Electromagnetism

Università degli studi di Roma "La Sapienza"
Physics and Astrophysics BSc

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Notes on Electromagnetism

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Written by

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Part I

Electrostatics

1 The Electric Field

§ 1.1 Electric Forces and the Electric Field

It has been found from the forefathers of Electrodynamics that, empirically, the force exerted between two charged objects has the following characteristics

1. It's directed from one object to the other
2. It depends on the *product* of the charges
3. It's proportional to the inverse squared of the distance between the objects d^{-2}

The experimental results brought with great clarity then, that for two point charges q_1, q_2, said $r = \left\|r_1^i - r_2^i\right\|$ and $\hat{r}^i$ the associated versor, the electrostatic force is

$$F^i = k_e \frac{q_1 q_2}{r^2} \hat{r}^i \tag{1.1}$$

Here, k_e is a coupling constant which takes different values for different choices of units. In the SI system we have

$$k_e = \frac{1}{4\pi\epsilon_0} \tag{1.2}$$

With ϵ_0 being the *permittivity of free space*, which has value

$$\epsilon_0 = 8.85 \cdot 10^{-12} \ \frac{\mathsf{C}}{\mathsf{Nm}^2} \tag{1.3}$$

These forces are obviously additive.
Suppose now that you have a set of n charges q_i and you add an imaginary test charge Q in order to *theoretically* test the force field generated by these charges. We have then

$$F^i = \sum_{j=1}^{n} f^i_{(j)} = \sum_{j=1}^{n} \frac{Q q_{(j)}}{4\pi\epsilon_0 r^2_{(j)}} \hat{r}^i_{(j)} = Q \sum_{j=1}^{n} \frac{q_{(j)}}{4\pi\epsilon_0 r^2_{(j)}} \hat{r}^i_{(j)} \tag{1.4}$$

The element inside the sum can be seen as the *field* generated by the single particle $q_{(j)}$, denoted as $E^i_{(j)}$. This field is the Electrostatic field. It's clear that then we can define a total field E^i by superposition of the single charge fields, and we can write, for a system of charges

$$F^i = Q \sum_{j=1}^{n} E^i_{(j)} = Q E^i \tag{1.5}$$

Then, in general, we can say

$$E^i = \frac{F^i}{Q} \tag{1.6}$$

For our system of n charges the previous calculation is pretty straightforward and we directly get

$$E^i = \sum_{i=1}^{n} \frac{q_{(i)}}{4\pi\epsilon_0} \frac{\hat{r}^i}{r^2} \tag{1.7}$$

The passage to continuous distributions of charge is straightforward. We define the following "transformations"

$$\begin{cases} q_i \longrightarrow \mathrm{d}q \\ \sum_i \longrightarrow \int \end{cases}$$

The electric field of such distribution is then

$$E^i = \frac{1}{4\pi\epsilon_0} \int \frac{\hat{r}^i}{r^2} \mathrm{d}q \tag{1.8}$$

In general, $\mathrm{d}q$ can be expressed mathematically with a charge density which can be linear, superficial or volumetric. I.e.

$$\mathrm{d}q \rightarrow \begin{cases} \lambda(\tilde{r}^i)\mathrm{d}l & \text{linear distribution} \\ \sigma(\tilde{r}^i)\mathrm{d}s & \text{superficial distribution} \\ \rho(\tilde{r}^i)\mathrm{d}^3\tilde{x} & \text{volumetric distribution} \end{cases} \tag{1.9}$$

The electric field will then be calculated with the integral (1.8) extended to the appropriate set (a curve, a surface or a volume)

§§ 1.1.1 Divergence of the Electrostatic Field

As we have defined previously the electric field it's clear that if the distribution is complicated enough the integrals might be hard to solve or straight up nonsolvable. We then want to find different ways for calculating the field.
In general a vector field is determined by both its divergence and its curl. We firstly remember the definition of the 3D Dirac delta function $\delta^3(r^i)$, which is simply

$$\delta^3(r^i) = \frac{1}{4\pi} \frac{\partial}{\partial x^i} \left(\frac{r^i}{r^2} \right) \tag{1.10}$$

We then take the definition of E^i for a continuous volumetric distribution and simply apply the divergence operator.

$$\partial_i E^i = \frac{1}{4\pi\epsilon_0} \frac{\partial}{\partial x^i} \iiint_V \rho(\tilde{r}^i) \frac{\hat{r}^i}{r^2} \mathrm{d}^3\tilde{x} \tag{1.11}$$

Noting that the integral is with respect to the primed coordinates (the ones with respect to the distribution) we can bring inside the divergence operator, and remembering that in this case $r = \left\| r^i - \tilde{r}^i \right\|$, with the definition of the 3D delta we get

$$\partial_i E^i = \frac{1}{4\pi\epsilon_0} \iiint_V \rho\left(\tilde{r}^i\right) \delta^3\left(r^i - \tilde{r}^i\right) \mathrm{d}^3\tilde{x} = \frac{1}{\epsilon_0} \rho\left(r^i\right)$$

Therefore, due to the generality of ρ we have that for *every* electrostatic field, the following equation holds

$$\partial_i E^i = \frac{\rho}{\epsilon_0} \tag{1.12}$$

This is Maxwell's first equation for the electrostatic field.
A really important property comes from this equation, *Gauss' law*. This law states that the flux of E^i is proportional to the total charge enclosed by the chosen volume V.
This is a direct consequence of Stokes' theorem for differential forms.
We choose a bounded volume $V \subset \mathbb{R}^3$ and integrate both sides of (1.12)

$$\iiint_V \partial_i E^i \mathrm{d}^3x = \oiint_{\partial V} E^i \hat{n}_i \mathrm{d}s = \frac{1}{\epsilon_0} \iiint_V \rho(r^i) \mathrm{d}^3x$$

Defining the flux of E^i as $\Phi_{\partial V}(E^i)$ we have, then

$$\oiint_{\partial V} E^i \hat{n}_i \mathrm{d}s = \Phi_{\partial V}(E^i) = \frac{Q_V}{\epsilon_0} \tag{1.13}$$

This is the mathematical expression of Gauss' law, where we have written

$$Q_V = \iiint_V \rho(r^i) \mathrm{d}^3x$$

Which is the total charge contained inside the volume V.
This theorem is *fundamental* for the solution of a myriad of electrostatic problems which would take a lot of calculations using (1.8). The main idea is that this can be used in conditions where there are particular symmetries of the system.

Example 1.1.1 (A charged sphere). Suppose that you have a charged sphere with radius R and total charge q and I want to know the electric field inside and outside the sphere. We begin by calculating the field outside using Gauss' law. Due to the radial symmetry of the problem we have that $\hat{n}^i = \hat{r}^i$ and therefore $E^i = E\hat{n}^i$ when we choose a spherical volume.
Let $\partial V = S_r^2$ be our "*gaussian surface*", a sphere of radius r, where the previous relation for E^i holds. We have that for any r

$$\Phi_{S_r^2}\left(E^i\right) = \oiint_{S_r^2} E^i \hat{n}_i \mathrm{d}s = E \oiint_{S_r^2} \mathrm{d}s = 4\pi r^2 E \tag{1.14}$$

The first part on the left of (1.13) is already evaluated. Then we need to calculate only the right side. Noting that there is no charge outside the sphere we have an internal volumetric density of charge $\rho = q/V$. Since V is a sphere we already know its volume, and the calculation it's quite easy

$$\iiint_{V_r} \rho(r^i) \mathrm{d}^3x = \frac{q}{V} \iiint_{V_r} \mathrm{d}^3x = \begin{cases} q\dfrac{V_r}{V} & r < R \\ q & r > R \end{cases} \tag{1.15}$$

Where V_r is the volume contained inside the gaussian sphere S_r^2. Remembering that $V_r = \frac{4}{3}\pi r^3$ and $V = \frac{4}{3}\pi R^3$ we have that

$$4\pi r^2 E = \begin{cases} \dfrac{q}{\epsilon_0}\left(\dfrac{r}{R}\right)^3 & r < R \\ \dfrac{q}{\epsilon_0} & r > R \end{cases} \tag{1.16}$$

Dividing by $4\pi r^2$ and remembering that $E^i = E\hat{n}^i = E\hat{r}^i$ we get the final solution for E^i, both inside and outside the charged sphere

$$E^i = \begin{cases} \dfrac{q}{4\pi\epsilon_0 R^3} r\hat{r}^i & r < R \\ \dfrac{q}{4\pi\epsilon_0 r^2}\hat{r}^i & r > R \end{cases} \tag{1.17}$$

Note how for $r < R$ the field grows linearly (we're adding charge increasing r), and it begins again falling like r^{-2} after we surpass the surface of the sphere at $r = R$. Curiously (but not at random) the field for a charged sphere with constant charge q is identical to the field produced by a point charge at the origin, it's like after we surpassed the surface of the sphere it collapsed all on the origin of the coordinates and became a point charge q at the origin.

The previous statements can be reformulated as a formal method

Method 1 (Gaussian Surfaces)**.** Given an electrostatic system with either spherical, cylindrical or planar symmetries. In order to solve (1.12) we need to choose an appropriate Gaussian surface G which encloses a bounded volume V for which $E^i \propto \hat{n}^i_G$. In this special case, integrating the equation (1.12) and applying Stokes' theorem we have

$$\oiint_G E^i \hat{n}^G_i \mathrm{d}s = E \oiint_G \mathrm{d}s = ES_G \tag{1.18}$$

Where S_G is the surface area of the gaussian surface. With this trick, if we call V the bounded volume such that $\partial V = G$ we have

$$E = \frac{1}{\epsilon_0 S_G} \iiint_V \rho(r^i)\mathrm{d}^3x$$

A rule of thumb for choosing G is the following:

- For spherical symmetry of E (like a point charge or a spherical distribution) G is the sphere of radius r
- For cylindrical symmetry (like a charged cable or a charged cylinder) G is the cylindrical surface of radius r
- for planar symmetry (like a charged plane) G is a "pillbox", i.e. simply a 3D rectangle

§§ 1.1.2 The Scalar Potential

A neat definition we can use is defining the *scalar potential* $V(r^i)$ of the electrostatic field. As usual a potential for a vector field is defined if and only if all closed path integrals of the field are 0 in a simply connected domain, i.e. that the curl of the field is zero in the selected domain.
We of course can choose this proof but it's much easier using this trick.
Take V as a bounded domain of $\mathbb{R}^3$ where there is some charge distribution $\rho(r^i)$ inside. The general formula for the electric field then is the following

$$E^i = \frac{1}{4\pi\epsilon_0} \iiint_V \rho(\tilde{r}^i)\frac{\hat{r}^i}{r^2}\mathrm{d}^3\tilde{x} \tag{1.19}$$

We immediately see that

$$\frac{\partial}{\partial x^i}\left(\frac{1}{r}\right) = -\frac{\hat{r}^i}{r^2}$$

Therefore, noting that the derivation acts only on the unprimed coordinates (i.e. it can go outside the integration without problems) we have

$$E^i = -\frac{\partial}{\partial x_i}\left(\frac{1}{4\pi\epsilon_0}\iiint_V \frac{\rho(\tilde{r}^i)}{r}\mathrm{d}^3\tilde{x}\right) \tag{1.20}$$

By definition of potential then, we can say that $E^i = -\partial^i V(r^i)$, where

$$V(r^i) = \frac{1}{4\pi\epsilon_0}\iiint_V \frac{\rho(\tilde{r}^i)}{r}\mathrm{d}^3\tilde{x} \tag{1.21}$$

This is known as the *scalar potential* of the electrostatic field.
Since by definition the curl of the gradient is always zero, we can immediately write a second constitutive equation for E^i

$$\epsilon^i_{jk}\partial^j E^k = 0 \tag{1.22}$$

This equation is Maxwell's third equation for static fields.
Defining $\mathbb{R}^\star = \mathbb{R}\cup\{\pm\infty\}$ and chosen two points $a, b \in \mathbb{R}^\star$, we have in the language of differential forms

$$\mathrm{d}V = E_i \mathrm{d}x^i \tag{1.23}$$

Therefore, with this definition, we can evaluate the work needed to move a charged particle through some path $\gamma : [a,b] \subset \mathbb{R}^\star \to \mathbb{R}^3$. We have

$$W = \int_\gamma F^i \hat{t}_i \mathrm{d}l = q\int_\gamma E^i \hat{t}_i \mathrm{d}l = -q\int_\gamma \partial^i V \hat{t}_i \mathrm{d}l$$

Writing $\hat{t}^i \mathrm{d}l = \mathrm{d}x^i$ we have

$$W = -q\int_\gamma \partial_i V \mathrm{d}x^i = -q\int_{V(a)}^{V(b)} \mathrm{d}V = q\Delta V \tag{1.24}$$

Therefore, $qV(r^i)$ can be imagined as a "potential energy" of the system. Via this definition, we have that the scalar potential has the following units in the SI system

$$[V] = \frac{[W]}{[q]} = \frac{\mathsf{J}}{\mathsf{C}} = \mathsf{V} \tag{1.25}$$

Where V are *Volts*. With this definition

$$1\ \mathsf{V} = 1\ \frac{\mathsf{J}}{\mathsf{C}} \tag{1.26}$$

From the definition of work we can immediately find a nice trick for evaluating the scalar potential of a distribution. Isolating the last two equalities in the first definition of work for the electric field we have

$$\int_\gamma E_i \mathrm{d}x^i = -\int_{V(a)}^{V(b)} \mathrm{d}V \tag{1.27}$$

Using the path independence of V we have by direct integration

$$V(b) - V(a) = -\int_\gamma E_i \mathrm{d}x_i \tag{1.28}$$

Due to the definition of V we know that it's defined up to a constant, and such constant can be chosen in order to have $V(a) = 0$. The point $a \in \mathbb{R}^\star$ is known as the *reference point* for the potential, and the appropriate choice depends from the charge distribution. The best choice is taking the point where the potential is 0
Suppose now we want to calculate the potential of a point charge in the origin. Since $E^i \to 0$ for $r \to \infty$ we take $a = \infty$, and therefore, since $V(r^i) \to 0$ for $r \to \infty$ we have at some distance $b = r$

$$V(r) = -\frac{q}{4\pi\epsilon_0}\int_\infty^r \frac{1}{r^2}\hat{r}^i \mathrm{d}x^i = -\frac{q}{4\pi\epsilon_0}\int_\infty^r \frac{1}{r^2}\mathrm{d}r = \frac{q}{4\pi\epsilon_0}\frac{1}{r} \tag{1.29}$$

Note that by linearity of the integral, for a system of point charges we have

$$V(r) = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^n \frac{q_i}{r_i} \tag{1.30}$$

Note that this trick doesn't work if the charge extends to infinity since the integral would diverge, in that case the reference point will be some other $a = r_0$

§§ 1.1.3 Maxwell Equations for Electrostatics and Boundary Conditions

So far we found two main equations for the E^i field, these are two coupled partial differential equations known as the *Maxwell equations for Electrostatics*. These equations are

$$\begin{cases} \partial_i E^i = \dfrac{\rho}{\epsilon_0} \\ \epsilon^i_{jk}\partial^j E^k = 0 \end{cases} \tag{1.31}$$

Or, in integral form for a bounded volume V and a regular surface Σ

$$\begin{cases} \oiint_{\partial V} E^i \hat{n}_i \mathrm{d}s = \dfrac{1}{\epsilon_0}\iiint_V \rho \mathrm{d}^3x \\ \oint_{\partial\Sigma} E^i \hat{t}_i \mathrm{d}l = 0 \end{cases} \tag{1.32}$$

Inserting the definition of the potential these two equations collapse in a single equation, which is the *Poisson equation* for the potential

$$\partial_i\partial^i V = -\frac{\rho}{\epsilon_0} \tag{1.33}$$

But, as for every partial differential equation, these make sense if and only if a boundary condition has been provided.
Without loss of generality we can consider an uniformly charged plane with surface density σ. We have using the Gaussian surface trick, choosing a pillbox with surface area A, that

$$E\oiint_G \mathrm{d}s = \frac{\sigma A}{\epsilon_0}$$

Noting that the contribute between the 4 sides is zero, only the two faces remain and $S_G = 2A$, and therefore

$$E = \frac{\sigma}{2\epsilon_0} \tag{1.34}$$

Since $E^i \propto \hat{n}^i$ we have

$$E^i = \frac{\sigma}{2\epsilon_0}\hat{n}^i \tag{1.35}$$

But the normal to the plane changes sign passing through its surface, therefore the field is discontinuous passing through its surface!
For the potential this is not true. By definition of potential we're checking the line integral along the tangent to the border of this Gaussian surface, which doesn't change sign when we pass through the surface. Therefore we have that $V \in C^2(V) \cup C^0(\partial V)$ while the field is discontinuous on the border. These conditions are valid for every regular surface.
Consider that, locally, every regular surface can be considered as "flat" or euclidean, therefore the Gaussian pillbox trick works well.
Noting that the outward normal of the pillbox above and below the "plane" is equal to $\pm\hat{n}^i$ where $\hat{n}^i$ is the normal to this plane. Therefore, by the previous calculations we must have that passing through the surface (locally)

$$\left(E^i_{in} + E^i_{out}\right)\hat{n}_i = 0 \tag{1.36}$$

I.e., the field outside this "plane" is opposite in sign to the field inside the "plane". Going back to the main general surface, via integration, we have that this result must hold generally, which emphasizes the discontinuity of the electric field.

§§ 1.1.4 Energy of the Electrostatic Field

Considering again the definition of work for a particle as $W = q\Delta V$ we can calculate it for a set of particles. Considering the interaction between particles we have that $W \propto q_i q_j$ where $i, j = 1, \cdots, n$, and noting that a charge doesn't self interact, i.e. $q_i q_j = 0$ for $i = j$ and that the usual multiplication between scalar is commutative, i.e. $q_i q_j = q_j q_i$ we have

$$W = \frac{1}{8\pi\epsilon_0}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\frac{q_i q_j}{r_{ij}^2} = \frac{1}{2}\sum_{i=1}^{n} q_i V(r_i) \tag{1.37}$$

Passing to continuous distributions we get

$$W = \frac{1}{2}\iiint_V \rho(r^i)V(r^i)\mathrm{d}^3x$$

Using the first Maxwell equation we have $\rho = \epsilon_0\partial_i E^i$, therefore

$$W = \frac{\epsilon_0}{2}\iiint_V V(r^i)\partial_i E^i \mathrm{d}^3x$$

Integrating by parts and applying Stokes' theorem we get

$$W = \frac{\epsilon_0}{2}\left(\oiint_{\partial V} VE^i \mathrm{d}s - \iiint_V E^i\partial_i V \mathrm{d}^3x\right) \tag{1.38}$$

Noting that E^i extends to infinity where it becomes zero, we have on the limit $V \to \mathbb{R}^3$, where we use $E^i = -\partial^i V$, that the total energy stored in a charge distribution is

$$W = \frac{\epsilon_0}{2}\iiint_{\mathbb{R}^3} E^2 \mathrm{d}^3x \tag{1.39}$$

§ 1.2 Conductors

The main real problem that somebody will encounter solving problems in electrostatics is problems with *conductors*. A conductor is a rigid body for which there are free charges which can move after the application of an electric field. An example of conductor is a metallic body in the rigid body approximation.
One main property of conductors is that inside of it the electric field is zero.
Imagine taking a neutral box conductor, and then apply an electric field parallel to the sides of the box. The free charges will then move due to the action of the electrostatic force towards the field (if $q > 0$) or against the field (if $q < 0$). Since the conductor was neutral and charges must be conserved since they cannot pop into existence randomly, we have that the field generated by the single negative and positive charges on the surface of the conductor will be equal in magnitude and opposite in sign, therefore the total field inside is 0, even though the field outside is nonzero.
A second property of conductors is that the charge density inside the conductor is 0 inside. Using Gauss law and the first property of conductors we have

$$\rho = \epsilon_0 \partial_i E^i = 0 \tag{1.40}$$

This is always true for conductors, since as we said before $E^i = 0$ inside.
One main explanation of this is that inside there is as much positive charge density ρ_+ and ρ_-. In fact, from Gauss' law we have

$$\rho = \rho_+ + \rho_- = 0 \implies \rho_+ = \rho_-$$

This indicates that the charges of the conductor will then be all on the surface, and therefore the conductor is an equipotential surface. In fact

$$\partial^i V_{in} = -E^i_{in} = 0 \implies V_{in} = k, \qquad k \in \mathbb{R} \tag{1.41}$$

In order to bring out other properties of the electric field in presence of conductors, we can consider the surface of separation between two materials. Consider a rectangular loop going through both materials. We have from the third Maxwell equation for electrostatics

$$\oint_A E^i \hat{t}_i \mathrm{d}l = 0 \tag{1.42}$$

Since the conductor is rectangular, separating the line integral into 4 integrals, where 2 go parallel to the surface and 2 are normal to it, we have that the two normal integrals taking a clockwise path must cancel each other and therefore we have

$$\oint_1 E^i_1 \hat{t}^1_i \mathrm{d}l + \oint_2 E^i_2 \hat{t}^2_i \mathrm{d}l = 0 \tag{1.43}$$

Since $\hat{t}^i_1 = -\hat{t}^i_2$ we have that, locally

$$\left(E^i_1 - E^i_2\right) \hat{t}^1_i = 0 \tag{1.44}$$

Therefore, the electric field tangent to the surface is continuous and therefore conserved. Since a charged conductor has a zero electric field inside and there is no external field, then

$$E^i_{in} = 0, \quad E^i_{in} \hat{t}_i = 0, \quad E^i_{ext} \hat{t}_i = 0$$

But since in general a vector can always be decomposed in a tangent component to the surface and a normal component to the surface we have

$$E^i_{in}\hat{n}_i = 0, \quad E^i_{ext}\hat{n}_i \neq 0$$

This because the conductor is charged. This means that there is a discontinuity in the field and the field itself must be normal to the surface of the conductor due to the continuity of the tangential component of the field. If E^i_c is the electric field generated by a conductor we have then

$$E^i_c = E\hat{n}^i \tag{1.45}$$

Consider now the potential inside and outside the conductor, V_{ext}, V_{in}. Considering that the charges we are moving are electrons with $q = -e$, where e is the fundamental charge

$$e = 1.6021766208(98) \cdot 10^{-19}\ \text{C} \tag{1.46}$$

We have that the work needed to bring outside the conductor our electron will be

$$\Delta U = -e\Delta V = -e\left(V_{ext} - V_{in}\right) \tag{1.47}$$

We define the work function as $L = \Delta U/e$ and it must obviously be positive since we're applying energy to the system in order to bring out an electron. We have

$$L = V_{in} - V_{ext} > 0 \implies V_{in} > V_{ext} \tag{1.48}$$

Due to all of these consideration, and noting that $\mathrm{d}V = -E_i \mathrm{d}x_i$ we have that the potential of a conductor will be defined as

$$V_0(r) = -\int_r^\infty E^i\hat{t}_i \mathrm{d}l = -\int_r^{r_0} E^i\hat{t}_i \mathrm{d}l \tag{1.49}$$

Where r_0 is the "first" radius immediately outside the conductor.

§§ 1.2.1 Coulomb Theorem

Consider now a conductor V and take a small cylinder orthogonal to its surface. Considering that the charge on a conductor is only on the surface we have using Gauss' law on the differential flux of E^i that

$$\mathrm{d}\Phi(E^i) = E^i\hat{n}_i \mathrm{d}S = \frac{\sigma}{\epsilon_0}\mathrm{d}S \tag{1.50}$$

Considering the equality in terms of norms of the E^i field and remembering that $E^{\parallel}\hat{n}^i$ we have that

$$E^i = \frac{\sigma}{\epsilon_0}\hat{n} \tag{1.51}$$

You can immediately see that this field is twice the field generated by a charged infinite plane. Let's consider what's happening with some more precision.
In that small cylinder $\mathrm{d}S$ we will have that the total external field will be composed from the contribution

of the charge inside the cylinder and the one outside. The same should be for the inside, but the inside field *must* be zero

$$E^i_{ext} = \left(E^i_{ext}\right)^{\mathrm{d}S} + \left(E^i_{ext}\right)^{S-\mathrm{d}S} \neq 0$$
$$E^i_{in} = \left(E^i_{in}\right)^{\mathrm{d}S} + \left(E^i_{in}\right)^{S-\mathrm{d}S} = 0$$

The field $\left(E^i\right)^{S-\mathrm{d}S}$ doesn't change and it must be the same as the field generated by $\mathrm{d}S$. Applying Gauss' theorem to the small surface element $\mathrm{d}S$ and noting that it must be the same of a plane with surface area $\mathrm{d}S$ we have

$$\left(E^i_{in}\right)^{S-\mathrm{d}S} = \left(E^i_{ext}\right)^{S-\mathrm{d}S} = -\left(E^i_{in}\right)^{\mathrm{d}S} = \frac{\sigma}{2\epsilon_0}\hat{n}^i$$

Therefore, finally

$$E^i_{ext} = \frac{\sigma}{2\epsilon_0}\hat{n} + \frac{\sigma}{2\epsilon_0}\hat{n} = \frac{\sigma}{\epsilon_0}\hat{n}$$

Where we used again that $\left(E^i_{ext}\right)^{\mathrm{d}S} = \frac{\sigma}{2\epsilon_0}\hat{n}$

§§ 1.2.2 Induced Charges

Consider now some conductor which is empty inside. Inside the first conductor we insert another conductor charged with charge Q. At $t = 0$ the external conductor is neutral, and therefore $Q_{ext} = 0$. Since charge must be conserved, we have that at $t > 0$ when we insert the new conductor inside the total charge must remain neutral, therefore

$$Q_{int} + Q_{ext} = 0$$

From Gauss' theorem, taking a surface inside the conductor that includes inside itself the internal surface of the conductor but not the external one. For Gauss we have

$$\Phi(E^i) = 0 = \frac{Q_V}{\epsilon_0} \implies Q + Q_{in} = 0$$

Therefore, there must be an *induced charge* Q_{in} on the internal surface of the conductor, such that

$$Q_{in} = -Q \tag{1.52}$$

From this, substituting before, we have that on the external surface we measure the charge we added inside the conductor, via the process of charge induction

$$Q_{ext} = -Q_{in} = Q \tag{1.53}$$

Note that this comes directly for having charge conservation.
Consider now the same empty conductor but don't add any charge inside of it, but rather charge the whole conductor with some positive charge Q. What happens inside the hole? Is there any charge?
By Gauss' theorem we have, since $E^i = 0$ inside the conductor, that the total charge inside the conductor is zero $Q_{in} = 0$.
There could still be a charge balance inside, where $Q^+_{in} - Q^-_{in} = Q_{in} = 0$. Supposing this true we can take a closed path that goes inside the hole. By definition of E^i the line integral on this path γ must be

zero. Divide the path into 1, that goes inside the hole, where there should be a field E^i between the two charges Q^+_{in} and Q^-_{in}, and path 2 which is inside the conductor. Then we would have

$$\oint_\gamma E^i \hat{t}_i \mathrm{d}l = \int_1 E^i \hat{t}^1_i \mathrm{d}l + \int_2 E^i \hat{t}^2_i \mathrm{d}l = \int_1 E^i \hat{t}^1_i \mathrm{d}l$$

Since the path is closed, call D the surface enclosed by the path, we have

$$\oint_\gamma E^i \hat{t}_i \mathrm{d}l = \oiint_D \epsilon^i_{jk} \partial^j E^k \hat{n}_i \mathrm{d}s = \int_1 E^i \hat{t}^1_i \mathrm{d}l \implies \epsilon^i_{jk} \partial^j E^k \neq 0 \tag{1.54}$$

This is in clear contradiction with Maxwell's equation for electrostatics (which we have already demonstrated that they generally hold), therefore all the charge is safely distributed on the *external* surface of the conductor, as we expected.

Exercise 1.2.1 (Two Charged Spheres)**.** Suppose that you have two metal spheres connected by a wire. One has radius R_1 and the other has radius R_2. At $t > 0$ we deposit some charge Q on the system. What will be the total charge distributed on the two spheres? (Q_1, Q_2)

S o l u t i o n

The potentials on the two spheres must be equal, and we know already from previous calculations that

$$\begin{aligned} V_1 &= \frac{1}{4\pi\epsilon_0}\frac{Q_1}{R_1} \\ V_2 &= \frac{1}{4\pi\epsilon_0}\frac{Q_2}{R_2} \\ V_1 &= V_2 \end{aligned} \tag{1.55}$$

From the previous equation we have that

$$Q_2 = \frac{R_2}{R_1} Q_1 \tag{1.56}$$

The total charge, on the other hand, will be $Q = Q_1 + Q_2$, therefore

$$Q = Q_1 + \frac{R_2}{R_1} Q_1 = \frac{R_1 + R_2}{R_1} Q_1 \implies Q_1 = \frac{R_1}{R_1 + R_2} Q \tag{1.57}$$

And, analogously

$$Q_2 = \frac{R_2}{R_1 + R_2} Q \tag{1.58}$$

From Gauss' theorem, if the spheres have surface charges σ_i, $i = 1, 2$ we must also have

$$\frac{Q_1}{R_1} = \frac{Q_2}{R_2} \implies \frac{4\pi R_1^2 \sigma_1}{R_1} = \frac{4\pi R_2^2 \sigma_2}{R_2} \tag{1.59}$$

I.e.

$$\sigma_1 R_1 = \sigma_2 R_2 \implies \sigma_1 = \frac{R_2}{R_1} \sigma_2$$

Since $\frac{R_1\sigma_1}{\epsilon_0} = \frac{R_2\sigma_2}{\epsilon_0}$, we must also have that the fluxes of the fields multiplied by R_i are equal, i.e. the electric fields are scaled as follows

$$E_2 = \frac{R_1}{R_2}E_1 \tag{1.60}$$

□

§§ 1.2.3 Capacity

Consider an isolated conductor on which there is some charge Q, distributed with density σ on its surface, such that the conductor is equipotential. We have that for every point in the conductor, by definition

$$\begin{aligned} V(r) &= \frac{1}{4\pi\epsilon_0}\iint_S \frac{\sigma}{r}\mathrm{d}s \\ Q &= \iint_S \sigma \mathrm{d}s \end{aligned} \tag{1.61}$$

It's clear that by this definition that if we vary σ to a new $\sigma' = \alpha\sigma$ with $\alpha \in \mathbb{R}$, we also have that $V' = \alpha V,\ Q' = \alpha Q$.
The following rate is then called the *capacity* of the conductor

$$C = \frac{Q}{V} \tag{1.62}$$

This is clearly only dependent on the geometry of the system. The capacity is measured in *Farads*, where

$$1\ \mathsf{F} = 1\ \frac{\mathsf{C}}{\mathsf{V}}$$

Example 1.2.1 (Capacity of a Spherical Conductor). Take now a spherical conductor with charge Q. We have

$$V = \frac{Q}{4\pi\epsilon_0 R}$$

Therefore

$$C = 4\pi\epsilon_0 R \tag{1.63}$$

This lets us redefine ϵ_0 in terms of Farads. In fact

$$[\epsilon_0] = \frac{[C]}{[R]} = \frac{\mathsf{F}}{\mathsf{m}}$$

Therefore

$$\epsilon_0 = 8.854\ \frac{\mathsf{F}}{\mathsf{m}} \tag{1.64}$$

In the case that we have multiple conductors one close to the other the problem gets slightly more complex.
Add a charge Q_1 to the first conductor, which will have potential V_1, which will induce a charge Q_2 and therefore a potential V_2 on the second. If I change the charge to $Q_1' = \alpha Q_1$ we will have a basically

identical result to the previous problem. Inverting the system and setting the charge on the second conductor Q_2 we will have a symmetrical system, for which we can write

$$V_i = \sum_{j=1}^{n} p_{ij} Q_j \tag{1.65}$$

The p_{ij} are the potential coefficients, for which holds $p_{ij} = p_{ji} > 0$, $p_{ii} \geq p_{ij}\ i \neq j$.
Due to the fact that the potential is unequivocally determined we must be able to solve the inverse problem, therefore we also know that $\det p_{ij} \neq 0$, and therefore

$$Q_i = \sum_{j=1}^{n} c_{ij} V_j \tag{1.66}$$

The matrix c_{ij} is known as the capacitance matrix, and we have $p_{ij} = c_{ij}^{-1}$. The diagonal elements c_{ii} are known as the *capacity coefficients*, while the off diagonal $c_{ij},\ i \neq j$ are known as the induction coefficients.
For this matrix hold the following properties, known as *Maxwell inequalities*

$$\begin{cases} c_{ij} = c_{ji} \\ c_{ii} > 0 \\ c_{ij} < 0 \quad i \neq j \\ \sum_{j=1}^{n} c_{ij} \geq 0 \end{cases} \tag{1.67}$$

§§ 1.2.4 Capacitors

Let's take again two conductors in total induction as for our previous system of two concentric conductors where one inside is set at a charge Q. Grounding the external surface we get that the external shell will be at a fixed $V = 0$, while the internal surface will have an induced charge $-Q$. Between these two surfaces there will be a potential difference ΔV, for which it's possible to evaluate the capacitance as

$$C = \frac{Q}{\Delta V}$$

Writing this in terms of the potential matrix $V_i = \sum_j p_{ij} Q_j$ we have the following system of equations, where $Q_1 = Q,\ Q_2 = -Q$

$$\begin{cases} V_1 = p_{11} Q - p_{12} Q \\ V_2 = p_{21} Q - p_{22} Q \end{cases}$$

Subtracting the second from the first we have

$$\Delta V = (p_{11} + p_{12} - 2p_{12})\, Q$$

Therefore

$$C = \frac{Q}{\Delta V} = \frac{1}{p_{11} + p_{12} - 2p_{21}} \tag{1.68}$$

Or in terms of the capacitance matrix c_{ij}

$$C = \frac{\det(c_{ij})}{c_{11} + c_{12} - 2c_{12}} \tag{1.69}$$

Finding the capacitance using these matrices tho is a quite long calculations, therefore we directly use the line integral of the E^i field for determining it, therefore, since

$$\Delta V_{12} = \int_2^1 E_i \mathrm{d}x^i$$

Example 1.2.2 (Spherical Capacitor). Consider now a spherical capacitor for which the outer shell is grounded, we have

$$E^i = \frac{Q}{4\pi\epsilon_0}\frac{\hat{r}^i}{r^2}$$

Therefore

$$\Delta V = \frac{Q}{4\pi\epsilon_0}\int_{r_2}^{r_1}\frac{1}{r^2}\mathrm{d}r = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

Therefore

$$C_s = \frac{4\pi\epsilon_0 r_1 r_2}{r_2 - r_1} \tag{1.70}$$

Example 1.2.3 (Cylindrical Capacitor). For a cylindrical capacitor made of two conducting cylinders of radius R_1 and R_2 and length $l >> R_2$ and total charge λl we have that the electric field is

$$2\pi l r E = \frac{\lambda l}{\epsilon_0}$$
$$E^i = \frac{\lambda}{2\pi\epsilon_0}\frac{\hat{r}}{r}$$

Therefore

$$\Delta V = \int_1^2 E_i \mathrm{d}x^i = \frac{\lambda}{2\pi\epsilon_0}\log\left(\frac{R_2}{R_1}\right)$$

Therefore, since $Q = \lambda l$, we have

$$C_c = \frac{2\pi\epsilon_0}{\log\left(\frac{R_2}{R_1}\right)} \tag{1.71}$$

Example 1.2.4 (Parallel Plane Capacitor). For two parallel plane conductors for which $d << \sqrt{S}$ where S is the surface area of the plane we have that

$$E = \frac{\sigma}{\epsilon_0},\ Q = S\sigma$$

Therefore

$$\Delta V = \frac{\sigma}{\epsilon_0}d$$

Where d is the distance between the plates, and therefore

$$C = \frac{\epsilon_0 S}{d} \tag{1.72}$$

§§ 1.2.5 Forces on a Conductor

Consider a charged conductor with surface area S. Considering a small element $\mathrm{d}S$ we have that the external field generated by the remaining surface is

$$\left(E^i_{ext}\right)^{S-\mathrm{d}S} = \frac{\sigma}{2\epsilon_0}\hat{n}$$

The total charge in $\mathrm{d}S$ is $\sigma \mathrm{d}S$, and therefore the (infinitesimal) force on the area element $\mathrm{d}S$ is, by definition of electrostatic force

$$\mathrm{d}F^i = \sigma \left(E^i\right)^{S-\mathrm{d}S} \mathrm{d}S = \frac{\sigma^2}{2\epsilon_0}\hat{n}^i \mathrm{d}S = \frac{1}{2}\epsilon_0 E^2 \hat{n}^i \mathrm{d}S = u\hat{n}\mathrm{d}S \tag{1.73}$$

Where we identified the energy density of the field u as

$$u = \frac{1}{2}\epsilon_0 E^2$$

Deriving everything by $\mathrm{d}S$, we have that the *electrostatic pressure* p^i on the infinitesimal element of the surface of the conductor is

$$p^i = \frac{\mathrm{d}F^i}{\mathrm{d}S} = u\hat{n}^i \tag{1.74}$$

Consider now a virtual displacement of the external surface of the conductor, where we move it by δr orthogonally to the previous surface, then, the (virtual) work necessary for such displacement is

$$\delta L = \delta F^i_{ext} \delta r_i = \delta U$$

Where we used that $\delta L = \delta U$, and F^i_{ext} as the "extraction force". Since $F^i_{ext} = -F^i$ we have that

$$\delta F_r = -\frac{\delta U}{\delta r}$$

But, by definition

$$\delta U = -\frac{1}{2}\epsilon_0 E^2 \delta r \mathrm{d}S$$

Therefore, as before

$$\delta F_r = u\mathrm{d}S$$

For constant charge, we might think to apply this to a charged parallel plate capacitor, for which we know that the infinitesimal work needed to charge it, i.e. to move the charges from infinity towards our capacitor, is

$$\mathrm{d}W = V\mathrm{d}q = \frac{q}{C}\mathrm{d}q \implies W = \frac{1}{2}\frac{Q^2}{C}$$

For a parallel plate capacitor therefore

$$U(x) = \frac{1}{2}\frac{Q^2 x}{\epsilon_0 S}$$

Therefore

$$F = -\frac{\partial U}{\partial x} = -\frac{1}{2}\frac{Q^2}{\epsilon_0 S}$$

This force is attractive (obvious from the system).
What if $V = cost.$ but the charge isn't constant? We know that

$$V = \frac{Q}{C}$$

And since both C, V are constants (one depends only on the geometry and the other is set constant by the system) Q can be the only one to have changed.
This means that there is some generator that charges up the capacitor, with work

$$\delta W_g = V\delta Q = V^2\delta C$$

Where we have $Q = VC$. From our previous relations we have

$$\delta W_{ext} = F_{ext}\delta x, \qquad \delta U = \delta W_g + \delta W_{ext}$$

Therefore, since $F_{ext} = -F$ we have that

$$\delta U = \delta W_g - \delta W$$

And, for the generator

$$\delta(CV^2) = \delta W + \delta\left(\frac{1}{2}CV^2\right)$$

Finally

$$\delta W = \delta U = F\delta x$$

Remembering that $U = CV^2/2$ and $C = S\epsilon_0/x$ we have through derivation that

$$F = -\frac{1}{2}\frac{S\epsilon_0 V^2}{x^2} = -\frac{1}{2}\frac{C^2V^2}{S\epsilon_0} = -\frac{1}{2}\frac{Q^2}{S\epsilon_0}$$

Which is the same result as before.
It's clear that for a charged conductor then the force is the mechanical moment of the system. It can be derived using the virtual work theorem, noting that $\delta L_{ext} = \delta U = -\delta L$, therefore

$$\delta L = F^i\delta x_i + L_i\delta\theta^i = -\delta U$$

Where, in the limit $\delta x, \delta\theta \to \mathrm{d}x, \mathrm{d}\theta$

$$F_x = -\frac{\partial U}{\partial x}$$
$$L_\theta = -\frac{\partial U}{\partial \theta}$$

2 The Electrostatic Potential

§ 2.1 Poisson and Laplace Equations

§§ 2.1.1 Green Identities

From the equations of Maxwell for electrostatics, we have seen that inserting the relation between the electrostatic field and the potential we get a second order partial differential equation known as the *Poisson equation*

$$\nabla^2 V = \partial^i \partial_i V(r^i) - \frac{\rho}{\epsilon_0} \tag{2.1}$$

And its homogeneous counterpart where $\rho = 0$, the *Laplace equation*

$$\partial^i \partial_i V = 0 \tag{2.2}$$

There are two fundamental theorems that we're gonna use for solving PDEs (Partial differential equations).

Theorem 2.1 (First Green Identity). *Given two functions $\varphi, \psi \in C^2(V)$ with V being a bounded set, we have*

$$\iiint_V \left(\varphi \partial^i \partial_i \psi + \partial^i \varphi \partial_i \psi\right) \mathrm{d}^3 x = \oiint_{\partial V} \varphi \frac{\partial \psi}{\partial x^i} \hat{n}^i \mathrm{d}s = \oiint_{\partial V} \varphi \frac{\partial \psi}{\partial n} \mathrm{d}s \tag{2.3}$$

Proof. Taken $A_i = \varphi \partial_i \psi$ we have that

$$\partial^i A_i = \partial^i \left(\varphi \partial_i \psi\right) = \varphi \partial^i \partial_i \psi + \partial^i \varphi \partial_i \psi$$

Therefore

$$\iiint_V \partial^i A_i \mathrm{d}^3 x = \iiint_V \left(\varphi \partial^i \partial_i \psi + \partial^i \varphi \partial_i \psi\right) \mathrm{d}^3 x = \oiint_{\partial V} \varphi \frac{\partial \psi}{\partial n} \mathrm{d}s = \oiint_{\partial V} A_i \hat{n}^i \mathrm{d}s$$

□

Theorem 2.2 (Second Green Identity). *Given two functions $\varphi, \psi \in C^2\left(\mathbb{R}^3\right)$, again from stokes theorem one has*

$$\iiint_V \left(\varphi \partial^i \partial_i \psi - \psi \partial^i \partial_i \varphi\right) \mathrm{d}^3 x = \oiint_{\partial V} \left(\varphi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \varphi}{\partial n}\right) \mathrm{d}s \tag{2.4}$$

Proof. Taken two vector fields $A_i = \varphi\partial_i\psi$, $B_i = \psi\partial_i\varphi$, analogously as before we have

$$\partial^i\left(A_i - B_i\right) = \varphi\partial^i\partial_i\psi - \psi\partial^i\partial_i\varphi$$

Applying Stokes' theorem to the previous definition we have the proof, since

$$A_i - B_i = \varphi\partial_i\psi - \psi\partial_i\varphi$$

□

With these two theorems, we can easily modify Poisson's equation into an integral equation which can help us find useful informations on the shape of V.
Taking (2.4) where we set

$$V(r^i) = \varphi$$
$$\frac{1}{r} = \psi$$

We get, remembering that $\partial^i\partial_i(r^{-1}) = -4\pi\delta^3(r^i - \tilde{r}^i)$ and $\partial^i\partial_i V = -\rho/\epsilon_0$

$$\iiint_V \left(V(\tilde{r}^i)\partial^i\partial_i\left(\frac{1}{r}\right) - \frac{1}{r}\partial^i\partial_i V\right)\mathrm{d}^3x = \iiint_V \left(-4\pi V\delta^3(r^i) + \frac{\rho}{r\epsilon_0}\right)\mathrm{d}^3x$$

Therefore

$$\iiint_V \left(-4\pi V(\tilde{r}^i)\delta^3(r^i) + \frac{\rho}{r\epsilon_0}\right)\mathrm{d}^3x = \oiint_{\partial V}\left(V(r^i)\frac{\partial}{\partial n}\left(\frac{1}{r}\right) - \frac{1}{r}\frac{\partial V}{\partial n}\right)\mathrm{d}s$$

Bringing to the left the surface integral and solving for $V(r^i)$ after having applied the Dirac delta we have

$$V(r^i) = \frac{1}{4\pi\epsilon_0}\iiint_V \frac{\rho}{r}\mathrm{d}^3x - \frac{1}{4\pi}\oiint_{\partial V}\left(V(r^i)\frac{\partial}{\partial n}\left(\frac{1}{r}\right) - \frac{1}{r}\frac{\partial V}{\partial n}\right)\mathrm{d}s \tag{2.5}$$

This result, is the general solution for Poisson's equation, consistent with a known volume charge ρ and a surface charge $\sigma = \epsilon_0\partial_n V$. Note how the solution depends on the boundary values of V.

§§ 2.1.2 Boundary Conditions and Uniqueness of the Solution

Given the general solution (2.5) how can we choose for appropriate boundary values such that the solution exists and is unique $\forall r^i \in V$ where V is a bounded and closed set?
One way is to specify V at the boundary, i.e. using Dirichlet boundary conditions, or to specify $E_n = E^i\hat{n}_i = -\partial_n V$ in the boundary ∂V, i.e. using Neumann boundary conditions.
Supposing Dirichlet boundary conditions for V we have that the solution is unique. Why?
Let V be the usual bounded set of $\mathbb{R}^3$ in which we have

$$\partial^i\partial_i V = -\frac{\rho}{\epsilon_0} \qquad \forall x^i \in V$$

Then, let $U = V_1 - V_2$ where V_1, V_2 are two solutions to Poisson's equation. By definition, then, U solves Laplace's equation

$$\partial^i\partial_i U = \partial^i\partial_i V_1 - \partial^i\partial_i V_2 = 0$$

At the boundary therefore we must have

$$U, \qquad \frac{\partial U}{\partial n} = 0 \qquad \forall x^i \in \partial V$$

From Green's 1st identity we also have that

$$\iiint_V U\partial^i\partial_i U + \partial^i U\partial_i U \mathrm{d}^3x = \oiint_{\partial V} U\frac{\partial U}{\partial n}\mathrm{d}s$$

Using that U must solve Laplace's equation and it must also be zero at the boundary, we have, writing $\partial^i U\partial_i U = |\partial U|^2$

$$\iiint_V |\partial U|^2 \mathrm{d}^3x = 0$$

This last integral implies that $|\partial U|^2 = 0$ and therefore $\partial_i U = 0\ \forall x^i \in V$, i.e. U is constant. Since $U \in C^2$ and it must be 0 in ∂V the constant must be 0 and therefore

$$V_1 = V_2$$

Which implies $\exists! V : V \to \mathbb{R}$ which solves Poisson's equation where V is defined on the boundary. With Neumann conditions this implies that the two solutions are linearly dependent, still implying the uniqueness of the solution.
It's also clear that using mixed Dirichlet/Neumann boundary conditions will give rise to a well behaved and unique solution.

§§ 2.1.3 Method of Images

A cool method for finding a solution of the Poisson and Laplace equations is the *method of images*, where we choose some imaginary charges put in some special positions such that the potential found solves the PDE and therefore is unique.

Example 2.1.1 (A Toy Problem). Suppose that some point charge q is held at some distance d from a grounded infinite conducting plane put at $z = 0$. What is $V(r^i)$ above the plane where there is q?
Note that it cannot be $q/4\pi\epsilon_0 r$ since there is an induced charge on the surface of the plane where $Q_i = -q$.
We imagine removing the plane and setting a charge $-q$ on the opposite side of the first charge. In this case the potential will be simply the sum of the two potentials of the single charge, where

$$V(x,y,z) = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{\sqrt{x^2+y^2+(z-d)^2}} - \frac{1}{\sqrt{x^2+y^2+(z+d)^2}}\right)$$

Note that this potential goes to 0 at infinity and it's 0 at $z = 0$ where there should be our plane. Due to the uniqueness of the solution we have that this is the solution to the first problem. We can also calculate the induced surface charge. We know that the surface charge will be proportional to the normal derivative of the potential at $z = 0$ (see the general solution of Poisson's equation), therefore, since the normal to the plane is the $\hat{z}^i$ versor, we have that

$$\frac{\partial V}{\partial n} = \frac{\partial V}{\partial z} = \frac{q}{4\pi\epsilon_0}\left(\frac{z+d}{(x^2+y^2+(z+d)^2)^{3/2}} - \frac{z-d}{(x^2+y^2+(z-d)^2)^{3/2}}\right)$$

Therefore, taking $z = 0$ and multiplying by $-\epsilon_0$ we have that the induced surface charge on the plane is:

$$\sigma(x, y) = -\frac{qd}{2\pi \left(x^2 + y^2 + d^2\right)^{3/2}}$$

Note that integrating σ over all the plane we get back that the total induced charge is $-q$ as expected.

The method of images is a particular method that uses the symmetries of the problem in order to carve out a solution to Poisson's equation and it can't be used in most situations. In those other situations we need to actually solve the partial differential equation and find the potential through integration, using a cool method that will be explained in the next section

§ 2.2 Separation of Variables

The main line of attack for Laplace's equation is the *separation of variables*, i.e. taking the following Ansatz for the potential $V(x, y, z)$

$$V(x, y, z) = f(x)g(y)h(z)$$

This Ansatz tho it's only usable when either the surface charge distribution σ or V are defined on the boundary of the set V, i.e. when our PDE has a defined boundary value problem with either Dirichlet or Neumann conditions.
Take as an example the following 2D problem.

Example 2.2.1 (Two Infinite Planes). Suppose that there are two infinite plates (grounded) parallel to each other and to the xz plane. One is at $y = 0$ and the other is at $y = a$. At $x = 0$ the left end of this strip is closed by an infinitely vertical strip at some fixed potential $V_0(y)$. Find $V(x, y, z)$ of the system. Since the system is independent from z we gotta solve the following differential equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

Where we have the following boundary conditions

$$\begin{cases} V(x, 0) = V(x, a) = 0 \\ V(0, y) = V_0(y) \\ \lim\limits_{x \to \infty} V(x, y) = 0 \end{cases}$$

We begin by separating the variables and writing $V(x, y) = f(x)g(y)$. We substitute into the differential equation and then divide by $f(x)g(y)$ and we get

$$\frac{1}{f(x)} \frac{\mathrm{d}^2 f}{\mathrm{d}x^2} + \frac{1}{g(y)} \frac{\mathrm{d}^2 g}{\mathrm{d}y^2} = 0$$

Note that now we have a sum of two functions depending on only one variable, i.e. $X(x) + Y(y) = 0$ This means that these functions must be equal, opposite in sign and constant, therefore the differential equation decouples into two ordinary differential equations

$$\begin{cases} \dfrac{\mathrm{d}^2 f}{\mathrm{d}x^2} = kf(x) \\ \dfrac{\mathrm{d}^2 g}{\mathrm{d}y^2} = -kg(y) \end{cases}$$

These two equations are of easy solution, and therefore we get

$$\begin{cases} f(x) = Ae^{kx} + Be^{-kx} \\ g(y) = C\cos(ky) + D\sin(ky) \end{cases}$$

Imposing the boundary conditions we get

$$\lim_{x\to\infty} f(x) = 0 \implies A = 0$$
$$g(0) = 0 \implies C = 0$$

The searched potential therefore has the following shape

$$V(x,y) = De^{-kx}\sin(ky)$$

Imposing $V(x,a) = 0$ we have the following constraint on the coupling constant k

$$V(x,a) = De^{-kx}\sin ka = 0 \implies k_n = \frac{n\pi}{a}$$

Therefore, we finally have

$$V_n(x,y) = D_n e^{-\frac{n\pi x}{a}}\sin\left(\frac{n\pi y}{a}\right)$$

The general solution of our problem then will be a linear superposition of *all* solutions, therefore

$$V(x,y) = \sum_{n=0}^{\infty} C_n e^{-\frac{n\pi x}{a}}\sin\left(\frac{n\pi y}{a}\right)$$

This is clearly the Fourier series solution of V, therefore the constants C_n will be found using Fourier's trick and multiplying on the left by $\sin(k_{n'}y)$ and integrating on the expansion interval, which for us is $[0,a]$. We have then, for $V(0,a) = V_0(y)$

$$\sum_{n=0}^{\infty} C_n \int_0^a \sin\left(\frac{k\pi y}{a}\right)\sin\left(\frac{n\pi y}{a}\right)\mathrm{d}y = \int_0^a V_0(y)\sin\left(\frac{k\pi y}{a}\right)\mathrm{d}y$$

Remembering that

$$\int_0^a \sin\left(\frac{k\pi y}{a}\right)\sin\left(\frac{n\pi y}{a}\right)\mathrm{d}y = \frac{a}{2}\delta_{kn}$$

We have

$$C_n = \frac{2}{a}\int_0^a V_0(y)\sin\left(\frac{n\pi y}{a}\right)\mathrm{d}y$$

I.e. C_n are the Fourier coefficients of the function $V_0(y)$. If $V_0(y) = V_0$ is constant the integral can be solved quickly, and we get

$$C_n = \frac{2V_0}{a}\left(1 - \cos(n\pi)\right) = \begin{cases} \frac{4V_0}{n\pi} & n \mod 2k = 0 \\ 0 & n \mod 2k+1 = 0 \end{cases}$$

And the complete solution is then

$$V(x,y) = \frac{4V_0}{\pi}\sum_{n=0}^{\infty}\frac{e^{-\frac{(2n+1)\pi x}{a}}}{2n+1}\sin\left(\frac{(2n+1)\pi y}{a}\right)$$

§§ 2.2.1 Laplace Equation in Spherical Coordinates

What happens when the boundaries exhibit spherical symmetry? We change to spherical coordinates!. The Laplacian in spherical coordinates is

$$\partial_i \partial^i = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2}{\partial \varphi^2}$$

The Laplace equation therefore becomes

$$\partial_i \partial^i V = \frac{1}{r^2}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 V}{\partial \varphi^2} = 0 \tag{2.6}$$

We suppose that the system has azimuthal symmetry, i.e. $\partial_\varphi V = 0$ and we solve the equation using the separation of variables, supposing $V(r,\theta) = f(r)g(\theta)$, then after dividing by V and multiplying by r^2 we get the following equation

$$\frac{1}{f(r)}\frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right) + \frac{1}{g(\theta)\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial g}{\partial \theta}\right) = 0$$

The equation can be then separated. Taking $c_1 = -c_2 = l(l+1)$ We get two ordinary differential equations

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}r}\left(r^2 \dfrac{\mathrm{d}f}{\mathrm{d}r}\right) = l(l+1)f(r) \\ \dfrac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta \dfrac{\mathrm{d}g}{\mathrm{d}\theta}\right) = -l(l+1)\sin\theta g(\theta) \end{cases} \tag{2.7}$$

The first equation has a power series solution, while the second is a special differential equation solved by the Legendre polynomials $P_l(\cos\theta)$, a complete and orthogonal set of polynomials defined by the recursive relation using the Rodrigues' formula

$$P_l(x) = \frac{1}{2^l l!}\frac{\mathrm{d}^l}{\mathrm{d}x^l}\left[(x^2-1)^l\right] \tag{2.8}$$

The solutions for the two differential equations are then

$$\begin{cases} f(r) = Ar^l + \dfrac{B}{r^{l+1}} \\ g_l(\theta) = P_l(\cos\theta) \end{cases} \tag{2.9}$$

The potential will then be, after superposition of all solutions in l, the following

$$V(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right)P_l(\cos\theta) \tag{2.10}$$

An example of using this solution is the following

Example 2.2.2 (A Hollow Sphere). Consider a hollow sphere with radius R, find V inside the sphere considering that the surface of the sphere is at some fixed potential $V_0(\theta)$.

The differential equation that must be solved here is the following

$$\begin{cases} \partial_i \partial^i V(r,\theta) = 0 \\ V(\theta, R) = V_0(\theta) \\ \lim_{r\to 0} V(r,\theta) = 0 \end{cases}$$

From the third condition we need that $B_l = 0$, if not the potential would blow up at the center, therefore the first sketch of the solution will be from the general solution (2.10)

$$V(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$$

From the second equation we have that at R it must be equal to $V_0(\theta)$. From (2.8) we can also get, using induction, the completeness relation for P_l.

$$\int_{-1}^{1} P_l(x) P_k(x) \mathrm{d}x = \int_0^{\pi} P_l(\cos\theta) P_k(\cos\theta) \sin\theta \mathrm{d}\theta = \frac{2}{2l+1}\delta_{lk} \tag{2.11}$$

Therefore, using Fourier's trick to the potential we found, we get that

$$A_l R^l \frac{2}{2l+1}\delta_{lk} = \int_0^{\pi} V_0(\theta) P_k(\cos\theta)\sin\theta \mathrm{d}\theta$$

This implies that the coefficients A_l we're searching are

$$A_l = \frac{2l+1}{2R^l}\int_0^{\pi} V_0(\theta) P_l(\cos\theta)\sin\theta \mathrm{d}\theta$$

The complete potential inside the sphere is then

$$V(r,\theta) = \sum_{l=0}^{\infty} \frac{2l+1}{2}\left(\frac{r}{R}\right)^l P_l(\cos\theta)\int_0^{\pi} V_0(\theta) P_l(\cos\theta)\sin\theta \mathrm{d}\theta$$

§ 2.3 Multipole Expansion of the Potential

§§ 2.3.1 Electric Dipoles

It's clear that from our calculations, at large distances from the distribution the electrostatic potential behaves approximatively like the potential of a single point charge

$$V(r) \approx \frac{q}{4\pi\epsilon_0}\frac{1}{r}$$

Note that if $Q_{tot} = 0$ we don't have necessarily that $V \approx 0$ at large distances! Take as an example the *electric dipole*. Take two point charges with charge $\pm q$ and position them at some distance d

between them. Writing r_+ and r_- as the distances of each charge from the point considered we can immediately write the potential of such system by superimposing the potentials of each single charge

$$V(r) = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{r_+} - \frac{1}{r_-}\right)$$

Noting that the distance d between the two charges and the distance from the origin of each writes a triangle, we can write

$$r_\pm^2 = r^2 + \frac{d^2}{4} \mp rd\cos\theta = r^2\left(1 + \frac{d^2}{4r^2} \mp \frac{d}{r}\cos\theta\right)$$

In our case $r_\pm >> d$ since we're far from the system, and therefore, approximating to the first order in $\frac{d}{r}$

$$\frac{1}{r_\pm} \approx \frac{1}{r}\left(1 \pm \frac{d}{2r}\cos\theta\right)$$

Therefore

$$\frac{1}{r_+} - \frac{1}{r_-} \approx \frac{d}{r^2}\cos\theta$$

Which, by substitution into our previous definition of the potential, gives

$$V(r) = \frac{1}{4\pi\epsilon_0}\frac{qd\cos\theta}{r^2} \tag{2.12}$$

The term on the right, qd, is known as the *electric dipole moment* of the distribution p.
In general, a potential can be approximated in a *multipole series*. The first term (the dominant one) is known as the *monopole term* of the potential, and it's equal to the potential of a single point charge

$$V_{mon}(r) = \frac{Q}{4\pi\epsilon_0 r} = \frac{1}{4\pi\epsilon_0}\frac{1}{r}\iiint_V \rho(\tilde{r}^i)\mathrm{d}^3\tilde{x}^i \tag{2.13}$$

If the total charge $Q = 0$, as for the previous case, the dominant term will be the *dipole term* of the potential

$$V_{dip}(r) = \frac{1}{4\pi\epsilon_0}\frac{p^i\hat{r}_i}{r^2} = \frac{1}{4\pi\epsilon_0}\frac{\hat{r}_i}{r^2}\iiint_V \tilde{r}^i\cos\theta\rho(\tilde{r}^i)\mathrm{d}^3\tilde{x} \tag{2.14}$$

The vector p^i is what we have defined as the dipole moment of the system, which is equal to

$$p^i = \iiint_V \tilde{r}^i\rho\left(\tilde{r}^i\right)\mathrm{d}^3\tilde{x} \tag{2.15}$$

For the previous case of the two charges, we easily have

$$p^i = qr_+^i - qr_-^i = qd^i \tag{2.16}$$

Where d^i is the vector connecting the two charges.
Note that in the case that the dipole moment of the potential is zero, there will be other terms that will

dominate the expansion, such as *quadrupole terms, octupole terms* and so on. The general formula for finding these coefficients can be extracted from the general shape of the potential in integral form

$$\frac{1}{4\pi\epsilon_0}\iiint_V \frac{\rho(\tilde{r}^i)}{\|r^i-\tilde{r}^i\|}\mathrm{d}^3\tilde{x}$$

Using

$$\|r^i-\tilde{r}^i\| = r^2\left(1+\left(\frac{\tilde{r}}{r}\right)^2-2\left(\frac{\tilde{r}}{r}\right)\cos\theta\right)$$

And supposing $\|r^i-\tilde{r}^i\| = r\sqrt{1+\varepsilon}$, where we choose ε as follows

$$\varepsilon = \left(\frac{\tilde{r}}{r}\right)\left(\frac{\tilde{r}}{r}-2\cos\theta\right)$$

We have, for $1+\varepsilon\to 0$, which is the case for long distances from the potential

$$\frac{1}{\|r^i-\tilde{r}^i\|} \approx \frac{1}{r}\left(1-\frac{1}{2}\varepsilon+\frac{3}{8}\varepsilon^2-\frac{5}{16}\varepsilon^3+\cdots\right) \tag{2.17}$$

Rewriting in terms of $\tilde{r}/r,\ \cos\theta$, we have on the right a series of cosines, which is known as the *Legendre Polynomials* in $\cos\theta$ $P_l(\cos\theta)$, which are the solutions to the angular part of the Laplace equation in polar coordinates. The function on the left of the series approximation is known as the *generating function* of the polynomials.

$$\frac{1}{\|r^i-\tilde{r}^i\|} = \frac{1}{r}\sum_{l=0}^{\infty}\left(\frac{\tilde{r}}{r}\right)^l P_l\left(\cos\theta\right) \tag{2.18}$$

In general, we have then that the complete multipole expansion of the electrostatic potential is

$$V_{mult}(r) = \frac{1}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\frac{1}{r^{l+1}}\iiint_V \tilde{r}^l P_l\left(\cos\theta\right)\rho\left(\tilde{r}^i\right)\mathrm{d}^3\tilde{x} \tag{2.19}$$

Note that this gives consistently that the potential goes as $1/r$ for monopoles, $1/r^2$ for dipoles, $1/r^3$ for quadrupoles and so on, and approximate charge distributions at great distances as a sum of simpler problems, a single point charge for the monopole, two point charges for the dipole, four point charges in a square for a quadrupole and so on.<++>

3 Electricity in Matter

§ 3.1 Polarization

So far we dealt with electrostatics with conductors only. We begin to consider non-conducting materials, which are known as *dielectrics*.
What changes from electrostatics with conductors? Experimentally it can be seen using capacitors.
Consider a parallel plane capacitor on which we put a charge Q and then fill the space between the places with some isotropic and homogeneous dielectric.
It can be seen that if V_0 is the potential without the dielectric, then in this case $\Delta V < \Delta V_0$. From the definition of capacitance then

$$C > C_0$$

Experimentally it's seen that, independently from the shape of the capacitor

$$\frac{C}{C_0} = \epsilon_r \tag{3.1}$$

This is known as the *relative dielectric constant*, which is from what we have seen greater than 1 and non-dimensional.
We can then write

$$C = \epsilon_r C_0 = \epsilon_r \frac{\epsilon_0 S}{d} = \frac{\epsilon S}{d} \tag{3.2}$$

Where we defined $\epsilon = \epsilon_r \epsilon_0$, which is the *dielectric constant of the medium*.
Using the known formulas for the capacitance we get that $\Delta V = \Delta V_0 / \epsilon_r$ and therefore $E = E_0 / \epsilon_r$, and this phenomenon can be explained as if we added a surface charge distribution on the two plates, and therefore

$$E = \frac{\sigma + \sigma'}{\epsilon_0}, \qquad E_0 = \frac{\sigma}{\epsilon_0}$$

I.e.

$$\sigma + \sigma' = \frac{\sigma}{\epsilon_r} \implies \sigma' = \frac{1 - \epsilon_r}{\epsilon_r}\sigma$$

We decide to cleverly distribute this charge on the positively charged plate as a negative charge distribution and vice versa on the other plate.
These charges are due to the *polarization* of the medium.

We also have that if we put a point charge inside a dielectric we get a new "scaled" Coulomb law

$$E = \frac{E_0}{\epsilon_r} = \frac{q}{4\pi\epsilon_0\epsilon_r}\frac{1}{r^2} = \frac{q}{4\pi\epsilon}\frac{1}{r^2} \tag{3.3}$$

§§ 3.1.1 The Polarization Field

Consider now an atom A. An atom in general it's a neutral object composed of a positively charged nucleus with charge $q = Ze$ and Z electrons with charge $q = -Ze$ Inserting it into a constant electric field we have that if it's not big enough to ionize the atom (making a conductor) it will move the nucleus and electrons till they get to a stable point, generating a dipole moment p^i. This process is known as *polarization* of the medium.
It's clear that this dipole moment is linearly coupled to the electric field with the following relationship

$$p^i = \alpha E^i \tag{3.4}$$

The coupling constant α is known as the *atomic polarizability* and depends on the chosen atom A. For anisotropic media, like molecules, this coupling constant becomes the *atomic polarizability tensor*, with the following relation

$$p^i = \alpha^i_k E^k \tag{3.5}$$

Consider now a molecule with a "built in" polarization, (i.e. *polar molecules*) like water. What happens when we apply an E field?
If E is uniform then the force on the positive charge cancels the one on the negative, $F_+ = -F_-$, however there is still a torque to consider

$$\tau^i = \epsilon^i{}_{jk} r^j F^k_+ + \epsilon^i{}_{jk} r^j F^k_- \tag{3.6}$$

Since $r^i = \pm d^i/2$ we have, substituting $F_\pm = \pm qE$

$$\tau^i = \frac{q}{2}\epsilon^i{}_{jk} d^j E^k + \frac{q}{2}\epsilon^i{}_{jk} d^j E^k$$

This is nonzero, in fact we have

$$\tau^i = q\epsilon^i{}_{jk} d^j E^k = \epsilon^i{}_{jk} p^j E^k \tag{3.7}$$

I.e., since $p^i = qd^i$ is the dipole moment of the molecule (which is nonzero), there is an induced torque when applying the field, which rotates the molecules until $p^i \parallel E^i$, and therefore $\tau^i = 0$.
Note that if the field is nonuniform we won't have anymore $F_+ = -F_-$, and we will have a net force applied to our dipole (the molecule)

$$F^i = F^i_+ + F^i_- = q\Delta E^i$$

For small dipoles, i.e. for small ΔE^i, we can approximate it to

$$\Delta E^i \approx d^i \partial_i E^j$$

And therefore the net force applied on the dipole is

$$F^i = qd^j \partial_j E^i = p^j \partial_j E^i \tag{3.8}$$

Now consider an element with an amount of molecules of the order of 10^{23}. All these tiny dipoles induced from the electric field or from the single molecule itself will sum up to a general dipole field, called the *polarization field* of the medium. By definition we have, that if V is some volume then

$$P^i = \lim_{V\to 0} \frac{1}{V} \sum_{\alpha=1}^{N} p^i_{(\alpha)} = \langle p^i \rangle \frac{\mathrm{d}N}{\mathrm{d}V} \tag{3.9}$$

Here we have indicated with $\langle p^i \rangle$ the average dipole moment of the system.
Now let's write the potential for a single molecule. Since the molecule can be approximated as a dipole, we know already then that

$$V(r) = \frac{p^i \hat{r}_i}{4\pi\epsilon_0 r^2}$$

From our previous definition of polarization field, then, integrating over all the dielectric and using $\mathrm{d}V \to \mathrm{d}^3x$ we have

$$V_{pol}(r) = \frac{1}{4\pi\epsilon_0} \iiint_V \frac{P^i \hat{r}_i}{r^2} \mathrm{d}^3\tilde{x} \tag{3.10}$$

Looking closely inside the integral, we can rewrite an identity inside that will ease our calculations, in fact

$$\frac{\hat{r}_i}{r^2} = \frac{\partial}{\partial x^i}\left(\frac{1}{r}\right)$$

With a clever trick then we can write, using product rules

$$P^i \frac{\partial}{\partial x^i}\left(\frac{1}{r}\right) = \frac{\partial}{\partial x^i}\left(\frac{P^i}{r}\right) - \frac{1}{r}\frac{\partial P^i}{\partial x^i}$$

Therefore, reinserting into our definition of V and applying Stokes when possible, we have

$$V_{pol}(r) = \frac{1}{4\pi\epsilon_0}\left[\oiint_{\partial V} \frac{P^i \hat{n}_i}{r} \mathrm{d}s - \iiint_V \frac{1}{r}\partial_i P^i \mathrm{d}^3x\right] \tag{3.11}$$

This potential resembles *a lot* the potential given from a volumetric charge plus some surface charge in some closed bound set, like

$$V_v(r) = \frac{1}{4\pi\epsilon_0}\left[\oiint_{\partial V} \frac{\sigma(r^i)}{r} \mathrm{d}s + \iiint_V \frac{\rho(r^i)}{r} \mathrm{d}^3x\right]$$

And, reinterpreting the polarization field as a field generated by a *bound charge*, we can define two simple equations that will make our V similar to V_v. Then, if

$$\begin{cases} P^i \hat{n}_i = \sigma_b \\ \partial_i P^i = -\rho_b \end{cases} \tag{3.12}$$

And defined as V_σ, V_ρ the two potentials generated by this "bound charge", we have that the total potential generated by a polarized medium is

$$V_{pol}(r^i) = V_\sigma(r^i) + V_\rho(r^i) \tag{3.13}$$

A nice observation from the second equation of (3.12) is that if the dielectric is homogeneous, the dipole moments inside the object will average to 0, and therefore P^i will be independent from the position inside the dielectric, i.e.

$$\partial_i P^i = 0 = -\rho_b$$

And all bound charges will be on the surface with distribution σ_b

§ 3.2 Perfect Dielectrics

§§ 3.2.1 Local Electric Field

So far we defined a dielectric as a cluster of molecules and atoms. It's clear so far that each atom and molecule has its little microscopic e^i field, therefore the electric field inside a dipole can change greatly between points, depending on where we measure the field, if near or far away from an electron (considering that the distances are $d \approx 10^{-10}$ m "far" can be a negligible quantity in relation to the dimension of the dielectric).
Take now a really small part of the dielectric, in this small element of dielectric we will have inside some sphere S molecules which are polarized when an external field E^i gets applied.
We consider 2 major cases:

1. There are no molecules inside S and therefore there will be only the bound surface charge $\sigma_b = P^i \hat{n}_i$ with $\hat{n}^i$ being the outward normal of the conductor
2. There are molecules inside S and therefore, there will also be a field generated by the polarization of the molecules

The field at the center of S, E^i_S will then be the sum of these three fields we considered, the external polarizing field E^i, the field $\tilde{E}^i$ generated by the bound surface charge, and the field E^i_{dip} generated by the molecular dipoles. Therefore

$$E^i_S = E^i + \tilde{E}^i + E^i_{dip} \tag{3.14}$$

Due to the homogeneity of the dielectric we must have that $\partial_i P^i = 0$, and therefore the field generated by the dipoles and the bound surface charge must balance themselves, $\tilde{E}^i + E^i_{dip} = 0$.
As we said before the molecule itself generates a small microscopic field e^i, therefore we define a *local field* or *Lorentz field* inside the dielectric by subtracting this e^i. We have that this field E^i_{loc} is

$$E^i_{loc} = E^i + \tilde{E}^i + E^i_{dip} - e^i = E^i + \tilde{E}^i + \underline{E}^i \tag{3.15}$$

Where we defined $\underline{E}^i = E^i_{dip} - e^i$. What's this field then?
We begin by evaluating $\tilde{E}^i$, which is the field generated by the surface charge. Then by definition of the E^i field itself we can immediately say

$$\mathrm{d}\tilde{E}^i = \frac{1}{4\pi\epsilon_0}\frac{\sigma_b \hat{r}^i}{r^2}\mathrm{d}s$$

Due to the symmetries imposed on the system (homogeneity of the dielectric,...) we have that $\mathrm{d}\tilde{E}^z = -\left\|\mathrm{d}\tilde{E}^i\right\| \cos\theta$, and therefore, remembering that $\sigma_b = P^i \hat{n}_i = -P\cos\theta$ ($\hat{n}^i$ is the *outward* normal)

$$\mathrm{d}\tilde{E}^z = -\frac{\sigma_b \cos\theta}{4\pi\epsilon_0 r^2}\mathrm{d}s$$

Since $\mathrm{d}s = r^2 \mathrm{d}\Omega$ we then have

$$\mathrm{d}\tilde{E}^z = -\frac{P\cos^2\theta}{4\pi\epsilon_0}\mathrm{d}\Omega \tag{3.16}$$

Integrating, we have

$$\tilde{E}^z = -\frac{P}{4\pi\epsilon_0}\int_0^{2\pi}\mathrm{d}\phi\int_{-\pi}^{\pi}\cos^2\theta\sin\theta\mathrm{d}\theta = \frac{P}{2\epsilon_0}\int_{-1}^{1}\cos^2\theta\mathrm{d}\left(\cos\theta\right) = \frac{P}{3\epsilon_0} \tag{3.17}$$

Therefore, we firstly found that

$$\tilde{E} = \frac{P}{3\epsilon_0} \tag{3.18}$$

We only miss evaluating the field generated by the dipoles minus the microscopic molecular electric field. We only need to know what's the field generated by an isotropic dipole.
We know already that an electric dipole has the following scalar potential

$$V(r) = \frac{p^i\hat{r}_i}{4\pi\epsilon_0 r^2} = \frac{p^i r_i}{4\pi\epsilon_0 r^3}$$

Taking the gradient we have

$$\frac{\partial V}{\partial x_i} = \frac{1}{4\pi\epsilon_0}\left(\frac{1}{r^3}\frac{\partial}{\partial x_i}\left(p^j r_j\right) + p^j r_j\frac{\partial}{\partial x_i}\left(\frac{1}{r^3}\right)\right)$$

Expanding and writing explicitly the gradient of a radial function with the usual formula, we have

$$\frac{\partial V}{\partial x_i} = \frac{1}{4\pi\epsilon_0}\left[\frac{1}{r^3}\left(\frac{\partial p^j}{\partial x_i}r_j + p^j\frac{\partial r_j}{\partial x_i}\right) - \frac{3(p^j r_j)r^i}{r^5}\right]$$

Using $\partial^i p^j = 0$ and $\partial^i r_j = \delta^i_j$ we have that

$$\frac{\partial}{\partial x_i}\left(p^j r_j\right) = p^j\delta^i_j = p^i$$

And therefore

$$\frac{\partial V}{\partial x_i} = \frac{1}{4\pi\epsilon_0}\left(\frac{p^i}{r^3} - \frac{3(p^j r_j)r^i}{r^5}\right)$$

Writing $\hat{r}^i = r^i/r$ we have finally, multiplying by -1

$$E^i = -\frac{\partial V}{\partial x_i} = \frac{1}{4\pi\epsilon_0 r^3}\left(3\left(p^j\hat{r}_j\right)\hat{r}^i - p^i\right) \tag{3.19}$$

In our special isotropic case inside a little sphere S, inside a dipole itself we have that this field will be oriented on the z axis, with a constant dipole moment of the $\alpha-$th molecule $p_{(\alpha)}$, then

$$\underline{E} = \underline{E}^z = \sum_{\alpha=1}^{N}\frac{p_{(\alpha)}\left(3z^2_{(\alpha)} - r^2_{(\alpha)}\right)}{4\pi\epsilon_0 r_{(\alpha)^5}} \tag{3.20}$$

(Note that here we took the opposite process and rewrote the non normalized vector r^i for ease of calculation).
Since the $p_{(\alpha)}$ are uniformly distributed around the dielectric we must have that

$$\sum_{\alpha=1}^{N} \frac{x_{(\alpha)}^2}{r_{(\alpha)}^5} = \sum_{\alpha=1}^{N} \frac{y_{(\alpha)}^2}{r_{(\alpha)}^5} = \sum_{\alpha=1}^{N} \frac{z_{(\alpha)}^2}{r_{(\alpha)}^5} = \frac{1}{3} \sum_{\alpha=1}^{N} \frac{r_\alpha^2}{r_{(\alpha)}^5} \tag{3.21}$$

Simply inserting it back into the definition of $\underline{E}$ we get then $\underline{E} = 0$.
The final result for the Lorentz field (the local field inside a dielectric), considering all the microscopic variables, is

$$E_{loc}^i = E^i + \frac{P^i}{3\epsilon_0} \tag{3.22}$$

I.e. it only depends on the external applied field E^i and the polarization of the dielectric P^i (divided by $3\epsilon_0$)

§§ 3.2.2 Susceptibility and the Clausius-Mossotti relation

So far we can finally conclude that with a good approximation the polarization of the dielectric P^i must depend on this local field E_{loc}^i, which basically decides how a certain material gets polarized. Therefore, using the definition of P^i and defining the numerical volumetric density of molecules $\frac{\mathrm{d}N}{\mathrm{d}V} = n$

$$P^i = n \left\langle p^i \right\rangle = n\alpha E_{loc}^i \tag{3.23}$$

§§§ 3.2.2.1 Gases and Vapors

Let's now consider different relations between the Lorentz field and the polarization field. The easiest case to consider is a gas. In this case, if we take the perfect gas approximation, i.e. the density is low enough, we can say that the molecules are too far apart in order for their fields to interact between each other, therefore $E_{loc}^i \approx E^i$.
We also have to consider thermal excitations of the molecules of the gas, and therefore the coupling constant α must be split in two parts. One, α_d, dependent on the molecule itself, and one α_t depending on the temperature of the gas and the specific polarization of the molecule

$$\alpha = \alpha_d + \alpha_t = \alpha_d + \frac{p_0^2}{3kT}$$
$$P^i = n\alpha E^i = n\left(\alpha_d + \frac{p_0}{3kT}\right) E^i$$

We then define the *electric susceptibility* of the medium χ via the following relation

$$P^i = \epsilon_0 \chi E^i \tag{3.24}$$

Therefore, for a gas

$$\chi(T) = \frac{n}{\epsilon_0}\left(\alpha_d + \frac{p_0}{3kT}\right) = \epsilon_r - 1 \tag{3.25}$$

Where ϵ_r is the relative permittivity of the substance, as we will see later

§§§ 3.2.2.2 Liquids and Amorphous Substances

For liquids everything changes a little bit. Since the density isn't low enough, the molecules will be packed and their local field will comprise of the external field applied plus the field generated by the polarization. We have

$$E^i_{loc} = E^i + \frac{P^i}{3\epsilon_0}$$
$$P^i = n\alpha E^i_{loc}$$

Then, by mere substitution

$$P^i = n\alpha\left(E^i + \frac{P^i}{3\epsilon_0}\right)$$

Solving for P^i (bringing it on the left and taking it outside the product with the constants) we have then

$$P^i = \frac{n\alpha}{1 - \frac{n\alpha}{3\epsilon_0}} E^i = \epsilon_0 \chi E^i \tag{3.26}$$

Now, solving for α, we have after some algebra, the *Clausius-Mossotti relation*, which ties α, a microscopic quantity, to ϵ_r via χ, a macroscopic quantity

$$\alpha = \frac{\epsilon_0}{n}\frac{3(\epsilon_r - 1)}{\epsilon_r + 2} \tag{3.27}$$

§§§ 3.2.2.3 Anisotropic Solids, Electrets and Piezoelectricity

In general when the solid is anisotropic, as we defined before the polarizability is not a simple constant but a tensor, where

$$P^i = \alpha^i_j E^j \tag{3.28}$$

For other materials, α can also be nonlinear. Take for example *electrets*. An *electret* or a *ferroelectric material* is a material which keeps a permanent polarization inside after turning off the external field, showing magnet-like behavior, like *hysteresis*. In this case α is non-unique.
Another example of a non-linear relation comes from *piezoelectric materials*. *Piezoelectricity* is a phenomenon given by substances that polarize under mechanical pressure, like quartz. In these materials α must depend on the mechanical pressure itself.

§§ 3.2.3 The Electric Displacement Field

So far, adding the theory on dielectrics, we can build multiple equations describing the polarization P^i, bound charges ρ_b, σ_b and the relation between P^i and E^i.
From Gauss' law we know that the divergence of the E^i field is equal to the (total) volumetric charge divided by ϵ_0. With dielectrics we then gotta consider also bound charges, therefore

$$\partial_i E^i = \frac{\rho + \rho_b}{\epsilon_0}$$

Remembering that the bound volumetric charge is tied to the polarization with the differential equation

$$\partial_i P^i = -\rho_b$$

We then have

$$\partial_i E^i = \frac{\rho}{\epsilon_0} - \frac{1}{\epsilon_0}\partial_i P^i$$

Multiplying by ϵ_0 and bringing $\partial_i P^i$ on the left and using the linearity of ∂_i we have

$$\partial_i \left(\epsilon_0 E^i + P^i\right) = \rho$$

We define the vector on the left as the *Electric Displacement field* D^i

$$D^i = \epsilon_0 E^i + P^i \tag{3.29}$$

And we immediately get from the previous equation, the equivalent Gauss law for this field

$$\partial_i D^i = \rho \tag{3.30}$$

With this field, the first and third Maxwell equations in dielectrics become two coupled partial differential equations for two different fields, E^i and D^i

$$\begin{cases} \partial_i D^i = \rho \\ \epsilon^i{}_{jk}\partial^j E^k = 0 \end{cases} \tag{3.31}$$

This is solvable only if we know the relations between D^i and E^i, or in general how P^i is related to E^i. In a perfect dielectric we have that the polarizability tensor α^i_j is independent of the position, time and electric field (note that a gas cannot be a perfect dielectric since α depends on the temperature).
We will study only isotropic perfect dielectrics, also known as *linear dielectrics*, where $\alpha^i_j = \alpha\delta^i_j$, and we can write for these, as we saw before

$$P^i = \alpha E^i = \epsilon_0 \chi E^i \tag{3.32}$$

Therefore, from our previous definition of D^i and noting that $\chi = \epsilon_r - 1$,

$$D^i = \epsilon_0 E^i + P^i = \epsilon_0 E^i + \epsilon_0 \chi E^i = \epsilon_0 \left(1 + \chi\right) E^i = \epsilon_0 \epsilon_r E^i \tag{3.33}$$

Using $\epsilon = \epsilon_0 \epsilon_r$ we have then, that in linear dielectrics the D^i field is linearly dependent on the E^i field, where

$$D^i = \epsilon E^i \tag{3.34}$$

Note that outside a dielectric (i.e. in free space) we must have $P^i = 0$, and therefore

$$D^i_f = \epsilon_0 E^i_f \tag{3.35}$$

Maxwell's equations for a linear dielectric then modify to a much simpler variant which differs from the usual electrostatic maxwell equations by simply setting $\epsilon_0 \to \epsilon$

$$\begin{cases} \partial_i E^i = \dfrac{\rho}{\epsilon} \\ \epsilon^i{}_{jk}\partial^j E^k = 0 \end{cases} \tag{3.36}$$

Note that in free space

$$\partial_i \left(\epsilon_0 E^i_f\right) = \rho$$

And in a dielectric

$$\partial_i \left(\epsilon E^i\right) = \rho$$

Then, we must have

$$\partial_i \left(\epsilon E^i\right) = \partial_i \left(\epsilon_0 E_f^i\right)$$

Integrating and using the first principle of the calculus of variation then it's obvious that

$$\epsilon_0 E_f^i = \epsilon E^i \implies E^i = \frac{1}{\epsilon_r} E_f^i \tag{3.37}$$

Where we used $\epsilon = \epsilon_r \epsilon_0$. This is the exact same experimental result that we found empirically before with the parallel plate capacitor

§ 3.3 Maxwell Equations for Electrostatics in Linear Dielectrics

We can now begin defining all the various laws we derived for electrostatic fields in free space in presence of dielectrics, using the linear relations that we found before.
From Gauss' law for the D^i field integrating we immediately have

$$\iiint_V \partial_i D^i \mathrm{d}^3 x = \oiint_{\partial V} D^i \hat{n}_i \mathrm{d}s = Q_{loc} = \iiint_V \rho \mathrm{d}^3 x \tag{3.38}$$

And, analogously, the Coulomb theorem for surface charges

$$D^i = \sigma \hat{n}^i \tag{3.39}$$

Note that we didn't indicate the *total* charge inside V, Q_V, since we're not considering the polarization bound charge Q_b! We're only considering the "free" charge, which is not due to polarization effects of the dielectric.
We have a bit of luck tho when dielectrics are linear, then with a simple multiplication of the third Maxwell equation by ϵ we also get a coupled set of equations for the D^i field

$$\begin{cases} \partial_i D^i = \rho \\ \epsilon^i{}_{jk} \partial^j D^k = 0 \end{cases} \tag{3.40}$$

Due to the clear linear relations between E^i and P^i it's also possible to know the polarization inside the medium, which is not always obvious and measurable (it's clear only for perfect dielectrics). Since $E^i = D^i / \epsilon_0 \epsilon_r$ and $\chi = \epsilon_r - 1$ we have

$$P^i = \epsilon_0 \chi E^i = \epsilon_0 (\epsilon_r - 1) \frac{D^i}{\epsilon_0 \epsilon_r} = \frac{\epsilon_r - 1}{\epsilon_r} D^i \tag{3.41}$$

Example 3.3.1 (A Charged Dielectric Sphere). Take as an example a sphere composed of dielectric material of radius R with charge Q.
From Gauss' theorem for D^i we have, for a spherical Gaussian surface with $r > R$

$$\Phi\left(D^i\right) = 4\pi R D = Q \implies D = \frac{Q}{4\pi r^2}$$

Since $E^i = \epsilon^{-1} D^i$ and $P^i = \epsilon_0 \chi E^i$ we have

$$P^i = \epsilon_0 \chi \frac{D^i}{\epsilon} = \frac{\epsilon_0(\epsilon_r - 1)}{\epsilon_0 \epsilon_r} D^i = \left(\frac{\epsilon_r - 1}{\epsilon_3} \right) \frac{Q}{4\pi r^2} \hat{r}^i$$

And

$$E^i = \frac{Q}{4\pi\epsilon r^2} \hat{r}^i$$

The bound polarization charge distributions are then found using the known formulas, and therefore for the surface polarization charge

$$\sigma_b = P^i \hat{n}_i = -P^i \hat{r}_i = -\frac{\epsilon_r - 1}{\epsilon_r} \frac{Q}{4\pi R^2} = -\frac{\epsilon_r - 1}{\epsilon_r} \sigma$$

The total polarization charge is

$$Q_b = 4\pi R^2 \sigma_b = -\frac{\epsilon_r - 1}{\epsilon_r} Q$$

And therefore the total charge is

$$Q_t = Q + Q_b = Q \left(1 - \frac{\epsilon_r - 1}{\epsilon_r} \right) = \frac{Q}{\epsilon_r}$$

While, for the volumetric polarization charge we have

$$\rho_b = -\partial_i P^i = \frac{1}{r^2} \frac{\mathrm{d}}{\mathrm{d}r} \left(r^2 P^r \right) = -\frac{1}{r^2} \frac{\mathrm{d}}{\mathrm{d}r} \left(\frac{\epsilon_r - 1}{\epsilon_r} \frac{Q}{4\pi} \right) = 0$$

I.e. $\rho_b = 0$ as we expected. Since the dielectric is neutral there also must be a charge $-Q_b > 0$ at $r \to \infty$.

Example 3.3.2 (A Parallel Plate Capacitor). This example is quite simple. We know from Gauss' theorem for the surface charge and D^i that

$$D = \sigma$$

Therefore

$$E = \frac{D}{\epsilon} = \frac{\sigma}{\epsilon}$$

The polarization field instead is

$$P = \epsilon_0 \chi E = \chi \epsilon_0 \frac{\sigma}{\epsilon} = \frac{\epsilon_r - 1}{\epsilon_r} \sigma$$

And, the polarization surface charge (remembering that we take the outer normal) is

$$\sigma_b = P^i \hat{n}_i = -P = -\frac{\epsilon_r - 1}{\epsilon_r} \sigma$$

The potential difference between the plates is simply

$$\Delta V = Ed = \frac{D}{\epsilon} d = \frac{\sigma}{\epsilon} d = \sigma S \frac{d}{\epsilon S} = Q \frac{d}{\epsilon S} = \frac{Q}{C}$$

Note that

$$C = \frac{S\epsilon}{d} = \epsilon_r C_0$$

As we expected.

§§ 3.3.1 Boundary Conditions

Suppose now that we have multiple dielectric regions. On the boundaries of these regions, passing from one dielectric to another, it's clear that the fields D^i, E^i have discontinuities and therefore we cannot use the differential equations anymore.
We might try by either solving the equations for every dielectric region, or instead by directly solving Poisson's equation with appropriate boundary conditions for each dielectric.
While we cannot use the differential equations (local) in the boundaries of the dielectrics we instead have that the integral relations still hold, where

$$\begin{cases} \oiint D^i \hat{n}_i \mathrm{d}s = 0 \\ \oiint E^i \hat{n}_i \mathrm{d}s \neq 0 \end{cases} \tag{3.42}$$

These imply that

1. The dielectric is neutral, $\sigma = 0$
2. The dielectric is polarized, $\sigma_b \neq 0$

Considering an infinitesimal cylinder centered on the boundary of two different dielectric regions, we have that inside the cylinder the flux of D must be 0, i.e.

$$D_1^i \hat{n}_i^1 \mathrm{d}s + D_2^i \hat{n}_2 \mathrm{d}s = 0$$

Therefore, noting that $\hat{n}_1^i = -\hat{n}_2^i$ we get

$$\left(D_1^i - D_2^i\right) \hat{n}_i^1 \mathrm{d}s = 0 \implies D_{n1} = D_{n2} \tag{3.43}$$

While, for E, using $D_{n1} = \epsilon_1 E_{n1}$

$$\frac{E_{n1}}{E_{n2}} = \frac{\epsilon_2}{\epsilon_1} \neq 1 \tag{3.44}$$

Considering that $\epsilon^i{}_{jk} \partial^j E^k = 0$ we can write instead, for the line integral on the closed curve describing the cylinder instead that

$$\begin{aligned} E_{t1} &= E_{t2} \\ \frac{D_{t1}}{D_{t2}} &= \frac{\epsilon_1}{\epsilon_2} \end{aligned} \tag{3.45}$$

The boundary conditions between two dielectrics then become the following connection relations

$$E_{t1} = E_{t2}, \qquad D_{n1} = D_{n2} \tag{3.46}$$

Example 3.3.3 (Parallel Plate Capacitor with 2 Dielectrics Inside). Consider now a parallel plate capacitor with surface area S, composed inside of two dielectrics, one thick d_1 with permeability ϵ_1 and one thick d_2 with permeability ϵ_2. If we smear on the plates a charge Q we have that by our previous definitions that D only sees the charge Q but not the polarization charges, that E sees. Since the field is normal to the plates we must have that between the two dielectrics

$$D_1 = D_2 = D$$

And, for what we've seen before

$$D = \sigma = \frac{Q}{S}$$

The potential difference is then

$$\Delta V = E_1 d_1 + E_2 d_2 = D\left(\frac{d_1}{\epsilon_1} + \frac{d_2}{\epsilon_2}\right) = \frac{Q}{S}\left(\frac{d_1}{\epsilon_1} + \frac{d_2}{\epsilon_2}\right)$$

Note that

$$\frac{\Delta V}{Q} = \frac{1}{C} = \frac{d_1}{\epsilon_1 S} + \frac{d_2}{\epsilon_2 S} = \frac{1}{C_1} + \frac{1}{C_2}$$

I.e. this parallel plate capacitor works exactly as a series of two capacitors! From what we've seen before we can write then the potential difference of these "2" capacitors

$$\Delta V_1 = E_1 d_1 = \frac{D}{\epsilon_1} d_1$$
$$\Delta V_2 = E_2 d_2 = \frac{D}{\epsilon_2} d_2$$

Or noting that

$$\Delta V = \frac{\epsilon_1 d_2 + \epsilon_2 d_1}{\epsilon_1 \epsilon_2} D$$

We can write

$$\Delta V_1 = \frac{\epsilon_2 d_1}{\epsilon_2 d_1 + \epsilon_1 d_2} \Delta V$$
$$\Delta V_2 = \frac{\epsilon_1 d_2}{\epsilon_2 d_1 + \epsilon_1 d_2} \Delta V$$

§ 3.4 Electrostatic Energy with Dielectrics

We know already that the electrostatic energy in free space is given by the following formula

$$U = \frac{1}{2} \iiint_{\mathbb{R}^3} \rho V \, \mathrm{d}^3 x$$

In presence of dielectrics this still holds if we consider that $\rho = \rho_b + \rho_f$ where ρ_b, ρ_f are the bound polarization charges and the free charges respectively.
Remembering that $\partial_i D^i = \rho$ and integrating by parts, we get for a volume V

$$U = \frac{1}{2} \iiint_V \frac{\partial}{\partial x^i} \left(D^i V\right) \mathrm{d}^3 x - \iiint_V D^i \partial_i V \, \mathrm{d}^3 x \tag{3.47}$$

Sending $V \to \mathbb{R}^3$ we get that the first integral is zero (it becomes a surface integral with Stokes' theorem and goes to 0), therefore, for a dielectric, remembering that $-\partial_i V = E_i$

$$U = \frac{1}{2} \iiint_{\mathbb{R}^3} D^i E_i \mathrm{d}^3 x \tag{3.48}$$

Which implies that the volumetric energy density for a dielectric is

$$u = \frac{1}{2} D^i E_i \tag{3.49}$$

For a perfect isotropic dielectric $D^i = \epsilon E^i$, therefore

$$u = \frac{1}{2} D^i E_i = \frac{\epsilon}{2} E^2 = \frac{1}{2} \frac{D^2}{\epsilon} \tag{3.50}$$

Which, if integrated, give the exact identical result for free space if we substitute $\epsilon_0 \to \epsilon$

Part II

Magnetostatics

4 The Magnetic Field

§ 4.1 Electric Currents

As for all materials (like dielectrics) we can see conductors as crystalline solids with atoms positioned at the nodes of the crystalline reticle, the valence electrons are free to roam on on all the volume of the conductor via thermal agitations.
Classically, this gas of electrons is in thermal equilibrium with kinetic energy

$$\frac{1}{2}m_e\left\langle v^2\right\rangle = \frac{3}{2}kT$$

This implies a classical root mean square velocity of

$$v_t = \sqrt{\langle v^2\rangle} = \sqrt{\frac{3kT}{m_e}} \approx 120\ \frac{\mathrm{km}}{\mathrm{s}}$$

If we add over this an electric field we get an ordinate motion of charges, that we call *current*.
Consider now a conductor where charges are moving and consider a generic section of it, if in this section S in some time interval $\mathrm{d}t$ a charge $\mathrm{d}q$ passes through that, we define the current I as

$$I = \frac{\mathrm{d}q}{\mathrm{d}t} \tag{4.1}$$

The units of current in the SI are therefore

$$[I] = \frac{[Q]}{[t]} = 1\ \frac{\mathrm{C}}{\mathrm{s}} = 1\ \mathrm{A} \tag{4.2}$$

These units are known as *Ampères*.
In a conductor, if a current is present, we have that if we have two sections with different potentials V_A and V_B the work needed to move charges between the plates is

$$\mathrm{d}W = V_A\mathrm{d}q - V_B\mathrm{d}q = I\Delta V\,\mathrm{d}t \implies \frac{\mathrm{d}W}{\mathrm{d}t} = I\Delta V \tag{4.3}$$

This represents the work employed by the electric field for moving the charges.
In general the electric field will induce a force F_e^i, which, for Newton's second law will be

$$F_e^i = qE^i = ma^i \implies a^i = \frac{qE^i}{m} \tag{4.4}$$

Remembering that for electrons $q = -e$ we have that the acceleration will be on the opposite direction from the field. The average velocity of electrons in a time Δt will then be, by definition

$$\left\langle v^i \right\rangle = \frac{1}{2}\Delta v^i = -\frac{eE^i}{2m}\Delta t$$

This, between collisions, that happen on average in a time $\Delta t = \lambda / v_t$, where λ is the mean free path of the electron and v_t is the thermal velocity of the electrons. Therefore

$$\left\langle v^i \right\rangle = -\frac{e}{2m}\frac{\lambda}{\sqrt{\frac{3kT}{m}}}E^i = -e\frac{\lambda}{\sqrt{3kTm}}E^i$$

I.e. $\left\langle v^i \right\rangle \propto E^i$.
Formally for a group of charges q inside a conductor where current is flowing, taken a differential cross-sectional surface $\mathrm{d}s$ we have that the charge flowing through that is, if n is the volumetric density of charges

$$\mathrm{d}Q = qn \left\langle v^i \right\rangle \hat{n}_i \mathrm{d}s\mathrm{d}t$$

We define the *current density* as the vector J^i, where

$$J^i = nq \left\langle v^i \right\rangle$$

Then, we have

$$\mathrm{d}Q = J^i \hat{n}_i \mathrm{d}s\mathrm{d}t \tag{4.5}$$

Therefore, the charge through this flux tube is

$$\mathrm{d}I = \frac{\mathrm{d}Q}{\mathrm{d}t} = J^i \hat{n}_i \mathrm{d}s$$

Integrating, we get

$$I = \int \mathrm{d}I = \oiint_S J^i \hat{n}_i \mathrm{d}s \tag{4.6}$$

It's clear that the current density then has the following units

$$[J] = [n][q][\langle v \rangle] = \ \mathrm{m}^{-3} \cdot \mathrm{C} \cdot \mathrm{ms}^{-1} = \ \frac{\mathrm{A}}{\mathrm{m}^2}$$

Considering that the average velocity of the charges is $\langle v \rangle \propto E$, since it's collinear with the E^i field (with direction depending on the sign of the charge) we have that with a special kind of conductors (known as *Ohmic conductors*) we can define what is known as *Ohm's law* which ties directly current flow density to the electric field applied

$$J^i = \sigma E^i \tag{4.7}$$

σ is known as the *electric conductibility*, and indicates how much one is able to move charges applying an electric field

§§ 4.1.1 Charge Conservation

Consider a bounded volume V with inside it some variable charge $Q(t)$. Since charge doesn't pop up randomically into existence (yet) we have that if $Q(t)$ changes there must be some charge going inside or coming outside V depending from the current itself. Then, if the current goes *out*, the charge must *reduce*. Then

$$-\mathrm{d}Q = I\mathrm{d}t = \oiint_{\partial V} J^i \hat{n}_i \mathrm{d}s\mathrm{d}t \tag{4.8}$$

In general we can write Q as a volume integral, and deriving the previous equation with respect to time, we have (bringing inside the integral the time derivative)

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = \iiint_V \frac{\partial \rho}{\partial t}\mathrm{d}^3x = -\oiint_{\partial V} J^i \hat{n}_i \mathrm{d}s$$

Using Stokes' theorem and bringing everything to the left then we have that

$$\iiint_V \left(\frac{\partial \rho}{\partial t} + \frac{\partial J^i}{\partial x^i}\right)\mathrm{d}^3x = 0 \tag{4.9}$$

This implies the *continuity equation*

$$\partial_i J^i + \partial_t \rho = 0 \tag{4.10}$$

We define now *stationary currents* as currents for which $\partial_t \rho = 0$, i.e., where the continuity equation becomes

$$\partial_i J^i = 0 \tag{4.11}$$

§ 4.2 Lorentz Force and Magnetic Fields

§§ 4.2.1 Lorentz Force Law

Consider a small (closed) test circuit where there is a stationary current and a mobile rod long $\mathrm{d}l$ with its tangent in the same direction of I. If we put it in a permanent magnetic field or next to cables where current is passing through we have that it will experience a force $\mathrm{d}F$ such that

$$\mathrm{d}F \propto I, \qquad \mathrm{d}F \propto \mathrm{d}l, \qquad \mathrm{d}F^i \perp \hat{t}^i \mathrm{d}l \tag{4.12}$$

Noting that there is a direction for which $F = 0$ which coincides with the direction of the magnetic needle in that point we can write what is known as *Lorentz's force law*, which introduces a new field, the *magnetic induction field* B^i which is oriented as the magnetic needle in the point considered

$$\mathrm{d}F^i = I\epsilon^i{}_{jk}\hat{t}^j B^k \mathrm{d}l \tag{4.13}$$

Rewriting $I\hat{t}^i\mathrm{d}l$ in terms of the current density J^i we have

$$I\hat{t}^i\mathrm{d}l = J\hat{t}^i\mathrm{d}S\mathrm{d}l = J^i\mathrm{d}^3x$$

Therefore

$$\mathrm{d}F^i = \epsilon^i{}_{jk}J^j B^k \mathrm{d}^3x \tag{4.14}$$

Or, writing $J^i = nqv^i$ and integrating

$$F^i = \iiint_V \epsilon^i{}_{jk} J^i B^k \mathrm{d}^3x = q \iiint_V \epsilon^i{}_{jk} v^i B^k n \mathrm{d}^3x$$

Writing again $n\mathrm{d}^3x = \mathrm{d}N$ we have then integrating the final and better known shape of Lorentz's force law

$$F^i = q\epsilon^i{}_{jk} v^j B^k \tag{4.15}$$

Note that by this definition, we have that *magnetic forces do no work*. In fact we have

$$\mathrm{d}W = F^i \mathrm{d}x_i = q\epsilon^i{}_{jk} v_i v^j B^k \mathrm{d}t = 0 \tag{4.16}$$

Note that this is obvious by the definition of the triple product $\epsilon^i{}_{jk} v_i v^j B^k$. In boldface notation this can be written as a determinant of the matrix which has as colums the vectors v^i, v^j, B^k, therefore since two colums are equal, it must be zero by definition of the determinant.
What's the unit for magnetic fields in the SI system?

$$[B] = \left[\frac{F}{qv}\right] = \frac{\mathrm{kg}\cdot\mathrm{m}\cdot\mathrm{s}^{-2}}{\mathrm{C}\cdot\mathrm{m}\cdot\mathrm{s}^{-1}} = \mathrm{kg}\cdot\mathrm{C}^{-1}\mathrm{s}^{-1} = \frac{\mathrm{Vs}}{\mathrm{m}^2} = \frac{\mathrm{Wb}}{\mathrm{m}^2} = \mathrm{T} \tag{4.17}$$

The unit Wb is known as *Weber* and it's a measure of *magnetic flux*, while the one on the right measures the intensity of the field and is known as *Tesla*.
The *total* force on an electric charge can then be written as a sum of the Coulomb force and the Lorentz force, where

$$F^i = qE^i + q\epsilon^i{}_{jk} v^j B^k \tag{4.18}$$

It's clear that if $v = 0$ there won't be any Lorentz force, and therefore B^i acts only on moving charges. Do another supposition: Take a set of moving charges with current I. If I do a relativistic frame change and I move to a frame where the charges have $v = 0$ then B disappears magically. It means that there must be an *electromagnetic field* which transforms covariantly with respect to Lorentz transformations, which can be seen as a mix of magnetic and electric fields.
Basically, the magnetic field can be seen as some kind of "relativistic effect" on charges

§§ 4.2.2 Motion of Charges in Magnetic Fields

Consider a charge with some velocity v in a magnetic field, we have then that in absolute value we have a *centripetal* force

$$F = qvB$$

Since the force is centripetal the motion will be circular on a plane, take it as the xy plane. In this plane

$$\frac{mv^2}{R} = qvB \implies \frac{v}{R} = \frac{qB}{m} = \omega$$

ω is the frequency of the circular motion, and the radius is

$$R = \frac{mv}{qB}$$

With period $T = 2\pi m/qB$ independent from the speed of the particle, and momentum $p = mv = qBR$. If the initial velocity goes along the z axis it's clear that the motion then is elicoidal.

Let's write this properly. Writing $v \to v^i$ in vector form, and $B^i = B\delta^i_3$ we have evaluating the cross product (use your preferred method, note that using Levi-Civita symbols it's easy to find when it's positive in 3 dimensions. Knowing that it's positive for even permutations, you can write 12312 as the first row and then writing 3 subsequent numbers removing the first one of the row. You'll get then three rows with values 123,231,312 which are all the positive elements of $\epsilon^i{}_{jk}$. obviously the negative ones will be 132,213,321 and the cross product vector can be then be written easily in terms of the components of the product of the two vectors multiplied, $v^i B^j$ in our case. Another way is to write the two vectors one over another and then work more or less like if it's a determinant starting from above. When we hide the x component of the first vector we will have the x component of the product, when we hide the second we will have *minus* the y component and at last the z coponent. Another way is to write a determinant in vector quaternion notation as follows

$$\epsilon^i{}_{jk} a^j b^k \to \det \left| \begin{pmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ a^i & a^2 & a^3 \\ b^1 & b^2 & b^3 \end{pmatrix} \right|$$

Remember that each way is valid as long as you get the right answer, especially for cross products which have their particularities) After this long parenthesis, we calculate our cross product with our favorite method, and we get, (Remember that $B^i \parallel z$, i.e. $B^i = B\delta^i_3$ this can ease calculations)

$$m\frac{\mathrm{d}v^i}{\mathrm{d}t} = q\epsilon^i{}_{jk} v^j B^k = qB\epsilon^i{}_{jk} v^j \delta^k_3 \to qB \begin{pmatrix} v_y \\ -v_x \\ 0 \end{pmatrix}$$

Note that $\frac{\mathrm{d}v^3}{\mathrm{d}t} = 0$ since the B field is acting only in the xy plane. Then

$$\begin{pmatrix} \frac{\mathrm{d}v^x}{\mathrm{d}t} \\ \frac{\mathrm{d}v^y}{\mathrm{d}t} \\ \frac{\mathrm{d}v^z}{\mathrm{d}t} \end{pmatrix} = \frac{qB}{m} \begin{pmatrix} v^y \\ -v^x \\ 0 \end{pmatrix} \tag{4.19}$$

Rewriting everything as a 2D problem in the xy plane and remembering that $v_z = v_0$ is constant and equal to the initial value of the velocity, we have

$$\begin{pmatrix} \frac{\mathrm{d}v^x}{\mathrm{d}t} \\ \frac{\mathrm{d}v^y}{\mathrm{d}t} \end{pmatrix} = \frac{qB}{m} \begin{pmatrix} v^y \\ -v^x \end{pmatrix} \tag{4.20}$$

From (4.20), deriving again we get

$$\begin{pmatrix} \frac{\mathrm{d}^2v^x}{\mathrm{d}t^2} \\ \frac{\mathrm{d}^2v^y}{\mathrm{d}t^2} \end{pmatrix} = \frac{qB}{m} \begin{pmatrix} \frac{\mathrm{d}v^y}{\mathrm{d}t} \\ -\frac{\mathrm{d}v^x}{\mathrm{d}t} \end{pmatrix}$$

Connecting this one to (4.20) we have two identical differential equations

$$\begin{pmatrix} \frac{\mathrm{d}^2v^x}{\mathrm{d}t^2} \\ \frac{\mathrm{d}^2v^y}{\mathrm{d}t^2} \end{pmatrix} = \left(\frac{qB}{m}\right)^2 \begin{pmatrix} \frac{\mathrm{d}v^x}{\mathrm{d}t} \\ \frac{\mathrm{d}v^y}{\mathrm{d}t} \end{pmatrix} \tag{4.21}$$

Solving these equations and inserting the initial values we have the final solutions

$$\begin{pmatrix} v^x \\ v^y \\ v^z \end{pmatrix} = \begin{pmatrix} v_{0\perp} \cos(\omega t) \\ -v_{0\perp} \sin(\omega t) \\ v_0 \end{pmatrix} \tag{4.22}$$

And this is the equation of an elicoidal motion, as we expected.

§§ 4.2.3 Mechanical Actions on Circuits

We know already a formula for magnetic forces, given from Lorentz's force law. In its differential form we have *Laplace's second formula*

$$\mathrm{d}F^i = I\epsilon^i{}_{jk}\mathrm{d}l^j B^k \tag{4.23}$$

Where $\mathrm{d}l^i = \hat{t}^i \mathrm{d}l$.
We always consider here *steady* currents, therefore in all our circuits there will be current generators that keep I constant. For a rigid circuit we have that the total force exerted on the charges is

$$F^i = I\oint \epsilon^i{}_{jk}\mathrm{d}l^j B^k = I\oint \epsilon^i{}_{jk}\hat{t}^j B^k \mathrm{d}l \tag{4.24}$$

Note that if B is uniform then, for a closed circuit the total force exerted on the charges is 0

$$F^i = I\oint \epsilon^i{}_{jk}\hat{t}^j B^k \mathrm{d}l = I\epsilon^i{}_{jk}B^k \oint \hat{t}^j \mathrm{d}l = 0$$

Since

$$\oint \hat{t}^i \mathrm{d}l = 0$$

$$M^i = I\oint \epsilon^i{}_{jk} r^j \epsilon^k{}_{lm}\hat{t}^l B^m \mathrm{d}l \tag{4.25}$$

Note that it's the usual classical formula $\mathbf{M} = \mathbf{r} \times \mathbf{F}$ extended to the whole circuit

Example 4.2.1 (A Closed Circuit). Consider a closed rectangular rigid circuit with a current I, submerged in an uniform B field.
If the rectangle has sides a and b where $a \perp b$ and the sides b are the ones "above" and "below". If the circuit isn't normal to the B field, and denominating with F_1, F_4 the forces acting on the sides long a, while F_3, F_1 the ones acting on the sides long b, we have

$$F_1 = F_3 = BIb\cos\theta$$

(Remember that the total current on one cable is Il where the cable is long l)
While

$$F_2 = F_4 = BIa$$

Using the right hand rule for cross products we know then that F_1, F_3 must be coplanar and therefore don't contribute to a force couple, while F_2, F_4 do. The angular momentum M^i depends on the orientation of B, and therefore we car write

$$M = BIab\sin\theta = BIS\sin\theta$$

Where $S = ab$ is the surface area. Therefore, rewriting $B\sin\theta = \|\mathbf{B} \times \hat{\mathbf{n}}\|$ we have

$$M^i = IS\epsilon^i{}_{jk}\hat{n}^j B^k = \epsilon^i{}_{jk}m^j B^k$$

Via analogy we define the *magnetic dipole moment* of the circuit m^i as

$$m^i = IS\hat{n}^i \tag{4.26}$$

§ 4.3 Maxwell's Equations for Magnetostatics

§§ 4.3.1 Biot-Savart's Law

In general, experimentally it has been found that the magnetic field follows the right hand rule and depends in magnitude from the current density and the inverse of the distance from the magnetic object. Experimentally it has been found that for a wire with static current density $\mathrm{d}I^i = I\mathrm{d}l^i = I^i\mathrm{d}l$ the magnetic field is

$$B^i = \frac{\mu_0}{4\pi}\int_\lambda \frac{\epsilon^i{}_{jk} I^j \hat{r}^k}{r^2}\mathrm{d}l \tag{4.27}$$

This equation is known as the law of *Biot and Savart*. The constant μ_0 is a coupling constant known as the *permeability of free space* and has units of Newton/Ampere squared. It has the following (exact) value in the SI system

$$\mu_0 = 4\pi\cdot 10^{-7}\ \frac{\mathrm{N}}{\mathrm{A}^2} \tag{4.28}$$

Note that this holds dimensionally with the B from what we found before from Lorentz's force law.

Example 4.3.1 (A Live Wire). Consider a wire along the x axis where a stationary current goes from right to left. Using Biot-Savart we can find the field B^i at a distance s from the wire. We have that for the symmetries of the system we're working on a 2D plane xy. The radius vector from the current to the point where we're evaluating the field describes a triangle in this plane, we indicate the angle between the height of the triangle (s) and the radius vector r as θ. Using trigonometry we have that the length of the basis of this triangle is $l = s\tan\theta$, therefore

$$\mathrm{d}l = \frac{s}{\cos^2\theta}\mathrm{d}\theta$$

Also, we have that in this triangle considering the angle α between the current and the radius vector we have that the cross product of the two has the following magnitude

$$\|\mathrm{d}\mathbf{l}\times\hat{\mathbf{r}}\| = r\sin\alpha\mathrm{d}l = r\frac{\sin\alpha}{\cos^2\theta}\mathrm{d}\theta$$

Using the fact that we're in an euclidean triangle (duh) we must have that $\alpha+\theta+\pi/2=\pi$, therefore $\alpha=\pi-\theta$. Still using trigonometry we can also say that

$$s = r\cos\theta \implies r = \frac{s}{\cos\theta} \implies \frac{1}{r^2} = \frac{\cos^2\theta}{s^2}$$

From Biot-Savart then

$$\mathrm{d}B = \frac{\mu_0}{4\pi}\frac{\cos\theta}{r^2}\mathrm{d}l = \frac{\mu_0\cos\theta}{4\pi}\left(\frac{s}{\cos^2\theta}\frac{\cos^2\theta}{s^2}\right)\mathrm{d}\theta$$

Therefore

$$\mathrm{d}B = \frac{\mu_0\cos\theta}{4\pi s}\mathrm{d}\theta \tag{4.29}$$

Considering that θ goes from some θ_0 to some θ_1 if the wire is finite, and $\theta_0=-\pi/2, \theta_1=\pi/2$ for an *infinite* wire, we have in the second case

$$B(s) = \frac{\mu_0}{4\pi s}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos\theta\mathrm{d}\theta = \frac{\mu_0}{2\pi s} \tag{4.30}$$

Using the right hand rule we can also find the direction of B, being always normal to both r and I we have that in 3 dimensions, using cylindrical coordinates

$$B^i = \frac{\mu_0}{2\pi s}\hat{\varphi}^i \tag{4.31}$$

§§ 4.3.2 Divergence and Curl of B

Consider now an infinite wire with a static current I^i going through it. If we take a closed loop C around the wire it's clear that if we integrate B in that loop the integral can't be 0. From Biot-Savart we have, using what we found previously for infinite wires

$$B = \frac{\mu_0 I}{2\pi r}\oint_C \mathrm{d}l = \mu_0 I \tag{4.32}$$

In general it's clear that for some loop C around multiple currents $I_{(i)}$, we will have that as for Gauss' law for electric fields with charges, that

$$\oint_C B^i \hat{t}_i \mathrm{d}l = \mu_0 I_{in} \tag{4.33}$$

Where I_{in} is the total current inside the loop. If we have a current density J^i it's clear that then,

$$I_{in} = \iint_S J^i \hat{n}_i \mathrm{d}s$$

Where S is the surface such that $\partial S = C$. Reconnecting and using Stokes' theorem on B we have

$$\iint_S \epsilon^i{}_{jk}\partial^j B^k \hat{n}_i \mathrm{d}s = \mu_0 \iint_S J^i \hat{n}_i \mathrm{d}s \tag{4.34}$$

We get then that *for straight wires*

$$\epsilon^i{}_{jk}\partial^j B^k = \mu_0 J^i \tag{4.35}$$

What about the divergence of B? We begin again with Biot-Savart law in a general setting with a volume current J^i. We have

$$B^i = \frac{\mu_0}{4\pi}\iiint_V \frac{1}{r^2}\epsilon^i{}_{jk}J^j \hat{r}^k \mathrm{d}^3x$$

We then apply the divergence operator to both sides

$$\partial_i B^i = \frac{\mu_0}{4\pi}\iiint_V \frac{\partial}{\partial x^i}\left(\epsilon^i{}_{jk}J^j\frac{\hat{r}^k}{r^2}\right)\mathrm{d}^3x$$

Using the antisymmetry of $\epsilon^i{}_{jk}$ we can rewrite the divergence of the cross product as follows

$$\frac{\partial}{\partial x^i}\epsilon^i{}_{jk}J^j\frac{\hat{r}^k}{r^2} = \frac{\hat{r}_i}{r^2}\epsilon^i{}_{jk}\partial^j J^k - J_i\epsilon^i{}_{jk}\partial^j\left(\frac{\hat{r}^k}{r^2}\right)$$

Remember: J^i doesn't depend from how far we're measuring the field, therefore $\partial_i J^k = 0$! The first part on the right hand side of the previous equation is zero, therefore we have

$$\frac{\partial}{\partial x^i}\epsilon^i{}_{jk}J^j\frac{\hat{r}^k}{r^2} = -J_i\epsilon^i{}_{jk}\partial^j\left(\frac{\hat{r}^k}{r^2}\right) = 0 \tag{4.36}$$

Where we used that the curl of $\hat{r}^i/r^2$ is 0 (it's the gradient of another function). Therefore we have that

$$\partial_i B^i = 0 \tag{4.37}$$

Note that this holds everywhere since we started from a generic current distribution! We can also recompute the curl of B from this generic current distribution. Remembering again that J depends only on the local coordinates of the distribution itself and not from how far we're measuring B we have, firstly, using the identities of $\epsilon^i{}_{jk}$, and juggling a bit

$$\epsilon^i{}_{jk}\partial^j\epsilon^k{}_{lm}J^l\frac{\hat{r}^m}{r^2} = J^i\partial_l\left(\frac{\hat{r}^l}{r^2}\right) - \left(J^l\partial_l\right)\frac{\hat{r}^i}{r^2} \tag{4.38}$$

Using the definition of the 3D Dirac delta for evaluating the first divergence we have

$$\epsilon^i{}_{jk}\epsilon^k{}_{lm}\partial^j J^l\frac{\hat{r}^m}{r^2} = 4\pi\delta^3(r)J^i - \left(J^l\partial_l\right)\frac{\hat{r}^i}{r^2} \tag{4.39}$$

Looking closely at the second side, we have

$$\frac{\partial}{\partial x^i}\left(J^i\frac{\hat{r}^j}{r^2}\right) = \frac{\hat{r}^j}{r^2}\partial_i J^i + \left(J^i\partial_i\right)\frac{\hat{r}^j}{r^2}$$

Noting that for steady currents $\partial_i J^i = 0$ we have then

$$\left(J^l\partial_l\right)\frac{\hat{r}^i}{r^2} = \frac{\partial}{\partial x^j}\left(J^i\frac{\hat{r}^j}{r^2}\right)$$

Reinserting everything into the definition of B^i we have

$$\epsilon^i{}_{jk}\partial^j B^k = \mu_0\iiint_V J^i\delta^3(r)\mathrm{d}^3x + \frac{\mu_0}{4\pi}\oiint_{\partial V}\frac{\hat{r}^i}{r^2}J^j\hat{n}_j\mathrm{d}s \tag{4.40}$$

The surface integral is safely equal to zero since all currents are safely inside the surface (by definition), and collapsing the first integral we have

$$\epsilon^i{}_{jk}\partial^j B^k = \mu_0 J^i \tag{4.41}$$

This relation is *general* and holds for all current distributions J^i. This is known formally as *Ampere's law for Magnetostatics*.

Analogously to Gauss' law for E, we have some tricks for evaluating B using Ampere's law. We draw some convenient closed loops for which B comes out easily from the integral. As we have Gaussian surfaces, we have for B *Amperian loops*.

Integrating both sides of (4.41) we have applying Stokes that if we chose a nice enough loop, i.e. one where B is parallel to the tangent of the loop it will come out from the integral and we will have on the right $\mu_0 I_{in}$, easing all calculations

§§ 4.3.3 The Parallel Between Magnetostatics and Electrostatics

We have now found 4 Maxwell equations, two for static charges and two for static currents. These equations are, in free space

$$\begin{cases} \partial_i E^i = \dfrac{\rho}{\epsilon_0} \\ \epsilon^i{}_{jk}\partial^j E^k = 0 \\ \partial_i B^i = 0 \\ \epsilon^i{}_{jk}\partial^j B^k = \mu_0 J^i \end{cases} \tag{4.42}$$

These equations are asymmetric! If we made a symmetric parallel between the two and supposed $\partial_i B^i = \mu_0 \rho_m$ where ρ_m is a "magnetic charge", the magnetic counterpart to the electric charge, these equation would be perfectly symmetric. The fact that $\partial_i B^i = 0$ always shows that magnetic charges, or magnetic monopoles, cannot exist in nature due to the actual shape of B.
This particular parallel, lets us think that there might also be a potential for B. We are already sure that it can't be a scalar potential in general, since the curl of B is nonzero.

§§§ 4.3.3.1 Magnetic Potentials

Let's look back at Biot-Savart's law, we have

$$B^i = \frac{\mu_0}{4\pi}\iiint_V \frac{1}{r^2}\epsilon^i{}_{jk}J^j\hat{r}^k \mathrm{d}^3x = \frac{\mu_0}{4\pi}\iiint_V \epsilon^i{}_{jk}J^j\partial^k\left(\frac{1}{r}\right)\mathrm{d}^3x$$

Using $\boldsymbol{\nabla}\times \mathsf{J} = 0$ we can use a little property of the epsilon symbol and bring outside the curl. We have

$$B^i = \frac{\mu_0}{4\pi}\epsilon^i{}_{jk}\frac{\partial}{\partial x_j}\iiint_V \frac{J^k}{r}\mathrm{d}^3x \tag{4.43}$$

I.e. we wrote B in terms of the curl of some other vector. This vector is the *magnetic vector potential* A^i and, writing

$$A^i = \frac{\mu_0}{4\pi}\iiint_V \frac{J^i}{r}\mathrm{d}^3x \tag{4.44}$$

It's clear that since the curl of a gradient is zero, this potential is defined up to gradients of continuous functions.
We will usually choose the *Coulomb Gauge*, where $\partial_i A^i = 0$, this means that since

$$A^i = A^i_0 + \partial^i\lambda \implies \partial_i A^i = 0 \iff \partial^i\partial_i\lambda = -\partial_i A^i_0$$

This is a Poisson equation and the solution is easily

$$\lambda(r) = \frac{1}{4\pi}\iiint_V \frac{\partial_i A^i_0}{r}\mathrm{d}^3x$$

In this potential formulation, if we take Ampere's law we have

$$\epsilon^i{}_{jk}\partial^j B^k = \epsilon^i{}_{jk}\partial^j\epsilon^k{}_{lm}\partial^l A^m = \partial_i\partial_l A^l - \partial^l\partial_l A^i = \mu_0 J^i \tag{4.45}$$

Taking the previous gauge choice we have 3 Poisson equations for each component of A

$$\partial_l \partial^l A^i = -\mu_0 J^i \tag{4.46}$$

If $J(r) \to 0$ we have then that the solution is, as before

$$A^i = \frac{\mu_0}{4\pi} \iiint_V \frac{J^i}{r} \mathrm{d}^3 x$$

§§ 4.3.4 Boundary Conditions for the Magnetic Field

Going back to electrostatic boundary conditions, we have that E is discontinuous on the surface *charge*. Instead, B, by analogy of the formulas is discontinuous on surface *currents*.
Checking again Maxwell's equations, we must have that if E is discontinuous on the *normal* to the surface, B must be discontinuous on the *tangent*, in fact, taken a pillbox P that goes through this surface current, then, since $\partial_i B^i = 0$ *always* we have

$$\iint_P B^i \hat{n}_i \mathrm{d}s = 0 \tag{4.47}$$

This immediately implies, noting that the normal vector above is opposite to the normal vector below the surface, that

$$B^i \hat{n}_i^{ab} = B^i \hat{n}_i^{bel} \tag{4.48}$$

I.e., the normal components of the magnetic field are continuous.
Take now an Amperian loop that goes through the surface. Then, by Ampere's law we get

$$\oint_A B^i \hat{t}_i \mathrm{d}l = \mu_0 I_{enc}$$

Where $I_{enc} = KI$ if K is our surface current density. Therefore, expanding the integral

$$B^i \hat{t}_i^{ab} - B^i \hat{t}_i^{bel} = \mu_0 K \tag{4.49}$$

Note that this holds since if we check the parallel components of the tangent of this loop, we must have $I_{enc} = 0$. Both these conditions can be summarized into a single equation.
Let K^i be the surface current density, then if we measure the magnetic field above (B^i_{ab}) and below B^i_{bel}, it must be continuous passing through the current surface and discontinuous on its tangent, i.e.

$$B^i_{ab} - B^i_{bel} = \mu_0 \epsilon^i{}_{jk} K^j \hat{n}^j \tag{4.50}$$

§ 4.4 Magnetic Dipoles

As for electrostatic potentials, it's possible to expand the magnetic potential into multipoles.
Remembering that if $r^i = r^i - \tilde{r}^i$, we can write r^{-1} in terms of Legendre functions

$$\frac{1}{r} = \frac{1}{\sqrt{r^2 + (\tilde{r})^2 - 2r\tilde{r}\cos\theta}} = \sum_{l=0}^{\infty} \left(\frac{\tilde{r}}{r}\right)^2 P_l(\cos\theta)$$

Therefore, for any Amperian loop A with current $I\hat{t}^i$ we have

$$A^i(r) = \frac{\mu_0 I}{4\pi}\sum_{l=0}^{\infty}\frac{1}{r^{l+1}}\oint_A (\tilde{r})^l\, P_l(\cos\theta)\hat{t}^i \mathrm{d}l \tag{4.51}$$

Expanding till quadrupoles ($l = 2$) we then can write

$$A^i \approx \frac{\mu_0 I}{4\pi r}\oint_A \hat{t}^i \mathrm{d}l + \frac{\mu_0 I}{4\pi r^2}\oint_A \tilde{r}\cos\theta\hat{t}^i \mathrm{d}l + \frac{\mu_0 I}{4\pi r^3}\oint_A \frac{(\tilde{r})^2}{2}\left(3\cos\theta - 1\right)\hat{t}^i \mathrm{d}l \tag{4.52}$$

Note that the monopole term ($l = 0$) is zero, as we expected! Again, there are no magnetic monopoles. We concentrate on the $l = 1$ term, the dipole term. We have then that the magnetic dipole is

$$A^i_{dip}(r) = \frac{\mu_0 I}{4\pi r^2}\oint_A \tilde{r}\cos\theta\hat{t}^i \mathrm{d}l \tag{4.53}$$

Remembering that by definition $\hat{r}^i\hat{\tilde{r}}_i = \cos\theta$ we can write everything in terms of scalar products. Since

$$\tilde{r}\cos\theta = \tilde{r}\hat{r}$$

We can write the integral in terms of a surface integral. Call S the surface enclosed by the loop, then

$$A^i_{dip}(r) = \frac{\mu_0 I}{4\pi r^2}\oint_A (\tilde{r}^j\hat{r}_j)\hat{t}^i \mathrm{d}l = \frac{\mu_0 I}{4\pi r^2}\left(-\epsilon^i{}_{jk}\hat{r}^j\iint_S \hat{n}^k \mathrm{d}s\right)$$

Remembering the definition of magnetic dipole, and switching the indexes on the cross product, we have

$$A^i_{dip}(r) = \frac{\mu_0}{4\pi r^2}\epsilon^i{}_{jk}m^j\hat{r}_k \tag{4.54}$$

Look how this is clearly equivalent to the field generated by a small dipole m^i generated by a current loop on a circuit A.

§§ 4.4.1 Interaction Between Currents

Consider two rigid circuits with some current I_i, $i = 1, 2$. Suppose these two circuits are small enough so that we can consider them as two magnetic dipoles.
In the second circuit, there is a force $\mathrm{d}F_{21}$ generated from the B_1 field of the first circuit. We have

$$\mathrm{d}F^i_{21} = I_2\epsilon^i{}_{jk}\mathrm{d}l^j_2 B^k_1 = I_2\epsilon^i{}_{jk}\mathrm{d}l^j_2\oint_1 \frac{\mu_0 I_1}{4\pi r^2_{12}}\epsilon^k{}_{lm}\mathrm{d}l^l_1\hat{r}^k_{12}$$

Where we took r_{12} as the distance between the two circuit elements $\mathrm{d}l_i$. The total force will be then, integrating

$$F^i_{12} = \frac{\mu_0 I_1 I_2}{4\pi}\oint_1\oint_2 \frac{1}{r^2_{12}}\epsilon^i{}_{jk}\epsilon^k{}_{lm}\mathrm{d}l^j_2\mathrm{d}l^l_1\hat{r}^m_{12} = \frac{\mu_0 I_1 I_2}{4\pi}\oint_1\oint_2\left(\mathrm{d}l^i_1\left(\frac{\hat{r}^l_{12}}{r^2_{12}}\mathrm{d}l^2_l\right) - \frac{\hat{r}^i_{12}}{r^2_{12}}\mathrm{d}l^l_1\mathrm{d}l^2_l\right) \tag{4.55}$$

Where we used the properties of $\epsilon^i{}_{jk}$ for computing the products. On the right hand side the first term can be seen as the curl of a vector (with respect to the 2nd line integral) integrated on the surface enclosed by the second loop. This curl is zero, and therefore we have finally

$$F_{12}^i = -\frac{\mu_0 I_1 I_2}{4\pi} \oint_1 \oint_2 \frac{\hat{r}_{12}^i}{r_{12}^2} \mathrm{d}l_1^l \mathrm{d}l_l^2 \tag{4.56}$$

Note that $F_{12} = -F_{21}$ simply because $r_{12} = -r_{21}$! Therefore what we found respects Newton's third law.
Consider now two infinite parallel wires with the same currents, we have that if $\mathrm{d}l_l^1 \mathrm{d}l_2^l \geq 0$ or $\mathrm{d}l_l^1 \mathrm{d}l_2^l \leq 0$ (i.e. the two currents have the same direction or opposite direction), we will get wither an attractive force or a repulsive force.
It's clear that this force will be orthogonal to the wires (see Lorentz's force law) and therefore we're interested to check only the orthogonal component.
If the two parallel wires are distant a from each other we can write immediately $r_{12}\cos\theta = a$, and $\mathrm{d}l_1 = \frac{a}{\cos^2\theta}\mathrm{d}\theta$. Noting that the two wires are infinite, using the transformation $\mathrm{d}l_1 \to \mathrm{d}\theta$ we get (dividing by $\mathrm{d}l_2$)

$$\frac{\mathrm{d}F_{21}^\perp}{\mathrm{d}l_2} = -\frac{\mu_0 I_1 I_2}{4\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{\cos^2\theta}{a^2}\right) \cos\theta \left(\frac{a}{\cos^2\theta}\mathrm{d}\theta\right) = -\frac{\mu_0 I_1 I_2}{4\pi a} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\theta \mathrm{d}\theta$$

Integrating, we immediately get

$$\frac{\mathrm{d}F_{12}^\perp}{\mathrm{d}l_2} = -\frac{\mu_0 I_1 I_2}{2\pi a} \tag{4.57}$$

Note this is negative only if $\mathrm{d}l_1^l \mathrm{d}l_l^2 > 0$, i.e. when the currents are parallel.

5 Magnetism in Matter

If we insert some material in a region where there is a B field there are three observed effects

1. Mechanical forces on the body are observed
2. The field around the bodies is modified by their presence
3. The bodies can be *magnetized*, i.e. they behave like a magnet

If we take as our experimental test field the one produced by a solenoid (a conductive spring where charges move in a closed loop) it can be verified immediately that all substances are distinguishable in three categories

1. Ferromagnets, which get attracted by the B field of the solenoid
2. Paramagnets, which get *weakly* attracted by the field
3. Diamagnets, which get *weakly* repulsed by the field

All these different behaviors are directly correlated from macroscopic proprieties.
Atoms themselves can be thought as small loop circuits (imagine electrons "going around" the nucleus), and therefore generate some magnetic dipole m^i. These dipoles interact with the field and tend to orient themselves in the same direction as B, i.e. the bodies get *magnetized*.

§ 5.1 Magnetization

The discussion of magnetism in matter is similar to the one on electricity in matter, and therefore it's good practice to begin with a microscopic approach to the problem.
Consider a small Hydrogen atom, one proton and one electron. Since $m_p \approx 2000 m_e$ we can consider the nucleus as locked in place, while electrons move around in a circular orbit with radius r_0.
The electron experiences the following centripetal coulomb force

$$F_c = m_e a_c = \frac{1}{4\pi\epsilon_0}\frac{e^2}{r_0^2}$$

I.e., using $a_c = v_0^2/r_0 = \omega_0^2 r_0$ we get

$$m_e \omega_0^2 r_0 = \frac{1}{4\pi\epsilon_0}\frac{e^2}{r_0^2}$$

Since $\omega_0 = 2\pi T_0^{-1}$ with T_0 being the period of the orbit we get

$$T_0 = \frac{4\pi}{e}\sqrt{\pi\epsilon_0 m_e r_0^3}$$

All these calculations are needed to find the magnetic dipole momentum of the electron, m^i. Using a bit of quantum mechanics and remembering that the electron is in a bound state ($E < 0$) we can find r_0 using the ionization energy (i.e. the energy needed to bring the electron to $r = \infty$ with $v = 0$) we have that

$$E = -I_e \implies r_0 = \frac{1}{8\pi\epsilon_0}\frac{e^2}{I} \approx 0.5 \text{ Å}$$

Note that we used *experimental data* (for now). In the setup we made we basically made a toy hydrogen atom, for which $I = 13.6$ eV, where $1 \text{ eV} = 1.6 \times 10^{-19}$ C $\cdot$ V $= 1.6 \times 10^{-19}$ J.
With this we get $T_0 \approx 1.5 \times 10^{-16}$ s as our orbit period, and the current associated with a single electron moving around a proton (a simple toy atom, hydrogen in this case) is

$$I = \frac{e}{T_0} \approx 1 \text{ mA}$$

Using $m^i = IS\hat{n}^i$ we have that the magnetic momentum of this system is

$$m = I\pi r_0^2 = \frac{e\pi r_0^2}{T_0} = 9.35 \times 10^{24} \text{ A} \cdot \text{m}$$

And the angular momentum is

$$L = m_e v_0 r_0 = m_e \frac{2\pi r_0^2}{T_0} \implies \frac{m}{L} = \frac{e}{2m_e}$$

The last constant is known as the *gyromagnetic factor* g of the electron and is a general result also valid in quantum mechanics.
Writing $L = \hbar l$ in a semiclassical fashion (you'll understand later, probably, or you already know) we get a new fundamental constant tied to the gyromagnetic factor g

$$m = gL = \hbar g l = \frac{\hbar e}{2m_e} l = \mu_B l$$

Where μ_B is known as *Bohr's magneton*, for which $\mu_B \approx 9.27 \cdot 10^{-24}$ A $\cdot$ m^2

§§ 5.1.1 The Magnetization Field

After the small "quantum" digression, we can get back to our classical treatment of Electrodynamics. We've seen that all atoms must have a magnetic dipole moment m^i tied to the "orbital" nature of bound electrons in nuclear fields. Analogously to dipole moments in dielectrics this must determine the magnetic properties in matter.
We define the *Magnetization intensity* M^i as follows

$$M^i = \lim_{\Delta V \to 0} \frac{\Delta N}{\Delta V} \langle m^i \rangle \tag{5.1}$$

Where ΔN is the numerical density of atoms.
In SI units we have

$$[M] = \frac{\mathrm{A}}{\mathrm{m}}$$

And rearranging a bit the previous terms, and using $\Delta V \to \mathrm{d}V$

$$\mathrm{d}m^i = M^i \mathrm{d}V \tag{5.2}$$

We begin by considering an uniform magnetization M^i inside a magnetized medium. It's clear that indide the body all the currents will cancel out and we'll be left only with surface effects, which will be magnetization-induced currents that will follow the right hand rule since there's no compensation outside the magnet.
Obviously, if M^i is not uniform, we will also have volumetric currents. Surface currents will be indicated with J_{ms} and volumetric currents with J_{mv}.
Using equation (4.54) we can see the relations between M^i and these currents. remembering equation (5.2) we can write for a magnetized body V

$$A^i = \frac{\mu_0}{4\pi} \int_V \frac{1}{r^3} \epsilon^i{}_{jk} M^j r^k \mathrm{d}^3 x' \tag{5.3}$$

Bringing $1/r^3$ inside the cross product and remembering that $r^i/r^3 = -\partial_i(r^{-1})$ and then applying a simple vector analysis identity ($\mathbf{v} \times \nabla f = f\nabla \times \mathbf{v} - \nabla \times (f\mathbf{v})$) we get two integrals

$$A^i = \frac{\mu_0}{4\pi} \left(\int_V \frac{\epsilon^i{}_{jk} \partial^j M^k}{r} \mathrm{d}^3 x' - \int_V \epsilon^i{}_{jk} \partial^j \left(\frac{M^k}{r} \right) \mathrm{d}^3 x' \right) \tag{5.4}$$

Using $\int_V \nabla \times \mathbf{v} \mathrm{d}V = -\int_{\partial V} \mathbf{v} \times \hat{n} \mathrm{d}s$ on the second one we get

$$A^i = \frac{\mu_0}{4\pi} \int_V \frac{\epsilon^i{}_{jk} \partial^j M^k}{r} \mathrm{d}^3 x' + \frac{\mu_0}{4\pi} \int_{\partial V} \frac{\epsilon^i{}_{jk} M^j n^k}{r} \mathrm{d}s' \tag{5.5}$$

Since the vector potential has an unique solution (it's defined from a Poisson equation with well defined conditions) We can interpret the first curl as our volumetric current density and the second cross product as our surface current densities, giving us the relations between the magnetization currents and the magnetization vector M^i

$$\begin{aligned} J^i_{mv} &= \epsilon^i{}_{jk} \partial^j M^k \\ J^i_{ms} &= \epsilon^i{}_{jk} M^j \hat{n}^k \end{aligned} \tag{5.6}$$

§ 5.2 Maxwell Equations for Magnetostatics in Magnetic Media

Taking back what we found for the B^i field we can try to build up again the Maxwell equation for magnetostatics in magnetized media.
As we already have found we have

$$\begin{aligned} \partial_i B^i &= 0 \\ \epsilon^i{}_{jk} \partial^j B^k &= \mu_0 J^i \end{aligned}$$

We now must consider that J^i indicates the total current, so we will consider it as the sum of "free" extra currents J^i_f and the previously found magnetization currents J^i_m.
Inside the magnetized volume V we can replace J^i_m with the curl of M^i and, bringing it to the left we can write guardando il forno e una

$$\epsilon^i{}_{jk}\partial^j\left(\frac{B^k}{\mu_0} - M^k\right) = J^i$$

We can define an auxiliary field inside this curl, which we will call the "magnetic field" H^i

$$H^i = \frac{B^i}{\mu_0} - M^i \tag{5.7}$$

Rewriting everything, we get Maxwell's equation for magnetostatics in media

$$\begin{cases} \partial_i B^i = 0 \\ \epsilon^i{}_{jk}\partial^j H^k = J^i_f \end{cases} \tag{5.8}$$

These equations can be solved only if we know the functional relations between B and H or M and B, or if we manage to find some conditions that can help us

§§ 5.2.1 Boundary Conditions

In order to solve these equations tho we need to consider what happens at the surface ∂V of the body. Suppose that we have two magnetized bodies separated by a surface S_s. Taken a small loop l on this separation surface, which encompasses a surface S, we can use the second equation of (5.8) we get

$$\oint_l H^i t_i \mathrm{d}l = \iint_S \epsilon^i{}_{jk}\partial^j H^k \hat{n}_i \mathrm{d}s = \iint_S J^i \hat{n}_i \mathrm{d}s = \sum_i I_i \tag{5.9}$$

Therefore, the closed line integral of H^i is the sum of the (free) currents enclosed by the loop.
Considering the same loop in the case where there are no free currents, equations (5.8) give the boundary conditions for B and H in matter.

$$\begin{cases} B_{n_1} = B_{n_2} \\ H_{t_1} = H_{t_2} \end{cases} \tag{5.10}$$

Where n_i, t_i are the normal and tangent components of the field between substance 1 and 2.
By definition of H^i we can see already that in vacuum $B^i_0 = \mu_0 H^i_0$ since $M^i = 0$. As for dielectrics in isotropic and homogeneous substances we can write $B^i = \mu H^i$ with $\mu_0 = \mu_0\mu_r$ where μ_r is the relative magnetic permeability.
For anisotropic substances μ can be described as a rank 2 tensor. Contrary to dielectrics, μ in general depends from the B field intensity, and is constant only for diamagnetic or paramagnetic substances. For ferromagnets $\mu = \mu(B)$.
With this definition, we can calculate the magnetization field of the body. We have $B = \mu H$ therefore

$$H^i = \mu_r H^i - M^i \implies M^i = (\mu_r - 1)H^i = \chi_m H^i \implies \mu M^i = \chi_m B^i \tag{5.11}$$

Where $\mu_r - 1 = \chi_m$ is the magnetic susceptibility.
Inserting that back to the definition of H^i we have

$$B^i = \mu_0 (1 + \chi_m) H^i \tag{5.12}$$

By definition, the value of χ_m defines the alignment of the magnetization with respect to the magnetic field. In general for values of χ_m between 10^{-5} to 10^{-3} we have an orientation of atomic magnetic dipoles and therefore paramagnetism.
For negative values we get diamagnetic effects and for very big positive effects we get ferromagnetic effects

§ 5.3 Ferromagnets and Hysteresis Cycles

In ferromagnets ($\chi_m >> 1$) the dependence $B(H)$ or $M(H)$ is really complex and the relations aren't unique and can change a lot for small changes on composition of the material.
For analyzing it we start with the unmagnetized material ($H = B = M = 0$) and place it inside a solenoid, for which we know already that, thanks to the Maxwell equations that $H = nI$, with n being the number of loops of the solenoid and I being the total current of the solenoid.
Changing I we have that B changes way quicker than H, with a strong contribution from the magnetization of the element through the relation

$$B = \mu_0 H + \mu_0 M$$

The growth is exponential until a saturation H_s value is reached. This growth is known as the "first magnetization curve". After this value the growth of B is linear in H till a maximum H_m due to a saturation in M, which reaches a saturation maximum M_s.
Shutting the current off ($I = 0$) we get to $H = 0$ and a residue magnetic induction field B_r can be measured.
Inverting the current's direction B goes down till 0, for $\mu_0 H = -\mu_0 M$, i.e. the magnetic field H reaches the coercive magnetic field value where $H_c = -M_c$. From here on, the fields quickly reaches a negative minimum at $H = -H_m$.
Making H grow again from the minimum the field B will reach $-B_r$ at $H = 0$ and will reconnect to the first cycle maximum at H_m.

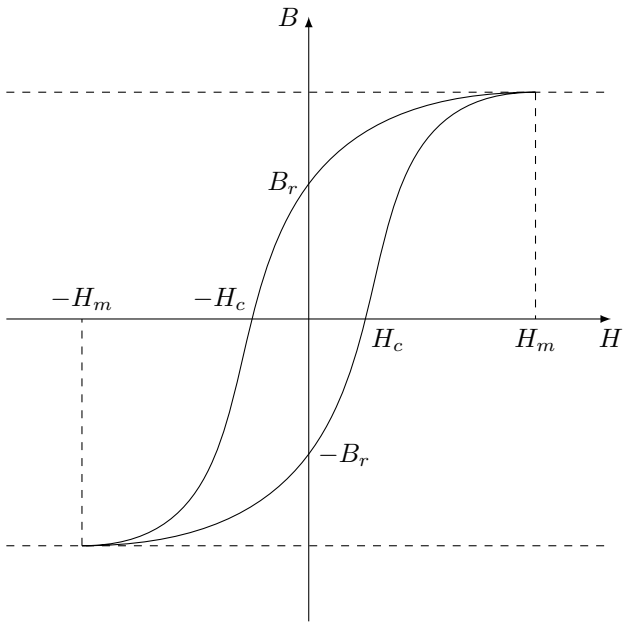

Figure 5.1: Example of an hysteresis cycle, without the first magnetization curve.

This full cycle is known as the magnetic hysteresis cycle and if it's drawn it's clear that $B(H)$ is not a function in the proper sense of it, since the value depends on what happened before to the material, and in general we have that

$$\mu(H) = \frac{B}{H} \tag{5.13}$$

It's also possible to draw a demagnetization cycle on the $B - H$ plane from any point making smaller and smaller hysteresis cycles, and with a simple analogy to $p - V$ planes in thermodynamics, one can calculate the work made per unit volume of the material with the relation $\mathrm{d}W = B\mathrm{d}H$ (remember that in thermodynamics $\mathrm{d}W = p\mathrm{d}V$ when there is no external work acting on the system).
From the relationship we found before for B one can write the differential magnetic permeability of a body as

$$\mu_d = \frac{\mathrm{d}B}{\mathrm{d}H}$$

Or its relative counterpart

$$\mu_{d_r} = \frac{\mu_d}{\mu_0} = \frac{1}{\mu_0}\frac{\mathrm{d}B}{\mathrm{d}H}$$

Another experimental result on ferromagnets is the *law of Curie-Weiss*, which states that for temperatures over a critical value T_c, a ferromagnet becomes a paramagnet, and its susceptibility goes as

$$\chi_m = \frac{k\rho}{T - T_c} \tag{5.14}$$

Where k is a constant and ρ is the material's density

§ 5.4 Local Magnetic Field

For evaluating the local counterpart of the magnetic field, since we can consider ourselves in vacuum, we're free to choose between using B and $H = B/\mu_0$. For notational ease H is "better".
Using the same exact path taken to find the Lorentz local field in dielectrics (3.22) we can say that the magnetic field around some atom, at its center is

$$H^i_{loc} = H^i + \frac{1}{3}M^i \tag{5.15}$$

This local field considers that the contribute of all the small dipoles inside the sphere around the atom sum to zero.
Since for paramagnetic and diamagnetic substances $M << H$ we could even write $H_{loc} \approx H$.
This doesn't hold for ferromagnets, and thanks to Weiss we get a reformulation of the local field

$$H^i_{loc,fm} = H^i + \gamma M^i \tag{5.16}$$

The constant γ is known as Weiss' constant, and $10^3 < \gamma < 10^4$. It has been justified by considering the ferromagnet as divided in multiple sectors where atomic dipoles have zones of common orientations, where the biggest zone is the one oriented with the magnetic field H. The zone engulfs the whole magnet then slowly.

§§ 5.4.1 Larmor Precession

Consider now a single atom, completely unaligned with the field. We have already found that its magnetic moment is

$$m^i_0 = -\frac{e}{2m_e}L^i$$

And its torque is

$$\tau^i = \epsilon^i{}_{jk} m^j_0 B^k_{loc}$$

With B_{loc} being our local B field.
By definition of torque τ we have

$$\frac{\mathrm{d}L^i}{\mathrm{d}t} = \epsilon^i{}_{jk} m^j_0 B^k_{loc} = \frac{e}{2m_e}\epsilon^i{}_{jk} B^j_{loc} L^k = \epsilon^i{}_{jk}\omega^i_L L^k \tag{5.17}$$

The last result gives us the Poisson formula for L, which indicates that it completes a precession motion with angular velocity ω_L, known commonly as Larmor precession. This speed is by definition parallel to the local field, and it's associated to a current given by this precession and the charged nature of the electron

$$I_L = -\frac{e}{T_L} = -\frac{e\omega_L}{2\pi} \tag{5.18}$$

This current is therefore tied to a magnetic moment, for which $m_L = I_L \tilde{S}$ where $\tilde{S}$ is the area of the orbit of the electron projected onto the same direction of the local field. Inserting a bit of numbers in the previous statement we have

$$m^i_L = -\frac{e}{2\pi}\tilde{S}\omega^i_L = -\frac{e^2}{4\pi m_e}\tilde{S}B^i_{loc}$$

Averaging the coordinates of the position of the electron we know already that

$$\langle x\rangle^2 = \langle y\rangle^2 = \langle z\rangle^2$$
$$\langle x\rangle^2 + \langle y\rangle^2 + \langle z\rangle^2 = \langle r\rangle^2$$

Therefore

$$\tilde{S} = \pi\left(\langle x\rangle^2 + \langle y\rangle^2\right) = \frac{2\pi}{3}\langle r\rangle^2$$

Therefore

$$m_L^i = -\frac{e^2}{6m_e}\langle r\rangle^2 B_{loc}^i \tag{5.19}$$

Summing for all Z electrons in an atom, and remembering that the average radius of an electron is a_B, the Bohr radius

$$m_L^i = -\frac{Ze^2a^2}{6m_e}B_{loc}^i \tag{5.20}$$

Note that we used $\omega_0 << \omega_L$ as an approximation, together with $B_l << 4\pi m_e T_0^{-1} e^{-1}$ ($B_{loc} << 5\cdot 10^5$ T), which is almost always verified.
This intrinsic atomic moment is always present by definition, and it always opposes the local field

§§§ 5.4.1.1 Microscopic Interpretation of Diamagnets

Considering atoms where there is no atomic magnetic moment we have only Larmor effects, and by definition therefore the magnetization will be

$$M^i = nm_L^i = -\frac{n\mu_0 Ze^2a^2}{6m_e}H_{loc}^i = \alpha_d H_{loc}^i \tag{5.21}$$

By definition $\alpha_d << 1$ and therefore, using (5.15) we write

$$M^i = \frac{3\alpha_d}{3-\alpha_d}H^i \approx \alpha_d H^i \tag{5.22}$$

By definition $M^i = \chi_m H^i$, i.e. $\chi_m \approx \alpha_d < 0$. This susceptibility doesn't depend on the temperature, is negative and for reasonable values of a, Z, n $\chi_m \approx -10^{-5}$ as we said before for diamagnets

§§ 5.4.2 Langevin Function

Going back to substances where its composing atoms have their own atomic magnetic moment m_0^i, we have that thermal agitation tends to bring them to a disorder in their orientation.
In order to evaluate this Langevin proposed to utilize a function which could be used to evaluate the average magnetic momentum. Called L this Langevin function we have

$$\langle m^i\rangle = \langle m_0^i\rangle L(y) \tag{5.23}$$

Where

$$L(y) = \coth(y) - \frac{1}{y} = \coth\left(\frac{m_0^i B_i^{loc}}{kT}\right) - \frac{kT}{m_0^i B_i^{loc}} \tag{5.24}$$

By definition we have that this function is limited at $\pm\infty$ by ± 1 and it's uneven ($L(y) = -L(-y)$)

§§§ 5.4.2.1 Paramagnets

For paramagnets we have atoms (or molecules, as always) with proper magnetic moment m_0, but in general $m_0 B_{loc} << kT$, i.e. $y << 1$ and we can use a power series approximation on Langevin's function at the first order, which implies the following statements

$$\langle m \rangle \approx m_0 \frac{y}{3} = \frac{m_0^2 \mu_0}{3kT} H_{loc} \implies M = \frac{n m_0^2 \mu_0}{3kT} H_l = \alpha_p H_{loc} \tag{5.25}$$

Using $\chi_m \approx \alpha_p$ and writing the number density of atoms $n = \rho N_A / A$ we have

$$\chi_m(T) = \frac{\rho N_A m_0^2 \mu_0}{3k} \frac{1}{T} \tag{5.26}$$

Which is Curie's law that we defined before, with the constant written out in full in this classical view of microscopic electromagnetism

§§§ 5.4.2.2 Ferromagnets

For ferromagnets the approximation $y << 1$ doesn't hold anymore since m_0 is big, and using Weiss' law for ferromagnets (5.16) and the definition of magnetization, remembering that $L(\infty) = 1$ indicates the saturation of the magnet, we have that the saturation magnetization will simply be $M_s = n m_0$, and we'll get

$$\begin{cases} M(y, H) = M_s L(y) \\ M(y, H) = \dfrac{kT}{m_0 \mu_0 \gamma} y - \dfrac{H}{\gamma} \end{cases} \tag{5.27}$$

Plotting the first equation we get the magnetization in terms of the parameter y, which looks something like this

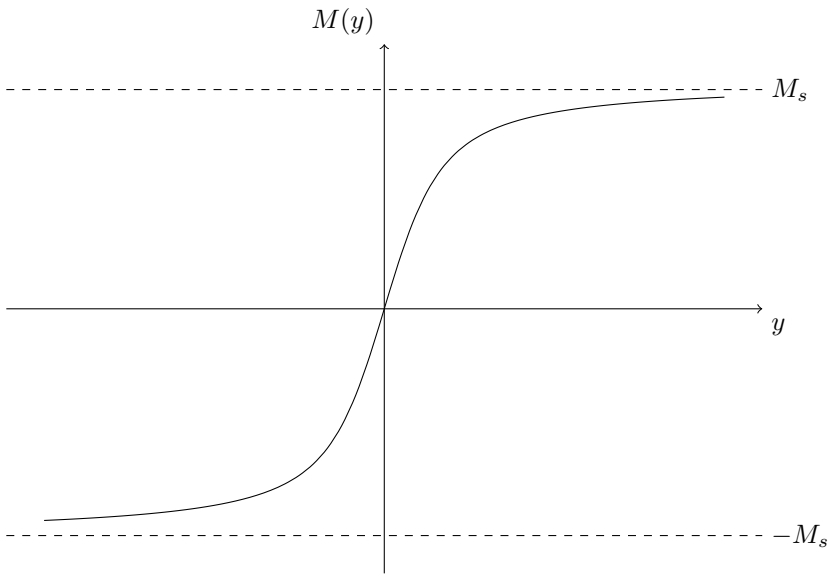

Figure 5.2: Langevin function for Magnetization

The second equation is a line tangent to $ML(y)$ at $y = 0$ intersecting the M axis at $-H/\gamma$, increasing the field H the intersection moves towards the right up until $M = M_s$.
Lowering the field until $H = H_c$ the line becomes tangent to $M_c L(y)$ for which we get two new intersections.
Inverting H (therefore I) the line reaches first M_c then $-M_s$, describing a magnetic hysteresis cycle (not drawn).
Reconsidering he system (5.27) we see that the line has angular coefficient $kT/m_0\mu_0\gamma$, therefore if T is high enough this coefficient is higher than $M_s/3$ of Langevin's curve $M(y)$. In this particular case there is only one intersection point, and therefore the substance becomes paramagnetic (there cannot be an hysteresis cycle).
Considering the derivative of the first and the second we have

$$\frac{kT}{m_0\mu_0 T} \geq \frac{M_s}{3} \implies T \geq \frac{\mu_0\gamma m_0 M_s}{3k} = T_c$$

Which gives the Curie temperature definition again. For $T > T_c$ we can approximate $L(y) \approx y/3$, therefore

$$\begin{aligned} M &= \frac{M_s y}{3} = \frac{n m_0^2 \mu_0}{3kT} H_{loc} = \frac{T_c}{\gamma T} H_{loc} \\ H_{loc} &= H + \gamma M \end{aligned} \tag{5.28}$$

Inserting the second equation in the first we have

$$M = \frac{T_c}{\gamma T}(H + \gamma M) = \frac{T_c}{\gamma (T - T_c)} H$$

Using again $\chi_m = M/H$ we have

$$\chi_m = \frac{1}{\gamma}\frac{T_c}{T - T_c} = \frac{\mu_0 n m_0^2}{3kT_c}\frac{T_c}{T - T_c}$$

And we get via simple algebra the Curie-Weiss law for ferromagnets

$$\chi_m(T) = \frac{\mu_0 m_0^2 n}{3k}\frac{1}{T - T_c}$$

Part III

Electrodynamics and Optics

6 Maxwell's Equations

§ 6.1 Faraday's Law

So far we managed to build four equations for the two fields, in *static* conditions. These are, whenever there no dielectrics and no magnets, are

$$\begin{cases} \partial_i E^i = \dfrac{\rho}{\epsilon_0} \\ \partial_i B^i = 0 \\ \epsilon^i{}_{jk} \partial^j E^k = 0 \\ \epsilon^i{}_{jk} \partial^j B^k = \mu_0 J^i \end{cases} \tag{6.1}$$

Faraday in his works had a major discovery while using two simple circuits.
Consider a closed circuit A with only a galvanometer, and a circuit B with a battery and a switch.
Using the switch for controlling the current flow on the circuit B (which is NOT connected to circuit A), he saw that the galvanometer measures a current while the current in B is changing.
Taking the same setup but keeping the switch closed on B, if either of the two circuits are in motion, there is current flow in A. The same happens if circuit B is substituted by a magnet!
Now take a non-rigid circuit immersed in a region with a constant magnetic field B^i. Deforming the circuit will also induce a current flow on it.
Calling f_{em} the *electromotive force* that drives the current in circuit A, Faraday deduced experimentally that

$$f_{em} = -\frac{\mathrm{d}\Phi}{\mathrm{d}t} \tag{6.2}$$

Where Φ is the magnetic flux passing inside the circuit.
Note that we know, by definition, that this electromotive force must be driven by an electric field E^i, where

$$f_{em} = \oint_C E^i_{ind} \hat{t}_i \mathrm{d}l \tag{6.3}$$

This field *can't* be conservative! Using the definition of the electric field as force per unit charge, and using Lorentz's force law we have a little hint that this induced field is the sum of a pure electric field plus a second field generated by the movement of charges

$$E^i_{ind} = E^i + \epsilon^i{}_{jk} v^j B^k \tag{6.4}$$

Noting that the charges are constrained to the circuit, we can divide the velocity v^i with a component parallel to the circuit $v^i_{\parallel}$ and a perpendicular component $v^i_{\perp}$. It's obvious then that

$$f_{em} = \oint_C \left(E^i + \epsilon^i{}_{jk} v^j_{\perp} B^k \right) \hat{t}_i \mathrm{d}l \tag{6.5}$$

Suppose now that we do not move the circuit, then $v^i = v^i_{\parallel}$ and $E^i_{ind} = E^i$, where this electric field is for sure not conservative.
All this jargon, condenses itself in one simple but powerful law, *Faraday-Neumann-Lenz's law*, which indicates exactly what Faraday discovered experimentally

Theorem 6.1 (Faraday-Neumann-Lenz, Electromagnetic Induction). *Given a time-dependent magnetic field $B^i(t, x^i)$, an electric field is induced by the variation of its flux, where*

$$\oint_{\partial S} E^i \hat{t}_i \mathrm{d}l = -\frac{\mathrm{d}}{\mathrm{d}t} \iint_S B^i \hat{n}_i \mathrm{d}s$$

Or, in its differential counterpart

$$\epsilon^i{}_{jk} \partial^j E^k = -\frac{\partial B^i}{\partial t} \tag{6.6}$$

Proof. Suppose that there is some circuit ∂S that spans some surface S inside of it, which is immersed in a time dependent magnetic field $B^i(t)$, then (6.2) holds, and therefore

$$\frac{\mathrm{d}\Phi}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t} \iint_{S(t)} B^i(x,t) \hat{n}_i \mathrm{d}s$$

The derivative on the right can be seen as the variation of the surface $S(t)$ when the B field is fixed in time at t_0, plus the integral over the surface $S(t_0)$ of the derivative of B with respect to time, i.e.

$$\frac{\mathrm{d}\Phi}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t_0} \iint_{S(t)} B^i(x,t_0) \hat{n}_i \mathrm{d}s + \iint_{S(t_0)} \frac{\partial B^i}{\partial t} \hat{n}_i \mathrm{d}s$$

Since we know already that (6.2) holds, we have that if f_{ind} is the induced f_{em}, we have that

$$f_{ind} = \oint_{\partial S(t_0)} E^i \hat{t}_i \mathrm{d}l = \oint_{\partial S(t_0)} E^i \hat{t}_i \mathrm{d}l = \iint_{S(t_0)} \frac{\partial B^i}{\partial t} \hat{n}_i \mathrm{d}s$$

But

$$\mathrm{d}\Phi = \iint_S B^i \frac{\mathrm{d}s}{\mathrm{d}t} \mathrm{d}t = \int \oint_{\partial S} \epsilon_{ijk} B^i v^j_D \hat{t}^k \mathrm{d}l \mathrm{d}t \tag{6.7}$$

Where v_D is the velocity in the direction of the movement of the circuit. Therefore, "dividing" by dt, we get

$$\frac{\mathrm{d}\Phi}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t} \iint_{S(t)} B^i(x,t_0) \hat{n}_i \mathrm{d}s = \oint_{\partial S} \epsilon^i{}_{jk} \hat{t}_i v^j_D B^k \mathrm{d}l$$

Since we know that an additional f_{ind} is given by the deformation of the circuit $\partial S(t_0) \to \partial S(t)$, we have in total

$$f_{em} = -\frac{\mathrm{d}\Phi}{\mathrm{d}t} = \oint_{\partial S} E^i + \epsilon^i{}_{jk} \hat{t}_i v^j_D B^k \mathrm{d}l$$

Since $v_D = v_\parallel + v_\perp$ and $v_\parallel \parallel \hat{t}$ we get that since $\mathbf{E}_{ind} = \mathbf{E} + \mathbf{v} \times \mathbf{B}$, $\mathbf{E}_{ind}$ has $\mathbf{B}$ as a source, and we can condense it all in a single integro-differential equation, which is Faraday-Neumann-Lenz's law

$$\nabla \times \mathbf{E} \cdot \hat{\mathbf{n}} \mathrm{d}s = -\frac{\mathrm{d}}{\mathrm{d}t} \iint_S \frac{\partial \mathbf{B}}{\partial t} \cdot \hat{\mathbf{n}} \mathrm{d}s$$

□

§§ 6.1.1 Self-Induction

Having deduced our previous results, the first thing that we might check is how a circuit behaves with itself.
Consider a closed circuit with some current $I(t)$ such that $\partial_t I \approx 0$. If the magnetic permeability of the body is constant we can apply Biot-Savart for evaluating the field, and we have

$$B^i(t) = \frac{\mu_0 I(t)}{4\pi} \oint_C \frac{\epsilon^i{}_{jk} \hat{t}^j r^k}{r^3} \mathrm{d}l \tag{6.8}$$

Evaluating the flux of this field we have, since the current is independent from the integrated variables, that $\Phi \propto I(t)$, and therefore

$$\Phi(t) = LI(t) \qquad L = \frac{\mu_0}{4\pi} \oint_C \iint_S \frac{\epsilon^i{}_{jk} \hat{t}^j r^k}{r^3} \mathrm{d}l \mathrm{d}s$$

The constant L only depends on the geometry of the circuit as it's easy to see from the integral, and it's known as the *self-induction coefficient* or also as *autoinduction coefficient*.
From Faraday's law, since this flux depends on time (through our current $I(t)$), it generates an electromotive force f_L, as follows

$$f_L = -\frac{\mathrm{d}\Phi}{\mathrm{d}t} = -L\frac{\mathrm{d}I}{\mathrm{d}t} \tag{6.9}$$

The autoinduction coefficient has units the following units:

$$[L] = \frac{[\Phi]}{[I]} = \frac{W}{A} = \frac{Vs}{A} = \Omega s = H$$

The SI unit H is known as *Henry* and it's equal to Watts/Ampere. Note that this can also be calculated via Ohm's law, noting that $V = RI$ and that $[RI] = \Omega A$

§§ 6.1.2 Mutual-Induction

Consider now a setup similar to the previous one, but with two circuits C_1 and C_2, which are close enough to each other such that the generated magnetic fluxes through each circuit are not negligible. The fluxes as before will be proportional to the currents, and without evaluating the self-induction of both circuits we have

$$\begin{aligned} \Phi_1(B_2) &= \iint_{S_1} B_2^i \hat{n}_i \mathrm{d}s \propto I_2(t) \\ \Phi_2(B_1) &= \iint_{S_2} B_1^i \hat{n}_i \mathrm{d}s \propto I_1(t) \end{aligned} \tag{6.10}$$

Using the previous considerations, we have then, in index form, that

$$\Phi_i = M_{ij} I_j(t) \tag{6.11}$$

The coefficients M_{ij} are known as the *mutual induction coefficients*. Obviously $M_{ij} = M_{ji}$.

§ 6.2 Magnetic Energy

With what we wrote before, we might consider a circuit with a given self-induction coefficient L and some time-dependent current $I(t)$ flowing through it. We can evaluate the work of that the magnetic force exerts on these charges as follows. Per unit time

$$\frac{\mathrm{d}w}{\mathrm{d}t} = -f_{em} I(t) \tag{6.12}$$

Using Faraday's law we know that

$$\frac{\mathrm{d}\Phi}{\mathrm{d}t} = -f_{em} = -L\frac{\mathrm{d}I}{\mathrm{d}t}$$

Therefore

$$\frac{\mathrm{d}w}{\mathrm{d}t} = L\left(\frac{\mathrm{d}I}{\mathrm{d}t}\right)^2 = \frac{1}{2}LI^2(t) \tag{6.13}$$

We can go forward with this calculus, noting that then, since $\Phi = LI$, and

$$\Phi = \iint \epsilon^i{}_{jk} \partial^j A^k \hat{n}_i \mathrm{d}s$$

Then, using Stokes' theorem

$$\oint A^i \hat{t}_i \mathrm{d}l = LI$$

And therefore, vectorizing the current as $I^i = I\hat{t}^i$

$$w = \frac{1}{2}\oint A^i I_i \mathrm{d}l$$

Or in general

$$w = \frac{1}{2}\iiint A^i J_i \mathrm{d}^3x = \frac{1}{2\mu_0}\iiint \epsilon^i{}_{jk} A_i \partial^j B^k \mathrm{d}^3x \tag{6.14}$$

Where we used Ampere's law to get the last integral. Playing around with the last curl, using vector notation, we have

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot \nabla \times \mathbf{A} - \mathbf{A} \cdot \nabla \times \mathbf{B} = B^2 - \mathbf{A} \cdot \nabla \times \mathbf{B}$$

Applying Stokes' theorem on the divergence and noting that the surface integral goes to 0 when we integrate over all space, we end up with the following result

$$w = \frac{1}{2\mu_0}\iiint_{\mathbb{R}^3} B^2 \mathrm{d}^3x$$

The parallelism with the energy of an electric field is astounding. Written side by side we have

$$\begin{aligned} W_{es} &= \frac{1}{2}\int_{\mathbb{R}^3} V\rho \mathrm{d}^3x = \frac{\epsilon_0}{2}\int_{\mathbb{R}^3} E^2 \mathrm{d}^3x \\ W_{ms} &= \frac{1}{2}\int_{\mathbb{R}^3} A^i J_i \mathrm{d}^3x = \frac{1}{2\mu_0}\int_{\mathbb{R}^3} B^2 \mathrm{d}^3x \end{aligned} \tag{6.15}$$

§ 6.3 Maxwell's Equations

So far, we found 2 pairs of coupled differential equations for the electric and magnetic field

$$\begin{cases} \partial_i E^i = \dfrac{\rho}{\epsilon_0} \\ \epsilon^i{}_{jk}\partial^j E^k = -\partial_t B^i \end{cases} \tag{6.16}$$

And

$$\begin{cases} \partial_i B^i = 0 \\ \epsilon^i{}_{jk}\partial^j B^k = \mu_0 J^i \end{cases} \tag{6.17}$$

The second pair of equations holds only if the current field is divergenceless, but using Gauss' law and the current conservation equation we have

$$\partial_i J^i + \partial_t \rho = \partial_i J^i + \epsilon_0 \partial_t \partial_i E^i = 0 \tag{6.18}$$

Grouping the divergences we see that the time derivative of the electric field behaves exactly like a current, commonly called the "displacement current". In order to fix all the equations now we can add this new current in the last couple of the Maxwell equations and get the well known fundamental equations of electromagnetism

$$\begin{cases} \partial_i E^i = \dfrac{\rho}{\epsilon_0} \\ \epsilon^i{}_{jk}\partial^j E^k = -\partial_t B^i \\ \partial_i B^i = 0 \\ \epsilon^i{}_{jk}\partial^j B^k = \mu_0 J^i + \epsilon_0\mu_0\partial_t E^i \end{cases} \tag{6.19}$$

These equations account for moving charges and are absolutely general in nature. It will be seen later that they're also Lorentz invariant, therefore they preserve between Lorentz transformations and therefore are relativistically covariant.
For linear dielectric and magnetic media we can rewrite easily with the already known rules, Maxwell's equations

$$\begin{cases} \partial_i D^i = \rho \\ \epsilon^i{}_{jk}\partial^j D^k = \epsilon\mu\partial_t H^i \\ \partial_i H^i = 0 \\ \epsilon^i{}_{jk}\partial^j H^k = J^i + \partial_t D^i \end{cases} \tag{6.20}$$

§§ 6.3.1 Poynting's Vector, Energy Conservation

As we have seen via previous calculations, the energies of the two separated fields are, calling them U_e and U_m

$$\begin{aligned} U_e &= \frac{\epsilon_0}{2}\int_V E^2 \mathrm{d}^3x \\ U_m &= \frac{1}{2\mu_0}\int_V B^2 \mathrm{d}^3x \end{aligned} \tag{6.21}$$

We can imagine that the energy of the combined electromagnetic field will be a sum of the two, i.e.

$$U_{em} = \frac{1}{2}\int_V \epsilon_0 E^2 + \frac{B^2}{\mu_0}\mathrm{d}^3x \tag{6.22}$$

We try to confirm this using the work done by some particle. Substituting into the force the formula for Lorentz's force we get

$$F^i \mathrm{d}l_i = q\left(E^i + \epsilon^i{}_{jk} v^j B^k\right)\mathrm{d}l_i = q\left(E^i + \epsilon^i{}_{jk} v^j B^k\right) v_i \mathrm{d}t = qE^i v_i \mathrm{d}t = \mathrm{d}W$$

Which is what we expected.
Going to a microscopic consideration we substitute $q = \rho \mathrm{d}^3x$, we get $J^i = \rho v^i$, and therefore, integrating with respect to time we have

$$W = \int_V E^i J_i \mathrm{d}^3x \mathrm{d}t \implies E^i J_i = \frac{\mathrm{d}w}{\mathrm{d}t}$$

But, from Maxwell equations we have

$$\epsilon^i{}_{jk}\partial^j B^k = \mu_0 J^i + \frac{1}{c^2}\frac{\partial E^i}{\partial t} \implies J^i = \frac{\epsilon^i{}_{jk}\partial^j B^k}{\mu_0} - \frac{1}{\mu_0 c^2}\frac{\partial E^i}{\partial t}$$

Where we used $\mu_0\epsilon_0 = c^-2$ (it will be clear later, for now check multiplying those two and see that it adds up)
Therefore, we rewrite $E^i J_i$ as follows

$$E^i J_i = \frac{1}{\mu_0} E_i \epsilon^i{}_{jk}\partial^j B^k - \epsilon_0 E_i \epsilon^i{}_{jk}\partial^j B^k$$

Note that, tho:

$$\partial_i \epsilon^i{}_{jk} E^j \frac{B^k}{\mu_0} = \frac{1}{\mu_0} B_i \epsilon^i{}_{jk}\partial^j E^k - E^i \epsilon^i{}_{jk}\partial^j \frac{B^k}{\mu_0}$$

Therefore

$$E_i \epsilon^i{}_{jk}\partial^j \frac{B^k}{\mu_0} = -\frac{1}{\mu_0} B_i \epsilon^i{}_{jk}\partial^j E^k - \partial_i \epsilon^i{}_{jk} E^j \frac{B^k}{\mu_0} = -\frac{B_i}{\mu_0}\frac{\partial B^i}{\partial t} - \partial_i \epsilon^i{}_{jk} E^j \frac{B^k}{\mu_0}$$

Which, gives us back

$$E^i J_i = -\frac{1}{2}\frac{\partial}{\partial t}\left(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2\right) - \frac{1}{\mu_0}\partial_i \epsilon^i{}_{jk} E^j B^k = \frac{\mathrm{d}w}{\mathrm{d}t} \tag{6.23}$$

We immediately recognize the volumetric density of energy of the electromagnetic field, let's denote it as u_{em}, and we get

$$E^i J_i = -\frac{\partial u_{em}}{\partial t} - \partial_i \epsilon^i{}_{jk} E^j \left(\frac{B^k}{\mu_0}\right) \tag{6.24}$$

We begin to have a better view of the phenomenon, we see a variation of energy on the right plus the divergence of some vector that we define now.

Definition 6.3.1 (Poynting Vector)**.** The *Poynting vector* is a vector defined as follows:

$$S^i = \frac{1}{\mu_0} \epsilon^i{}_{jk} E^j B^k \tag{6.25}$$

It has dimensions of a flux of energy, as we will see.

With the previous definition, everything becomes much clearer in terms of notation, in fact

$$E^i J_i = -\frac{\partial u_{em}}{\partial t} - \partial_i S^i$$

Integrating in a random volume V we get, as said before, our flux of energy!

$$\int_V E^i J_i \mathrm{d}^3 x = -\frac{\partial}{\partial t} \int_V u_{em} \mathrm{d}^3 x - \oiint_{\partial V} S^i \hat{n}_i \mathrm{d}s$$

It's clear that in order to make sense we must sum energies with energies, giving the previously stated dimensions of the Poynting vector as an energy flux.
Rewriting $E^i J_i$ as our work variation we have that it's nothing else than the time derivative of the volumetric density of mechanical energy, and writing $u_{em} + u_{mech}$ as our total energy variation, we have

$$\int_V \frac{\partial u}{\partial t} \mathrm{d}^3 x = -\oiint_{\partial V} S^i \hat{n}_i \mathrm{d}s \tag{6.26}$$

The associated PDE is clearly the conservation of energy of the whole system

$$\frac{\partial u}{\partial t} + \partial_i S^i = 0 \tag{6.27}$$

This shape also gives the real idea of what's Poynting's vector: an energy "current"

§§ 6.3.2 Stress Tensor, Momentum Conservation

Lorentz'force as we have seen, in terms of microscopic evaluations is written (it's a force density in this case) as

$$f^i = \rho E^i + \epsilon^i{}_{jk} J^j B^k \tag{6.28}$$

We rewrite it in terms of fields only using the two following Maxwell equations

$$\begin{aligned} \partial_i E^i = \frac{\rho}{\epsilon_0} \quad &\Longrightarrow \quad \rho = \frac{1}{\epsilon_0} \partial_i E^i \\ \epsilon^i{}_{jk} \partial^j B^k = \mu_0 J^i + \frac{1}{c^2} \frac{\partial E^i}{\partial t} \quad &\Longrightarrow \quad J^i = \frac{1}{\mu_0} \epsilon^i{}_{jk} \partial^j B^k - \frac{1}{\mu_0 c^2} \frac{\partial E^i}{\partial t} \end{aligned} \tag{6.29}$$

Therefore, Lorentz's force becomes, using $\epsilon_0 = (\mu_0 c^2)^{-1}$

$$f^i = \epsilon_0 E^i \partial_j E^j + \epsilon^i{}_{jk} \left(\frac{1}{\mu_0} \epsilon^j{}_{ml} \partial^l B^m - \epsilon_0 \frac{\partial E^j}{\partial t} \right) B^k$$

Or, moving inside the cross product for clarity

$$f^i = \epsilon_0 E^i \partial_j E^j + \frac{1}{\mu_0} \epsilon^i{}_{jk} B^k \epsilon^j{}_{lm} \partial^l B^m - \epsilon_0 \epsilon^i{}_{jk} \partial_t E^j B^k$$

Using the product rule on the time derivative at the last factor we have that

$$\partial_t \epsilon^i{}_{jk} E^j B^k = \epsilon^i{}_{jk} \partial_t (E^j) B^k + \epsilon^i{}_{jk} E^j \partial_t (B^k) \implies \epsilon^i{}_{jk} \partial_t (E^j) B^k = \partial_t \epsilon^i{}_{jk} E^j B^k - \epsilon^i{}_{jk} E^j \partial_t (B^k)$$

From the second Maxwell equation tho we have

$$\epsilon^i{}_{jk} \partial^j E^k = -\partial_t B^i$$

Therefore

$$-\epsilon^i{}_{jk} E^j \partial_t (B^k) = \epsilon^i{}_{jk} E^j \epsilon^k{}_{lm} \partial^l B^m$$

And everything comes back to

$$\epsilon^i{}_{jk} \partial_t (E^j) B^k = \partial_t \left(\epsilon^i{}_{jk} E^j B^k \right) + \epsilon^i{}_{jk} E^j \epsilon^k{}_{lm} \partial^l E^m$$

And inserting it back into the Lorentz force density gives

$$f^i = \epsilon_0 E^i \partial_j E^j + \frac{1}{\mu_0} \epsilon^i{}_{jk} B^k \epsilon^j{}_{lm} \partial^l B^m - \epsilon_0 \frac{\partial}{\partial t} \left(\epsilon^i{}_{jk} E^j B^k \right) + \epsilon_0 \epsilon^i{}_{jk} E^j \epsilon^k{}_{lm} \partial^l E^m \tag{6.30}$$

Or, rearranging the two fields

$$f^i = \epsilon_0 \left(E^i \partial_j E^j - \epsilon^i{}_{jk} \epsilon^k{}_{lm} E^j \partial^l E^m \right) + \frac{1}{\mu_0} \left(B^i \partial_j B^j - \epsilon^i{}_{jk} \epsilon^j{}_{lm} \partial^l B^m B^k \right) - \epsilon_0 \frac{\partial}{\partial t} \left(\epsilon^i{}_{jk} E^j B^k \right) \tag{6.31}$$

Where we used $\partial_i B^i = 0$ in order to symmetrize the shape of the equation.
Using the properties of the Levi-Civita symbol we have that (note that $g_{ij} = \delta_{ij}$ in this metric)

$$\epsilon^i{}_{jk} \epsilon^k{}_{lm} = \delta^i_l \delta_{jm} - \delta_{jl} \delta^i_l \tag{6.32}$$

We have (using a generic vector here, it can be either B^i or E^i)

$$\epsilon^i{}_{jk} \epsilon^k{}_{lm} A^j \partial^l A^m = A_m \partial^i A^m - A^l \partial_l A^i$$

So

$$f^i = \epsilon_0 \left(E^i \partial_j E^j + E^l \partial_l E^i - E^m \partial^i E_m \right) + \frac{1}{\mu_0} \left(B^i \partial_j B^j + B^l \partial_l B^i - B^m \partial^i B_m \right) - \frac{1}{c^2} \frac{\partial S^i}{\partial t}$$

Looking closely we see that the big mess inside the parentheses is simply

$$\partial_j \left(E^i E^j \right) - \frac{1}{2} \partial_j \left(E^k E_k \right) = \partial_j \left(E^i E^j - \frac{1}{2} \delta^{ij} E^k E_k \right)$$

Seen this, we define the following symmetric rank-2 tensor σ^{ij}

Definition 6.3.2 (Maxwell Stress Tensor)**.** The Maxwell stress tensor is defined as follows:

$$\sigma^{ij} = \epsilon_0 \left(E^i E^j - \frac{1}{2} \delta^{ij} E^k E_k \right) + \frac{1}{\mu_0} \left(B^i B^j - \frac{1}{2} \delta^{ij} B^k B_k \right) \tag{6.33}$$

It's a rank 2 tensor and it's obviously symmetric

The Lorentz force density becomes then the following simply

$$f^i = \frac{\partial \sigma^{ij}}{\partial x^k} - \frac{1}{c^2} \frac{\partial S^i}{\partial t} \tag{6.34}$$

7 Potentials and Fields

§ 7.1 Maxwell's Equation for Potentials

As we have seen already, Maxwell's equations are the following

$$\begin{cases} \partial_i E^i = \dfrac{\rho}{\epsilon_0} \\ \epsilon^i{}_{jk}\partial^j E^k = -\dfrac{\partial B^i}{\partial t} \\ \partial_i B^i = 0 \\ \epsilon^i{}_{jk}\partial^j B^k = \mu_0 J^i + \dfrac{1}{c^2}\dfrac{\partial E^i}{\partial t} \end{cases} \tag{7.1}$$

If we wanted to write the potential formulation of this equation, we must know that in general, the potentials might be different. In fact, Coulomb's law and Biot-Savart only work in the static case, i.e. where $\partial_t E = \partial_t B = 0$.
Using that $\partial_i B^i = 0$ from the third equation we can say for sure that

$$\epsilon^i{}_{jk}\partial^j A^k = B^i$$

And inserting it in the second we get

$$\epsilon^i{}_{jk}\partial^j E^k = -\frac{\partial}{\partial t}\epsilon^i{}_{jk}\partial^j A^k$$
$$\epsilon^i{}_{jk}\partial^j \left(E^k + \frac{\partial A^k}{\partial t} \right) = 0$$

The second line immediately tells us that the vector field inside must be the gradient of some scalar field!
Using $\partial^i V = -E^i$ then we can rewrite the electric potential as a sum of the time variation of the vector potential and the scalar potential, which gives us

$$E^i(x^i, t) = -\frac{\partial V}{\partial x_i} - \frac{\partial A^i}{\partial t} \tag{7.2}$$

Now that we have the potentials for the dynamic case we know that the Poisson equation for the electric field then becomes

$$\partial_i E^i = -\partial_i \partial^i V - \frac{\partial}{\partial t}\partial_i A^i = \frac{\rho}{\epsilon_0}$$

Or, written better

$$\partial_i \partial^i V + \frac{\partial}{\partial t} \partial_i A^i = -\frac{\rho}{\epsilon_0} \tag{7.3}$$

Also, for the equivalent vectorial Poisson equation for the A^i field

$$\epsilon^i{}_{jk} \epsilon^k{}_{lm} \partial^l A^m = \mu_0 J^i - \frac{1}{c^2} \frac{\partial}{\partial t} \left(\frac{\partial V}{\partial x_i} - \frac{1}{c^2} \frac{\partial A^i}{\partial t} \right)$$

Rewriting the first double cross product as

$$\partial^i \partial_j A^j - \partial_j \partial^j A^i$$

And bringing the time derivative of the gradient of V to the left, while grouping it inside the ∂^i operator (it's linear), and bringing with it the second time derivative on A^i we have (note also that I changed sign on both sides)

$$\partial_j \partial^j A^i - \frac{\partial}{\partial x_i} \left(\frac{1}{c^2} \frac{\partial V}{\partial t} + \partial_j A^j \right) - \frac{1}{c^2} \frac{\partial^2 A^i}{\partial t^2} = -\mu_0 J^i \tag{7.4}$$

Both together give us the most general possible way to formulate Maxwell equations for potentials, which reduce to two coupled non-homogeneous second order PDEs

$$\begin{cases} \partial_j \partial^j V + \dfrac{\partial}{\partial t} \partial_j A^j = -\dfrac{\rho}{\epsilon_0} \\ \partial_j \partial^j A^i - \dfrac{1}{c^2} \dfrac{\partial^2 A^i}{\partial t^2} - \dfrac{\partial}{\partial x_i} \left(\dfrac{1}{c^2} \dfrac{\partial V}{\partial t} + \partial_j A^j \right) = -\mu_0 J^i \end{cases} \tag{7.5}$$

§§ 7.1.1 Gauge Freedom

What we've learned before about electromagnetic potentials is that *they're gauge-modifiable*. Depending on what we really need we might choose between any given gauge, since Maxwell's equation are gauge-invariant.
The first gauge we will use is the most common one, it's useful when dealing with magnetostatics or when we really need to find V. It's *Coulomb's gauge*.
Here we set the divergence of A^i to zero, and the first equation of (7.5) reduces back to a Poisson's equation. The second simplifies a bit, but it's not easy to solve…

$$\begin{cases} \partial_i \partial^i V = -\dfrac{\rho}{\epsilon_0} \\ \partial_j \partial^j A^i - \dfrac{1}{c^2} \dfrac{\partial^2 A^i}{\partial t^2} - \dfrac{1}{c^2} \dfrac{\partial^2 V}{\partial x_i \partial t} = -\mu_0 J^i \end{cases} \tag{7.6}$$

Another thing to note here is that V *cannot* be observable, moving charges change ρ which changes V istantaneously, it's not Lorentz invariant.
The second most important gauge we can define it's Lorenz's[1] gauge, which defines the divergence of A as follows

$$\partial_i A^i = \frac{1}{c^2} \frac{\partial V}{\partial t} \tag{7.7}$$

[1] Lorenz, not Lorentz, apparently

Then, reinserting it back to (7.5) we get by immediate substitution two uncoupled non-homogeneous wave equations

$$\begin{cases} \partial_j \partial^j V - \dfrac{1}{c^2}\dfrac{\partial^2 V}{\partial t^2} = -\dfrac{\rho}{\epsilon_0} \\ \partial_j \partial^j A^i - \dfrac{1}{c^2}\dfrac{\partial^2 A^i}{\partial t^2} = -\mu_0 J^i \end{cases} \tag{7.8}$$

This is a *relativistic wave equation* with sources ρ and J^i.
Note that if we define the four-gradient as the 4-vector composed by the following components ($\mu = 0, \cdots, 3$)

$$\partial_\mu = \left(\frac{1}{c}\partial_t, -\partial_i\right) \tag{7.9}$$

We have that, formally

$$\partial_\mu \partial^\mu = \frac{1}{c^2}\partial_t^2 - \partial_i \partial^i = \Box$$

Where the box operator is known as the *D'Alambertian*, which is the equivalent of the Laplacian in 4 spacetime dimensions. Therefore we can also write

$$\begin{cases} \Box V = \partial^\mu \partial_\mu V = \dfrac{\rho}{\epsilon_0} \\ \Box A^i = \partial^\mu \partial_\mu A^i = \mu_0 J^i \end{cases} \tag{7.10}$$

§ 7.2 Retarded Potentials

Using (7.10) and setting the time derivatives as 0, we get back Poisson's equations for both potentials, for which we know already the general solution for a volume V.

$$V(x^j) = \frac{1}{4\pi\epsilon_0}\int_V \frac{\rho(\tilde{x}^j)}{r}\mathrm{d}^3\tilde{x}$$
$$A^i(x^j) = \frac{\mu_0}{4\pi}\int_V \frac{J^i(\tilde{x}^j)}{r}\mathrm{d}^3\tilde{x}$$

We can say, from the previous equations, that the interaction travels at speed c, therefore we might imagine the time progression of the interaction as "retarded in time" by a factor of r/c. We then define the *retarded time* as

$$t_r = t - \frac{r}{c} \tag{7.11}$$

We therefore can imagine a solution to those equations as

$$\begin{aligned} V(x^j, t_r) &= \frac{1}{4\pi\epsilon_0}\int_V \frac{\rho(\tilde{x}^j, t_r)}{r}\mathrm{d}^3\tilde{x} \\ A^i(x^j, t_r) &= \frac{\mu_0}{4\pi}\int_V \frac{J^i(\tilde{x}^j, t_r)}{r}\mathrm{d}^3\tilde{x} \end{aligned} \tag{7.12}$$

Note that we could imagine this solution only due to the mathematical shape of the equation, it cannot be done the same way for the fields.
Now let's check if this idea we had is a solution for the relativistic equations. Noting that

$$\partial_i t_r = -\frac{\hat{x}^i}{c}$$

We have, after using the chain rule

$$\frac{\partial \rho}{\partial x^i} = -\frac{1}{c}\frac{\partial \rho}{\partial t}\hat{x}^i$$

And therefore

$$\partial_i V = -\frac{1}{4\pi\epsilon_0}\int_V \frac{1}{c}\frac{\partial \rho}{\partial t}\frac{\hat{x}^i}{r} - \rho\frac{\hat{x}^i}{r^2}\mathrm{d}^3x$$

Applying again the del operator we get

$$\partial^i\partial_i V = -\frac{1}{4\pi\epsilon_0}\int_V \frac{1}{c}\left(\frac{\hat{x}^i}{r}\frac{\partial^2\rho}{\partial t\partial x^i} + \frac{\partial\rho}{\partial t}\frac{\partial_i\hat{x}^i}{r} + \frac{\partial\rho}{\partial t}\hat{x}^i\partial_i\left(\frac{1}{r}\right)\right) + \left(\frac{\hat{x}^i}{r^2}\frac{\partial\rho}{\partial x^i} + \rho\frac{\partial}{\partial x^i}\left(\frac{\hat{x}^i}{r^2}\right)\right)\mathrm{d}^3x$$

But

$$\frac{\partial^2\rho}{\partial t\partial x^i} = -\frac{1}{c}\frac{\partial^2\rho}{\partial t^2}\hat{x}^i$$

And

$$\partial_i\left(\frac{\hat{x}^i}{r}\right) = \frac{1}{r^2}, \qquad \partial_i\left(\frac{\hat{x}^i}{r^2}\right) = 4\pi\delta^3(x^i)$$

Therefore, finally

$$\partial^i\partial_i V = \frac{1}{4\pi\epsilon_0 c^2}\int_V \frac{1}{r}\frac{\partial^2\rho}{\partial t^2}\mathrm{d}^3x - \frac{\rho}{\epsilon_0}\delta^3(x^i) \tag{7.13}$$

Seeing immediately on the right the time second time derivative with respect to ct of V, bringing it to the left and playing with minuses we get again the awaited Maxwell equation.

$$\Box V = \frac{\rho}{\epsilon_0}$$

The calculation for A^i is completely analogous. Note that we could also have chosen an *advanced time* t_a defined as

$$t_a = t + \frac{r}{c}$$

Everything comes back to the two Maxwell equations, but the physical sense gets lost since the potentials we found don't respect causality, they sense the change *before* it actually happens in the chosen reference frame.

§ 7.3 Jefimenko's Equations

Given the two retarded potentials defined in (7.12), we could imagine to determine the electric and magnetic field generated by both. Since the retarded potentials, as we have shown, solve *generally*

Maxwell's equations (7.10), the fields will also solve generally Maxwell's equations for the fields. We begin by finding E^i. We know that

$$E^i = -\partial^i V - \partial_t A^i$$

Therefore

$$E^i = -\frac{1}{4\pi\epsilon_0}\int_V \partial^i\left(\frac{\rho}{r}\right)\mathrm{d}^3x - \frac{\mu_0}{4\pi}\int_V \frac{1}{r}\frac{\partial J^i}{\partial t}\mathrm{d}^3x \tag{7.14}$$

From the previous calculations, we already know that

$$\partial^i\left(\frac{\rho}{r}\right) = -\frac{1}{c}\frac{\partial\rho}{\partial t}\hat{x}^i - \rho\frac{\hat{x}^i}{r^2}$$

And we get easily the first Jefimenko equation for the E field

$$E^i(x^j,t) = \frac{1}{4\pi\epsilon_0}\int_V\left(\frac{1}{c}\frac{\partial\rho}{\partial t} + \frac{\rho}{r}\right)\frac{\hat{x}^i}{r}\mathrm{d}^3x - \frac{\mu_0}{4\pi}\int_V\frac{1}{r}\frac{\partial J^i}{\partial t}\mathrm{d}^3x$$

Using $\epsilon_0 = (\mu_0 c^2)^{-1}$ we can group everything in a clearer equation

$$E^i(x^j,t) = \frac{1}{4\pi\epsilon_0}\int_V\left(\frac{1}{c}\frac{\partial\rho}{\partial t} + \frac{\rho}{r}\right)\frac{\hat{x}^i}{r} - \frac{1}{rc^2}\frac{\partial J^i}{\partial t}\mathrm{d}^3x \tag{7.15}$$

For B^i the calculations are slightly harder due to the presence of the curl, but with some discipline are doable. We have that

$$B^i = \epsilon^i{}_{jk}\partial^j A^k = \frac{\mu_0}{4\pi}\int_V\frac{1}{r}\epsilon^i{}_{jk}\partial^j J^k + \epsilon^i{}_{jk}J^i\partial^k\left(\frac{1}{r}\right)\mathrm{d}^3x$$

But, by definition we have

$$\partial_i J^k = \left(\partial_t J^k\partial_k\right)t_r = -\frac{1}{c}\frac{\partial J^k}{\partial t}\frac{\partial r}{\partial x^k} = \frac{1}{c}\frac{\partial J^k}{\partial t}\hat{x}_k$$

So, the cross product is simply

$$\epsilon^i{}_{jk}\partial^j J^k = \frac{1}{c}\epsilon^i{}_{jk}\partial_t J^j\hat{x}^k$$

The second part instead comes immediately from the gradient of r^{-1}, and we have

$$B^i(x^j,t) = \frac{\mu_0}{4\pi}\int_V\frac{1}{r}\epsilon^i{}_{jk}\left(\frac{J^j}{r} + \frac{1}{c}\frac{\partial J^j}{\partial t}\right)\hat{x}^k\mathrm{d}^3x \tag{7.16}$$

Both Jefimenko equations grouped are, therefore (and finally)

$$\begin{cases} E^i(x^k,t) = \dfrac{1}{4\pi\epsilon_0}\displaystyle\int_V\frac{1}{r}\left(\frac{1}{c}\frac{\partial\rho}{\partial t}\hat{x}^i + \rho\frac{\hat{x}^i}{r} - \frac{1}{c^2}\frac{\partial J^i}{\partial t}\right)\mathrm{d}^3\tilde{x} \\ B^i(x^k,t) = \dfrac{\mu_0}{4\pi}\displaystyle\int_V\frac{1}{r}\epsilon^i{}_{jk}\left(\frac{J^j}{r} + \frac{1}{c}\frac{\partial J^j}{\partial t}\right)\hat{x}^k\mathrm{d}^3\tilde{x} \end{cases} \tag{7.17}$$

8 Electromagnetic Waves

§ 8.1 Propagation of Electromagnetic Waves

Consider a random point in spacetime, where only an electromagnetic field is present.
Here we have

$$\rho = 0$$
$$J^i = 0$$

Using H^i instead of B^i, since, in free space

$$B^i = \mu_0 H^i$$

We get Maxwell's equation for what's known as a free electromagnetic field

$$\begin{cases} \partial_i E^i = 0 \\ \epsilon^i{}_{jk} \partial^j E^k = -\mu_0 \dfrac{\partial H^i}{\partial t} \\ \partial_i H^i = 0 \\ \epsilon^i{}_{jk} \partial^j H^k = \epsilon_0 \dfrac{\partial E^i}{\partial t} \end{cases} \tag{8.1}$$

The absence of sources here is given by the two divergence relations. Note also that these coupled PDEs are valid both in the dynamic and static case.
These equations, although it might not be that clear from the system, are completely separable. Taken the two curl equations we have, using the relations found before, that

$$\begin{cases} \epsilon^i{}_{jk} \epsilon^k{}_{lm} \partial^j \partial^l E^m = -\mu_0 \epsilon_0 \dfrac{\partial^2 E^i}{\partial t^2} \\ \epsilon^i{}_{jk} \epsilon^k{}_{lm} \partial^j \partial^l H^m = -\mu_0 \epsilon_0 \dfrac{\partial^2 H^i}{\partial t^2} \end{cases}$$

Using the Levi-Civita identity we have that the double curl becomes

$$\partial^i \left(\partial_j E^j \right) - \partial^j \partial_j E^i$$

Which, if substituted inside the previous system, using that $\partial_i(H^i, E^i) = 0$ gives back the already well known wave equation

$$\begin{aligned} \frac{1}{c^2}\frac{\partial^2 E^i}{\partial t^2} - \partial^j \partial_j E^i = \Box E^i = 0 \\ \frac{1}{c^2}\frac{\partial^2 H^i}{\partial t^2} - \partial^j \partial_j H^i = \Box H^i = 0 \end{aligned} \tag{8.2}$$

The solution will be what's known as an *electromagnetic wave*, a wave moving in space composed by both an electric and magnetic field. From the wave equation we can already say that it's moving at speed c, as it should. To be precise it's what we know as *light*.

§§ 8.1.1 Electromagnetic Waves in Dielectric Media

In case that we're dealing with the movement of this wave in dielectric or magnetic media, we must remember that the fields *will be different*. The equations will be the same in shape, but will need some tweaking.
The tweaking is not actually in the fields, but in the constants. Given the speed of an electromagnetic wave in the vacuum is c, where

$$c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$$

In a media, we will have the *speed of propagation* of the waves, u, given by the substitution of $\mu_0\epsilon_0$ with $\mu\epsilon$ (remember that $\mu = \mu_r\mu_0$ and $\epsilon_r\epsilon_0$)

$$u = \frac{1}{\sqrt{\mu\epsilon}} = \frac{1}{\sqrt{\mu_r\mu\epsilon_r\mu_0}} = \frac{c}{\sqrt{\mu_r\epsilon_r}} \tag{8.3}$$

From this we define the *index of refraction* of a medium, n as

$$n = \frac{c}{u} = \sqrt{\mu_r\epsilon_r} \tag{8.4}$$

So that, the propagation speed can be written also

$$u = \frac{c}{n}$$

And the wave equations become

$$\begin{cases} \frac{n^2}{c^2}\frac{\partial^2 E^i}{\partial t^2} - \partial^j \partial_j E^i = 0 \\ \frac{n^2}{c^2}\frac{\partial^2 H^i}{\partial t^2} - \partial^j \partial_j H^i = 0 \end{cases} \tag{8.5}$$

In general, it's clear that both fields solve the same basic wave equation

$$\partial^i \partial_i f = \frac{1}{u^2}\frac{\partial^2 f}{\partial t^2}$$

The solution of this equation (in terms of complex exponentials) is

$$f(x^i, t) = e^{i(k^i x_i - \omega t)} \tag{8.6}$$

What happens when we apply the same differential operators that appear in Maxwell's equation to this solution?
Well we simply take the derivatives and see that we're actually dealing with an eigenfunction of the differential operator, with eigenvalues

$$\begin{cases} \partial_i f = ik_i f \\ \partial_t f = -i\omega f \end{cases} \tag{8.7}$$

Thus, in terms of electromagnetic fields we can write Maxwell's equations as follows

$$\begin{cases} k^i E_i = 0 \\ \epsilon^i{}_{jk} k^j E^k = \mu\omega H^i \\ k^i B_i = 0 \\ \epsilon^i{}_{jk} k^j H^k = -\epsilon\omega E^i \end{cases} \tag{8.8}$$

It's clear that these three vectors compose a mutually orthogonal triad. Therefore, if we consider the magnitudes of these vectors from the third equation, we get

$$H = \frac{\epsilon\omega}{k} E = \epsilon u E \tag{8.9}$$

Where we used $\omega/k = u$. If we rewrite them in terms of the refraction index $n = c/u$, defined $Z_0 = \sqrt{\mu_0/\epsilon_0}$ the "free space impedance" we also have

$$H = \frac{n}{Z_0} E$$

§§ 8.1.2 Energy Flow

As we saw before, the Poynting vector is defined as the cross product of E with H, and is a representation of the flux of electromagnetic energy per unit area. If we take an electromagnetic field which solves the wave equation, as

$$\begin{aligned} E^i &= E_0^i \cos\left(k^i x_i - \omega t\right) \\ H^i &= H_0^i \cos\left(k^i x_i - \omega t\right) \end{aligned} \tag{8.10}$$

We have that the Poynting vector for such fields is

$$S^i = \epsilon^i{}_{jk} E^j H^k = S_0^i \cos^2\left(k^i x_i - \omega t\right)$$

Where $S_0 = E_0 \times H_0$. Since, as we saw before k^i is perpendicular to both E and H, it must be parallel to the Poynting vector. Taken the average (remember that $\langle \cos^2(\theta) \rangle = 1/2$) we get

$$\left\langle S^i \right\rangle = \frac{1}{2} S_0^i = I \frac{k^i}{k} = I\hat{n} \tag{8.11}$$

Where I is known as the *irradiance* and has, clearly, value

$$I = \frac{1}{2} E_0 H_0 = \frac{n}{2Z_0} E_0^2$$

This is known also as *Malus' Law*.
The irradiance, defined as before, is nothing more than the rate of flow of energy, and it's proportional to the square of the amplitude of the electric field.
For isotropic media, then the direction of the energy flux is defined by both S^i and k^i.

§ 8.2 Polarization of Electromagnetic Waves

§§ 8.2.1 Linear Polarization

Definition 8.2.1 (Linear Polarization)**.** Consider a general plane harmonic wave with the following solution to Maxwell's equations:

$$E^i = E_0^i e^{i(k^i r_i - \omega t)}$$
$$H^i = H_0^i e^{i(k^i r_i - \omega t)}$$

If both $E_0^i,\ H_0^i \in \mathbb{R}^3$ are constant (real constant vectors), then the wave is said to be *linearly polarized*.

Definition 8.2.2 (Polarizer)**.** A *polarizer* is an optical element that generates linearly polarized light. One of such instrument is the Polaroid filter.
A polarizer is said to have two main axes, one transmission axis and one blocking axis. The transmission axis is the one that will let the component of the E^i field pass, therefore by definition polarizing the light wave. If the polarizer is *completely* transparent to the incoming light parallel to the transmission axis it's known as an *ideal polarizer*

Consider now some randomly polarized light that passes through an ideal linear polarizer, and suppose that it arrives such that E^i arrives at an angle θ with respect to the transmission axis. The transmitted magnitude then will simply be the projection of E^i onto the axis.
Called τ our axis, then

$$E_\tau = E\cos\theta$$

And since $I \propto E^2$, the transmitted intensity is

$$I_\tau = I\cos^2\theta$$

By the same reasoning as before, for unpolarized light then, since $\langle\cos^2\theta\rangle = 1/2$ we have that, the transmitted intensity through this polarizer is exactly half the incoming intensity, i.e.

$$I_\tau = \frac{1}{2}I_u$$

Definition 8.2.3 (Partial Polarization)**.** A light wave is said to be *partially polarized* if it's made by a mixture of polarized and unpolarized light. The *degree of polarization* P is defined as the ratio between the intensity of polarized light I_p and the total intensity I_T (the sum of the intensity of polarized and unpolarized light)

$$P = \frac{I_p}{I_T}$$

For the special case of partial *linear* polarization, it can be calculate as the difference between the maximum intensity minus the minimum intensity of light, normalized with respect to the total intensity

$$P_{pl} = \frac{I_{max} - I_{min}}{I_{max} + I_{min}}$$

It's clear that $P \in [0, 1]$, where $P = 0$ indicates completely unpolarized light and $P = 1$ indicates completely polarized light

§§ 8.2.2 Circular and Elliptical Polarization

Consider two orthogonal waves $\psi_1,\ \psi_2$ with equal amplitudes $A_1 = A_2$. Due to their orthogonality there's a $\pi/2$ phase shift between the two, and can be written as:

$$\psi_1 = E_0 \cos(kz - \omega t)\,\hat{x}^i \psi_2 \quad = E_0 \sin(kz - \omega t)\,\hat{y}^i \tag{8.12}$$

Using the principle of superposition, then, the total field (i.e., total wave) is then

$$E^i = E_0^i \left(\cos(kz - \omega t)\,\hat{x}^i + \sin(kz - \omega t)\,\hat{y}^i\right) \tag{8.13}$$

This is clearly a solution to the Maxwell equation $\Box E^i = 0$ and therefore can be seen as a valid electromagnetic wave, especially one which rotates around the z axis with angular frequency ω. The direction of rotation defines if it's a *right circularly polarized wave* $(-y)$ or a *left circularly polarized wave* $(+y)$.
Using complex notation and noting that the $\pi/2$ shift can be defined with a multiplication by i of the y component, hence

$$E^i = \left(\hat{x}^i \pm i\hat{y}^i\right) e^{i(kz - \omega t)} \tag{8.14}$$

The circular polarization is just a special case of an elliptical polarization, where $A_1 \neq A_2$, where

$$E_0^i = E_0 \hat{x}^i + iE_1 \hat{y}^i \quad \Longrightarrow \quad E^i = E_0^i e^{i(kz - \omega t)} \tag{8.15}$$

It's also clear how this notation generalizes polarization altogether, in fact, if E_0^i is a real vector we get back our usual linear polarization that we treated before.
A question arises now, what optical element can create elliptical polarization from a general light wave? The answer is *quarter-wave plates*.

Definition 8.2.4 (Quarter Wave Plate)**.** A *quarter wave plate*, also known as a $\lambda/4$*-plate* is an optical element made of two different refracting transparent crystals, which combined give a polarizer with two different transmission axes.
The axis with the highest refraction index (n_1) is known as the *fast axis*, while the axist with the smallest refraction index (n_2), is known as the *slow axis*.
The instrument is built such that the optical thickness (nd, where d is the thickness), obeys the following relation

$$n_1 d - n_2 d = \frac{\lambda_0}{4}$$

With λ_0 being the vacuum wavelength of the considered wave. Solving for d, we then have the quarter-wave plate relation

$$d = \frac{\lambda_0}{4(n_1 - n_2)} \tag{8.16}$$

§§ 8.2.3 Jones Calculus

Consider a general plane harmonic electromagnetic wave, i.e.

$$E_0^i = E_{0x}\hat{x}^i + E_{0y}\hat{y}^i \tag{8.17}$$

Since in the most general case $E_{0x}, E_{0y} \in \mathbb{C}$ we might choose to use the $\hat{x}^i, \hat{y}^i$ basis and work directly in $\mathbb{C}^2$. We can define then the *Jones Vector* as follows

Definition 8.2.5 (Jones Vector)**.** We define a *Jones vector* as a vector $v^i \in \mathbb{C}^2$ defined as follows:

$$\begin{pmatrix} E_{0x} \\ E_{0y} \end{pmatrix} = \begin{pmatrix} |E_{0x}|e^{i\phi_x} \\ |E_{0y}|e^{i\phi_y} \end{pmatrix} \tag{8.18}$$

Then, this vector is the most general way of defining a plane harmonic electromagnetic wave. As we have seen before, then we have

1. For linearly polarized waves in one direction (x or y):

$$A\begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad A\begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{8.19}$$

2. For linearly polarized waves (45 degrees):

$$A\begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{8.20}$$

3. For left-circularly polarized waves:

$$A\begin{pmatrix} 1 \\ i \end{pmatrix} \tag{8.21}$$

4. For right-circularly polarized waves:

$$A\begin{pmatrix} 1 \\ -i \end{pmatrix} \tag{8.22}$$

This representation in terms of complex vectors is really useful for calculating the polarization (and amplitude) of the wave resulting from the superposition of two differently polarized waves. I.e.: suppose you have a RCP (Right Circularly Polarized) wave and a LCP (Left Circularly Polarized) wave with unitary amplitude in some units. After the superposition we get:

$$\begin{pmatrix} 1 \\ -i \end{pmatrix} + \begin{pmatrix} 1 \\ i \end{pmatrix} = 2\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Then, RCP+LCP=x-Linear polarization.
The action of optical elements can then be seen as the action of a matrix upon these vectors. Then, as an example, we can represent a linear polarizer on the x direction as:

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{8.23}$$

And so on for other optical elements.
In general, we have:

1. Linear Polarizers
   - Horizontal polarizers
   $$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{8.24}$$
   - Vertical polarizers
   $$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{8.25}$$
   - $\pm 45^\circ$ polarizers
   $$\frac{1}{2}\begin{pmatrix} 1 & \pm 1 \\ \pm 1 & 1 \end{pmatrix} \tag{8.26}$$
2. $\lambda/4$ Plates
   - Vertical fast axis
   $$\begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \tag{8.27}$$
   - Horizontal fast axis
   $$\begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix} \tag{8.28}$$
   - $\pm 45^\circ$ fast axis
   $$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & \pm i \\ \pm i & 1 \end{pmatrix} \tag{8.29}$$
3. $\lambda/2$ Plates
   - Horizontal or vertical fast axis
   $$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{8.30}$$
4. Retarders
   $$\begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{i\phi} \end{pmatrix} \tag{8.31}$$
5. Phase Changers
   $$\begin{pmatrix} e^{i\phi_x} & 0 \\ 0 & e^{i\phi_y} \end{pmatrix} \tag{8.32}$$
6. Circular Polarizers

- Left Circular Polarizer

$$\frac{1}{2}\begin{pmatrix}1 & -i\\ i & 1\end{pmatrix} \tag{8.33}$$

- Right Circular Polarizer

$$\frac{1}{2}\begin{pmatrix}1 & i\\ -i & 1\end{pmatrix} \tag{8.34}$$

In general, for a train of optical devices $(A^i_j)_1, \cdots, (A^i_j)_n$, where A_i are complex matrices, we have that the resulting wave R^i will be simply the following product, given I^i as our incident wave:

$$\prod_{\alpha=1}^{n} (A^i_j)_\alpha I^j = R^i \tag{8.35}$$

§§ 8.2.4 Orthogonal Polarization

Given two waves $E^i_1, E^i_2 \in \mathbb{C}^2$, they're said to be *orthogonally polarized* if, the complex scalar product between the two is null.
I.e.

$$\langle E^i_1, E^j_2 \rangle = E^i_1 \overline{E}^2_i = 0 \tag{8.36}$$

9 Reflection and Refraction

§ 9.1 Reflection and Refraction

§§ 9.1.1 Snell's Law

The experimental effect of reflection and refraction can be studied in two ways mainly:

1. Using wave mechanics
2. Using electromagnetism

The first treatment is pretty simple to guess. Experimentally, what we see is an incoming (incident) wave coming from a place with some refraction index n_1, passing through a boundary plane, before reaching the second place where there is a new different refraction index $n_2 \neq n_1$. Experimentally we might consider the final result as three waves, ψ, the incident wave, ψ_{rr}, the refracted wave, and ψ_{re}, the reflected wave.
Being all these waves, we can write them in terms of solutions to the wave equation in absence of sources, complex exponentials.
Ignoring the amplitude factor we have

$$\begin{aligned} \psi(x^i, t) &= e^{i(k^i x_i - \omega t)} \\ \psi_{rr}(x^i, t) &= e^{i(k^i_{rr} x_i - \omega t)} \\ \psi_{re}(x^i, t) &= e^{i(k^i_{re} x_i - \omega t)} \end{aligned} \tag{9.1}$$

Since ωt is the same for every wave (as it should) we must have

$$k^i x_i = k^i_{rr} x_i = k^i_{re} x_i$$

And, if we put ourselves in the reference frame of the incidence plane (perpendicular to the boundary plane between the two zones), calling the angle of incidence θ, the angle of reflection θ' and the angle of refraction ϕ, this condenses to

$$k \sin\theta = k_{re} \sin\theta' = k_{rr} \sin\phi \tag{9.2}$$

By definition, k and k_{re} are traveling in the same medium with n_1 as the refraction index, hence $k = k_{re}$ and we get through the previous equality

$$\theta' = \theta$$

We're left with a single equation then

$$k \sin\theta = k_{rr} \sin\phi$$

By definition of k, we can also say

$$\frac{k_{rr}}{k} = \frac{\omega/u_2}{\omega/u} = \frac{\omega n_2}{c}\frac{c}{\omega n_1} = \frac{n_2}{n_1} = n$$

I.e., the ratio between the modulus of the wavevector of the incoming wave and the modulus of the wavevector of the refracted wave is equal to the ratio of the two refraction indexes, aso known as the *average refraction index* n.
Combining it all we get *Snell's law*

$$\sin\phi = n \sin\theta \tag{9.3}$$

§§ 9.1.2 Fresnel Equations

Snell's law can let us calculate with ease the reflection and refraction angles, but by definition it doesn't convey any information about the magnitudes of the waves. For this, we need electromagnetic theory. Using the second equation of (8.8) we know that the magnetic vectors are orthogonal both to the electric and wavevector fields, and they're equal to

$$H^i = \frac{1}{\mu\omega}\epsilon^i{}_{jk} k^j E^k \tag{9.4}$$

Consider now two possible cases.

1. TE polarization (or S, σ polarization), where the E^i field is perpendicular to the incidence plane
2. TM polarization (or P, π polarization), where the H^i field is perpendicular to the incidence plane (hence E^i is parallel to the plane)

Considering the two section as dielectrics, it's possible to use the boundary conditions for E and H, and knowing how the normal and tangential components behave while passing through the plane we can have more information on the amplitude of the two resulting waves (reflected and refracted).
For TE (S, σ) polarization we therefore have, remembering that the tangential component of the electric field is conserved

$$\begin{cases} E + E_{re} = E_{rr} \\ H_{re}\cos\theta - H\cos\theta = -H_{rr}\cos\phi \\ -kE\cos\theta + k_{re}E_{re}\cos\theta = -k_{rr}E_{rr}\cos\phi \end{cases} \tag{9.5}$$

While for TM (P, π) polarization, we have

$$\begin{cases} H - H_{re} = H_{rr} \\ E\cos\theta + E_{re}\cos\theta = E_{rr}\cos\phi \\ kE - k_{re}E_{re} = k_{rr}E_{rr} \end{cases} \tag{9.6}$$

Solving the system of equations, we can define *four* coefficients, two for TE polarization and one for TM polarization. Indicating them as $t_{s/p}$ and $r_{s/p}$ (respectively s/p-polarization transmission coefficient

and s/p-reflection coefficient), as the ratio of the intensity of the transmitted/reflected electric field with the initial incident field, i.e.

$$\begin{aligned} t_s &= \left(\frac{E_{rr}}{E}\right)_{TE} \quad r_s = \left(\frac{E_{re}}{E}\right)_{TE} \\ t_p &= \left(\frac{E_{rr}}{E}\right)_{TE} \quad r_p = \left(\frac{E_{re}}{E}\right)_{TM} \end{aligned} \tag{9.7}$$

Solving the equations, we get for the r coefficient

$$\begin{aligned} r_s &= \frac{\cos\theta - n\cos\phi}{\cos\theta + n\cos\phi} \\ r_p &= \frac{\cos\phi - n\cos\theta}{n\cos\theta + \cos\phi} \end{aligned} \tag{9.8}$$

Where, as usual, $n = n_2/n_1$ is the average refraction index.
A special case comes from *normal incidence*, i.e. when $\theta = \phi = 0$. The two coefficients become

$$r_s = r_p ? \frac{1-n}{1+n} \tag{9.9}$$

It's clear now, since n can be also greater than one, that two main cases must exists, one where $n < 1$ and therefore the phase mustn't change, and one where $n > 1$, where phase must change by π. It's easy to believe this is true in the realm of complex numbers, where we can write $e^{i\pi} = -1$, which gives the aforementioned phase change by π. The second case, where $n > 1$ will be treated with more detail later in the section on *total internal reflection*.
With these coefficients, using Snell's law, we can get *Fresnel's equations*, which give us a way to calculate the amplitudes of the electromagnetic waves which get reflected/refracted. Hence, using $n = \sin\theta/sin\phi$ and cleaning up what we get via trigonometric identities, we have

$$\begin{aligned} r_s &= -\frac{\sin(\theta-\phi)}{\sin(\theta+\phi)} \\ t_s &= \frac{2\cos\theta\sin\phi}{\sin(\theta+\phi)} \\ r_p &= -\frac{\tan(\theta-\phi)}{\tan(\theta+\phi)} \\ t_p &= \frac{2\cos\theta\sin\phi}{\sin(\theta+\phi)\cos(\theta-\phi)} \end{aligned} \tag{9.10}$$

Or, by eliminating ϕ using Snell's law

$$\begin{aligned} r_s &= \frac{\cos\theta - \sqrt{n^2 - \sin^2\theta}}{\cos\theta + \sqrt{n^2 - \sin^2\theta}} \\ r_p &= \frac{\sqrt{n^2 - \sin^2\theta} - \cos\theta}{\cos\theta - \sqrt{n^2 - \sin^2\theta}} \end{aligned} \tag{9.11}$$

Defined these coefficients, for what we have already said about intensity and by the shape of these coefficients, we can define a new parameter as follows

Definition 9.1.1 (Reflectance). Given the reflection coefficients (for s or p polarization) we define the *reflectance* as the fraction of light energy that gets reflected, i.e.

$$\begin{aligned} R_s &= |r_s|^2 = \left|\frac{E_{re}}{E}\right|^2_{TE} \propto I_r^{TE} \\ R_p &= |r_p|^2 = \left|\frac{E_{re}}{E}\right|^2_{TM} \propto I_r^{TM} \end{aligned} \tag{9.12}$$

For normal incidence, therefore, it just reduces to the following

$$R_s = R_p = \left(\frac{n-1}{n+1}\right)^2 \tag{9.13}$$

§ 9.2 Total Internal Reflection

§§ 9.2.1 External and Internal Reflection

Going back to our two possible cases of outcome for reflection in normal incidence, we can define the following

1. Internal reflection, for $n > 1$
2. External reflection, for $n < 1$

It's clear from Fresnel's equations that when $n > 1$, all values of θ are possible and both r and t coefficients are real, while for $n < 1$ there exists a θ_c, known as the *critical angle* for which $\sin\theta_c = n$, where for $\theta < \theta_c$ transmission *can happen*.
For $\theta > \theta_c$ the coefficients $r_s, r_p \in \mathbb{C}$ and trasmission *doesn't happen*, since it cannot. Here, in this case, since $\sin\theta > n$, we get

$$\begin{aligned} r_s &= \frac{\cos\theta - i\sqrt{\sin^2\theta - n^2}}{\cos\theta + i\sqrt{sin^2\theta - n^2}} \\ r_p &= \frac{i\sqrt{sin^2\theta - n^2} - n^2\cos\theta}{n^2\cos\theta + i\sqrt{\sin^2\theta - n^2}} \end{aligned} \tag{9.14}$$

Note that, by definition then:

$$R_s = R_p = \overline{r}_s r_s = \overline{r}_p r_p = 1 \tag{9.15}$$

Also, if it wasn't clear, the critical angle can be calculated with ease as follows

$$\theta_c = \arcsin(n) \tag{9.16}$$

Hence we are in a regime of so called *total reflection*, or, to be precise, *total internal reflection*, which we will call *TIR* for ease.
It's possible to use TIR experimentally for transmitting light through continous fibers (see fiber optics), and create *light circuits*.
Consider a single fiber as a solid cylindrical dielectric, immersed in a medium for which $n_{ext} < n_{int}$.

We find ourselves in the case of $n < 1$, and if the angle of incidence of light to the fiber optic α follows $\alpha > \theta_c$, the light is trapped inside the dielectric in a regime of TIR.
By definition, then

$$\alpha = \arcsin\left(\sqrt{n_{int}^2 - n_{ext}^2}\right) \tag{9.17}$$

§§ 9.2.2 Brewster Angle

For p (or TM) polarization, there exists another "critical angle" for which $r_p = 0$. From the Fresnel formula in TIR regime for r_p, this angle, known as *Brewster angle* or *polarization angle* θ_B is defined as

$$\theta_B = \arctan(n) \tag{9.18}$$

For $\theta = \theta_B$ all unpolarized light gets refracted with TE (or s) polarization. ($r_p = 0$, there is no polarization)
A cool application of this principle is the *Brewster window*, an optical element for which the incoming light is incident at $\theta = \theta_B$ on some glass plate with two parallel faces. No light gets reflected, and what gets refracted is TE polarized.

§§ 9.2.3 Evanescent Wave

Although the electromagnetic wave in TIR regime gets reflected completely, there exists an electromagnetic field beyond the boundary of TIR, this field defines what's known as the *evanescent wave*. Consider the mathematical function which defines a refracted (transmitted) wave:

$$E_t^i = E_{rr}^i e^{i(k_{rr}^i r - \omega t)}$$

Choosing a reference system with the x axis orthogonal to the transmission axis we have

$$k_{rr}^i r_i = k_{rr} x \sin\phi - k_{rr} y \cos\phi$$

Using Snell's law for writing $\cos\phi$ in terms of $\sin\theta$ we have, simplifying k_{rr} to k

$$k^i r_i = kx \sin\phi - iky\sqrt{\frac{\sin^2\theta}{n^2} - 1} \tag{9.19}$$

Inserting it back to our transmitted E field, we get, given the following definitions

$$\begin{cases} \alpha = k\sqrt{\dfrac{\sin^2\theta}{n} - 1} \\ k_1 = \dfrac{k}{n}\sin\theta \end{cases}$$

An evanescent wave decaying on the y direction

$$E_t^i = E_{rr}^i e^{-\alpha|y|} e^{i(k_1 x - \omega t)} \tag{9.20}$$

§§ 9.2.4 Phase Changes

In the case of TIR, as we implied before, the complex nature of the reflection coefficients imply a phase change in function of the incidence angle.
By definition in TIR $|r_s| = |r_p| = 1$, and since $r_s, r_p \in \mathbb{C}$, by definition of complex angle we write both in terms of complex exponentials as follows

$$\begin{cases} r_s = e^{i\delta_s} = \dfrac{ae^{-i\alpha}}{ae^{i\alpha}} \\ r_p = e^{i\delta_p} = \dfrac{be^{-i\beta}}{be^{i\beta}} \end{cases} \tag{9.21}$$

The arguments of the two complex numbers, δ_s, δ_p, for how we defined them are exactly equal to 2α and 2β for how we defined them.
In terms of tangents, we have therefore

$$\tan\alpha = \tan\frac{\delta_s}{2}$$
$$\tan\beta = \tan\frac{\delta_p}{2}$$

Defining the tangent in terms of r and its complex conjugate we have, since

$$z + \overline{z} = 2\mathfrak{Re}(z)$$
$$z - \overline{z} = 2i\mathfrak{Im}(z)$$

That

$$\begin{aligned} \tan\alpha &= \frac{\mathfrak{Im}(r_s)}{\mathfrak{Re}(r_s)} = \frac{\sqrt{\sin^2\theta - n^2}}{\cos\theta} \\ \tan\beta &= \frac{\mathfrak{Im}(r_p)}{\mathfrak{Re}(r_p)} = \frac{\sqrt{\sin^2\theta - n^2}}{n^2\cos\theta} \end{aligned} \tag{9.22}$$

Defining the *phase difference* Δ as follows

$$\Delta = \delta_p - \delta_s$$

We have

$$\tan\Delta = \frac{\cos\theta\sqrt{\sin^2\theta - n^2}}{\sin^2\theta} \tag{9.23}$$

§ 9.3 Reflection Matrix

Using Jones calculus in the regime of reflection and refraction, we can imagine the s and p polarization components of the electromagnetic wave as a basis for $\mathbb{C}^2$, with the most general vector given by (p, s). Here we can define a (isotropic) *reflection matrix* R^i_j as:

$$R^i_j = \begin{pmatrix} -r_p & 0 \\ 0 & r_s \end{pmatrix} \tag{9.24}$$

And a *transmission matrix* T_j^i analogously as

$$T_j^i = \begin{pmatrix} t_p & 0 \\ 0 & t_s \end{pmatrix} \tag{9.25}$$

In the various regimes of reflection this matrix can be more or less complex.
Taken the case of normal incidence, as we saw we have

$$r_s = r_p = \frac{1-n}{1+n}$$

And therefore

$$R_j^i = \frac{1-n}{1+n} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{9.26}$$

For what's known as *near grazing incidence* (i.e. when $r_p = r_s$ for external reflection and $r_p = -r_s$ for internal reflection) we have

$$R_j^i = \pm \delta_j^i \tag{9.27}$$

While for TIR, where $r_s = e^{i\delta_s}$ and $r_p = e^{i\delta_p}$, we have

$$R_j^i = \begin{pmatrix} e^{-i\delta_p} & 0 \\ 0 & e^{-i\delta_s} \end{pmatrix}$$

Note that, when a vector gets reflected in TIR regime, we have the following result:

$$\begin{pmatrix} A' \\ B' \end{pmatrix} = \begin{pmatrix} e^{-i\delta_p} & 0 \\ 0 & e^{-i\delta_s} \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = e^{-i\delta_p} \begin{pmatrix} A \\ Be^{i\Delta} \end{pmatrix} \tag{9.28}$$

I.e., the p component of the (generally elliptically polarized wave) gets a phase change by exactly $\Delta = \delta_p - \delta_s$

10 Coherence and Interference

§ 10.1 Coherent Waves

§§ 10.1.1 Interference

Consider two interacting waves with electric fields $E^i_{(1)}$ and $E^i_{(2)}$. In general these two waves are solutions of the Maxwell wave equations, therefore, the most general shape they can have is the following:

$$\begin{cases} E^i_{(1)} = E^i_1 e^{ik^i_1 r_i - i\omega t + i\phi_1} \\ E^i_{(2)} = E^i_2 e^{ik^i_2 r_i - i\omega t + i\phi_2} \end{cases} \tag{10.1}$$

Where $\phi_1, \phi_2 \in [0, 2\pi]$ are two general phase factors.

Definition 10.1.1 (Mutually Coherent Waves)**.** Given two general waves, they're said to be *coherent* if and only if, given the phases ϕ_1 and ϕ_2, we have

$$\phi_1 - \phi_2 = k \tag{10.2}$$

Where $k \in \mathbb{R}$ is a constant real number.

Getting back to our interference pattern, we have that the total field that will be measured must be a linear superposition of these two general field. Therefore, the measured total intensity I is, (considering two linearly polarized waves, for ease of calculation))

$$I = \|E\|^2 = \left(E^i_{(1)} + E^i_{(2)}\right)\overline{\left(E^{(1)}_i + E^{(2)}_i\right)} = E^2_1 + E^2_2 + 2\mathfrak{Re}\left(E^i_{(1)}\overline{E^{(2)}_i}\right)$$

Evaluating the third addendum on the previous expression, using the general wave solution (10.1), we get

$$2\mathfrak{Re}\left(E^i_{(1)}\overline{E^{(2)}_i}\right) = 2E^i_1 E^2_i \mathfrak{Re} e^{i(k^i_1 - k^i_2)r_i + i(\phi_1 - \phi_2)}$$

Evaluating the right hand side of the previous equation, we have that the intensity will in general be calculated (for linearly polarized waves) as

$$I = I_1 + I_2 + 2E^i_1 E^2_i \cos\left[\left(k^i_1 - k^i_2\right) r_i + i(\phi_1 - \phi_2)\right] \tag{10.3}$$

Where I_1 and I_2 are the partial intensities of the single waves. The third term, is known as the *interference term*, which, since it's comprised between $[-1, 1]$, makes the intensity oscillate between $I_1 + I_2 \leq I \leq I_1 + I_2$ depending on

$$\theta(r^i) = i(k_1^i - k_2^i)r_i + i(\phi_1 - \phi_2) \tag{10.4}$$

The result are interference fringes. It's also clear that this can happen only and only if the waves are mutually coherent. If they weren't $\phi_1 - \phi_2$ would vary randomly, and, by definition

$$\langle \cos\left(\theta(r^i)\right) \rangle = 0$$

Which implies that the interference term is in general null, and $I = I_1 + I_2$ always.
Interference fringes can be obtained also through spatial variations, since the argument of the cosine depends both on the phase difference and the distance r.
Interference fringes also cannot appear in the case of two orthogonally polarized waves, because by definition their scalar product is zero, eliminating the interference term, independently from their mutual coherence.

§§ 10.1.2 Young's Double Slit Experiment

In 1802 dr. Young et al. managed to build an experiment which managed to show the interference pattern caused by the interference of two coherent electromagnetic waves.
The experimental setup is pretty simple, a coherent source (point-like) S emits a single electromagnetic wave, which passes through two slits S_1 and S_2, placing a detector (in case of light a simple wall is enough), it's possible to observe regular peaks of intensity.
Theoretically it all depends on the phase difference between the two waves that come out of each slit. Taken a random point P on the screen, we can say that it will be at a distance d_1 from S_1 and d_2 from S_2. The phase difference will obviously depend on the difference of the two distances from the chosen point.
For convenience, we search for the points of max intensity, (i.e., when $\cos\theta(r) = 1$ and the interference term is the maximum possible). Due to coherence between the two fields, we can say without any problem that

$$\theta(r) = k^i r_i = k_0(d_2 - d_1)$$

Hence, our intensity minimums will be

$$\max(I) = I|_{\cos\theta(r)=1} \implies k_0(d_2 - d_1) = \pm 2n\pi, \quad n \in \mathbb{N}$$

Remembering that $k_0 = 2\pi\lambda^{-1}$, we get the final simple result

$$|d_2 - d_1| = n\lambda \tag{10.5}$$

I.e., the distance between the intensity peaks is strictly tied to the distance between the slits and the wavelength of the electromagnetic wave, where the successive peaks will always be at integer multiples of the wavelenght.
In general, for two slits distant h between each other, chosen a reference frame for which the origin is the middle point between the two slits and the screen is distant x from this, taken a general point (x, y) on it we will have that

$$|d_2 - d_1| = \sqrt{x^2 + \left(y + \frac{h}{2}\right)^2} + \sqrt{x^2 + \left(y - \frac{h}{2}\right)^2}$$

Therefore, solving the previous equation, searching for the intensity peaks, we have, after a second order approximation that

$$|d_2 - d_1| \approx x\left(1 + \frac{1}{2x^2}\left(y + \frac{h}{2}\right)^2\right) - x\left(1 + \frac{1}{2x^2}\left(y - \frac{h}{2}\right)^2\right) \approx n\lambda$$

And solving

$$y \approx \frac{nx\lambda}{h} \tag{10.6}$$

I.e., the distance between peaks is approximately an integer multiple of the distance from the slits times the wavelength, over the distance between the two slits themselves.

§§ 10.1.3 Michelson Interferometer

Another slightly more complex system which analyzes the interference phenomena, is the Michelson interferometer. Michelson's interferometer was developed in 1880 by Michelson et al.
It's known as a *division of amplitude* interferometer, for the simple reason that it divides the beam in two (ideally equal) parts after passing through a so called *beam splitter*. A beam splitter is a particular optical device, which divides an electromagnetic wave in two (ideally equal) parts, sending one in the transverse direction.
For building up this interferometer, the setup is a source of coherent electromagnetic waves, a beam splitter, two mirrors and a detector. Placed the beam splitter at the center of the setup, with one of the two mirrors M_2 placed at a distance d_2 from the splitter, and the other one, M_1 at a distance d_1 on the other transmission axis of the splitter. Opposite to the mirror there will be our detector D, which will measure the interference fringes caused by the interaction of the two beams when they end up interacting again before the reflection at the beam splitter.
The observed fringes directly depend on the optical path of the two mirrors, which is modified by the simple motion of one of the two.
This setup is equivalent to having two point-like sources distant $d = |d_2 - d_1|$ from each other, interfering and reaching a detector.
Due to the two waves being ideally identical due to the inner workings of the beam splitter, (i.e., in this case $k_1 = k_2$) the measured intensity at the detector will depend only on the phase shift induced by the optical distance traveled, and

$$I = I_1 + I_2 + 2\sqrt{I_1 I_2}\cos(\phi_1 - \phi_2)$$

Again, the interference peaks will be when $\cos(\phi_1 - \phi_2) = 1$, i.e., when

$$\phi_1 - \phi_2 = k_0 d = \frac{2\pi d}{\lambda} \tag{10.7}$$

With intensity

$$I = I_1 + I_2 + 2\sqrt{I_1 I_2} \tag{10.8}$$

§ 10.2 Partial Coherence

In reality, what we treated before is an approximation of reality. In fact, in general, two interfering electromagnetic waves *aren't necessarily coherent*, or at least they aren't always coherent.
By definition of coherence, this means that the phases of the signals vary randomly through time, which brings forward the idea of averaging the total amplitude.

Definition 10.2.1 (Time Average)**.** We define the *time average* of a periodic function $f(t)$, as the following improper integral

$$\langle f \rangle = \lim_{T\to\infty} \frac{1}{T} \int_0^T f(t)\mathrm{d}t \tag{10.9}$$

Note how the final result doesn't depend on time.

Applied to what we found for the amplitude, we then have

$$\langle I \rangle = I_1 + I_2 + 2\mathfrak{Re}\left(\left\langle E_1^i \overline{E_i^2} \right\rangle\right) \tag{10.10}$$

Where we conveniently defined $I_1 = \langle E_1^2 \rangle$ and $I_2 = \langle E_2^2 \rangle$.
Looking back at the previous experimental setups, it's clear how interference depends on the difference of length of the optical paths of the two interfering waves, due to our choice of time averaging we can see the difference as an added needed time to reach the detector, let's say τ, for which, if E_1 reaches it at a time t, E_2 will reach it at a time $t+\tau$. Considering that the interference is seen when both waves have reached the detector, we have that the interference term is the following

$$2\mathfrak{Re}\left(\left\langle E_1^i(t)\overline{E_i^2(t+\tau)} \right\rangle\right) \tag{10.11}$$

The integral inside the real part operator, can then be redefined as follow, as a new function depending on τ

Definition 10.2.2 (Correlation Function)**.** The *correlation function* $\Gamma_{12}(\tau)$ is defined as follows, forgetting the vectorial nature of electromagnetic waves for a second,

$$\Gamma_{12}(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_0^T E_1(t)\overline{E_2(t+\tau)}\,\mathrm{d}t \tag{10.12}$$

Clearly, $\Gamma_{12}(\tau) = \Gamma_{21}(\tau)$, and from this we can also define the *autocorrelation function* $\Gamma_{11}(\tau)$ and $\Gamma_{22}(\tau)$. It's also evident that

$$\begin{aligned} \Gamma_{11}(0) &= I_1 \\ \Gamma_{22}(0) &= I_2 \end{aligned} \tag{10.13}$$

Another, more comfortable definition is the following

Definition 10.2.3 (Degree of Spatial Coherence)**.** Defined the correlation function $\Gamma_{12}(\tau)$ as before, we define the *degree of spatial coherence* $\gamma_{12}(\tau)$ as the normalized correlation function

$$\gamma_{12}(\tau) = \frac{\Gamma_{12}(\tau)}{\sqrt{\Gamma_{11}(0)\Gamma_{22}(0)}} = \frac{\Gamma_{12}(\tau)}{\sqrt{I_1 I_2}} \tag{10.14}$$

Clearly, $-1 \leq \gamma_{12}(\tau) \leq 1$

Then, the irradiance or amplitude I, can be rewritten as follows

$$I = I_1 + I_2 + 2\sqrt{I_1 I_2}\mathfrak{Re}\left(\gamma_{12}(\tau)\right) \tag{10.15}$$

In general, as we'll see later, γ_{12} is a complex-valued function periodic in τ. It's also obvious that the interference bands will appear only and only if $\gamma_{12}(\tau) \neq 0$.
Two waves will then said to be

1. Completely coherent, if $|\gamma_{12}(\tau)| = 1$
2. Completely incoherent, if $|\gamma_{12}(\tau)| = 0$
3. Partially coherent, if $0 \leq |\gamma_{12}(\tau)| \leq 1$

The minimum and maximum values of amplitude in a regime of partial or complete coherence then, can be found as follows

$$\begin{aligned} I_{max} &= I_1 + I_2 + 2\sqrt{I_1 I_2}|\gamma_{12}(0)| \\ I_{min} &= I_1 + I_2 - 2\sqrt{I_1 I_2}|\gamma_{12}(0)| \end{aligned} \tag{10.16}$$

Definition 10.2.4 (Fringe Visibility)**.** Another important value in the analysis of interference is the *visibility of fringes* $\mathcal{V}$, defined as follows

$$\mathcal{V} = \frac{I_{max} - I_{min}}{I_{max} + I_{min}} \tag{10.17}$$

Or

$$\mathcal{V} = \frac{2\sqrt{I_1 I_2}|\gamma_{12}(0)|}{I_1 + I_2} \tag{10.18}$$

Note that, by definition, then if $I_1 = I_2 = I$, the fringe visibility is simply the absolute value of the degree of coherence

$$\mathcal{V} = |\gamma_{12}(0)|$$

This constant, by definition, is $0 \leq \mathcal{V} \leq 1$, where a value of 0 implies that there is no interference, while a value of 1 implies the maximum possible interference between the waves

§§ 10.2.1 Coherence Time and Coherence Length

In order to understand better the regime of partial coherence, we take a quasimonochromatic electromagnetic wave, with a random phase $\phi(t)$, which is a periodic step function which changes value every *coherence time* τ_0.
Suppose that this electromagnetic wave gets split at some point, and the two resulting waves travel two different optical paths long d_1, d_2, and rejoin at some point creating an interference pattern. In general, the two waves will have the following mathematical shape

$$E^i(t) = E_0^i e^{-i\omega t + i\phi(t)}$$

Indicating the amplitude $I_{1,2} = |E_0|^2$ we get that then, the *self-degree of coherence* is

$$\gamma(\tau) = \frac{\left\langle E^i(t) E_i(t+\tau)\right\rangle}{I} = \lim_{T\to\infty}\frac{1}{T}\int_0^T e^{-i\omega t + i\phi(t)} e^{i\omega(t+\tau) - i\phi(t+\tau)}\mathrm{d}t$$

Simplifying the exponentials inside we get

$$\gamma(\tau) = e^{i\omega\tau} \lim_{T\to\infty} \frac{1}{T} \int_0^T e^{i(\phi(t)-\phi(t+\tau))} \mathrm{d}t$$

For evaluating this integral we need to define the behavior of $\phi(t) - \phi(t+\tau)$. Firstly we know that it's a step function which varies randomly periodically with period τ_0, hence, the difference will be zero until we find ourselves with $\phi(t+\tau)$ (or vice-versa, $\phi(t)$) in a different step. Mathematically, for the first period

$$\begin{cases} \phi(t) - \phi(t+\tau) = 0 & 0 < t < \tau_0 - \tau \\ \phi(t) - \phi(t+\tau) = \Delta & \tau_0 - \tau < t < \tau_0 \end{cases}$$

With $\Delta \in [0, 2\pi]$ being a random value. Then, for a single period, the integral before becomes

$$\frac{e^{i\omega\tau}}{\tau_0} \left(\int_0^{\tau_0-\tau} \mathrm{d}t + \int_{\tau_0-\tau}^{\tau_0} e^{i\Delta} \mathrm{d}t \right) = \left(\frac{\tau_0 - \tau}{\tau_0} e^{i\omega\tau} + \frac{\tau}{\tau_0} e^{i\Delta} \right)$$

Time-averaging the result, we get the normalized autocorrelation function for these general waves. Remembering the randomness of $e^{i\Delta}$ we have that its average will be 0, leaving only the first part, which, simplified becomes

$$\gamma(\tau) = \begin{cases} \left(1 - \frac{\tau}{\tau_0}\right) e^{i\omega\tau} & \tau < \tau_0 \\ 0 & \tau \geq \tau_0 \end{cases} \tag{10.19}$$

In this case, $I_1 = I_2$ (by definition), hence $|\gamma| = \mathcal{V}$, i.e.

$$\mathcal{V} = |\gamma(\tau)| = \begin{cases} 1 - \frac{\tau}{\tau_0} & \tau < \tau_0 \\ 0 & \tau \geq \tau_0 \end{cases} \tag{10.20}$$

This clearly gives the name *correlation time* to τ_0, it's evident how the waves become completely uncorrelated after for $\tau \geq \tau_0$, and the fringes are not visible! Analogously, we can define a related length called *coherence length*, as follows

$$l_c = c\tau_0 \tag{10.21}$$

This value corresponds to the maximum length that of an uninterrupted wave train

§§ 10.2.2 Spectral Resolution of Finite Wave Trains

In nature, strictly monochromatic sources of electromagnetic waves do not exist, and the best we can hope for is a finite wave train with frequencies spread around some mean value $\langle\omega\rangle = \omega_0$.
By definition of finite wave train, there should be some way to analyze the relationship between the frequency spread (or line width) and the coherence of a source.
Fourier calculus comes in our help for this task. Called $f(t)$ our wavefunction we define the Fourier pair $g(\omega)$ as

$$g(\omega) = \hat{\mathcal{F}}[f]$$

In our case, the Fourier transform, transforms a wavefunction from the time space to the frequency space. As we should know from mathematics, the Fourier operator is invertible, and the frequency-time pair can be defined in integral form as follows

$$f(t) = \hat{\mathcal{F}}^{-1}[g] = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} g(\omega) e^{-i\omega t} \mathrm{d}\omega$$
$$g(\omega) = \hat{\mathcal{F}}[f] = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(t) e^{i\omega t} \mathrm{d}t$$

Getting back to our analysis, consider $f(t)$ being the wavefunction of some finite wave train with coherence time τ_0. By definition then, we have

$$f(t) = \begin{cases} e^{-i\omega_0 t} & -\frac{\tau_0}{2} < t < \frac{\tau_0}{2} \\ 0 & |t| > \frac{\tau_0}{2} \end{cases} \tag{10.22}$$

Its Fourier transform is

$$g(\omega) = \mathcal{F}[f] = \frac{1}{\sqrt{2\pi}} \int_{-\frac{\tau_0}{2}}^{\frac{\tau_0}{2}} e^{i(\omega-\omega_0)t} \mathrm{d}t = \frac{1}{\sqrt{2\pi}} \left[\frac{e^{i(\omega-\omega_0)t}}{i(\omega-\omega_0)} \right]_{\frac{\tau_0}{2}}^{\frac{\tau_0}{2}} \tag{10.23}$$

Evaluating the integral and remembering the complex exponential formula for $\sin(z)$, we have

$$\hat{\mathcal{F}}[f] = \frac{1}{i(\omega-\omega_0)\sqrt{2\pi}} \left(e^{\frac{i(\omega-\omega_0)\tau_0}{2}} - e^{-\frac{i(\omega-\omega_0)\tau_0}{2}} \right) = \frac{1}{\omega-\omega_0} \sqrt{\frac{2}{\pi}} \sin\left(\frac{(\omega-\omega_0)\tau_0}{2} \right) \tag{10.24}$$

Or, using for convenience the sinc function we have

$$\hat{\mathcal{F}}[f] = \frac{1}{\omega_0-\omega} \frac{2}{\sqrt{\pi}} \operatorname{sinc}\left(\frac{(\omega-\omega_0)\tau_0}{2} \right)$$

Definition 10.2.5 (Power Spectrum)**.** The utility of defining the Fourier transform of a train waves comes in handy for defining a new entity, the so called *Power spectrum* of the wave, commonly indicated as $G(\omega)$, defined as the square modulus of the Fourier transform, i.e.

$$G(\omega) = \left| \hat{\mathcal{F}}[f] \right|^2 = |g(\omega)|^2 \tag{10.25}$$

Having defined the power spectrum of a wave, we have for our generic wave train then

$$G(\omega) = \frac{2}{\pi(\omega-\omega_0)^2} \sin^2\left(\frac{\omega-\omega_0}{2} \tau_0 \right) \tag{10.26}$$

The utility of defining the power spectrum comes from its mathematical properties, in fact we have $G(\omega) = 0$ if and only if $\omega = \omega_0$, and its maximum is, when

$$\max(G(\omega)) = G(\omega_M) \qquad \omega_M = \frac{\pi}{2\tau_0} \pm \omega_0 \tag{10.27}$$

Therefore, the width of the frequency distribution, $\Delta\omega$ is then

$$\Delta\omega = \frac{2\pi}{\tau_0} \tag{10.28}$$

From this, in terms of frequency, we have that

$$\Delta\nu = \frac{1}{t_0}$$

I.e., a sequence of wave trains lasting τ_0 will have the same exact power spectrum of the single pulse. In general, if the pulses also have random τ_0, we can say without loss of generality that it's approximately equal to $\langle\tau_0\rangle^{-1}$.
Reasoning in the opposite direction, given a spectral source with line width $\Delta\nu$, the coherence time of the wave trains $\langle\tau_0\rangle$ can be estimated as follows

$$\langle\tau_0\rangle = \frac{1}{\Delta\nu}$$

Which, in terms of coherence length l_c it gives us

$$l_c = \frac{c}{\Delta\nu}$$

Using also that $\Delta\nu/\nu = |\Delta\lambda|/\lambda$, we get that

$$l_c = \frac{\lambda^2}{\Delta\lambda} \tag{10.29}$$

With $\Delta\lambda$ being the width of the spectral line in terms of wavelength.

§ 10.3 Spatial Coherence

In this section we will analyze a different case. Before we dealt only with two fields that reach a single fixed point in space, from here on we will study the coherence of two or more fields reaching two or more *different* points in space.
Suppose again a quasimonochromatic point-like source S and 3 receiving points P_i, with P_3 being on the same director from the point P_1 and P_2 disposed in a different director from the source at the same distance d from it.
The coherence for the fields E_1, E_3 will be known as the *Longitudinal Spatial Coherence*, while the coherence between E_1, E_2 will be known as the *Transverse Spatial Coherence*. It's already clear for what we said before that the longitudinal spatial coherence will depend only on how far is P_1 from P_3, called this distance d_{13} we can also say that it depends on the value

$$t_{13} = \frac{d_{13}}{c}$$

Compared to the coherence time τ_0. It's clear that for whatever $E_1(t)$ we have, $E_3(t)$ will have the same time dependence of E_1, with a retardation of t_{13}.
If $t_{13} << \tau_0$ the two fields will be coherent, while for $t_{13} >> \tau_0$ the two fields will be little or completely

not coherent between the two.
For the fields E_1 and E_2 instead the time dependence will be the same, hence they will always be completely mutually coherent ($t_{12} = 0$), but only if the source S is point-like. It's clear that in reality there are no real point sources, and for an extended source (i.e., with spatial resolution) the size of the source itself must also be accounted for.
In general these extended sources can be considered as a cluster of point sources, and therefore we can immediately consider the more general case of multiple sources (two for ease of calculation) with two measuring points P_1, P_2, distant $d_{1a}, d_{1b}, d_{2a}, d_{2b}$ from the sources, in a regime of transverse spatial coherence.
Called these two sources S_a, S_b, we can define the two fields E_1, E_2 as the sum of the two fields going from the $i-$th source to the $j-$th point. I.e.

$$\begin{aligned} E_1 &= E_{1a} + E_{1b} \\ E_2 &= E_{2a} + E_{2b} \end{aligned} \tag{10.30}$$

Noting that the fields emitted from the source S_a *will not* be coherent (actually, will be completely incoherent) with the fields emitted from the source S_b, the correlation function between the two fields at the two receiving points will be

$$\gamma_{12}(\tau) = \frac{\left\langle E_1(t)\overline{E_2}(t+\tau)\right\rangle}{\sqrt{I_1 I_2}} = \frac{1}{\sqrt{I_1 I_2}}\left(\left\langle E_{1a}(t)\overline{E_{2a}}(t+\tau)\right\rangle + \left\langle E_{1b}(t)\overline{E_{2b}}(t+\tau)\right\rangle\right) \tag{10.31}$$

If the two fields are generic train waves, we already know that the self coherence of such pulse is

$$\gamma(\tau) = \left(1 - \frac{\tau}{\tau_0}\right) e^{i\omega\tau}$$

And considered the two different optical paths with different travel times for the waves, then, in our case

$$\gamma_{12}(\tau) = \frac{1}{2}\gamma(\tau_a) + \frac{1}{2}\gamma(\tau_b)$$

Where, obviously, we have

$$\begin{aligned} \tau_a &= \frac{d_{2a} - d_{1a}}{c} + \tau \\ \tau_b &= \frac{d_{2b} - d_{1b}}{c} + \tau \end{aligned}$$

After some algebra which I won't do so bear with me here, we have

$$|\gamma_{12}(\tau)|^2 = \left(1 - \frac{\tau_a}{\tau_0}\right)\left(1 - \frac{\tau_b}{\tau_0}\right)\left(\frac{1 + \cos\left[\omega\left(\tau_b - \tau_a\right)\right]}{2}\right) \tag{10.32}$$

Where the approximation $\tau_a - \tau_b << \tau_a, \tau_b$ has been used.
It's clear then that the visibility and coherence of waves in such system also depends on the difference of travel times for the waves, from the sources to the points, also in a periodic manner.
This clearly means that at the two points the waves will show periodically interference (i.e., coherence, $\gamma_{12} \neq 0$) also if the two sources emit electromagnetic radiation which in which is completely incoherent. This is called a *periodic spatial dependence*.

Consider now the special case of having the two points disposed in a symmetrical manner with respect to the sources, hence with $d_{1a} = d_{1b}$. Then

$$\tau_b - \tau_a = \frac{d_{2a} - d_{1b}}{c} \approx \frac{sl}{2cd} \tag{10.33}$$

Where, s is the distance between the sources, l is the distances between the points, d is the mean distance between the points and the sources. Here we used the approximation $r >> s, l$.
Or, extending the calculations, we have

$$r_{2a,b} = \sqrt{\left(l \pm \frac{s}{2}\right)^2 + d^2} = d\sqrt{1 + \frac{1}{d^2}\left(l \pm \frac{s}{2}\right)^2}$$

Approximating the root using the same approximation idea we had before, we have

$$d_{2a,b} \approx d + \frac{1}{2d}\left(l^2 + \frac{s^2}{4} \pm ls\right) \implies \tau_b - \tau_a \approx \frac{r}{c} + \frac{l^2}{2dc} + \frac{s^2}{8dc} \pm \frac{ls}{2dc}$$

The experimental setup in this case is really similar to the Young interferometer's case, and the variation of lateral coherence between the two will be described by a periodic bell shaped curve.
The dips of this curve, will be then for $|\gamma_{12}|^2 = 0$. Considering the main peak at the center, these peaks will be reached at some distance l_t, which is approximately the length of the area of maximum coherence. Then

$$\frac{\omega l_t s}{2dc} = \pi$$

Using $\omega/c = k_0 = 2\pi/\lambda$, which implies $\omega = 2\pi c/\lambda$ we have

$$\frac{2\pi l_t s}{2\lambda d} = \pi \implies l_t = \frac{\lambda d}{s} \tag{10.34}$$

Or, in terms of angular separation $\theta_s \approx s/d$

$$l_t = \frac{\lambda}{\theta_s} \tag{10.35}$$

The value l_t is known as the *transverse coherence width*.

§ 10.4 Multiple Beam Interference

The most general treatment of coherence comes from the study of multi-beam interference. For the treatment of coherence here, the division of amplitude is needed, and one way of obtaining this is by using multiple reflection between two semi-reflecting parallel plates.
Set up the experiment, with the two parallel plates fixed at some distance d between each other, we send the initial beam with amplitude E_0 towards them. Due to their being semi-reflecting, at contact with the first one the beam will be reflected at some angle θ and amplitude $E_0 r$, while the second will travel through it with amplitude $E_0 t$.
The process continues when the beam hits the second semi-reflecting surface, which will again repeat the same process, sending towards the first surface a beam with amplitude $E_0 tr$ and transmitting a beam with amplitude $E_0 t^2$.

Studying the process for n iterations, we get the following succession of amplitudes for the beams found inside the two surfaces:

$$E_0 t, E_0 t r, E_0 t r^2, \cdots, E_0 t r^n$$

While, the transmitted ones *outside* the second surface, will follow the following succession

$$E_0 t^2, E_0 t^2 r^2, E_0 t^2 r^4, \cdots, E_0 t^2 r^{2n}$$

Each of the transmission will have an added phase change to the wave depending on the added optical path of the wave. In our case it comes from the $2n$-th reflection ($n \in \mathbb{N},\ n > 1$), since we want to see the actual phase displacement of the outgoing final beam, which we will then measure.
Hence, supposing that between the two plates $n = n_2/n_1 = 1$ we have

$$\delta = 2kd\cos\theta = \frac{4\pi}{\lambda} d\cos\theta \tag{10.36}$$

Or, in general, if $n \neq 1$

$$\delta = 2k_0 dn\cos\theta = \frac{4\pi n}{\lambda_0} d\cos\theta \tag{10.37}$$

Then, accounting the phase shift, the final total amplitude of the beam will be the superposition (i.e. the sum) of the amplitudes of every beam that gets transmitted post multiple reflection, with a multiplied phase shift, i.e.

$$E_T = E_0 t^2 + E_0 t^2 r^2 e^{i\delta} + E_0 t^2 r^4 e^{2i\delta} + \cdots$$

At the limit of infinite reflection, or, in practical terms approximating for many reflections, we have a geometric series for the final result of the amplitude, which is easily calculable with the usual mathematical result

$$E_T = E_0 t^2 \sum_{j=0}^{\infty} r^{2j} e^{ij\delta} = \frac{E_0 t^2}{1 - r^2 e^{i\delta}} \tag{10.38}$$

In terms of intensity $I_T = |E_T|^2$, and remembering that in general $r, t \in \mathbb{C}$, we have

$$I_T = |E_T|^2 = \frac{I_0 |t|^4}{|1 - r^2 e^{i\delta}|^2} \tag{10.39}$$

Called $\delta_r/2$ the phase shift for a single reflection (the half unity factor is just for ease of calculation later), we have that

$$r = |r| e^{i\frac{\delta_r}{2}}$$

And, writing $\Delta = \delta + \delta_r$ (the half factor here gets elided since we're considering only pairs of reflections), we have in terms of reflectance R and transmittance T

$$I_T = I_0 \frac{T^2}{|1 - R e^{i\Delta}|^2}$$

Calculating the absolute value in the denominator as $z\overline{z} = |z|^2$ we have, after some simple algebra

$$\left|1 - R e^{i\Delta}\right|^2 = 1 - R\left(e^{i\Delta} + e^{-i\Delta}\right) + R^2 = 1 - 2R\cos(\Delta) + R^2$$

Rewriting the cosine as one minus sine squared of half the argument, we have

$$\left|1 - Re^{i\Delta}\right| = 1 - 2R + 4R\sin^2\left(\frac{\Delta}{2}\right) + R^2 = (1-R)^2\left(1 + \frac{4R}{(1-R)^2}\sin^2\left(\frac{\Delta}{2}\right)\right)$$

Defined the *coefficient of Finesse* F as

$$F = \frac{4R}{(1-R)^2} \tag{10.40}$$

We have that finally the total intensity of the different beams is

$$I_T = \frac{I_0 T^2}{(1-R)^2}\frac{1}{1 + F\sin^2\left(\frac{\Delta}{2}\right)} \tag{10.41}$$

The second factor on the product is known as Airy's function, which will define (up to an intensity coefficient) the interference lines observed from the convergence of all these beams. The physical meaning of the finesse coefficient then can be seen as a value defining the sharpness of the fringes. Note that, also, if $\Delta/2$ is a whole multiple of π, Airy's function will always be equal to one for all values of F. Keep this in mind since it will be an useful concept for when we'll treat the Fabry-Pérot interferometer and the concept of *free spectral range* of a Fabry-Pérot interferometer (or etalon, depending on the experimental configuration).
As it's clear from the shape of the function and the previous observation, we have that the condition for having the maximum of Airy's function is then

$$\frac{\Delta_{max}}{2} = N\pi$$

Where $N \in \mathbb{N}$ is the *order of interference*, i.e. it's an indicator of the path difference between two successive beams. In fact, from the definition of Δ we have

$$\Delta_{max} = 2N\pi = \frac{4\pi}{\lambda_0} nd\cos\theta + \delta_r \tag{10.42}$$

In the most general case, for which the reflecting properties are not equal, we they'll contribute with two different reflection coefficients $r_1, r_2 \in \mathbb{C}$, which will contribute each with a phase shift δ_1, δ_2, i.e.

$$\begin{aligned} r_1 &= |r_1|e^{i\delta_1} \\ r_2 &= |r_2|e^{i\delta_2} \end{aligned} \tag{10.43}$$

All the previous derivations hold if and only if we define the transmittance and reflectance as follows

$$\begin{aligned} T &= |t_1||t_2| = \sqrt{T_1 T_2} \\ R &= |r_1||r_2| = \sqrt{R_1 R_2} \end{aligned} \tag{10.44}$$

Going back to our study of the intensity function for multi-beam interference, defined $\mathcal{I} = I_T/I_0$ then we have that the maximum and minimum values for intensity will be the following

$$\begin{aligned} \mathcal{I}_{max} &= \frac{T^2}{(1-R)^2} \\ \mathcal{I}_{min} &= \frac{T^2}{(1+R)^2} \end{aligned} \tag{10.45}$$

Which, in the realistic case of absorption of energy, defined an absorption function A for which

$$A + T + R = 1$$

We can rewrite the maximum of $\mathcal{I}$ in terms of reflectance and absorption as

$$\mathcal{I}_{max} = \frac{(1 - A - R)^2}{(1 - R)^2}$$

Where we used the simple substitution $T = 1 - A - R$

§ 10.5 Fabry-Pérot Instruments

An instruments which uses the exact concept that we defined in the previous section is the *Fabry-Pérot Interferometer* and the *Fabry-Pérot Etalon*, invented by C.Fabry and A.Pérot in 1889.
The instrument is composed, excluding the source, by one collimating lens which redirects the electromagnetic waves towards two semi-reflecting plates, and a focusing lens which focuses all the transmitted waves towards a single point, in which we would usually put a photomultiplier, an amplifier and then a detector or recorder.
The main difference between the Etalon configuration and the Interferometer configuration of a Fabry-Pérot instrument is whether the semi reflecting plates are movable or fixed. The Etalon configuration has fixed plates, while an interferometer or *scanning Fabry-Pérot* has moving plates.
These plates are usually made of glass or quartz, and the reflecting surfaces are parallel and *as smooth as possible*. The flatness required for having a working Fabry-Pérot is from at least $\lambda/20$, up to $\lambda/100$, with λ being the wavelength of the studied beam.
Commonly, scanning Fabry-Pérot are used with point sources, and the transmitted electromagnetic waves are focused to a pinhole, while the etalon is used with broad sources, the final beams here get focused to a single point on the focal plane.
The result of the interference created in a Fabry-Pérot instrument are circular interference fringes. Each ring corresponds to values of constant θ (remember that $\delta = \delta(\theta)$), and they're also known as *fringes of equal inclination*.
We continue by defining a fundamental concept in Fabry-Pérot instruments, the *free spectral range*.

Definition 10.5.1 (Free Spectral Range)**.** The *free spectral range* of a Fabry-Pérot is defined as the phase separation between two adjacent orders of interference N, as for $N, N+1$. I.e., the phase values Δ inside this range satisfy

$$\Delta_{N+1} - \Delta_N = 2\pi \tag{10.46}$$

Substituting $\Delta_N = 2N\pi$ and $\Delta_{N+1} = 2(N+1)\pi = \Delta_N + 2\pi$, and their definitions in terms of $\delta(\theta)$, we get

$$\frac{4\pi}{\lambda_{N+1}} nd\cos\theta - \frac{4\pi}{\lambda_N} nd\cos\theta = 2\pi$$

Solving in terms of $\lambda_{N+1}^{-1} - \lambda_N^{-1}$ we have

$$\frac{1}{\lambda_{N+1}} - \frac{1}{\lambda_N} = \frac{1}{2nd\cos\theta} \tag{10.47}$$

Or, substituting $\omega = 2\pi c/\lambda$, in terms of angular frequency

$$\omega_{N+1} - \omega_N = \frac{\pi c}{nd\cos\theta} \tag{10.48}$$

Or, also using $\omega = 2\pi\nu$, in terms of frequency

$$\nu_{N+1} - \nu_N = \frac{c}{2nd\cos\theta} \tag{10.49}$$

Or, again, using $k_0 = \omega/c = 2\pi/\lambda = 2\pi\nu/c$, in terms of wavenumber

$$k_{N+1} - k_N = \frac{\pi}{nd\cos\theta} \tag{10.50}$$

§§ 10.5.1 Resolution of Fabry-Pérot Instruments

We will now treat the resolution of interference fringes with Fabry-Pérot instruments.
For simplicity in calculation we will consider a non-monochromatic wave for which the spectrum is composed by two exact frequencies ω, ω_1, and this spectrum will be studied with a Fabry-Pérot.
Each frequency will contribute to the total intensity, therefore the fringe intensity pattern will be given by the sum of the two. Assuming that the initial intensity I_0 is the same for both, we get the total intensity as the sum of the two Airy functions

$$I_T = \frac{I_0}{1 + F\sin^2\left(\frac{\Delta}{2}\right)} + \frac{I_0}{1 + F\sin^2\left(\frac{\Delta_1}{2}\right)} \tag{10.51}$$

As usual, the phase Δ is

$$\Delta = \delta(\theta) + \delta_r = \frac{4\pi n}{\lambda_0} d\cos\theta + \delta_r$$

Which, if we assume an almost-normal incidence of the electromagnetic wave to the plates ($\theta << 1$), we have at first order and in terms of frequencies

$$\Delta = \frac{2\omega d}{c} + \delta_r + \mathcal{O}\left(\theta^3\right)$$
$$\Delta_1 = \frac{2\omega_1 d}{c} + \delta_r + \mathcal{O}\left(\theta^3\right)$$

And here comes in our help what's known as Taylor's criterion, which states that two *equal* lines are resolved if and only if the individual curves cross at the point of half intensity of each, for which then the intensity at the saddle is equal to exactly twice the initial intensity.
Since we're exactly smack in the middle between the two single peaks, considered the range between the two $\Delta - \Delta_1$, the total intensity at this saddle will be the usual Airy function times $2I_0$ this time, considered at half range, i.e. at $(\Delta - \Delta_1)/2$, i.e.

$$I_T|_{\frac{\Delta-\Delta_1}{2}} = \frac{2I_0}{1 + F\sin^2\left(\frac{\Delta-\Delta_1}{4}\right)} = I_0 \tag{10.52}$$

Solving for $\Delta - \Delta_1$, we're left with

$$F \sin^2\left(\frac{\Delta - \Delta_1}{4}\right) = 1$$

Assumed (we hope so for a good resolution!) $\Delta - \Delta_1 << 1$, we can approximate to the first order the sine function, getting

$$F\left(\left(\frac{\Delta - \Delta_1}{4}\right)^2 + \mathcal{O}\left((\Delta - \Delta_1)^6\right)\right) = 1 \implies |\Delta - \Delta_1| \approx \frac{4}{\sqrt{F}} = 2\frac{1-R}{\sqrt{R}} \tag{10.53}$$

I.e., the minimum resolvable range depends directly to the inverse of the square root of the finesse coefficient, which is only determined by the physical characteristics of the instrument!
In terms of frequencies, remembering that $\Delta \approx \frac{2d}{c}\omega$, we have that the smallest resolvable interval of frequencies is then

$$|\omega - \omega_1| \approx \frac{2c}{d\sqrt{F}} = \frac{c}{d}\frac{1-R}{\sqrt{R}} \tag{10.54}$$

Therefore, the smallest resolvable interval of frequencies for a Fabry-Pérot instrument depends only on the reflectance of the plates R and their distance d.
We also give a new definition for two elements often used when treating Fabry-Pérot instruments

Definition 10.5.2 (Reflecting Finesse)**.** Given a Fabry-Pérot instrument, the *reflecting finesse* $\mathcal{F}$, is defined as the free spectral range divided by the smallest resolvable interval, i.e.

$$\mathcal{F} = \frac{\Delta_{N+1} - \Delta_N}{|\Delta - \Delta_1|} = \frac{\pi}{2}\sqrt{F} = \frac{\pi}{2}\frac{\sqrt{R}}{1-R} \tag{10.55}$$

Definition 10.5.3 (Resolving Power)**.** The *resolving power* RP of a Fabry-Pérot instrument is defined as the inverse of the minimum resolvable interval times the scanned frequency (or angular frequency, or also wavelength), i.e.

$$RP = \frac{\omega}{|\omega - \omega_1|} = \frac{\nu}{|\nu - \nu_1|} = \frac{\lambda}{|\lambda - \lambda_1|} \tag{10.56}$$

It's clear, by direct substitution and some algebra, that the resolving power is directly tied to the reflecting finesse as

$$RP = N\mathcal{F} = N\pi\frac{\sqrt{R}}{1-R} \tag{10.57}$$

I.e., it's directly proportional to the reflecting finesse and the order of interference. Clearly, the resolving power can be made arbitrarily big simply by choosing higher interference orders. Note that by definition of N, this can be accomplished by increasing the mirror separation, which unfortunately reduces the free spectral range of the instrument.
Another idea for increasing the RP is by increasing $\mathcal{F}$, bringing the reflectance closer and closer to 1. There's a clear limit to this, in fact it's physically limited by the absorption of the used material, which reduces the intensity of the transmitted fringes.

11 Diffraction

§ 11.1 Helmholtz Equation

The basis for understanding diffraction, comes obviously from Maxwell's equations.
Suppose that we're in a medium for which there are no *free* charges and no currents whatsoever ($M^i = 0$). Since in the most general case we treat non-monochromatic waves, we propose an Ansatz. Given any field (E,B,H.D), it can be decomposed in *phasors* (complex exponentials), times an amplitude part that in general depends from frequency ω.
Therefore, for any given ω we have

$$E^i(x^i, t) = E_\omega(x^i)e^{i\omega t} \tag{11.1}$$

Where E^i_ω are the so-called monochromatic components of the fields, and $\epsilon_\omega(x^i) = \epsilon_0\,(1 + \chi_\omega(r))$ is the dielectric permittivity.
Then, Maxwell's equation become, remembering that $D = \epsilon(x,t)E$ and $\mu_0 H = B$ ($D_\omega = \epsilon_\omega(x^i)E_\omega$ and $\mu_0 H_\omega = B_\omega$ with our Ansatz) in this case,

$$\begin{cases} \partial_i D^i = 0 \\ \partial_i H^i = 0 \\ \epsilon^i{}_{jk}\partial^j E^k = -\mu_0 \partial_t H^i \\ \epsilon^i{}_{jk}\partial^j H^k = \partial_t D^i \end{cases}$$

The "time-independent" monochromatic version then becomes

$$\begin{cases} \partial_i E^i_\omega = -\dfrac{E^i_\omega \partial_i \epsilon_\omega}{\epsilon_\omega} = -2E^i_\omega \partial_i \log(n_\omega) \\ \partial_i H^i_\omega = 0 \\ \epsilon^i{}_{jk}\partial^j E^k_\omega = -i\omega\mu_0 H^i_\omega \\ \epsilon^i{}_{jk}\partial^j H^k_\omega = i\omega\epsilon_\omega E^i_\omega \end{cases} \tag{11.2}$$

Where we defined the usual refraction index as follows

$$n_\omega(x^i) = \sqrt{\frac{\epsilon_\omega(x^i)}{\epsilon_0}}$$

Now, solving only for E^i_ω, using the properties of the $\epsilon^i{}_{jk}$ symbol, we get the following partial differential equation

$$\partial_j \partial^j E^i_\omega + 2\partial_j \left(E^i_\omega \partial_i \log(n_\omega) \right) + \omega^2 \mu_0 \epsilon_\omega E^i_\omega = 0 \tag{11.3}$$

We can rewrite $\epsilon_\omega \mu_0 \omega^2 = \omega^2/c^2 n^2_\omega = k_0^2 n^2_\omega = k^2$, and imposing that the refraction index varies slowly ($\Delta n_\omega / n_\omega << 1$), we get *Helmholtz's equation*, a time-independent counterpart to the usual wave equation, which can also be seen as an eigenvalue problem of the Laplace operator

$$\partial^j \partial_j E^i_\omega + k_0^2 n_\omega E^i_\omega = 0 \tag{11.4}$$

§§ 11.1.1 Normal Modes of the Electric Field

The solution of this differential equation isn't straightforward in general sets, so we begin to impose another Ansatz. Given the monochromatic field E_ω, we suppose that the solutions will be exponential solutions depending on the wavevector k^i, as follows

$$E^i_\omega(x^i) = E^i_{\mathbf{k}} e^{-ik^i x_i}$$

It's clear that in order to satisfy Gauss' equation we must have

$$\partial_i E^i_\omega = -ik_i E^i_{\mathbf{k}} e^{-ik^i x_i} = 0 \implies k_i E^i_\omega = 0$$

This clearly means that the direction of propagation of the waves is parallel to the wavevector, i.e. we have transverse propagation. Supposing that this direction coincides with the direction of the z axis, we can propose a new notation.

$$x^i = (r^\alpha_\perp, z)$$
$$k^i = (k^\alpha_\perp, k_z)$$

Where $\alpha = 1, 2$. Rewriting k_z in terms of $k_\perp$ we have

$$k^2 = k^2_\perp + k^2_z \implies \quad k_z = \sqrt{k^2 - k^2_\perp}$$

Which gives us a constraint on k, as

$$k \geq k_\perp$$

Note that there exist values for which $k_z \in \mathbb{C}$, they'll be treated later. Therefore, the general solution is

$$E^i_\omega(x^i) = E^i_{\mathbf{k}_\perp} e^{-ik^i_\perp r_{i,\perp} - iz\sqrt{k^2 - k^2_\perp}} \tag{11.5}$$

This is our well known plane wave propagation formula. A similar formula for H^i_ω can be retrieved by applying the same manipulations to Maxwell's equations, and the resulting Helmholtz equation will describe a field which vibrates in a direction orthogonal to E. The complete monochromatic solution to Helmholtz's equation will then be a superposition of every solution with $k_\perp \leq k$

$$E^i(x^i, t) = \iint_{k_\perp \leq k} E^i_{\mathbf{k}_\perp}(\omega) e^{i\omega t - ik^i_\perp r_{i,\perp} - iz\sqrt{k^2 - k^2_\perp}} \, \mathrm{d}^2 k_\perp \tag{11.6}$$

As we said, in general $k \in \mathbb{C}$ and the solutions found are said to be *evanescent modes*. All other solutions where $k \in \mathbb{R}$ are said to be *radiation modes*. Therefore, the integral can be extended without problems to all the possible values of $k_\perp$, i.e. $\mathbb{R}^2$

§ 11.2 Scalar Diffraction Theory

§§ 11.2.1 Green's Functions and Kirchhoff Diffraction

The previous section deals with Helmholtz's equation and its general solution. Usually tho, in real world application the using that solution is counterproductive, as it's not always readily integrable.
Looking again at Helmholtz's equation, it's clear that it's a set of three *uncoupled* partial differential equations. Said $u(r)$ as the generic component of the field, the equation is

$$\partial^i \partial_i u(r) + k^2 u(r) = 0 \tag{11.7}$$

Now, let's say we want to solve it in some particular (compact) set V, which has a random shape.
Due to the improbable boundary conditions we could choose in order to fit our problem inside the chosen set, we use *Green's functions*, for which, a solution $u(r)$ can then be expressed as follows, for some differential equation $\hat{L}h = f$

$$h(x) = \int G(x - x_0) f(x_0) \, \mathrm{d}x_0$$

Here G is Green's function for the differential operator $\hat{L}$, i.e. a function such that $\hat{L}G = \delta_{x_0}$, where $\delta_{x_0} = \delta(x - x_0)$ is Dirac's delta distribution.
Going back to our partial differential equation, if the source point of the electromagnetic waves is $r_0 \in V \subset \mathbb{R}^3$, the field $u(r)$ diverges at r_0, so it's a good idea to remove an infintesimal ball of radius ϵ from the set, i.e. we define a new set

$$\tilde{V} = V \setminus B_\epsilon(r_0)$$

This differential equation doesn't properly follow what we wrote before about Green's functions, since here Green's function would just be a solution of the problem, since

$$\hat{L}G = \left(\partial^i \partial_i + k^2\right) G(r) = 0$$

Therefore, Green's function is not unique. We will choose the simplest solution, i.e. a spherical wave emitting from r_0

$$G(r - r_0) = \frac{e^{ik(r - r_0)}}{|r - r_0|} \tag{11.8}$$

We can now use Green's theorem for finding a really important relation

$$\iiint_{\tilde{V}} \left(u(r)\partial^i \partial_i G - G(r - r_0)\partial^i \partial_i u\right) \mathrm{d}^3 x = -\iiint_{\tilde{V}} \left(uk^2 G - uk^2 G\right) \mathrm{d}^3 x = 0 \tag{11.9}$$

Where we applied Helmholtz's equation on the integral.
But also, for Green's theorem, said $\partial\tilde{V} = \partial V \cup \partial B_\epsilon(r_0)$, then

$$\oiint_{\partial\tilde{V}} (u(r)\partial_i G - G(r - r_0)\partial_i u)\, \hat{n}^i \, \mathrm{d}^2 s = \oiint_{\partial V \cup \partial B_\epsilon(r_0)} (u(r)\partial_i G - G(r - r_0)\partial_i u)\, \hat{n}^i \, \mathrm{d}^2 s = 0 \tag{11.10}$$

Or, using the properties of integrals and measures

$$\oiint_{\partial B_\epsilon(r_0)} (u(r)\partial_i G - G(r - r_0)\partial_i u)\, \hat{n}^i \, \mathrm{d}^2 s = -\oiint_{\partial V} (u(r)\partial_i G - G(r - r_0)\partial_i u)\, \hat{n}^i \, \mathrm{d}^2 s = 0 \tag{11.11}$$

The propagation problem now just reduced to calculating the integral on the left, which is analytically solvable. In $\partial B_\epsilon(r_0)$ it's clear that for our formulation of the problem $\hat{n}^i = -\hat{r}^i$, and therefore,

$$\frac{\partial G}{\partial n} = \partial_r G = -\left(ik - \frac{1}{|r-r_0|}\right)\frac{e^{ik(r-r_0)}}{|r-r_0|}$$

Therefore, Substituting into both integrals this result, and integrating on the left and sending $\epsilon \to 0$ (basically, going to r_0), we have

$$\lim_{\epsilon\to 0}\left[4\pi\epsilon^2\left(u(r_0)\left(\frac{1}{\epsilon} - ik\right)\frac{e^{ik\epsilon}}{\epsilon} - \hat{r}^i\partial_i u(r_0)\frac{e^{ik\epsilon}}{\epsilon}\right)\right] = 4\pi u(r_0)$$

I.e., the integral on the ball collapses to a constant times the field evaluated at the source. Fixing everything together we get *Kirchhoff-Helmholtz's integral theorem*

$$u(r_0) = \frac{1}{4\pi}\oiint_{\partial V}\left(\frac{\partial u}{\partial n}G(r-r_0) - \frac{\partial G}{\partial n}u(r)\right)\mathrm{d}^2 s \tag{11.12}$$

This theorem, clearly states that, at any point $r_0 \in V$, $u(r)$ can be determined only in terms of its boundary values. Writing explicitly Green's function, we have

$$u(r_0) = \frac{1}{4\pi}\oiint_{\partial V}\left(\frac{\partial u}{\partial n}\frac{e^{ik(r-r_0)}}{|r-r_0|} - u(r)\left(ik - \frac{1}{|r-r_0|}\right)\frac{e^{ik(r-r_0)}}{|r-r_0|}\right)\mathrm{d}^2 s \tag{11.13}$$

§§ 11.2.2 Fresnel-Kirchhoff Diffraction Formula

Since, as we calculated before, the wave at some point is defined from the values of the same at the boundary, we need to specify those.
Suppose that there's an opaque infinite wall with an aperture Σ. In order to evaluate the diffracted field we need to choose a useful surface ∂V over which we will perform the integration.
Chosen a point r_0 at which we will evaluate the field, we take $\partial V = S_1 + S_R$, where S_1 is a plane just after the aperture, and S_R is a spherical cap of radius R centered in r_0.
On S_R, we have that, if $R \to \infty$

$$\left.\frac{\partial G}{\partial n}\right|_{S_R} = \left(ik - \frac{1}{R}\right)\frac{e^{ikR}}{R} \approx ikG(r-r_0) \tag{11.14}$$

Then

$$\iint_{S_R}\left(G(r-r_0)\frac{\partial u}{\partial n} - u(r)\frac{\partial G}{\partial n}\right)\mathrm{d}^2 s \simeq \iint_{S_R} G(r-r_0)\left(\frac{\partial u}{\partial n} - iku(r)\right)\mathrm{d}^2 s$$

Written $\mathrm{d}^2 s = R^2 \mathrm{d}\Omega$, with Ω being the solid angle, we have that due to the functional shape of G, the function $|RG(r-r_0)|$ is uniformly bounded, which implies that the integral vanishes only if

$$\lim_{R\to\infty}\left[R\left(\frac{\partial u}{\partial n} - iku(r)\right)\right] = 0 \tag{11.15}$$

I.e., if the field $u(r)$ vanishes as dast as a diverging wave. This condition is known as «Sommerfield radiation condition», and *guarantees* that we're dealing only with outgoing waves from Σ and not

incoming. Note that this is clearly respected for spherical scalar waves.
We're left with only one integral now, which is the following

$$u(r_0) = \frac{1}{4\pi} \iint_{S_1} \left(G(r - r_0) \frac{\partial u}{\partial n} - u(r) \frac{\partial G}{\partial n} \right) \mathrm{d}^2 s \tag{11.16}$$

We now proceed to have two assumptions on the boundary behavior of u.

1. The values at the aperture of u are the same *with or without* the screen.
2. The values of u at the opaque wall is identically 0.

Then, the integral can be approximated to the following

$$u(r_0) = \frac{1}{4\pi} \iint_{\Sigma} \left(G(r - r_0) \frac{\partial u}{\partial n} - u(r) \frac{\partial G}{\partial n} \right) \mathrm{d}^2 s \tag{11.17}$$

This approximation holds *only* if $\lambda >> d$, with d being the distance from the aperture. If this is true, we can again use the approximation (11.14), and obtain the following simplified integral

$$u(r_0) = \frac{1}{4\pi} \iint_{\Sigma} \frac{e^{ik(r-r_0)}}{|r - r_0|} \left(\frac{\partial u}{\partial n} - u(r) \frac{\hat{n}^i r_i}{|r - r_0|} \right) \mathrm{d}^2 s$$

We suppose that $u(r)$ is a spherical wave coming from a secondary source r_1 with amplitude A, for which holds $\lambda >> d_1$, with d_1 being the distance from the source to the aperture, then, if we write r' as the distance between the source and the screen and r the distance between the source and the observation point, we have firstly that

$$\begin{cases} u(r) = \dfrac{A}{r'} e^{ikr' + i\omega t} \\ \dfrac{\partial u}{\partial n} \simeq \dfrac{ikA}{r'} e^{ikr' + i\omega t} \end{cases}$$

Then we have that

$$u(r_0) = \frac{ikAe^{i\omega t}}{4\pi} \iint_{\Sigma} \frac{e^{ik(r+r')}}{rr'} \left(\frac{r'^i \hat{n}_i}{r'} - \frac{r^i \hat{n}_i}{r} \right) \mathrm{d}^2 s$$

Rewriting the scalar products on the right as $\cos(\hat{n}, r)$, i.e. the cosines angles between the two vectors in question, we get, remembering that $k = 2\pi/\lambda$

$$u(r_0) = \frac{Ae^{i\omega t}}{2i\lambda} \iint_{\Sigma} \frac{e^{ik(r+r')}}{rr'} \left(\cos(\hat{n}, r) - \cos(\hat{n}, r') \right) \mathrm{d}^2 s \tag{11.18}$$

This integral is known as the «Fresnel-Kirchhoff diffraction formula». Note how this is a mathematical version of Huygens' principle, which states that a wavefront is generated at each point of it by spherical waves.
What we see in this integral is exactly that, the measured field at r_0 can be seen simply as a superposition of spherical waves times a correction factor, called «obliquity», which corrects the calculation in case of waves that do not arrive frontally to the aperture.
It's also important to note that if we switch the observer with the source and vice-versa, the result doesn't change. This is well known as «Helmholtz's reciprocity theorem».

§§ 11.2.3 Rayleigh-Sommerfeld Diffraction

Another formulation of diffraction comes from Rayleigh and Sommerfeld, which idea comes from the fact that if we chose a wave that, on Σ has $\partial_n u = u = 0$, then, being u harmonic, it must be zero everywhere. Therefore it's clear that Kirchhoff's formulation of diffraction, even though it's precise in predicting diffraction patterns, it still implies in its boundary condition that there is no field before the aperture.
Sommerfeld then, in order to solve this problem imposing a different choice of Green's function. Supposed that

1. Scalar theory holds, i.e. $\Delta n_\omega / n_\omega << 1$ (dispersion is negligible)
2. $u(r), G(r)$ satisfy Helmholtz's equation
3. Sommerfeld's radiation condition holds, i.e. $\lim_{R\to\infty} R\left(\frac{\partial u}{\partial n} - iku(r)\right) = 0$

We can choose a Green's function such that either $G(r)$ or its normal derivative are equal to zero on S_1, so that we don't have to impose conditions on both u and its normal derivative.
Suppose then that $G(r)$ is generated at the observation point, and there is a secondary source mirrored at the other side of the screen. Assuming that the wavelength of the waves emitted by these two sources are equal, we can say for sure that their phases are shifted by π or they're in phase. Thus, chosen the Green's function with a phase difference of π, we have

$$G_-(r) = \frac{1}{|r - r_0|} e^{ik(r-r_0)} - \frac{1}{|r - \tilde{r}_0|} e^{ik(r-\tilde{r})} \tag{11.19}$$

It's clear that $G(\Sigma) = 0$, hence the Kirchhoff integral reduces to the first Rayleigh-Sommerfeld solution

$$u_I(r_0) = -\frac{1}{4\pi} \iint_\Sigma u(r) \frac{\partial G_-}{\partial n} \, \mathrm{d}^2 s \tag{11.20}$$

With our definition, if again $d >> \lambda$ we have that $G_-(r) \approx 0$ everywhere, and

$$\frac{\partial G_-}{\partial n} \approx 2ik \cos(\hat{n}, r) \frac{e^{ik(r-r_0)}}{|r - r_0|} \tag{11.21}$$

If we chose the Green's function with no phase difference between the imaginary source and the observation point, we'd have

$$G_+(r) = \frac{1}{|r - r_0|} e^{ik(r-r_0)} + \frac{1}{|r - \tilde{r}_0|} e^{ik(r-\tilde{r})} \tag{11.22}$$

Which implies that now the normal derivative vanishes on Σ, and the second Rayleigh-Sommerfeld solution is

$$u_{II}(r) = \frac{1}{4\pi} \iint_\Sigma G_+(r) \frac{\partial u}{\partial n} \, \mathrm{d}^2 s \tag{11.23}$$

Taken the Fresnel-Kirchhoff diffraction formula (11.18), it's pretty easy to get back the two Rayleigh-Sommerfeld solutions by simple substitution.

Given that $d >> \lambda$ and $\tilde{d} >> \lambda$, with $\tilde{d}$ being the distance of the imaginary source from the aperture, we have that in this approximation, called $G_K(r)$ the Kirchhoff-Fresnel Green's function

$$\begin{cases} \dfrac{\partial G_-}{\partial n} = 2ik\cos(\hat{n}, r)\dfrac{e^{ik(r-r_0)}}{|r-r_0|} = 2\dfrac{\partial G_K}{\partial n} \\ G_+(r) = 2\dfrac{e^{ik(r-\tilde{r}_0}}{|r-\tilde{r_0}|} = 2G_K(r) \end{cases} \tag{11.24}$$

Which implies that

$$\begin{cases} u_I(r_0) = \dfrac{1}{i\lambda}\iint_\Sigma \dfrac{u(r)}{|r-r_0|}e^{ik(r-r_0)}\cos(\hat{n}, r)\,\mathrm{d}^2 s \\ u_{II}(r_0) = \dfrac{1}{2\pi}\iint_\Sigma \dfrac{1}{|r-\tilde{r}_0|}\dfrac{\partial u}{\partial n}e^{ik(r-\tilde{r}_0)}\cos(\hat{n}, \tilde{r})\,\mathrm{d}^2 s \end{cases} \tag{11.25}$$

Which, for a diverging spherical wave become

$$\begin{cases} u_I(r_0) = \dfrac{Ae^{i\omega t}}{i\lambda}\iint_\Sigma \dfrac{e^{ik(r+r')}}{rr'}\cos(\hat{n}, r)\,\mathrm{d}^2 s \\ u_{II}(r_0) = \dfrac{iAe^{i\omega t}}{\lambda}\iint_\Sigma \dfrac{e^{ik(r+r')}}{rr'}\cos\hat{n}, r\,\mathrm{d}^2 s \end{cases} \tag{11.26}$$

Note that if we compare the two theories, and define an *obliquity* factor $\psi(\theta, \theta')$ as

$$\psi(\theta, \theta') = \begin{cases} \psi_K = \dfrac{1}{2}\left(\cos(\theta) - \cos(\theta')\right) \\ \psi_I = \cos\theta \\ \psi_{II} = -\cos\theta' \end{cases} \tag{11.27}$$

The two theories converge, where Kirchhoff theory is the mathematical average of the two. In this situation the general diffraction integral is written as

$$u(r_0) = \frac{Ae^{i\omega t}}{i\lambda}\iint_\Sigma \frac{e^{ik(r+r')}}{rr'}\psi(\theta, \theta')\,\mathrm{d}^2 s \tag{11.28}$$

It's also important to note that, in order to work properly, both Rayleigh-Sommerfeld solutions need that the source and/or the measuring point are far away from the aperture (d»λ), also known as *far field approximation*, for which we can consider that the waves are *plane* and not spherical, thing that we cannot avoid with Kirchhoff's formulation.
Also, Rayleigh-Sommerfeld theory needs planar diffraction screens, which is not always the case.

§§ 11.2.4 Non-monochromatic Case

Considered Rayleigh-Sommerfeld theory in the case of non-monochromatic waves $u(r, t)$, the first thing that comes to mind is firstly decomposing the wave into monochromatic components and then superimpose all the components into an integral. It's clear that this integral is actually a Fourier transform of the field.

$$u(r, t) = \int_{\mathbb{R}} \hat{u}(r, \nu)e^{2\pi i\nu t}\,\mathrm{d}t$$

Where $\hat{u}$ is u's transform.
Introducing a parity transformation in frequency space $\nu \to -\nu$, and said r_0 and r_1 respectively the distances from the measuring point and from the source to the screen, we get from (11.18) that, using $\lambda\nu = c/n = u$, $k = 2\pi/\lambda = 2\pi\nu/u$

$$\hat{u}(r_0, -\nu) = -\frac{i\nu}{u}\int_{\mathbb{R}} \hat{u}(r_1, -\nu)\frac{e^{-2\pi i\nu\frac{r-r_0}{u}}}{|r-r_0|}\cos(\hat{n}, r)\,\mathrm{d}^2 s \tag{11.29}$$

Therefore, integrating with respect to time

$$u(r_0, t) = \iint_\Sigma \frac{\cos(\hat{n}, r)}{2\pi u|r-r_0|}\int_{\mathbb{R}} -2\pi i\nu\hat{u}(r_1, -\nu)e^{-2\pi i\nu\left(t-\frac{r-r_0}{u}\right)}\,\mathrm{d}t\,\mathrm{d}^2 s \tag{11.30}$$

Since the measuring point and the source aren't moving, using the properties of the Fourier transform, we can say that, the time integral is just the derivative of a transform, where, if we indicate the Fourier inverse operator as $\hat{\mathcal{F}}^{-1}$, gives

$$u(r_0, t) = \frac{1}{2\pi u}\iint_\Sigma \frac{\partial}{\partial t}\hat{\mathcal{F}}^{-1}\left[\hat{u}\right]\left(t - \frac{r-r_0}{u}\right)\frac{\cos(\hat{n}, r)}{|r-r_0|}\,\mathrm{d}^2 s \tag{11.31}$$

Writing explicitly the transform as it's defined, we then have

$$u(r_0, t) = \frac{1}{2\pi u}\iint_\Sigma \left.\frac{\partial u}{\partial t}\right|_{t=t_r}\frac{\cos(\hat{n}, r)}{|r-r_0|}\,\mathrm{d}^2 s \tag{11.32}$$

Where we evaluate the time derivative at the retarded time $t_r = t - r/u$, therefore also conserving causality.

§ 11.3 Fresnel Diffraction

Let's put ourselves again in the paraxial case, and assume that there are two planes, one containing the observed diffraction pattern, with coordinates (x, y), and another one containing the aperture, with coordinates (ξ, η). Using Rayleigh-Sommerfeld's first solution and imposing the paraxial approximation, for which, since we chose z as our direction of propagation, we could say, called θ our inclination angle, with respect to z

$$u(r_0) = \frac{z}{i\lambda}\iint_\Sigma u(r_{01})\frac{e^{ikr_{01}}}{r_{01}^2}\,\mathrm{d}\xi\,\mathrm{d}\eta$$

Where we have

$$r_{01} = \sqrt{z^2 + (x-\xi)^2 + (y-\eta)^2} \tag{11.33}$$

It's clear, that in order to have this approximation work we need two approximations

1. Scalar theory
2. $r_{01} >> \lambda$

The third approximation comes from the definition of r_{01} itself. Approximating the power series to the first order, in order to count for the slight curvature of the waves, we have

$$r_{01} \approx z\left(1+\frac{1}{2}\left[\left(\frac{x-\xi}{z}\right)^2+\left(\frac{y-\eta}{z}\right)^2\right]\right) = z+\frac{1}{2z}(x-\xi)^2+\frac{1}{2z}(y-\eta)^2 \tag{11.34}$$

The choice of keeping first order factors is based on the exponential factor itself. In fact, since $k \propto \lambda^{-1}$, for wavelengths of visible light, we have

$$kr_{01} \propto \frac{r_{01}}{\lambda} \propto 10^7$$

Therefore, rendering the approximation invalid. It should be noted tho that for a far field approximation we can say without problems that the term r_{01} at the denominator is approximately equal to z, while at the exponential we have to keep the first order. Then

$$\begin{cases} r_{01}^{-1} \approx z^{-1} \\ e^{ikr_{01}} \approx e^{ikz}e^{\frac{ik}{2z}\left[(x-\xi)^2+(y-\eta)^2\right]} \end{cases}$$

Fixing everything, and noting that since in this approximation the contributions come only from the aperture, we rewrite the Rayleigh-Sommerfeld integral as follows

$$u(x,y) = \frac{e^{ikz}}{i\lambda z}\iint_{\mathbb{R}^2} u(\xi,\eta)e^{\frac{ik}{2z}\left[(x-\xi)^2+(y-\eta)^2\right]}\,\mathrm{d}\xi\,\mathrm{d}\eta \tag{11.35}$$

Where $u(\xi,\eta)$ is the so called «aperture function».
Note that if we define the *kernel* $h(x,y)$ as follows

$$h(x,y) = \frac{e^{i(kz+\omega t)}}{i\lambda z}e^{\frac{ik}{2z}\left[(x-\xi)^2+(y-\eta)^2\right]} \tag{11.36}$$

Then, the field measured at the screen is simply the convolution between the aperture function and the kernel

$$u(x,y) = u(\xi,\eta) * h(x.y) \tag{11.37}$$

Working again on the expression inside the integral, it's possible to expand the squares and bring outside all common factors, yielding

$$\frac{ik}{2z}\left[(x-\xi)^2+(y-\eta)^2\right] = \frac{ik}{2z}(x^2+y^2)+\frac{ik}{2z}(\xi^2+\eta^2)-\frac{ik}{z}(x\xi+y\eta)$$

And the integral for a monochromatic source becomes

$$u_\omega(x.y) = \frac{e^{i(kz+\omega t)}e^{\frac{ik}{2z}(x^2+y^2)}}{i\lambda z}\iint_{\mathbb{R}^2}\left[u(\xi,\eta)e^{\frac{ik}{2z}(\xi^2+\eta^2)}\right]e^{-\frac{2\pi i}{\lambda z}(x\xi+y\eta)}\,\mathrm{d}\xi\,\mathrm{d}\eta = \frac{e^{i(kz+\omega t)}e^{\frac{ik}{2z}(x^2+y^2)}}{i\lambda z}\hat{\mathcal{F}}\left[u(\xi,\eta)e^{\frac{ik}{2z}(\xi^2+\eta^2)}\right](x,y) \tag{11.38}$$

Which then describes the observed pattern as a bidimensional Fourier transform of the aperture function times a quadratic phase factor.
This integral is known as the *Fresnel diffraction integral*. Remember that this integral is valid only and only if:

1. Scalar theory holds
2. The screen is at many wavelengths of distance from the aperture
3. The second order terms in the square root power expansion can be neglected

The third condition is satisfied, only if, expanding the root to the second order, as

$$ikr_{01} \approx ikz + \frac{ik}{2z}\left[(x-\xi)^2+(y-\eta)^2\right] + \frac{ik}{8z^3}\left[(x-\xi)^2+(y-\eta)^2\right]^2 \tag{11.39}$$

And, the last term is negligible, i.e., in terms of wavelength

$$\frac{\pi}{4\lambda z^3}\left[(x-\xi)^2+(y-\eta)^2\right]^2 << 1 \implies z^3 >> \frac{\pi}{4\lambda}\left[(x-\xi)^2+(y-\eta)^2\right]^2 \tag{11.40}$$

As an example, for a circular aperture of diameter 1 cm, an observation region of size 1 cm and electromagnetic waves with wavelenghts of 0.5 μm, the distance $z >> 25$cm for rendering this approximation valid. This can be reduced by imposing that the second order terms are adiabatic invariants for the integral. This approximation is also known as the *near field approximation* for this reason.

§§ 11.3.1 Fresnel Zones and Zone Plates

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§ 11.4 Fraunhofer Diffraction

Fraunhofer diffraction imposes an even stricter approximation on the exponential, by ditching completely the first order terms, therefore, if

$$z >> \frac{k}{2}\left(\xi^2+\eta^2\right) \tag{11.41}$$

Then the integral gets even more simplified, and the resulting diffracted field becomes the exact Fourier transform of the aperture function $u(\xi,\eta)$, i.e., said $\nu_x = x/(\lambda z)$ and $\nu_y = y/(\lambda z)$, we have

$$u(x,y) = \frac{e^{i(kz-\omega t)}e^{\frac{ik}{2z}(x^2+y^2)}}{i\lambda z}\hat{\mathcal{F}}\left[u\right](\nu_x,\nu_y)$$

Or, explicitly

$$u(x,y) = \frac{e^{i(kz-\omega t)}e^{\frac{ik}{2z}(x^2+y^2)}}{i\lambda z}\iint_\Sigma u(\xi,\eta)e^{-\frac{2\pi i}{\lambda z}(x\xi+y\eta)}\,\mathrm{d}\xi\,\mathrm{d}\eta \tag{11.42}$$

This approximation, due to its zeroth order nature is known as the *far field approximation*, in fact, if we wanted to use this approximation with electromagnetic waves with wavelength of 0.6 μm (red light) and an aperture width of 2.5 cm the observation distance $z >> 1600$m. Another less stringent condition is the so called *antenna designer's formula*, for which, given an aperture of linear dimension D, the Fraunhofer approximation will be valid for

$$z > \frac{2D^2}{\lambda} \tag{11.43}$$

Still, also with this, in the previous case $z > 2000$m, but the requirement is less stringent.

§ 11.5 Examples of Fraunhofer Diffraction Integrals

§§ 11.5.1 Single Slit

§§ 11.5.2 Rectangular Aperture

§§ 11.5.3 Circular Aperture

§§ 11.5.4 Diffraction Gratings

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§ 12.1 Diffraction

A Waves

§ A.1 Plane Harmonic Waves

The wave equation in 4d is the following second order partial differential equation

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \frac{1}{u^2}\frac{\partial^2 f}{\partial t^2} \tag{A.1}$$

Taken the simpler case of a wave equation moving through a single spatial dimension, i.e.

$$\frac{\partial^2 f}{\partial x^2} = \frac{1}{u^2}\frac{\partial^2 f}{\partial t^2}$$

We already know that it has a simple solution in terms of cosines (or sines)

$$f(x,t) = f_0 \cos(kx - \omega t) \tag{A.2}$$

Given that

$$\frac{\omega}{k} = u$$

The solution (A.2) is of particular importance in treating electromagnetic waves (or waves in general). It tells us that the wave $f(x,t)$ varies sinusoidally with the distance x and it's harmonic in t for any given point in space.
Fixing the position and moving only through time, going to a new time $t + \Delta t$, we have that due to the previous constraint the point x will have moved by $\Delta x = u\Delta t$.
It's exactly like adding a phase to the wave (hence the name *phase velocity* of u). Note that if we used another solution, say

$$f_r(x,t) = f_0 \cos(kx + \omega t)$$

We would describe a wave going back, with displacement $\Delta x = -u\Delta t$ for some given time.
In terms of electrodynamics, all these constants have names:

- u is the phase velocity
- λ is the wavelength
- ω is the angular frequency
- k is the angular wavenumber

There are also other derived values that have given name, which are

- T the period, for which $\lambda = uT = \frac{2\pi}{k}$
- ν the frequency, for which $\nu = \frac{u}{\lambda} = \frac{\omega}{2\pi} = T^{-1}$
- σ the spectroscopic wavenumber, for which $\sigma = \frac{1}{\lambda}$

Going back to waves in 3 spatial dimensions, the general solution of the wave equation (A.1) is known as a *plane harmonic wave*, and has the following mathematical shape

$$f(x, y, z, t) = f_0 \cos\left(k^i x_i - \omega t\right) \tag{A.3}$$

Here the angular wavenumber (or wavenumber for simplicity), is a vector, and is known as the *propagation vector* k^i. The magnitude of this vector *is the actual wavenumber*, i.e.

$$k = \sqrt{k^i k_i}$$

The physical meaning of this solution resides mostly inside the cosine argument, $k^i x_i - \omega t$.
Setting it as constant we get the equation of a plane in space, which are called the *surfaces of constant phase*. The normals of these surfaces (planes) are proportional and perpendicular to the wavevector, and these planes "move" in that direction at a rate equal to the phase velocity u

§§ A.1.1 Alternative Representations of the Wavefunction

We can think immediately about alternative representations of the wavefunction (A.3). One of these is by using the definition of the constant phase surfaces we gave before, in fact, as we said $k^i = k\hat{n}^i$, and therefore

$$f(x^i, t) = f_0 \cos\left[(x^i \hat{n}_i - ut)k\right]$$

Remember that $ku = \omega$.
Another one, is by using complex functions. Using the Euler identity for the complex exponential we can write

$$f(x^i, t) = F_0 e^{i(k^i x_i - \omega t)}$$

Note that since generally $F_0 \in \mathbb{C}$, the actual physical quantity is the real part of f.
One use of the complex wavefunction is immediate when dealing with spherical waves, where $x^i = r$ and

$$f(r, t) = \frac{1}{r}\cos(kr - \omega t) = \mathfrak{Re}\left\{\frac{1}{r}e^{i(kr-\omega t)}\right\} \tag{A.4}$$

Ignoring the real part on the right it's clear that the second will be extremely easier to manipulate.

§ A.2 Group Velocity

Suppose now that we have two harmonic waves ϕ, ψ with different angular frequencies $\omega_1 = \omega + \Delta\omega$ and $\omega_2 = \omega - \Delta\omega$.
In general also the wavenumbers will differ, and we'll call them for simplicity $k_1 = k + \Delta k$, $k_2 = k - \Delta k$.

Supposing that the waves have the same amplitude A and are traveling in the same direction (say z), the superposition of the two will be

$$\Psi = \psi + \phi = A\left(e^{i((k+\Delta k)z-(\omega+\Delta\omega)t)} + e^{i((k-\Delta k)z-(\omega-\Delta\omega)t)}\right)$$

Collecting terms

$$\Psi = Ae^{i(kz-\omega t)}\left(e^{i(z\Delta k-t\Delta\omega)} + e^{-i(z\Delta k-t\Delta\omega)}\right)$$

Recognizing a $2\cos(\cdots)$ on the right we have that

$$\Psi = 2Ae^{i(kz-\omega t)}\cos\left(z\Delta k - t\Delta\omega\right)$$

The result of this superposition is a new wave η, with amplitude $B = 2A$ multiplied by a "modulation envelope" given by the cosine.
From the previous results it's clear that this envelope doesn't travel at the phase velocity, in fact, we have a new propagation velocity, known as the *group velocity*

$$u_g = \frac{\Delta\omega}{\Delta k} \tag{A.5}$$

At the limit, we have that $\omega'(k) = u_g$, and since $\omega = ku$ or $\omega = kc/n$ we have

$$u_g = \frac{\mathrm{d}}{\mathrm{d}k}\left(\frac{kc}{n}\right) = \frac{c}{n} - \frac{ck}{n^2}\frac{\mathrm{d}n}{\mathrm{d}k} = u\left(1 - \frac{k}{n}\frac{\mathrm{d}n}{\mathrm{d}k}\right)$$

Note that we wrote $n = n(k)$! The refraction index is in general dependent on the angular frequency ω, and therefore on k! As an example, you can see how different wavelengths of light behave passing through glass. They go through when λ is in the range of visible light, but are completely opaque in infrared, i.e. n *must* vary with frequency (or wavelength, or wavenumber, or …)
From the previous monologue and from the definitions we can then say that

$$\begin{aligned} u_g &= u\left(1 - \frac{k}{n}\frac{\mathrm{d}n}{\mathrm{d}k}\right) \\ u_g &= u - \lambda\frac{\mathrm{d}u}{\mathrm{d}\lambda} \\ \frac{1}{u_g} &= \frac{1}{u} - \frac{\lambda_0}{c}\frac{\mathrm{d}n}{\mathrm{d}\lambda_0} \end{aligned} \tag{A.6}$$

Where λ_0 is the vacuum wavelength.
Phase velocity and group velocity can only be equal in vacuum ($n = 0$), where $u_g = u = c$.

B Special Relativity

§ B.1 Principle of Relativity

The principle of relativity states a quite simple but deep affirmation: *All interaction propagate at a constant speed independent from the chosen frame of reference*. This speed is usually denoted as c and it's informally known as the speed of light, which has the following value (in SI units)

$$c = 2.998 \times 10^8 \text{ m/s} \tag{B.1}$$

In the part on classical mechanics we always intended between the lines that all interactions are instantaneous and therefore we'd have $c \to \infty$ formally. This can be interpreted as taking classical mechanics as an approximation of Einstein's relativity for which $v/c << 1$, which is the case for our really slow classical particles.
Note that this constant speed of propagation precludes that time isn't universal, and it is frame dependent. In order to understand this it's useful to get two coordinate frames K and $\tilde{K}$, where one is moving with respect to the other with a constant speed V.
Suppose now that a point A emits a signal towards two other points B and C

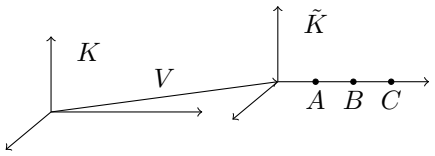

Figure B.1: The two frames K and $\tilde{K}$

In the frame $\tilde{K}$, where A is at rest, we see that the signal reaches both points at the same time, but the same CANNOT be true for the other system, since the relativity principle would be violated. Thinking in a different way, suppose that you're standing at the origin of the K system. If the velocity of the signal is constant in all reference frames we can for sure say that it's so where we're standing, therefore we end up seeing B moving towards the signal and C moving away from it, both with speed V. In this system we therefore must see a delay in when the two points receive such signal.
Although counterintuitive we're experimentally more than sure that this is actually a better approximation of nature than our beloved Newtonian mechanics.

§ B.2 Spacetime

Since time it's not anymore an universal thing and behaves itself as a coordinate, we can now think of our universe as a $4D$ manifold with time as a new coordinate. This is known as *Minkowsky Spacetime* or in short as *Spacetime*. This new definition follows:

Definition B.2.1 (Event)**.** Given a spacetime with coordinates (ct, x, y, z) with c the speed of light, we define a point in spacetime as an *event* in such.
Since time only "flows" one way, we have that for every particle corresponds a wordline which connects all the events pertaining to such. Note that events are also known as *universe points*

Given the principle of relativity one might also ask rightfully how to formulate mathematically all of this, bringing out some invariants that might help with further derivations. Take again the previous system and call l the distance traveled by the signal after being emitted from A. Calling t_1 and t_2 the emission time and the arrival time respectively, we have that for obvious reasons

$$l = c(t_1 - t_2) \tag{B.2}$$

But, we can also write as follows

$$l = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \tag{B.3}$$

With (x_1, y_1, z_1) being the departure coordinates and (x_2, y_2, z_2) the arrival coordinates in K In $\tilde{K}$, analogously we have

$$\begin{aligned} \tilde{l} &= c(\tilde{t}_2 - \tilde{t}_1) \\ \tilde{l} &= \sqrt{(\tilde{x}_2 - \tilde{x}_1)^2 + (\tilde{y}_2 - \tilde{y}_1)^2 + (\tilde{z}_2 - \tilde{z}_1)^2} \end{aligned} \tag{B.4}$$

Tying up both equations we end with the following result

$$\begin{cases} c^2(t_2 - t_1)^2 - (x_2 - x_1)^2 - (y_2 - y_1)^2 - (z_2 - z_1)^2 = 0 \\ c^2(\tilde{t}_2 - \tilde{t}_1)^2 - (\tilde{x}_2 - \tilde{x}_1)^2 + (\tilde{y}_2 - \tilde{y}_1)^2 + (\tilde{z}_2 - \tilde{z}_1)^2 = 0 \end{cases} \tag{B.5}$$

In "layman" words this basically means, that the following quantity

$$s_{12}^2 = c^2(t_2 - t_1)^2 - (x_2 - x_1)^2 - (y_2 - y_1)^2 - (z_2 - z_1)^2 \tag{B.6}$$

Called, *interval*, is a *relativistic invariant*, and therefore invariant with respect to changes of coordinate frames in the context of special relativity.
From (B.5) we have that if the two points are infinitesimally close to eachother we can define the infinitesimal interval as

$$\mathrm{d}s^2 = c^2\mathrm{d}t - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2 \tag{B.7}$$

The invariance of such differential quantity is easy to show considering the previous case we stated where $\mathrm{d}s = \mathrm{d}\tilde{s} = 0$ we have, using basic intuition that

$$\mathrm{d}s^2 = a(V)\mathrm{d}\tilde{s}^2 \tag{B.8}$$

Where $a(V)$ is a function of the relative velocity between the two considered frames. It cannot depend on direction due to the isotropy of space.

Consider now three inertial reference frames K, K_1, K_2, and let V_1, V_2 be the velocities of the frames K_1, K_2. We can therefore say, using (B.8) that

$$\begin{aligned} \mathrm{d}s^2 &= a(V_1)\mathrm{d}s_1^2 = a(V_2)\mathrm{d}s_2^2 \\ \mathrm{d}s_1^2 &= a(V_{12})\mathrm{d}s_2^2 \end{aligned} \tag{B.9}$$

Where we defined the velocity between K_1, K_2 as V_{12}. Rewriting the equation we have

$$\mathrm{d}s^2 = a(V_1)a(V_{12})\mathrm{d}s_2^2 = a(V_2)\mathrm{d}s_2^2$$

Equating the coefficients of the differential $\mathrm{d}s_2$, we have

$$a(V_{12}) = \frac{a(V_2)}{a(V_1)} \tag{B.10}$$

The previous equation then might be true if and only if $a(V_{12})$ depends only on the angle between the velocities V_1, V_2. This cannot be true due to the isotropy of spacetime, as we stated for the previous problem, and therefore $a(V)$ might only be a constant function. Taking $a(V_{12}) = 1$ for consistency between frames of reference, we have finally demonstrated that the differential spacetime interval is invariant

$$\mathrm{d}s = \mathrm{d}\tilde{s} \tag{B.11}$$

This definition of $\mathrm{d}s$ gives rise to three kinds of intervals:

1. *Spacelike intervals* if $s_{12}^2 < 0$
2. *Timelike intervals* if $s_{12}^2 > 0$
3. *Light-like intervals* if $s_{12}^2 = 0$

These three distinctions let us answer two previously impossible questions: is it possible to find a reference frame where two events happen at the same time or at the same place in our three-dimensional perception?. The answer is surprisingly *yes*. It depends on the kind of the interval between the two points.
Let's work with the first assumption, taken two events in spacetime E_1, E_2, defined $t_{12} = t_2 - t_1$ and l_{12} as our usual 3D distance between the events, we have

$$s_{12}^2 = c^2 t_{12}^2 - l_{12}^2$$

Let's now search a system where $l'_{12} = 0$. In order to have this, using that $s_{12} = s'_{12}$ we have

$$s_{12} = c^2 t_{12} - l_{12}^2 = c^2 t_{12}^{'2} = s_{12}^{'2} > 0$$

I.e. the spacetime interval between the frame of reference at rest with respect to the two events and the new unknown frame of reference is timelike.
Analogously, if we wanted to find a new system where the two events happen at the same time, we might have set $t'_{12} = 0$, therefore getting

$$s_{12} = c^2 t_{12} - l_{12}^2 = l_{12}^{'2} = s_{12}^{'2} < 0 \tag{B.12}$$

§§ B.2.1 Spacetime Diagrams

The idea of spacetime and absoluteness of the velocity of interactions can be described well by a 2D spacetime diagram. Taken an origin for our system of coordinates (ct, x) we have that, considering v as the slope of a constant wordline, that $|v| < c$.

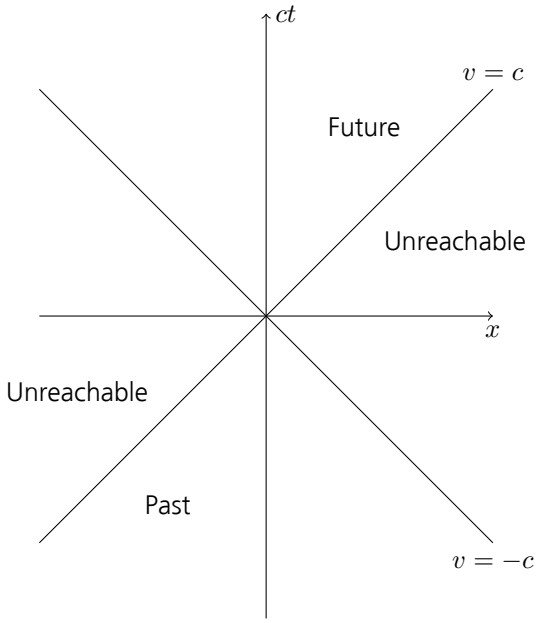

Figure B.2: Simple spacetime diagram. Note how all the events beyond the asymptote (or *horizon*) $v = \pm c$ are inaccessible from 0

Thought in higher dimensions we have that all the past and future of an event are enclosed inside a cone bordered by our horizon $|v| = c$ which separates physical impossibilities from the actual physical past and future of what we're considering.
Note that if $v = \pm c$ we must have $x = \pm ct$, giving us a spacelike interval for our diagram.
Considering instead past and future it's also easy to see that the past is always spacelike, since $c^2t^2 - x^2 < 0$, and that the future is always timelike. Note also that past and future must be absolute

§ B.3 Proper Time

Since time is not a relativistic invariant, we need to search for a good substitute of it. Given a clock fixed at the origin of some inertial frame K'. After some time $\mathrm{d}t$, the clock has moved (in our system) by the following quantity

$$\sqrt{\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2}$$

By definition, in K' this clock is at rest, therefore we have

$$\mathrm{d}x' = \mathrm{d}y' = \mathrm{d}z' = 0$$

Imposing the invariance of intervals we have that

$$\mathrm{d}s^2 = c^2\mathrm{d}t^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2 = c^2\mathrm{d}t'^2 \tag{B.13}$$

Therefore, it must be true that

$$\mathrm{d}t' = \mathrm{d}t\sqrt{1 - \frac{\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2}{c^2\mathrm{d}t^2}} \tag{B.14}$$

This is the expression for the passing of time in the system where the clock is at rest, and it's called the *proper time* of the clock, usually indicated with τ. Writing the sum of differentials as $\mathrm{d}r^2$ and using the definition of v^2, we have that

$$\mathrm{d}\tau = \mathrm{d}t\sqrt{1 - \frac{v^2}{c^2}} = \frac{\mathrm{d}s}{c} \tag{B.15}$$

Integrating and using the fundamental theorem of calculus, we have that a given time interval will be "felt" differently by the clock, where

$$\Delta\tau = \int_{\tau_1}^{\tau_2} \sqrt{1 - \frac{v^2}{c^2}}\mathrm{d}t < \Delta t \tag{B.16}$$

This tells us that a moving clock will tick slower than a clock at rest (note also on how this definition depends directly on the chosen frame).
This difference of measured time is known as *time dilation*.

§ B.4 Formalization of the Principle of Relativity

All of what we found before can be crammed into the most fundamental element of relativity: coordinate transformations.
Consider two reference frames K, (ct, x, y, z) and K', (ct, x, y, z). Mathematically, what we call interval is the usual 4D distance in a seminegative definite metric, and due to its invariance we must have that all coordinate transformations between these two systems must be rototraslations (isometries). Translations can be immediately ignored since they only move the origin of the system, and therefore we choose our faithful rotations in order to find these coordinate transformation laws.
All the possible rotations are between the planes xy, xz, yz and tx, ty, tz. All rotations xy, xz, yz are our usual 3D rotations and are of no use, therefore we choose the rotations tx, ty, tz. Taking tx as the chosen one we have that the spacetime interval is

$$s^2 = c^2t^2 - x^2$$

Therefore, all searched rotations *must preserve* this relationship. The first idea one might have is to look at the symmetry of the system and deduce immediately that such rotation must be hyperbolic in nature. We therefore define the following

$$\begin{pmatrix} x \\ ct \end{pmatrix} = \begin{pmatrix} \cosh\psi & \sinh\psi \\ \sinh\psi & \cosh\psi \end{pmatrix} \begin{pmatrix} x' \\ ct' \end{pmatrix} \tag{B.17}$$

Taking $x' = 0$ it all reduces to this single equation

$$\frac{x}{ct} = \frac{V}{c} = \tanh\psi \tag{B.18}$$

It's common to indicate such value with the pure number β, called the *Lorentz Boost*, where

$$\beta = \frac{V}{c}$$

Solving (B.18) we have that

$$\beta = \frac{\sinh\psi}{\sqrt{1+\sinh^2\psi}} = 0 \implies \sinh^2\psi = \frac{\beta}{\sqrt{1-\beta^2}} \tag{B.19}$$

And

$$\cosh^2\psi = 1+\sinh^2\psi \implies \cosh\psi = \frac{1}{\sqrt{1-\beta^2}} = \gamma \tag{B.20}$$

Where γ is known as the *Lorentz/Gamma Factor*.
Substituting back into (B.18) we have back our searched transformations

$$\begin{pmatrix} x \\ ct \end{pmatrix} = \begin{pmatrix} \gamma & \beta\gamma \\ \beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} x' \\ ct' \end{pmatrix} \tag{B.21}$$

Note that the inverse transformation is simply given imposing $\beta \to -\beta$.
The complete transformation between the two reference frames will finally be a 4D linear system as follows

$$\begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} \tag{B.22}$$

These transformations are known as *Lorentz Transformations* and are the fundamental transformations between frames of reference in special relativity. These transformations formalize the principle of relativity. For $v << c$ these transformations bring back the usual Galilean transformations corrected by a first order factor in c, as we expected

$$\begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & \beta & 0 & 0 \\ \beta & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} \tag{B.23}$$

§§ B.4.1 Length Contraction and Time Dilation

Using Lorentz transformations it's possible to mathematically formalize all relativistic effects. One of such is known as *length contraction*, where the measured length of an object depends on the chosen reference frame.
As a matter of example take a "rigid" rod in a system K, long Δx, and consider the system K' where the rod is at rest. In this system we have

$$\Delta x' = x'_1 - x'_2 = \gamma(x_2 - x_1) - \gamma\beta c(t_2 - t_1) = \gamma\Delta x - \gamma\beta c\Delta t \tag{B.24}$$

Since we're measuring the length directly, we can say without problems that $\Delta t = 0$, and we get

$$\Delta x' = \gamma \Delta x = \frac{\Delta x}{\sqrt{1-\beta^2}} = \frac{\Delta x}{\sqrt{1-\frac{v^2}{c^2}}} \tag{B.25}$$

Therefore, for $\beta \neq 0$ we have $\Delta x' < \Delta x$. We call $\Delta x = l_0$ as the proper lenght of this rod.
Note that a major consequence of this is that a rigid body in the classical sense of the term cannot be conceived in Special Relativity.
A second effect that we stated before and didn't formalize properly is that of time dilation. Taken a clock at rest in a system K' and two events happening at some coordinate (x', y', z') of K'. We have that the time elapsed between the two events will be $\Delta t' = t'_2 - t'_1$, and therefore, using Lorentz transformations we get, in K

$$\Delta t = \gamma \left(t'_1 + \frac{\beta}{c} \Delta x' \right) \tag{B.26}$$

Imposing that the events happen at the same place (x', y', z') we have $\Delta x' = 0$ and therefore

$$\Delta t = \gamma \Delta t' \tag{B.27}$$

Therefore, the clock in the still frame is measuring smaller time intervals, and the time measured is dilated.

§§ B.4.2 Velocity Transformations

As we have seen velocities have an upper bound which is the speed of light. It's possible to find the transformations of velocities from the transformations (B.21) and applying them to differentials.
We have

$$\begin{pmatrix} \mathrm{d}t \\ \mathrm{d}x \\ \mathrm{d}y \\ \mathrm{d}z \end{pmatrix} = \begin{pmatrix} \gamma & \frac{\beta\gamma}{c} & 0 & 0 \\ \frac{\beta\gamma}{c} & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathrm{d}t' \\ \mathrm{d}x' \\ \mathrm{d}y' \\ \mathrm{d}z' \end{pmatrix} \tag{B.28}$$

Rearranging the terms we have finally

$$\begin{cases} v_x = \dfrac{v'_x + \beta c}{1 + \frac{\beta}{c} v'_x} \\ v_y = \dfrac{v'_y}{\gamma \left(1 + \frac{\beta}{c} v'_x \right)} \\ v_z = \dfrac{v'_z}{\gamma \left(1 + \frac{\beta}{c} v'_x \right)} \end{cases} \tag{B.29}$$

Approximating for $v << c$ we get the usual velocity composition formula with an added relativistic correction

$$\begin{cases} v_x \approx v'_x + V \left(1 - \dfrac{v_x^{2'}}{c^2} \right) \\ v_y \approx v'_y - v'_x v'_y \dfrac{\beta}{c} \\ v_z \approx v'_z - v'_x v'_z \dfrac{\beta}{c} \end{cases} \tag{B.30}$$

Or, in vector form

$$v^i = v^{i'} + V^i - \frac{v^{i'}}{c^2}(V^i v'_i) \tag{B.31}$$

Note how v and v' are tied asymetrically in the transformation. Consider now a simple planar motion in the xy plane, where $v^i = (v_x, v_y, 0)$, we can find the law of transformation of angles considering that v^i can be rewritten in polar coordinates, as follows

$$\begin{cases} v_x = v\cos\theta \\ v_y = v\sin\theta \\ v_z = 0 \end{cases}$$

Applying the transformations, we have

$$\begin{cases} v\cos\theta = \dfrac{v'\cos\theta' + \beta c}{1 + \frac{\beta}{c}v'\cos\theta'} \\ v\sin\theta = \dfrac{v'\sin\theta'}{\gamma\left(1 + \frac{\beta}{c}v'\cos\theta'\right)} \end{cases} \tag{B.32}$$

Where we used that the motion in the new system will be still planar.
Rewritten in other terms, we have

$$\tan\theta = \frac{\frac{v'\sin\theta'}{\gamma\left(1+\frac{\beta}{c}v'\cos\theta'\right)}}{\frac{v'\cos\theta'+\beta c}{1+\frac{\beta}{c}v'\cos\theta'}} = \frac{v'\sin\theta'}{\gamma\left(v'\cos\theta' + \beta c\right)} \tag{B.33}$$

Which explicitates the change of direction of velocity between different coordinate systems.

§ B.5 4-Vectors

[1]

As we have already suggested before, the 4-tuple $x^\mu = (ct, x, y, z)$ can be seen as a set of coordinates in spacetime, or as a radius vector. The square of vectors in spacetime can be seen as a non-euclidean scalar product as follows

$$x^\mu x_\mu = g_{\mu\nu}x^\mu x^\nu = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 \tag{B.34}$$

Where $g_{\mu\nu}$ is the metric tensor of spacetime

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{B.35}$$

From what we wrote for special relativity itself, we have a new definition

[1] From here on, all greek indexes $(\mu, \nu, \sigma, \cdots)$ are to be intended as spacetime indexes, and latin indexes $(i, j, k, \cdots)$ as usual 3D indexes if not otherwise stated

Definition B.5.1 (4-Vector). A *4-vector* is a 4-tuple that transforms between coordinate frames using Lorentz transformations, as

$$a^\mu = \Lambda^\mu_\nu a^\nu \tag{B.36}$$

Where λ^μ_ν is the already defined transformation matrix of the Lorentz transformations.

Using the metric tensor one can transforms between covariant vectors and contravariant vectors using $a_\mu = g_{\mu\nu}a^\nu$, and due to the semidefinite signature of the metric one has that $a^i = -a_i$, where a^i is the spatial part of the vector. Note also that inserting it into the formula for a scalar product ($a^\mu b_\mu$) one gets back what we had defined before.
It's also possiible to define 4-scalars, which are relativistic invariants. One of such 4-scalars is the square of a 4-vector or the scalar product between 2 4-vectors.
Another way of writing 4-vectors is with a tuple composed as follows

$$a^\mu = (a^0, a^i) \tag{B.37}$$

Where the first component is known as the *polar* component of the 4-vector, and the second is known as the *axial* component of the 4-vector. Therefore we can write

$$\begin{aligned} x^\mu &= (ct, x^i) \\ x_\mu &= (ct, -x_i) \end{aligned} \tag{B.38}$$

§§ B.5.1 4-Velocity and 4-Acceleration

It's possible to define a 4-vector analogue to the velocity of a particle. Indicating with τ the proper time we define the 4-velocity u^μ as

$$u^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} \tag{B.39}$$

Since $\mathrm{d}\tau = \frac{c}{\gamma}\mathrm{d}t$ we have

$$u^\mu = \frac{\gamma}{c}\frac{\mathrm{d}x^\mu}{\mathrm{d}t}$$

In other words

$$u^\mu = \left(\gamma, \frac{\gamma}{c}v^i\right)$$

Note that the square of u^μ is a relativistic invariant and special in nature due to its unitary value, in fact

$$u^\mu u_\mu = \gamma^2 - \gamma^2\frac{v^2}{c^2} = 1$$

The 4-acceleration w^μ is defined analogously derivating again with respect to the proper time, hence

$$w^\mu = \frac{\gamma}{c}\,\mathrm{d}u^\mu\, t = \left(\frac{\gamma}{c}\frac{\mathrm{d}\gamma}{\mathrm{d}t}, \frac{\gamma}{c^2}\frac{\mathrm{d}\gamma v^i}{\mathrm{d}t}\right) \tag{B.40}$$

Deriving with respect to time we have firstly that

$$\frac{\mathrm{d}\gamma}{\mathrm{d}t} = \frac{v^i a_i}{\left(1-\frac{v^2}{c^2}\right)^{\frac{3}{2}}} = \frac{\gamma^3}{c^2}v^i a_i$$

And therefore

$$w^\mu = \frac{\mathrm{d}u^\mu}{\mathrm{d}\tau} = \frac{\gamma}{c}\left(\frac{\gamma^3}{c^2}v^i a_i, \frac{\gamma}{c^3}v^j a_j v^i + \frac{\gamma}{c}a^i\right) \tag{B.41}$$

It's possible to demonstrate that $w^\mu u_\mu = 0$, i.e. that 4-velocity and 4-acceleration are always mutually orthogonal. In fact

$$\frac{\mathrm{d}}{\mathrm{d}\tau}u^\mu w_\mu = \frac{\mathrm{d}u^\mu}{\mathrm{d}\tau}u_\mu + \frac{\mathrm{d}u_\mu}{\mathrm{d}\tau}u^\mu = 2u^\mu w_\mu = 0$$

§ B.6 Exercises

Exercise B.6.1 (Uniformly Accelerated Motion)**.** Solve the motion of an uniformly accelerated particle in the context of Special Relativity.
Consider that the 4-acceleration is constant only in the frame comoving with the particle.
S O L U T I O N
We have that in the comoving frame $\gamma = 1$ and $v = 0$, therefore

$$w^\mu = \left(0, \frac{\dot{v}^i}{c^2}\right)$$

Since a is constant we rotate the 3D system in order to get $a \parallel x$, therefore getting

$$w^\mu = \left(0, \frac{a}{c^2}, 0, 0\right)$$

Note that we can also define a 4-scalar

$$w^\mu w_\mu = -\frac{a^2}{c^2}$$

Changing to the fixed frame of reference, we have

$$w^{\mu'} = \frac{\gamma}{c}\left(\frac{\gamma^3}{c^2}v^i\dot{v}_i, \frac{\gamma^3}{c^2}v^j\dot{v}_j v^i + \frac{\gamma}{c}\dot{v}^i\right) = \frac{\gamma^4}{c^2}\left(\frac{v^i\dot{v}_i}{c}, \frac{v^2}{c^2}\dot{v}^i + \frac{\dot{v}^i}{\gamma^2}\right)$$

Using that

$$\left(\frac{v^2}{c^2} + \frac{1}{\gamma^2}\right)\dot{v}^i = \dot{v}^i$$

We end up with the following simplified result

$$w^{\mu'} = \frac{\gamma^4}{c^2}\left(\frac{1}{c}\dot{v}^i v_i\right)$$

Which gives us the following differential equation

$$w^\mu w_\mu = \frac{\gamma^8}{c^4}\left(\frac{1}{c^2}(v^i\dot{v}_i)^2\right) - \frac{\gamma^8}{c^4}\dot{v}^2 = -\frac{a^2}{c^4}$$

Simplifying the LHS we get

$$\frac{\gamma^8}{c^4}\left(\frac{v^2}{c^2}\dot{v}^2 - \dot{v}^2\right) = \frac{\gamma^8}{c^4}\left(\frac{v^2}{c^2} - 1\right) = -\frac{\gamma^6}{c^4}\dot{v}^2$$

Therefore, putting it back into the first equation, we get

$$-\gamma^6\dot{v}^2 = -a^2 \implies \gamma^3\frac{\mathrm{d}v}{\mathrm{d}t} = a$$

Note that using the derivative of γ with respect to time we can rewrite the LHS as the derivative of a product, in fact

$$\frac{\mathrm{d}(\gamma v)}{\mathrm{d}t} = \frac{\gamma^3}{c^2}v^2\dot{v} + \gamma\dot{v} = \dot{v}\left(\frac{\gamma^3}{c^2}v^2 + \gamma\right) = \gamma^3\dot{v}\left(\frac{v^2}{c^2} + \frac{1}{\gamma^2}\right) = \gamma^3\dot{v}$$

Therefore, finally

$$\frac{\mathrm{d}(\gamma t)}{\mathrm{d}t} = a \implies \gamma v(t) = at + c$$

Imposing that $v(0) = 0$ we get $c = 0$ and therefore, solving for $v(t)$, we have

$$\frac{v(t)}{\sqrt{1-\frac{v^2}{c^2}}} = at \implies v^2 = a^2t^2 - \frac{a^2t^2}{c^2}v^2 \implies v^2 = a^2t^2\left(1 + \frac{a^2t^2}{c^2}\right)^{-1}$$

Therefore

$$v(t) = \frac{at}{\sqrt{1+\frac{a^2t^2}{c^2}}}$$

Then, by direct integration we can find $x(t)$

$$x(t) = \int \frac{at}{\sqrt{1+\frac{a^2t^2}{c^2}}}\mathrm{d}t = \frac{c^2}{2a}\int \frac{1}{\sqrt{1+w^2}}\mathrm{d}w = \frac{c^2}{2a}\left(2\sqrt{1+w} + k\right)$$

Where we used the substitution $w = \frac{a^2t^2}{c^2}$. Imposing the initial condition that $x(0) = 0$ we get $k = -1$, and therefore

$$x(t) = \frac{c^2}{a}\left(2\sqrt{1+\frac{a^2t^2}{c^2}} - 1\right)$$

The proper time of the particle is

$$\tau = \frac{1}{c}\int_{s_0}^{s}\mathrm{d}s = \int_{t_0}^{t}\frac{1}{\gamma}\mathrm{d}t = \int_0^t\sqrt{1-\frac{v^2}{c^2}}\mathrm{d}t$$

From the definition of $v(t)$ we have that

$$\gamma = \frac{1}{1-\frac{a^2t^2}{c^2\left(1+\frac{a^2t^2}{c^2}\right)}}$$

Therefore our integral becomes

$$\tau = \int_0^t \sqrt{1 - \frac{a^2 t^2}{c^2\left(1 + \frac{a^2 t^2}{c^2}\right)}}\mathrm{d}t = \frac{a}{c}\int_0^{\frac{a}{c}t} \sqrt{1 - \frac{z^2}{1+z^2}}\mathrm{d}z = \frac{a}{c}\int_0^{\frac{a}{c}t} \frac{1}{\sqrt{1+z^2}}\mathrm{d}z = \frac{a}{c}\arcsin\left(\frac{at}{c}\right)$$

Where we used the substitution $\frac{at}{c} = z$

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