§ 0.1 Measure Theory

Definition 0.1.1 (Lower and Upper Sums)**.** We define the *upper* and *lower Riemann sums* as follows.
Let $f(x)$ be a function, then

$$\begin{cases} \mathcal{U}(f,x) := \sum_{i=1}^{n} \sup_{t\in[x_k,x_{k+1}]} (f(t)) \\ \mathcal{L}(f,x) := \sum_{i=1}^{n} \inf_{t\in[x_k,x_{k+1}]} (f(t)) \end{cases} \tag{1}$$

A function is said to be Riemann integrable if $\lim_{n\to\infty}(\mathcal{L}(f,x) - \mathcal{U}(f,x)) = 0$

Definition 0.1.2 (Set Function)**.** Let A be a set. We define the following function $\mathbb{1}_A(x)$ as follows

$$\mathbb{1}_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases} \tag{2}$$

Theorem 0.1. *The function $\mathbb{1}_{\mathbb{Q}}$ is not integrable over the set $[0,1]$ with the usual definition of the integral (Riemann sums)*

Proof. Indicating the integral I as usual

$$I = \int_0^1 \mathbb{1}_{\mathbb{Q}}(x)\,\mathrm{d}x$$

We see immediately that

$$\mathcal{U}(\mathbb{1}_{\mathbb{Q}}, x) = 1$$
$$\mathcal{L}(\mathbb{1}_{\mathbb{Q}}, x) = 0$$

Therefore $\mathbb{1}_{\mathbb{Q}}(x)$ is not integrable in $[0,1]$ (with the Riemann integral) □

Definition 0.1.3 (Measure)**.** Let $A \subset X$ be a subset of a metric space. We define the measure of the set A, $\mu(A)$ as follows

$$\mu(A) = \int_X \mathbb{1}_A(x)\,\mathrm{d}x \tag{3}$$

Basically, what we did before, was demonstrating that the set $\mathbb{Q} \cap [0,1]$ is not measurable in the Riemann integration theory. This is commonly indicated with saying that the set $\mathbb{Q} \cap [0,1]$ *is not Jordan measurable.*
For clarity, let K be some measure theory. We will say that a set is K*-measurable* if the following calculation exists

$$\mu_K(A) = \int_X \mathbb{1}_A(x)\,\mathrm{d}x \tag{4}$$

Definition 0.1.4 (Algebra)**.** Let $X \neq \{\}$ be a set. An *algebra* $\mathcal{A}$ over X is a collection of subsets of X such that

1. $\{\} \in \mathcal{A}$

2. $X \in \mathcal{A}$
3. $A \in \mathcal{A} \implies A^{\mathrm{c}} \in \mathcal{A}$
4. $A_1, \cdots, A_n \in \mathcal{A} \implies \bigcup_{i=1}^n A_i, \bigcap_{i_1}^n A_i \in \mathcal{A}$

Example 0.1.1 (Simple Set Algebra). Let $X = \mathbb{R}^2$ and call R the set of all rectangles $I_i \subset \mathbb{R}^\star \times \mathbb{R}^\star$, where $\mathbb{R}^\star = \mathbb{R} \cup \{\pm\infty\}$. It's easy to see that this is not an algebra, since, by taking $[0,1] \in R$, we have that $[0,1]^{\mathrm{c}} \notin R$, hence it cannot be an algebra.
But, taken $\mathcal{S}$ as follows

$$\mathcal{S} := \left\{ A \subset \mathbb{R}^2 \middle| \, A = \bigcup_{i=1}^n I_i \quad I_i \in R \right\}$$

We can see easily, using De Morgan law, that $\mathcal{S}$ is an algebra.

§§ 0.1.1 Jordan Measure

Definition 0.1.5 (Disjoint Union)**.** Taken two sets A, B, we define their *disjoint union* the binary operation $A \sqcup B$ as follows

$$A \sqcup B := A \cup B \setminus A \cap B \tag{5}$$

Definition 0.1.6 (Simple Set)**.** A set A is a *simple set* iff, for some $R_i \in \mathcal{S}$, we have

$$A = \bigsqcup_{i=1}^n R_i$$

Definition 0.1.7 (Measure of a Simple Set)**.** Let A be a simple set, the *Jordan measure* of a simple set is given by the sum of the measure of the rectangles, i.e. the "area" of A is given by the sum of the area of each rectangle R_i

$$\mu_J(A) = \sum_{i=1}^n \mu_J(R_i) \tag{6}$$

Definition 0.1.8 (External and Internal Measure)**.** We define the external measure $\overline{\mu}_J$ and the internal measure $\underline{\mu}_J$ as follows.
Taken a limited set B and a simple set A we have

$$\begin{aligned} \overline{\mu}_J(B) &= \inf\{\, \mu_J(A) |\, B \subset A\} \\ \underline{\mu}_J(B) &= \sup\{\, \mu_J(A) |\, A \subset B\} \end{aligned} \tag{7}$$

A set is said to be *Jordan measurable* iff

$$\overline{\mu}_J(B) = \underline{\mu}_J(B) = \mu_J(B)$$

Remark (A Non Measurable Set). A good example for showing that the Jordan measure is the set we were trying to measure, the set $\mathbb{Q} \cap [0,1]$. We can easily see that

$$\begin{aligned} \overline{\mu}_J(\mathbb{Q} \cap [0,1]) &= 1 \\ \underline{\mu}_J(\mathbb{Q} \cap [0,1]) &= 0 \end{aligned}$$

Therefore it's not Jordan measurable.
From this we can jump to a new definition of measure, which is the *Lebesgue measure* where instead of covering $\mathbb{Q} \cap [0,1]$ with a *finite* number of simple sets, we use sets which are formed from the union of *countable infinite* simple sets.
We can define

$$\mathbb{Q} \cap [0,1] := \{q_1, q_2, \cdots\}$$

We then take $\epsilon > 0$ and define the following set

$$A = \bigcup_{n=1}^{\infty} \left[q_n - \frac{\epsilon}{2^n}, q_n + \frac{\epsilon}{2^n} \right]$$

We have that

$$\mu(A) \leq \sum_{n=1}^{\infty} \frac{\epsilon}{2^{n-1}} = 2\epsilon$$

But $\overline{\mu}(\mathbb{Q} \cap [0,1]) \leq \mu(A) \leq 2\epsilon \to 0$, therefore $\mathbb{Q} \cap [0,1]$ is measurable with $\mu(\mathbb{Q} \cap [0,1]) = 0$

§§ 0.1.2 Lebesgue Measure

Definition 0.1.9 ($\sigma-$Algebras and Borel Spaces)**.** Given a non empty set X a $\sigma-$*algebra* on X is a collection of subsets $\mathcal{F}$ such that

1. $\forall A \in \mathcal{F},\ A \subset X$
2. Let $A_i \in \mathcal{F},\ i \in \mathcal{I} : |\mathcal{I}| = \aleph_0$ then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$

The couple $(X, \mathcal{F})$ is called a *Borel space* or also a *measurable space*

Definition 0.1.10 (Measure)**.** Given a Borel space $(X, \mathcal{F})$, we can define an application

$$\mu : \mathcal{F} \longrightarrow [0, \infty] = \mathbb{R}_+^\star \tag{8}$$

Which satisfies the following properties

1. σ-additivity, given $A_i \in \mathcal{F}$ with $i \in I \subset \mathbb{N},\ |I| \leq \aleph_0$, such that $A_n \cap A_k = \{\}$ for $n \neq k$

$$\mu\left(\bigsqcup_{i \in I} A_i\right) = \sum_{i \in I} \mu(A_i)$$

2. If $Y_j \subset X$, with $j \in J \subseteq \mathbb{N},\ \mu(Y_j) < \infty$ then $X = \bigcup_{j=1}^{\infty} Y_j$

Definition 0.1.11 (Measure Space)**.** A *measure space* is a triplet $(X, \mathcal{F}, \mu)$ with $\mathcal{F}$ a $\sigma-$algebra and μ a measure.

Remark. The empty set has null measure.

Proof. Due to $\sigma-$additivity we have that

$$\mu(\{\}) = \mu(\{\} \cup \{\}) = \mu(\{\}) + \mu(\{\})$$

Therefore, $\mu(\{\}) = 0$ necessarily. $\square$

Definition 0.1.12 (Lebesgue Measure)**.** Consider again $X = \mathbb{R}^2$ and $\mathcal{S}$ the algebra of simple sets. The *external Lebesgue measure* of a set $B \subset \mathbb{R}^2$ is then defined as follows

$$\overline{\mu}_L(B) := \inf \left\{ \sum_{i=1}^{\infty} \text{Area}(R_i) \middle| R_i \in \mathcal{S},\ B \subset \bigcup_{i=1}^{\infty} R_i \right\} \tag{9}$$

The set B is said to be *Lebesgue measurable* iff, $\forall C \subset \mathbb{R}^2$

$$\overline{\mu}_L(C) = \overline{\mu}_L(C \cap B) + \overline{\mu}_L(C \setminus B) \tag{10}$$

If it's measurable, then, $\overline{\mu}_L(B) = \mu_L(B)$ and it's called the *Lebesgue measure* of the set.
In other words $\exists \epsilon > 0 \, : \, \exists A, C \subset \mathbb{R}^2$, with $A = A^\circ,\ C = \overline{C}$ such that

$$C \subset B \subset A \ \vee \ \overline{\mu}_L(A \setminus C) < \epsilon \tag{11}$$

Definition 0.1.13 (Borel Algebra)**.** Let R be the set of all rectangles. The smallest $\sigma-$algebra containing R is called the *Borel algebra* and it's indicated as $\mathcal{B}$

Definition 0.1.14 (Lebesgue Algebra)**.** The set of (Lebesgue) measurable sets is a $\sigma-$algebra, which we will indicate as $\mathcal{L}$. In particular, we have that, if I is a rectangle, $I \in \mathcal{L}$.
If we add the fact that in $\mathcal{B}$ there are null measure sets which have subsets which aren't part of $\mathcal{B}$, we end up with the conclusion that $\mathcal{B} \subset \mathcal{L}$

Definition 0.1.15 (Null Measure Sets)**.** A set with null measure is a set $X \subset \mathcal{F}$ such that

$$\mu(X) = 0 \tag{12}$$

Where μ is a measure function.
It's obvious that sets formed by a single point have null measure.
I.e take a set $A = \{a\}$, then it can be seen as a rectangle with 0 area, and therefore

$$\mu\left(\{a\}\right) = 0 \tag{13}$$

Theorem 0.2. *Every set such that* $|A| = \aleph_0$ *has null measure*

Corollary 0.1.1. Every line in $\mathbb{R}^2$ has null measure

Proof. Take the set $A = \{a_1, a_2, a_3, \cdots\}$. Then, due to $\sigma-$additivity, we have

$$\mu\left(\{a_1, a_2, a_3, \cdots\}\right) = \mu\left(\bigsqcup_{k=1}^{\infty} \{a_k\}\right) = \sum_{k=1}^{\infty} \mu\left(\{a_k\}\right) = 0 \tag{14}$$

For the corollary, it's obvious if the line is thought as a rectangle in $\mathbb{R}^2$ with null area ∎

§ 0.2 Integration

Definition 0.2.1 (Measurable Function)**.** Given a Borel space $(X, \mathcal{F})$ a *measurable function* is a function $f : X \longrightarrow \mathbb{F}$ such that, $\forall k \in \mathbb{F}$ the following set is measurable

$$I_f := \{\, k \in \mathbb{F} |\, f(x) < k\} \tag{15}$$

Or, in other words $I_f \in \mathcal{F}$, with $\mathcal{F}$ the given $\sigma-$algebra of the Borel space.
The space of all measurable functions on X will be identified as $\mathcal{M}(X)$

Theorem 0.3. *Given a set $A \in \mathcal{F}$ with $\mathcal{F}$ a $\sigma-$algebra, the function $\mathbb{1}_A(x)$ is measurable*

Proof. We have that

$$I_{\mathbb{1}_A} = \begin{cases} A & k > 1 \\ \{\} & t \leq 1 \end{cases}$$

Therefore $I_{\mathbb{1}_A} \in \mathcal{F}$ and $\mathbb{1}_A(x)$ is measurable □

Definition 0.2.2 (Simple Measurable Function)**.** Given a Borel space $(X, \mathcal{F})$, a *simple measurable function* is a function $f : X \longrightarrow \mathbb{F}$ which can be written as follows

$$f(x) = \sum_{k=1}^{n} c_k \mathbb{1}_{A_k}(x) \tag{16}$$

Where $A_k \in \mathcal{F},\ c_k \in \mathbb{F} \quad 0 \leq k \leq n$

Definition 0.2.3 (Integral)**.** Given a measure space $(X, \mathcal{F}, \mu)$ and a simple function $f(x)$, we can define the *integral* of the function f with respect to the measure μ over the set X as follows

$$\int_X f(x)\mu\,(\mathrm{d}x) = \sum_{k=1}^{n} c_k \mu(A_k) \tag{17}$$

For non negative functions we define the integral as follows

$$\int_X f(x)\mu\,(\mathrm{d}x) = \sup \left\{ \int_X g(x)\mu\,(\mathrm{d}x) \right\} \tag{18}$$

Where $g(x)$ is a simple measurable function such that $0 \leq g \leq f$.
If f assumes both negative and positive values we can write

$$f = f^+ - f^- \tag{19}$$

Where

$$\begin{cases} f^+ = \max f, 0 \\ f^- = \max -f, 0 \end{cases} \tag{20}$$

The integral, due to linearity, then will be

$$\int_X f(x)\mu\,(\mathrm{d}x) = \int_X f^+(x)\mu\,(\mathrm{d}x) - \int_X f^-(x)\mu\,(\mathrm{d}x) \tag{21}$$

With the only constraint that the function $f(x)$ must be misurable in the σ-algebra $\mathcal{F}$

§§ 0.2.1 Lebesgue Spaces

Definition 0.2.4 ($\mathcal{L}^p$ spaces)**.** With the previous definitions, we can define an *infinite dimensional function space* with the following properties
Given a measure space $(X, \mathcal{F}, \mu)$ we have the following definition

$$\mathcal{L}^p\,(X, \mathcal{F}, \mu) = \mathcal{L}^p(\mu) := \left\{ f : X \longrightarrow \mathbb{F} |\ I_f \in \mathcal{F} \wedge \int_X |f|^p \mu\,(\mathrm{d}x) < \infty \right\} \tag{22}$$

Defining the integral as an *operator* $\hat{K}_\mu[f]$ we can see easily that this is a vector spaces due to the properties of $\hat{K}_\mu$.
It's easy to note that if the chosen $\sigma-$algebra and measure are the Lebesgue ones, then this integral is simply an extension of the usual Riemann integral.
It's important to note that a norm in $\mathcal{L}^p(\mu)$ can't be defined as an usual integral $p-$norm, since there are nonzero functions which have actually measure zero.

Definition 0.2.5 (Almost Everywhere Equality)**.** Taken two functions $f, g \in \mathcal{L}^p(\mu)$ we say that these two function are *almost everywhere equal* if, given a set $A := \{x \in X |\, f(x) \neq g(x)\}$ has null measure. Therefore

$$f \sim g \iff \mu(A) = 0 \tag{23}$$

This equivalence relation creates equivalence classes of functions compatible with the vector space properties of $\mathcal{L}^p(\mu)$.

Definition 0.2.6 (L^p-Spaces)**.** With the definition of the almost everywhere equality we can then define a quotient space as follows

$$L^p(\mu) = \mathcal{L}^p(\mu) \backslash \sim \tag{24}$$

This is a vector space, obviously, where the elements are the equivalence classes of functions $f \in \mathcal{L}^p(\mu)$, indicated as $[f]$.
If we define our $\sigma-$algebra and measure as the Lebesgue ones, this space is called the *Lebesgue space* $L^p(X)$, where an integral $p-$norm can be defined.

§§ 0.2.2 Lebesgue Integration

Note:
In this section the differential $\mathrm{d}x$ will actually indicate the Lebesgue measure $\mu\left(\mathrm{d}x\right)$ used previously, unless stated otherwise.

Theorem 0.4. *Let $f : E \longrightarrow \mathbb{F}$ be a measurable function over E.*
Given

$$F_{+\infty} = x \in E |\, f(x) = +\infty \ \wedge \ F_{-\infty} = x \in E |\, f(x) = -\infty$$

Assuming $E \subset X$, with $(X, \mathcal{L}, \mu)$ a Lebesgue measure space, we have that

$$\mu\left(F_{+\infty}\right) = \mu\left(F_{-\infty}\right) = 0$$

Proof. We can immediately say that

$$F_{+\infty} = \bigcap_{k \geq 0} F_k \in \mathcal{L}$$

Letting $r > 0$ we will indicate with $\mathbb{1}_r(x)$ the set function of the set $F_{+\infty} \cap B_r(0)$, therefore we have that

$$f^+(x) \geq k\mathbb{1}_r(x) \quad \forall k \in \mathbb{N}$$

Therefore

$$\mu\left(F_{+\infty} \cap B_r(0)\right) = \int \mathbb{1}_r(x)\,\mathrm{d}x \leq \frac{1}{k}\int_E f^+(x)\,\mathrm{d}x \longrightarrow 0$$

The proof is analogous for $F_{-\infty}$ $\square$

Theorem 0.5. *Let $(X, \mathcal{L}, \mu)$ be a measure space, where $\mathcal{L}$ is the Lebesgue $\sigma-$algebra and μ is the Lebesgue measure. Given a function $f \in L^1(X)$ we have that*

$$\int_X f(x)\,\mathrm{d}x = 0 \iff f \sim 0 \tag{25}$$

Proof. Let $F_0 = x \in X|\, f(x) > 0 = \bigcap_{k \geq 0} F_{1/k}$.
Since $f(x) > 1/k,\ \forall x \in F_{1/k}$, we have that, $\forall k \in \mathbb{N}$

$$\mu(F_{1/k}) \leq \int_X f(x)\,\mathrm{d}x = 0$$

Through induction, we obtain that $\mu(F_0) = 0$ □

Theorem 0.6 (Monotone Convergence (B. Levi))**.** *Let $(f)_k$ be a sequence of measurable functions over a Borel space E, such that*

$$0 \leq f_1(x) \leq \cdots \leq f_k(x) \leq \cdots \quad \forall x \in F \subset E,\ \mu(F) = 0$$

If $f_k(x) \to f(x)$, we have that

$$\int_E f(x)\,\mathrm{d}x = \lim_{k \to \infty} \int_E f_k(x)\,\mathrm{d}x \tag{26}$$

Or, in another notation

$$\int_E f_k(x)\,\mathrm{d}x \to \int_E f(x)\,\mathrm{d}x \tag{27}$$

Proof. Let $F_{0k} = 0 < y < f_k(x)$ and $F_0 = 0 < y < f(x)$ be two sets defined as seen. They are all measurable since $f_k(x), f(x)$ are measurable, and due to the monotony of $f_k(x)$ we have that

$$F_{01} \subset F_{02} \subset \cdots \subset F_{0k} \subset \cdots \ \wedge\ F_0 = \bigsqcup_{k=1}^{\infty} F_{0k}$$

Due to $\sigma-$additivity of the measure function, we have that F_0 is measurable, and that

$$\mu\left(F_0\right) = \sum_{k=1}^{\infty} \mu\left(F_{0k}\right) \quad \therefore \mu\left(F_0\right) = \lim_{k \longrightarrow \infty} \mu\left(F_{0k}\right)$$

□

Notation (For Almost All)**.** We now introduce a new (unconventional) symbol in order to avoid writing too much, which would complicate the already difficult to understand theorems.
In order to indicate that we're picking *almost all* elements of a set we will use a new quantifier, which means that we're picking all elements of a null measure subset of the set in question. The quantifier in question will be the following

$$\forall^\dagger \tag{28}$$

Corollary 0.2.1. Let $f_k(x)$ be a sequence of non-negative measurable functions over a measurable set E, then $\forall^\dagger\, x \in E$

$$\int_E \sum_{k \geq 0} f_k(x)\,\mathrm{d}x = \sum_{k \geq 0} \int_E f_k(x)\,\mathrm{d}x \tag{29}$$

Theorem 0.7 (Fatou). *Let $f_k(x)$ be a sequence of measurable functions over a measurable set E, such that $\forall^\dagger\, x \in E\ \exists\Phi(x)$ measurable: $f_k(x) > \Phi(x)$, then*

$$\int_E \liminf_{k\to\infty} f_k(x)\,\mathrm{d}x \leq \liminf_{k\to\infty} \int_E f_k(x)\,\mathrm{d}x$$

Analogously happens with the $\limsup$ *of the sequence*

Proof. Let $h_k(x) = f_k(x) - \Phi(x) \geq 0\ \ \forall^\dagger\, x \in E$ and $g_j(x) = \inf_{k\geq k} h_k(x)$, then $\forall k \geq j$ we have

$$\int_E g_j(x)\,\mathrm{d}x \leq \int_E h_k(x)\,\mathrm{d}x$$

It's also (obviously) true taking the $\limsup$ of the RHS, and for the theorem on the monotone convergence, we have that

$$\int_E \lim_{j\to\infty} g_j(x)\,\mathrm{d}x = \lim_{j\to\infty} \int_E g_j(x)\,\mathrm{d}x \leq \int_E h_k(x)\,\mathrm{d}x$$
$$\therefore \lim_{j\to\infty} g_j(x) = \sup_j g_j(x) = \sup_j \inf_{k\geq j} h_k(x) = \liminf_{k\to\infty} h_k(x)$$

□

Theorem 0.8 (Dominated Convergence (Lebesgue)). *Let $h(x) \geq 0$ be a measurable function on the measurable set E such that for a sequence of measurable functions $f_k(x)$ we have that*

$$|f_k(x)| \leq h(x) \quad \forall^\dagger\, x \in E$$

And

$$f(x) = \lim_{k\to\infty} f_k(x) \quad \forall^\dagger\, x \in E$$

Then

$$\int_E f(x)\,\mathrm{d}x = \lim_{k\to\infty} \int_E f_k(x)\,\mathrm{d}x$$

Proof. By definition we have that $-h(x) \leq f_k(x) \leq h(x)\ \ \forall^\dagger\, x \in E$, and we can apply Fatou's theorem

$$\int_E f(x)\,\mathrm{d}x \leq \liminf_{k\to\infty} \int_E f_k(x)\,\mathrm{d}x \leq \limsup_{k\to\infty} \int_E f_k(x)\,\mathrm{d}x \leq \int_E f(x)\,\mathrm{d}x$$

□

Corollary 0.2.2. Let E be a measurable set such that $\mu(E) < \infty$ and let $f_k(x)$ be a sequence of functions in E such that $|f_k(x)| \leq M\ \ \forall^\dagger\, x \in E$ and $f_k(x) \to f(x),\ \forall^\dagger\, x \in E$. Then the theorem (0.8) is valid.

Example 0.2.1. Take the sequence of functions $f_k(x) = kxe^{-kx}$ over $E = [0,1]$. We already know that $f_k(x) \longrightarrow f(x) = 0$ for $x \in E$, but $f_k(x) \not\rightrightarrows f(x)$ in E.
We have that

$$\sup_E f_k(x) = e^{-1} = h(x) \neq f(x)$$

We have that $h(x)$ is measurable in E and we can apply the theorem (0.8)

Definition 0.2.7 (Carathéodory Function)**.** Let $(X, \mathcal{L}, \mu)$ be a measure space and $A \subset \mathbb{R}^n$. $f : X \times A \longrightarrow \mathbb{R}$ is a *Carathéodory function* iff $f(x^\mu, a^\nu) \in C(A)\ \forall a^\nu \in A$ and $f(x^\mu, a^\nu) \in \mathcal{M}(X)\ \forall^\dagger\, x^\mu \in X$

Definition 0.2.8 (Locally Uniformly Integrably Bounded)**.** Let $f : X \times A \longrightarrow \mathbb{R}$ be a Carathéodory function. It's said to be *locally uniformly integrably bounded* if $\forall a^\nu \in A\ \exists h_{a^\nu} : X \longrightarrow \mathbb{R}$ measurable, and $\exists B_\epsilon(a^\nu) \subset A$, such that

$$\forall y^\nu \in B_\epsilon(x^\mu)\ |f(x^\mu, y^\nu)| \le h_{a^\nu}(x^\mu)$$

Note that if μ is a finite measure, then f bounded $\implies$ f locally uniformly integrably bounded or LUIB.

Theorem 0.9 (Leibniz's Derivation Rule)**.** *Let $(X, \mathcal{F}, \mu)$ be a measure space and $A \subset \mathbb{R}^n$ an open set. If $f : X \times A \longrightarrow \mathbb{R}$ is a LUIB Carathéodory function we can define*

$$g(a^\mu) = \int_X f(x^\nu, a^\mu)\, \mathrm{d}\mu\,(x^\sigma) \in C(A)$$

Then

$$\partial_{x^\mu} f(x^\nu, a^\sigma) \in C(A)$$

Is LUIB, and therefore

$$g(a^\mu) \in C^1(A)$$

And

$$\partial_\mu g = \int_X \partial_{a^\mu} f(a^\nu, x^\sigma)\, \mathrm{d}\mu\,(x^\gamma)$$

In other terms

$$\partial_{a^\mu} \int_X f(a^\nu, x^\sigma)\, \mathrm{d}\mu(x^\gamma) = \int_X \partial_{a^\mu} f(a^\nu, x^\sigma)\, \mathrm{d}\mu(x^\gamma) \tag{30}$$

Proof. Since f is a LUIB Carathéodory function we have that $\exists h_{a^\mu}(x^\nu) : X \longrightarrow \mathbb{R}$ and $B_\epsilon(a^\mu) \subset A\ :\ \forall y^\mu \in B_\epsilon(a^\nu)$

$$|f(y^\mu, x^\nu)| \le h_{a^\mu}(x^\nu)$$

Therefore

$$|g(a^\mu)| \le \int_X h_{a^\mu}(x^\nu)\, \mathrm{d}\mu(x^\sigma) < \infty$$

Now take a sequence $(a^\mu)_n\ :\ (a^\mu)_n \to a^\mu$, then $f \in C(A) \implies f(a^\mu_n, x^\nu) \to f(a^\mu, x^\nu)\ \forall^\dagger\, x^\mu \in X, \forall a^\mu_n \in B_\epsilon(a^\mu)$

$$\therefore \exists N \in \mathbb{N}\ :\ \forall n \ge N\ |f(a^\mu_n, x^\nu)| \le h_{a^\mu}(x^\nu)$$

Then

$$g(a^\mu_n) = \int_X f(a^\mu_n, x^\nu)\, \mathrm{d}\mu(x^\sigma) \to \int_X f(a^\mu, x^\nu)\, \mathrm{d}\mu(x^\sigma) = g(a^\mu)$$

Since f is differentiable and its derivative is measurable, we have for the mean value theorem

$$f(a^\mu + te^\mu, x^\nu) - f(a^\mu, x^\nu) = t\partial_\mu f(\xi^\nu(t, x^\sigma), x^\gamma)$$

If $\xi^\mu(t, x^\nu) \in B_\epsilon(a^\mu)$ we have that

$$|t\partial_\mu f(\xi^\nu(t, x^\sigma), x^\gamma)| \le h_{a^\mu}(x^\nu)$$

And therefore
$$\frac{g(a^\mu + te^\mu) - g(a^\mu)}{t} = \frac{1}{t}\int_X t\partial_\mu f(\xi^\nu(t, x^\sigma), x^\gamma)\,\mathrm{d}\mu(x^\delta)$$
For $t \to 0$ $\partial_\mu f(\xi^\nu, x^\sigma) \to \partial_\mu f(a^\nu, x^\sigma)$, and the LHS is simply the gradient of g. Therefore for theorem (0.8)
$$\partial_\mu g(a^\nu) = \frac{\partial}{\partial a^\mu}\int_X f(a^\nu, x^\sigma)\,\mathrm{d}\mu(x^\gamma) = \int_X \partial_\mu f(a^\nu, x^\sigma)\,\mathrm{d}\mu(x^\gamma)$$
□

§ 0.3 Calculus of Integrals in $\mathbb{R}^2$ and $\mathbb{R}^3$

§§ 0.3.1 Double Integration

Theorem 0.10. *Let $E \subset \mathbb{R}^2$ and $F \subset \mathbb{R}^3$. Define $E_x := \{y \in \mathbb{R}|\, (x, y) \in E\}$ the sections of E parallel to the y axis, then*
$$\mu(E) = \int_{\mathbb{R}} \mu_1(E_x)\,\mathrm{d}y \tag{31}$$
Where with μ_i we indicate the $i-$dimensional measure on $\mathbb{R}^n$.
Analogously, we define $F_z := \left\{ (x, y) \in \mathbb{R}^2 \middle|\, (x, y, z) \in F \right\}$ then
$$\mu(F) = \int_{\mathbb{R}} \mu_2(F_z)\,\mathrm{d}z \tag{32}$$
If we define $F_{xy} := \{z \in \mathbb{R}|\, (x, y, z) \in F\}$ we have
$$\mu(F) = \iint_{\mathbb{R}^2} \mu_1(F_{xy})\,\mathrm{d}x\,\mathrm{d}y \tag{33}$$

Proof. Let $A \subset \mathbb{R}^2$ open, and let $Y_k \subset \mathbb{R}^2$ be rectangles such that
$$Y_1 \subset Y_2 \subset Y_3 \subset \cdots$$
$$A = \bigsqcup_{k=1}^{\infty} Y_k$$
Then, due to $\sigma-$additivity, we have
$$\mu_2(A) = \lim_{k\to\infty} \mu_2(Y_k) = \lim_{k\to\infty}\int_{\mathbb{R}} \mu_1(Y_{kx})\,\mathrm{d}x$$
But
$$Y_{1x} \subset Y_{2x} \subset \cdots$$
$$A_x = \bigsqcup_{k=1}^{\infty} Y_{kx}$$
Due to $\sigma-$additivity and the Beppo-Levi theorem we have that
$$\int_{\mathbb{R}} \mu_1(A_x)\,\mathrm{d}x = \lim_{k\to\infty}\int_{\mathbb{R}} \mu_1(Y_{kx})\,\mathrm{d}x$$

Let $E \subset \mathbb{R}^2$ be a measurable set. Define a sequence of compact sets K_i and a sequence of open sets A_j such that

$$K_1 \subset \cdots \subset K_j \subset E \subset A_j \subset \cdots \subset A_1$$

We have that $\lim_{j\to\infty} \mu_2(A_j) = \lim_{j\to\infty} \mu_2(K_j) = \mu_2(E)$ and that $K_{jx} \subset E \subset A_{jx}$.
From the previous derivation we can write that

$$\lim_{j\to\infty} \int_{\mathbb{R}} (\mu_1(A_{jx}) - \mu_1(K_{jx}))\,\mathrm{d}x = 0$$

Building a sequence of non-negative functions $f_j(x) = \mu_1(A_{jx}) - \mu_1(K_{jx})$ we have that $f_j(x) \leq f_{j-1}(x)$ and due to Beppo-Levi we have that

$$\lim_{j\to\infty} \int_{\mathbb{R}} f_j(x)\,\mathrm{d}x = \int_{\mathbb{R}} \lim_{j\to\infty} f_j(x)\,\mathrm{d}x$$

And therefore $\mu_1(K_{jx}) = \mu_1(A_{jx})$, and

$$\forall^\dagger\, x \in \mathbb{R} \quad \mu_2(K_j) = \int_{\mathbb{R}} \mu_1(K_{jx})\,\mathrm{d}x \leq \int_{\mathbb{R}} \overline{\mu_1}(E_x)\,\mathrm{d}x \leq \int_{\mathbb{R}} \mu_1(A_{jx}) = \mu_2(A_j)$$

□

Theorem 0.11 (Fubini)**.** *Let $f(x,y)$ be a measurable function in $\mathbb{R}^2$, then*

1. $\forall^\dagger\, x \in \mathbb{R} \quad y \mapsto f(x,y)$ *is measurable in* $\mathbb{R}$
2. $g(x) = \int_{\mathbb{R}} f(x,y)\,\mathrm{d}y$ *is measurable in* $\mathbb{R}$
3. $\iint_{\mathbb{R}^2} f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_{\mathbb{R}} \int_{\mathbb{R}} f(x,y)\,\mathrm{d}x\,\mathrm{d}y$

Proof. Let $f(x,y) \geq 0$. Defining $F_0 := \{\,(x,y) \in E \times \mathbb{R} |\, 0 < z < f(x,y)\} \subset \mathbb{R}^3$, we have that F_0 is measurable, and

$$\mu_3(F_0) = \iint_{\mathbb{R}^2} f(x,y)\,\mathrm{d}x\,\mathrm{d}y$$

But F_{0x} is also measurable $\forall^\dagger\, x \in \mathbb{R}$ and therefore

$$\mu_3(F_0) = \int_{\mathbb{R}} \mu_2(F_{0x})\,\mathrm{d}x = \int_{\mathbb{R}} \int_{\mathbb{R}} f(x,y)\,\mathrm{d}x\,\mathrm{d}y$$

□

Theorem 0.12 (Tonelli)**.** *Let $f(x,y)$ be a measurable function and $E \subset \mathbb{R}^2$ be a measurable set. If one of these integrals exists, the others also exist and have the same value*

$$\iint_{\mathbb{R}^2} f(x,y)\,\mathrm{d}x\,\mathrm{d}y \quad \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(x,y)\,\mathrm{d}x \right) \mathrm{d}y \quad \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(x,y)\,\mathrm{d}y \right) \mathrm{d}x$$

Theorem 0.13 (Integration Over Rectangles)**.** *Let $R = [a,b] \times [c,d] \subset \mathbb{R}^2$ be a rectangle, and $f(x,y)$ a measurable function over R. Then*

1. *If* $\forall^\dagger x \in [a,b]\ \exists G(x) = \int_c^d f(x,y)\,\mathrm{d}y$*, the function* $G(x)$ *is measurable in* $[a,b]$ *and*

$$\iint_R f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_a^b G(x)\,\mathrm{d}x = \int_a^b \int_c^d f(x,y)\,\mathrm{d}y\,\mathrm{d}x$$

2. *If* $\forall^\dagger y \in [c,d]\ \exists F(y) = \int_a^b f(x,y)\,\mathrm{d}x$*, the function* $F(y)$ *is measurable in* $[c,d]$ *and*

$$\int_{\mathbb{R}^2} f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_c^d F(y)\,\mathrm{d}y = \int_c^d \int_a^b f(x,y)\,\mathrm{d}x\,\mathrm{d}y$$

If both are true, then

$$\int_R f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_a^b \mathrm{d}x \int_c^d f(x,y)\,\mathrm{d}y = \int_c^d \mathrm{d}y \int_a^b f(x,y)\,\mathrm{d}x \tag{34}$$

Definition 0.3.1 (Normal Set)**.** A set $E \subset \mathbb{R}^2$ is said to be *normal* with respect to the x axis if

$$E = \left\{ (x,y) \in \mathbb{R}^2 \middle| \, a \le x \le b \ \alpha(x) \le y \le \beta(x) \right\}$$

The definition is analogous for the other axes.

Theorem 0.14 (Integration over Normal Sets)**.** *Let* $E \subset \mathbb{R}^2$ *be a normal set with respect to the* x *axis, and* $f(x,y)$ *is a measurable function over* E*. Then*

$$\int_E f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_a^b \mathrm{d}x \int_{\alpha(x)}^{\beta(x)} f(x,y)\,\mathrm{d}y \tag{35}$$

Theorem 0.15 (Dirichlet Inversion Formula)**.** *Take the triangle* $T := \left\{ (x,y) \in \mathbb{R}^2 \middle| \, a \le y \le x \le b \right\}$*. It can be considered normal with respect to both axes, and we can use the inversion formula*

$$\iint_T f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_a^b \mathrm{d}x \int_a^x f(x,y)\,\mathrm{d}y = \int_a^b \mathrm{d}y \int_y^b f(x,y)\,\mathrm{d}x \tag{36}$$

§§ 0.3.2 Triple Integration

Theorem 0.16 (Wire Integration)**.** *Let* $E \subset \mathbb{R}^3$ *be a normal set with respect to the* z *axis. If* $f(x,y,z)$ *is measurable in* E *we have*

$$\iiint_E f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \iint_D \mathrm{d}x\,\mathrm{d}y \int_{h(x,y)}^{g(x,y)} f(x,y,z)\,\mathrm{d}z \tag{37}$$

This is called the wire integration formula

Theorem 0.17 (Section Integration)**.** *Let* $F \subset \mathbb{R}^3$ *be a measurable set bounded by the planes* $z = a$ *and* $z = b$ *with* $a < b$*. Taken* $z \in [a,b]$ *we can define* F_z *and we have*

$$\iiint_F f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \int_a^b \mathrm{d}z \iint_{F_z} f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y \tag{38}$$

This is called the section integration formula

Theorem 0.18 (Center of Mass)**.** *Take a plane $E \subseteq \mathbb{R}^2$ with surface density $\rho(x, y) > 0$. We define the total mass M as follows*

$$M = \iint_E \rho(x, y) \, \mathrm{d}x \, \mathrm{d}y \tag{39}$$

The coordinates of the center of mass will be the following

$$\begin{aligned} x_G &= \frac{1}{M} \iint_E \rho(x, y) x \, \mathrm{d}x \, \mathrm{d}y \\ y_G &= \frac{1}{M} \iint_E \rho(x, y) y \, \mathrm{d}x \, \mathrm{d}y \end{aligned} \tag{40}$$

Theorem 0.19 (Moment of Inertia)**.** *Taken the same plane E, we define the moment of inertia with respect to a line r as the following integral*

$$I_r = \iint_E \rho(x, y) \left(d(p^\mu, r)\right)^2 \mathrm{d}x \, \mathrm{d}y \tag{41}$$

Where $d(p^\mu, r)$ is the distance function between the point (x, y) and the rotation axis r.
Both formulas are easily generalizable in $\mathbb{R}^3$

§§ 0.3.3 Change of Variables

Definition 0.3.2 (Diffeomorphism)**.** Let $M, N \subset X$ be two subsets of a metric space X. The two sets are said to be *diffeomorphic* if $\exists f : M \xrightarrow{\sim} N$ an isomorphism such that $f \in C^1(M)$ and $f^{-1} \in C^1(N)$. The application f is called a *diffeomorphism.*
Two diffeomorphic sets are indicated as follows

$$M \simeq N$$

Theorem 0.20. *Let $A, B \subset \mathbb{R}^n$ be two open sets and $\varphi^\mu : A \xrightarrow{\sim} B$ a diffeomorphism, such that*

$$\varphi^\mu(E) = F$$

If $f : E \subset B \longrightarrow \mathbb{R}$ is measurable, we have that

$$\int_E f(y^\mu) \, \mathrm{d}y^\mu = \int_{\varphi^{-1}(E)} f(\varphi^\mu(x^\nu)) \left| \det_{\mu\nu} \partial_\mu \varphi^\nu \right| \mathrm{d}x^\mu = \int_F f(\varphi^\mu) \left| \det_{\mu\nu} \partial_\mu \varphi^\nu \right| \mathrm{d}x^\mu$$

Theorem 0.21 (Change of Variables)**.** *Let $\varphi^\mu : \mathbb{R}^n \xrightarrow{\sim} \mathbb{R}^n$ be a diffeomorphism such that*

$$\varphi^\mu(x^\nu) = x^\mu \quad \forall \|x^\mu\|_\mu > 1$$

And $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ a function such that $\operatorname{supp} f = K \subset \mathbb{R}^n$ *is a compact set. If f is measurable, we have that*

$$\int_{\mathbb{R}^n} f(y^\mu) \, \mathrm{d}y^\mu = \int_{\mathbb{R}^n} f(\varphi^\mu(x^\nu)) \left| \det_{\mu\nu} \partial_\mu \varphi^\nu \right| \mathrm{d}x^\mu \tag{42}$$

Proof. Take $n = 2$ without loss of generality. We can immediately write that

$$g(y^1, y^2) = \int_{-\infty}^{y^1} f(\eta, y^2) \, \mathrm{d}\eta$$

Then, for the fundamental theorem of integral calculus

$$\partial_1 g(y^1, y^2) = f(y^1, y^2)$$

Taken $c \in \mathbb{R},\ c > 1 \,:\, K \subset Q = [-c, c] \times [-c, c]$, we have that $\varphi^\mu(x^\nu) = \delta^\mu_\nu \ \forall \|x^\mu\|_\mu > 1 \ \wedge \ f(x^\mu) = 0 \ \forall x^\mu \notin Q$.
Therefore $f(\varphi^\mu) = 0$ also and we have

$$\int_{\mathbb{R}^n} f(\varphi^\mu) \det_{\mu\nu} \partial_\mu \varphi^\nu \, \mathrm{d}x^\gamma = \int_Q f(\varphi^\mu) \det_{\mu\nu} \partial_\mu \varphi^\nu \, \mathrm{d}x^\gamma = \int_Q \partial_1 g(\varphi^\mu) \det_{\mu\nu} \partial_\mu \varphi^\nu \, \mathrm{d}x^\gamma$$

But we have that

$$g(y^\mu) = 0 \quad \forall \left|y^1\right| \geq c \ \vee \ \left|y^1\right| < -c$$

Define the following matrix $H_{\mu\nu}$

$$H_{\mu\nu} = \begin{pmatrix} \partial_\mu g(\phi^\gamma) \\ \partial_\mu \varphi^2 \end{pmatrix}$$

Then we have that

$$\det_{\mu\nu} H_{\mu\nu} = \partial_1 g(\varphi^\mu) \det_{\mu\nu} \partial_\mu \varphi^\nu$$

Writing $g(\varphi^\mu) = G(x^\mu)$ we have

$$\det_{\mu\nu} H_{\mu\nu} = \partial_1 G \partial_2 \varphi^2 - \partial_2 G \partial_1 \varphi^2$$

Thanks to the integration formula (34) we can then write

$$\int_Q \det_{\mu\nu} H_{\mu\nu} \, \mathrm{d}x^\gamma = \int_{-c}^{c} \mathrm{d}x^2 \int_{-c}^{c} \partial_1 G \partial_2 \varphi^2 \, \mathrm{d}x^\nu$$

Integrating by parts we get

$$\int_Q \det_{\mu\nu} H_{\mu\nu} \, \mathrm{d}x^\gamma = \left. G\partial_2 \varphi^2 \right|_{-c}^{c} - \int_{-c}^{c} G \partial^2_{21} \varphi^2 \, \mathrm{d}x^1 - \left. G \partial_1 \varphi^2 \right|_{-c}^{c} - \int_{-c}^{c} G \partial^2_{12} \varphi^2 \, \mathrm{d}x^2$$

But $\forall x^\mu \in \partial Q \quad \varphi^\mu(x^\nu) = x^\mu \implies G(-c, x^2) = g(-c, x^2) = 0 \ \wedge \ G(c, x^2) = g(c, x^2)$

$$\therefore \int_Q \det_{\mu\nu} H_{\mu\nu} \, \mathrm{d}x^\gamma = \int_Q f(x^\mu) \, \mathrm{d}x^\gamma$$

□

Theorem 0.22 (Common Coordinate Transformation in $\mathbb{R}^2$ and $\mathbb{R}^3$). *1. Polar Coordinates*

$$\varphi^\mu(x^\nu) = \begin{cases} x(\rho, \theta) = \rho \cos\theta & \rho \in \mathbb{R}^+ \\ y(\rho, \theta) = \rho \sin\theta & \theta \in [0, 2\pi) \end{cases} \tag{43a}$$

$$\begin{aligned} \partial_\mu \varphi^\nu &= \begin{pmatrix} \cos\theta & -\rho\sin\theta \\ \sin\theta & \rho\cos\theta \end{pmatrix} \\ \det_{\mu\nu} \partial_\mu \varphi^\nu &= \rho \end{aligned} \tag{43b}$$

2. Spherical Coordinates

$$\varphi^\mu(x^\nu) = \begin{cases} x(\rho,\theta,\phi) = \rho\sin\phi\cos\theta & \rho \in \mathbb{R}^+ \\ y(\rho,\theta,\phi) = \rho\sin\phi\sin\theta & \theta \in [0, 2\pi) \\ z(\rho,\theta,\phi) = \rho\cos\phi & \phi \in [0, \pi] \end{cases} \tag{44a}$$

$$\begin{aligned} \partial_\mu \varphi^\nu &= \begin{pmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta & \rho\cos\phi\cos\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta & \rho\cos\phi\sin\theta \\ \cos\phi & 0 & -\rho\sin\phi \end{pmatrix} \\ \det_{\mu\nu} \partial_\mu \varphi^\nu &= \rho^2 \sin\phi \end{aligned} \tag{44b}$$

3. Cylindrical Coordinates

$$\varphi^\mu(x^\nu) = \begin{cases} x(\rho,\theta,z) = \rho\cos\theta & \rho \in \mathbb{R}^+ \\ y(\rho,\theta,z) = \rho\sin\theta & \theta \in [0, 2\pi) \\ z(\rho,\theta,z) = z & z \in \mathbb{R} \end{cases} \tag{45a}$$

$$\det_{\mu\nu} \partial_\mu \varphi^\nu = \rho \tag{45b}$$

Definition 0.3.3 (Rotation Solids)**.** Let $D \subset \mathbb{R}^2$ be a bounded measurable set contained in the half-plane $y = 0, x > 0$. Suppose we let D "pop up" into $\mathbb{R}^3$ through a rotation by an angle θ_0 around the z axis. What has been obtained is a *rotation solid* $E \subset \mathbb{R}^3$. We have that

$$\mu(E) = \iiint_E \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \iint_D \int_0^{\theta_0} \rho\,\mathrm{d}\rho\,\mathrm{d}\theta\,\mathrm{d}z = \theta_0 \iint_D \rho\,\mathrm{d}\rho\,\mathrm{d}z = \theta_0 \iint_D x\,\mathrm{d}x\,\mathrm{d}y \tag{46}$$

Or

$$\mu(E) = \theta_0 x_G \mu_2(D)$$

Theorem 0.23 (Guldino)**.** *The measure of a rotation solid is given by the measure of the rotated figure times the circumference described by the center of mass of the solid.*
This is exactly the previous formula.

§§ 0.3.4 Line Integrals

Definition 0.3.4 (Line Integral of the First Kind)**.** Given a scalar field $f : A \subset \mathbb{R}^3 \longrightarrow \mathbb{R}$ and a smooth curve $\{\gamma\} \subset \mathbb{R}^3$, we define the *line integral of the first kind* as follows

$$\int_\gamma f\,\mathrm{d}s = \int_a^b f(\gamma^\mu) \left\| \frac{\mathrm{d}\gamma^\mu}{\mathrm{d}t} \right\|_\mu \mathrm{d}t \tag{47}$$

Theorem 0.24 (Center of Mass of a Curve). *Given a curve $\gamma^\mu : [a,b] \longrightarrow \mathbb{R}^3$ with linear mass density $m : \{\gamma\} \longrightarrow \mathbb{R}$, we define the total mass of γ as follows*

$$M = \int_\gamma m \, \mathrm{d}s = \int_a^b m(\gamma^\mu) \left\| \frac{\mathrm{d}\gamma^\mu}{\mathrm{d}t} \right\|_\mu \mathrm{d}t \tag{48}$$

The center of mass is then defined as follows

$$x_G^\mu = \frac{1}{M} \int_\gamma x^\mu m(x^\nu) \, \mathrm{d}s \tag{49}$$

Definition 0.3.5 (Line Integral of the Second Kind). Given a vector field $f^\mu : A \longrightarrow \mathbb{R}^3$ and a smooth curve $\gamma^\mu : [a,b] \longrightarrow A \subset \mathbb{R}^3$ we define the *line integral of the second kind* as follows

$$\int_\gamma f^\mu T_\mu \, \mathrm{d}s = \int_a^b f^\mu(\gamma^\nu) \frac{\mathrm{d}\gamma_\mu}{\mathrm{d}t} \, \mathrm{d}t \tag{50}$$

Defining a differential form $\omega = f^\mu \, \mathrm{d}x_\mu$ we can also see this integral as follows

$$\int_\gamma \omega = \int_\gamma f^\mu T_\mu \, \mathrm{d}s \tag{51}$$

Where T^μ is the tangent vector of the curve

Definition 0.3.6 (Conservative Field). Let $f^\mu : A \longrightarrow \mathbb{R}^3$ be a vector field such that $f^\mu \in C^1(A)$ and A is open and connected. This field is said to be *conservative*, if $\forall x^\mu \in A$

$$\exists U(x^\mu) \in C^2(A) \, : \, f^\mu = -\partial^\mu U \tag{52}$$

The function $U(x^\mu)$ is called the *potential* of the field.

Theorem 0.25 (Line Integral of a Conservative Field). *Given a conservative field $f^\mu : A \longrightarrow \mathbb{R}^3$ and a smooth curve $\{\gamma\} \subset A, \ \gamma^\mu : [a,b] \longrightarrow \mathbb{R}^3$ with A open and connected, we have that*

$$\int_\gamma f^\mu T_\mu \, \mathrm{d}s = U(\gamma(a)) - U(\gamma(b)) \tag{53}$$

Where $U(x^\mu)$ is the potential of the vector field.

Definition 0.3.7 (Rotor). Given a vector field $f^\mu : A \longrightarrow \mathbb{R}^3$ with $f^\mu \in C^1(A)$, we define the *rotor* of the vector field as follows

$$\mathrm{rot}(f^\mu) = \epsilon^\mu_{\nu\gamma} \partial^\nu f^\gamma \tag{54}$$

Theorem 0.26. *Given f^μ a conservative vector field on an open connected set A, we have that*

$$\epsilon^\mu_{\nu\gamma} \partial^\nu f^\gamma = 0 \tag{55}$$

Alternatively, if $\gamma^\mu : [a,b] \longrightarrow \mathbb{R}^3$ is the parameterization of a smooth closed curve, we have that

$$\oint_\gamma f^\mu T_\mu \, \mathrm{d}s = 0 \tag{56}$$

§§ 0.3.5 Surface Integrals

Definition 0.3.8 (Area of a Surface)**.** Given $r^\mu : K \subset \mathbb{R}^2 \longrightarrow \Sigma \subset \mathbb{R}^3$ a smooth surface, we have that given its metric tensor $g_{\mu\nu}(u, v)$ we have that

$$\mu\left(\Sigma\right) = \int_\Sigma \mathrm{d}\sigma = \iint_K \sqrt{\det_{\mu\nu} g_{\mu\nu}}\, \mathrm{d}u\, \mathrm{d}v = \iint_K \sqrt{EG - F^2}\, \mathrm{d}u\, \mathrm{d}v \tag{57}$$

For a cartesian surface S we have that

$$\mu\left(S\right) = \int_S \mathrm{d}s = \iint_K \sqrt{1 + \left(\|\partial_\mu f\|_\mu\right)^2}\, \mathrm{d}x\, \mathrm{d}y \tag{58}$$

Definition 0.3.9 (Rotation Surface)**.** Given a smooth curve $\gamma^\mu : [a, b] \longrightarrow \mathbb{R}^3$, the rotation of this curve around the $z-$axis generates a smooth surface Σ with the following parameterization

$$r^\mu(t, \theta) = \begin{cases} \gamma^1(t) \cos\theta \\ \gamma^2(t) \sin\theta \qquad (t, \theta) \in [a, b] \times [0, \theta_0] \\ \gamma^3(t) \end{cases} \tag{59}$$

The area of a rotation surface is calculated as follows

$$\mu\left(\Sigma\right) = \theta_0 \int_a^b \gamma^1(t) \sqrt{\left(\frac{\mathrm{d}\gamma^1}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}\gamma^2}{\mathrm{d}t}\right)^2}\, \mathrm{d}t \tag{60}$$

Theorem 0.27 (Guldino II)**.** *Given Σ a smooth rotation surface defined as before, we have that its area will be*

$$\mu\left(\Sigma\right) = \theta_0 \int_\gamma x^1\, \mathrm{d}s = \theta_0 x_G^1 L_\gamma \tag{61}$$

Where x_G^1 is the first coordinate of the center of mass of the curve, calculated as follows

$$x_G^1 = \frac{1}{L_\gamma} \int_\gamma x^1\, \mathrm{d}s$$

Definition 0.3.10 (Surface Integral)**.** Given a smooth surface $\Sigma \subset \mathbb{R}^3$ with parameterization $r^\mu : K \longrightarrow \Sigma$ and a scalar field $h : \mathbb{R}^3 \longrightarrow \mathbb{R}$, we define the *surface integral* of h as follows

$$\int_\Sigma h(x^\mu)\, \mathrm{d}\sigma = \iint_K h(r^\mu) \sqrt{\det_{\mu\nu} g_{\mu\nu}}\, \mathrm{d}u\, \mathrm{d}v \tag{62}$$

If Σ is a cartesian surface, we have

$$\int_\Sigma h(x^\mu)\, \mathrm{d}\sigma = \iint_K h(x^1, x^2, f) \sqrt{1 + (\|\partial_\mu f\|^\mu)^2}\, \mathrm{d}x\, \mathrm{d}y \tag{63}$$

Definition 0.3.11 (Center of Mass of a Surface)**.** Given a smooth surface Σ with parameterization $r^\mu(u, v)$ and mass density δ, we define its total mass as follows

$$M = \int_\Sigma \delta\, \mathrm{d}\sigma \tag{64}$$

Its center of mass x_G^μ will be calculated as follows

$$x_G^\mu = \frac{1}{M} \int_\Sigma x^\mu \delta(x^\nu)\, \mathrm{d}\sigma \tag{65}$$

Definition 0.3.12 (Moment of Inertia of a Surface). Given a smooth surface Σ with parameterization $r^\mu(u,v)$ and mass density δ we define its moment of inertia around an axis r, I, as the following integral

$$I = \int_\Sigma \delta(x^\mu)\left(d(p^\mu, r)\right)^2 \mathrm{d}\sigma \quad p^\mu \in \Sigma \tag{66}$$

Definition 0.3.13 (Orientable Surface). A smooth surface with parameterization $r^\mu : K \subset \mathbb{R}^2 \longrightarrow \Sigma \subset \mathbb{R}^3$ is said to be *orientable* if $\forall \gamma : [a,b] \longrightarrow \Sigma$ smooth closed curve, we have, given n^μ the normal vector of the surface

$$n^\mu(\gamma^\nu(a)) = n^\mu(\gamma^\nu(b)) \tag{67}$$

Another way of formulating it is

$$n^\mu(x^\nu) \in C(K) \tag{68}$$

Definition 0.3.14 (Boundary of a Surface). Given a smooth surface as before, we define the *boundary* $\partial\Sigma$ as follows

$$\partial\Sigma = \overline{\Sigma} \setminus \Sigma \tag{69}$$

Note how, given the parameterization r^μ, we have $r^\mu(\partial K) = \partial\Sigma$

Definition 0.3.15 (Closed Surface). A surface $\Sigma \subset \mathbb{R}^3$ is said to be *closed* iff $\partial\Sigma = \{\}$

Definition 0.3.16 (Flux). Given a vector field $f^\mu : A \subset \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ and a smooth orientable surface $\Sigma \subset A$, we define the *flux* of the vector field f^μ on the surface as follows

$$\Phi_\Sigma(f^\mu) = \int_\Sigma f^\mu n_\mu \,\mathrm{d}\sigma = \iint_K f^\mu(r^\nu)\epsilon_{\mu\gamma\sigma}\partial_1 r^\gamma \partial_2 r^\sigma \,\mathrm{d}u\,\mathrm{d}v \tag{70}$$

§ 0.4 Integration in $\mathbb{C}$

Definition 0.4.1 (Piecewise Continuous Function). Let $\gamma : [a,b] \longrightarrow \mathbb{C}$ be a piecewise continuous curve such that $\{\gamma\} \subset D \subset \mathbb{C}$, and $f : D \longrightarrow \mathbb{C}, \quad f \in C(D)$. Then the function $(f \circ \gamma)\,\gamma'(t) : [a,b] \longrightarrow \mathbb{C}$ is a *piecewise continuous function*

Definition 0.4.2 (Line Integral in $\mathbb{C}$). Let $\gamma : [a,b] \longrightarrow D \subset \mathbb{C}$ be a piecewise continuous curve and $f : D \longrightarrow \mathbb{C}$ a measurable function $f \in C(D)$.
We define the *line integral over* γ the result of the application of the integral operator $\hat{K}_\gamma[f]$, where

$$\hat{K}_\gamma[f] = \int_\gamma f(z)\,\mathrm{d}z = \int_a^b (f \circ \gamma)\,\gamma'(t)\,\mathrm{d}t \tag{71}$$

Where $\forall^\dagger\, z \in \{\gamma\}\ f(z)$ is defined

Theorem 0.28 (Properties of the Line Integral). *Let $z, w, t \in \mathbb{C}$, $f, g \in \mathcal{M}(\mathbb{C})$ and $\{\gamma\}, \{\eta\}, \{\kappa\}$ three smooth curves, then*

1. $\hat{K}_\gamma[zf + wg] = z\hat{K}_\gamma[f] + w\hat{K}_\gamma[g]$
2. $\gamma \sim \eta \implies \hat{K}_\gamma[f] = \hat{K}_\eta[f]$

3. $\gamma = \eta + \kappa \implies \hat{K}_\gamma[f] = \hat{K}_{\eta+\kappa}[f] = \hat{K}_\eta[f] + \hat{K}_\kappa[f]$

4. $\hat{K}_{\gamma+w}[f(z)] = \hat{K}_\gamma[f(z+w)]$

Notation. If a measurable function $f(z)$ has the same value of the integral for different curves between two points $z_1, z_2 \in \mathbb{C}$, we will write directly

$$\int_\gamma f(z)\,\mathrm{d}z = \int_{z_1}^{z_2} f(z)\,\mathrm{d}z$$

Theorem 0.29 (Darboux Inequality)**.** *Let $f : D \longrightarrow \mathbb{C}$ be a measurable function and $\gamma : [a,b] \longrightarrow \{\gamma\} \subset D \subseteq \mathbb{C}$ piecewise smooth. Then*

$$\left\|\int_\gamma f(z)\,\mathrm{d}z\right\| \leq L_\gamma \sup_{z\in\{\gamma\}} \|f(z)\|$$

Proof. The proof is quite straightforward using the definition given for the line integral

$$\left\|\int_\gamma f(z)\,\mathrm{d}z\right\| = \left\|\int_a^b (f\circ\gamma)\,\gamma'(t)\,\mathrm{d}t\right\| \leq \int_a^b \|(f\circ\gamma)\,\gamma'(t)\|\,\mathrm{d}t \leq$$
$$\leq \sup_{z\in\{\gamma\}} \|f(z)\| \int_a^b \|\gamma'(t)\|\,\mathrm{d}t = L_\gamma \sup_{z\in\{\gamma\}} \|f(z)\|$$

□

§§ 0.4.1 Integration of Holomorphic Functions

Definition 0.4.3 (Primitive)**.** Let $f : D \longrightarrow \mathbb{C}$ and $F : D \longrightarrow \mathbb{C}$ be two functions and $D \subset \mathbb{C}$ an open and connected set. $F(z)$ is said to be the *primitive function* or *antiderivative* of f in D if

$$\frac{\mathrm{d}F}{\mathrm{d}z} = f(z) \quad \forall z \in D \tag{72}$$

Notation. Given a closed curve γ and a measurable function $f(z)$ we define the following notation

$$\int_\gamma f(z)\,\mathrm{d}z = \oint_\gamma f(z)\,\mathrm{d}z$$

Theorem 0.30 (Existence of the Primitive Function)**.** *Let $f : D \longrightarrow \mathbb{C} \quad f \in C(D)$ with $D \subset \mathbb{C}$ open and connected. Then these statements are equivalent*

1. $\exists F : D \longrightarrow \mathbb{C} \,:\, F'(z) = f(z)$

2. $\forall z_1, z_2 \in D,\ \forall \{\gamma\} \subset D$ *piecewise smooth* $\int_\gamma f(z)\,\mathrm{d}z = \int_{z_1}^{z_2} f(z)\,\mathrm{d}z$

3. $\forall \gamma : [a,b] \longrightarrow \{\gamma\} \subset D$ *closed piecewise smooth* $\oint_\gamma f(z)\,\mathrm{d}z = 0$

Proof. 1 $\implies$ 2. As with the hypothesis we have that $\exists F : D \longrightarrow \mathbb{C} \,:\, F'(z) = f(z)\ \forall z \in D$. Given two points $z_1, z_2 \in D$ and taken a smooth curve $\gamma : [a,b] \longrightarrow D \,:\, \gamma(a) = z_1 \ \wedge\ \gamma(b) = z_2$. Therefore

$$\int_\gamma f(z)\,\mathrm{d}z = \int_a^b (f \circ \gamma)\,\gamma'(t)\,\mathrm{d}t = \int_a^b (F' \circ \gamma)\,\gamma'(t)\,\mathrm{d}t$$

The result of the integral is obviously $F(z_2) - F(z_1)$, therefore we can immediately write that, if

$$\exists F : D \longrightarrow \mathbb{C} \,:\, F'(z) = f(z) \implies \int_\gamma f(z)\,\mathrm{d}z = \int_{z_1}^{z_2} f(z)\,\mathrm{d}z$$

2 $\implies$ 1 Taken a point $z_0 \in D$, any point $z \in D$ can be connected with a polygonal to z_0 since D is connected. The integral of f over this polygonal is obviously path-independent, hence we can define the following function

$$F(z) = \int_{z_0}^{z} f(w)\,\mathrm{d}w$$

Since D is open we can define $\delta_z \in \mathbb{R},\ \delta_z > 0 \ \wedge\ \exists B_{\delta_1}(z) \subset D$. Taken $\Delta z \in \mathbb{C} \,:\, \|\Delta z\| < \delta_1$ we have that

$$F(z + \Delta z) - F(z) = \int_z^{z+\Delta z} f(w)\,\mathrm{d}w$$

Dividing by Δz and taking the limit as $\Delta z \to 0$ we have that using the Darboux inequality we get that

$$\left\| \frac{F(z+\Delta z) - F(z)}{\Delta z} - f(z) \right\| = \frac{1}{\|\Delta z\|} \left\| \int_z^{z+\Delta z} f(w)\,\mathrm{d}w \right\| \leq \epsilon$$

2 $\implies$ 3. Taken an arbitrary piecewise smooth curve γ and $z_1 \neq z_2 \in \{\gamma\}$. We can now find two curves such that $\gamma(t) = \gamma_1(t) - \gamma_2(t)$. Since the integral of f is path independent, we get

$$\int_\gamma f(z)\,\mathrm{d}z = \int_{\gamma_1} f(z)\,\mathrm{d}z - \int_{\gamma_2} f(z)\,\mathrm{d}z = 0$$

3 $\implies$ 2 is exactly as before but with the opposite reasoning. □

Example 0.4.1. Let's calculate the integral of functions $f_n(x) = z^{-n} \quad n \in \mathbb{N}$ for a closed simple piecewise smooth curve γ such that $0 \notin \{\gamma\}$.
For $n > 1$ we have that $f \in C(D)$ where $D = \mathbb{C} \setminus \{0\}$, and we have that

$$\int \frac{1}{z^n}\,\mathrm{d}z = -\frac{z^{-(n-1)}}{n-1} + w \quad w \in \mathbb{C}$$

Therefore, for every closed simple piecewise smooth curve $\gamma \,:\, 0 \notin \{\gamma\}$ we have

$$\oint_\gamma \frac{1}{z^n}\,\mathrm{d}z = 0$$

For $n = 1$ we still have that $f \in C(D)$ but $\nexists F(z) : D \longrightarrow \mathbb{C}$ primitive of $f_1(z)$, but there exists one in the domain G of holomorphy of the logarithm.

Although we have that $G \subset D$, and we can take a curve $\gamma : 0 \in \operatorname{extr}\gamma$, and therefore $\{\gamma\} \subset G$ and we have that

$$\oint_\gamma \frac{1}{z}\,\mathrm{d}z = 0$$

If we otherwise have $0 \in \gamma^\circ$ the integral is non-zero.
Take a branch of the logarithm σ and a curve η has only one point of intersection with such branch at $z_i = u_0 e^{i\alpha}$. Taken $\eta(a) = \eta(b) = u_0 e^{i\alpha}$, we define $\eta_\epsilon : [a+\epsilon, b+\epsilon] \longrightarrow \mathbb{C}$ with $\epsilon > 0 : \eta_\epsilon(t) = \eta(t)\ \forall t \in [a+\epsilon, b+\epsilon]$, then

$$\oint_\eta \frac{1}{z}\,\mathrm{d}z = \lim_{\epsilon\to 0}\oint_{\eta_\epsilon}\frac{1}{z}\,\mathrm{d}z$$

Therefore, $\forall z \in \mathbb{C}\setminus\{\sigma\}$ we have that

$$\frac{\mathrm{d}\log z}{\mathrm{d}z} = \frac{1}{z}$$

And therefore

$$\oint_{\eta_\epsilon}\frac{1}{z}\,\mathrm{d}z = \log\left(\eta(b-\epsilon)\right) - \log\left(\eta\left(a+\epsilon\right)\right)$$

For $\epsilon \to 0$ we have

$$\int_\eta \frac{1}{z}\,\mathrm{d}z = (\log(u_0) + i(\alpha+2\pi)) - (\log(u_0) + i\alpha) = 2\pi i$$

Example 0.4.2. Let's calculate the integral of $f(z) = \sqrt{z}$ along a closed simple piecewise smooth curve $\gamma : [a,b] \longrightarrow \mathbb{C} : 0 \in \gamma^\circ$ and it intersects the line $\sigma_\alpha = u_0 e^{i\alpha}$, where

$$\sqrt{z} = \sqrt{r}e^{i\frac{\theta}{2}} \quad r \in \mathbb{R}^+,\ \theta \in (\alpha, \alpha+2\pi],\ \alpha \in \mathbb{R}$$

Taken a parametrization $\gamma(t) : \gamma(a) = \gamma(b) = u_0 e^{i\alpha}$ we have that $f(z) \in H(D)$ where $D = \mathbb{C}\setminus\{\sigma_\alpha\}$. Proceding as before, we have

$$\oint_\gamma \sqrt{z}\,\mathrm{d}z = \lim_{\epsilon\to 0}\oint_{\gamma_\epsilon}\sqrt{z}\,\mathrm{d}z$$

Since it has a primitive in D we can write

$$\lim_{\epsilon\to 0}\oint_{\gamma_\epsilon}\sqrt{z}\,\mathrm{d}z = \frac{2}{3}\lim_{\epsilon\to 0} z\sqrt{z}\Big|_{\gamma_\epsilon(a+\epsilon)}^{\gamma_\epsilon(b-\epsilon)} = \frac{2}{3}u_0\sqrt{u_0}e^{\frac{3}{2}i(\alpha+2\pi)} - \frac{2}{3}u_0\sqrt{u_0}e^{\frac{3}{2}i\alpha} = -\frac{4}{3}u_0\sqrt{u_0}e^{\frac{3}{2}i\alpha}$$

Lemma 0.4.1. Taken a closed simple pointwise smooth curve $\gamma : [a,b] \longrightarrow \mathbb{C}$ and taken $D = \{\gamma\}^\circ \cup \gamma = \overline{\{\gamma\}^\circ}$ and a function $f \in H(D)$, for a finite cover of D, $\mathcal{Q}$ composed by squares $Q_j \in \mathcal{Q}\ \forall j \in [1, N] \subset \mathbb{N}$, we have that

$$\exists z_j \in Q_j \cap \overline{\{\gamma\}^\circ} : \left\|\frac{f(z) - f(z_j)}{z - z_j} - \frac{\mathrm{d}f}{\mathrm{d}z}\Big|_{z_j}\right\| < \epsilon\ \forall z \in Q_j \cap \overline{\{\gamma\}^\circ}\setminus\{z_j\}$$

Proof. Going by contradiction, let's say that

$$\exists \epsilon > 0 : \nexists z_j \in Q_j \cap \overline{\{\gamma\}^\circ}$$

Taken a finite subcover $\mathcal{Q}_n$ where $\operatorname{diam}(Q_j^n) = \frac{d}{2^n}\ \forall Q_j \in \mathcal{Q}$ we can define for some $k \in K \subset \mathbb{N}$

$$A_n = \bigcup_{k \in K} Q_k^n \cap \overline{\{\gamma\}^\circ} \quad \forall n \in \mathbb{N}$$

We have that $A_{n+1} \subset A_n$, and taking a sequence $(w)_n \in \overline{\{\gamma\}^\circ}$ we have due to the compactness of $\overline{\{\gamma\}^\circ}$ that $\exists (w)_{n_j} \to w \in \overline{\{\gamma\}^\circ}$. Since $f \in H(\overline{\{\gamma\}^\circ})$ we have that f is holomorphic in w, therefore

$$\forall \epsilon > 0\ \exists \delta_\epsilon > 0\ :\ \left\| \frac{f(z) - f(w)}{z - w} - \left.\frac{\mathrm{d}f}{\mathrm{d}z}\right|_w \right\| < \epsilon\ \forall z \in B_{\delta_\epsilon}(w) \setminus \{w\}$$

Taken an $\tilde{n}$ such that $\operatorname{diam}(Q_j^{\tilde{n}}) = \frac{\sqrt{2}}{2^{\tilde{n}}} d < \delta$ we have that still $w \in A_n\ \forall n \in \mathbb{N}$, and due to its closedness we can also say

$$\exists N_j \in \mathbb{N}\ :\ \forall n_j > N_j\ (w)_{n_j} \in A_n$$

Therefore

$$\exists k_0 \in \mathbb{N}\ :\ w \in Q_{k_0}^{\tilde{n}} \cap \overline{\{\gamma\}^\circ} \subset A_{\tilde{n}}\ \lightning$$

□

Theorem 0.31 (Cauchy-Goursat)**.** *Taken $\gamma : [a, b] \longrightarrow \mathbb{C}$ a closed simple piecewise smooth curve and $D = \{\gamma\} \cup \{\gamma\}^\circ$ and a function $f \in H(D)$, we have*

$$\oint_\gamma f(z)\,\mathrm{d}z = 0 \tag{73}$$

Proof. Using the previous lemma we can say that for a finite cover $\{\gamma\}, Q_j \in \mathcal{Q}\ \exists z_j \in Q_j \cap \overline{\{\gamma\}^\circ}$ and a function

$$\delta_j(z) = \begin{cases} \dfrac{f(z) - f(z_j)}{z - z_j} - f'(z_j) & z \neq z_j \\ 0 & z = z_j \end{cases}$$

Which is countinuous and $\delta_j(z) < \epsilon\ \forall z \in Q_j \cap \overline{\{\gamma\}^\circ}$.
Taken a curve $\{\eta_j\} = \partial\left(Q_j \cap \overline{\{\gamma\}^\circ}\right)$, and the expansion of $f(z)$ in the region, we have that

$$\begin{gathered} f(z) = f(z_j) + f'(z_j)(z - z_j) + \delta_j(z)(z - z_j) \\ \oint_{\eta_j} f(z)\,\mathrm{d}z = (f(z_j) - z_j f'(z_j)) \oint_{\eta_j} \mathrm{d}z + f'(z_j) \oint_{\eta_j} z\,\mathrm{d}z + \oint_{\eta_j} \delta_j(z)(z - z_j)\,\mathrm{d}z \end{gathered}$$

The first two integrals on the second line are null, and we have therefore

$$\oint_{\eta_j} f(z)\,\mathrm{d}z = \oint_{\eta_j} \delta_j(z)(z - z_j)\,\mathrm{d}z$$

By definition $\{\gamma\} = \bigcup_{j=1}^N \{\eta_j\}$ and therefore

$$\oint_\gamma f(z)\,\mathrm{d}z = \sum_{j=1}^N \oint_{\eta_j} \delta_j(z)(z - z_j)\,\mathrm{d}z$$

Using the Darboux inequality we have immediately that

$$\left\|\oint_\gamma f(z)\,\mathrm{d}z\right\| \le \sum_{j=1}^{N}\left\|\oint_{\eta_j}\delta_j(z)(z-z_j)\,\mathrm{d}z\right\| \le \sum_{j=1}^{N}\epsilon\sqrt{2}d(4d+L_j)$$

Using the theorem on the Jordan curve, we have that $\exists Q_n \in \mathcal{Q}$ such that $\{\gamma\} \subset Q_n$. Taken $\operatorname{diam}(Q_n) = D$

$$\left\|\oint_\gamma f(z)\,\mathrm{d}z\right\| \le \sum_{j=1}^{N}\epsilon\sqrt{2}D(4D+L) \to 0$$

□

Definition 0.4.4 (Simple Connected Set)**.** An open set $G \subset X$ with X some metric space, is said to be *simply connected* iff $\forall\{\gamma_j\} \subset G$ simple curves we have that $\gamma_j \sim 0$.
$\gamma \sim 0$ implies that the curve is homotopic to a point

Theorem 0.32 (Cauchy-Goursat II)**.** *Let $G \subset \mathbb{C}$ open and simply connected. Then, $\forall f \in H(G), \{\gamma\} \subset G$ with γ simple closed and smooth*

$$\oint_\gamma f(z)\,\mathrm{d}z = 0$$

Proof.

1. The curve γ doesn't intersect itself.

$$\oint_\gamma f(z)\,\mathrm{d}z = \oint_0 f(z)\,\mathrm{d}z = 0$$

2. The curve γ intersects itself $n-1$ times.
   Then $\{\gamma\} = \bigcup_{k=1}^{n}\{\gamma_k\}$ with γ_k simple smooth non intersecting curves. Since $\{\gamma_k\} \subset G\ \forall k = 1,\cdots,n,\ \{\gamma_k\} \sim 0$, we have

$$\oint_\gamma f(z)\,\mathrm{d}z = \sum_{k=1}^{n}\oint_{\gamma_k} f(z)\,\mathrm{d}z = 0$$

□

Theorem 0.33. *Let $G \subset \mathbb{C}$ be a simply connected open set. If $f \in H(G)$, then there exists a primitive for $f(z)$*

§§ 0.4.2 Integral Representation of Holomorphic Functions

Definition 0.4.5 (Positively Oriented Curve)**.** The parametrization of a curve in $\mathbb{C}$ is said to be *positively oriented* if its parametrization is taken such the path taken results counterclockwise.

Notation. The integral over a closed positively oriented parametrization of a curve γ is indicated as follows

$$\oint_\gamma$$

Theorem 0.34 (Cauchy Integral Representation)**.** *Taken a positively oriented closed simple piecewise smooth curve* $\gamma : [a, b] \longrightarrow \mathbb{C}$ *and a function* $f : G \subset \mathbb{C} \longrightarrow \mathbb{C}$ *such that if* $D = \{\gamma\} \cup \{\gamma\}^\circ \subset G,\ f \in H(D)$*, we have that*

$$f(z) = \frac{1}{2\pi i} \oint_\gamma \frac{f(w)}{w - z}\, \mathrm{d}w \quad \forall w \in \{\gamma\}^\circ \tag{74}$$

Proof. Taken $\gamma_\rho(\theta) = z + \rho e^{i\theta}$ such that $\gamma_\rho \sim \gamma,\ \{\gamma_\rho\} \subset \{\gamma\}^\circ$ is a simple curve, we have

$$\oint_\gamma \frac{f(w)}{w - z}\, \mathrm{d}w = \oint_{\gamma_\rho} \frac{f(w)}{w - z}\, \mathrm{d}w$$

Then, using that

$$\oint_\gamma \frac{1}{w - z}\, \mathrm{d}w = 2\pi i$$

We get

$$\oint_\gamma \frac{f(z)}{w - z}\, \mathrm{d}w - 2\pi i f(z) = \oint_{\gamma_\rho} \frac{f(w) - f(z)}{w - z}\, \mathrm{d}w$$

Since $f \in H(\{\gamma\}^\circ)$ we have that

$$\forall \epsilon > 0\ \exists \delta_\epsilon > 0\ :\ \|z - w\| < \delta_\epsilon \implies \|f(z) - f(w)\| < \epsilon$$

Taken $\rho < \delta_\epsilon$ we get, using the Darboux inequality

$$\left\| \oint_{\gamma_\rho} \frac{f(w) - f(z)}{w - z}\, \mathrm{d}w \right\| \leq 2\pi\epsilon \implies \oint_\gamma \frac{f(w) - f(z)}{w - z}\, \mathrm{d}w = 0$$

□

Theorem 0.35 (Derivatives of a Holomorphic Function)**.** *Let* $D \subset \mathbb{C}$ *be an open set and* $f : D \longrightarrow \mathbb{C}$ *a function* $f \in H(D)$*, then* $f \in C^\infty(D)$ *and*

$$\frac{\mathrm{d}^n f}{\mathrm{d}z^n} = \frac{n!}{2\pi i} \oint_\gamma \frac{f(w)}{(w - z)^{n+1}}\, \mathrm{d}w \tag{75}$$

Where γ *is a closed simple piecewise smooth curve such that* $z \in \{\gamma\}^\circ$ *and* $\overline{\{\gamma\}} \subset D$

Corollary 0.4.1. Let $f \in H(D)$, then

$$\forall n \in \mathbb{N} \quad \frac{\mathrm{d}^n f}{\mathrm{d}z^n} \in H(D)$$

Theorem 0.36 (Morera)**.** *Let* $D \subset \mathbb{C}$ *be an open and connected set. Take* $f : D \longrightarrow \mathbb{C}\ :\ f \in C(D)$*. Then, if* $\forall \{\gamma\} \subset D$ *closed piecewise smooth*

$$\oint_\gamma f(z)\, \mathrm{d}z = 0 \implies f \in H(D) \tag{76}$$

Proof. Since $f \in C(D)$ $\exists F(z) \in C^1(D) : f(z) = F'(z)$. Since $C^1(\mathbb{C}) \simeq H(\mathbb{C})$ we have that, due to the previous corollary

$$\frac{\mathrm{d}F}{\mathrm{d}z} = f(z) \in H(D)$$

□

Theorem 0.37 (Cauchy Inequality)**.** *Let $f \in H(B_R(z_0))$ with $z_0 \in \mathbb{C}$. If $\|f(z)\| \leq M$ $\forall z \in B_R(z_0)$*

$$\left\| \left.\frac{\mathrm{d}f}{\mathrm{d}z}\right|_{z_0} \right\| \leq \frac{n!M}{R^n} \tag{77}$$

Proof. Take $\gamma_r(\theta) = z_0 + re^{i\theta}$ with $\theta \in [0, 2\pi]$, $r > R$, then the derivative $\left.\frac{\mathrm{d}^n f}{\mathrm{d}z^n}\right|_{z_0}$ can be written using the Cauchy integral representation, since $f \in H(B_r(z_0))$

$$\left.\frac{\mathrm{d}^n f}{\mathrm{d}z^n}\right|_{z_0} = \frac{n!}{2\pi i} \oint_{\gamma_r} \frac{f(w)}{(w - z_0)^{n+1}} \,\mathrm{d}w$$

Using the Darboux inequality we have then

$$\left\| \left.\frac{\mathrm{d}^n f}{\mathrm{d}z^n}\right|_{z_0} \right\| \leq \frac{n!}{r^n} \sup_{z \in \{\gamma_r\}} \|f(z)\| \leq \frac{n!M}{r^n}$$

Since $r < R$ we therefore have

$$\left\| \left.\frac{\mathrm{d}^n f}{\mathrm{d}z^n}\right|_{z_0} \right\| \leq \frac{n!M}{R^n}$$

□

Theorem 0.38 (Liouville)**.** *Let $f : \mathbb{C} \longrightarrow \mathbb{C}$ a function such that $f \in H(\mathbb{C})$, i.e. whole. If $\exists M > 0 : \|f(z)\| \leq M$ $\forall z \in \mathbb{C}$ the function $f(z)$ is constant*

Proof. $f \in H(\mathbb{C})$, $\|f(z)\| \leq M$ and we can write, taken $\gamma_R(\theta) = z + Re^{i\theta}$ with $\theta \in [0, 2\pi]$

$$f'(z) = \frac{1}{2\pi i} \oint_{\gamma_R} \frac{f(w)}{(w - z)^2} \,\mathrm{d}z$$

For Darboux

$$\|f'(z)\| \leq \frac{1}{2\pi} \left\| \oint_{\gamma_R} \frac{f(w)}{(w - z)^2} \right\| \mathrm{d}z \leq \frac{\sup_{z \in \{\gamma_R\}} \|f(z)\|}{R} \leq \frac{M}{R}$$

Since $R > 0$ is arbitrary, we can say directly that $\|f'(z)\| = 0$ and therefore $f(z)$ is constant $\forall z \in \mathbb{C}$. □

Theorem 0.39 (Fundamental Theorem of Algebra)**.** *Take a polynomial $P_n(z) \in \mathbb{C}_n[z]$, where $\mathbb{C}_n[z]$ is the space of complex polynomials with variable z and degree n. If we have*

$$P_n(z) = \sum_{k=0}^{n} a_k z^k, \quad z, a_k \in \mathbb{C},\ a_n \neq 0$$

We can say that $\exists z_0 \in \mathbb{C} : P_n(z_0) = 0$

Proof. As an absurd, say that $\forall z \in \mathbb{C},\ P_n(z) \neq 0$. Then $f(z) = 1/P_n(z) \in H(\mathbb{C})$.
Since $\lim_{z\to\infty} P_n(z) = \infty$, we have that $\|f(z)\| \le M\ \forall z \in \mathbb{C}$, and $\lim_{z\to\infty} f(z) = 0$.
Therefore $\exists R > 0 : \forall \|z\| > R,\ \|f(z)\| < 1$. Since $f \in H(\mathbb{C})$, we have that $f \in C(\overline{B}_R(z))$. Due to the Liouville theorem we have that $f(z)$ is constant $\lightning$ $\square$

§ 0.5 Integral Theorems in $\mathbb{R}^2$ and $\mathbb{R}^3$

Theorem 0.40 (Gauss-Green). *Given $D \subset \mathbb{R}^2$ a set with a piecewise smooth parameterization of ∂D and two functions $\alpha, \beta : A \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$ and $\overline{D} \subset A$*

$$\iint_D \partial_x \beta \,\mathrm{d}x\,\mathrm{d}y = \int_{\partial^+ D} \beta(x,y)\,\mathrm{d}y, \quad \iint_D \partial_y \alpha \,\mathrm{d}x\,\mathrm{d}y = -\int_{\partial^D} \alpha(x,y)\,\mathrm{d}x\,\mathrm{d}y \tag{78}$$

Theorem 0.41 (Stokes). *Given $D \subset \mathbb{R}^2$ an open set with ∂D piecewise smooth and a vector field $f^\mu : A \longrightarrow \mathbb{R}^2$ with $D \subset A$*

$$\int_D \epsilon_{3\mu\nu} \partial^\mu f^\nu \,\mathrm{d}x\,\mathrm{d}y = \int_{\partial^+ D} f^\mu t_\mu \,\mathrm{d}s \tag{79}$$

Where t^μ is the vector tangent to $\partial^+ D$

Theorem 0.42 (Gauss 1). *Given $D \subset \mathbb{R}^n$ open set with ∂D piecewise smooth and a vector field $f^\mu : A \longrightarrow \mathbb{R}^n$ with $D \subset A$*

$$\iiint_D \partial_\mu f^\mu \,\mathrm{d}x\,\mathrm{d}y = \int_{\partial^+ D} f^\mu n_\mu \,\mathrm{d}s \tag{80}$$

Where n^μ is the normal vector to $\partial^+ D$

Theorem 0.43 (Stokes for Surfaces). *Given a smooth surface $\Sigma \subset \mathbb{R}^3$ with parameterization r^μ and a vector field $f^\mu : A \longrightarrow \mathbb{R}^3$ with $\Sigma \subseteq A$*

$$\int_\Sigma n^\mu \epsilon_{\mu\nu\gamma} \partial^\nu f^\gamma \,\mathrm{d}\sigma = \int_{\partial^+ \Sigma} f^\mu t_\mu \,\mathrm{d}s \tag{81}$$

Where t^μ is the tangent vector to the border of the surface

Theorem 0.44 (Useful Identities). *Given $u, v \in C^2(\Omega)$ and a vector field $f^\mu \in C^2(\Omega, \mathbb{R}^3)$*

$$\begin{aligned}
\int_\Omega \partial_\mu \partial^\mu v \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z &= \int_{\partial\Omega} n^\mu \partial_\mu v \,\mathrm{d}\sigma \\
\int_\Omega u \partial_\mu f^\mu \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z &= -\int_\Omega f^\mu \partial_\mu w \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z + \int_{\partial\Omega} u f^\mu n_\mu \,\mathrm{d}\sigma \\
\int_\Omega u \partial_\mu \partial^\mu v \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z &= -\int_\Omega \partial_\mu u \partial^\mu v \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z + \int_{\partial\Omega} u n^\mu \partial_\mu v \,\mathrm{d}\sigma \\
\int_\Omega (u \partial_\mu \partial^\mu v - w \partial_\mu \partial^\mu u) \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z &= \int_{\partial\Omega} (u n^\mu \partial_\mu v - w n^\mu \partial_\mu u) \,\mathrm{d}\sigma
\end{aligned} \tag{82}$$

We can analogously write these theorems in the language of differential forms and manifolds, after giving a couple of definitions

Definition 0.5.1 (Volume Element)**.** Given a $k-$dimensional compact oriented manifold M with boundary and $\omega \in \Lambda^k(M)$ a $k-$differential form on M, we define the *volume* of M as follows

$$V(M) = \int_M \mathrm{d}V = \int_M \omega \tag{83}$$

Where $\mathrm{d}V$ is the *volume element* of the manifold, given by the unique $\omega \in \Lambda^k(M)$, defined as follows

$$\omega = f \, \mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_k} \tag{84}$$

With f an unique function.
For $M \subset \mathbb{R}^3$ with n^μ as outer normal and $\omega \in \Lambda^2(M)$ we can write immediately, by definition

$$\omega_{\mu\nu} v^\mu w^\nu = n^\mu \epsilon_{\mu\nu\gamma} v^\nu w^\gamma = \mathrm{d}A$$

Therefore

$$\mathrm{d}A = \|\epsilon_{\mu\nu\gamma} v^\nu w^\gamma\|^\mu \tag{85}$$

Which is the already known formula.
For a $2-$manifold we can write immediately the following formulas

$$\mathrm{d}A = n^1 \, \mathrm{d}y \wedge \mathrm{d}z + n^2 \, \mathrm{d}z \wedge \mathrm{d}x + n^3 \, \mathrm{d}x \wedge \mathrm{d}y \tag{86}$$

And, on M

$$\begin{cases} n^1 \, \mathrm{d}A = \mathrm{d}y \wedge \mathrm{d}z \\ n^2 \, \mathrm{d}A = \mathrm{d}z \wedge \mathrm{d}x \\ n^3 \, \mathrm{d}A = \mathrm{d}x \wedge \mathrm{d}y \end{cases} \tag{87}$$

Theorem 0.45 (Gauss-Green-Stokes-Ostogradskij)**.** *Given M a smooth manifold with boundary, c a $p-$cube in M and $\omega \in \Lambda(M)$ we have*

$$\int_c \mathrm{d}\omega = \int_{[0,1]^p} c^\star \, \mathrm{d}\omega = \int_{\partial c} \omega \tag{88}$$

In general, we can write

$$\int_M \mathrm{d}\omega = \int_{\partial M} \omega \tag{89}$$

Definition 0.5.2 (Gauss-Green, Differential Forms)**.** Given $M \subset \mathbb{R}^2$ a compact $2-$manifold with boundary and two functions $\alpha, \beta : M \longrightarrow \mathbb{R}$ with $\alpha, \beta \in C^1(M)$ defining

$$\omega = \alpha \, \mathrm{d}x + \beta \, \mathrm{d}y \tag{90}$$

We have

$$\int_{\partial M} \alpha \, \mathrm{d}x + \beta \, \mathrm{d}y = \int_{\partial M} \omega = \int_M \mathrm{d}\omega = \iint_M \left(\frac{\partial \beta}{\partial x} - \frac{\partial \alpha}{\partial y} \right) \mathrm{d}x \wedge \mathrm{d}y \tag{91}$$

Proof. Take $\omega = \alpha \, \mathrm{d}x + \beta \, \mathrm{d}y$, then

$$\mathrm{d}\omega = \mathrm{d}(\alpha \, \mathrm{d}x + \beta \, \mathrm{d}y) = \left(\frac{\partial \beta}{\partial x} - \frac{\partial \alpha}{\partial y} \right) \mathrm{d}x \wedge \mathrm{d}y$$

□

Theorem 0.46 (Gauss, Differential Forms)**.** *Given M a $3-$manifold smooth with boundary and compact with outer normal n^μ and a vector field $f^\mu \in C^1(M)$, we have*

$$\int_M \partial_\mu f^\mu \,\mathrm{d}V = \int_{\partial M} f^\mu n_\mu \,\mathrm{d}A \tag{92}$$

Proof. Taken the following differential form

$$\omega = f^1 \,\mathrm{d}y \wedge \mathrm{d}z + f^2 \,\mathrm{d}z \wedge \mathrm{d}x + f^3 \,\mathrm{d}x \wedge \mathrm{d}y$$

We have, using the formulas (87)

$$\omega = f^\mu n_\mu \,\mathrm{d}A$$

And

$$\mathrm{d}\omega = \partial_\mu f^\mu \,\mathrm{d}V$$

Therefore

$$\int_M \partial_\mu f^\mu \,\mathrm{d}V = \int_M \mathrm{d}\omega = \int_{\partial M} \omega = \int_{\partial M} f^\mu n_\mu \,\mathrm{d}A$$

□

Theorem 0.47 (Stokes, Differential Forms)**.** *Given $M \subset \mathbb{R}^3$ a compact oriented smooth $2-$manifold with boundary with n^μ as outer normal and t^μ as tangent vector in ∂M, given a vector field $f^\mu \in C^1(A)$ where $M \subset A$, we have*

$$\int_M n^\mu \epsilon_{\mu\nu\gamma} \partial^\nu f^\gamma \,\mathrm{d}A = \int_{\partial M} f^\mu t_\mu \,\mathrm{d}s \tag{93}$$

Proof. Taking the following differential form

$$\omega = f^\mu \,\mathrm{d}x_\mu$$

We have that

$$\mathrm{d}\omega = (\partial_2 f^3 - \partial_3 f^2) \,\mathrm{d}y \wedge \mathrm{d}z + (\partial_3 f^1 - \partial_1 f^3) \,\mathrm{d}z \wedge \mathrm{d}x + (\partial_1 f^2 - \partial_2 f^1) \,\mathrm{d}x \wedge \mathrm{d}y$$

Using the formulas (87) we have

$$\mathrm{d}\omega = n^\mu \epsilon_{\mu\nu\gamma} \partial^\nu f^\gamma \,\mathrm{d}A$$

Since in $\mathbb{R}^2$ we have $t^\mu \,\mathrm{d}s = \mathrm{d}x^\mu$ we therefore have

$$f^\mu t_\mu \,\mathrm{d}s = f^\mu \,\mathrm{d}x_\mu = \omega$$

And therefore

$$\int_M n^\mu \epsilon_{\mu\nu\gamma} \partial^\nu f^\gamma \,\mathrm{d}A = \int_M \mathrm{d}\omega = \int_{\partial M} \omega = \int_{\partial M} f^\mu t_\mu \,\mathrm{d}s$$

□

These last formulas are a good example on how they can be generalized through the use of differential forms, bringing an easy way of calculus in $\mathbb{R}^n$ of the various integral theorems, all condensed in one formula, the *Gauss-Green-Stokes-Ostogradskij theorem*