

## § 0.1 Linear Functionals

### §§ 0.1.1 Dual Spaces and Functionals

**Definition 0.1.1** (Dual Space II). Given  $(\mathcal{V}, \|\cdot\|)$  a normed vector space over a field  $\mathbb{F}$ , we define the following set

$$\mathcal{V}^* := \{f : \mathcal{V} \longrightarrow \mathbb{F} \mid v \mapsto f(v)\}$$

An element  $f \in \mathcal{V}^*$  is called a *continuous linear functional*, with the following properties,  $\forall f, g \in \mathcal{V}^*$ ,  $u, v \in \mathcal{V}$ ,  $\lambda, \lambda_1, \lambda \in \mathbb{F}$

$$\begin{aligned} f(\lambda_1 u + \lambda_2 v) &= \lambda_1 f(u) + \lambda_2 f(v) \\ (f + g)(u) &= f(u) + g(u) \\ (\lambda f)(u) &= f(\lambda u) = \lambda f(u) \end{aligned} \tag{1}$$

With these properties the set  $\mathcal{V}^*$  has a vector space structure, and it's called the *dual space*

**Definition 0.1.2** (Bounded Linear Functional). Given  $f \in \mathcal{V}^*$ , we call  $f$  a *bounded linear functional* if,  $\forall x \in \mathcal{V}$

$$\sup_{\|x\| \leq 1} |f(x)| < \infty \tag{2}$$

**Definition 0.1.3** (Dual Norm). Given a normed vector space  $(\mathcal{V}, \|\cdot\|)$  we can give a structure of normed vector space to its dual with the couple  $(\mathcal{V}^*, \|\cdot\|_*)$ , where the application  $\|\cdot\|_*$  is called the *dual norm*

$$\|\cdot\|_* : \mathcal{V}^* \longrightarrow \mathbb{F}$$

The dual norm is defined in two ways.  $\forall f \in \mathcal{V}^*$ ,  $v \in \mathcal{V}$

$$\|f\|_{*1} = \sup_{v \neq 0} \frac{|f(v)|}{\|v\|} \tag{3a}$$

$$\|f\|_{*2} = \sup_{\|v\| \leq 1} |f(v)| \tag{3b}$$

Where  $f \in \mathcal{V}^*$  is a bounded linear functional.

**Theorem 0.1.** For a bounded linear functional  $f \in \mathcal{V}^*$  the two definitions of the dual norm coincide, i.e.

$$\|f\|_{*1} = \|f\|_{*2} = \|f\|_*$$

And we can define the following inequality

$$|f(v)| \leq \|f\|_* \|v\| \quad \forall v \in \mathcal{V} \tag{4}$$

*Proof.* Since  $f \in \mathcal{V}^*$  is bounded we have that both norms exist and must be finite. We can therefore write, thanks to the homogeneity of  $f$ , taken  $v \in \mathcal{V}$

$$\|f\|_{*1} = \sup_{v \neq 0} \left| f\left(\frac{v}{\|v\|}\right) \right| = \sup_{\|v\|=1} |f(v)| \leq \sup_{\|v\| \leq 1} |f(v)| = \|f\|_{*2}$$

I.e.  $\|f\|_{\star_1} \leq \|f\|_{\star_2}$ , analogously

$$\|f\|_{\star_2} = \sup_{\|v\| \leq 1} |f(v)| \leq \sup_{0 < \|v\| \leq 1} \frac{|f(v)|}{\|v\|} \leq \sup_{v \neq 0} \frac{|f(v)|}{\|v\|} = \|f\|_{\star_2}$$

I.e.  $\|f\|_{\star_1} \geq \|f\|_{\star_2}$ , therefore we have

$$\|f\|_{\star_1} = \|f\|_{\star_2} = \|f\|_{\star}$$

The inequality is obvious taken the definition of supremum □

**Theorem 0.2.** *Given  $f \in \mathcal{V}^*$  with  $\mathcal{V}$  normed vector space, we have that the following assumptions are equivalent*

1.  $f$  is continuous
2.  $f$  is continuous at the origin
3.  $f$  is bounded

*Proof.* 1)  $\implies$  2)

Since  $f \in \mathcal{V}^*$  is linear by definition, we have that it's also continuous and injective, therefore

$$\lim_{v \rightarrow 0} f(v) = f(0) = 0$$

2)  $\implies$  3)

Since  $f$  is continuous at the origin, we have by definition of continuity and limit

$$\forall \epsilon > 0 \exists \delta > 0 : \|x\| < \delta \implies |f(x)| < \epsilon \quad x \in \mathcal{V}$$

Taken  $u = \delta x \in \mathcal{V}$ , we have  $\|u\| = |\delta| \|x\|$  and

$$|f(x)| = \left| f\left(\frac{u}{\delta}\right) \right| = \frac{1}{|\delta|} |f(u)| \leq \epsilon$$

Therefore, if  $\|x\| \leq 1$  we have that  $|f(x)| \leq \delta \epsilon$ , therefore

$$|f(u)| \leq \|f\|_{\star} \|u\| \implies \|f\|_{\star} \|x\| \leq \|f\|_{\star} \leq \epsilon$$

3)  $\implies$  1) By definition of continuity and boundedness we have

$$\forall u, w \in \mathcal{V}, \forall \epsilon > 0 \exists \delta > 0 : \|u - w\| < \delta \implies |f(u) - f(w)| < \epsilon$$

Through the linearity of  $f$  we have

$$|f(u) - f(w)| = |f(u - w)| \leq \|f\|_{\star} \|u - w\| \leq \delta \|f\|_{\star}$$

Taken  $\delta = \epsilon \|f\|_{\star}^{-1}$ , we have

$$|f(u) - f(w)| \leq \delta \|f\|_{\star} = \epsilon$$

□

**Corollary 0.1.1.** Given  $f \in \mathcal{V}^*$ , we have that, given  $C \in \mathbb{F}$  a constant

$$\|f\|_* = C \iff \begin{cases} \forall x \in \mathcal{V} & |f(x)| \leq C\|x\| \\ \forall \epsilon > 0 \exists x \in \mathcal{V} : & |f(x)| \geq (C - \epsilon)\|x\| \end{cases}$$

*Proof.* We have, by definition of  $\|f\|_*$

$$\|f\|_* = \sup_{\|x\| \leq 1} |f(x)| \leq \sup_{\|x\| \leq 1} |c\|x\|| = C$$

In the second case, supposing  $\|f\|_* < C$  and taken  $\epsilon = 1/2(C - \|f\|_*)$

$$\exists x \in \mathcal{V} : |f(x)| \geq (C - \|f\|_*)\|x\| \quad \nexists$$

□

**Definition 0.1.4** (Kernel). Given a linear functional  $f \in \mathcal{V}^*$  we define the *kernel* of  $f$  as the set of zeros of  $f$ , i.e.

$$\ker f = \{v \in \mathcal{V} \mid f(v) = 0\} \quad (5)$$

**Theorem 0.3** (Riesz Representation Theorem). *Given  $(\mathcal{V}, \langle \cdot, \cdot \rangle)$  a Hilbert space, we can uniquely define its dual  $\mathcal{V}^*$  through the isomorphism  $\phi_v : \mathcal{V} \xrightarrow{\sim} \mathbb{F}$  defined as follows*

$$\forall u, v \in \mathcal{V} \quad \phi_v(u) = \langle u, v \rangle$$

*I.e.  $v \mapsto \langle \cdot, v \rangle$ .*

*This isomorphism has the following properties*

$$\begin{aligned} 1) & \forall v \in \mathcal{V} \quad \phi_v \in \mathcal{V}^*, \quad \|\phi_v\|_* = \|v\| \\ 2) & \forall c_1, c_2 \in \mathbb{F}, \quad v_1, v_2 \in \mathcal{V} \quad \phi_{c_1 v_1 + c_2 v_2} = \overline{c_1} \phi_{v_1} + \overline{c_2} \phi_{v_2} \\ 3) & f \in \mathcal{V}^* \implies \exists v \in \mathcal{V} : f = \phi_v \end{aligned} \quad (6)$$

*Proof.* Taken  $w, v, z \in \mathcal{V}$  and  $c, s \in \mathbb{F}$  we have by definition

$$\begin{aligned} \phi_v(w + z) &= \langle w + z, v \rangle = \langle w, v \rangle + \langle z, v \rangle = \phi_v(w) + \phi_v(z) \\ \phi_v(cw) &= \langle cw, v \rangle = c\langle w, v \rangle = c\phi_v(w) \\ \phi_{cw + sz}(v) &= \langle v, cw + sz \rangle = \overline{c}\langle v, w \rangle + \overline{s}\langle v, z \rangle = \overline{c}\phi_w(v) + \overline{s}\phi_z(v) \end{aligned}$$

We also have, thanks to Cauchy-Schwartz

$$\begin{aligned} |\phi_v(z)| &= |\langle z, v \rangle| \leq \|z\| \|v\| \\ |\phi_v(v)| &= |\langle v, v \rangle| \leq \|v\|^2 \quad \therefore \|\phi_v\|_* = \|v\| \end{aligned}$$

For the last one we take  $f \in \mathcal{V}^*$ . If  $\ker f = \mathcal{V}$  we have by definition that

$$f(v) = 0 \quad \forall v \in \mathcal{V} \quad \therefore f = 0$$

Taken  $v = 0$  we then have

$$\phi_0(v) = \langle 0, v \rangle = 0 = f(v) \quad \forall v \in \mathcal{V}$$

Supposing now  $\ker f \neq \mathcal{V}$  there must be  $w \in \ker f^\perp : w \neq 0$ . Taken  $u \in \mathcal{V}$  we can write

$$u = u_1 + \frac{f(u)}{f(w)}w = u - \frac{f(u)}{f(w)}w + \frac{f(u)}{f(w)}w$$

Therefore, through the linearity of  $f$

$$f(u_1) = f\left(u - \frac{f(u)}{f(w)}w\right) = f(u) - f(w) = f(u - w) = 0 \quad \therefore u_1 \in \ker f$$

Since  $w \in \ker f^\perp$ , we have that

$$\langle u_1, w \rangle = 0$$

By definition of  $\phi_v$ , then

$$\phi_v(u) = \langle v, u_1 \rangle + \langle v, \frac{f(u)}{f(w)}w \rangle = \langle \frac{f(w)}{\|w\|^2}w, u_1 \rangle + \langle \frac{f(w)}{\|w\|^2}w, \frac{f(u)}{f(w)}w \rangle$$

Then

$$\phi_v(u) = \frac{f(w)}{\|w\|^2} \langle w, u_1 \rangle + \frac{f(w)}{\|w\|^2} \frac{f(u)}{f(w)} \langle w, w \rangle = f(u)$$

Therefore

$$\phi_v(u) = f(u) \quad \forall f \in \mathcal{V}^*, \forall u \in \mathcal{V}$$

□

## § 0.2 Distributions

### §§ 0.2.1 Local Integrability

**Definition 0.2.1** (Weak Convergence). Given a normed vector space  $(\mathcal{V}, \|\cdot\|)$ , a sequence  $(v)_{k \in \mathbb{N}} \in \mathcal{V}$  is said to be *weakly convergent* to  $v \in \mathcal{V}$  and it's indicated as  $v_k \rightharpoonup v$  if

$$\forall f \in \mathcal{V}^* \quad \lim_{k \rightarrow \infty} f(v_k) = f(v)$$

**Theorem 0.4.** Given a normed vector space  $\mathcal{V}$  and a sequence  $(v)_{k \in \mathbb{N}} \in \mathcal{V}$  such that  $v_k \rightarrow v \in \mathcal{V}$  (i.e. converges strongly), we have that

$$v_k \rightarrow v \implies v_k \rightharpoonup v$$

*Proof.* Given  $v_k \rightarrow v$ , we have by definition that

$$v_k \rightarrow v \iff \lim_{k \rightarrow \infty} \|v_k - v\| = 0$$

Now, writing the definition of weak convergence and applying the linearity of the limit

$$v_k \rightharpoonup v \iff \forall f \in \mathcal{V}^* \quad \lim_{k \rightarrow \infty} (f(v_k) - f(v)) = 0$$

Therefore

$$\forall f \in \mathcal{V}^* \quad \lim_{k \rightarrow \infty} (f(v_k - v)) \leq \|f\|_* \lim_{k \rightarrow \infty} \|v_k - v\|$$

Therefore we have that, since  $f$  is bounded

$$\exists C \in \mathbb{R} : \lim_{k \rightarrow \infty} (f(v_k - v)) \leq C \lim_{k \rightarrow \infty} \|v_k - v\| \quad \forall f \in \mathcal{V}^*$$

And therefore

$$v_k \rightarrow v \implies v_k \rightharpoonup_w v$$

□

*Remark.* The opposite isn't necessarily true.

*Proof.* Take  $(\mathcal{V}, \|\cdot\|) = (\ell^2, \|\cdot\|_2)$  and take the standard sequence  $(e_i)_j$  where

$$(e_i)_j = (0, 0, \dots, 0, \underbrace{(1)_j}_{i\text{-th}}, 0, \dots)$$

And  $(1)_j$  is the identity sequence.

We have that  $(e_i)_j \not\rightharpoonup (a)_i$  since

$$\|(e_i)_j - (e_i)_k\|_2 = \sqrt{2} \quad \forall j \neq k$$

Now, considering that  $(\ell^2, \|\cdot\|_2)$  is a Hilbert space, we have that, due to Riesz representation theorem

$$\forall f \in \ell^{2*} \quad f = \langle \cdot, a \rangle = \phi_a \quad a \in \ell^2$$

Therefore

$$f(e_{ij}) = \langle e_{ij}, a_i \rangle = \sum_{i=1}^{\infty} a_i e_{ij} = a_j \rightarrow 0$$

Therefore

$$\forall (a)_j \in \ell^2 \quad \forall f \in \ell^{2*} \quad \lim_{j \rightarrow \infty} f(e_{ij}) = 0$$

Hence, by definition  $e_{ij} \rightharpoonup_w 0$

□

**Definition 0.2.2** (Isolated Singularities II). Given a function  $f : \mathbb{R} \rightarrow \mathbb{C}$ , it's said to have *isolated singularities* if

$$S_f := \left\{ x \in \mathbb{R} \mid \lim_{y \rightarrow x} f(y) \neq f(x) \right\}$$

Doesn't have accumulation points, i.e. if  $|S_f| < \infty$

**Definition 0.2.3** (Piecewise Continuity). A function  $f : [a, b] \subset \mathbb{R} \rightarrow \mathbb{C}$  is said to be *piecewise continuous* if

$$1. S_f \neq \{\} \text{ and } |S_f| < \infty$$

$$2. \forall u \in S_f$$

$$\begin{aligned} \exists \lim_{x \rightarrow u^+} f(x) &= f(u^+) \\ \exists \lim_{x \rightarrow u^-} f(x) &= f(u^-) \end{aligned} \tag{7}$$

3.

$$\begin{aligned} \exists \lim_{x \rightarrow a^+} f(x) &= f(a^+) \\ \exists \lim_{x \rightarrow b^-} f(x) &= f(b^-) \end{aligned} \quad (8)$$

**Definition 0.2.4** (Jump). Given a piecewise continuous function  $f$  we define the *jump* of the function at a discontinuity  $x \in S_f$  as follows

$$\Delta f(x) = f(x^+) - f(x^-)$$

**Definition 0.2.5** (Piecewise Differentiability). A piecewise continuous function  $f : [a, b] \rightarrow \mathbb{C}$  is said to be *piecewise differentiable* if and only if

1.  $f$  is piecewise continuous in  $[a, b]$
2.  $f'$  is piecewise continuous in  $[a, b]$

**Definition 0.2.6** (Local Integrability). A function  $f : \mathbb{R} \rightarrow \mathbb{C}$  is said to be *locally integrable* if

1.  $S_f \neq \{\}$  and  $|S_f| < \infty$
2.  $\forall a, b \in \mathbb{R}, a < b$  we have that

$$\int_a^b |f(x)| dx < \infty$$

The set of locally integrable functions forms a subspace of  $L^1(\mathbb{R})$  and it's indicated as  $L^1_{loc}(\mathbb{R})$

**Theorem 0.5.** Let  $f : \mathbb{R} \rightarrow \mathbb{C}$  be a piecewise continuous function, then  $f \in L^1_{loc}(\mathbb{R})$

*Proof.* Take  $f(x)$  a piecewise continuous function, then  $\forall a, b \in \mathbb{R} \exists (u)_0 \leq k \leq n \in \mathbb{R} : u_0 = a, u_n = b$  for which we have  $f \in C((u_{k-1}, u_k)) \forall 0 \leq k \leq n$ . I can therefore define  $\tilde{f}_k : [u_{k-1}, u_k] \rightarrow \mathbb{C}$ , where

$$\tilde{f}_k(x) = \begin{cases} f(x) & x \in (u_{k-1}, u_k) \\ f(u_{k-1}^+) & x = u_{k-1} \\ f(u_k^-) & x = u_k \end{cases}$$

We have that

$$f(x) = \sum_{k=0}^{\infty} \tilde{f}_k(x)$$

And therefore

$$\int_a^b |f(x)| dx = \sum_{k=0}^n \int_{u_{k-1}}^{u_k} |f(x)| dx = \sum_{k=0}^n \int_{u_{k-1}}^{u_k} |\tilde{f}_k(x)| dx$$

Therefore  $f(x) \in L^1_{loc}(\mathbb{R})$  and the theorem is proved  $\square$

**Theorem 0.6** (Integration by Parts in  $L^1_{loc}(\mathbb{R})$ ). Let  $f, g \in L^1_{loc}([a, b])$ , then indicating the evaluation of a function at two points as  $[f(x)]_z^w$  we have

$$\begin{aligned} [fg]_{a^+}^{b^-} &= \int_a^b (f'(x)g(x) + f(x)g'(x)) dx + \\ &+ \sum_{x \in S_f \cup S_g} (f(x^-)\Delta g(x) + g(x^-)\Delta f(x) + \Delta f(x)\Delta g(x)) \end{aligned} \quad (9)$$

*Proof.* Take  $H : [a, b] \rightarrow \mathbb{C}$ , then for the fundamental theorem of calculus

$$H(b^-) - H(a^+) = \int_a^b \frac{dH}{dx} dx + \sum_{x \in S_H} \Delta H(x)$$

Taken  $H(x) = f(x)g(x)$  we have

$$\Delta(fg)(x) = f(x^+)g(x^+) - f(x^-)g(x^-)$$

With some manipulation we have

$$\begin{aligned} \Delta(fg)(x) &= f(x^+)g(x^+) - f(x^+)g(x^-) + f(x^+)g(x^-) - f(x^-)g(x^-) = f(x^+)\Delta g(x) + g(x^-)\Delta f(x) = \\ &= (f(x^-) + \Delta f(x))\Delta g(x) + g(x^-)\Delta f(x) = f(x^-)\Delta g(x) + g(x^-)\Delta f(x) + \Delta f(x)\Delta g(x) \end{aligned}$$

Therefore

$$(fg)(b^-) - (fg)(a^+) = \int_a^b (fg)'(x) dx + \sum_{x \in S_f \cup S_g} (f(x^-)\Delta g(x) + g(x^-)\Delta f(x) + \Delta f(x)\Delta g(x))$$

□

### §§ 0.2.2 Regular and Singular Distributions

**Definition 0.2.7** (Test Functions). A function  $f : \mathbb{R} \rightarrow \mathbb{C}$  is said to be a *test function* if  $\overline{\text{supp}\{f\}}$  is compact and  $f \in C^\infty(\mathbb{R})$ , i.e.  $f \in C_c^\infty(\mathbb{R})$

This space is usually denoted as follows  $C_c^\infty(\mathbb{R}) = \mathcal{K}$

**Definition 0.2.8** ( $\mathcal{K}$ -convergence). Given a sequence  $(f)_{n \in \mathbb{N}} \in \mathcal{K}$ , it's said to be  $\mathcal{K}$ -convergent if

1.  $\exists I \subset \mathbb{R} : \forall x \in I^c \ f_n(x) = 0$
2.  $\forall k \in \mathbb{N} \ f_n^{(k)}(x) \Rightarrow f^{(k)}(x)$

Then, it's indicated as  $f_n \rightarrow_{\mathcal{K}} f$

**Definition 0.2.9** (Distribution). A *distribution* is a continuous linear functional  $\varphi : \mathcal{K} \rightarrow \mathbb{C}$ , i.e.

$$\forall (f)_n \in \mathcal{K}, \ f_n \rightarrow_{\mathcal{K}} f \implies \varphi(f_n) = \varphi(f)$$

By definition of dual space, we have that  $\varphi \in \mathcal{K}^*$

**Theorem 0.7.** Given  $g \in L_{loc}^1(\mathbb{R})$  and  $\varphi_g : \mathcal{K} \rightarrow \mathbb{C}$ , if we have

$$\varphi_g(f) = \int_{\mathbb{R}} g(x)f(x) dx \quad \forall f \in \mathcal{K}$$

Then  $\varphi_g \in \mathcal{K}^*$

*Proof.* Using the fact that  $f \in \mathcal{K}$  we can immediately say that  $|f(x)| \leq M \forall x \in [-a, a] \subset \mathbb{R}$ , therefore

$$\int_{\mathbb{R}} |g(x)f(x)| dx = \int_{-a}^a |g(x)f(x)| dx \leq M \int_{-a}^a |g(x)| dx < \infty$$

Alternatively, using the definition of integral we can say that

$$\varphi_g(\alpha f + \beta h) = \alpha \varphi_g(f) + \beta \varphi_g(h) \quad \forall \alpha, \beta \in \mathbb{C}, \forall f, h \in \mathcal{K}$$

We only need to show that this application is  $\mathcal{K}$ -continuous. I take  $f_n \rightarrow_{\mathcal{K}} f$ , with  $f_n \in \mathcal{K}$ . It's obvious that  $f_n \Rightarrow f$ , and using the linearity and that  $g \in L_{loc}^1(\mathbb{R})$ , calling  $A = \|g\|$ , we have

$$|\varphi_g(f) - \varphi_g(f_n)| \leq A \|f - f_n\|_u \rightarrow 0$$

Therefore  $\varphi_g(f_n) \rightarrow \varphi_g(f) \forall f_n \in \mathcal{K}, f_n \rightarrow_{\mathcal{K}} f$  □

**Definition 0.2.10** (Regular Distribution). A distribution  $f \in \mathcal{K}^*$  is said to be *regular* if  $\exists g \in L_{loc}^1(\mathbb{R})$  such that

$$f = \langle \cdot, g \rangle = \varphi_g$$

I.e.

$$f(h) = \varphi_g(h) = \langle h, g \rangle = \int_{\mathbb{R}} h(x)g(x) dx$$

**Definition 0.2.11** (Ceiling and Floor Functions). We define the *ceiling function*  $\lceil \cdot \rceil : \mathbb{R} \rightarrow \mathbb{Z}$  and the *floor functions*  $\lfloor \cdot \rfloor : \mathbb{R} \rightarrow \mathbb{Z}$  as follows

$$\begin{aligned} \lfloor x \rfloor &= \max \{ m \in \mathbb{Z} \mid m \leq x \} \\ \lceil x \rceil &= \min \{ n \in \mathbb{Z} \mid n \geq x \} \end{aligned} \tag{10}$$

**Definition 0.2.12** (Heaviside Function). We define the *Heaviside function* as a function  $H : \mathbb{R} \rightarrow \mathbb{R}$  as follows

$$H(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$$

It's obviously piecewise continuous and therefore  $H \in L_{loc}^1(\mathbb{R})$

A secondary definition is the one that follows  $\tilde{H} : \mathbb{R} \rightarrow \mathbb{R}$

$$\tilde{H}(x) = \begin{cases} 1 & x > 0 \\ 0 & x \leq 0 \end{cases}$$

*Example 0.2.1* (Floor and Ceiling Distributions). Since both the floor and ceiling distributions are locally integrable, we can build a regular distribution  $\varphi \in \mathcal{K}^*$  as follows

$$\begin{aligned} \varphi_{\lfloor x \rfloor}(f) &= \int_{\mathbb{R}} f(x) \lfloor x \rfloor dx = \sum_{k \in \mathbb{Z}} \int_k^{k+1} k f(x) dx \\ \varphi_{\lceil x \rceil}(f) &= \int_{\mathbb{R}} f(x) \lceil x \rceil dx = \sum_{k \in \mathbb{Z}} \int_k^{k+1} (k+1) f(x) dx \end{aligned}$$



*Example 0.2.2* (Theta Distribution). Given  $H \in L^1_{loc}(\mathbb{R})$  we can define the associated *theta distribution*  $\varphi_H = \vartheta \in \mathcal{K}$ , as follows

$$\vartheta(f) = \int_{\mathbb{R}} f(x)H(x) \, dx = \int_{\mathbb{R}^+} f(x) \, dx$$

It's already obvious that  $\varphi_{\tilde{H}} = \varphi_H$ , but it's better to formalize it in the following theorem

**Theorem 0.8.** *Let  $f, g \in L^1_{loc}(\mathbb{R})$  such that  $\{x \in \mathbb{R} \mid f(x) \neq g(x)\} = \{u_1, \dots, u_n\}$ . Then we have  $\varphi_g = \varphi_f \in \mathcal{K}^*$*

*Proof.* We have by Riesz theorem that

$$\varphi_g(f) = \int_{\mathbb{R}} g(x)f(x) \, dx = \int_{-\infty}^{u_1} f(x)g(x) \, dx + \sum_{k=2}^{n-1} \int_{u_k}^{u_{k+1}} f(x)g(x) \, dx + \int_{u_n}^{\infty} f(x)g(x) \, dx$$

But, since  $f(x) = g(x)$ ,  $\forall x \in (u_k, u_{k+1})$  we have

$$\begin{aligned} \int_{\mathbb{R}} f(x)g(x) \, dx &= \int_{-\infty}^{u_1} f(x)g(x) \, dx + \sum_{k=2}^{n-1} \int_{u_k}^{u_{k+1}} f(x)g(x) \, dx + \int_{u_n}^{\infty} f(x)g(x) \, dx = \\ &= \int_{-\infty}^{u_1} f(x)h(x) \, dx + \sum_{k=2}^{n-1} \int_{u_k}^{u_{k+1}} f(x)h(x) \, dx + \int_{u_n}^{\infty} f(x)h(x) \, dx = \int_{\mathbb{R}} f(x)h(x) \, dx \end{aligned}$$

Therefore

$$\varphi_g = \varphi_h$$

□

**Definition 0.2.13** (Singular Distribution). A *singular distribution* is a distribution  $f \in \mathcal{K}^*$  for which, given  $g \in L^1_{loc}(\mathbb{R})$ ,  $f \neq \varphi_g$ , where

$$\varphi_g(h) = \int_{\mathbb{R}} h(x)g(x) \, dx$$

**Definition 0.2.14** (Dirac Delta Distribution). An example of singular distribution is the *Dirac delta distribution*  $\delta_a \in \mathcal{K}^*$ . This distribution is defined as follows

$$\delta_a(f) = f(a) \quad \forall f \in \mathcal{K}, \quad a \in \mathbb{R}$$

**Theorem 0.9.** *Given the Dirac delta distribution  $\delta_a$ ,  $\nexists \delta(x) \in L^1_{loc}(\mathbb{R})$  such that (taken  $a = 0$  without loss of generality)*

$$\delta_0(f) = \int_{\mathbb{R}} f(x)\delta(x) \, dx$$

*Proof.* Let's say that  $\exists \delta(x) \in L^1_{loc}(\mathbb{R})$ , therefore, we could define  $\delta_0 = \langle \cdot, \delta(x) \rangle$ . This function therefore must have these properties

$$\delta(x) \neq 0 \iff x = 0, \quad x \in S_\delta$$

But, since for  $b \in \mathbb{R}$ ,  $b \neq 0$ , we have that  $\delta(x) \rightarrow \delta(b)$  continuously, i.e.  $\delta(b) = A > 0$ , therefore

$$\exists \epsilon > 0 : \delta(x) \geq \frac{A}{2} \quad \forall x \in [b - \epsilon, b + \epsilon]$$

But, then  $\exists f \in \mathcal{K}$  such that  $f(x) \rightarrow f(b)$  continuously,  $f > 0$ ,  $f(b) = 1$ . Taken  $\epsilon < b$  we can say

$$\exists \tilde{\epsilon} \in (0, \epsilon) : f(x) \geq \frac{1}{2} \quad \forall x \in [b - \tilde{\epsilon}, b + \tilde{\epsilon}]$$

Therefore, we have that

$$\int_{\mathbb{R}} f(x) \delta(x) dx = \int_{-\epsilon}^{\epsilon} f(x) \delta(x) dx \geq \int_{-\epsilon}^{\epsilon} \frac{A}{2} \frac{1}{2} dx = \frac{\epsilon' A}{2}$$

But, by definition  $f(0) = 0$ , therefore

$$\int_{\mathbb{R}} f(x) \delta(x) dx = 0 \implies \varphi_{\delta}(f) = \varphi_0(f) \quad \nmid$$

Therefore, the distribution  $\delta_0 \neq \langle \cdot, \delta(x) \rangle$  □

**Notation** (Common Abuse of Notation). A common abuse of notation in the usage of the Dirac delta distribution, is supposing that it's writable like a regular distribution, i.e. supposing that  $\exists \delta(x) \in L^1_{loc}(\mathbb{R})$ , the “Dirac delta function” such that

$$\delta_a(f) = \int_{\mathbb{R}} f(x) \delta(x - a) dx = f(a) \quad \forall f \in \mathcal{K}$$

This is obtainable only if the “delta function” is defined as follows

$$\delta(x) = \begin{cases} +\infty & x = 0 \\ 0 & x \neq 0 \end{cases} \quad (11)$$

Note that

$$\delta_0(1) = \int_{\mathbb{R}} \delta(x) dx = 1$$

**Theorem 0.10.** Given  $g_n \in L^1_{loc}(\mathbb{R})$  such that  $g_n \in C[-a_n, a_n]$  with  $a_n \rightarrow 0$ ,  $a_n > 0 \quad \forall n \in \mathbb{N}$ , we have that, if

$$\int_{\mathbb{R}} g_n(x) dx = 1$$

Then

$$\lim_{n \rightarrow \infty} \varphi_{g_n}(f) = \delta_0(f)$$

*Proof.* We use the definition of limit, therefore we have

$$\left| \int_{\mathbb{R}} f(x) g_n(x) dx - \delta_0(f) \right| \leq \int_{-a_n}^{a_n} |f(x) - f(0)| g_n(x) dx$$

Using that the integral over the real axis of  $g_n$  is unitary

$$\left| \int_{\mathbb{R}} f(x)g_n(x) \, dx - \delta_0(f) \right| \leq \sup_{|x| \leq a_n} |f(x) - f(0)| \rightarrow 0$$

Therefore

$$\lim_{n \rightarrow \infty} \varphi_{g_n}(f) = \delta_0(f)$$

□

**Corollary 0.2.1.** Given  $g \in C_1(\mathbb{R})$  a non-negative function  $g \geq 0$  such that

$$\int_{\mathbb{R}} g(x) \, dx = 1$$

Then, if we put  $g_n(x) = ng(nx)$  we have that

$$\lim_{n \rightarrow \infty} \varphi_{g_n}(f) = \delta_0(f)$$

*Proof.* As before, we use the definition, and therefore we have

$$\left| \int_{\mathbb{R}} f(x)g_n(x) \, dx - \delta_0(f) \right| \leq \int_{\mathbb{R}} |f(x) - f(0)|g_n(x) \, dx = n \int_{\mathbb{R}} |f(x) - f(0)|g(nx) \, dx$$

Using the substitution  $u = nx$  we therefore get

$$\int_{\mathbb{R}} |f(x) - f(0)|g(nx)n \, dx = \int_{\mathbb{R}} \left| f\left(\frac{u}{n}\right) - f(0) \right| g(u) \, du$$

But, by definition of  $g(x)$ , we have that  $\exists L > 0 : \exists \epsilon > 0$  for which

$$1 - \epsilon \leq \int_{-L}^L g(u) \, du \leq 1 \quad \wedge \quad \int_{|u| \geq L} g(u) \, du \leq \epsilon$$

And therefore

$$\int_{\mathbb{R}} \left| f\left(\frac{u}{n}\right) - f(0) \right| g(u) \, du = \int_{-L}^L \left| f\left(\frac{u}{n}\right) - f(0) \right| g(u) \, du + \int_{|u| \geq L} \left| f\left(\frac{u}{n}\right) - f(0) \right| g(u) \, du$$

By using the properties of the integral, we know that

$$\int_{\mathbb{R}} \left| f\left(\frac{u}{n}\right) - f(0) \right| g(u) \, du \leq \sup_{x \leq \frac{L}{n}} |f(x) - f(0)| + \epsilon \left( \sup_{u \in \mathbb{R}} \left| f\left(\frac{u}{n}\right) \right| - |f(0)| \right)$$

Since  $f(x) \rightarrow f(0)$  continuously, due to the fact that  $f \in \mathcal{K}$ , we have that

$$\int_{\mathbb{R}} |f(x) - f(0)|g(nx)n \, dx \leq 2\epsilon \|f\|_u$$

Therefore

$$\lim_{n \rightarrow \infty} \left| \int_{\mathbb{R}} f(x)g_n(x) \, dx - \delta_0(f) \right| = 0$$

□

**Definition 0.2.15** (PV  $(x^{-n})$ -Distribution). Taken  $n > 0$ ,  $n \in \mathbb{N}$ , we have that  $x^{-n} \notin L^1_{loc}(x)$  and therefore there is no associated distribution  $\varphi_{x^{-n}} \in \mathcal{K}^*$ .

A useful thing we could do is utilizing the definition of the Cauchy principal value of the function. Therefore, we define the following singular distribution

$$\text{PV} \left( \frac{1}{x^n} \right) [f] = \lim_{a \rightarrow \infty} \int_{-a}^a \frac{1}{x^n} \left( f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} x^k \right) dx \quad (12)$$

**Theorem 0.11.** The application  $\text{PV}(x^{-n}) : \mathcal{K} \longrightarrow \mathbb{R}$  defined as before is a distribution, hence  $\text{PV}(x^{-n}) \in \mathcal{K}^*$

*Proof.* We already know that this distribution is linear, since  $\forall f, g \in \mathcal{K}, \forall c, d \in \mathbb{R}$

$$\begin{aligned} \text{PV} \left( \frac{1}{x^n} \right) [cf + dg] &= \lim_{a \rightarrow \infty} \int_{-a}^a \frac{1}{x^n} \left( cf(x) + dg(x) + \sum_{k=0}^{n-1} \frac{1}{k!} \left( cf^{(k)}(0) + dg^{(k)}(0) \right) x^k \right) dx = \\ &= c \text{PV} \left( \frac{1}{x^n} \right) [f] + d \text{PV} \left( \frac{1}{x^n} \right) [g] \end{aligned}$$

Secondly we must show that the integral is well defined.

Let  $f \in \mathcal{K}$ , then

$$I_R^n(f) = \int_{-R}^R \frac{1}{x^n} \left( f(x) - \sum_{k=0}^{n-1} \frac{1}{k!} f^{(k)}(0) x^k \right) dx, \quad n \in \mathbb{N}, R > 0$$

Using the Lagrange formulation for the remainder of the McLaurin expansion, we have

$$R_{n-1}(x) = f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} x^k = x^n r_{n-1}(x)$$

And therefore we have

$$I_R^n(x) = \int_{-R}^R r_{n-1}(x) dx$$

Since  $r_{n-1}(x) \in C^\infty(\mathbb{R})$  (it's a polynomial) and  $\text{supp } f \subset [-b, b]$  for some  $b \in \mathbb{R}$ ,  $b > 0$ , we have that for  $R > b$

$$I_R^n(f) = I_b^n(f) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} \int_{b \leq |x| \leq R} \frac{1}{x^{n-k}} dx$$

The last integral evaluates to

$$\int_{b \leq |x| \leq R} \frac{1}{x^{n-k}} dx = \begin{cases} \frac{2}{n-k-1} \left[ \frac{1}{b^{n-k-1}} - \frac{1}{R^{n-k-1}} \right] & n-k \mod 2 = 0, n-k > 2 \\ 0 & \text{else} \end{cases}$$

Therefore, introducing the following dummy index  $j = n - k$  where  $j \mod 2 = 0$ , we have

$$\text{PV} \left( \frac{1}{x^n} \right) (f) = I_b^n(f) - \sum_{j=0}^{n-1} \frac{f^{(j)}(0)}{j!} \frac{2}{(j-1)b^{j-1}}$$

Therefore the integral defining the distribution is well defined.

Lastly we need to prove that  $\text{PV}(x^{-n})(f_n) \rightarrow_{\mathcal{K}} \text{PV}(x^{-n})(f)$ ,  $\forall (f)_n \in \mathcal{K} : f_n \rightarrow f$ , i.e. that it's a continuous functional.

Going back to the previous definition we have that

$$r_{n-1}(x) = \frac{R_{n-1}(x)}{x^n} = \frac{1}{x^n} \frac{d^{n-1}f}{dx^{n-1}}(\xi(x)) \quad \xi(x) \in (0, x)$$

Therefore

$$|I_R^n(f)| \leq 2 \sup_{x \in [-R, R]} |r_{n-1}(x)| \leq \frac{2R}{n!} \|f\|_u$$

Which means

$$\left| \text{PV} \left( \frac{1}{x^n} \right) (f) \right| \leq \frac{2b}{n!} \|f^{(n)}\|_u + \sum_{j=0}^{n-1} \frac{|f^{(j)}(0)|}{j!} \frac{2}{(j-1)b^{j-1}}$$

Taken  $b_1 = \max\{b, 1\}$  we have that  $\text{supp } f \subset [-b_1, b_1]$  and whenever  $j-1 > 1$ ,  $b_1 \geq 1$

$$\frac{2}{(j-1)b_1^{j-1}} \leq 2$$

Therefore, finally

$$\left| \text{PV} \left( \frac{1}{x^n} \right) (f) \right| \leq \frac{2}{n!} \max\{b, 1\} \|f^{(n)}\|_u + 2 \sum_{j=0}^{n-1} \frac{|f^{(j)}(0)|}{j!}$$

Now, defined  $f_j \in \mathcal{K}$  and  $h_j(x) = f(x) - f_j(x)$  with  $f_j \rightarrow f$  we have that  $\text{supp } h_j \subset [-b, b]$ ,  $\forall j \in \mathbb{N}$ .

By definition we have that  $h_j \rightrightarrows 0$  and  $h_j^{(n)} \rightrightarrows 0 \forall n \in \mathbb{N}$

$$\text{PV} \left( \frac{1}{x^n} \right) (h_j) \leq \frac{2}{n!} \max\{b, 1\} \|h_j^{(n)}\|_u + 2 \sum_{j=0}^{n-1} \frac{|h_j^{(j)}(0)|}{j!}$$

This means

$$\forall k \in \mathbb{N}, \forall \epsilon > 0, \exists N_R \in \mathbb{N} : \forall k \geq N_j \quad \left| h_j^{(k)}(x) \right| \leq \frac{\epsilon}{2(b+1+\epsilon)}$$

Chosen  $N = \max_{0 \leq k \leq n-1} N_k$  we have that  $\forall j \geq N$

$$\left| \text{PV} \left( \frac{1}{x^n} \right) (h_j) \right| \leq \frac{2\epsilon}{2(b+1+\epsilon)} \left[ \max\{b, 1\} + \sum_{j=0}^{n-1} \frac{1}{j!} \right] \leq \epsilon \rightarrow 0$$

Therefore,  $\text{PV}(x^{-n}) \in \mathcal{K}^*$  □

### §§ 0.2.3 Operations with Distributions

**Definition 0.2.16** (Weak Derivative). Given  $u \in L^1([a, b])$ , we define the weak derivative  $v \in L^1([a, b])$  if,  $\forall h \in C_c^\infty([a, b])$  we have

$$\int_a^b u(x) h'(x) dx = - \int_a^b v(x) h(x) dx$$

The function  $v(x)$  will then be identified as follows

$$v(x) = D u(x)$$

**Theorem 0.12** (Operations with Distributions). *Given  $\varphi, \gamma \in \mathcal{K}^*$ ,  $f \in \mathcal{K}$ ,  $h \in C^\infty(\mathbb{R})$  and  $c \in \mathbb{C}$  we define the following operations in  $\mathcal{K}^*$*

$$\begin{aligned} + : \mathcal{K}^* \times \mathcal{K}^* &\longrightarrow \mathcal{K}^* \\ \cdot : \mathbb{C} \times \mathcal{K}^* &\longrightarrow \mathcal{K}^* \\ \circ : C^\infty \times \mathcal{K}^* &\longrightarrow \mathcal{K}^* \\ D : \mathcal{K}^* &\longrightarrow \mathcal{K}^* \end{aligned} \tag{13}$$

Where, they act as follows

$$\begin{aligned} +(\varphi(f), \gamma(f)) &= (\varphi + \gamma)(f) = \varphi(f) + \gamma(f) \\ \cdot(c, \varphi(f)) &= (c\varphi)(f) = c\varphi(f) \\ \circ(h, \varphi(f)) &= (h\varphi)(f) = \varphi(hf) \\ D\varphi(f) &= -\varphi(f') \end{aligned}$$

The last operation is the distributional derivative.

**Theorem 0.13.** *Given  $g(x) \in L^1_{loc}(\mathbb{R})$  and  $\varphi_g \in \mathcal{K}^*$  its associated distribution. Then,  $\forall h \in C^\infty$ ,  $f \in \mathcal{K}$*

$$(h\varphi_g)(f) = \varphi_g(hf) = \varphi_{hg}(f) \tag{14}$$

*Proof.* The proof is quite straightforward

$$(h\varphi_g)(f) = \varphi_g(hf) = \int_{\mathbb{R}} g(x) (h(x)f(x)) dx = \int_{\mathbb{R}} f(x) (g(x)h(x)) dx = \varphi_{hg}(f)$$

□

**Theorem 0.14.** *Taken  $g \in L^1_{loc}(\mathbb{R})$  and  $g \mapsto \varphi_g \in \mathcal{K}^*$  its associated distribution, we have that*

$$D\varphi_g = \varphi_{Dg}$$

Where  $Dg$  is the weak derivative of  $g$

*Proof.*  $\forall f \in \mathcal{K}$  we have that

$$D\varphi_g(f) = -\varphi_g(f') = -\int_{\mathbb{R}} f'(x)g(x) dx = \int_{\mathbb{R}} f(x) Dg(x) dx = \varphi_{Dg}(f)$$

□

**Theorem 0.15.**  $\forall g, f \in L^1_{loc}(\mathbb{R})$  and given  $\varphi_g, \varphi_f \in \mathcal{K}^*$ ,  $c \in \mathbb{C}$  their associated distributions, we have that

$$\begin{aligned} \varphi_g + \varphi_f &= \varphi_{g+f} \\ \varphi_{cf} &= c\varphi_f \end{aligned}$$

*Proof.* The proof is obvious using the linearity of the integral operator  $\square$

**Theorem 0.16** (Useful Identities). *Here is a list of some useful identities*

$$\begin{aligned} x\delta_0 &= 0 \\ x\text{PV}\left(\frac{1}{x}\right) &= \varphi_1 \\ xD\delta_0 &= -\delta_0 \end{aligned} \tag{15}$$

*Proof.* 1)  $x\delta_0 = 0$ .

Taken  $f \in \mathcal{K}$

$$x\delta_0(f) = \delta_0(xf) = 0f(0) = 0$$

2)  $x\text{PV}\left(\frac{1}{x^n}\right) = \varphi_1$

Again, taken  $g \in \mathcal{K}$

$$x\text{PV}\left(\frac{1}{x}\right)(g) = \text{PV}\left(\frac{1}{x}\right)(xg) = \lim_{\epsilon \rightarrow 0} \int_{|x| \geq \epsilon} f(x) dx = \varphi_1$$

For the last one, taken  $h \in \mathcal{K}$

$$xD\delta_0(f) = D\delta_0(xf) = -\delta_0(f + xf') = -f(0) - 0f(0) = -f(0) = -\delta_0(f)$$

$\square$

**Notation** (Abuse of Notation). Given  $g \in L^1_{loc}(\mathbb{R})$  and its associated distribution  $g \mapsto \varphi_g$ , is quite common to use the original function to indicate actually the distribution. Together with this, the distributional derivative  $D$  gets indicated as an usual derivative, therefore

$$\begin{aligned} \varphi_g &\rightarrow g(x) \\ D\varphi_g &\rightarrow g'(x) = \frac{dg}{dx} \end{aligned}$$

Therefore it isn't uncommon to see identities like this

$$x\text{PV}\left(\frac{1}{x}\right) = 1$$

Where actually  $1 \rightarrow \varphi_1$  is the identity distribution.

Or

$$\delta'_0 = -\delta_0$$

This makes an easy notation for calculating distributional derivatives and have some calculations, but one should watch out to this common abuse of notation

**Definition 0.2.17** ( $\mathcal{K}^*$  Convergence). Given  $(\sigma)_{n \in \mathbb{N}} \in \mathcal{K}^*$  a sequence of distributions, it's say to  $\mathcal{K}^*$ -converge if

$$(\sigma)_n(f) \rightarrow \sigma(f) \in \mathcal{K}^*$$

It's indicated as follows

$$(\sigma)_n \rightarrow_\star \sigma$$

**Theorem 0.17.** Given  $\kappa \in \mathcal{K}^*$  a distribution, we have that  $f \in C^\infty(\mathcal{K})$  in the sense of distributional derivatives, where the  $n$ -th derivative is defined as follows

$$D^n \kappa(f) = (-1)^n \kappa(f^{(n)}) \quad \forall f \in \mathcal{K}$$

*Proof.* Taken  $f \in \mathcal{K}$  we have that  $f \in C_c^\infty(\mathbb{R}) \subset C^\infty(\mathbb{R})$  hence  $f$  is smooth, therefore we have, for  $\kappa \in \mathcal{K}^*$

$$D \kappa(f) = -\kappa(f')$$

Iterating, we get for  $n = 2$

$$D^2 \kappa(f) = -D \kappa(f') = \kappa(f'')$$

Iterating till  $n$  we get finally

$$D^n \kappa(f) = (-1)^n \kappa(f^{(n)})$$

Hence the derivability depends only on the test function  $f$ , and since it's smooth, the distributional derivatives can be defined  $\forall n \in \mathbb{N}$  □

*Example 0.2.3.* Take for example the delta distribution  $\delta_0 \in \mathcal{K}^*$ , given  $f \in \mathcal{K}$  we want to calculate the following value

$$D^n \delta_0(f)$$

Using the previous identity, we have that

$$D^n \delta_0(f) = (-1)^n \delta_0[f^{(n)}] = (-1)^n f^{(n)}(0)$$

**Theorem 0.18** (Chain Rule). Given  $h \in C^\infty(\mathbb{R})$  and  $\eta \in \mathcal{K}^*$ , we have that

$$D(h\eta) = h'\eta + h D\eta$$

Where the distributional derivative is identical to the usual derivative for  $h$  and it's the usual distributional derivative for  $\eta$

*Proof.* Taken  $h \in C^\infty$ ,  $\eta \in \mathcal{K}^*$  and  $f \in \mathcal{K}$  we have

$$(h\eta)(f) = \eta(fh)$$

Therefore, using the identity for derivating a distribution we have that

$$\begin{aligned} D(h\eta)[f] &= -(h\eta)[f'] = \\ &= -\eta[hf'] = -\eta[(hf)' - h'f] = -\eta[(hf)'] + \eta[h'f] = \\ &= (h D\eta)[f] + (h'\eta)[f] \end{aligned}$$

□

**Theorem 0.19.** Given  $g_n(x) : \mathbb{R} \rightarrow \mathbb{C}$  a continuous sequence of functions such that  $g_n \rightrightarrows g \in C(\mathbb{R})$ . Then, taken  $g_n \mapsto \varphi_{g_n} \in \mathcal{K}^*$  the associated distribution, we have that

$$\varphi_{g_n} \rightarrow_\star \varphi_g \in \mathcal{K}^*$$



*Proof.* By definition of convergence, we have that, given  $f \in \mathcal{K}$ , where  $\text{supp } f \subset [-a, a]$

$$|\varphi_{g_n}(f) - \varphi_g(f)| \leq \varphi_{|g_n - g|}(|f|) \leq 2a \|g_n - g\|_u \|f\|_u \rightarrow 0$$

Therefore,  $\varphi_{g_n} \rightarrow_\star \varphi_g$  □

**Theorem 0.20.** *Given  $(\eta)_n \in \mathcal{K}^*$  a sequence of distribution, then if  $(\eta)_n \rightarrow_\star \eta$  we have that*

$$D \eta_n \rightarrow_\star D \eta$$

*Proof.* By definition, we have

$$\lim_{n \rightarrow \infty} D \eta_n(f) = - \lim_{n \rightarrow \infty} \eta_n(f') = -\eta(f') = D \eta(f)$$

□

*Example 0.2.4* (The Absolute Value Distribution). Taken  $g(x) = |x|$  we have that (obviously)  $|x| \in L^1_{loc}(\mathbb{R})$ , therefore there exists a distribution  $\varphi_{|x|} \in \mathcal{K}^*$  defined as follows

$$\varphi_{|x|}(f) = \int_{\mathbb{R}} |x| f(x) dx$$

We have that the distributional derivative it's actually the function  $\text{sgn}(x)$ , (in this case, since it's a locally integrable function it coincides with its weak derivative), therefore it's not unusual to see expressions like this

$$\frac{d}{dx} |x| = \text{sgn}(x) \quad \text{weakly/distributionally}$$

The proof of this is quite easy. Taken  $f \in \mathcal{K}$

$$\begin{aligned} D \varphi_{|x|}(f) &= -\varphi_{|x|}(f') = - \int_{\mathbb{R}} |x| f'(x) dx = \\ &= [xf(x)]_{\mathbb{R}^-} - [xf(x)]_{\mathbb{R}^+} + \int_{\mathbb{R}^+} f(x) dx - \int_{\mathbb{R}^-} f(x) dx = \\ &= \int_{\mathbb{R}} \text{sgn}(x) f(x) dx = \varphi_{\text{sgn}(x)}(f) \end{aligned}$$

Since we have that

$$D \varphi_g = \varphi_{Dg}$$

Where  $Dg$  is intended as the weak derivative of  $g$ , we have that

$$D |x| = \text{sgn}(x)$$

Where in order to emphasize that this is a weak derivative, one could use the notation

$$D_w |x| = \text{sgn}(x)$$

Note that in literature it's common to use the abuse of notation written beforehand in general and in this particular case.

Also note that if the function we had used would have been an ordinarily derivable function, the weak derivative would have coincided with the ordinary derivative.

**Theorem 0.21.** *In general, given  $g \in C^1(\mathbb{R})$ , we can say that  $g(|x|)$  has the following weak derivative*

$$Dg(|x|) = Dg(|x|) \operatorname{sgn}(x)$$

*Proof.* Since  $g(|x|) \in L^1_{loc}(\mathbb{R})$  we can say that exists  $g \mapsto \varphi_g \in \mathcal{K}^*$  such that

$$D\varphi_{g(|x|)} = \varphi_{Dg(|x|)}$$

Where  $Dg$  is the weak derivative of  $g(|x|)$ . Since it's a derivative (and it's also demonstrable) we can use the rules for composite derivation, and knowing that  $D|x| = \operatorname{sgn}(x)$  we have

$$D\varphi_{g(|x|)} = \varphi_{Dg(|x|)} = \varphi_{Dg D|x|} = \varphi_{\operatorname{sgn} Dg}$$

□

**Theorem 0.22** (Derivative of the  $\vartheta$  Distribution). *Given  $\vartheta \in \mathcal{K}^*$  the theta distribution, we define*

$$\vartheta(f) = \int_{\mathbb{R}} f(x)H(x) \, dx = \int_{\mathbb{R}^+} f(x) \, dx$$

*The derivative of this distribution is*

$$D\vartheta = \delta_0$$

*Proof.* We have  $\forall f \in \mathcal{K}$

$$D\vartheta(f) = -\vartheta(f') = -\int_{\mathbb{R}} f'(x)H(x) \, dx = -\int_{\mathbb{R}^+} f'(x) \, dx = f(0) = \delta_0(f)$$

Therefore

$$D\vartheta = \delta_0$$

Or, using a common abuse of notation

$$\vartheta = \vartheta(x) = H(x), \quad DH(x) = \delta(x)$$

□

**Notation** (Piecewise Derivative). Given  $g : \mathbb{R} \rightarrow \mathbb{C}$  a piecewise differentiable function, we define the differential operator  $D_\circ$  as follows

$$D_\circ g(x) = \begin{cases} g'(x) & \exists g'(x) \\ 0 & \nexists g'(x) \end{cases} \quad (16)$$

**Theorem 0.23.** *Given  $f : \mathbb{R} \rightarrow \mathbb{C}$  a piecewise differentiable function and  $S_f := \{u_1, \dots, u_k\}$  isolated singularities, then*

$$Df(x) = D_\circ f(x) + \sum_{i=1}^{\infty} \Delta f(u_i) \delta(u_i)$$

*Proof.* Taken  $S_f := \{u\}$  without loss of generality, we have that

$$\begin{aligned}\lim_{x \rightarrow u^+} f(x) &= f(u^+) \\ \lim_{x \rightarrow u^-} f(x) &= f(u^-) \\ \Delta f(u) &= f(u^+) - f(u^-)\end{aligned}$$

And,  $\forall g \in \mathcal{K}$ , taken  $f \mapsto \varphi_f \in \mathcal{K}^*$

$$D \varphi_f(g) = -\varphi_f(g') = -\int_{-\infty}^u g'(x)f(x) dx - \int_u^{\infty} g(x)f(x) dx$$

Integrating by parts and rebuilding the definition of  $\Delta f(u)$  we have

$$D \varphi_f(g) = g(u)\Delta f(u) + \int_{\mathbb{R}} g(x) D_{\circ} f(x) dx = \varphi_{D_{\circ} g}(f) + \Delta f(u)\delta_u(f)$$

And therefore

$$D f(x) = D_{\circ} f + \Delta f(u)\delta(u)$$

□

*Example 0.2.5* (Deriving the Sign Function). Take  $a \in \mathbb{R}$  and the function  $\text{sgn}(x - a)$ , using the previous formula we have that  $S_{\text{sgn}(x-a)} = \{a\}$  and  $\Delta \text{sgn}(a) = 2$ , therefore

$$D \text{sgn}(x - a) = D_{\circ} \text{sgn}(x - a) + 2\delta(x - a)$$

Since  $D_{\circ} \text{sgn}(x - a) = 0$  we have finally

$$D \text{sgn}(x - a) = 2\delta(x - a)$$

And equivalently

$$D \text{sgn}(a - x) = -2\delta(x - a)$$

*Example 0.2.6* (A General Piecewise Differentiable Function). Take the function  $g : \mathbb{R} \rightarrow \mathbb{R}$  defined as follows

$$g(x) = \begin{cases} 1 & x < 0 \\ x - 2 & x \geq 0 \end{cases}$$

We have that

$$D_{\circ} g(x) = H(x) = \begin{cases} 0 & x \leq 0 \\ 1 & x > 0 \end{cases}$$

Since the discontinuity is in the origin and we have  $\Delta g(0) = -3$  we have

$$D g(x) = H(x) - 3\delta(x)$$

*Example 0.2.7* (Derivative of the Floor Function). Given the floor function  $[x] : \mathbb{R} \rightarrow \mathbb{Z}$  we have that  $S_{[x]} = \mathbb{Z}$ ,  $\Delta [x] = 1$  and  $D_{\circ} [x] = 0$  therefore

$$D [x] = \sum_{k \in \mathbb{Z}} \delta(x - k) = \sum_{k=-\infty}^{\infty} \delta(x - k)$$

Given instead a different function,  $\lfloor x^2 \rfloor$  we have that  $D_\circ \lfloor x^2 \rfloor = 0$  and  $S_{\lfloor x^2 \rfloor} = \{k \in \mathbb{Z} \mid \pm \sqrt{k}\} \subset \mathbb{R}$  with  $\Delta[\pm \sqrt{k}] = \pm 1$ , therefore

$$D \lfloor x^2 \rfloor = \sum_{k \geq 1} \delta(x - \sqrt{k}) - \delta(x - \sqrt{-k}) = \sum_{k=1}^{\infty} \delta(x - \sqrt{k}) - \delta(x - \sqrt{-k})$$

**Theorem 0.24.** *Given  $h \in C^\infty(\mathbb{R})$  a smooth function and  $a \in \mathbb{R}$ , we have that*

$$h(x) D^n \delta(x - a) = \sum_{k=0}^n (-1)^k \binom{n}{k} h^{(k)}(a) D^{n-k} \delta(x - a)$$

*Proof.* Taken  $f \in \mathcal{K}$  we have that

$$(h(x) D^n \delta_a)(f) = D^n \delta_a(hf) = \delta_a((hf)^{(n)})$$

Using now Leibnitz's composite derivation rule, we have

$$\frac{d^n}{dx^n} h(x) f(x) = \sum_{k=0}^n \binom{n}{k} \frac{d^k h}{dx^k} \frac{d^{n-k} f}{dx^{n-k}}$$

We have that

$$\begin{aligned} (h(x) D^n \delta_a)(f) &= \delta_a \left[ (-1)^n \sum_{k=0}^n \binom{n}{k} h^{(k)}(x) f^{(n-k)}(x) \right] = \\ &= \sum_{k=0}^n \binom{n}{k} (-1)^k h^{(k)}(x) D^{n-k} \delta_a(f) \end{aligned}$$

□

*Example 0.2.8.* Take as an example the following distributional derivative

$$D^7 (xe^x D^2 \delta_0)$$

We have from that formula that

$$xe^x D^2 \delta_0 = \frac{d^2}{dx^2} (xe^x) \delta_0 = \frac{d}{dx} (e^x + xe^x) \delta_0 = (2 + x) e^x \delta_0$$

And therefore  $h'(0) = 1$ ,  $h''(0) = 2$

Which means that

$$xe^x D^2 \delta_0 = \binom{2}{2} h(0) D^2 \delta_0 - \binom{2}{1} h'(0) D \delta_0 + \binom{2}{0} h''(0) \delta_0$$

Inserting the values, we get

$$xe^x D^2 \delta_0 = 2 D \delta_0 - 2 \delta_0$$

The derivative is now trivial, and we get

$$D^7 (xe^x D^2 \delta_0) = 2 D^7 (\delta_0 - D \delta_0) = 2 (D^7 \delta_0 - D^8 \delta_0)$$

**Theorem 0.25.** Given  $k, m \in \mathbb{N}$  and a function  $f \in C^k(\mathbb{R})$  we have that

$$D^k(x^m f)(0) = \begin{cases} 0 & k < m \\ \binom{k}{m} m! f^{(k-m)}(0) & k \geq m \end{cases}$$

And

$$D^i(x^m)(0) = m! \delta_m^i$$

*Proof.* We apply immediately the Leibnitz chain rule and we have

$$D^k(x^m f)(0) = \sum_{j=0}^m \binom{k}{j} f^{(k-j)}(0) m! \delta_{jm} = m! \binom{k}{m} f^{(k-m)}(0) \quad \forall k \geq m$$

□

**Theorem 0.26** (Properties of the  $PV(x^{-n})$  Distribution). *Here there will be listed some properties of the  $PV(x^{-n})$  distribution*

1.  $D \log |x| = PV(x^{-n})$
2. Given  $n, m \in \mathbb{N}$ , then

$$x^m PV\left(\frac{1}{x^n}\right) = \begin{cases} x^{m-n} & m \geq n \\ PV\left(\frac{1}{x^{n-m}}\right) & m < n \end{cases}$$

3.  $D^m PV(x^{-n}) = (-m)! PV(x^{-n-m})$

*Proof.* 1) Taken  $\log |x| \in L_{loc}^1(\mathbb{R})$  we know that  $\exists \varphi_{\log |x|} \in \mathcal{K}^*$  such that  $\log |x| \mapsto \varphi_{\log |x|}$  and therefore we can write as follows the previous derivative, that  $\forall f \in \mathcal{K}$

$$D \varphi_{\log |x|}(f) = -\varphi_{\log |x|}(f') = -\lim_{\epsilon \rightarrow 0^-} \int_{-\infty}^{\epsilon} f'(x) \log(-x) dx - \lim_{\epsilon \rightarrow 0^+} \int_{\epsilon}^{\infty} f(x) \log(x) dx$$

Integrating by parts we get

$$D \varphi_{\log |x|}(f) = \lim_{\epsilon \rightarrow 0^+} \log \epsilon (f(\epsilon) - f(-\epsilon)) + \lim_{\epsilon \rightarrow 0^+} \int_{|x| \geq \epsilon} \frac{f(x)}{x} dx = PV\left(\frac{1}{x^n}\right)(f)$$

Or

$$D \log |x| = PV\left(\frac{1}{x^n}\right)$$

- 2) Taken  $f \in \mathcal{K}$

$$\left(x^m PV\left(\frac{1}{x^n}\right)\right)(f) = PV\left(\frac{1}{x^n}\right)(x^m f) = \lim_{R \rightarrow \infty} \int_{-R}^R \frac{1}{x^n} \left(x^m f(x) - \sum_{k=0}^{n-1} \frac{D^k(x^m f)(0)}{k!} x^k\right) dx$$

Taken  $m \geq n$  we have that the sum is null  $\forall k \leq m$ , therefore since  $k \leq n-1 < m$  it sums to 0  
Therefore

$$\lim_{R \rightarrow \infty} I_R^n(x^m f) = \text{PV} \left( \frac{1}{x^n} \right) (x^m f) = \int_{\mathbb{R}} x^{m-n} f(x) dx = \varphi_{x^{m-n}}(f)$$

Taken  $m < n$  we have that

$$\sum_{k=0}^{n-1} \frac{x^k}{k!} D^k(x^m f)(0) = \sum_{k=m}^{n-1} \binom{k}{m} \frac{m!}{k!} f^{(k-m)}(0) = \sum_{k=0}^{n-m-1} \frac{f^{(k)}(0)}{k!} x^k$$

Therefore

$$\lim_{R \rightarrow \infty} I_R^n(x^m f) = \lim_{R \rightarrow \infty} \lim_{\epsilon \rightarrow 0} \int_{-R}^R \frac{1}{x^{n-m}} \left( f(x) - \sum_{k=0}^{n-m-1} \frac{f^{(k)}(0)}{k!} x^k \right) dx = \text{PV} \left( \frac{1}{x^{n-m}} \right) (f)$$

3) Taken  $n \in \mathbb{N}$ ,  $n > 0$ ,  $m = 1$  we have

$$D \text{PV} \left( \frac{1}{x^n} \right) (f) = -\text{PV} \left( \frac{1}{x^n} \right) (f') = -\lim_{R \rightarrow \infty} I_R^n(f')$$

Integrating by parts we have

$$\begin{aligned} I_R^n(f') &= \left[ \frac{1}{x^n} \left( f(x) - f(0) - \sum_{k=0}^{n-1} \frac{f^{(k+1)}(0)}{(k+1)!} x^{k+1} \right) \right]_{-R}^R + \\ &+ \int_{-R}^R \frac{n}{x^{n+1}} \left( f(x) - f(0) - \sum_{k=0}^{n-1} \frac{f^{(k+1)}(0)}{(k+1)!} x^{k+1} \right) dx = \\ &= \frac{1}{R^n} \left( f(R) - \sum_{k=0}^n \frac{f^{(k)}(0)}{k!} R^k \right) - \frac{1}{(-R)^n} \left( f(-R) - \sum_{k=0}^n \frac{f^{(k)}(0)}{k!} (-R)^k \right) + n I_R^{n+1}(f) \end{aligned}$$

Since  $\text{supp } f \subset [-b, b]$  we have two cases,  $R > b$  and  $R < b$ .

Supposing  $R > b$  we have

$$I_R^n(f') = n I_R^{n+1}(f) - \sum_{k=0}^n \frac{f^{(k)}(0)}{k!} \left( \frac{1}{(-R)^{n-k}} - \frac{1}{R^{n-k}} \right)$$

Taken a new index  $j = n - k$  such that  $j \bmod 2 = 0$  we have

$$I_R^n(f') = n I_R^{n+1}(f) - 2 \sum_{j=0}^n \frac{f^{(j)}(0)}{j!} \frac{1}{R^{n-k}} \rightarrow -n \text{PV} \left( \frac{1}{x^{n+1}} \right) (f)$$

In case that  $m > 1$ , by iteration we get

$$D^m \text{PV} \left( \frac{1}{x^n} \right) = (-m)! \text{PV} \left( \frac{1}{x^{n+m}} \right)$$

□

§§ 0.2.4 A Physicist's Trick, The Dirac  $\delta$  "Function"

**Definition 0.2.18** (Composite Delta). Supposing that a Dirac  $\delta$  exists also as a function, we can imagine describing it as a composite function  $(\delta \circ f)(x) = \delta(f(x))$ .

Watch out, since  $\delta(f(x)) \neq \delta_0(f)$

**Definition 0.2.19** (Giving it Some Meaning). Taken  $g_n \in L^1_{loc}(\mathbb{R})$  a sequence of functions such that  $g_n \in C[-a_n, a_n]$  with  $a_n > 0$ ,  $a_n \rightarrow 0$ , we have that  $\exists! g_n \mapsto \varphi_{g_n}$  such that  $\varphi_{g_n} \rightarrow_\star \delta_0$ . A good choice for this sequence would be the Gaussian function  $\gamma_n$  defined as follows

$$\gamma_n(x) = \frac{n}{\sqrt{\pi}} e^{-n^2 x^2} \quad (17)$$

Then, we have that  $\varphi_{\gamma_n} \rightarrow_\star \delta_0$ , or written in a clear abuse of notation with non-existing functions

$$\lim_{n \rightarrow \infty} \gamma_n(x) = \delta(x)$$

Which, actually means that

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}} \gamma_n(x) f(x) dx = f(0) = \delta_0(f) \quad \forall f \in \mathcal{K}$$

Then, letting  $b(x) \in C^1(\mathbb{R})$  we can define the following quantity

$$\int_{\mathbb{R}} f(x) \delta(b(x)) dx = \lim_{n \rightarrow \infty} \int_{\mathbb{R}} \gamma_n(b(x)) f(x) dx \quad \forall f \in \mathcal{K}$$

**Theorem 0.27.** Let  $g \in C^1(\mathbb{R})$  be a function with isolated and simple zeros  $Z_g := \{x_1, x_2, x_3, \dots\}$ , then  $\forall f \in \mathcal{K}$  we have

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}} \gamma_n(g(x)) f(x) dx = \int_{\mathbb{R}} f(x) \delta(g(x)) dx = \sum_{x_k \in Z_g} \frac{\delta_{x_k}(f)}{|g'(x_k)|} \quad (18)$$

*Proof.* Taken  $g(x) \in C^1(\mathbb{R})$  such that  $g(x) \geq g(y) \forall x \geq y$ , supposing that  $Z_g = \{x_0\}$  we have, since it's a simple zero, that  $g'(x_0) \neq 0$

Taking in the previous integral the substitution  $y = g(x)$ , we have that  $x = g^{-1}(y)$  and  $dx = Dg^{-1}(y) dy$ , therefore

$$I = \lim_{n \rightarrow \infty} \int_{\mathbb{R}} \gamma_n(g(x)) f(x) dx = \lim_{n \rightarrow \infty} \int_{\inf(g)}^{\sup(g)} \gamma_n(y) \frac{f(g^{-1}(y))}{g'(g(y))} dy$$

Evaluating the integral we have

$$I = \frac{f(g^{-1}(0))}{g'(g^{-1}(0))} = \frac{f(x_0)}{g'(x_0)} = \frac{1}{|g'(x_0)|} \delta_{x_0}$$

This is easily generalizable if there is more than one zero □

*Example 0.2.9* (Composition with a Constant). Take  $\delta(ax)$  with  $g(x) = ax$  and  $a \neq 0$ . Since  $Z_g = \{0\}$  using the previous formula we have  $g'(0) = a$  and therefore

$$\delta(ax) = \frac{1}{|a|} \delta(x)$$

*Example 0.2.10.* Taken  $g(x) = \text{atan}(x)$  we have that  $Z_g := \{0\}$  and

$$g'(x) = \frac{1}{1+x^2} \quad g'(0) = 1$$

Therefore

$$\delta(\text{atan}(x)) = \delta(x)$$

*Example 0.2.11* (Composition with a Polynomial). Take  $g(x) = (x^2 - a^2)$ , then  $Z_g := \{-a, a\}$  and  $g'(\pm a) = \pm 2a$ , therefore, if  $a \neq 0$

$$\delta(x^2 - a^2) = \frac{1}{2|a|} (\delta(x - a) + \delta(x + a))$$

*Example 0.2.12* (Composition with an Exponential). Taken  $g(x) = e^x$  we have that  $Z_g = \{\}$  therefore  $\delta(e^x) = 0$

*Example 0.2.13* (Composition with a Trigonometric Function). Taken now  $g(x) = \cos(x)$  we have that  $Z_g = \{k \in \mathbb{Z} | k\pi/2\}$  and  $g'(x) = -\sin(x)$ , therefore

$$g'(x_k) = -\sin\left(\frac{k\pi}{2}\right) = (-1)^{k+1}$$

Therefore

$$\delta(\cos(x)) = \sum_{k \in \mathbb{Z}} \delta\left(x - \frac{k\pi}{2}\right) = \sum_{k=-\infty}^{\infty} \delta\left(x - \frac{k\pi}{2}\right)$$

*Example 0.2.14* (Calculating an Integral with the Composite Delta). Take the following integral

$$\int_{\mathbb{R}} e^{-|x|} \delta(\sin(2x)) \, dx$$

In order for solving it we expand the composite delta. We have that  $Z_g = \{k \in \mathbb{Z} | k\pi/2\}$  and  $g'(x_k) = (-1)^k 2$ , therefore

$$\int_{\mathbb{R}} e^{-|x|} \delta(\sin(2x)) \, dx = \frac{1}{2} \sum_{k=-\infty}^{\infty} e^{-|x|} \delta\left(x - \frac{k\pi}{2}\right) = \frac{1}{2} \sum_{k=-\infty}^{\infty} e^{-|\frac{k\pi}{2}|}$$

Fixing the sum and summing, we have

$$\int_{\mathbb{R}} e^{-|x|} \delta(\sin(2x)) \, dx = \sum_{k \in \mathbb{N}} e^{-\frac{k\pi}{2}} - \frac{1}{2} = \frac{1}{1 - e^{\frac{\pi}{2}}} - \frac{1}{2}$$

## § 0.3 Integral Representation of Distributions

### §§ 0.3.1 Dirac Delta and Heaviside Theta

**Theorem 0.28** (Integral Representation of the Delta Distribution). *Taken the non-existent  $\delta(x)$  function, we can represent it as follows*

$$\delta(x - y) = \frac{1}{2\pi} \iint_{\mathbb{R}^2} e^{ik(x-y)} f(x) \, dk \, dx \quad (19)$$



Which means

$$\delta_y(f) = \frac{1}{2\pi} \iint_{\mathbb{R}^2} e^{ik(x-y)} f(x) dk dx$$

**Theorem 0.29** (Integral Representation of the Theta Distribution). *Taken  $\vartheta(x)$  the Heaviside regular distribution we can write  $\forall x \in \mathbb{R} \setminus \{0\}$*

$$\vartheta(x) = \frac{1}{2\pi i} \text{PV} \int_{\mathbb{R}} \frac{e^{ikx}}{k - i\epsilon} dk \quad (20)$$

*Proof.* This equation has two cases, one with  $x > 0$  and one with  $x < 0$ . Therefore taking  $x > 0$  we have

$$I(x) = \text{PV} \int_{\mathbb{R}} \frac{e^{ikx}}{k - i\epsilon} dk$$

This integral can be evaluated using the residue theorem. We have defining the function  $f(k)$  as follows

$$f(k) = \frac{1}{k - i\epsilon}$$

The integral can be seen as follows

$$I(x) = \text{PV} \int_{\mathbb{R}} e^{ikx} f(k) dk$$

This can be written as a complex integral, writing the transformation  $x \rightarrow z$  with  $z \in \mathbb{C}$

$$I(x) \rightarrow J(x) = \lim_{R \rightarrow \infty} \oint_{\gamma_R} \frac{e^{izz}}{z - i\epsilon} dz$$

Where the path is the following

$$\{\gamma_R\} = C_R^+ \cup [-R, R]$$

Writing it all out explicitly we have that

$$J(x) = \lim_{R \rightarrow \infty} \int_{C_R} \frac{e^{izz}}{z - i\epsilon} dz + \lim_{R \rightarrow \infty} \int_{-R}^R \frac{e^{izz}}{z - i\epsilon} dz = J_R + I(\Re z)$$

Due to the Jordan lemma we know for sure that  $J_R = 0$  therefore we have that, writing  $\Re(z) = x$ , thanks to the residue theorem

$$J(x) = I(x) = 2\pi i \sum_{z_k \in S_f} \text{Res}_{z=z_k} \frac{e^{izz}}{z - i\epsilon}$$

The set of singularities of  $f$  is formed by a single point  $S_f = \{i\epsilon\}$  which is a simple pole, and therefore, since  $i\epsilon \in \{\gamma_R\}^\circ$  we have

$$J(x) = 2\pi i \text{Res}_{z=i\epsilon} \frac{e^{izz}}{z - i\epsilon} = 2\pi i \lim_{z \rightarrow i\epsilon} (z - i\epsilon) \frac{e^{izz}}{z - i\epsilon} = e^{-\epsilon x}$$

Therefore, we get

$$J(x) = 2\pi i e^{-\epsilon x}$$

For getting back to the first integral, we have

$$\frac{1}{2\pi i} I(x) = \lim_{\epsilon \rightarrow 0^+} e^{-\epsilon x} = 1 = \vartheta(x) \quad x > 0$$

In the second case, for  $x < 0$  we have that the new curve will be

$$\{\gamma_R^-\} = C_R^- \cup [-R, R]$$

Since  $S_f \not\subset \{\gamma_R^-\}^\circ$  we have thanks to Cauchy-Goursat that

$$\frac{1}{2\pi i} I(x) = \frac{1}{2\pi i} J^-(z) = 0 = \vartheta(x) \quad x < 0$$

Proving our assumption □

### §§ 0.3.2 Distributions in $\mathbb{R}^n$

**Definition 0.3.1** (Scalar Test Fields). Given a scalar field  $f : \mathbb{R}^n \rightarrow \mathbb{C}$  one can define the space of scalar test fields  $\mathcal{K}(\mathbb{R}^n)$  as follows

$$\mathcal{K}(\mathbb{R}^n) = C_c^\infty(\mathbb{R}^n) \tag{21}$$

A function  $f \in C_c^\infty(\mathbb{R}^n)$  is a function such that  $\text{supp } f \subset A \subset \mathbb{R}^n$  with  $A$  a compact set and  $\exists \partial^\alpha f \quad \forall |\alpha| \in \mathbb{N}$

**Definition 0.3.2** ( $\mathcal{K}(\mathbb{R}^n)$ –Convergence). Given  $f_n \in \mathcal{K}(\mathbb{R}^n)$  a sequence of scalar fields.  $f_n$  converges to  $f \in \mathcal{K}(\mathbb{R}^n)$  if

1.  $\exists A \subset \mathbb{R}^n$  compact set such that  $f_n(A^c) = \{0\}$
2.  $\forall \alpha \in \mathbb{N}^n$  multi-index  $\partial^\alpha f_n \Rightarrow \partial^\alpha f, \forall x \in A$

It's indicated as

$$f_n \rightarrow_{\mathcal{K}_n} f$$

**Definition 0.3.3** (Multidimensional Distribution). A  $n$ –dimensional distribution is a continuous functional  $\varphi : \mathcal{K}(\mathbb{R}^n) \rightarrow \mathbb{C}$  such that

$$\forall g_n \rightarrow_{\mathcal{K}_n} g \quad \varphi(g_n) \rightarrow \varphi(g) \tag{22}$$

Then  $\varphi \in \mathcal{K}^*(\mathbb{R}^n)$

**Theorem 0.30** (Operations in  $\mathcal{K}^*(\mathbb{R}^n)$ ). We can define the usual distributional operations in  $\mathcal{K}(\mathbb{R}^n)$ . The usual operations defined for  $\mathcal{K}^*$  are defined in the same way, the only difference is given by the definition of the derivative Given  $\alpha \in \mathbb{N}^n, f \in \mathcal{K}(\mathbb{R}^n), \varphi \in \mathcal{K}^*(\mathbb{R}^n)$

$$\partial^\alpha \varphi(f) = (-1)^{|\alpha|} \varphi(\partial^\alpha f) \tag{23}$$

*Example 0.3.1* (Laplacian of a Distribution). Taken  $\partial^\alpha = \partial^2 = \partial_\mu \partial^\mu$

$$\partial_\mu \partial^\mu \varphi(f) = \varphi(\partial_\mu \partial^\mu f)$$

*Remark* (Local Integrability). The local integrability of some functions is different in  $\mathbb{R}^n$ . In fact taken the spherical  $(n-1)$ -dimensional coordinate transformation we have

$$d^n x = r^{n-1} dr dS_{n-1}$$

And taken the function  $g(x^\nu) = \|x^\mu\|_\mu^{-a}$  we get

$$\int_{\mathbb{R}^n} \frac{f(x)}{\|x^\mu\|_\mu^a} d^n x = \int_{\mathbb{R}} \int_{S_{n-1}} f(x) dS_{n-1} \frac{r^{n-1}}{r^a} dr$$

Which means that  $\|x^\mu\|_\mu^{-a} \in L_{loc}^1(\mathbb{R}^n) \quad \forall a < n$

## § 0.4 Some Applications in Physics

### §§ 0.4.1 Electrostatics

*Example* 0.4.1 (Point Charge). Taken  $q$  a point charge in the origin  $(0,0,0)$  we define the potential generated by this charge as follows (in cgs)

$$\partial_\mu \partial^\mu V(x^\nu) = -4\pi \rho(x)$$

Where  $\rho(x)$  is the charge density, defined as

$$\rho(x) = q\delta^3(x^\nu)$$

With  $\delta^3$  being the 3-d Dirac delta  
Therefore

$$\partial_\mu \partial^\mu \left( \frac{1}{\|x^\mu\|_\mu} \right) = -4\pi \delta^3(x^\nu)$$

Which means actually, that  $\forall f \in \mathcal{K}(\mathbb{R}^3)$

$$\iiint_{\mathbb{R}^3} \frac{\partial_\mu \partial^\mu f}{\|x^\mu\|_\mu} d^3 x = -4\pi f(0)$$

*Proof.* Since  $f \in \mathcal{K}(\mathbb{R}^3)$  we have that  $\exists R > 0 : f(x^\mu) = 0 \forall x \notin B_{\frac{R}{2}}(0)$   
Therefore we get that

$$\iiint_{\mathbb{R}^3} \frac{\partial_\mu \partial^\mu f}{\|x^\mu\|_\mu} d^3 x = \lim_{\epsilon \rightarrow 0} \int_{\epsilon \leq \|x^\mu\|_\mu \leq R} \frac{\partial_\mu \partial^\mu f}{\|x^\mu\|_\mu} d^3 x$$

Taken  $A_{\epsilon R}(0) = \{x^\mu \in \mathbb{R}^3 \mid \epsilon \leq \|x^\mu\|_\mu \leq R\}$  We have that the second integral becomes, using Stokes' theorem

$$\iiint_{A_{\epsilon R}} f(x) \partial_\mu \partial^\mu \left( \frac{1}{\|x^\mu\|_\mu} \right) f(x) d^3 x + \iint_{\partial A_{\epsilon R}} \left[ \frac{1}{\|x^\mu\|_\mu} \partial_\mu f - f \partial_\mu \left( \frac{1}{\|x^\mu\|_\mu} \right) \right] n^\mu d\sigma$$

Changing to spherical coordinates we have that

$$\partial_\mu \partial^\mu \left( \frac{1}{\|x^\mu\|_\mu} \right) = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \left( \frac{1}{r} \right) \right) = 0$$

Therefore we get

$$\iiint_{A_{\epsilon R}} \frac{1}{\|x^\mu\|_\mu} \partial_\mu \partial^\mu f \, d^3 x = - \iint_{\|x^\mu\|_\mu = \epsilon} \left( \frac{1}{r} \partial_\mu f - f \partial_\mu \left( \frac{1}{r} \right) \right) r^\mu \, dS_2$$

Which, taken  $\partial_\mu r^{-1} = -\hat{r}^\mu r^{-2}$  we have

$$\iiint_{A_{\epsilon R}} \frac{1}{\|x^\mu\|_\mu} \partial_\mu \partial^\mu f \, d^3 x = - \iint_{\|x^\mu\|_\mu = \epsilon} \hat{r}^\mu \partial_\mu f r \, dS_2 - \iint_{\|x^\mu\|_\mu = \epsilon} f(x) \, dS_2$$

We have for the first integral that

$$\left\| \iint_{\|x^\mu\|_\mu = \epsilon} \hat{r}^\mu \partial_\mu f r \, dS_2 \right\| \leq 4\pi\epsilon \sup_{\mathbb{R}^3} \|\partial_\mu f\|$$

And for the second

$$\left\| \iint_{\|x^\mu\|_\mu = \epsilon} f(x) \, dS_2 - 4\pi f(0) \right\| \leq 4\pi \sup_{\|x^\mu\|_\mu = \epsilon} |f(x^\nu) - f(0)|$$

Since  $\epsilon \rightarrow 0^+$  and since  $f \in \mathcal{K}(\mathbb{R}^3)$  we have that the first and second integral converge to 0, and therefore

$$\iiint_{\mathbb{R}^3} \frac{1}{\|x^\mu\|_\mu} \partial_\mu \partial^\mu f \, d^3 x = - \lim_{\epsilon \rightarrow 0} \iint_{\|x^\mu\|_\mu = \epsilon} f(x) \, dS_2 = -4\pi f(0)$$

□