§ 0.1 Complex Numbers

Definition 0.1.1 (Complex Numbers)**.** Define with $\mathbb{C}$ the set of *complex numbers*, i.e. the set of numbers $z \in \mathbb{C} \, : \, z = (x, y)$ and $x, y \in \mathbb{R}$.
We define the *real and imaginary parts* of z as follows

$$\begin{aligned} \mathfrak{Re}(z) &= x \\ \mathfrak{Im}(z) &= y \end{aligned} \tag{1}$$

Definition 0.1.2 (Operations in $\mathbb{C}$)**.** Take $z_1, z_2 \in \mathbb{C}$, then we define

$$\begin{aligned} z_1 = z_2 &\iff \mathfrak{Re}(z_1) = \mathfrak{Re}(z_2), \; \mathfrak{Im}(z_1) = \mathfrak{Im}(z_2) \\ z_1 + z_2 &= (\mathfrak{Re}(z_1) + \mathfrak{Re}(z_2), \mathfrak{Im}(z_1) + \mathfrak{Im}(z_2)) \\ z_1 z_2 &= (\mathfrak{Re}(z_1)\mathfrak{Re}(z_2) - \mathfrak{Im}(z_1)\mathfrak{Im}(z_2), \mathfrak{Re}(z_1)\mathfrak{Im}(z_2) + \mathfrak{Im}(z_1)\mathfrak{Re}(z_2)) \end{aligned}$$

Theorem 0.1. *With the previous definitions the set $\mathbb{C}$ forms a field.*

Definition 0.1.3 (Imaginary Unit)**.** We define the imaginary unit $i = (0, 1) \in \mathbb{C}$. From this definition and the definition of product of two complex numbers, we have that $i^2 = -1$
With this definition, we have

$$\forall z \in \mathbb{C} \quad z = \mathfrak{Re}(z) + i\mathfrak{Im}(z) \tag{2}$$

Definition 0.1.4 (Complex Conjugate)**.** Taken $z \in \mathbb{C}$, we call the *complex conjugate of z* the number w such that

$$w = \mathfrak{Re}(z) - i\mathfrak{Im}(z) \tag{3}$$

This number is denoted as $\overline{z}$

Definition 0.1.5 (Complex Module)**.** We define the *module* or *norm* of a complex number, the following operator.

$$\|z\| = \sqrt{z\overline{z}} = \sqrt{\mathfrak{Re}^2(z) + \mathfrak{Im}^2(z)} \tag{4}$$

Definition 0.1.6 (Complex Inverse)**.** The inverse of a complex number $z \in \mathbb{C}$ is defined as z^{-1} and it's calculated as follows

$$z^{-1} = \frac{\overline{z}}{\|z\|^2} \tag{5}$$

Definition 0.1.7 (Polar Form)**.** Taken a complex number $z \in \mathbb{C}$ one can define it in polar form with its modulus r and its *argument* θ. We have that, if $z = x + iy$

$$\begin{aligned} r &= \sqrt{x^2 + y^2} = \|z\|^2 \\ \tan(\theta) &= \frac{y}{x} \end{aligned} \tag{6}$$

Definition 0.1.8 (Principal Argument)**.** Taken $\arg(z) = \theta$ we can define two different arguments, due to the periodicity of the tan function.

1. $\mathrm{Arg}(z) \in (-\pi, \pi]$ called the *principal argument*
2. $\arg(z) = \mathrm{Arg}(z) + 2k\pi, \; k \in \mathbb{Z}$ called the *argument*

As a rule of thumb, using the previous definition of argument of a complex number $z = x + iy$, we have

$$\mathrm{Arg}(z) = \begin{cases} \arctan(y/x) - \pi & x < 0,\ y < 0 \\ \arctan(y/x) & x \geq 0,\ z \neq 0 \\ \arctan(y/x) + \pi & x < 0,\ y \geq 0 \end{cases} \tag{7}$$

Definition 0.1.9 ($\arg_+$)**.** Given $z \in \mathbb{C}$ we define the $\arg_+(z)$ as the only value of $\arg(z)$ such that $0 \leq \theta < 2\pi$.
In case we have a polydromic function, in order to specify we're using this argument, there will be a $+$ as index.
I.e. $\log_+(z), [z^a]^+, \sqrt{z}^+, \cdots$ and so on.

Theorem 0.2 (De Moivre Formula)**.** *A complex number $z \in \mathbb{C}$ in polar form can be written with complex exponential and sine and cosine function as follows.*

$$z = \|z\|^2 e^{i \arg z} = \|z\|^2 (\cos(\arg z) + i \sin(\arg z)) \tag{8}$$

This formula easily generalizes the calculus of exponentials of complex numbers. With this definition, it's obvious that the $n-$th root of a complex number $\sqrt[n]{z}$ has actually $n-1$ results, given the $2\pi-$periodicity of the $\arg(z)$ *function.*

§ 0.2 Regions in $\mathbb{C}$

Definition 0.2.1 (Line)**.** A line λ in $\mathbb{C}$, from z_1, z_2 can be written as follows

$$\lambda(t) = z_1 + t(z_2 - z_1) \quad t \in [0, 1] \tag{9}$$

If $t \in \mathbb{R}$ this defines the line lying between z_1, z_2. Its non-parametric representation is the following

$$\{\lambda\} := \left\{ z \in \mathbb{C} |\ \mathfrak{Im}\left(\frac{z - z_1}{z_2 - z_1}\right) = 0 \right\} \tag{10}$$

Where $z = \lambda(t)$.

Definition 0.2.2 (Circumference)**.** A circumference γ centered in a point $z_0 \in \mathbb{C}$ with radius R is defined as follows

$$\gamma(\theta) = z_0 + Re^{i\theta} \quad \theta \in [0, 2\pi] \tag{11}$$

Non parametrically, it can be defined as follows

$$\{\gamma\} := \{ z \in \mathbb{C} |\ \|z - z_0\| = R\} \tag{12}$$

§§ 0.2.1 Extended Complex Plane $\hat{\mathbb{C}}$

Definition 0.2.3 (Extended Complex Plane)**.** We define the *extended complex plane* $\hat{\mathbb{C}}$ as follows

$$\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\} \tag{13}$$

This can be imagined by projecting $\mathbb{C}$ into the Riemann sphere centered in the origin.

Definition 0.2.4 (Points in $\hat{\mathbb{C}}$)**.** Given a point $z \in \mathbb{C},\ z = x + iy$ we can find its coordinates with the following transformation

$$\hat{z} = (xt, yt, 1-t) \in \hat{\mathbb{C}} \tag{14}$$

Where the condition $\|\hat{z}\| = 1$ must hold, defining the value of $t \in \mathbb{R}$ Inversely, given $\hat{z} = (x_1, x_2, x_3) \in \hat{\mathbb{C}}$ one finds

$$z = \frac{x_1 + ix_2}{1 - x_3} \tag{15}$$

§ 0.3 Elementary Functions

Definition 0.3.1 (Exponential)**.** The exponential function $z \mapsto e^z$ with $z \in \mathbb{C}$ is defined as follows

$$e^z = e^{\mathfrak{Re}(z) + i\mathfrak{Im}(z)} = e^{\mathfrak{Re} z}\left(\cos(\mathfrak{Im}(z)) + i\sin(\mathfrak{Im}(z))\right) \tag{16}$$

This gives

$$\begin{aligned} \|e^z\| &= \left|e^{\mathfrak{Re}(z)}\right| \\ \arg(e^z) &= \mathfrak{Im}(z) + 2\pi k \quad k \in \mathbb{Z} \end{aligned} \tag{17}$$

We have therefore, for $z, w \in \mathbb{C}$

$$\begin{aligned} e^z e^w &= e^{z+w} \\ \frac{e^z}{e^w} &= e^{z-w} \end{aligned} \tag{18}$$

Definition 0.3.2 (Logarithm)**.** We define the logarithm function $z \mapsto \log z$ as follows

$$\log(z) = \log\|z\| + i\arg(z) \tag{19}$$

It's evident how this function has multiple values for the same z value, and therefore is known as a *polydromic function*, like the square root. We also define the principal branch of the logarithm as $\mathrm{Log}(z)$

$$\mathrm{Log}(z) = \log\|z\| + i\,\mathrm{Arg}(z) \tag{20}$$

Lastly we define the $\log_+(z)$ as follows

$$\log_+(z) = \log(\|z\|) + i\arg_+(z) \tag{21}$$

Definition 0.3.3 (Branch of the Logarithm)**.** A general branch of the log function is defined as the function $f(z) : D \subset \mathbb{C} \longrightarrow \mathbb{C}$ such that

$$e^{f(z)} = z \tag{22}$$

§§ 0.3.1 Complex Exponentiation

Definition 0.3.4 (Complex Exponential)**.** Taken $s, z \in \mathbb{C}$, we define the complex exponential a follows, taken z a variable

$$z^s = e^{s\log(z)} \quad z \neq 0 \tag{23}$$

Its derivative has the following value

$$\frac{\mathrm{d}}{\mathrm{d}z} z^s = se^{(s-1)\log(z)} = sz^{s-1} \tag{24}$$

Alternatively, we define

$$s^z = e^{z\log(s)} \tag{25}$$

§§ 0.3.2 Properties of Trigonometric Functions

Definition 0.3.5 (Trigonometric Functions)**.** Using De Moivre's formula, we define

$$\begin{aligned} \sin(z) &= \frac{1}{2i}\left(e^{iz} - e^{-iz}\right) \\ \cos(z) &= \frac{1}{2}\left(e^{iz} + e^{-iz}\right) \end{aligned} \tag{26}$$

Definition 0.3.6 (Hyperbolic Functions)**.** We define the hyperbolic functions as follows, given $z = iy$

$$\begin{aligned} \sinh(y) &= -i\sin(iy) \\ \cosh(y) &= \cos(iy) \end{aligned} \tag{27}$$

For a general value of z, we define

$$\begin{aligned} \sinh(z) &= \frac{1}{2}\left(e^{z} - e^{-z}\right) \\ \cosh(z) &= \frac{1}{2}\left(e^{z} + e^{-z}\right) \end{aligned} \tag{28}$$

Theorem 0.3 (Trigonometric Identities)**.** *Given $z, z_1, z_2 \in \mathbb{C}$ we have*

$$\begin{aligned} \sin^2(z) + \cos^2(z) &= 1 \\ \sin(z_1 \pm z_2) &= \sin(z_1)\cos(z_2) \pm \cos(z_1)\sin(z_2) \\ \cos(z_1 \pm z_2) &= \cos(z_1)\cos(z_2) \mp \sin(z_1)\sin(z_2) \\ \sin(z) &= \sin\left(\mathfrak{Re}(z)\right)\cosh\left(\mathfrak{Im}(z)\right) + i\cos\left(\mathfrak{Re}(z)\right)\sinh\left(\mathfrak{Im}(y)\right) \\ \cos(z) &= \cos\left(\mathfrak{Re}(z)\right)\cosh\left(\mathfrak{Im}(z)\right) - i\sin\left(\mathfrak{Re}(z)\right)\sinh\left(\mathfrak{Im}(y)\right) \\ \|\sin(z)\|^2 &= \sin^2\left(\mathfrak{Re}(x)\right) + \sinh^2\left(\mathfrak{Im}(y)\right) \\ \|\cos(z)\|^2 &= \cos^2\left(\mathfrak{Re}(x)\right) + \sinh^2\left(\mathfrak{Im}(y)\right) \\ \cosh^2(z) - \sinh^2(z) &= 1 \\ \sinh(z_1 \pm z_2) &= \sinh(z_1)\cosh(z_2) \pm \cosh(z_1)\sinh(z_2) \\ \cos(z_1 \pm z_2) &= \cosh(z_1)\cosh(z_2) \pm \sinh(z_1)\sinh(z_2) \\ \sinh(z) &= \sinh\left(\mathfrak{Re}(z)\right)\cos\left(\mathfrak{Im}(z)\right) + i\cosh\left(\mathfrak{Re}(z)\right)\sin\left(\mathfrak{Im}(y)\right) \\ \cos(z) &= \cosh\left(\mathfrak{Re}(z)\right)\cos\left(\mathfrak{Im}(z)\right) + i\sinh\left(\mathfrak{Re}(z)\right)\sin\left(\mathfrak{Im}(y)\right) \\ \|\sin(z)\|^2 &= \sinh^2\left(\mathfrak{Re}(x)\right) + \sin^2\left(\mathfrak{Im}(y)\right) \\ \|\cos(z)\|^2 &= \cosh^2\left(\mathfrak{Re}(x)\right) + \sin^2\left(\mathfrak{Im}(y)\right) \end{aligned} \tag{29}$$

Definition 0.3.7 (Inverse Trigonometric Functions)**.** Given $z \in \mathbb{C}$ we define

$$\begin{aligned} \arcsin(z) &= -i\log\left(iz + \sqrt{1 - z^2}\right) \\ \arccos(z) &= -i\log\left(z + i\sqrt{1 - z^2}\right) \\ \arctan(z) &= -\frac{i}{2}\log\left(\frac{i - z}{i + z}\right) \end{aligned} \tag{30}$$

Definition 0.3.8 (Inverse Hyperbolic Functions)**.** Given $z \in \mathbb{C}$ we define

$$\begin{aligned} \operatorname{asinh}(z) &= \log\left(z + \sqrt{z^2+1}\right) \\ \arccos(z) &= \log\left(z + \sqrt{z^2-1}\right) \\ \operatorname{atanh}(z) &= \frac{1}{2}\log\left(\frac{1+z}{1-z}\right) \end{aligned} \tag{31}$$