§ 0.1 Stability and Free Oscillations

One of the most important topics of mechanics is the idea of stability and small oscillations around these "stable points". The idea of stable and unstable points stems from the main idea of equilibrium.

Definition 0.1.1 (Mechanical Equilibrium, Stability and Instability)**.** Given a system interacting with a certain potential $\mathcal{U}$, equilibrium points of such are defined as the «critical points» of the potential $\mathcal{U}$. The stability of such points is then defined by their nature, where minimums of the potential function are said to be «stable equilibrium points» and maximums as «unstable equilibrium points» of the system.

In order to evaluate the stability of equilibrium points, given the definition, immediately uses optimization theory applied on the potential function.
With this, for a system with n degrees of freedom, given a critical point q_0^μ of $\mathcal{U}(q^\mu)$, and having defined the Hessian matrix of $\mathcal{U}$ as $\partial^2_{\mu\nu}\mathcal{U} = \mathcal{U}_{\mu\nu}$ one has two main results.

1. If $\mathcal{U}_{\mu\nu}(q_0^\mu)$ is positive definite, the critical point is a minimal and therefore a stable equilibrium point
2. If $\mathcal{U}_{\mu\nu}(q_0^\mu)$ is negative definite, then it's a maximal and therefore an unstable equilibrium point

In case that the Hessian matrix is undefined, nothing can be derived through this criterion.
Note that using Sylvester's criterion for determining the signature of the eigenvalues of a matrix, in two dimensions this reduces to evaluating sign of the determinant of $\mathcal{U}_{\mu\nu}$ and the sign of the first entry of the matrix, $\mathcal{U}_{11}$, where

1. $|\mathcal{U}_{\mu\nu}| > 0$ and $\mathcal{U}_{11} > 0$ imply a stable equilibrium point
2. $|\mathcal{U}_{\mu\nu}| > 0$ and $\mathcal{U}_{11} < 0$ imply an unstable equilibrium point
3. $|\mathcal{U}_{\mu\nu}| < 0$ imply a saddle (unstable) point of equilibrium
4. $|\mathcal{U}_{\mu\nu}| = 0$ implies that $\mathcal{U}_{\mu\nu}$ is indefinite and therefore no clear conclusion can be given on the type of equilibrium at the considered point

§§ 0.1.1 Free Oscillations in 1 Dimension

Suppose having some system with one degree of freedom interacting in a field with potential $\mathcal{U}(q)$, which has q_0 as a local minimum point. Therefore we have

$$\left(\frac{\mathrm{d}\mathcal{U}}{\mathrm{d}q}\right)_{q_0} = 0 \tag{1}$$

For evaluating small oscillations we expand with a power series to the second order this potential around q_0

$$\mathcal{U}(q) \simeq \mathcal{U}(q_0) + \frac{1}{2}\left(\frac{\mathrm{d}^2\mathcal{U}}{\mathrm{d}q^2}\right)_{q_0} (q-q_0)^2 + \mathcal{O}\big((q-q_0)^3\big) \tag{2}$$

Imposing $\mathcal{U}(q_0) = 0$ and writing $\mathcal{U}''(q_0) = k$ we have that our potential can be approximated to a harmonic oscillator potential up to second order, yielding

$$\mathcal{U}(q) \approx \frac{1}{2}k(q-q_0)^2 \tag{3}$$

Substituting $q - q_0 = x$ we can then write the approximate harmonic Lagrangian as the usual harmonic oscillator Lagrangian

$$\mathcal{L}(x, \dot{x}) = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2 \tag{4}$$

Deriving the Lagrangian we get the usual Euler-Lagrange equations for the harmonic oscillator, and substituting $\omega^2 = k/m$ we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{q}} - \frac{\partial \mathcal{L}}{\partial q} = \ddot{x} + \omega^2 x = 0 \tag{5}$$

This differential equation has two possible solutions

$$\begin{aligned} x(t) &= c_1 \cos(\omega t) + c_2 \sin(\omega t) \\ x(t) &= A\cos(\omega t + \phi) \end{aligned} \tag{6}$$

Where $c_1, c_2, A, \phi \in \mathbb{R}$. The second solution can be obtained using the formula

$$\cos(\omega t + \phi) = \cos(\omega t)\cos\phi - \sin(\omega t)\sin\phi$$

Which gives us the expression of A and ϕ in terms of the two constants c_1, c_2 as follows

$$A = \sqrt{c_1^2 + c_2^2} \qquad \tan\phi = -\frac{c_2}{c_1} \tag{7}$$

A is known as teh «amplitude» of the motion, while ϕ is the initial value of the phase with frequency ω.
Writing the mechanical energy of an oscillatory system and substituting the solution $x(t)$ we have

$$E = \frac{1}{2}m\omega^2 A^2 \tag{8}$$

Which gives us $E \propto A^2$. Noting also how $x(t)$ is shaped, we can also write it using Euler's identity as the real part of a complex function, as

$$x(t) = \mathfrak{Re}\left\{ae^{i\omega t}\right\} \tag{9}$$

Where we set $a = Ae^{i\phi}$ as the complex amplitude, which has the amplitude as modulus and the phase as argument.

§ 0.2 Free Oscillations in n Degrees of Freedom

Working analogously in n dimensions, for studying these oscillations we choose a certain critical point q_0^μ of the potential, using $x^\mu = q^\mu - q_0^\mu$ we approximate the potential as

$$\mathcal{U}(x^\mu) \approx \frac{1}{2}\partial^2_{\mu\nu}\mathcal{U}(0)x^\mu x^\nu = \frac{1}{2}k_{\mu\nu}x^\mu x^\nu \tag{10}$$

The matrix $k_{\mu\nu}$ is the Hessian matrix of $\mathcal{U}$, and therefore, since $\mathcal{U} \in C^2(\mathbb{R}^n)$, $k_{\mu\nu} = k_{\nu\mu}$.
Analogously, for the kinetic energy we have

$$T = \frac{1}{2}a_{\mu\nu}(q_0^\gamma)\dot{q}^\mu\dot{q}^\nu \tag{11}$$

Writing $a_{\mu\nu}(q_0^\gamma)$ as $m_{\mu\nu}$ and using the fact that $T \in C^2(\mathbb{R}^n)$ we have the linearized Lagrangian for a system with n degrees of freedom

$$\mathcal{L}(\dot{x}^\mu, x^\mu) = \frac{1}{2}\left(m_{\mu\nu}\dot{x}^\mu\dot{x}^\nu - k_{\mu\nu}x^\mu x^\nu\right) \tag{12}$$

For finding the equations of motion of this system we write the total differential of the Lagrangian, getting

$$\mathrm{d}\mathcal{L} = m_{\mu\nu}\dot{x}^\mu \mathrm{d}\dot{x}^\nu - k_{\mu\nu}x^\mu \mathrm{d}x^\nu = \frac{\partial \mathcal{L}}{\partial \dot{x}^\nu}\mathrm{d}\dot{x}^\nu + \frac{\partial \mathcal{L}}{\partial x^\nu}\mathrm{d}x^\nu \tag{13}$$

Therefore, the searched equations of motion will be a system of ODEs, which will be

$$m_{\mu\nu}\ddot{x}^\mu + k_{\mu\nu}x^\mu = 0 \tag{14}$$

Keeping in mind what we found before for oscillating system we suppose the solution as a complex vector where

$$x^\mu(t) = A^\mu e^{i\omega t} \quad A^\mu \in \mathbb{C}^n$$

Deriving and substituting back this function we get a linear system with A^μ as our unknown vector

$$\left(-\omega^2 m_{\mu\nu} + k_{\mu\nu}\right) A^\mu = 0 \tag{15}$$

The solution of this system boils down to searching simultaneous eigenvectors and eigenvalues for the two matrices, and therefore this is possible if and only if

$$\det\left|-\omega^2 m_{\mu\nu} + k_{\mu\nu}\right| = 0 \tag{16}$$

Since both $m_{\mu\nu}, k_{\mu\nu}$ are symmetric matrices we must have that $\omega^2 \in \mathbb{R}$, i.e. that all the eigenvalues of the combined system are real. This also implies that, if Δ_a^μ is the $a-$th minor of the composite matrix $d_{\mu\nu} = -\omega^2 m_{\mu\nu} + k_{\mu\nu}$, then $A^\mu \propto \Delta_a^\mu$, which implies that the general solution therefore is

$$x_a^\mu(t) = \Delta_a^\mu C_a e^{i\omega_a t} \tag{17}$$

Where x_a^μ is the solution associated with the $a-$th eigenvalue ω_a. The complete solution will therefore be a real superposition of the previous one, which boils down to the following function

$$x^\mu(t) = \sum_{a=1}^{n} \Delta_a^\mu \mathfrak{Re}\left\{C_a e^{i\omega_a t}\right\} = \sum_{a=1}^{n} \Delta_a^\mu \Theta_a(t) \tag{18}$$

Where with $\Theta_a(t)$ we have indicated a simple periodic oscillation with frequency ω_a. These simple oscillations are called the «normal oscillations» of the system and they are linearly independent from one another, forming an orthogonal basis in which the matrix $d_{\mu\nu}$ is diagonal. Obviously they also solve the ODE

$$\ddot{\Theta}_a + \omega_a^2\Theta_a = 0 \tag{19}$$

With a basis transformation, the linearized Lagrangian (12) becomes as follows

$$\mathcal{L}\left(\dot{\Theta}, \Theta\right) = \sum_a \frac{1}{2} m_a \left(\dot{\Theta}_a^2 - \omega^2\Theta_a^2\right) \tag{20}$$

These basis vectors can be immediately normalized to a new basis, where $Q_a = \sqrt{m_a \Theta_a}$, which changes slightly our Lagrangian to the new version

$$\mathcal{L}(\dot{Q}, Q) = \frac{1}{2}\sum_{a=1}^{n}\left(\dot{Q}_a^2 - \omega_a^2 Q_a^2\right) \tag{21}$$