§ 0.1 Bessel Inequality and Fourier Coefficients

Definition 0.1.1 (Fourier Coefficients)**.** Suppose $(u_k)_{k\in\mathbb{N}} = \mathcal{U} \subset \mathcal{V}$, with $(\mathcal{V}, \langle\cdot,\cdot\rangle)$ an euclidean space, taken $v \in \mathcal{V}$ we can define an operator $\widehat{\mathcal{F}}_{\mathcal{U}} : \mathcal{V} \longrightarrow \mathbb{C}^{\mathbb{N}}$ such that

$$\forall v \in \mathcal{V} \quad \widehat{\mathcal{F}}_{\mathcal{U}}(v) = c \in \mathbb{C}^{\mathbb{N}}$$

Where

$$c = (\langle v, u_1\rangle, \langle v, u_2\rangle, \cdots) \in \mathbb{C}^{\mathbb{N}}$$

The coefficients $\langle v, u_k\rangle \in \mathbb{C}$ are called the *Fourier coefficients* of the vector $v \in \mathcal{V}$

Theorem 0.1 (Bessel Inequality & Parseval's Theorem)**.** *Given $(u_k)_{k\in\mathbb{N}} = \mathcal{U} \subset \mathcal{V}$ an orthonormal system and $\mathcal{V}$ an euclidean space with $v \in \mathcal{V}$. Taken $\alpha_1, \cdots, \alpha_k \in \mathbb{C}$ some coefficients and defined the two following sums*

$$S_n = \sum_{k=1}^{n} \langle v, u_k\rangle u_k = \sum_{k=1}^{n} c_k u_k$$
$$S_n^{\alpha} = \sum_{k=1}^{n} \alpha_k u_k$$

Then

$$\|v - S_n\| \leq \|v - S_n^{\alpha}\|$$
$$\sum_{k=1}^{\infty} \|c_k\|^2 \leq \|v\|^2$$

The last inequality is known as Bessel's inequality
Lastly we also have Parseval's equality or Parseval's theorem, which states

$$v = \sum_{k=1}^{\infty} \langle v, u_k\rangle u_k = \sum_{k=1}^{\infty} c_k u_k \iff \sum_{k=1}^{\infty} \|c_k\|^2 = \|v\|^2$$

Due to this the operator $\widehat{\mathcal{F}}_{\mathcal{U}}$ actually acts into $\ell^2(\mathbb{C})$, i.e.

$$\widehat{\mathcal{F}}_{\mathcal{U}} : \mathcal{V} \longrightarrow \ell^2(\mathbb{C})$$

Proof. By definition of euclidean norm and using the bilinearity of the scalar product we have

$$\begin{aligned} 0 \leq \|v - S_n^{\alpha}\| &= \|v\|^2 - 2\mathfrak{Re}\left(\langle v, S_n^{\alpha}\rangle\right) + \|S_n^{\alpha}\|^2 = \\ &= \|v\|^2 - 2\mathfrak{Re}\left(\sum_{k=1}^{n} \langle v, u_k\rangle \overline{\alpha_k}\right) + \sum_{k=1}^{n} \|\alpha_k\|^2 \end{aligned}$$

Therefore

$$0 \leq \|v\|^2 - \sum_{k=1}^{n} \|c_k\|^2 + \sum_{k=1}^{n} \|\alpha_k - c_k\|^2$$

The minimum on the left is given for $\alpha_k = c_k$ and therefore, since $S_n^c = S_n$ we have

$$\|v - S_n\| \leq \|v - S_n^{\alpha}\|$$

And, using the non-negativity of the norm operator, putting $n \to \infty$ we have

$$0 \leq \|v - S_n\| = \|v\|^2 - \sum_{k=1}^{n} \|c_k\|^2 \implies \sum_{k=1}^{n} \|c_k\|^2 \leq \|v\|^2$$

Therefore

$$\sum_{k=1}^{\infty} \|\langle v, u_k \rangle\|^2 \leq \|v\|^2$$

Which means that the sum on the left converges uniformly, and therefore $c_k \in \ell^2(\mathbb{C})$. This demonstrates that $\widehat{\mathcal{F}}_{\mathcal{U}} : \mathcal{V} \longrightarrow \ell^2(\mathbb{C})$ and Bessel's inequality.
This also gives Parseval's equality, since, for $n \to \infty$

$$\left\| v - \sum_{k=1}^{\infty} \langle v, u_k \rangle u_k \right\|^2 = 0 \iff \|v\|^2 = \sum_{k=1}^{\infty} \|\langle v, u_k \rangle\|^2$$

Due to the uniform convergence in $\mathcal{V}$ we have therefore

$$\widehat{\mathcal{F}}_{\mathcal{U}}(v) = \sum_{k=1}^{\infty} \langle v, u_k \rangle u_k$$

□

Definition 0.1.2 (Closed System). An system $(u_k)_{k\in\mathbb{N}} \in \mathcal{V}$ is said to be *closed* iff $\forall v \in \mathcal{V}$

$$\|v\|^2 = \sum_{k=1}^{\infty} \|\langle v, u_k \rangle\|^2$$
$$v = \sum_{k=1}^{\infty} \langle v, u_k \rangle u_k$$

Theorem 0.2 (Closeness and Completeness). *Given an orthonormal system $\mathcal{U} = (u_k)_{k\in\mathbb{N}} \in \mathcal{V}$ with $\mathcal{V}$ an euclidean space, we have that $\mathcal{U}$ is a complete set if and only if $\mathcal{U}$ is a closed system.*
If $\mathcal{U}$ is complete or closed, $\mathcal{V}$ is separable

Proof. Defined S_n the partial sums of the Fourier representation of v (ndr the series that represents v with respect to the system (u_k)), we have that for the theorem to be true the following two things must hold

$$\lim_{n\to\infty} S_n = v \qquad \overline{\mathrm{span}(u_k)} = \mathcal{V}$$

I.e. $\forall \epsilon > 0\ \exists N \in \mathbb{N},\ \alpha_1, \cdots, \alpha_N \in \mathbb{C}$ such that $\|v - S_n^\alpha\| < \epsilon$. Using Bessel-Parseval we have

$$0 \leq \|v - S_N\| \leq \|v - S_N^\alpha\| < \epsilon$$

Proving the closure of the system if the space $\mathcal{V}$ is complete.
Taken $(u_k)_{k\in\mathbb{N}}$ a closed system, we have that $S_n \to v$, therefore $v \in \mathrm{ad}\,(\mathrm{span}(\mathcal{U}))$, which implies

$$v \in \overline{\mathrm{span}(\mathcal{U})} \implies \mathcal{V} = \overline{\mathrm{span}(\mathcal{U})}$$

The last implication is given by the fact that $v \in \mathcal{V}$ is arbitrary, and it implies the completeness of $\mathcal{U}$ and the separability of $\mathcal{V}$ □

Theorem 0.3 (Riesz-Fisher)**.** *Given $\mathcal{V}$ a hilbert space and $\mathcal{U} = (u_k)_{k\in\mathbb{N}} \in \mathcal{V}$ an orthonormal system, therefore $\forall c \in \ell^2 \ \exists v \in \mathcal{V} \, : \, \widehat{\mathcal{F}}_{\mathcal{U}}[v] = c$ and*

$$c_k = \langle v, u_k \rangle$$

$$\|v\|^2 = \|c\|_2^2 = \sum_{k=1}^{\infty} \|c_k\|^2$$

$$v = \sum_{k=1}^{\infty} \langle v, u_k \rangle u_k$$

Proof. Taken a sequence $(v_k) \in \mathcal{V}$ defined as follows

$$v_n = \sum_{k=1}^{m} c_k u_k$$

This sequence is a Cauchy sequence, therefore it converges to $v \in \mathcal{V}$, since

$$\|v_n - v_m\|^2 = \left\| \sum_{k=n+1}^{m} c_k u_k \right\|^2 = \langle \sum_{k=n+1}^{m} c_k u_k, \sum_{k=n+1}^{m} c_k u_k \rangle =$$

$$= \sum_{k=n+1}^{m} \sum_{i=n+1}^{m} c_k \overline{c_i} \langle u_i, u_k \rangle = \sum_{k=n+1}^{m} \|c_k\|^2$$

By definition, since $c \in \ell^2$, the sum on the right converges, therefore

$$\|v_n - v_m\|^2 \leq \sum_{k=n+1}^{\infty} \|c_k\|^2 < \infty$$

Which means, that $\forall \epsilon > 0 \ \exists N \in \mathbb{N}$ such that

$$\|v_n - v_m\|^2 \leq \sum_{k=n+1}^{\infty} \|c_k\|^2 < \epsilon \quad \forall n \geq N$$

Which implies that $v_n \to v$ and

$$v = \sum_{k=1}^{\infty} c_k u_k$$

We can now write $\langle v, u_k \rangle = \langle v_n, u_k \rangle + \langle v - v_n, u_k \rangle$.
We have

$$\forall n \geq k \ \langle v_n, u_k \rangle = \sum_{i=1}^{n} c_i \langle u_i, u_k \rangle = c_k$$

For Cauchy-Schwartz we also have that

$$\|\langle v - v_n, u_k \rangle\| \leq \|v - v_n\| \to 0$$

Which implies that $c_k = \langle v, u_k \rangle$ and therefore

$$v = \sum_{k=1}^{\infty} \langle v, u_k \rangle u_k = \sum_{k=1}^{\infty} c_k u_k$$

□

§ 0.2 Fourier Series

§§ 0.2.1 Fourier Series in $L^2[-\pi,\pi]$

Definition 0.2.1 (Fourier Series)**.** Given a function $f \in L^2[-\pi,\pi]$ we define the *Fourier series expansion* of this function the following expression

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(kx) + b_k \sin(kx) \tag{1}$$

Where

$$\begin{cases} a_k = \dfrac{1}{\pi} \displaystyle\int_{-\pi}^{\pi} f(x)\cos(kx)\,\mathrm{d}x \\ b_k = \dfrac{1}{\pi} \displaystyle\int_{-\pi}^{\pi} f(x)\sin(kx)\,\mathrm{d}x \end{cases} \tag{2}$$

The notation $\sim$ indicates that the Fourier series of the function *converges to* the function $f(x)$. Usually an abuse of notation is used, where the function is actually set as equal to the Fourier expansion.

Definition 0.2.2 (Trigonometric Polynomial)**.** A function $p \in L^2[-\pi,\pi]$ is said to be a *trigonometric polynomial* if, for some coefficients α_k, β_k we have

$$p(x) = \frac{\alpha_0}{2} + \sum_{k=1}^{\infty} \alpha_k \cos(kx) + \beta_k \sin(kx) \tag{3}$$

Theorem 0.4 (Completeness of Trigonometric Functions)**.** *Given* $(u_k), (v_k) \in L^2[-\pi,\pi]$ *two sequences of functions, where*

$$\begin{cases} u_k(x) = \cos(kx) \\ v_k(x) = \sin(kx) \end{cases}$$

The set $\{u_k, v_k\}$ *is orthogonal and complete, i.e. a basis in* $L^2[-\pi,\pi]$

Remark. These trigonometric identities always hold, $\forall n, k \in \mathbb{N}$, $n \neq k$

$$\begin{aligned} \cos(nx)\cos(kx) &= \frac{1}{2}\left(\cos[(n+k)x] + \cos[(n-k)x]\right) \\ \sin(nx)\sin(kx) &= \frac{1}{2}\left(\cos[(n-k)x] - \cos[(n+k)x]\right) \\ \cos(nx)\sin(kx) &= \frac{1}{2}\left(\sin[(n+k)x] - \sin[(n-k)x]\right) \end{aligned} \tag{4}$$

Proof. We begin by demonstrating that the two function sequences u_k, v_k are orthogonal in $L^2[-\pi,\pi]$. Therefore, by explicitly writing the scalar product, we have, for $k \neq n$

$$\langle u_n, u_k \rangle = \int_{-\pi}^{\pi} \cos(nx)\sin(kx)\,\mathrm{d}x = \frac{1}{2}\int_{-\pi}^{\pi} \cos[(n+k)x] + \cos[(n-k)x]\,\mathrm{d}x$$

Therefore
$$\langle u_n, u_k \rangle = \frac{1}{2}\left[\frac{\sin[(n+k)x]}{n+k} + \frac{\sin[(n-k)x]}{n-k}\right]_{-\pi}^{\pi} = 0$$
Analogously
$$\langle v_n, v_k \rangle = \frac{1}{2}\left[\frac{\sin[(n-k)x]}{n-k} - \frac{\sin[(n+k)x]}{n+k}\right]_{-\pi}^{\pi} = 0$$
And, finally
$$\langle u_n, v_k \rangle = \frac{1}{2}\left[\frac{\cos[(n+k)x]}{n+k} - \frac{\cos[(n-k)x]}{n-k}\right]_{-\pi}^{\pi} = \frac{1}{2} - \frac{1}{2} = 0$$
Which demonstrates that, for $k \neq n$ $u_k \perp u_n,\ v_k \perp v_n,\ u_k \perp v_k$.
Now, taken a trigonometric polynomial $p(x) \in L^2[-\pi,\pi]$ we need to prove that $\overline{\operatorname{span}\{u_k, v_k\}} = L^2[-\pi,\pi]$, i.e.
$$\forall \epsilon > 0\ \forall f \in L^2[-\pi,\pi] \quad \|p - f\|_2 < \epsilon$$
We have already that $\overline{C[-\pi,\pi]} = L^2[-\pi,\pi]$ and that for a Weierstrass theorem (without proof), every periodic function with period 2π is the uniform limit of a trigonometric polynomial.
Using these two results, given $f \in L^2[-\pi,\pi]$, $\exists g \in C[-\pi,\pi] : \|f - g\|_2 < \epsilon/3$. Taken $\hat{g}(x)$ as the periodic extension of $g(x)$, for Weierstrass we have
$$\|g - \hat{g}\|_2 < \frac{\epsilon}{3} \quad \|p - \hat{g}\|_2 < \frac{\epsilon}{3} \implies \|p - \hat{g}\|_u < \frac{\epsilon}{3\sqrt{2\pi}}$$
Therefore, finally $\|f - p\|_2 < \epsilon$ □

Theorem 0.5 (Parseval Identity)**.** *Given $f \in L^2[-\pi,\pi]$ we have that*
$$\int_{-\pi}^{\pi} |f(x)|^2 \,\mathrm{d}x = \frac{|a_0|^2}{2} + \sum_{k=1}^{\infty} |a_k|^2 + |b_k|^2$$

Proof. The proof is quite straightforward, since trigonometric polynomials form a basis for $L^2[-\pi,\pi]$ we have that this is simply the already known Parseval identity, since
$$\|f\|_2^2 = \sum_{k=0}^{\infty} \|c_k\|^2$$
Writing $c_k = a_k + b_k$ we have
$$\|f\|_2^2 = \frac{\|a_0\|^2}{2} + \sum_{k=1}^{\infty} \|a_k\|^2 + \|b_k\|^2$$
□

§§ 0.2.2 Fourier Series in $L^2[a,b]$

Definition 0.2.3 (Basis of the Space)**.** In order to define a trigonometric basis in $L^2[a,b]$ with $a \neq b$, we can use a simple coordinate transformation onto the $\{(u_k),(v_k)\}$ basis of the space $L^2[-\pi,\pi]$. Therefore, taken

$$y(x) = \frac{\pi}{b-a}(2x-a-b)$$

The new basis on $L^2[a,b]$ will be

$$\begin{cases} (u_k(y(x))) = \cos(ky(x)) = \cos\left(\dfrac{k\pi}{b-a}(2x-a-b)\right) \\ (v_k(y(x))) = \sin(ky(x)) = \sin\left(\dfrac{k\pi}{b-a}\,(2x-a-b)\right) \end{cases}$$

The completeness of this basis is given by the fact that, this change of coordinates is a smooth diffeomorphism between $L^2[-\pi,\pi], L^2[a,b]$.

Definition 0.2.4 (General Fourier Series)**.** With the previous definition, the Fourier series of a function $f \in L^2[a,b]$ is given as follows

$$f(x) \sim \frac{\tilde{a}_0}{2} + \sum_{k=1}^{\infty} \tilde{a}_k \cos\left(\frac{k\pi}{b-a}(2x-a-b)\right) + \tilde{b}_k \sin\left(\frac{k\pi}{b-a}(2x-b-a)\right) \tag{5}$$

Where

$$\begin{cases} \tilde{a}_k = \dfrac{1}{b-a}\displaystyle\int_a^b f(x)\cos\left(\dfrac{k\pi}{b-a}\,(2x-b-a)\right)\mathrm{d}x \\ \tilde{b}_k = \dfrac{1}{b-a}\displaystyle\int_a^b f(x)\sin\left(\dfrac{k\pi}{b-a}\,(2x-b-a)\right)\mathrm{d}x \end{cases} \tag{6}$$

§§ 0.2.3 Fourier Series in Symmetric Intervals, Expansion in Only Sines and Cosines

Definition 0.2.5. We firstly begin finding the Fourier series of a function in $L^2[-l,l]$. Using the previous general case in $L^2[a,b]$ and setting $a=-l,\ b=l$ we have $\forall f \in L^2[-l,l]$

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos\left(\frac{k\pi x}{l}\right) + b_k \sin\left(\frac{k\pi x}{l}\right) \tag{7}$$

With coefficients

$$\begin{cases} a_k = \dfrac{1}{l}\displaystyle\int_{-l}^{l} f(x)\cos\left(\dfrac{k\pi x}{l}\right)\mathrm{d}x \\ b_k = \dfrac{1}{l}\displaystyle\int_{-l}^{l} f(x)\sin\left(\dfrac{k\pi x}{l}\right)\mathrm{d}x \end{cases} \tag{8}$$

Theorem 0.6. *Taken the space $L^2[0,\pi]$ we have that both trigonometric sequences $(u_k(x))$ and $(v_k(x))$ are orthogonal bases in this space, and the following equalities hold.*

$\forall f \in L^2[0, \pi]$

$$f(x) \sim \frac{a_0'}{2} + \sum_{k=1}^{\infty} a_k' \cos(kx)$$

$$f(x) \sim \sum_{k=1}^{\infty} b_k' \sin(kx)$$

Where

$$a_k' = \frac{2}{\pi} \int_0^{\pi} f(x) \cos(kx) \, \mathrm{d}x$$

$$b_k' = \frac{2}{\pi} \int_0^{\pi} f(x) \sin(kx) \, \mathrm{d}x$$

Proof. The proof of this theorem is straightforward, we firstly define the even and uneven extensions of the function $f(x)$ in $L^2[-\pi, \pi]$ as follows

$$f^e(x) = \begin{cases} f(x) & x \in [0, \pi] \\ f(-x) & x \in [-\pi, 0) \end{cases}$$

And

$$f^u(x) = \begin{cases} f(x) & x \in [0, \pi] \\ -f(-x) & x \in [-\pi, 0) \end{cases}$$

Expanding both these functions in $[-\pi, \pi]$ we have that, indicating the coefficients of each as $a_k^e, \ b_k^e, \ a_k^u, \ b_k^u$

$$a_k^e = \frac{1}{\pi} \int_{-\pi}^{\pi} f^e(x) \cos(kx) \, \mathrm{d}x = \frac{2}{\pi} \int_0^{\pi} f(x) \cos(kx) \, \mathrm{d}x = a_k'$$

$$b_k^e = 0$$

$$a_k^u = 0$$

$$b_k^u = \frac{1}{\pi} \int_{-\pi}^{\pi} f^u(x) \sin(kx) \, \mathrm{d}x = \frac{2}{\pi} \int_0^{\pi} f(x) \sin(kx) \, \mathrm{d}x = b_k'$$

Therefore

$$f^e(x) \sim \frac{a_0'}{2} + \sum_{k=1}^{\infty} a_k' \cos(kx)$$

$$f^u(x) \sim \sum_{k=1}^{\infty} b_k' \sin(kx)$$

Which implies that

$$\|f^e - S_n\|_2^2 = 2\|f - S_n\|_{[0,\pi]}^2 \to 0$$

And

$$\|f^u - S_n\|_2^2 = 2\|f - S_n\|_{[0,\pi]}^2 \to 0$$

Proving the theorem. $\square$

Example 0.2.1. Taken the function $f(x) = x^2 \quad x \in [-l, l]$ we want to find the Fourier expansion of this function.
Since x^2 is even, thanks to the previous theorem we know that the coefficients $b_k = 0$ in all the set of definition, therefore

$$x^2 \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos\left(\frac{k\pi x}{l}\right)$$

We firstly calculate the coefficient a_0 of the expansion

$$a_0 = \frac{1}{l}\int_{-l}^{l} x^2 \,\mathrm{d}x = \frac{2l^2}{3}$$

The coefficients a_k can be calculated using the fact that x^2 is even, and therefore we have

$$\begin{aligned} a_k &= \frac{1}{l}\int_{-l}^{l} x^2 \cos\left(\frac{k\pi x}{l}\right) \mathrm{d}x = \frac{1}{l}\left[x^2 \sin\left(\frac{k\pi x}{l}\right)\frac{l}{k\pi}\right]_{-l}^{l} - \frac{4}{k\pi}\int_0^l x \sin\left(\frac{k\pi x}{l}\right) \mathrm{d}x = \\ &= \frac{4l}{(k\pi)^2}\left[x\cos\left(\frac{k\pi x}{l}\right)\right]_{-l}^{l} - \frac{4l}{(k\pi)^2}\int_0^l \sin\left(\frac{k\pi x}{l}\right) \mathrm{d}x \end{aligned}$$

Since the last integral is 0 we have

$$a_k = \frac{4l}{(k\pi)^2}\left[x\cos\left(\frac{k\pi x}{l}\right)\right]_{-l}^{l} = \frac{(-1)^k l^2}{(k\pi)^2}$$

The searched Fourier expansion is therefore

$$x^2 \sim \frac{l^2}{3} + \frac{4l^2}{\pi^2}\sum_{k=1}^{\infty}\frac{(-1)^k}{k^2}\cos\left(\frac{k\pi x}{l}\right)$$

Example 0.2.2 (Parseval's Equality)*.* Having now found the Fourier expansion for x^2, we can use Parseval's equality in order to calculate the sum

$$\sum_{k=1}^{\infty}\frac{1}{k^2}$$

Thanks to Parseval, we therefore have

$$\left\|x^2\right\|_2^2 = \frac{1}{l}\int_{-l}^{l} x^4 \,\mathrm{d}x = \frac{2l^4}{9} + \frac{16l^2}{\pi^4}\sum_{k=1}^{\infty}\frac{1}{k^4}$$

The integral on the left is obvious, and moving the terms around we finally have

$$\sum_{k=1}^{\infty}\frac{1}{k^4} = \frac{\pi^4}{16l^5}\int_{-l}^{l} x^4 \,\mathrm{d}x - \frac{\pi^4}{36} = \frac{\pi^4}{16}\left(\frac{2}{5} - \frac{2}{9}\right)$$

Therefore

$$\sum_{k=1}^{\infty}\frac{1}{k^4} = \frac{\pi^4}{90}$$

§§ 0.2.4 Complex Fourier Series

Theorem 0.7 (Complex Exponential Basis). *Taken the space $L^2[-\pi,\pi]$, and defining a system $(e_k)_{k\in\mathbb{Z}} = e^{ikx}$, this system is an orthogonal basis for the space.*

Proof. Using Euler's formula for complex exponentials we have

$$(e_k)_{k\in\mathbb{Z}} = e^{ikx} = \cos(kx) + i\sin(kx) = u_k(x) + iv_k(x)$$

Therefore, due to the linearity of the scalar product, these functions are orthogonal to each other, and due to linearity we also have

$$\mathrm{span}\{e^{ikx}\} = \mathrm{span}\{\cos(kx), \sin(kx)\}$$

Which, implies

$$\overline{\mathrm{span}\{e^{ikx}\}} = L^2[-\pi,\pi]$$

Note that

$$\left\|e^{ikx}\right\|_2^2 = 2\pi$$

□

Definition 0.2.6 (Complex Fourier Series). Given $f \in L^2[-\pi,\pi]$ we can now define a Fourier expansion in complex exponentials as follows

$$f(x) \sim \sum_{k=-\infty}^{\infty} \frac{\langle f(x), e^{ikx}\rangle}{\|e^{ikx}\|_2^2} e^{ikx} = \sum_{k\in\mathbb{Z}} c_k e^{ikx} \tag{9}$$

Where, the coefficients will be

$$c_k = \frac{\langle f(x), e^{ikx}\rangle}{\|e^{ikx}\|_2^2} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{ikx}\,\mathrm{d}x \tag{10}$$

Note that if $f(x) : \mathbb{R} \longrightarrow \mathbb{R}$ we have

$$c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{ikx}\,\mathrm{d}x = \frac{1}{2\pi}\int_{-\pi}^{\pi} \overline{f(x)e^{ikx}}\,\mathrm{d}x = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-ikx}\,\mathrm{d}x = c_{-k}$$

Therefore, for a real valued function

$$f(x) \sim c_0 + \sum_{k=1}^{\infty} c_k e^{ikx} + c_{-k}e^{-ikx} = c_0 + 2\sum_{k=1}^{\infty} \mathfrak{Re}\left\{c_k e^{ikx}\right\} \tag{11}$$

Example 0.2.3. Taken the function $f(x) = e^x$, $x \in [-\pi,\pi]$, we want to find the Fourier series in terms of complex exponentials. Since $f(x)$ is a real valued function, we have

$$e^x \sim c_0 + 2\sum_{k=1}^{\infty} \mathfrak{Re}\left\{c_k e^{ikx}\right\}$$

The coefficients will be

$$c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^x \,\mathrm{d}x = \frac{1}{2\pi}\left(e^{\pi} - e^{-\pi}\right) = \frac{\sinh(\pi)}{\pi}$$
$$c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{(1-ik)x} \,\mathrm{d}x = \frac{1}{2\pi}\left[\frac{1}{1-ik}\left(e^{\pi(1-ik)} - e^{-\pi(1-ik)}\right)\right]$$

The second expression can be seen as follows

$$c_k = \frac{1}{2\pi(1-ik)}\left(e^{ik\pi}e^{\pi} - e^{-ik\pi}e^{-\pi}\right) = \frac{(-1)^k}{\pi(1-ik)}\sinh(\pi)$$

The final expansion will then be given from finding the real part of this coefficient times the basis vector, i.e.

$$\mathfrak{Re}\left\{\frac{(-1)^k\sinh(\pi)}{\pi(1-ik)}e^{ikx}\right\} = \frac{(-1)^k\sinh(\pi)}{1+k^2}\mathfrak{Re}\left((1+ik)(\cos(kx)+i\sin(kx))\right)$$

The last calculation is obvious, and we therefore have

$$\mathfrak{Re}\left\{\frac{(-1)^k\sinh(\pi)}{\pi(1-ik)}e^{ikx}\right\} = \frac{(-1)^k\sinh(\pi)}{1+k^2}\left(\cos(kx) - k\sin(kx)\right)$$

And the final solution will be

$$e^x \sim \frac{\sinh(\pi)}{\pi} + \frac{2\sinh(\pi)}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^k}{1+k^2}\left(\cos(kx) - k\sin(kx)\right)$$

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§§ 0.2.5 Piecewise Derivability and Pointwise Convergence of Fourier Series

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§ 0.3 Fourier Transform

§§ 0.3.1 Convolution