment. The electrons emitted from the cathode C travel through a cylindrical capacitor long L at a constant velocity v_x. Interacting with the electric field the path of the radiation variates and this variation is measured in a screen S at the end

In this experiment, we have $V_A >> V_C$ and therefore the electrons are accelerated from C to A, from which continue traveling with constant velocity v_x. The y component of the velocity gets perturbed by an electrostatic force, which gives us, inside the cylinder

$$\begin{cases} a_x = 0 \\ a_y = \dfrac{q}{m}E \end{cases} \tag{2.1}$$

Which implies

$$\begin{cases} x(t) = v_x t \\ y(t) = \dfrac{q}{2m}Et^2 \end{cases}$$

Since the capacitor is long L we have that when exiting the capacitor at a time $t_L = L/v_x$ the particle will be at the following y position with v_y velocity

$$\begin{cases} y(t_L) = \dfrac{q}{2m}\dfrac{EL^2}{v_x^2} \\ v_y(t_L) = \dfrac{q}{m}\dfrac{EL}{v_x} \end{cases}$$

The final measured y position on the screen, y_s will be given by an easy computation

$$y_s = y(t_L) + v_y(t_L)\frac{\Delta x}{v_x}$$

Where Δx is the horizontal distance between the end of the capacitor and the screen.
We get therefore

$$y_s = \frac{qE}{2m}\frac{L^2}{v_x^2} + \frac{qE}{m}\frac{L\Delta x}{v_x^2} = \frac{qE}{m}\frac{L}{v_x^2}\left(\frac{L}{2} + \Delta x\right) \tag{2.2}$$

The problem with this equation is that v_x is unknown and therefore it's not possible to find q/m.
Accounting for an orthogonal magnetic field B and applying the Lorentz force it's possible to find that

$$v_x = \frac{E}{B}$$

And therefore

$$\frac{q}{m} = \frac{y_s}{L\left(\frac{L}{2} + \Delta x\right)}\frac{E}{B} \tag{2.3}$$

This is a property of the projectile since it's invariant with the gas and the components of the anodes, and Nobel in 1906 calculated

$$\frac{q}{m} = 1.76 \cdot 10^{11}\ \mathrm{C/kg}$$

A subsequent experiment by Milliken et al. in 1923 used a falling drop of oil in order to calculate the charge of this new particle. The setup of the experiment was as follows

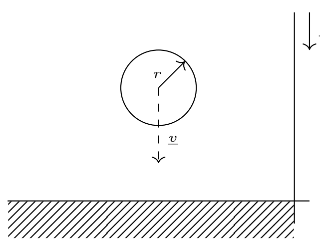


Figure 2.3: Experimental setup of the falling droplet of oil

This experiment used a falling drop of oil. The mass of the drop is quickly calculated assuming a spherical drop

$$m_p = \frac{4\pi}{3}r^3\rho_{oil}$$

Using hydrodynamics the drag force's modulus is quickly calculated accounting for the viscosity η as follows

$$F_d = -6\pi\eta r v_0$$

Where v_0 is experimentally measured as the terminal velocity of the droplet. The classical equilibrium is reached when the weight of the droplet is balanced by the hydrodynamic drag

$$\frac{4\pi}{3}\pi r^3\rho_{oil}g = 6\pi r v_0$$

Solving for r we get a measurable formula for the radius of the droplet, i.e.

$$r = \sqrt[3]{\frac{\eta v_0}{2\rho_{oil} g}} \tag{2.4}$$

Applying now an electric field opposite to the motion of the droplet we get a new terminal velocity v_1 which can be experimentally measured, gives a relation between the known measured variables and the charge, as follows

$$qE = 2\pi r \left(\frac{2r^2}{3} \rho_{oil} g - 3\eta v_1 \right)$$

Substituting for r and with some algebra we get

$$qE = 18\eta\pi \sqrt{\frac{\eta v_0}{2\rho_{oil} g}} (v_0 - v_1)$$

Since E is known and the variation in terminal velocities of the droplet is directly measured we get that the charge of this new particle is

$$q = \frac{18\eta\pi}{E} \sqrt{\frac{\eta v_0}{2\rho_{oil} g}} \Delta v = 1.59 \cdot 10^{-19} \text{ C} \tag{2.5}$$

The final value reached from Milliken is so precise that its relative error is in the 1% from the modern known value.
Mixing the result of these two experiment it's possible to evaluate the mass of these beta particles, which gives

$$m_\beta = 0.911 \cdot 10^{-30} \text{ Kg} \approx 511 \text{ KeV} \tag{2.6}$$

This beta particle is now a well known fundamental particle that we treated already in depth in the previous chapters, which is the electron.

§§ 2.1.2 Discovery of the Nucleus

After the discovery of the electron and its properties with β radiation, Rutherford et al. proceeded with a new experiment trying to discern the physics behind α particles. It was known that for some atom, with A big enough, we can have a radioactive decay through the emission of an α particle, in the following reaction

$$^{A}_{Z}\text{N} \to ^{A'}_{Z'}\text{M} + \alpha \tag{2.7}$$

Note that today it's known that $\alpha = ^4_2\text{He}$, $m_\alpha = 3.7$ GeV and $A' = A - 4$, $Z' = Z - 2$.
These α particles emitted naturally by these heavy nuclei have a center of mass energy of around 5 MeV. Using special relativity we know that they aren't relativistic, in fact we have

$$E_{cm} = E_\alpha - m_\alpha = (\gamma - 1) m_\alpha \Longrightarrow \gamma \approx 2.35 \cdot 10^{-3}$$

Rutherford's experiment was initially in order to determine which of the nuclear models was true. At the time there were two main contending ideas, one being Thompson's idea, where the electrons $q < 0$ were situated inside this nucleus with $q_n > 0$ and the total atom was neutral (the so called Pancake nucleus) whereas Rutherford supposed of an atom with the positive charges all in the center and electrons orbited these positive charges, rendering in general the atom neutral.

This experiment is extremely similar to the setup (1.5), where the target is our nucleus with $q > 0$.
The experimental setup that Rutherford et al. used consisted of a radioactive α emitter bar targeting these particles to a foil of gold ${}_{79}\mathrm{Au}$ which is surrounded by a $150°$ spherical detector, which would measure the differences in the deflection angles.
The source chosen was Radium Bromide, RaBr_2, an alpha emitter radioactive compound.

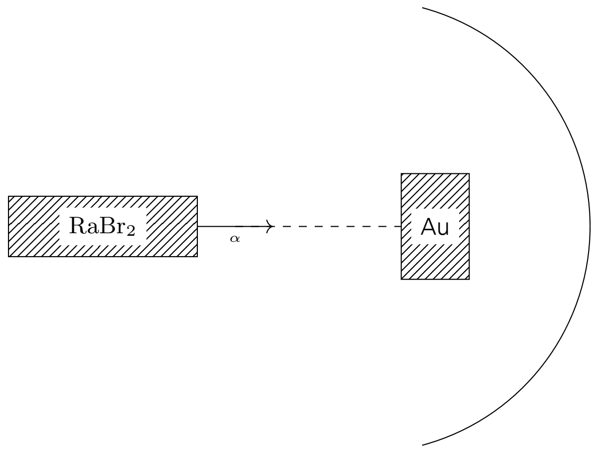


Figure 2.4: The setup was as follows, the radioactive source was placed at some distance from a gold plate at the center of a semi-spherical detector, which would then be used to determine the deviation angles of the deflected α particles

The main scope of the experiment was to measure the numbers of reaction with respect to the deviation angle.
This can be done by firstly finding the differential cross section. The experimental setup makes it much easier since the spherical symmetry of the system easies the calculation.
Since the angular momentum of the particles and energy is conserved we can begin by writing the Lagrangian of the system

$$\mathcal{L}_\alpha = \frac{m_\alpha}{2}\left(\dot{r}^2 + r^2\dot{\varphi}^2\right) - \mathcal{U}(r) \tag{2.8}$$

Thanks to the spherical symmetry we have that $\dot{\varphi}$ is cyclical and its momentum will be conserved, i.e.

$$p_\varphi = mr^2\dot{\varphi} = L$$

And the energy of the system is

$$E = p\dot{q} - \mathcal{L}_\alpha = \frac{1}{2}m_\alpha\dot{r}^2 + \frac{1}{2}m_\alpha r^2\dot{\varphi}^2 + \mathcal{U}(r)$$

Substituting the cyclical coordinate with the conserved quantity L we have

$$E = \frac{1}{2}m\dot{r}^2 + \frac{L^2}{2mr^2} + \mathcal{U}(r) = T(\dot{r}) + \mathcal{U}_{eff}(r) \tag{2.9}$$

Where $\mathcal{U}_{eff}(r)$ is an effective potential that includes the centrifugal potential.

Solving for $\dot{r}$ we get thanks to the conservation of energy the following differential equation

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \sqrt{\frac{2}{m_\alpha}}\sqrt{E - \frac{L^2}{2mr^2} - \mathcal{U}(r)}$$

In order to evaluate in terms of the deflection angle, using p_φ we can substitute $\mathrm{d}t$ with $\mathrm{d}\varphi$, using the straightforward substitution

$$\mathrm{d}\varphi = \frac{L}{m_\alpha r^2}\mathrm{d}t$$

Which gives

$$\mathrm{d}\varphi = \frac{\frac{L}{m_\alpha r^2}}{\sqrt{2m_\alpha}\sqrt{(E - \mathcal{U}(r)) - \frac{L^2}{r^2}}}\mathrm{d}r$$

It's now useful to pass everything to the impact parameters for the α particle.
The conservation of energy permits us to write without problems that

$$\begin{cases} p_i = m_\alpha v_\infty \\ E_\infty = E_0 = \dfrac{1}{2}m_\alpha v_\infty^2 \\ L = \|r_\infty \wedge p_\infty\| = b m_\alpha v_\infty \end{cases} \tag{2.10}$$

Where r_∞ is the distance of the particle from the nucleus of gold, v_∞ is the velocity "at infinity", i.e. the velocity of the non-interacting particle, and b is the already known impact parameter.
Substituting into the differential of φ we have that

$$2m_\alpha\left(\frac{1}{2}m_\alpha v_\infty^2 - \mathcal{U}(r) - \frac{b^2 m^2 v_\infty^2}{r^2}\right) = m_\alpha^2 v_\infty^2\left(1 - \frac{\mathcal{U}(r)}{E_\infty} - \frac{b^2}{r^2}\right)$$

Substituting it back and integrating we have

$$\varphi(r) = \int_{r_{min}}^{\infty}\frac{b}{r\sqrt{1 - \frac{\mathcal{U}(r)}{E_\infty} - \frac{b^2}{r^2}}}\mathrm{d}r$$

The potential is the usual Coulomb potential, which in natural units is

$$\mathcal{U}(r) = \frac{\alpha Z_p Z_n}{r} = \frac{A}{r}$$

Substituting it back into the integral we get a solvable integral

$$\varphi(r) = \frac{r_{min}}{\infty}\frac{b}{r\sqrt{1 - \frac{A}{E_\infty r} - \frac{b^2}{r^2}}}\mathrm{d}r = \arccos\left[\frac{\frac{A}{2E_\infty b}}{\sqrt{1 + \left(\frac{A^2}{2E_\infty b}\right)}}\right] \tag{2.11}$$

Writing $B = A/2E_\infty b$ lets us invert the function in terms of the impact parameter, giving us

$$b(\varphi) = \frac{A}{2E_\infty}\tan(\varphi) \tag{2.12}$$

It's quick to see that $\varphi = \pi/2 - \theta/2$, therefore

$$b(\theta) = \frac{A}{2E_\infty} \cot\left(\frac{\theta}{2}\right) \tag{2.13}$$

Deriving with respect to θ and inserting it into the equation for the differential cross section we get

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\alpha^2 Z_p^2 Z_n^2}{16E_\infty^2} \csc^4\left(\frac{\theta}{2}\right) \tag{2.14}$$

Rutherford confirmed this cross section, and went forward estimating the value of r_{min}. Using $E_\infty = E_\alpha = 5$ MeV and the conservation of energy we have

$$\mathcal{U}(r_{min}) = \frac{\alpha Z_p Z_n}{r_{min}} = \frac{1}{2} m_\alpha v_\infty^2 = 5\ \mathrm{MeV}$$

I.e.

$$r_{min} = \frac{\alpha Z_p Z_n}{5}\ \mathrm{MeV}^{-1} = 0.23\ \mathrm{MeV}^{-1}$$

Converting into more usual units we have that $r_{min} = 46$ fm, which from the experiment it was confirmed that $r_{min} < 30$ fm.

§§ 2.1.3 Discovery of the Proton and of the Neutron

Again from Rutherford et al. in 1918, still using α radiation, the proton was discovered.
Consider the following nuclear reaction

$$\alpha + {}^{14}_{7}\mathrm{N} \to {}^{17}_{8}\mathrm{O} + X$$

This reaction included an artificial nuclear transmutation and the emission of an unknown particle X on which a spectrography was executed and the measured q/m was compatible with H^+. This particle was called the proton, with symbol p. The reaction was now completed

$$\alpha + {}^{14}_{7}\mathrm{N} \to {}^{17}_{8}\mathrm{O} + p$$

Continuing on this path, Chadwick et al. studied another reaction, for which an unknown neutral particle was discovered. The reaction is

$$\alpha + {}^{9}_{4}\mathrm{Be} \to {}^{12}_{6}\mathrm{C} + X$$

The creation of this particle was observed also with ${}^{9}_{4}\mathrm{Li}$ and ${}^{9}_{4}\mathrm{B}$ This particle is heavily piercing, and therefore two main hypotheses were considered

1. The particle is a photon
2. It's a new heavy neutral particle

The idea for evaluating which is true was to consider the scattering with a proton from which to measure the impulse of p and therefore evaluate the mass of this unknown particle.
The measured impulse of the proton after the scattering event had velocities $\beta \approx 0.1$ and therefore can be considered weakly relativistic.
Curie et al. went forward proposing the idea that the X particle was a photon with an energy of around 50 MeV, which tho contradicted previous experiments for which the energy of such photon would have been of the order of one MeV.
Chadwick et al. finally concluded that this was a new neutral massive particle, for which $m_n \approx m_p \pm 10\%$.
This particle was called the neutron, and is between the early fundamental blocks of nuclear physics together with p, e^- and α particles.

§§ 2.1.4 Modern Considerations and Experiments

In our modern understanding of nuclear physics we know 7 main properties of atomic nuclei

- Mass A
- Charge Z, number of protons
- Number of neutrons $N = A - Z$
- Nuclear spin
- Magnetic moment μ
- Electric moments and quadrupole moments $'m$
- Isospin

All the chemical properties are tied to the charge of the nucleus Z and therefore compose the periodic table.
With these properties one can use the two main ones, A, Z in order to build a couple and from there identify a nucleus. In general, a nuclear object with mass and charge (A, Z) will be called a «*nuclide*».
An «*isotope*» is a nuclide of some determined element (A', Z) for which although the mass is different, the charge is the same. Vice-versa one can define «*isobars*», or a nuclide of some determined element (A, Z') for which charge is different but the mass is the same.
The properties of a nucleus can be found out with different experiments.
Starting for mass one can repeat the Thompson experiment using a mass spectrometer, while for charge one can use an X ray spectroscopy of the internal electron shells of the atom. Other experiments include tests for finding the radius of a nuclide, repeating the Chadwick-Rutherford experiment with either highly energetic electrons in order to have a smaller resolution, or using μ-mesic studies of the atom with muons instead of electrons, which all give finer details on the nucleus.
The final conclusion from all the various experiments are that

- The matter distribution is proportional to the charge distribution, i.e. $A \propto Z$
- The nucleus is in good approximation a sphere, for which $R \approx R_0 \sqrt[3]{A}$ with $R_0 = 1.1$ fm
- The volume enclosed in a nucleus is proportional to the mass of such, i.e. $V_n \propto A$

§ 2.2 Nuclear Binding Energy

§§ 2.2.1 Stability and Radioactivity

Suppose having some nuclide (A, Z) with unknown mass. In general, we can say without doubt that the total mass of the nuclide will be smaller than the sum of the masses of the components.

$$M(A,Z) < Zm_p + (A-Z)m_n$$

This is immediately obvious in a relativistic context, in fact we're not yet accounting for the binding energy of the nucleons, which is for sure negative, and together with that, we're not accounting for the electronic binding energy.
With some simple calculations and noting that obviously the binding energy of the electrons can be neglected, we have that the binding energy B of such nucleon will be

$$B(A,Z) = Zm_p + (A-Z)m_n - M(A,Z) \tag{2.15}$$

The experimental determination of this value can be made via a spectrometer for some stable nucleus, and using nuclear reactions for unstable radioactive nuclei.
The shape of the binding energy function for nuclei, must include a stable region of nuclides for $A \approx 60$. This corresponds to the fact that Fe is the most stable nuclide, which is well known from astrophysical processes in stars.
In general we have that for

- $A \geq 30$, $\frac{B}{A} \approx 8$ MeV
- $A = 60$, $\frac{B}{A} \approx 8.5$ MeV
- $A > 60$, $\frac{B}{A} \to 7.5$ MeV

But,what is precisely B/A? Quantum mechanically nuclides are bound states of neutrons and protons, that behave exactly as a quantum bound state would.
The main caveat of this is that since the nuclei are quite energetic, all excited states emit high energy photons. This is the proper origin of gamma radiation.
Take some nuclide in some excited state $|E^\star\rangle$. This state is unstable and will decay towards a $|E\rangle$ ground state.
Suppose that the level corresponding to $|E^\star\rangle$ is as in the following diagram

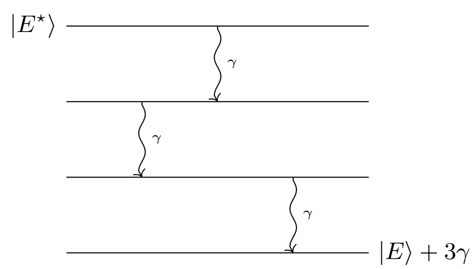


Figure 2.5: Level diagram for a gamma emission from a nuclide

The half-life of an excited nuclear state can be $10^{-17}\mathrm{s} \lesssim \tau^{\star}_{1/2} \lesssim 10^{2}\mathrm{y}$. In general the gamma decay has an exponential nature. Given some nuclide ${}^{A}_{Z}\mathrm{X}$ we have

$$N\left({}^{A}_{Z}\mathrm{X}^{\star}\right) = N_0 e^{-\frac{t}{\tau}}$$

The half life will then be half e-folding time of the previous equation.
It's important to note that not all ground states of nuclides are stable, in fact there exist various radionuclides, or naturally radioactive elements, like ${}^{3}_{1}\mathrm{H}$, also known as Tritium, ${}^{238}_{92}\mathrm{U}$, etc.
Due to the composition of nuclei, made with Z positive charges and $A - Z$ neutral particles, it's already obvious that electromagnetism doesn't explain their existence, therefore we must account for a new force that we now call the «*strong force*».
Going back to stable and unstable nuclei that, if Z_s is the stable charge value and N is the number of neutrons we already can determine two possible radioactive decays.

1. Nuclei with $N > Z_s$. In these nuclei a neutron decays into a proton via β^- decay, with the following reaction
$$ {}^{A}_{Z}\mathrm{X}_N \to {}^{A}_{Z+1}\mathrm{Y}_{N-1} + e^- + \overline{\nu}_e \tag{2.16}$$

2. Nuclei with $N < Z_s$. In these nuclei a proton decays into a neutro via β^+ decay.
$$ {}^{A}_{Z}\mathrm{X}_N \to {}^{A}_{Z-1}\mathrm{Y}_{N+1} + e^+ + \nu_e \tag{2.17}$$

3. Another possibility is the inverse-β process, also known as electron capture, which is not a decay.
$$ {}^{A}_{Z}\mathrm{X}_N + e^- \to {}^{A}_{Z-1}\mathrm{Y}_{N-1} + \nu_e \tag{2.18}$$

Another radioactive process, together with γ and β radiation, which a nucleus can use to reach stability, is α decay. This type of decay happens usually for $A \gtrsim 180$ and is typical for $A > 200$ nuclides. This process corresponds to the emission of a ${}^{4}_{2}\mathrm{He}$ nucleus, as in the following reaction

$$ {}^{A}_{Z}\mathrm{X}_N \to {}^{A-4}_{Z-2}\mathrm{Y}_{N-2} + {}^{4}_{2}\mathrm{He}_2 \tag{2.19}$$

An example would be the decay of ${}^{238}_{92}\mathrm{U}$ as in

$$ {}^{238}_{92}\mathrm{U} \to {}^{234}_{90}\mathrm{Th} + {}^{4}_{2}\mathrm{He}$$

In general we have that $E_\alpha \approx 4.2$ Mev.

§§ 2.2.2 Binding Energy and Nuclear Structure

For calculating the binding energy of a nucleus we need to account for various correlation factors

- Strong interactions
- Electromagnetic interactions

- Quantum mechanical effects

The last ones are directly tied to the Heisenberg indetermination principle and Pauli's exclusion principle, due to the nucleons being $s = 1/2$ particles.
The indetermination principle makes sure that these nucleons can't be still in one point and therefore are freely moving inside a spherical potential barrier that corresponds to the nuclear boundary.
Supposing $T > 0$ we have that $E_k >> k_B T \neq 0$. This energy is deeply tied to the Fermi energy of the ensemble, and we can say that the potential barrier is

$$\mathcal{U} = \epsilon_F + \frac{B}{A}$$

For studying the effect on nucleons we directly delve into a quantum mechanical study of a spherical well.
Solving the Schrödinger equation we get, in natural units

$$\psi(x) = A\sin(px)$$

Imposing the boundary conditions at the center and border of the well we get in 3 dimensions that the permitted values of momentum for a nucleon are

$$\begin{aligned} p_x &= \frac{\pi n_x}{L} \\ p_y &= \frac{\pi n_y}{L} \\ p_z &= \frac{\pi n_z}{L} \end{aligned} \tag{2.20}$$

Inserting into the energy of a free particle (i.e., inside the barrier), we have

$$E_n = \frac{\pi^2}{2mL^2} n^2 \tag{2.21}$$

In order to find the Fermi energy of the system we must know how many states for a given value of momentum or energy.
Taking a spherical slice of phase space, we have that for $p \in [p + \mathrm{d}p]$ there will be $\mathrm{d}n$ states, which accounting for the spherical symmetry of the system give

$$\mathrm{d}n = \frac{1}{8}\frac{4\pi p^2 \mathrm{d}p}{\pi^3/L^3} = \frac{V}{(2\pi)^3} 4\pi p^2 \mathrm{d}p \tag{2.22}$$

Dividing both sides by $\mathrm{d}p$ we get that

$$\frac{\mathrm{d}n}{\mathrm{d}p} \propto p^2$$

Which implies that the derivative of the particle number in the infinitesimal shell with respect to the energy is actually an energy density

$$\frac{\mathrm{d}n}{\mathrm{d}E} = \rho(E) \propto \sqrt{E}$$

Integrating this energy density we must have, since the particles are A fermions with $s = 1/2$ and $g_s = 2$

$$n = A = g\int_0^{\epsilon_F} \mathrm{d}n$$

Dividing for electrons and protons we have

$$n_p = Z = \int_0^{\epsilon_F^p} \mathrm{d}n_p$$
$$n_n = A - Z = \int_0^{\epsilon_F^n} \mathrm{d}n_n$$

For which, integrating, we have

$$Z = \frac{2V}{(2\pi)^2} \int_0^{p_F^p} 4\pi p^2 \mathrm{d}p = \frac{2V}{(2\pi)^3} \frac{4\pi p_F^3}{3} \tag{2.23}$$

Using $R = R_0 A^{1/3}$ we have

$$Z = \frac{4}{9}\pi p_F^3 A R_0^3$$

Which implies

$$p_{F,p} = \frac{1}{R_0} \sqrt[3]{\frac{9Z}{4\pi A}} \tag{2.24}$$

And for neutrons, substituting with $n_p = A - Z$

$$p_{F,n} = \frac{1}{R_0} \sqrt[3]{\frac{9(A-Z)}{4\pi A}} \tag{2.25}$$

Inserting into $E = p^2/2m$ we get that the Fermi energy of the nucleus will be

$$\epsilon_F = \frac{p_F^2}{2m} \approx 30 \text{ MeV} \tag{2.26}$$

The average kinetic energy will then be the integral of the energy density with respect to the number of particles divided by the number of particles A

$$\langle K \rangle = \frac{2}{A} \left(\int_0^{p_{Fp}} \frac{p^2}{2m_p} \mathrm{d}n_p + \int_0^{p_{Fn}} \frac{p^2}{2m_n} \mathrm{d}n_n \right) \tag{2.27}$$

Using

$$\mathrm{d}n = \frac{2V}{(2\pi)^3} 4\pi p^2 \mathrm{d}p$$

We get the following integral

$$\langle K \rangle = \frac{4\pi V}{A(2\pi)^3} \left(\frac{1}{m_p} \int_0^{p_{F,p}} p^4 \mathrm{d}p + \frac{1}{m_n} \int_0^{p_{F,n}} p^4 \mathrm{d}p \right) \tag{2.28}$$

The integral is of direct solution, giving us

$$\langle K \rangle = \frac{V}{2\pi^2 A} \left(\frac{p_{F,p}^5}{5m_p} + \frac{p_{F,n}^5}{5m_n} \right) = \frac{4R_0^3}{3\pi} \left(\frac{p_{F,p}^5}{10m_p} + \frac{p_{F,n}^5}{10m_n} \right) \tag{2.29}$$

Where note that we used $V = (4/3)\pi A R_0^3$
Substituting what we found for the Fermi momentums of neutrons and protons and using $m_n \simeq m_p = m \approx 1$ GeV we get

$$\langle K \rangle = \frac{2}{15 m \pi R_0^2} \left(\frac{9}{4\pi}\right)^{\frac{5}{3}} \left[\left(\frac{Z}{A}\right)^{\frac{5}{3}} + \left(\frac{A-Z}{A}\right)^{\frac{5}{3}} \right] \tag{2.30}$$

The last equation can be expanded with power series into the following approximate result

$$\langle K \rangle \approx k \left(A + \frac{5\,(A-2Z)^2}{9A} \right) \tag{2.31}$$

§§ 2.2.3 Nuclear Drop Model

In order to setup all the possible correlations to the binding energy of a nucleus it's possible to rewrite the binding energy formula including everything.
All these possible contributors include

1. A volume term
2. An electromagnetic interaction term
3. A surface term
4. Semiclassical electromagnetic corrections
5. Kinetic corrections

This model of the nucleus used is known as the «*liquid drop model*» of the nucleus and gives us a semi-empirical formula for finding the binding energy of the nucleus.
Starting from the volume term, we have that $V \propto A$, therefore our first term will be

$$B_V = a_V A$$

For the electromagnetic interaction term, considering that $\mathcal{U}_{EM} \simeq 2^{-1} A(A-1)$ we have

$$B_C = a_C A^2$$

The third term, the surface term, considers a loss of energy on the surface of the nucleus, and noting that $S \propto A^{2/3}$ we have

$$B_S = a_S A^{\frac{2}{3}}$$

The fourth term is slightly more complex. Consider a charged classical sphere with charge Z and radius $R = R_0 A^{1/3}$. The (constant) charge density inside the volume will be

$$\rho = \frac{Ze}{V}$$

This implies that the electromagnetic energy will be

$$E_{EM} = \int \rho V(r) \mathrm{d}^3 r \propto R^5$$

Calculating properly the integral we have

$$E_{EM} = \frac{Z^2 e^2}{15} \frac{9}{(4\pi)^2 R_0 A^{\frac{1}{3}}}$$

I.e. this energetic corrections gives us $E_{EM} \propto Z^2 A^{-1/3}$, giving us our electromagnetic correction term

$$B_{EM} = -a_{EM} Z^2 A^{-\frac{1}{3}}$$

The final term comes directly from the formula (2.31), which immediately gives the following correction term

$$B_F = -a_F \frac{(A-2Z)^2}{A}$$

Adding all contributions we get the «*Bethe-Weiszacker formula*», a semi-empirical formula for evaluating the nuclear binding energies of nucleons (A, Z)

$$B(A,Z) = a_V A - a_S A^{\frac{2}{3}} - a_{EM} Z^2 A^{-\frac{1}{3}} - a_F \frac{(A-2Z)^2}{A} \tag{2.32}$$

The constants are found through fitting from experimental results, and have the following approximate values.

$$\begin{aligned} a_V &\approx 16 \text{ MeV} \\ a_S &\approx 18 \text{ MeV} \\ a_{EM} &\approx 0.7 \text{ MeV} \\ a_F &=\approx 93 \text{ MeV} \end{aligned}$$

The formula tho, is systematically different from the experimental results, noting that a full term accounting for spin is missing. This term will be either positive or negative depending on the values of A and Z. In general

$$\delta = \begin{cases} +\delta & A, Z, A-Z \text{ even} \\ 0 & A \text{ uneven} \\ -\delta & A \text{ even}, Z, A-Z \text{ uneven} \end{cases}$$

The principal characteristics for this formula is that for isobars it follows a parabolic path

$$B(A,Z) \to B(Z, Z^2)$$

Using the mass formula

$$M(A,Z) = Z m_p + (A-Z) m_n - B(Z, Z^2)$$

We can now calculate the maximum of the mass function, in order to find that the minimum Z with A constant for the mass is

$$Z_{min} \simeq \frac{A}{2} \frac{1}{1 + 0.0076 A^{2/3}} \tag{2.33}$$

Connecting this to our previous relationship between Z and β decays we have that

- $Z < Z_{min}$ implies a β^-−active radionuclide

$$^A_Z\text{X} \to {}^{A}_{Z+1}\text{Y} + e^- + \overline{\nu_e}$$

- $Z > Z_{min}$ implies a β^+−active radionuclide

$$^A_Z\text{X} \to {}^{A}_{Z-1}\text{Y} + e^+ + \nu_e$$

§ 2.3 Alpha Decay

The alpha decay of nucleus is a radioactive process that happens for $A > 200$, for which $E_\alpha \approx 5$ MeV. The reaction is a two body decay

$$^A_Z\text{X} \to {}^{A-4}_{Z-2}\text{Y} + \alpha$$

The decay half-life of an α-active nucleus is a strong function of the kinetic energy of the particle $K_\alpha \approx 5$ MeV. We have empirically that this half-life can be evaluated via the «*Geiger-Nuttall law*»

$$\log \tau_{1/2} = a - b \log \sqrt{K_\alpha} \tag{2.34}$$

Since K_α can go from 4 MeV to extremely large values $\tau_{1/2}$ can also change by various orders of magnitude.
Let's evaluate the kinematics of this decay in the center of mass of ^A_ZX. For the conservation of 4-momentum we get

$$(M_X, 0) = \left(M_Y + m_\alpha + K_\alpha + K_Y, \underline{p}_\alpha + \underline{p}_Y \right)$$

From this we immediately get $p_Y = p_\alpha = 0$ and that

$$Q = M_X - M_Y - m_\alpha = K_\alpha + K_Y = \frac{p^2}{2m_\alpha}\left(1 + \frac{m_\alpha}{M_Y}\right)$$

Noting that $M_Y >> m_\alpha$ since we're considering nuclei with $A \simeq 200$ we can approximate and then write an expression for K_α

$$K_\alpha = \frac{Q}{1 + \frac{m_\alpha}{M_Y}} \simeq Q\left(1 - \frac{4}{A}\right) \tag{2.35}$$

Where we used $M \simeq A$
It's obvious that for this reaction to happen we need $Q > 0$. Since Q is an energy, also the binding energy has an important role in this decay. We therefore have

$$Q = B_Y(A-4, Z-2) + B_\alpha(4,2) - B_X(A,Z) \tag{2.36}$$

The constraint $Q > 0$ imposes that

$$B_X(A,Z) < B_Y(A-4, Z-2) + B_\alpha(4,2) \approx B_Y(A-4, Z-2) + 28\ \text{MeV}$$

From what we found we can also say that

$$\frac{\partial B}{\partial A} < 0 \qquad \text{for } A > 60$$

Ignoring $B_\alpha(4,2)$ we have that an alpha decay can happen already for $A > 60$, which is not experimentally supported since it's seen only for nuclides with $A > 200$.
Using Geiger-Nuttall we have an empirical table which connects Q with the half-life of the radionuclide $\tau_{1/2}$

$$\begin{cases} A \simeq 140 & Q \simeq 0 \Longrightarrow \tau_{1/2} \to \infty \\ A \simeq 200 & Q \simeq 4\ \mathrm{MeV} \\ A \simeq 240 & Q \simeq 8\ \mathrm{MeV} \end{cases}$$

I.e. for $200 < A < 240$ alpha decay is possible, but with a long $\tau_{1/2}$. Spontaneous alpha decays are experimentally seen from $^{209}\mathrm{Bi}$, which is the radionuclide with the longest known half-life.

§§ 2.3.1 Quantum Tunneling and α Decay

A quantum mechanical model for alpha decay can be constructed starting from the fact that α is a strongly bound state with $B_\alpha \simeq 28$ MeV and the decaying nucleus is a heavy nuclide with $A \simeq 200$.
The decay can be thought as having the α particle bound in a potential well created by $^{A-4}_{Z-2}\mathrm{Y}$, where

$$\mathcal{U}_Y(r) = \begin{cases} -\mathcal{U}_0 & r < R_0 A^{1/3} \\ \dfrac{2\alpha(Z-2)e^2}{r} & r > R_0 A^{1/3} \end{cases} \tag{2.37}$$

This can be graphed with an energy/distance graph as follows

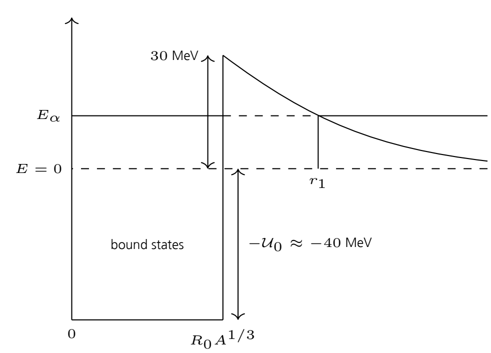


Figure 2.6: Energy level diagram of the potential well inside $^{A}_{Z}\mathrm{Y}$ and the coulombian barrier at $r = R_0 A^{1/3}$

Approximating the coulombian barrier as a step potential from R to r_1 with height $\mathcal{U}_0$ we can solve the Schrödinger equation inside and outside the well, getting two free-particle solutions and a decaying exponential solution inside.

In the first and third zone (outside the barrier) the solution will obviously be the free particle one, and using $\hbar = c = 1$ we can immediately write the Schrödinger equation for both

$$\frac{1}{2m_\alpha}\frac{\mathrm{d}^2\psi_{1,3}}{\mathrm{d}r^2} + E_\alpha\psi_{1,3}(r) = 0$$

Which gives the following solutions

$$\begin{aligned}\psi_1(r) &= Ae^{ipr} + Be^{-ipr}\\ \psi_3(r) &= Ee^{ipr}\end{aligned}$$

Note that we don't have a wave traveling backwards outside the barrier, since we're interested only in the transmitted particle.
In the second region, instead we get the following Schrödinger equation

$$-\frac{1}{2m_\alpha}\frac{\mathrm{d}^2\psi_2}{\mathrm{d}r^2} + (\mathcal{U}_0 - E_\alpha)\,\psi_2(r) = 0$$

With exponential solution

$$\psi_2(r) = Ce^{ikr} + De^{-ikr}$$

Note that since in the first and third regions we have a free particle we have

$$E_\alpha = \frac{p^2}{2m_\alpha} \Longrightarrow p = \sqrt{2m_\alpha E_\alpha}$$

In the second region instead

$$E_\alpha = \mathcal{U}_0 - \frac{k^2}{2m_\alpha} \Longrightarrow k = \sqrt{2m_\alpha(\mathcal{U}_0 - E_\alpha)}$$

The boundary conditions needed in order for ψ_i to be a wavefunction are that $\psi \in L^2$ inside and outside the well. This means that the wavefunction and its first derivative must be continuous on the walls of the potential. More specifically, using $R = R_0A^{1/3}$

$$\begin{cases}\psi_1(R) = \psi_2(R)\\ \psi_1'(R) = \psi_2'(R)\\ \psi_2(r_1) = \psi_3(r_1)\\ \psi_2'(r_1) = \psi_3'(r_1)\end{cases} \tag{2.38}$$

Shifting the potential barrier by R and writing $R - r_1 = L$ as the length of the barrier, we get by substituting the wavefunctions inside the system

$$\begin{cases}A + B = C + D\\ ip\,(A - B) = k\,(C - D)\\ Ce^{kL} + De^{-kL} = Ee^{ipL}\\ k\left(Ce^{kL} - De^{-kL}\right) = ipEe^{ipL}\end{cases} \tag{2.39}$$

The tunneling probability T will be then the norm squared of the amplitude of the outgoing particle E divided by the norm squared of the amplitude of the incoming particle A.
The solution of this system is a long and tedious algebra task, which we will now undertake. No unfortunately the answer isn't obvious so here's the calculations.
Firstly we begin by rearranging the equations. Dividing the second row by ip and summing it to the first we get a new way of writing the continuity condition at the first wall

$$A + B = C + D$$
$$2A = \left(1 + \frac{k}{ip}\right) C + \left(1 - \frac{k}{ip}\right) D$$

We continue by taking the fourth row and dividing it by k, and then summing and subtracting it from the third row, getting a new (nice) condition for C and D

$$2Ce^{kL} = Ee^{ipL}\left(1 + \frac{ip}{k}\right)$$
$$2De^{-kL} = Ee^{-ipL}\left(1 - \frac{ip}{k}\right)$$

Dividing out the constants on these we get the new third and fourth rows of the system, leaving us with this intermediate solution

$$\begin{cases} A + B = C + D \\ 2A = \left(1 + \frac{k}{ip}\right) C + \left(1 - \frac{k}{ip}\right) D \\ C = \frac{E}{2} e^{ipL - kL}\left(1 + \frac{ip}{k}\right) \\ D = \frac{E}{2} e^{ipL + kL}\left(1 - \frac{ip}{k}\right) \end{cases} \tag{2.40}$$

We continue by plugging in the second row the third and fourth of the new system, getting this algebraic monster

$$2A = \left(1 + \frac{k}{ip}\right)\left(1 + \frac{ip}{k}\right)\frac{E}{2}e^{ipL-kL} + \left(1 - \frac{k}{ip}\right)\left(1 - \frac{ip}{k}\right)\frac{E}{2}e^{ipL+kL}$$

Rearranging after some algebra, we have

$$2A = \frac{E}{2}e^{ipL}\left[\left(2 + \frac{k}{ip} + \frac{ip}{k}\right)e^{-kL} + \left(2 - \frac{k}{ip} - \frac{ip}{k}\right)e^{kL}\right]$$

Fixing the exponentials inside the square brackets we have

$$2A = \frac{E}{2}e^{ipL}\left[2\left(e^{kL} + e^{-kL}\right) + \left(\frac{k}{ip} + \frac{ip}{k}\right)e^{-kL} - \left(\frac{k}{ip} + \frac{ip}{k}\right)e^{kL}\right]$$

Working on the sums inside the parentheses we have

$$\frac{k}{ip} + \frac{ip}{k} = \frac{k^2 - p^2}{ipk} = -\frac{i\left(k^2 - p^2\right)}{pk}$$

Which, reinserted back into the equation, gives

$$2A = \frac{E}{2}e^{ipL}\left[2\left(e^{kL}+e^{-kL}\right)+\frac{i(k^2-p^2)}{kp}\left(e^{kL}-e^{-kL}\right)\right]$$

Which in terms of hyperbolic functions is

$$2A = \frac{E}{2}e^{ipL}\left[4\cosh(kL)+2i\frac{k^2-p^2}{kp}\sinh(kL)\right]$$

Dividing by $2E$ both sides we have

$$\frac{A}{E} = e^{ipL}\left[\cosh(kL)+i\frac{k^2-p^2}{2kp}\sinh(kL)\right]$$

Taking the square modulus of this we get the inverse of the tunneling probability T, which remembering how complex numbers behave is

$$T^{-1} = \left|\frac{A}{E}\right|^2 = \cosh^2(kL)+\frac{(k^2-p^2)^2}{4k^2p^2}\sinh^2(kL)$$

Using $\cosh^2(x) = \sinh^2(x)+1$ we can rewrite it as follows

$$T^{-1} = 1+\left(1+\frac{(k^2-p^2)^2}{4k^2p^2}\right)\sinh^2(kL) \tag{2.41}$$

Continuing the calculations on the term inside the parentheses we have

$$1+\frac{(k^2-p^2)}{4k^2p^2} = \frac{k^4+p^4-2k^2p^2+4k^2p^2}{4k^2p^2} = \frac{\left(k^2+p^2\right)^2}{4k^2p^2}$$

Substituting $k = \sqrt{2m_\alpha(\mathcal{U}_0-E_\alpha)}$ and $p = \sqrt{2m_\alpha E_\alpha}$ we have

$$\frac{(k^2+p^2)^2}{4k^2p^2} = \frac{4m_\alpha^2\mathcal{U}_0^2}{16m_\alpha^2E_\alpha(\mathcal{U}_0-E_\alpha)} = \frac{\mathcal{U}_0^2}{4E_\alpha(\mathcal{U}_0-E_\alpha)}$$

Which gives

$$T^{-1} = 1+\frac{\mathcal{U}_0^2}{4E_\alpha(\mathcal{U}_0-E_\alpha)}\sinh^2(kL)$$

Which gives our final tunneling probability for an alpha particle jumping a coulombian potential barrier

$$T = \frac{1}{1+\frac{\mathcal{U}_0^2}{4E_\alpha(\mathcal{U}_0-E_\alpha)}\sinh^2\left(L\sqrt{2m_\alpha(\mathcal{U}_0-E_\alpha)}\right)} \tag{2.42}$$

Gamow, which was the first to propose quantum tunneling as an answer to alpha decay, continued the calculation approximating the tunnel probability for small values of kL (note that $k \simeq 430$ MeV and $L \approx 40$ fm), and using

$$\sinh^2(kL) \approx \frac{1}{4}e^{2kL}$$

We get that

$$T \approx 4\frac{\mathcal{U}_0^2 - (2E_\alpha - \mathcal{U}_0)^2}{\mathcal{U}_0}e^{-2kL}$$

Or, expressing the multiplicative constant in terms of momenta as

$$\begin{cases} \mathcal{U}_0 = \dfrac{1}{2m_\alpha}(p^2 + k^2) \\ E_\alpha = \dfrac{p^2}{2m_\alpha} \end{cases}$$

We get

$$T \approx 16\frac{k^2p^2}{(k^2+p^2)^2}e^{-2kL} = Ae^{-2G}$$

The constant $G = kL$ is the so called «*Gamow factor*». Integrating for all possible r_1 we have that the total tunnel probability will be the product of all probabilities, we have

$$T = B\prod_{i=1}^{n} T_i$$

And therefore

$$G = \int_{r_0}^{r_1} \sqrt{2m_\alpha(\mathcal{U}_0 - E_\alpha)}\mathrm{d}r \tag{2.43}$$

§§ 2.3.2 Radioactivity and Units

Given a radioactive material we have two main useful informations from which to base the units we need

1. Activity, i.e. the number of decays per second
2. Effects on biological tissue from α, β, γ radiation

The first can be seen in two ways: one being the actual number of decays per second, which is completely unrelated to energy, and the second the energy produced by the decay products.
For the first the most common used units are two, one being the Becquerel (Bq), which corresponds to 1 decay per second, and the second being the Curie (Ci), with the following definition

$$1\ \mathrm{Bq} = 1\ \mathrm{dec/s}$$
$$1\ \mathrm{Ci} = 37\ \mathrm{GBq} = 3.7 \cdot 10^{10}\ \mathrm{Bq}$$

The effects on biological tissues is then evaluated by the actual ionizing power of the radioactive products, in units of Coulomb per kg of ionized air. A derived unit used commonly is the Röntgen, which is defined as follows

$$1\ \mathrm{R} = 2.58 \cdot 10^{-4}\ \mathrm{C/kg}$$

This units gives a measure on the exposure to ionizing radiation.
More useful on determining possible biological effects is the absorbed dose and the equivalent dose,

the first being the amount of energy per kg actually absorbed by the body, independent of the type of radiation. The two most common units are the Gray (Gy) and the Radiation Absorbed Dose (rad), which are defined as follows

$$1\ \mathrm{Gy} = 1\ \mathrm{J/kg}$$
$$1\ \mathrm{rad} = 0.01\ \mathrm{Gy}$$

Including a radiation weighting factor (W_R) to these two units we can now distinguish between the various radiation kinds and the potential damage caused by ionizing radiation, the two derived units are the Sievert (Sv) and the Röntgen Equivalent Man (rem), which are by definition

$$1\ \mathrm{Sv} = W_R \cdot 1\ \mathrm{Gy}$$
$$1\ \mathrm{rem} = W_R \cdot 1\ \mathrm{rad}$$

All these units can be then multiplied by a second factor W_T which weights the radiation dose per each different tissue.
As an example, note that $W_R = 20$ for alpha particles, therefore 1 Gy of exposure to alpha radiation corresponds to a weighted equivalent dose of 20 Sv In order to have a deeper understanding of the dosage units it's useful to check the lethal dose of radiation.
The lethal dose is defined as a radiation dose (expressed in Sieverts) expected to cause death in 50% of an exposed population within 30 days if received within 30 days, and it's denoted as $LD_{50/30}$. Such dose is in the following range of

$$LD_{50/30} = 400 \to 450\ \mathrm{rem} = 4 \to 5\ \mathrm{Sv}$$

Note that a yearly dose of natural background radiation corresponds to around 2.4 mSv, which is roughly $0.27\ \mu\mathrm{Sv/h}$

§ 2.4 Nuclear Reactions

Let a, b be two nuclides and c, d the resulting nuclides formed from a reaction. In general a nuclear reaction takes the following shape

$$a + b \to c + d + Q \tag{2.44}$$

Q is known as the «*Q-value*» of a reaction and corresponds to the energy absorbed from the environment or emitted into the environment after the reaction takes place, and for the previous general reaction evaluates to

$$Q = m_a + m_b - m_c - m_d$$

More generally, if there are R reagents with masses m_i and N resulting particles with masses f_i, we have that

$$Q = \sum_{i=1}^{R} m_i - \sum_{i=1}^{N} f_i \tag{2.45}$$

All reactions can be divided into two subcategories depending on the sign of Q.

1. Exothermic reactions, with $Q > 0$
2. Endothermic reactions, with $Q < 0$

As we have demonstrated before, a decay process is necessarily exothermic, since $Q > 0$ corresponds to the threshold energy for such reaction to happen.
Considering an exothermic reaction, henceforth $Q > 0$, we have that the masses of the reagent particles partially convert to kinetic energy of the resulting particles, whereas if the reaction is endothermic ($Q < 0$), the kinetic energy of the reacting particles converts to mass for the resulting particles.
Consider now a really famous endothermic reaction

$$\alpha + {}^{14}_{7}\mathrm{N} \to {}^{17}_{8}\mathrm{O} + p$$

This reaction has $Q = -1.19$ MeV and needs $K_\alpha \gtrsim 5$ MeV in order to take place, which is the energy needed for the α particle to overcome the Coulomb barrier. This is the main reason for the need of accelerating particles.
In order to accelerate charged particles it's possible to use electric fields, but the main problem comes with neutral particles like neutrons.
A base reaction we can take as an example is the following

$$p + {}^{3}_{7}\mathrm{Li} \to \alpha + \alpha$$

In this reaction $Q = 17.5$ MeV, which implies that the reaction is exothermic and therefore the kinetic energy of the proton is transferred into the kinetic energy of the alpha particles. This corresponds to an indirect acceleration of α.
For neutrons one such reaction is the one used by Chadwick in 1932

$$\alpha + {}^{9}_{4}\mathrm{Be} \to {}^{12}_{6}\mathrm{C} + n$$

Here $Q = 5.71$ MeV and both the Beryllium and Carbon isotopes are produced at rest, which implies an indirect acceleration of the neutron. This discovery led the beginning of the studies of nuclear reactions with neutrons, which don't interact with the Coulomb field.
The main results obtained in the early days of neutron physics were that the reaction

$$n + {}^{A}_{Z}\mathrm{X}$$

Had three main results: elastic scattering, neutron capture or an induced fission of the nuclide X

§§ 2.4.1 Uranium Fission and Nuclear Power Plants

The most known result of neutron physics was the discovery of nuclear fission, and in particular the fission of Uranium.
The first observation of Uranium fission happened in 1938, where Hahn and Strassmann observed the following reaction

$$n + {}^{238}_{92}\mathrm{U} \to {}^{137}_{56}\mathrm{Ba} + {}^{84}_{36}\mathrm{Kr} + 17n$$

Since heavier nuclei have more neutrons than needed for stability, the end result of this reaction is a net release of neutrons.
The next year Meitner and Frisch continued the study on the more general decay

$${}^{A}_{Z}\mathrm{N} \to {}^{A-A_2}_{Z-Z_2}\mathrm{X} + {}^{A_2}_{Z_2}\mathrm{Y}$$

The main question remained on why this reaction isn't spontaneous, since for $A > 60$, $\partial_A \overline{B} < 0$ which should favor a spontaneous fission.

Considering the case of Uranium fission we have that $Q \approx 210$ MeV, but still spontaneous fission reactions are observed mostly for $A \geq 300$.
A proposed explanation for the fission of uranium was nuclear deformation, where the mostly spherical nucleon gets deformed into an ellipsoid containing the two child products.
The final volume of the deformed nucleus is

$$V_d = \frac{4\pi}{3} ab^2$$

Where $a = R(1+\epsilon)$ is the radial deformation. This gives $ab^2 = R^3 \Longrightarrow b = R(1+\epsilon)^{-1}$.
Reevaluating the Bethe-Weiszäcker formula for the binding energy of the nucleon with this we get that the variation of the binding energy is

$$\Delta B = -a_S A^{2/3} \left(\frac{2}{5}\epsilon^2\right) - a_C Z^2 A^{1/3} \left(-\frac{1}{5}\epsilon^2\right)$$

In order to then have a fission of the nucleus it's necessary to insert in the system an energy $E_f ? \Delta B$ in order to let the system overcome this energy barrier.
Imposing $\Delta B \approx 0$ we get that

$$\frac{Z^2}{A} \approx 50$$

Suggesting that the fission can be induced with really little energy, or it can also happen through quantum tunneling across the barrier.
Considering again the $Q-$value for a fission reaction, we generally have

1. $A \approx 300$, $Q - E > 0$ and the fission is spontaneous
2. $A \approx 240$, $E - Q \approx 6\,\mathrm{MeV}$, the fission can happen with little energy or through quantum tunneling
3. $A \approx 100$, $E - Q \approx 60$ MeV, the fission reaction doesn't occur.

For Uranium in particular we have $A = 238$ (99.3% in nature, $A = 235$ 0.7% in nature) and therefore the fission reaction can happen both through quantum tunneling and via low energy neutron bombardment.
Considering a neutron bombardment of $^{235}_{92}\mathrm{U}$ we have the following self sustaining chain reaction

$$n + {}^{235}_{92}\mathrm{U} \rightarrow {}^{236}_{92}\mathrm{U}^\star \rightarrow {}^{139}_{56}\mathrm{Ba} + {}^{94}_{32}\mathrm{Kr} + 3n$$

The neutron absorption reaction has $Q = 6.5$ MeV and $E = 6.2$ MeV, with cross section $\sigma_{^{235}\mathrm{U},n} \approx 580\,\mathrm{b}$, for which $K_n = 0.025\,\mathrm{eV}$ are needed. This makes $^{235}\mathrm{U}$ an easily fissile material. Considering the more abundant $^{238}\mathrm{U}$ we get $Q_{238} = 4.8$ MeV, $E = 6.6$ MeV, and therefore $K_n = 1.8$ MeV is needed to make the reaction happen.
In general, for a complete fission reaction of an uranium nucleus we have $Q = 200$ MeV.
Considering $m_p \approx m_n \approx 1$ GeV we have that $M_\mathrm{U} \approx 240$ GeV, and in this reaction we have an efficiency of energy released

$$\eta = \frac{Q}{M_\mathrm{U}} = \frac{200 \text{ MeV}}{240 \text{ GeV}} \approx 10^{-3}$$

This corresponds that the possible energetic output from the fission of 1 g of uranium is

$$Q = 200 \text{ MeV} \cdot \left(\frac{1 \text{ g}}{A} N_A\right) \approx 5 \cdot 10^{23} \text{ MeV} \approx 10^{11} \text{ J}$$

Note that this corresponds to 3 times the energy output from the combustion of $1\ \mathrm{ton}$ of carbon.
This high energy potential output gives the possibility of energy production using nuclear reactors with fissile fuels.
In modern nuclear reactors the chain reaction is stabilized using either heavy or light water, which acts as a neutron moderator (slows the fast neutron from the fission in order to thermalize them and let them sustain a chain reaction).
This is useful since the neutron-proton and the neutron-neutron cross section is quite high, implying that the loss of energy is major.

§ 2.5 Nuclear Fusion

§§ 2.5.1 Quantum Tunneling and Fusion

For lighter nuclei ($A = a < 60$) the opposite of the fission reaction is possible, i.e. a «*nuclear fusion*». The reaction in question has the following shape

$$ {}^{a}_{Z_1}\mathrm{X} + {}^{b}_{Z_2}\mathrm{Y} \to {}^{a+b}_{Z_1+Z_2}\mathrm{N} + Q \tag{2.46} $$

The main idea for having a successful fusion reaction is that a coulomb barrier with maximum as $R = R_X + R_Y$ must be overcame, i.e.

$$ \mathcal{U}_{max}(R) = \frac{\alpha Z_1 Z_2}{R_X + R_Y} $$

Considering a $p + p$ fusion reaction we have that $\mathcal{U}_{max} \propto \frac{\alpha}{2R_0} = 550\ \mathrm{keV}$, which is our minimal kinetic energy needed by one proton to jump the coulomb barrier of the other proton and complete the reaction.
Take now a second reaction with $A > 1$ nuclei.

$$ {}^{12}_{6}\mathrm{C} + {}^{12}_{6}\mathrm{C} \to {}^{24}_{12}\mathrm{Mg} $$

Here $Q = 13.9\ \mathrm{MeV}$. The barrier evaluates to $9.3\ \mathrm{MeV}$, which gets us to reason if it's really a convenient fusion reaction when the net gain of energy is $4.3\ \mathrm{MeV}$. The efficiency of this kind of reaction (for this one is $\eta \propto 10^{-4}$) is proportional to Q and inversely proportional to the mass of the sum of the fusing nuclei, which lets us hope that Q decreases slowly when mass decreases.
Fortunately this is mostly the case and we can talk about nucleosynthesis in stars.

§§ 2.5.2 Stellar Fusion and Nucleosynthesis

In stars of the main sequence the most common fusion reaction observed is the $p - p$ chain reaction, where proton are fused creating Helium and energy which keeps the star from collapsing.
The chain reaction is summarized as follows

$$ \begin{aligned} p + p &\to {}^{2}_{1}\mathrm{H} + e^{+} + \nu_e + 0.42\ \mathrm{MeV} \\ p + {}^{2}_{1}\mathrm{H} &\to {}^{3}_{2}\mathrm{He} + \gamma + 5.49\ \mathrm{MeV} \\ {}^{3}_{2}\mathrm{He} + {}^{3}_{2}\mathrm{He} &\to {}^{4}_{2}\mathrm{He} + p + p + 12.86\ \mathrm{MeV} \end{aligned} \tag{2.47} $$

The last reaction continues the cycle and produces a stable Helium nucleus together with around $13\ \mathrm{MeV}$.
The total reaction is

$$4p \to {}^4_2\mathrm{He} + 2e^+ + 2\nu_e + 2\gamma + 24.7\ \mathrm{MeV}$$

Considering he total Q-value of this reaction is $Q \approx 24\ \mathrm{MeV}$ since we need to consider the extremely weak interaction of neutrinos, which basically amounts to a loss energy of $2m_\nu$.
It's important to note that stellar cores are rich in atoms, therefore the total $Q \approx 26\ \mathrm{MeV}$ since it's necessary to consider electron-positron interactions.
All of this considered, we have

$$\eta_{pp} = \frac{Q}{4m_p} \approx 6.5 \cdot 10^{-3}$$

Making this an efficient fusion reaction, which explains why it's so common in stars.
Since this reaction creates alpha particles, when the proton fuel for the reaction ends and the density of helium is enough, the pp chain reaction ends, causing a contraction in the star and an increase of temperature which permits a new chain fusion, the α chain fusion process.
This process bases itself on the following single reaction

$$\alpha + \alpha \to {}^8_4\mathrm{Be} \to \alpha + \alpha$$

The first reaction has $Q \approx -100\ \mathrm{keV}$, and the nuclide of ${}^8\mathrm{Be}$ is highly unstable with half life $\tau_{1/2} \approx 8 \cdot 10^{-17} s$. This nuclide is fundamental for the reaction, since we have the following chain emerging from the $\alpha-$Beryllium fusion

$$\begin{aligned}
\alpha + {}^8_4\mathrm{Be} &\to {}^{12}_6\mathrm{C} + \gamma \\
\alpha + {}^{12}_6\mathrm{C} &\to {}^{16}_8\mathrm{O} + \gamma \\
\alpha + {}^{16}_8\mathrm{O} &\to {}^{20}_{10}\mathrm{Ne} + \gamma \\
\alpha + {}^{20}_{10}\mathrm{Ne} &\to {}^{24}_{12}\mathrm{Mg} + \gamma
\end{aligned} \tag{2.48}$$

The next tightly bound nucleus is ${}^{12}_6\mathrm{C}$ and it's the next nucleus to be fused inside the stellar core.
This process continues up until $A = 60$ with ${}^{60}\mathrm{Fe}$. The iron-iron fusion is endothermic and therefore needs energy to proceed, causing a stellar core-collapse.
Heavier nuclides are created then with supernovas and neutron capture

§ 2.6 Beta Decay

Beta decay is a kind of decay that at first was thought to be a two-body decay with the emission of an electron in the following reaction

$${}^A_Z\mathrm{X} \to {}^{A}_{Z+1}\mathrm{Y} + e^-$$

Plotting the experimental results for decays per energy, we clearly see that it's not compatible with a 2-body decay.
A quick relativistic calculation gives in the center of mass of X

$$\sqrt{s} = M_x = E_Y + E_e \qquad p_Y = p_e = p$$

Substituting $E = \sqrt{m^2 + p^2}$ we have

$$M_x = \sqrt{M_Y^2 + p^2} + \sqrt{m_e^2 + p^2}$$

And solving for p^2

$$p^2 = \frac{\left(M_X^2 + M_Y^2 - m_e^2\right)^2}{4M_X^2} - M_Y^2$$

Using the approximation $M_X \approx M_Y \approx Am_p >> m_e$ we have $p^2 \approx 0$ and therefore $E_Y \approx M_Y + K_Y$ and

$$E_X = M_Y + K_Y + E_e$$

And therefore $E_e \approx M_X - M_Y$ which implies that energy isn't conserved in this process.
Another way of evaluating that this reaction is not possible is by checking the angular momentum of the particles in this reaction. We have

$$\hbar L_X = \hbar L_Y + \frac{\hbar}{2}$$

Here angular momentum is obviously not conserved.
An idea for solving this problem was adding a new child particle, firstly it was though that it was a photon, but since $S_\gamma = \hbar$ this is not the case.
Pauli in 1930 supposed the existence of a new fermion ($S = \hbar/2$) with a small mass and a neutral charge in order to balance the reaction.
Fermi proceeded to the theorization of the interaction of this new particle, called the neutrino ν for its properties.
In Fermi's theory, beta decay then became a three body process with the following reaction

$$^A_Z\mathrm{X} \to {}^{A}_{Z+1}\mathrm{Y} + e^- + \overline{\nu} \tag{2.49}$$

This kind of interaction, due to its weak nature was called the «*weak interaction*», where firstly the following kind of reactions are studied using Fermi's theory and also explain the process of electron capture.

§§ 2.6.1 Unstable States and Decay Rates

Consider a group of particles N_0 situated in an unstable state. As we already know, after a time t an exponential number of particles will have decayed, giving

$$N(t) = N_0 e^{-\frac{t}{\tau}} = N_0 e^{-\Gamma t}$$

The value $\Gamma = 1/\tau$ is known as the decay rate of the unstable state.
Suppose that this state is an exponential solution of the Schrödinger equation with imaginary energy $E_0 + i\Gamma$ where E_0 is the unperturbed state.
The evolved state will then be

$$\psi(\underline{r}, t) = \psi_0(\underline{r}) e^{-iE_n t} e^{-\frac{\Gamma}{2} t}$$

Where ψ_0 is the solution to the TISE unperturbed Hamiltonian.
We're interested to see the behavior of the wavefunction in the energy space, and therefore we perform an inverse Fourier transform of the following quantity

$$e^{-i(E_0 + i\frac{\Gamma}{2})t} = \int_{\mathbb{R}} \varphi(E) e^{-iEt} \mathrm{d}E$$

Where $\varphi(E)$ is the searched wavefunction in the energy space. Antitrasforming we have

$$\varphi(E) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-i(E-E_0)t+\frac{\Gamma}{2}t}\mathrm{d}t \tag{2.50}$$

Considering that for $t < 0$ the unstable state doesn't exist and modifying accordingly the integration bounds we get

$$\varphi(E) = \frac{1}{2\pi}\left(\frac{1}{\frac{\Gamma}{2} - i(E-E_0)}\right)$$

The probability density function for this energy wavefunction will be a Breit-Wigner Lorentzian distribution

$$|\varphi(E)|^2 = \frac{1}{4\pi^2}\frac{1}{\frac{\Gamma^2}{4} + (E-E_0)^2}$$

Which will draw a bell curve with width Γ. This property gives the decay rate Γ its second name, the «*decay width*».
Note that for $\Gamma/m << 1$ the particle is clearly stable, since for the indetermination principle $\Gamma\tau = \hbar$.
Suppose now having an unstable particle in its center of mass, which for $t > 0$ decays into two child particles

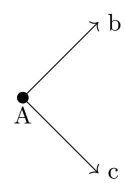


Figure 2.7: Example drawing of the $\mathrm{A} \to \mathrm{b} + \mathrm{c}$ two body decay

For this decay we have

$$\sqrt{s} = M_A = \sqrt{(p^\mu p_\mu)_a + (p^\mu p_\mu)_c}$$

This $\sqrt{s}$ is well determined since for $\Gamma \to 0$ $|\varphi|^2 = \delta(E - m)$.
We continue by writing the unstable state as a weak perturbation $\hat{\mathcal{H}}_I$ on a Hamiltonian $\hat{\mathcal{H}}_0$ for which the TISE holds for the unperturbed state $|n\rangle$

$$\hat{\mathcal{H}}_0 |n\rangle = E_n |n\rangle$$

We suppose that $|n\rangle$ is an orthonormal eigenvector basis for this Hamiltonian, and we begin evaluating the new perturbed state using undetermined coefficients which depend on time, after evolving the unperturbed state

$$\hat{\mathcal{H}}|\psi\rangle = \left(\hat{\mathcal{H}}_0 + \hat{\mathcal{H}}_I\right)|\psi\rangle = \hat{E}|\psi\rangle$$
$$|\psi\rangle = \sum_n a_n(t)\hat{\mathcal{U}}|n\rangle$$

We proceed by inserting the new state inside a TDSE, using $\hat{E} = i\partial_t$, getting

$$\hat{E}|\psi\rangle = i\sum_n \dot{a}_n\hat{\mathcal{U}}|n\rangle + \sum_n a_n(t)E_n\hat{\mathcal{U}}|n\rangle = \hat{\mathcal{H}}|\psi\rangle$$

Using the first result on the unperturbed Hamiltonian and $\hat{\mathcal{H}} = \hat{\mathcal{H}}_0 + \hat{\mathcal{H}}_I$ we have immediately confronting the two results that

$$i\sum_n \dot{a}_n(t)\hat{\mathcal{U}}\left|n\right\rangle = \hat{\mathcal{H}}_I\left|\psi\right\rangle$$

Using the orthonormality of $\left|n\right\rangle$ ($\left\langle k|n\right\rangle = \delta_{nk}$), we have by multiplying both sides by $\left\langle k\right|$ that

$$i\sum_n \dot{a}_n(t)\hat{\mathcal{U}}\left\langle k|n\right\rangle = \left\langle k\right|\hat{\mathcal{H}}_I\left|\psi\right\rangle = \sum_n a_n(t)\left\langle k\right|\hat{\mathcal{H}}_I\hat{\mathcal{U}}\left|n\right\rangle$$

Rearranging things and evaluating the sums, and defining $\left\langle n\right|\hat{\mathcal{H}}_I\left|n\right\rangle = \hat{V}_{nk}$ we have

$$i\dot{a}_n(t)e^{-iE_kt} = \sum_n a_n(t)\hat{V}_{nk}e^{-iE_nt}$$

We define for convenience $\hat{\mathcal{M}} = -i\hat{V}$, which gives us the final differential equation for the variation

$$\frac{\mathrm{d}a_k}{\mathrm{d}t} = \hat{\mathcal{M}}_{kn}e^{-i(E_n-E_k)t} \tag{2.51}$$

Where

$$\hat{\mathcal{M}}_{kn} = \left\langle k\right|\hat{\mathcal{M}}\left|n\right\rangle = -i\left\langle k\right|\hat{V}\left|n\right\rangle = -i\left\langle k\right|\hat{\mathcal{H}}_I\left|n\right\rangle \tag{2.52}$$

Considering that the perturbation is small we can say without problems that $a_n(0) = \delta_{mk}$ for some initial state m, and considering a weak time dependence, we can also say that $a_m(t) \approx 1$
With these hypotheses $\hat{\mathcal{M}}_{mk}$ can be considered weakly dependent on time (adiabatic) and the integration will give

$$a_k(t) = \hat{\mathcal{M}}_{mk}\int_0^T e^{i(E_k-E_m)t}\mathrm{d}t$$

Taking k as our final state and m our initial state the transition probability per unit time will be

$$P_{i\to f} = \lim_{T\to\infty}\frac{\left|a_f(t)\right|^2}{T} \tag{2.53}$$

Which is

$$\lim_{T\to\infty}\frac{\left|\hat{\mathcal{M}}_{mk}\right|^2}{T}\int_0^T e^{i(E_f-E_i)t}\mathrm{d}t\int_0^T e^{-i(E_f-E_i)\tau}\mathrm{d}\tau$$

Executing a change of variables $T \to T - T/2$ the integrals on the right become easily identifiable as the Fourier transforms of a complex exponential, which gives $\hat{f}(x) = 2\pi\delta(x)$, and therefore

$$P_{i\to f} = \lim_{T\to\infty}\frac{\left|\hat{\mathcal{M}}_{mk}\right|^2}{T}2\pi T\delta(E_f-E_i)$$

Which finally gives «*Fermi's golden rule*» for time dependent perturbations

$$P_{i\to f} = 2\pi\left|\hat{\mathcal{M}}_{mk}\right|^2\delta(E_f-E_i) \tag{2.54}$$

Since there are various possibilities for this two body decay, it's necessary to then consider a statistical approach for the energies, and we need to evaluate the density of states in the phase space, for the volume $E_f - E_i$. In this case, considering then all possible decays, we have finally the transition rate between the initial and the final state

$$\Gamma_{fi} = \int P_{i\to f}\mathrm{d}n = \int P_{i\to f}\frac{\mathrm{d}n}{\mathrm{d}E_f}\mathrm{d}E_f = 2\pi\left|\hat{\mathcal{M}}_{fi}\right|^2\int\delta(E_f - E_i)\frac{\mathrm{d}n}{\mathrm{d}E_f}\mathrm{d}E_f$$

Integrating the delta on the right we have finally

$$\Gamma = 2\pi\left|\hat{\mathcal{M}}_{fi}\right|^2\rho(E_i) \tag{2.55}$$

Where $\hat{\mathcal{M}}_{fi}$ depends on the interaction Hamiltonian, whereas $\rho(E_i)$ is the density of states in the phase space for the two body decay.

§§ 2.6.2 Beta Decay Rate

Using what we found before for unstable states we can immediately imagine to apply it to Fermi's theory of β decay. In this case we have a reaction of the type

$$n \to p + e^- + \overline{\nu}$$

Here, we have $|i\rangle = |n\rangle$ and $|f\rangle = |pe^-\overline{\nu}\rangle$. The decay rate as for Fermi's golden rule is

$$\Gamma = 2\pi\left|\hat{\mathcal{M}}_{fi}\right|^2\rho(E)$$

Where, in this case

$$\hat{\mathcal{M}}_{fi} = -i\int\overline{\psi}_p\overline{\psi}_e\overline{\psi}_\nu G_F\psi_n\mathrm{d}^3r \tag{2.56}$$

Where G_F is our interaction parameter.
Suppose to normalize the wavefunctions on a nuclear volume V, $\psi \propto V^{-1/2}$. With this normalization we also have that $[G_F] = E^{-2}$.
Considering the total decay the wavefunctions for $e^-, \overline{\nu}$ can be considered without loss of generality as plane waves due to their non-interacting nature after the decay, therefore

$$\begin{aligned}\psi_e &= \frac{1}{\sqrt{V}}e^{-i\underline{p}\underline{r}}\\ \psi_\nu &= \frac{1}{\sqrt{V}}e^{-i\underline{q}\underline{r}}\end{aligned} \tag{2.57}$$

Where we chose $\underline{p}_e = \underline{p}$ and $\underline{p}_\nu = \underline{q}$.
Approximating the interaction factor G_F as a constant as for Fermi's theory we have finally

$$\hat{\mathcal{M}}_{fi} = -i\frac{G_F}{V}\int\overline{\psi}_p\psi_n e^{i(\underline{p}+\underline{q})\underline{r}}\mathrm{d}^3r$$

Another approximation can be made by checking that $Q = m_n - m_p - m_e - m_\nu \le 1$ and therefore $p \simeq q \le 1\ \mathrm{MeV}$, and $r \approx 1\ \mathrm{fm}$, which gives $(p+q)r \approx 5/1000$.

Considering this the exponential in this integral can be expanded with a power series and approximated to the first order, giving

$$\hat{\mathcal{M}}_{fi} = -i\frac{G_F}{V}\int \overline{\psi}_p\psi_n \mathrm{d}^3r = -\frac{iG_F N}{V} \tag{2.58}$$

Where N is the so called «*nuclear term*» $N = \langle p|n\rangle$.
Reinserting it into the decay rate equation we have

$$\Gamma = \frac{2\pi G_F^2 |N|^2}{V^2}\rho(E)$$

We only need to find $\rho(E)$ in order to complete the calculations. Considering that for the conservation of 3-momentum $\underline{p}_e + \underline{p}_\nu + \underline{p}_X = 0$, and for the conservation of energy

$$\begin{aligned} \rho(E) &= \left(\frac{\mathrm{d}n}{\mathrm{d}E_f}\right)_{E_i} = \int \delta(E_f - E_i)\mathrm{d}n \\ \mathrm{d}n &= \frac{V^2}{(2\pi)^6}\mathrm{d}^3p\mathrm{d}^3q \end{aligned} \tag{2.59}$$

In the center of mass of X we have then

$$E_i = \sqrt{s} = M_X, \quad E_f = E_Y + E_e + E_\nu = M_Y + K_Y + E_e + E_\nu$$

Considering that the masses of X and Y are much greater than the mass of the electron and neutrino we have that $K_Y \approx 0$ and we have $M_X \approx M_Y + E_e + E_\nu$. The total available energy for the reaction will then be

$$E_T = M_X - M_Y = E_e + E_\nu$$

And, therefore

$$Q = M_X - M_Y - m_e - m_\nu \approx E_T$$

In the limit case where $\mathrm{X} = p$ and $\mathrm{Y} = n$ we have $E_T \approx 1\ \mathrm{MeV}$, and

$$\delta(E_f - E_i) = \delta(E_T - E_e - E_\nu)$$

Approximating $m_\nu = 0$ we have $E_\nu = q$ and therefore $q^2\mathrm{d}q = E_\nu^2\mathrm{d}E_\nu$. For the electron we have instead

$$p = \sqrt{E_e^2 - m_e^2} \qquad p^2\mathrm{d}p = pE_e\mathrm{d}E_e$$

Substituting in the delta integral we have

$$(4\pi)^2\iint \delta(E_f - E_i)p^2q^2\mathrm{d}p\mathrm{d}q = (4\pi)^2\iint \delta(E_T - E_e - E_\nu)E_\nu^2 E_e p\mathrm{d}E_\nu\mathrm{d}E_e = (4\pi)^2\int_0^{E_T} pE_e(E_T - E_e)^2\mathrm{d}E_e$$

Substituting p we get finally for the decay rate

$$\Gamma_\beta = \frac{G_F^2|N|^2}{2\pi^3}\int_0^{E_T} E_e\sqrt{E_e^2 - m_e^2}(E_T - E_e)^2\mathrm{d}E_e$$

For the limit case of a hyperrelativistic electron we can say with ease that $m_e << E_e$ and therefroe we can approximate the integral as follows

$$\int_0^{E_T} E_e^2(E_T - E_e)^2 \mathrm{d}E_e = \frac{E_T^5}{3} + \frac{E_T^5}{5} - \frac{E_T^5}{2}$$

Summing up and inserting into the decay rate we get «*Sargent's rule*» for beta decay with highly energetic electrons

$$\Gamma = \frac{G_F^2|N|^2}{60\pi} E_T^5 \tag{2.60}$$

Since here $E_T \approx M_X - M_Y$ we have that the phase space must grow with $M_X - M_Y$

§§ 2.6.3 Experimental Estimate of m_ν

So far the hypotheses we managed to stack up from beta decay are

1. $m_\nu \neq 0$
2. $\Gamma \propto pE_e(E_T - E_e)^2$, $E_T = M_X - M_Y - m_\nu - m_e = E_e + E_\nu$

It's clear that reducing E_T we have the maximum possible E_e. We define here the Kurie function $K(E)$ as

$$K(E) = \sqrt{\frac{1}{pE_e}\frac{\mathrm{d}\Gamma}{\mathrm{d}E}} \propto (E_T - E_e)^2$$

Evaluating this function for a minimal E_T it's possible to resolve $E_T - m_\nu$ and evaluate the neutrino mass.
Note that $K(E) \propto (E_T - E_e)$ and therefore we expect a linear decay of this function up until E_e^{max}. Experimentally it has been observed that at $\min(E_T)$ instead the function evaluates to $E_T - m_\nu$ effectively giving an estimate for the mass of the neutrino

§§ 2.6.4 Cross Section for Beta Decays

The reaction considered up until now

$$n \to p + e^- + \overline{\nu}$$

Can also be reversed, indicating that another weak reaction is possible

$$\overline{\nu} + p \to n + e^+ \tag{2.61}$$

This scattering reaction is experimentally observed, but we must

- generate neutrinos
- choose a p-rich target
- observe n, e^+ and count the number of events
- measure the scattering cross section σ

The first question we get is: Can Fermi's 4-fermion theory evaluate theoretically σ?
Starting again from cross sections we have, by definition that the number of observed reactions per unit time is

$$\frac{\mathrm{d}N_R}{\mathrm{d}t} = \sigma n_T \frac{\mathrm{d}N_p}{\mathrm{d}t}\mathrm{d}x$$

Where N_p is the number of incoming projectiles, and n_T is the particle density of the target.
Evaluating everything we have

$$\frac{\mathrm{d}N_R}{\mathrm{d}t} = \frac{\sigma}{S}\underbrace{n_T S \mathrm{d}x}_{N_T}$$

Also using $S^{-1}\mathrm{d}N_p/\mathrm{d}t = \phi_p$, i.e. the flux of incoming projectiles, we have

$$\frac{1}{N_T}\frac{\mathrm{d}N_R}{\mathrm{d}t} = \sigma\phi_p = \sigma v_p n_p = \sigma\frac{N_p}{V}$$

Where n_p, N_p are respectively the number density and total number of particles of the projectile beam, and v_p is the relative velocity of the projectiles with respect to the target.
Fixing the previous equation we have

$$\frac{1}{N_T N_p}\frac{\mathrm{d}N_R}{\mathrm{d}t} = \sigma\frac{v_p}{V}$$

Using $v_p = p_p/E_p$ and noting that on the left we have the number of reaction per unit time per single projectile on a single target, we have

$$\frac{1}{N_T N_p}\frac{\mathrm{d}N_R}{\mathrm{d}t} = \sigma\frac{p_p}{V E_p} = \Gamma(a+b\to c+d) = 2\pi\left|\hat{\mathcal{M}}_{fi}\right|^2\rho(E)$$

Therefore we have

$$\sigma(a+b\to c+d) = 2\pi\left|\hat{\mathcal{M}}_{fi}\right|^2\frac{V}{v_p}\rho(E) \tag{2.62}$$

For our reaction we have the following scattering reaction

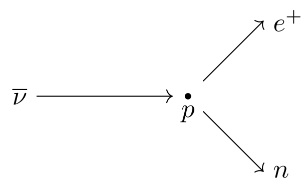


Figure 2.8: Scattering process in study

Moving to the center of mass of the initial $\overline{\nu}+p$ state we have

$$\beta\gamma = \frac{p}{\sqrt{s}} = \frac{E_\nu}{\sqrt{m_p^2 + 2E_p m_\nu}}$$

Where we approximated $m_\nu = 0$ therefore $p_\nu = E_\nu$.
The velocity of the ν with respect to the p targets is therefore

$$v_i = \frac{p_\nu^\star}{E_\nu^\star} + \frac{p_p^\star}{E_p^\star}$$

Inserting it all into the cross-section equation we get

$$\sigma = \frac{2\pi G_F^2 |N|^2}{V} \frac{E_\nu^\star E_p^\star}{p_\nu^\star E_p^\star + p_p^\star E_\nu^\star} \rho(E_f)$$

In order to calculate the density of states in the final decayed state we need to consider the conservation of 3-momentum, which implies $p_n^\star = p_e^\star = p^\star$, and accounting that

$$\rho(E_f) = \frac{\mathrm{d}n}{\mathrm{d}E_f} = \frac{V}{2\pi^2} p^2 \frac{\mathrm{d}p}{\mathrm{d}E_f}$$

Using $E_f = \sqrt{p^2 + m_e^2} + \sqrt{p^2 + m_n^2}$ gives

$$\frac{\mathrm{d}E_f}{\mathrm{d}p^\star} = \frac{p(E_n^\star + E_e^\star)}{E_n^\star E_e^\star} = v_f$$

Where v_f is the velocity of the positrons with respect to the neutrons. Inverting and inserting it into the cross-section equation, using $E_f = E_n^\star + E_e^\star$ we have

$$\sigma = \frac{G_F^2 |N|^2 (p^\star)^2}{\pi v_i v_f} = \frac{G_F^2 |N|^2}{\pi} \frac{E_p^\star E_\nu^\star E_p^\star E_e^\star}{E_f \left(p_\nu^\star E_p^\star + p_p^\star E_p^\star\right)} p^\star \tag{2.63}$$

Where $p^\star = p_e^\star = p_\nu^\star$ is the momentum of the final two decay products, the positron and the neutron. Inserting the experimental values of $G_F = 1.17 \cdot 10^{-5}\ \mathrm{GeV}^{-2}$, $v_i \approx v_f \approx 1$, we get

$$\sigma (p^\star)^2 \approx 10^{-37}\ \mathrm{cm}^2/\mathrm{GeV}^2$$

Considering that the $Q-$value for this reaction is $Q \approx 1.8\ \mathrm{MeV}$ we have that this process is possible only if $E_\nu > 1.8\ \mathrm{MeV}$ which gives, for the cross-section at threshold level

$$\sigma (p^\star)^2 \approx 10^{-43}\ \mathrm{cm}^2/\mathrm{GeV}^2$$

In order to evaluate what it means for experimental tests of this decay we need to account for the mean free path of the reaction. Supposing a light water target, for which $A_{\mathrm{H_2O}} = 18\ \mathrm{g/mol}$ we have $n_T = A^{-1} N_A$, for which we get

$$\frac{1}{\lambda} = \sigma n_T = E \cdot 10^{-43} \cdot 3 \cdot 10^{22}\ \mathrm{MeV}^{-1}\mathrm{cm}^{-1}$$

Using $E \approx 1\ \mathrm{MeV}$ we get an estimate of $\lambda \approx 10^{19} m$ which is an absurdly large mean free path for such reaction.
What is needed therefore for building experiments on this reaction is to note that accelerated neutrinos will have a higher E_ν, and therefore will contribute to lower this. A second fix is the usage of high

density targets together with heavy neutrinos emitters.
One way of accomplish this is considering Uranium fission. In 1956 Reines, Cowen et al. were the first to build a nuclear reactor for the production of energy, with a thermal power output of $1000\ \mathrm{MW} = 10^9\ \mathrm{J/s} = 6 \cdot 10^{27}\ \mathrm{eV/s}$.
Considering $Q_{fis} = 200\ \mathrm{MeV}$ and a production of around $N_\nu = 6$ neutrinos per reaction we have

$$N_R = \frac{P_{reactor}}{Q} = 3 \cdot 10^{19}\ \mathrm{Hz}$$

Which implies an average production of $10^{20}\ \mathrm{Hz}$ neutrinos, with $E_\nu \approx 3\ \mathrm{MeV}$.
Not too far from the reactor core we have that the flux of neutrinos per solid angle of detector is

$$\phi_\nu = \frac{\mathrm{d}N_\nu}{\mathrm{d}t\mathrm{d}S} = 10^{13}\ \mathrm{cm}^{-2}\mathrm{s}^{-1}$$

Using a $\mathrm{CaCl_2} + \mathrm{H_2O}$ target the reaction we expect to observe is an extra production of $\gamma + \gamma$ corresponding to the following reaction between the decay products and the electrons of the target

$$\begin{aligned} \overline{\nu} + p &\to n + e^+ \\ e^+ + e^- &\to \gamma + \gamma \end{aligned}$$

There is a second $\gamma + \gamma$ peak that needs to be accounted, since neutrons thermalize in the collisions with the fluid, it's possible to also have a neutron capture reaction inside the target, as follows

$$n + {}^A_Z\mathrm{X} \to {}^{A+1}_{Z}\mathrm{Y}^\star \to {}^{A+1}_{Z}\mathrm{Y} + \gamma + \gamma$$

Where $E_\gamma \approx 6\ \mathrm{MeV}$. Turning on the reactor it's possible to see a sharp increase of $\gamma + \gamma$ reactions, corresponding to the neutrino scattering, confirming the existence of the neutrino.

3 Electromagnetic Interactions

§ 3.1 Particle Detection

We have described nuclear physics with three main interactions: Electromagnetic, Strong (Nuclear) and Weak (Fermi's Theory).
All experimental results consisted in measuring the impulse of decay products, the energy of photons produced and the detection of neutrons. All these measures obviously need the existence of a particle detector, i.e. a device for a quantitative measure of some information or trace of particles.
The usual trace detected is the energy released by the particles, although ideally a particle shouldn't lose energy for being detected.
Suppose that a particle has and energy E and loses a ΔE. There are three possible scenarios

1. The particle loses a small $\Delta E/E$ fraction, this loss can be ignored and the particle can be detected using a charged particle tracker
2. The particle loses a considerable fraction of energy, in this case it's possible to study the process and find the correct value of E
3. $\Delta E/E = 1$ and the particle loses *all* energy. In this case we use «*calorimetry*»

Consider now the detector. The detector itself, being made of matter, interacts with a projectile particle. Depending on the interaction type there are three important considerations to make

1. EM interactions have an infinite radius of interaction
2. Strong interactions need a $b \approx 1\ \mathrm{fm}$
3. Weak interactions have an infinitesimally small radius of interaction and therefore they're *almost* negligible

From these three considerations we can say without doubt $\sigma_{EM} >> \sigma_S >> \sigma_W$. Note that for $q = 0$ and $q \neq 0$ require different detection strategies

§§ 3.1.1 Charged Particles and Ionization

Charged particles with $q \neq 0$ itσpossible to define a clear distinction

- Hadrons, which interact strongly

- Leptons, which interact weakly

Leptons include quarks q, muons μ, tauons τ and electrons e. The detection strategy for these particles is using elastic collisions as in Rutherford's experiment.
The two kinds of collisions we can consider are collision with nuclei and nuclear collisions with electrons.
In the first collision the nucleus behaves like a wall for the electron, therefore $\Delta E_e << 1$, where

$$\Delta E = \frac{\Delta p^2}{2m_N}$$

The second collision we have the nucleus as a projectile, for which $m_p = M >> m_e$ and the electron ricochets with a really big energy variation

$$\Delta E_e >> 1$$

Here the interaction with the electron is dominant.
The two main results of this collision are the following

1. The electron collects energy and the atom or molecule excites, when de-exciting it emits a photon
2. If $M >> m_e$ and therefore $\Delta E_p >> 1$ and the atom can be ionized and it's then possible to measure the charge q, for which $q \propto \Delta E_p \propto E_p$

The main loss of energy for charged particles is ionization.
The loss of energy for ionization can be described as a function of density ρ, momentum $\underline{p}$, Energy E, charge q, electron momenta $\underline{p}_e$ and average ionization charge $\langle I \rangle$, all depending on the traveled space Δx made by the charged particle inside the target.
The ionization loss per unit length was studied in depth by Bethe Bloch et al. using QED in 1930, but a non relativistic limit can be studied using Bohr's atom.
Considering that collisions are a stochastic process depending on Δx, for the central limit theorem we can expect that the loss of energy follows a Gaussian distribution.
For small targets this distribution becomes the Landau distribution and it follows closely a Poisson statistics.

§§ 3.1.2 Bohr Formula

Considering a non relativistic case for an atom with charge Ze we have that energy transitions are the usual

$$\Delta E = \hbar(\omega_2 - \omega_1)$$

Where the energy levels are

$$E_n = -\frac{\alpha Z e}{2n^2}$$

Using a semiclassical approximation we can say that the classical radius of the electron is, from $m_e c^2 = \mathcal{U}_{EM}$ $r_e = \alpha / m_e$.
Putting ourselves in the reference frame of the projectile we have that the transferred momentum to the electron is

$$\Delta \underline{p} = \int \underline{F} \mathrm{d}t = \int \frac{\underline{F}}{v} \mathrm{d}x$$

Dividing the force into the parallel and orthogonal components we have that the parallel component is zero due to symmetry evaluations, and therefore

$$\Delta p_\perp = \frac{1}{v}\int \underline{F}_\perp \mathrm{d}x = \frac{e}{v}\int \underline{E}_\perp \mathrm{d}x \tag{3.1}$$

Considering a cylinder long L with radius b where b is the impact parameter will then be the surface integral of the perpendicular component of the electric field, which will then be divided into the sum of the top and bottom circles of the cylinders and the boundary of such The flux of the field is then

$$\phi(\underline{E}_\perp) = \int_{\Sigma_1} \underline{E}_\perp \mathrm{d}\underline{S} + \int_{\Sigma_2} \underline{E}_\perp \mathrm{d}\underline{S} = 2\pi b \int E_\perp \mathrm{d}x$$

Solvin the integral we have that

$$\Delta p_\perp = \frac{2Ze^2}{4\pi\epsilon_0 bv} = Z\frac{e^2}{4\pi\epsilon_0 b^2}\frac{2b}{v}$$

The parameter $2b/v$ is known as the «*collision time*».
In the non-relativistic approximation we have that the energy variation T is

$$T = \frac{\Delta p * 2}{2m_e} = \left(\frac{e^2}{4\pi\epsilon_0}\right)^2 \frac{4}{b^2v^2}\frac{Z^2}{2m_e}$$

Using $r_e = \frac{e^2}{4\pi\epsilon_0}\frac{1}{m_ec^2}$ we get

$$T = 2m_ec^2\frac{Z^2r_e^2}{b^2\beta^2} \tag{3.2}$$

This equation gives the relation between the impact parameter b and the transferred energy T.
For a fact we know that this transferred energy is equal to $-\Delta E_p$ which is the lost energy by the projectile, in our case the nucleus.
Inverting for the impact parameter we get

$$b^2 = 2m_ec^2v_e^2\frac{Z^2}{\beta^2 T} \tag{3.3}$$

Since $\mathrm{d}\sigma = 2\pi b\mathrm{d}b$ we have

$$\mathrm{d}\sigma = 2\pi m_ec^2r_e^2\frac{Z^2}{\beta^2T^2}\mathrm{d}T \tag{3.4}$$

Which is the cross section for the process in study.
It's obvious that $\sigma'(T) \propto T^{-2}$ and therefore processes with small T are more probable. Note that $T \propto b^{-2}$.
Since T is the lost energy in a single collision, the total lost energy will be $\Delta E_t = \Delta EN$ where N is the total number of collisions, and

$$N = 2\pi bn_e\mathrm{d}b\mathrm{d}x$$

And therefore

$$\frac{\mathrm{d}^2E}{\mathrm{d}b\mathrm{d}x} = 2\pi n_3 b\Delta E = 4\pi m_ec^2r_e^2n_e\frac{Z^2}{\beta^2 b} \tag{3.5}$$

In the end, the searched energy loss is

$$\frac{\mathrm{d}E}{\mathrm{d}x} = A \log\left(\frac{b_{max}}{b_{min}}\right)$$

Therefore, we have

$$\frac{\mathrm{d}E}{\mathrm{d}x} \propto \frac{Z^2}{\beta^2} \log\left(\frac{b_{min}}{b_{max}}\right)$$

Bohr continued this calculation evaluating this in the quantum mechanical case of the atom considering that $b/v \propto t_{ion}$, i.e. the b/v is proportional to the ionization time of the electron.
Considering therefore the electron as if it was free we have

$$\frac{b}{v} \approx \gamma t_{det} = \frac{\gamma}{\langle\omega\rangle} = \gamma \frac{\hbar}{I}$$

Where t_{det} is the detection time, with I being the average ionization energy.
Therefore

$$b_{max} = \gamma \frac{\hbar v}{I} = \frac{\hbar \gamma \beta c}{I}$$

For the indetermination principle we can say that $\Delta x \approx \hbar / p_e$ and therefore

$$b_{min} \approx \Delta x = \frac{\hbar}{p_e} = \frac{\hbar}{\gamma \beta m c}$$

Substituting it into (3.5) we have

$$-\frac{\mathrm{d}E}{\mathrm{d}x} = 4\pi m_e c^2 r_e^2 n_e \frac{Z^2}{\beta^2} \log \frac{\gamma^2 \beta^2 m_e c^2}{I}$$

Using $n_e = \rho A N_A Z/A$ we get

$$-\frac{\mathrm{d}E}{\mathrm{d}x} = 4\pi m_e c^2 r_e^2 N_A \rho \frac{Z^3}{\beta^2 A} \log \frac{\gamma^2 \beta^2 m_e c^2}{I} \tag{3.6}$$

Note that the minus sign is there since the energy gained comes from the loss of energy of the projectile, due to conservation of energy.
Approximating the constants to $0.9\ \mathrm{MeV}$ and dividing $-\frac{\mathrm{d}E}{\mathrm{d}x}$ by the density of the material we get «*Bohr's formula*»

$$-\frac{1}{\rho}\frac{\mathrm{d}E}{\mathrm{d}x} = \frac{0.3}{2}\ \mathrm{MeV} \frac{Z^2}{\beta^2} \log \frac{\beta^2 \gamma^2 m_e c^2}{I} \tag{3.7}$$

This formula has major problems, since it doesn't account for quantum mechanics and relativity and isn't universal for all materials.
Bethe and Bloch, then reevaluated the calculation using Quantum Electrodynamics, finding that

$$-\frac{1}{\rho}\frac{\mathrm{d}E_{QFD}}{\mathrm{d}x} = (0.3\ \mathrm{MeV}) \frac{Z^3}{A\beta^2} \left(\log \frac{2 m_e \beta^2 \gamma^2 c^2 \omega_{max}}{I^2} - \beta^2 - \frac{\delta}{2}\right) \tag{3.8}$$

Where $\omega_{max} \propto \max E_e^\star$, i.e. $\omega_{max} = 2 m_e c^2 \beta^2 \gamma^2$.
We immediately notice $-\beta^2$ as a relativistic correction and $\frac{\delta}{2}$ as a correction for density and polarization effects of the object, all together corrected by an electric screening factor.
This formula, permits us with a measurement of $\frac{\mathrm{d}E}{\mathrm{d}x}$ and p permits to identify the projectile particle, with a formula that holds up for $\beta\gamma$ from $0.1 \to 1000$

§§ 3.1.3 Residual Path

Roche proceeded to use the Bethe-Bloch equation for evaluating the so called «*residual path*», i.e. the path for which the projectile loses all its energy. This is given by

$$R(E) = \int_E^0 \mathrm{d}x = \int_E^0 \frac{\mathrm{d}E}{-\frac{\mathrm{d}E_{QED}}{\mathrm{d}x}} = \int_0^E \frac{\mathrm{d}E}{\frac{\mathrm{d}E_{QED}}{\mathrm{d}x}} \tag{3.9}$$

From Bethe-Bloch's equation it's quick to deduce that $\beta\gamma$ small include heavy losses of energy.
It's of note that before the actual stopping of the projectile particle, the Bethe-Bloch formula finds its maximum in what's known the «*Bragg peak*».
This spike in energy is quite useful in tumor treating in what's known as Hadrontherapy, where hadrons, usually p or $^{12}\mathrm{C}$, are shot in a localized region and by energy dissipation releases a sharp amount of energy where shot.
In general, considering an elastic collision with nuclei, $\frac{\mathrm{d}E}{\mathrm{d}x} \approx 0$ due to collisions being elastic, but this leads to a big Δp, which implies an angular deviation as in Rutherford's experiment. The angular deviations are random, and it makes this a stochastic process.
Consider a thick target, for the theorem of the central limit we can say that the distribution of angular deviation is proportional to a Gaussian distribution

$$f(\Delta\theta) \propto G(\langle\theta\rangle = 0, \sigma = \sqrt{\langle\theta^2\rangle})$$

The $n-$th moment of the distribution will be

$$\langle\theta^n\rangle = \frac{\int \theta^n \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\mathrm{d}\Omega}{\int \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}}$$

Where σ is the Rutherford cross section, which is proportional to $\csc^2(\theta/2)$. Using $\mathrm{d}\Omega \approx 2\pi\theta\mathrm{d}\theta$ we can say without many doubts that

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} \propto \frac{\mathrm{d}\sigma}{\mathrm{d}\theta}$$

Calculating the second momenta we get the expected value for a multiple coulombian scattering, which is

$$\langle\theta^2\rangle = 21\ \mathrm{MeV}\frac{Z}{\beta cp}\sqrt{\frac{x}{\chi_0}}$$

Where x is the depth traveled in the medium (in cm), and χ_0 is the «*radiation length*», which is

$$\frac{1}{\chi_0} = 4r_e^2\alpha\rho\frac{N_A Z^2}{A}\log\left(183Z^{-1/3}\right) \tag{3.10}$$

§ 3.2 Cherenkov Effect

The Cherenkov effect is an effect similar to a sonic boom, for which a massive particle passes through a medium at a velocity higher than the speed of light in that medium $c_n = c/n$. This effect is due to polarization effects.

Consider a projectile moving at $v_x > c_n$ in some medium. The Cherenkov radiation is accompanied by the emission of photons perpendicular to the light cone at some angle θ_c.
Consider now the process after a time t. The radius of the boom circle created by this "superluminal" particle is $L = v_y t = ct/n$, supposing the particle has traveled some space $\Delta x = \beta c \Delta t$ we have

$$L = \Delta x \cos\theta_c \Longrightarrow \cos\theta_c = \frac{1}{\beta n} \tag{3.11}$$

The angle θ_c can be experimentally measured, and a measure of θ_c and p.
Note that the constraint $\cos\theta_c < 1$ we get that this process exists if and only if $\beta \geq 1/n$
Tying this angle to the momentum of the particle we begin by noting that

$$\beta\gamma = \frac{\beta}{\sqrt{1-\beta^2}} = \frac{p}{m}$$

We have

$$\frac{1}{\beta} = \sqrt{1 + \frac{m^2}{p^2}}$$

Substituting what we found before for $\cos\theta_c$ we have

$$\cos\theta_c = \frac{1}{\beta n} = \frac{1}{n}\sqrt{1 + \frac{m^2}{p^2}} > 1 \tag{3.12}$$

Which implies that the second constraint on Cherenkov radiation is that $p \geq p_{th}$. In the limit case of $o_c = p + m$, $\beta_{th} = n^{-1}$ we get

$$\frac{1}{n} = \sqrt{1 + \frac{m^2}{p_{th}^2}}$$

Which, since $n > 1$ gives

$$p_{th} = \frac{m}{\sqrt{n^2 - 1}} \tag{3.13}$$

For $p >> p_{th}$ it's clear from formulas and experimental measurements that it's not possible to distinguish between different particles from their Cherenkov angle.
In order to find the number of Cherenkov photons emitted in the process we can use classical electrodynamics, getting

$$\frac{\mathrm{d}^2 N}{\mathrm{d}x \mathrm{d}E} = \frac{\alpha Z^2}{\hbar c}\left(1 - \frac{1}{\beta^2 n^2(E)}\right) = \frac{\alpha Z^2}{\hbar c}\sin^2\theta_c \tag{3.14}$$

Where $n(E) = n(\lambda)$ is the refraction index of the medium.
Using $E = 2\pi\hbar c/\lambda$ we have changing variables

$$\frac{\mathrm{d}^2 N}{\mathrm{d}x \mathrm{d}\lambda} = \frac{2\pi\hbar c}{\lambda^2}\frac{\mathrm{d}^2 N}{\mathrm{d}x \mathrm{d}E} = \frac{2\pi\alpha Z^2}{\lambda^2}\sin^2\left(\theta_c(\lambda)\right) \tag{3.15}$$

In the visible spectrum we have $E \approx 2\ \mathrm{eV}$ $(1.8 \to 3.1\ \mathrm{eV})$

§ 3.3 Loss of Energy for Electrons and Positrons

Since we're dealing with the loss of energy of charged particles we have to use the Bethe-Bloch loss of energy equation.
We have that tat the minimum of the functon $\beta\gamma \approx 3$ and therefore $p \approx 1.5\ \mathrm{MeV}$ for electrons.
This impilies that the relativistic climb of the electrons is really slow.
Considering heavy projectiles like nuclei. The electrons will get strongly accelerated and will emit radiation, as for the Larmor formula we have that the radiated power is

$$P_L = \frac{2}{3}\frac{e^2}{m^2c^3}|\dot{\underline{v}}|^2 \tag{3.16}$$

Extending it to a Lorentz invariant formulation we have

$$P_L = -\frac{2}{3}\frac{e^2}{m^2c^3}\frac{\mathrm{d}p_\mu}{\mathrm{d}\tau}\frac{\mathrm{d}p^\mu}{\mathrm{d}\tau}$$

Using

$$\frac{\mathrm{d}p_\mu}{\mathrm{d}\tau}\frac{\mathrm{d}p^\mu}{\mathrm{d}\tau} = \frac{1}{c^2}\left|\frac{\mathrm{d}E}{\mathrm{d}\tau}\right|^2 - \left|\frac{\mathrm{d}\underline{p}}{\mathrm{d}\tau}\right|^2$$

And substituting $E = \gamma mc^2$, $\underline{p} = \gamma mc\underline{\beta}$ we have

$$P_L = \frac{2e^2}{3c}\gamma^6\left(\left(\frac{\mathrm{d}\underline{\beta}}{\mathrm{d}t}\right)^2 - \left(\underline{\beta}\times\frac{\mathrm{d}\underline{\beta}}{\mathrm{d}\tau}\right)^2\right) \tag{3.17}$$

Which, β, γ are the relativistic parameters of a charged particle.
There are two limit cases, one of linear acceleration, and one of perpendicular acceleration of the particles.
For linear acceleration we have

$$\begin{aligned}\dot{\underline{\beta}}&||\underline{\beta}\\ \dot{\underline{\beta}}\times\underline{\beta}&=0\\ P_L&=\frac{2e^2}{3c}\gamma^6\dot{\beta}^2\end{aligned}$$

And for perpendicular acceleration

$$\begin{aligned}\dot{\underline{\beta}}&\perp\underline{\beta}\\ P_L&=\frac{2e^2}{c}\gamma^6\left(\dot{\beta}^2-\beta^2\dot{\beta}^2\right)\end{aligned}$$

Writing the addition on the parentheses as $\beta\dot{\beta}^2/\gamma$ we get that the Larmor radiation power of the electron on a perpendicular acceleration motion (circular motion, ndr.) is proportional to the energy of the particle divided by its mass

$$P_L \propto \gamma^4 = \frac{E^4}{m^4}$$

This is important for the determination of the particle emitting this Larmor radiation.
Suppose you have two charged projectiles with $m_1 \neq m_2$ and $E_1 = E_2$, since $P_L \propto m^{-4}$ we have a

huge increase in the Larmor power for the lighter (charged) particle.
As an example take E fixed and evaluate the radiated power of protons and electrons, we have

$$\frac{P_L^e}{P_L^p} = \frac{m_e^4}{m_p^4} \approx 1.6 \cdot 10^{13}$$

I.e. electrons emit radiation 13 orders of magnitude more powerful than the one emitted by protons

§ 3.4 Bremsstrahlung

Bremsstrahlung, or «*braking radiation*» translated from German, is an effect dominant in electrons, and approximately 0 for $m > m_e$ as long as E isn't big enough.
From Bethe-Bloch's formula we have that the lost energy is

$$-\frac{\mathrm{d}E}{\mathrm{d}x} \propto \frac{E}{\chi_0}$$

Where χ_0 is defined in (3.10) and it's a property of the medium. Solving the approximate ODE we have

$$E(x) = E_0 e^{-\frac{x}{\chi_0}} \tag{3.18}$$

At $x = \chi_0$ we have E_0/e which implies that energy decreases by about 30%. In a standard length χ_0 there therefore around 63% of loss of energy. This loss of energy is known as «*Bremmstrahlung*».
Confronting it with the ionization energy loss for electrons and positrons we have

$$-\frac{\mathrm{d}E_{tot}}{\mathrm{d}x} = -\left(\frac{\mathrm{d}E_{ion}}{\mathrm{d}x} + \frac{\mathrm{d}E_{Brem}}{\mathrm{d}x}\right)$$

Inserting the Bethe-Bloch formula we have

$$R_{B/i} = \frac{\frac{\mathrm{d}E_{Brem}}{\mathrm{d}x}}{\frac{\mathrm{d}E_{ion}}{\mathrm{d}x}} \approx \frac{K_e Z}{1200 m_e}$$

Where $K_e = E_e - m_e \approx E_e$. It's defined as «*critical energy of the medium*» the value of energy such that $R_{B/i} = 1$. For electrons we have

$$E_c = \frac{600}{Z}\,\mathrm{MeV}$$

This is useful for evaluating the ionization minimum for materials, as

$$I_{min} = \frac{E_c}{\chi_0} \approx (3.5\,\mathrm{MeV})\frac{Z}{A}$$

§ 3.5 Photons in Matter

There are three major processes for photons that depend directly on the energy of the photon γ

1. At low energy, the photoelectric effect dominates, with reaction

$$\gamma + {}^{A}_{Z}\mathrm{X} \to {}^{A}_{Z-1}\mathrm{X} + e^- \tag{3.19}$$

2. At a higher energy, the Compton scattering process dominates, with reaction

$$\gamma + e^- \to \gamma + e^- \tag{3.20}$$

3. At high energies, the pair production effect dominates, with reaction

$$\gamma + {}^{A}_{Z}\mathrm{X} \to e^+ + e^- + {}^{A}_{Z}\mathrm{X} \tag{3.21}$$

The total cross-section for photon interaction will obviously be a function depending on E_γ and Z of the interacting nucleus, since at different energies (and also at different Z) different processes can happen

§§ 3.5.1 Photoelectric Effect

Considering the photoelectric effect we have a reaction of the kind

$$\gamma + {}^{A}_{Z}\mathrm{X} \to {}^{A}_{Z-1}\mathrm{X} + e^-$$

In this effect we have

$$\sqrt{s} = \sqrt{m_e^2 + m_X^2 + 2\left(E_e E_X - \underline{p}_e \underline{p}_X\right)}$$

The second part $m_X^2 + 2\left(E_e E_X - \underline{p}_e \underline{p}_X\right)$ is the impulse absorbed by the nucleus. Since $m_X >> m_e$ we have $\Delta E_X = 0$ and all the impulse of γ gets transferred to the electron.
The quantum effect to take note in this case is the ionization energy of the electron, which gets absorbed by e^- in order to escape the atomic bound state, therefore the gained energy is

$$E_e = E_\gamma - I$$

Inverting the relation in terms of E_γ and noting that $E_e = K_e$ we have

$$E_\gamma = K_e + I$$

This effect is dominant for $E_\gamma < m_e \approx 100\ \mathrm{keV}$, and the cross section is

$$\sigma_{\gamma X} \propto \begin{cases} \dfrac{Z^5}{E_\gamma^3} & E_\gamma < m_e \\ \dfrac{Z^5}{E_\gamma} & E_\gamma > m_e \end{cases} \tag{3.22}$$

§§ 3.5.2 Compton Effect

For photon energies $E_\gamma >> I$ we have that $\sigma_{\gamma X}$ gets particularly small and the Compton effect becomes the dominant EM process.

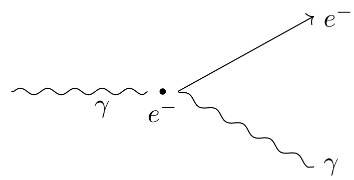


Figure 3.1: Sketch of the Compton scattering, the photon and the electron get scattered by an angle θ

Using the transformations of angle equation in special relativity we have that the new energy of the photon after the scattering is

$$E'_\gamma = \frac{E_\gamma}{1 + \frac{E_\gamma}{m_e}(1 - \cos\theta)} \tag{3.23}$$

The maximum energy transfer will happen at $\theta_{max} = \pi$, for which

$$E_\gamma(\theta = \pi) = \frac{E_\gamma}{1 + \frac{2E_\gamma}{m_e}} \tag{3.24}$$

And when $E_\gamma \approx m_e$, where solving we have

$$E'_\gamma \approx \frac{E_\gamma}{3} = \frac{m_e}{3} \tag{3.25}$$

In this last case, for $\theta = \pi, E_\gamma \approx m_e$, the photon loses around 66% of its initial energy, transferring it all to the electron.
This effect is dominant for $E_\gamma > m_e$ and its cross section is

$$\sigma_{\gamma e^-} \propto \frac{Z}{E_\gamma} \tag{3.26}$$

§§ 3.5.3 Pair Production

Considering the pair production process we might think to write it as a process where a photon decays into an electron and a positron

$$\gamma \to e^- + e^+$$

Evaluating $\sqrt{s}$ for the LHS and RHS of the process we see clearly that this is impossible, since

$$\sqrt{s} = m_\gamma = 0 \neq m_{e^-}^2 + m_{e^+} + 2\left(E_{e^-}E_{e^+} - \underline{p}_{e^-}\underline{p}_{e^+}\right)$$

Adding a nucleus ${}^A_Z\mathrm{X}$ on both sides we instead get

$$\sqrt{s} = m_X^2 + 2E_\gamma m_X$$

Where $2E_\gamma m_X$ is a recoil term of the nucleus given by the conservation of momentum.
It's obvious that there is a threshold energy for this process, for which

$$E_\gamma \geq m_{e^-} + m_{e^+} \approx 1.022\ \mathrm{MeV} \tag{3.27}$$

Taken all these process at once we have that for $10\ \mathrm{eV} \leq E_\gamma \leq 100\ \mathrm{keV}$ the photoelectric effect is dominant, for $100\ \mathrm{keV} < E_\gamma \leq 10\ \mathrm{MeV}$ the Compton effect is dominant with a maximum for $E_\gamma = m_e$. And lastly for $E_\gamma > 10\ \mathrm{MeV}$ the pair production effect is the most predominant effect, starting from $E_\gamma = 1.022\ \mathrm{MeV}$

§§ 3.5.4 Attenuation Length for Photons

Consider now a beam of photons going through some medium. We have that the intensity of the beam will depend on the distance traveled inside the medium as for the equation

$$I(x) = I_0 e^{-\frac{x}{\lambda}} = I_0 e^{-\mu x} \tag{3.28}$$

Where μ is the well known attenuation coefficient. Considering the relation between the cross section and the attenuation coefficient we have that

$$\mu = \mu_{e^+e^-} + \mu_{\gamma e^-} + \mu_{\gamma X} = \sigma n = \left(\sum_i \sigma_i\right) n \tag{3.29}$$

Where σ is the total cross section of the photon interaction. We have that for $E > 100\ \mathrm{MeV}$ σ is mostly constant, and therefore

$$\frac{1}{x_\gamma} = \sigma n \approx \frac{7}{9}\frac{1}{\chi_0} \tag{3.30}$$

Where $1/x_\gamma$ is the attenuation length of the photon, which is deeply tied to the radiation length of the medium. Note that n is the refraction coefficient.
Since we're in the range of $E_\gamma > 10\ \mathrm{MeV}$ the dominant process is pair production, and from this we can calculate the cross section for pair production as

$$\sigma_{e^+e^-} = \frac{7A}{9\rho\chi_0 N_A} \propto Z^2 \log\left(183 Z^{-1/3}\right) \tag{3.31}$$

Note that positrons have the exact same parameters of electrons, excluding the charge which is opposite.

§§ 3.5.5 Electromagnetic Showers in Mediums

Consider a beam of high energy e^+, e^- in some medium, these particles will produce high energy photons through Bremsstrahlung radiation. These high energy photons will then produce pairs of e^+, e^- creating a shower.
Note that the positron-electron pair will lose 30% of their initial energy due to Bremsstrahlung and the photons lose around 60% of their initial energy due to the previous listed scattering processes.
This process is a stochastic shower which continues up until $E_i > E_c$. This process is determined by χ_0, which gives the average distance traveled before the doubling of particles, which cause E to half, and it will continue up until there is no more Bremsstrahlung effect (E_i reaches the level of Compton scattering).

This process is studied in function of the depth $t = x/\chi_0$ and $-E_0^{-1}\frac{\mathrm{d}E}{\mathrm{d}x}$. The profile recovered for this stochastic process is

$$F(t) \propto \frac{E_0}{E_c} t^a e^{-bt} \tag{3.32}$$

And from this we have that the maximum depth the shower will reach is

$$t_{max} = \log\left(\frac{E_0}{E_c}\right) + c \tag{3.33}$$

Where a, b, c are parameters given by energetic corrections and fitting of data.
The cone created by the shower is given by multiple coulomb scattering, which gives no energy loss (elastic scattering) but only an angular reaction. The radius of the cylinder containing 90% of the particles interacting in the shower is

$$R_M = \frac{21}{E_c}\chi_0 \tag{3.34}$$

Which is known as «*Moliere radius*».

§§ 3.5.6 Hadronic Interactions

Consider now the case of hadrons, particles that interact both electromagnetically and strongly, like $p, \pi^{\pm}, K^{\pm}, n$ (protons, pions, kaons, neutrons). These particles can interact strongly with the nuclei in the medium. In general

1. For low energies ($2\ \mathrm{GeV} \leq E \leq 5\ \mathrm{GeV}$) the scattering is elastic and there is no energy loss
2. For intermediate energies ($5\ \mathrm{GeV} \leq E \leq 100\ \mathrm{GeV}$) there is an EM interaction with the medium which transfers around $100\ \mathrm{MeV}$ of energy
3. For high energies ($E > 100\ \mathrm{GeV}$) the hadrons interact strongly with the nucleus, and similarly to the EM case, a shower happens

For the intermediate energy levels we have that $\sigma_n \approx \pi R_N^2 \approx \pi\ \mathrm{fm}^2 \approx 30\ \mathrm{mb}$ and the cross section is inelastic, since the collision is inelastic. σ grows with energy, $\sigma \approx \sigma_0 A^{2/3}$.
For the high energy case, considering a beam of hadrons with intensity $I(x) = I_0 \exp(-x/\lambda)$ we can say that

$$\frac{1}{A} = \sigma_h n = \sigma\rho\frac{N_A}{A}$$

Where λ is the nuclear interaction length, also known as the usual mean free path between the inelastic collisions. In general $\sigma_n \approx 1\ \mathrm{b} < \sigma_{EM}$ therefore $\lambda > \chi_0$

4 Particle Detectors

Consider a general reaction

$$p + p \to H + q_1 + q_2$$

Where H is the Higgs boson which decays as usual into two Z^0 bosons which decay in electron-positron or muon-antimuon couples, and q_1, q_2 are hadron swarms.
The ideal objective is to measure the 4-momentum of all particles. For this various kinds of detectors are used

§ 4.1 Trackers

Tracker detectors for charged particles which measure p through ionization. They rebuild the trace of the particle and use the radius of curvature of the path and use it for finding p.
In this case $\frac{\mathrm{d}E}{\mathrm{d}x}$ must be measurable.
In order to have this reduction of E from ionization we have from Bethe-Bloch that

$$-\frac{\mathrm{d}E}{\mathrm{d}x} \propto C\rho\frac{Z}{A}f(\beta\gamma)$$

Inside the detector we have that the reduction of energy is

$$-\mathrm{d}E \propto \rho\mathrm{d}x$$

And therefore we need ρ small (i.e. a gas) and a thin $\mathrm{d}x$

§§ 4.1.1 Cloud Chambers

Cloud chambers are a simple kind of tracker detector composed by a container filled with saturated vapor.
The charged particles passing through the vapor ionize it, creating bubbles around the ions. After that the track can be photographed and then the needed values can be measured.
This type of particle detector was used to discover the positron. In that cloud chamber a $6\ \mathrm{mm}$ thick slab of Pb was used as a target to slow the particles and a magnetic field was applied.

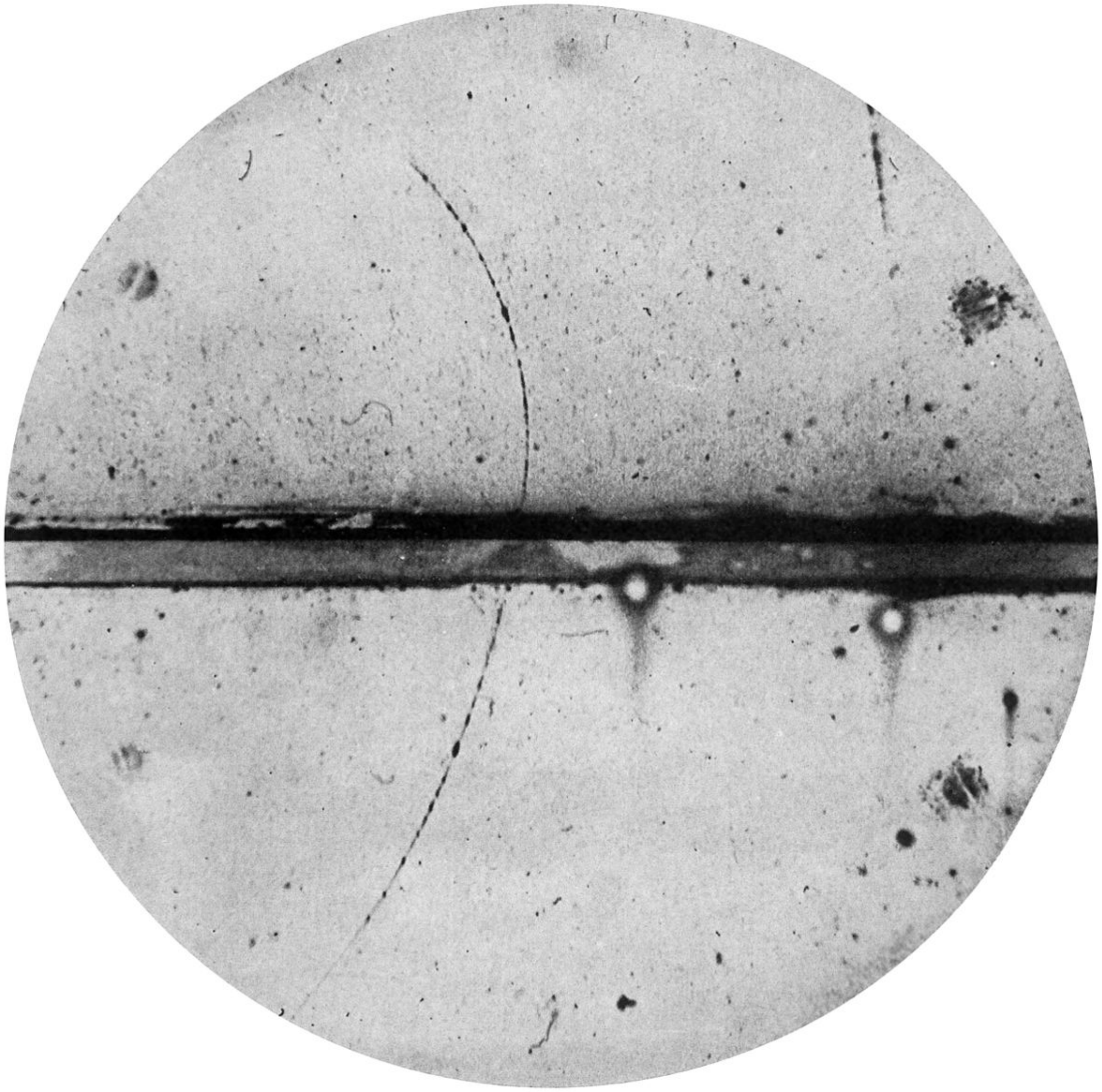


Figure 4.1: The picture taken to the cloud chamber in 1932 by Anderson et al. that shown the production of a positron

From the photo it's possible to identify a particle going through the lead from the bottom, with $p_i = 63\ \mathrm{MeV}$ and with $p_f = 23\ \mathrm{MeV}$. Using $\chi_{0,\mathrm{Pb}} = 5.6\ \mathrm{mm}$ we have that

$$E(\chi_0) \approx \frac{E_0}{3}$$

Supposing that the particle is a proton we would have

$$\beta\gamma = \frac{p}{m_p} = \frac{63}{1000} \approx 0.06$$

Since it's clear from the experiment that the particle passed more than $6\ \mathrm{cm}$, we have that it must be an unknown particle, the positron in this case. Doing the same calculation with m_e we get $\beta\gamma \approx 120$.
A slight modification of this is the bubble chamber, which uses high pressure liquids to create the saturated vapor. This detector has a much greater spatial resolution with $\delta x \approx 100\ \mu\mathrm{m}$

§§ 4.1.2 Nuclear Emulsions

Another kind of tracker detector used is the nuclear emulsion, where gelatinous slabs of silver bromide AgBr are in a suspension on the slab.
The ionization of charged particles releases electrons that make silver shine engraving the track permanently on the slab.
This type of detector has a resolution of $\delta x \approx 1\ \mu\mathrm{m}$, but it's slow to analyze, and therefore it's useful for processes with low frequencies.

§ 4.2 Ionization Detectors

Ionization detectors function by catching the ionization charge of the particles passing by. The measured signal depends on $\frac{\mathrm{d}E}{\mathrm{d}x}$ and on the potential difference ΔV created by two plates that enclose a noble gas.
Chapak in 1962 proposed the construction of such detector with cathode planes and parallel anode wires. The spectral resolution of such detector is $\delta x \approx 300\ \mu\mathrm{m}$

§§ 4.2.1 Drift Chambers

Drift chambers are ionization detectors made by a multiwire proportional chamber which measures the time needed for a signal to arrive from one side to another

§§ 4.2.2 Silicon Detectors

Silicon detectors are another kind of ionization detectors. They are really quick and have a high resolution. They use semiconductors.
Considering that ρ_{Si} is big, the detectors are quite thin, with a resolution of $\delta x \approx 10\ \mu\mathrm{m}$. Note that their resolution is deeply tied to the magnetic field B and thickness of the detector L, in fact we have

$$\frac{\delta P}{P} = \frac{P}{0.3BL^2}\delta x \tag{4.1}$$

I.e., the resolution gets smaller when P grows. Note that

$$P \approx 0.3BR$$

Where P is expressed in GeV, B in T and R in m

§ 4.3 Calorimeters and Energy Measures

The passage of photons, electron positron pairs and hadrons creates EM or hadronic swarms inside mediums. Their ionization can excite molecules or atoms inside this medium.
One kind of calorimeter is the scintillator, which records the de-excitation photons of the swarms (in the visible, close UV spectra). There are two major kinds of scintillators

1. Organic or plastic scintillators made mainly from anthracene, with a response time of 10^{-8} s
2. Inorganic or crystalline scintillators, made mainly from $\mathrm{NaI}, \mathrm{CsI}, \mathrm{PbWO_4}$ with a response time of 10^{-6} s

The main problem with scintillators is that the medium reabsorbs part of these de-excitation photons, giving fewer photons to measure, they activate mainly in the close UV region.
The main solution for solving the first problem is using doping materials in order to increase the scintillation photons.

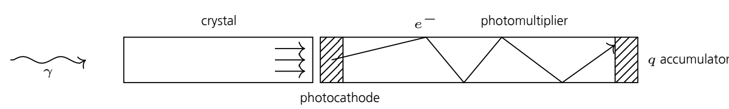


Figure 4.2: Tiny scheme describing how a scintillator functions

As in the picture, the photocatode is a piece sensible to photons and photoelectric effects. It creates a landslide effects on diodes that amplify the photo-electron (photoelectric process' child particle).
For around 10 nodes it's possible to have a gain of 10^4 till 10^7 depending on the value of ΔV. The charge measured q is proportional to the number of photo-electrons and therefore proportional to the energy of the photons.
A detector made only of scintillators and photomultipliers is known as a «*homogeneous calorimeter*».
The calorimeters use long crystals, such that $N\chi_0 \approx 20\chi_0$ in order to contain more or less all the energy of the EM swarm. Therefore, for a crystal with $\chi_0 = 3$ cm in a scintillator it'd be long at least 50 cm.
Hadronic swarms, on the counterpart are regulated by the interaction length $\lambda >> \chi_0$, therefore using homogeneous calorimeters would be prohibitive since they'd need to be really long.
For hadrons usually «*sample calorimeters*» which use a first scintillator block and multiple absorbent blocks, permitting an absorption of the hadrons, since $\lambda^{-1} = \rho\sigma$. Inside the absorber the swarm develops faster and we have

$$\delta q \propto \sqrt{N} \propto \sqrt{N}$$

In calorimeters we have

$$\frac{\delta E}{E} \approx \frac{a}{\sqrt{E}} \tag{4.2}$$

Where a is a measured constant depending on the calorimeter used. This constant is known as the characteristic constant of the calorimeter, and it's known that $a_{hom} < a_{sample}$.
Note also that a higher energy input grants a better resolution, as clear from (4.2)

§ 4.4 Particle Accelerators

A great example, and the first, of particle accelerators, is the well known cathodic tube, which accelerates electrons using a variation of tension. It's a typical example of linear accelerator.
Other famous linear accelerators are PEP-II and BaBAR which managed to reach a $\sqrt{s} = 90 \text{ GeV}$

§§ 4.4.1 Cyclotrons

Cyclotron accelerators are a type of accelerators which use a magnetic field in order to accelerate circularly charged particles. It was first suggested by Lawrence in 1959 and accelerate ions emitted at the center of a circular object composed by two semicircular "Dee".
The frequency of a cyclotron is readily calculable using classical electromagnetism, and it's equal to

$$\nu_c = \frac{qB}{2\pi m} \tag{4.3}$$

This comes since inside the cyclotron there is an uniformly accelerated circular motion caused by ΔV.
Using Newton's second law we have

$$F = ma = m\frac{v^2}{r} = qvB \Longrightarrow \frac{v}{r} = \frac{qB}{m}$$

The period of a revolution is fixed by B, and equals

$$\Delta T = \frac{1}{\nu_c} = \frac{2\pi m}{qB}$$

Note that this formula is not relativistic. In the relativistic formulation, using $\Delta t = \gamma \delta\tau$ we have

$$\nu_c = \frac{qB}{2\pi\gamma m} \qquad \gamma = \frac{E}{m} \tag{4.4}$$

It's quick to see that the cyclotron frequency depends on the velocity of the particle (in the relativistic case), and therefore ultrarelativistc e^- are not suitable for a cyclotron.
Ions, being much heavier and slower, are a better particle candidate for use inside cyclotrons.
We have

$$T_{max} = \frac{1}{2}mv_{max}^2 = \frac{1}{2}m\omega^2 r^2$$

And therefore, setting $r = R$ with R being the radius of the cyclotron, we get that the maximum kinetic energy reached by the ions inside the cyclotron is

$$T_{max} = \frac{(qBR)^2}{2m} \tag{4.5}$$

For relativistic particles, since if v grows $1/\gamma$ decreases, we might suppose to decrease the cyclotron's frequency in order to keep valid the previous relation, keeping B constant
This kind of variable-frequency cyclotron is known as a synchro-cyclotron, which use variable E fields to reach this result.
Another solution is to variate B while keeping ν_c constant in order to compensate for γ^{-1}, these accelerators are known as synchrotrons.
Synchrocyclotrons are usually used for accelerating particles from 10 to 900 MeV, which is a relatively small acceleration

§§ 4.4.2 Synchrotrons

Synchrotrons are the relativistic counterpart of cyclotrons. They work by fixing the radius R and variating B for compensating for γ^{-1} in the synchrotron frequency formula

$$\nu_s = \frac{qB}{2\pi\gamma m} \tag{4.6}$$

This kind of accelerator doesn't need to create poles like the cyclotrons and its "Dees" and a uniform B field is not needed in all the accelerator, therefore there are multiple dipoles along the synchrotrons. The principal limit of synchrotron is Larmor radiation, also known as synchrotron radiation, with power

$$P_L = \frac{e^2}{6\pi\epsilon_0 c^3}\gamma^4 aP2 \tag{4.7}$$

Using $\gamma = E/m$ and $p = v^2/R$ we get

$$P_L = \frac{e^2 E^4}{6\pi\epsilon_0 R^2 m^4} \tag{4.8}$$

Which, for a period $\Delta T = 2\pi R/c$ gives that the energy lost in synchrotron radiation is

$$\Delta E_{lost} \propto \frac{E^4}{m^4 R^2}$$

In order to balance this energy loss it's needed to make bigger synchrotrons. The biggest (so far) is the Large Hadron Collider in Geneva, a synchrotron with $R = 4.3\ \mathrm{km}$. For electrons in the LHC ($m_e \approx 500\ \mathrm{keV}$) we have that the lost energy is proportional to

$$\Delta E_{e,LHC} \approx 88.5\frac{E^4}{4300} \tag{4.9}$$

Which imposes that for high energy electrons all energy is used to compensate for radiative losses inside the synchrotron.
For LEP@CERN, an experiment lasted from 1988 till the early 2000, there was $\sqrt{s} = 90\ \mathrm{GeV}$, for a single beam energy of $45\ \mathrm{GeV}$. It used the same tunnel of LHC and its radiative losses per lap were

$$\Delta E_{LEP} = 84\ \mathrm{MeV/lap}$$

Note that

$$\nu_{LEP} = \frac{c}{2\pi R} = \frac{c}{27\ \mathrm{km}} \approx 10^6\ \mathrm{Hz}$$

Note that for what we have seen it's impossible to accelerate electrons to TeV ranges without increasing the radius of the synchrotron. From 2000 onwards, using protons, a $\sqrt{s} = 13\ \mathrm{TeV}$ has been reached, which corresponds to $6.5\ \mathrm{MeV}$ per beam.
A new project is planned, the FCC, a supercollider with a circumference of $100\ \mathrm{km}$ built in Geneva. With these radius it's possible to reach $\sqrt{s} = 50\ \mathrm{TeV}$.

5 Basics of Quantum Field Theory

§ 5.1 Perturbations and Feynman Diagrams

The first question that comes to mind when talking about particle accelerators is why.
The idea is quite simple and comes from $E^2 = m^2 + p^2$, therefore we can use higher E for converting it into mass.
This lets us discover heavier particles, create more particles via inelastic collisions and to obviously discover new particles.
In general we treat a reaction of the kind

$$a + b \to c + d(+f + g + \cdot)$$

We need to evaluate the initial and final states for such reactions, and Fermi's golden rule comes in handy for this. For FGR, given a transition probability P_{fi} between an initial state and a final state is

$$\lim_{T\to 0} \frac{P_{fi}}{T} = 2\pi \left|\hat{\mathcal{M}}_{fi}\right|^2 \delta\left(E_f - E_i\right) \tag{5.1}$$

Note that different disposition of particles in the final state changes only the kinematics of the final state, since the 4-momentum is always conserved.
What this actually mean is that $\hat{\mathcal{M}}_{fi}$ is independent from the kinematics of the process.
Remember that the decay rate of a reaction is

$$\Gamma_{fi} = \int 2\pi \left|\hat{\mathcal{M}}_{fi}\right|^2 \delta(E_f - E_i) \mathrm{d}n = 2\pi \left|\hat{\mathcal{M}}_{fi}\right|^2 \rho(E) \tag{5.2}$$

Where $\rho(E)$ is the density of states in the phase space and is equal to

$$\rho(E) = \frac{\mathrm{d}n}{\mathrm{d}E_f} \tag{5.3}$$

It's important to note that

$$\hat{\mathcal{M}}_{fi} = -i \left\langle f\right| \hat{\mathcal{H}}_I \left|f\right\rangle \tag{5.4}$$

Where $\hat{\mathcal{H}}_I$ is the perturbation Hamiltonian for a system $\hat{\mathcal{H}} = \hat{\mathcal{H}}_0 + \hat{\mathcal{H}}_I$.
Using perturbation theory for such system we have

$$\begin{aligned} \hat{\mathcal{H}}_0 \psi_n = E_n \psi_n i \frac{\partial \psi}{\partial t} = \left(\hat{\mathcal{H}}_0 + \hat{\mathcal{H}}_I\right) \psi \\ \psi = \sum_n a_n(t) \psi_n e^{-iE_n t} \end{aligned} \tag{5.5}$$

For a transition $|i\rangle \to |f\rangle$ we impose the following approximation conditions on the functions $a_n(t)$

$$\begin{cases} a_i(t) = 1 & k = i \\ a_k(0) = 0 & k \neq i \end{cases} \tag{5.6}$$

Taking the second equation in (5.5) and multiplying on the left by $\langle k|$ we get the following Schrödinger equation for a $k-$th state

$$\dot{a}_k(t) = -i \int \overline{\psi}_k \hat{\mathcal{H}}_I \psi_i e^{i(E_k - E_i)t} \mathrm{d}^3 r \tag{5.7}$$

Integrating and rewriting on the RHS the definition of $\hat{\mathcal{M}}_{ki}$, and imposing $|k\rangle = |f\rangle$ we have that

$$P_{fi} = |a_f(t)|^2 = \left| \int_0^t \hat{\mathcal{M}}_{fi} e^{i(E_f - E_i)t} \mathrm{d}t \right|^2 \tag{5.8}$$

Going over to the second order terms of the perturbation and reinserting everything inside the Schrödinger equation, where we write the $a_n(t)$ we found before, we have

$$\dot{a}_k(t) = -i\hat{V}_{ki} e^{i(E_k - E_i)t} + (-i)^3 \sum_{n \neq i} \frac{\hat{V}_{kn}\hat{V}_{ni}}{E_n - E_i} e^{i(E_k - E_i)t} = -i\hat{V}_{ki}^{(2)} e^{i(E_k - E_i)t} \tag{5.9}$$

Where $\hat{V}_{ki}^{(2)}$ is the second order perturbation and is equal to

$$\begin{aligned} \hat{V}_{ki}^{(2)} &= \hat{V}_{ki} + (-i)^3 \sum_{n \neq i} \frac{\hat{V}_{kn}\hat{V}_{ni}}{E_i - E_n} \\ \hat{V}_{ki} &= \langle k| \hat{\mathcal{H}}_I |i\rangle \end{aligned} \tag{5.10}$$

The second order transition probability is then

$$\Gamma_{fi}^{(2)} = 2\pi \left| \hat{\mathcal{M}}_{fi}^{(2)} \right|^2 \rho(E) \tag{5.11}$$

Using FGR we have for the conservation of energy $\delta(E_f - E_i) \Longrightarrow E_f = E_i$ but only to the first order. Considering a scattering like Rutherford scattering we have considering second order corrections

$$\hat{\mathcal{M}}_{fi} \sim \sum_{n \neq i} \frac{\hat{V}_{fn}\hat{V}_{ni}}{E_i - E_n}$$

The system in this approximation will jump from the first state $|i\rangle$ to accessible states $|n\rangle$ to the final state $|f\rangle$ via the transition matrices $\hat{V}_{ni}$, $\hat{V}_{fn}$.
By definition we have $E_n \neq E_i \neq E_f$, therefore energy is not conserved in this situation. This is fixed by imposing that states with $E_n >> E_i$ and $E_n >> E_i$ are improbable.
A good example of showing graphically these perturbations is using «*Feynman diagrams*», graphs where each vertex corresponds to a degree of perturbation.
At the first order of perturbation we can write as an example the following transition

$$\left| e^+ e^- \right\rangle \to |\gamma\rangle$$

Which corresponds to a pair annihilation reaction $e^+ + e^- \to \gamma$. The first order perturbation will be graphed as

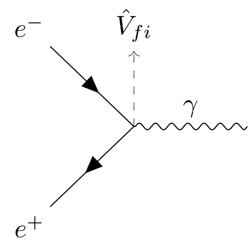


Figure 5.1: Feynman representation of the first order perturbation $\hat{V}_{fi}$.

This graph is a ≈ 1 dimensional graph. The only dimension accounted here is time, which flows from right to left. Matter is drawn as arrows flowing with time and antimatter (see e^+) is drawn with arrows that flow against time. The vertices represent the actual interaction matrix $\hat{V}_{fi}$, in this case only for a 1st order perturbation.
With a quick check of this process we see that

1. The incoming particles are a positron and an electron

2. The only outgoing particle is a photon

And therefore

$$\sqrt{s_i} = 2m_e^2 \neq 0 = \sqrt{s_f}$$

Therefore the conservation of energy given by $\delta(E_f - E_i)$ is not valid.
More generally, with a reaction $a + b \to X$, the diagram would be drawn as

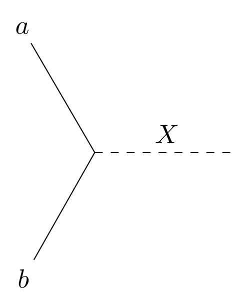


Figure 5.2: Feynman diagram for the process $a + b \to X$

Note that this process can only happen if $s_i = m_a^2 + m_b^2 = m_X^2 = s_f$, and therefore doesn't happen for $s_i > s_f$.
For second order processes the diagram for the interaction described in (5.1) becomes

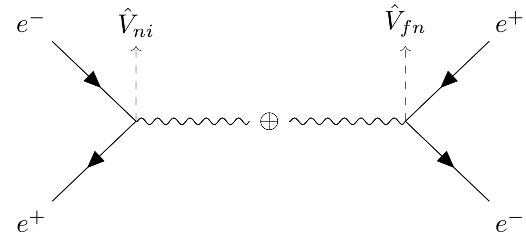
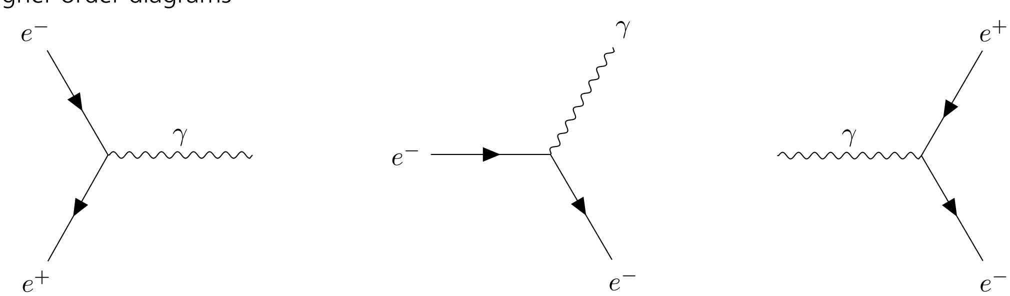


Figure 5.3: Second order diagram considered as the sum of the two vertexes corresponding to the transitions $|i\rangle \to |n\rangle$ and $|n\rangle \to |f\rangle$. The photon connecting the two diagram is known as «*virtual*» due to its non-physical and non measurable energy E_n

This diagram is exactly drawn as the sum of two single vertex diagrams corresponding to a transition $|i\rangle \to |n\rangle \to |f\rangle$. The photon inside can have E_n that aren't possible otherwise, such as $E_n \neq p^2 + m^2$, the so called «*off shell*» energies.
What happened in this scattering process, a $e^+e^- \to e^+e^-$ elastic scattering, is that a virtual photon mediates the process. Basically the electron and positron annihilate creating a virtual photon which re-decays into an electron and a positron, which get measured as outgoing particles.
Note that for Heisenberg this is possible. In fact $\Delta E \Delta t \approx \hbar$ imply that $\Delta t \leq \hbar/\Delta E$, and therefore for $t < \Delta t$ the ΔE violations are possible, as long as charge and quantum numbers are conserved.

§§ 5.1.1 Electrodynamic Processes

For describing electromagnetic processes we can build three basic vertexes which can be used to build up higher order diagrams



All these three basic vertexes share one main thing: charge is conserved.
Consider now Rutherford scattering, this process can be described via the following diagram

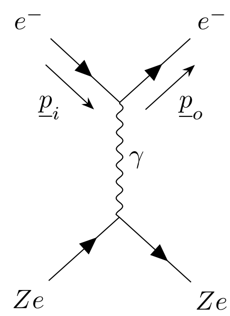


Figure 5.4: Feynman diagram for Rutherford scattering

From this diagram we can immediately see that $Q_i = e(Z-1) = Q_f$, since we used two fundamental vertexes. In order to get something more from this diagram, especially how to grasp the perturbations from the vertexes we need to do some calculations on the initial state $|i\rangle$ and the final state $|f\rangle$. We begin by Born-approximating the wavefunction of the incoming electron as a planar wave, therefore

$$\begin{aligned} \psi_i &= \frac{1}{\sqrt{V}} e^{i\underline{p}_i \underline{r}} \\ \psi_0 &= \frac{1}{\sqrt{V}} e^{i\underline{p}_o \underline{r}} \end{aligned} \tag{5.12}$$

Therefore, having an electromagnetic perturbation given by the potential of the nucleus, we have that our perturbation is

$$\hat{V}_{fi} = -\langle f| \frac{Z\alpha}{r} |i\rangle = -\frac{Z\alpha}{V} \int \frac{1}{r} e^{i(\underline{p}_i - \underline{p}_o)\underline{r}} \mathrm{d}^3 r$$

Writing $\underline{q} = \underline{p}_i - \underline{p}_f$ and $\underline{q}\underline{r} = qr\cos\theta$ and transforming the integral into spherical coordinates we have

$$\hat{V}_{fi} = -\frac{Z\alpha}{V} \int_0^\infty r\mathrm{d}r \int_0^{2\pi} \mathrm{d}\varphi \int_0^\pi e^{iqr\cos\theta} \sin\theta \mathrm{d}\theta$$

Taking the third integral and writing $\mathrm{d}\cos\theta = -\sin\theta$ we have

$$\int_0^\pi \sin\theta e^{iqr\cos\theta} \mathrm{d}\theta = \frac{1}{iqr} \int_{-1}^1 e^{iqr\cos\theta} \mathrm{d}\cos\theta = \frac{1}{iqr} \left(e^{iqr} - e^{-iqr}\right)$$

Reinserting it into the integral and integrating with respect to φ we have

$$\hat{V}_{fi} = -\frac{2\pi Z\alpha}{iqV} \int_0^\infty \left(e^{iqr} - e^{-iqr}\right) \mathrm{d}r$$

The last integral can be calculated using something similar to Feynman's integration trick, by multiplying the function by a dummy function $e^{-\varepsilon r}$ and taking the limit for $\varepsilon \to 0$ for getting back the last result. Substituting and integrating we have

$$\hat{V}_{fi} = -\frac{2Z\pi\alpha}{iqV} \lim_{\varepsilon \to 0} \left(-\frac{1}{iq - \varepsilon} - \frac{1}{iq + \varepsilon} \right) = -\frac{2Z\pi\alpha}{iqV} \frac{2iq}{q^2}$$

Simplifying, we get

$$\hat{V}_{fi} = -\frac{-4Z\pi\alpha}{Vq^2} \tag{5.13}$$

Using $\hat{\mathcal{M}}_{fi} = -i\hat{V}_{fi}$ we get the transition matrix, and the transition probability as

$$\begin{aligned}\hat{\mathcal{M}}_{fi} &= \frac{4iZ\pi\alpha}{Vq^2}\\ \left|\hat{\mathcal{M}}_{fi}\right|^2 &= \frac{16Z^2\pi^2\alpha^2}{V^2q^4}\end{aligned} \tag{5.14}$$

Which implies $\sigma \propto Z^2\alpha^2q^{-4}$.
Going back to the diagram and remembering that each vertex represents a perturbation we have, for the vertex of the nucleus a charge of Ze and a contribute of $\sqrt{\alpha}$, for the other one we have another contribution $\sqrt{\alpha}$ with charge e, connected by a virtual photon, which contributes for the moment with a so called «*propagator*». The photon propagator is proportional to q^{-2}, and therefore, simply by looking at the two vertexes in natural units ($e = 1$), we get

$$\hat{\mathcal{M}}_{fi} = \sqrt{\alpha}\frac{1}{q^2}Z\sqrt{\alpha} = \frac{Z\alpha}{q^2} \tag{5.15}$$

Which is what we found up to a factor of $-i4\pi$.
Consider now as a second example of these rules the process of Compton scattering.
The diagram for this process will be

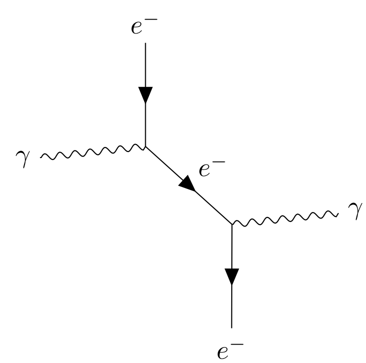


Figure 5.5: Second order diagram for Compton scattering

Using Feynman rules on vertexes we have that $\hat{\mathcal{M}}_{fi} \approx \sqrt{\alpha}\sqrt{\alpha} = \alpha$, and therefore we can immediately suppose that $\sigma \sim \left|\hat{\mathcal{M}}_{fi}\right|^2 \approx \alpha^2$.
One quick thought about conservation laws makes clear that if the virtual particle in the diagram *must* be an electron, because the number of leptons must be conserved in all vertexes. Note that if the virtual particle was a baryon like a proton, it also wouldn't be right since the number of baryons isn't conserved.
Also consider a quick thing, if the outgoing particles were switched in the diagram, the result would be

the same, although we must consider this diagram's contribution in the final calculations.
Take now the Bremsstrahlung radiation, this process corresponds to the following third order diagram
For this diagram we have $\hat{\mathcal{M}}_{fi} \propto Z\sqrt{\alpha}\sqrt{\alpha}\sqrt{\alpha}\frac{1}{q^2}$, therefore $\sigma \propto Z^2\alpha^3/q^4$.

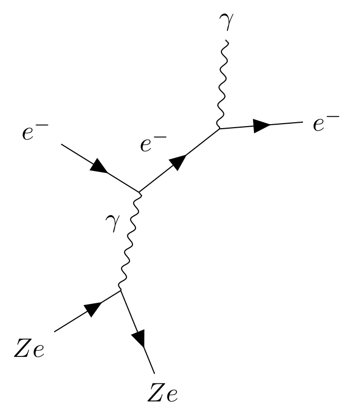


Figure 5.6: Bremsstrahlung effect third order Feynman diagram

Confronting the obtained cross section to the one for Rutherford scattering we get $\sigma_R \propto \alpha^2$, and $\sigma_{Brem} \propto \alpha^3$, with $\alpha = 1/137$.
Going down this path of describing electrodynamic processes with Feynman diagrams we impact ourselves in a new Feynman rule for electromagnetic interactions: vertexes can't have multiple photons reaching them, therefore this diagram is impossible

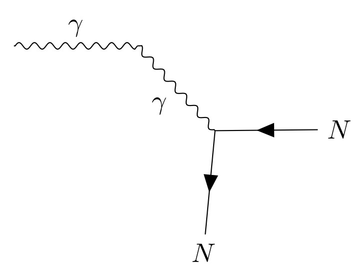


Figure 5.7: An impossible diagram

And now one might think, how can I build a pair production diagram? The answer is: add a virtual particle between the photonic vertexes. The searched diagram then is

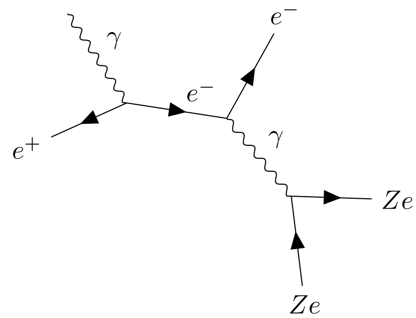


Figure 5.8: Maybe fix this? idk not sure probably pair production

By just looking at the diagram we have $\hat{\mathcal{M}}_{fi} \propto Z\sqrt{\alpha}\sqrt{\alpha}\sqrt{\alpha}q^{-2} = Z\alpha^{3/2}/q^2$.
Another interesting process is Bhabha scattering, $e^+e^- \to e^+e^-$. This diagram is pretty simple to draw

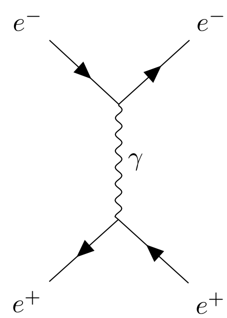


Figure 5.9: Bhabha scattering diagram

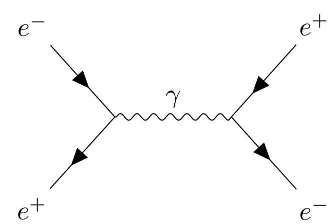


Figure 5.10: A symmetric version of the same bhabha scattering diagram

By checking the nodes of this diagram we get immediately that $\sigma \propto \alpha^2/q^4$

§ 5.2 Klein-Gordon Equation and the Yukawa Potential

Yukawa in 1935, using the idea of virtual mediator particles went on trying to explain the nuclear force between nucleons. Experimetally it had been seen that it was a short range force, and that there is a symmetry between neutrons and protons.
From classical EM we know that the electrostatic potential generated by a pointlike charge at the origin solves the inhomogeneous Poisson equation

$$\nabla^2 V = -e\delta^3(r) \tag{5.16}$$

The solution is an integral retarded potential

$$V(r) = \int_V \frac{\rho(\underline{r}')}{|\underline{r} - \underline{r}'|} \mathrm{d}^3 r' \tag{5.17}$$

For time dependent potentials, defining $\Box = \partial^\mu \partial_\mu = \partial_t^2 - \nabla^2$ the Maxwell equations are

$$\Box\left(\underline{E}, \underline{B}\right) = 0 \tag{5.18}$$

Where, in the potential formulation become, writing a 4-potential $A_\mu = (\phi, \underline{A})$

$$\begin{aligned} \frac{\partial^2 \phi}{\partial t^2} - \nabla^2 \phi = \rho \\ \frac{\partial^2 \underline{A}}{\partial t^2} - \nabla^2 \underline{A} = \underline{J} \end{aligned} \tag{5.19}$$

Taking only the first of the two equations, we might think to quantize this equation imposing $i\partial_t \to \hat{E}$ and $-i\nabla \to \underline{\hat{p}}$, getting the following equation

$$\Box\phi = \left(\hat{E}^2 - \hat{p}^2\right)\phi = 0 \tag{5.20}$$

It's immediately clear that we must have $E = p$, and therefore this equation works only for massless particles.
Since we're dealing with massive particles we might immediately think to substitute $E^2 = m^2 + p^2$, getting what is known as the «*Klein-Gordon equation*», which satisfies a massive mediation of the potential

$$\left(\hat{E}^2 - \hat{p}^2 + m^2\right)\phi = \left(\Box + m^2\right)\phi = 0 \tag{5.21}$$

Fitting it into the stationary case of the nucleon with a nuclear charge $g \neq 0$ at $\underline{r} = 0$ we get the following equation

$$\left(\nabla^2 - m^2\right)\phi(\underline{r}) = -g\delta^3(\underline{r}) \tag{5.22}$$

Which has for solution the Yukawa potential for the strong interaction

$$\phi(r) = -\frac{g}{4\pi r}e^{-mr} \tag{5.23}$$

This potential corresponds to a shielded couloomb interaction.
In natural units we have that mr is adimensional, and using $\Delta E \Delta t \approx \hbar$ with $'DE \approx mc^2, \Delta t \approx R/c$ with $R \approx 1.5 \text{ fm}$ we get that

$$mc^2 \approx \frac{\hbar c}{R} \approx 150 \text{ MeV}$$

The mediating particle for this process must have a mass of $m \approx 150 \text{ MeV}$, so Yukawa theorized the existence of a mesotron particle which mediates the nuclear force.
From experiments we now know that this mesotron, or better known as pion, isn't an elementary particle, and therefore Yukawa's model is effective in explaining these interactions but it's not a fundamental one.
In general a charged pion π^2 is a meson (quark-antiquark bound state), with state $\left|u\overline{d}\right\rangle$.
We might think to find this pion propagator using Yukawa's potential as a perturbation to Born states, which gives

$$\langle f| - \frac{g}{4\pi r}e^{-mr}|i\rangle = -i\frac{g^2}{4\pi}\frac{1}{q^2 + m^2} \tag{5.24}$$

This propagator is similar to the photon propagator, and using $\alpha_{EM} = e^2/4\pi$ we might think to construct a "fine structure constant" for strong interaction, which is $\alpha_S = g^2/4\pi$.
In general for a massive potential we have a propagator of the following kind

$$\langle f|\hat{\mathcal{H}}_I|i\rangle = -i\alpha\frac{1}{q^2 + m^2} \tag{5.25}$$

Supposing a force with $m^2 >> q^2$ we can see immediately that the momentum exchanged between the interacting particles is negligible with respect to the mass of the mediating particle, which lets us approximate the propagator to

$$\langle f | \hat{\mathcal{H}}_I | i \rangle \approx -i\alpha \frac{1}{m^2} \tag{5.26}$$

§§ 5.2.1 Weak Interactions

We might think to apply this approximation to Fermi's interactions, with $\hat{\mathcal{H}}_I = G_F$. The weak interaction that Fermi studied is a finite range interaction. Let's suppose that this interaction is mediated by a massive particle with charge g_w and mass m_w, the potential for such interaction is analoguous to Yukawa's potential

$$V_w = -\frac{g_w}{4\pi r} e^{-m_w r} \tag{5.27}$$

Writing $\hat{\mathcal{H}}_I = g_w V_w = G_F$ we get from experimental values

$$G_F = \frac{g_w^2}{4\pi m_w^2} = 1.16 \cdot 10^{-5} \text{ MeV}^{-2} \tag{5.28}$$

Which suggests a mass of the weak mediator, in terms of measurable quantities, of

$$m_w = \left(\frac{g_w}{e}\right)^2 \frac{\alpha_{EM}}{G_F} \approx \left(\frac{g_w}{e}\right)^2 10^2 \text{ MeV} \tag{5.29}$$

In order to discover this new W particle a new particle accelerator was built at LEP, which was the combination of a proton-synchrotron and an antiproton-synchrotron, which was used to verify the following theoretical reaction

$$p + \overline{p} \to W^+ + X \tag{5.30}$$

Considering a cross beam interaction with two targets we have

$$E_p = E_{\overline{p}} = 270 \text{ GeV}, \qquad \sqrt{s} = 2E_p = 540 \text{ GeV}$$

The confirmation of this reaction awarded a Nobel prize in 1984 to Rubbiz and Van der Meer.
So far we have listed three fundamental forces: Electromagnetism, strong force and the weak force, all mediated by particles, respectively $\gamma, g, W^\pm/Z^0$. The second mediator is known as the «*gluon*» and it's a massless particle, which is not predicted by Yukawa's theory, although it's effective to explain nuclear phenomena.
Only one fundamental force is missing a mediator particle, which is gravity. The idea of a «*graviton*» particle which for now there haven't been any experimental verifications.

§ 5.3 Symmetries

As we know from Nöther's theorem, each constant of motion corresponds to a simmetry of the system. Considering reactions $a + b \to c + d$, we immediately know that a conserved quantity in the reaction is a symmetry of the transition Hamiltonian $\hat{\mathcal{H}}_I$.
There are 4 kinds of symmetry we might consider

1. Continuous symmetries
   - Temporal traslations, which correspond to E conservation
   - Spatial translations, which correspond to $\underline{p}$ conservation
   - Spatial rotations, which correspond to $\underline{L}$ conservation
2. Gauge symmetries, which correspond to q conservation
3. Fundamental symmetries, like the lepton number conservation, baryon number conservation, etc.
4. Non-spatial rotations, which correspond to the conservation of a new quantity known as Isospin $\underline{\hat{I}}$

§§ 5.3.1 Discrete Transformations

Going to the quantum world with our symmetries we can list immediately three discrete symmetries

- Parity reflection, $\hat{P} : \hat{r} \to -\hat{r}$
- Charge conjugation, $\hat{C} : q \to -q$
- Time inversion, $\hat{T} : t \to -t$

These laws are multiplicative, and as an example the matter-antimatter symmetry corresponds to a $\hat{C}\hat{P}$ transformation.

§§ 5.3.2 Leptonic Number

Consider a weak interaction like the β decay, with reaction

$$n \to p + e^- + X$$

Considering Reines-Cowan's experiment which supposes a reaction $X + p \to n + e^+$, a possible reaction with $Q > 0$, we have a problem, and the reaction was never observed.
It was theorized the existence of a leptonic number associated with the electron L_e, for which $\Delta L_e = 0$ in reactions.
Leptonic matter accounts for $L = 1$, while leptonic antimatter for $L = -1$. In leptonic matter also neutrinos are accounted, and we can make a list of couples lepton/neutrino.

$$\begin{pmatrix} e^- \\ \nu_e \end{pmatrix} L_e = 1 \quad \begin{pmatrix} \mu^- \\ \nu_\mu \end{pmatrix} L_\mu = 1 \quad \begin{pmatrix} \tau^- \\ \nu_\tau \end{pmatrix} L_\tau = 1 \tag{5.31}$$

And antilepton/antineutrino

$$\begin{pmatrix} e^+ \\ \overline{\nu}_e \end{pmatrix} L_e = -1 \quad \begin{pmatrix} \mu^+ \\ \overline{\nu}_\mu \end{pmatrix} L_\mu = -1 \quad \begin{pmatrix} \tau^+ \\ \overline{\nu}_\tau \end{pmatrix} L_\tau = -1 \tag{5.32}$$

For all interactions, a violation of the conservation of the leptonic number was never observed, and therefore it's possible to assume

$$\Delta L_i = 0 \qquad i = e, \mu, \tau \tag{5.33}$$

§§ 5.3.3 Baryonic Number and Isospin

Segré with its experiment discovered the antiproton, was searching for the proof of the existence of the following reaction

$$p + p \to p + p + p + \overline{p} \tag{5.34}$$

This reaction corresponds to an inelastic scattering of two protons, but there are also other reactions possible

$$\begin{aligned} &p + p \to p + p + \pi^+ + \pi^- \\ &p + p \to p + p + \pi^0 \\ &p + p \to p + p \end{aligned}$$

From his eperiment it was discovered that $m_p = m_{\overline{p}}$, but also a new question arises. Why is this reaction not observed?

$$p + p \to p + \overline{p} + \pi^+ + \pi^+$$

This non-possibility of the previous reaction brings us to the conservation of the baryonic number B.
A baryon (antibaryon) is a composite particle composed by three quarks (antiquarks)

$$B = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} \quad \overline{B} = \begin{pmatrix} \overline{q}_1 \\ \overline{q}_2 \\ \overline{q}_3 \end{pmatrix} \tag{5.35}$$

For each baryon a baryonic number N is associated, which evaluates to ± 1 if the particle considered is either a baryon or an antibaryon. Particles like the pions $\pi^{\pm}, \pi^0$ are known as «*mesons*» and are composed of a quark/antiquark bound state

$$M = \begin{pmatrix} q_1 \\ \overline{q}_2 \end{pmatrix} \tag{5.36}$$

Counting $1/3$ for each quark and $-1/3$ for each antiquark we have that mesons contribute with $N_B = 0$ and each baryon with $N_B = \pm 1$ as we said before.
The question remains, why doesn't that reaction happen, since the baryonic number is conserved? In order to explain this we have to dwelve deeper into the physics of this.
Since $m_n - m_p \approx 1\ \mathrm{MeV}$, nuclear interactions *do not* distinguish neutrons from protons. Due to the non-degeneration theorem we know that there exists a new degree of freedom of the system with an associated quantum number in order to account to this strong symmetry between n, p.
This new degree of freedom is the «*Isospin*», which its algebra corresponds exactly to the spin algebra, a rotation group.
We have for protons and neutrons

$$\begin{aligned} |p\rangle &= |I, I_3\rangle = \left| \frac{1}{2}, \frac{1}{2} \right\rangle \\ |n\rangle &= |I, I_3\rangle = \left| \frac{1}{2}, -\frac{1}{2} \right\rangle \end{aligned} \tag{5.37}$$

Accounting for this, nuclear interactions are invariant under isospin transformations.
Note that for pions

$$\begin{aligned} \left|\pi^+\right\rangle &= |11\rangle \\ \left|\pi^0\right\rangle &= |10\rangle \\ \left|\pi^-\right\rangle &= |1-1\rangle \end{aligned} \tag{5.38}$$

§ 5.4 Isospin

Isospin was a property first introduced by Heisenberg in order to explain strong nuclear interactions between protons and neutrons.
This property behaves algebrically as angular momentum and it can be used for classifying known Hadrons, estimate strong cross section and to theorize states which are not yet observed.
In strong interactions isospin is conserved, indicating that it's a symmetry of the strong interaction Hamiltonian.
Let's begin considering a bound neutron-proton state, deuterium ${}^2_1\mathrm{H}$. As we know from before we have

$$\begin{aligned} |p\rangle &= |I, I_z\rangle = \left|\frac{1}{2}, \frac{1}{2}\right\rangle \\ |n\rangle &= |I, I_z\rangle = \left|\frac{1}{2}, -\frac{1}{2}\right\rangle \end{aligned}$$

Using the angular momentum summation rules we have that, for a state $|np\rangle$ we have $I = 0, 1$ which implies the existence of three states plus one with total $I_z = -1, 0, 1$, these triplet states correspond to the couple $|pp\rangle, |np\rangle, |nn\rangle$ plus $|np\rangle$ and are

$$\begin{aligned} |pp\rangle &= |p\rangle |p\rangle = |1, 1\rangle \\ |np\rangle &= \frac{1}{\sqrt{2}} \left(|n\rangle |p\rangle + |n\rangle |p\rangle\right) = |0, 1\rangle \\ |nn\rangle &= |n\rangle |n\rangle = |1, -1\rangle \end{aligned} \tag{5.39}$$

The previous states are known, as for spin, triplet symmetric isospin states. The additional remaining state is the singlet antisymmetric state

$$|np\rangle = \frac{1}{\sqrt{2}} \left(|n\rangle |p\rangle - |n\rangle |p\rangle\right) = |0, 0\rangle \tag{5.40}$$

Experimentally triplet states are not observed, therefore we can safely assume that for a deuterium nucleus, (deuteron, d) we have

$$|d\rangle = |np\rangle = |00\rangle \tag{5.41}$$

§§ 5.4.1 Pion-Nucleon Scattering

Consider the scattering between pions and nucleons

$$\pi^{\pm,0} + (p, n) \to \pi^{\pm,0} + (p, n) \tag{5.42}$$

Considering the isospin for the system and noting that $I_\pi = 1$, $I_{p,n} = 1/2$ we have that for the initial state

$$|i\rangle = |1, a\rangle + \left|\frac{1}{2}, b\right\rangle = \alpha \left|\frac{3}{2}, a + b\right\rangle + \beta \left|\frac{1}{2}, a + b\right\rangle \tag{5.43}$$

The scattering process can be seen as the addition of an interaction Hamiltonian $\hat{\mathcal{H}}_S$ which is invariant for isospin, and considering a final state

$$|f\rangle = \gamma \left|\frac{3}{2}\right\rangle + \delta \left|\frac{1}{2}\right\rangle \tag{5.44}$$

We have that, using the selection rule $\Delta I = 0$

$$\langle f|\, \hat{\mathcal{H}}_S \,|i\rangle = a \left\langle \frac{3}{2}\right| \hat{\mathcal{H}}_S \left|\frac{3}{2}\right\rangle + b \left\langle \frac{1}{2}\right| \hat{\mathcal{H}}_S \left|\frac{1}{2}\right\rangle = a\hat{\mathcal{M}}_3 + b\hat{\mathcal{M}}_1 \tag{5.45}$$

The final cross section will then be

$$\sigma \propto |a|^2 \left|\hat{\mathcal{M}}_3\right|^2 + |b|^2 \left|\hat{\mathcal{M}}_1\right|^2 + ab\hat{\mathcal{M}}_1\hat{\mathcal{M}}_3 \tag{5.46}$$

Using Clebsch-Gordan coefficients we can already see how these transition matrices will decompose. Noting that pions are isospin 1 and nucleons are isospin $1/2$ system we have a $\mathbf{3} \otimes \mathbf{2}$ system. From the table ?? we have that such state will decouple in a symmetric quadruplet and an antisymmetric doublet (remembering that $I_z^\pi = 1, 0-1$ and $I_z^{(n,p)} = 1/2, -1/2$) giving for $\pi^+ + p \to \pi^+ + p$ and $\pi^- + n \to \pi^- + n$

$$\sigma \propto \left|\hat{\mathcal{M}}_3\right|^2 \tag{5.47}$$

Due to symmetry one immediately expects that $N(\pi^- + p) = N(\pi^+ + n)$ where

$$\sigma_{\pi^- p} \propto \frac{1}{3}\left|\hat{\mathcal{M}}_3\right|^2 + \frac{2}{3}\left|\hat{\mathcal{M}}_1\right|^2 + \frac{2}{9}\hat{\mathcal{M}}_1\hat{M}_3 \tag{5.48}$$

Expanding for the other possible reaction and remembering that $\hat{\mathcal{H}}_S$ is orthonormal in the total isospin basis the remaining cross sections (and therefore also the number of reactions) are easily calculable

§§ 5.4.2 Isospin and Charge

Consider a nucleus (A, Z), using $I_z^p = 1/2,\ I_z^n = -1/2$ we have that the total $z-$projection of isospin will be

$$I_z = \frac{Z}{2} - \frac{A-Z}{2} \tag{5.49}$$

Inverting for Z we have a direct connection between isospin and charge

$$Z = I_z + \frac{A}{2} \tag{5.50}$$

Considering that A is the total number of neutrons and protons in the nucleus, we have from Gell-Mann-Nishijima that for a generic system with baryonic number B the total charge will be given by the following equation

$$Q = I_3 + \frac{B}{2} \tag{5.51}$$

This clearly also works for mesons, in fact taking π^+ as an example, for which $Q = 1$

$$Q_{\pi^+} = 1 + \frac{0}{2} = 1$$

The later discovery of strange particle broke this formula, which was fixed by defining a *hypercharge*, which is defined as

$$Y = B + S \tag{5.52}$$

With B being the baryonic number and S the strangeness number. The introduction of the quark model with charm, strange, top and bottom quarks then defined the hypercharge as the sum of the quantum numbers of all the previous quarks, giving for the Nishijima equation

$$Q = I_z + \frac{B + s + c + b + t}{2} = I_z + \frac{Y}{2} \tag{5.53}$$

Note that that strange and bottom quarks reduce s, b by 1 like antitop and anticharm quarks reduce charm and top quantum numbers. Remember that this is purely a convention