§ 0.1 Metric Spaces

§§ 0.1.1 Topology

Definition 0.1.1 (Metric Space)**.** Let X be a non-empty set equipped with an application d, defined as follows

$$\begin{aligned} d : X \times X &\longrightarrow \mathbb{F} \\ (x, y) &\rightarrow d(x, y) \end{aligned} \tag{1}$$

Where $\mathbb{F}$ is an ordered field.
The couple (X, d) is said to be a *metric space*, if and only if $\forall x, y, z \in X$ the application d satisfies the following properties

1. $d(x, y) \geq 0$
2. $d(x, x) = 0$
3. $d(x, y) = d(y, x)$
4. $d(x, y) \leq d(x, z) + d(z, y)$

Definition 0.1.2 (Ball)**.** Let (X, d) be a metric space. We then define the *open ball of radius* r, centered in x in X (B_r^X), and the *closed ball of radius* r centered in x $(\overline{B_r^X})$ as follows

$$\begin{aligned} B_r^X(x) &:= \{ u \in X |\, d(u, x) < r \} \\ \overline{B_r^X}(x) &:= \{ u \in X |\, d(u, x) \leq r \} \end{aligned} \tag{2}$$

When there won't be doubts on on where the ball is defined, the superscript indicating the set of reference will be omitted.

We're now ready to define the *topology* on a metric space

Definition 0.1.3 (Open Set)**.** Let (X, d) be a metric space, and $A \subseteq X$ a subset. A is said to be an *open set* if and only if

$$\forall x \in X \ \exists B_r^X(x) \subset A \tag{3}$$

Definition 0.1.4 (Complementary Set)**.** Let A be a generic set, then the set A^{c} is defined as follows

$$A^{\mathrm{c}} := \{ a \notin A \} \tag{4}$$

This set is said to be the *complementary set* of A.
It's also obvious that $A \cap A^{\mathrm{c}} = \{\}$

Definition 0.1.5 (Closed Set)**.** Alternatively to the notion of open set, we can say that $E \subseteq X$ is a *closed set*, if and only if

$$\forall x \in E^{\mathrm{c}} \cap X \ \exists B_r^X(x) \subset E^{\mathrm{c}} \cap X \tag{5}$$

Remark. A set isn't necessarily open nor closed!

Proposition 1. 1. The set $B_r^X(x)$ is open

2. The set $\overline{B_r^X}(x)$ is closed

Proof. Let $A = B_r^X(x)$. If A is open, we have therefore, applying the definition of open set, that

$$\forall x \in A\ \exists \epsilon > 0\ :\ B_\epsilon^X(x) \subset A$$

So

$$x_0 \in A \implies d(x, x_0) < r$$
$$\therefore \epsilon = r - d(x, x_0) > 0$$

Then, by definition of open ball we have

$$y \in B_\epsilon^X(x) \implies d(x, y) < \epsilon$$

Then, we can say that

$$d(y, x_0) \leq d(y, x) + d(x, x_0) < \epsilon + d(x, x_0) = r$$
$$\therefore y \in B_\epsilon^X(x) \implies y \in B_r^X(x_0) \subset A$$

The demonstration of the second point is exactly the same, whereby we take E as our closed ball and $A = E^c$ □

Proposition 2. Let (X, d)

1. The sets $\{\}, X$ are open
2. The sets $\{\}, X$ are closed
3. If $\{A_i\}_{i=1}^n$ is a collection of open sets, then $A = \bigcap_{i=1}^n A_i$ is open
4. If $\{C_i\}_{i=1}^n$ is a collection of closed sets, then $C = \bigcup_{i=1}^n C_i$ is closed
5. Let $I \subset \mathbb{N}$ be an index set, then
   (a) If $\{A_\alpha\}_{\alpha \in I}$ is a collection of open sets, then $B = \bigcup_{\alpha \in I} A_\alpha$ is open
   (b) If $\{C_\alpha\}_{\alpha \in I}$ is a collection of closed sets, then $D = \bigcap_{\alpha \in I} C_\alpha$ is closed

Proof. The first two statements are of easy proof. Let $B_\epsilon^X \subset \{\}$. This means that B_ϵ^X is empty and therefore $B_\epsilon^X = \{\}$, which makes it open by definition. Therefore we have that $\{\}^c = X$, and X must be closed, but if we reason a bit, we can say that $\forall x \in X\ B_\epsilon^X(x) \subset X$, which means that X is open, thus $X^c = \{\}$ must be closed.
Since we gave a proof for $\{\}$ and X being open, we have that these two sets are both open and closed. These two sets are said to be *clopen*.
For the other statements we use the De Morgan laws on set calculus, therefore we have

$$x \in \bigcap_{i=1}^n A_i \implies x \in A_i$$
$$\therefore \exists \epsilon_i\ :\ B_{\epsilon_i}^X(x) \subset A_i$$

Taking $\epsilon = \min_{i \in I} \epsilon_i$ we have

$$B_\epsilon^X(x) \subset B_\epsilon^X(x) \implies B_\epsilon^X(x) \subset \bigcap_{i=1}^n A_i = A$$

And A is open
If we let $C = A^c$ we have that

$$C = A^c = \left(\bigcap_{i=1}^{n} A_i\right)^c = \bigcup_{i=1}^{n} A_i^c$$
$$\therefore C \text{ is closed}$$

For the last two we proceed as follows

$$x \in A_\alpha \implies \exists \alpha_0 \in I : x \in A_{\alpha_0}$$
$$\therefore \exists \epsilon > 0 : B_\epsilon^X(x) \subset A_{\alpha_0} \subset \bigcup_{\alpha \in I} = B$$

For the last one, we use the De Morgan laws and the proposition is demonstrated □

Definition 0.1.6 (Internal Points, Closure, Border)**.** Let (X, d) be a metric space and $A \subset X$ a subset.
We define the following sets from A

1. $A^\circ = \bigcup_{\alpha \in I} G_\alpha$ is the set of internal points of A, where I is an index set and $G_\alpha \subset A$ are open
2. $\overline{A} = \bigcap_{\beta \in J} F_\beta$ is the closure of A, where J is another index set and $F_\beta \subset A$ are closed
3. $\partial A = \overline{A} \setminus A^\circ = \overline{A} \cup (A^\circ)^c$ is the border of A

Proposition 3. 1. A is an open set iff $A = A^\circ$

2. A is closed iff $A = \overline{A}$
3. $A^\circ = \overline{(A^\circ)}^c$
4. $\overline{A} = \left[(A^c)^\circ\right]^c$
5. $(A \cap B)^\circ = A^\circ \cap B^\circ$
6. $\overline{A \cap B} = \overline{A} \cup \overline{B}$

Proof. Let $\mathcal{O}(A)$ be a collection of open sets, such that $\forall G \in \mathcal{O}(A) \implies G \subset A$, then

$$A = A^\circ \implies A = \bigcup_{G \in \mathcal{O}(A)} G$$

Therefore, being a union of a finite number of open sets, A is open.
For the same reason as before and the previous proposition, we have that $\overline{A}$ is closed
For the third proposition, we have

$$\left(\overline{A^c}\right)^c = \left(\bigcap_{A^c \subset F} F\right)^c = \bigcup_{A^c \subset F} F^c = \bigcup_{G \in \mathcal{O}(A)} G = A^\circ$$

The others are easily demonstrated throw this process, iteratively □

Proposition 4. Let (X, d) be a metric space, and $A \subset X$, $x \in X$

1. $x \in A \iff \exists \epsilon > 0 : B_\epsilon(x) \subset A$
2. $x \in \overline{A} \iff \forall \epsilon > 0 \; B_\epsilon(x) \cap A \neq \{\}$
3. $x \in \partial A \iff \forall \epsilon > 0 \; B_\epsilon(x) \cap A \neq \{\} \wedge B_\epsilon(x) \cap \overline{A} \neq \{\}$

Proof. $\boxed{1}$ Let $I(A) := \{x \in X | \, \exists \epsilon > 0 : B_\epsilon(x) \subset A$.

$$x \in I(A) \implies \exists \epsilon > 0 : B_\epsilon(x) \subset A, \; \therefore x \in \bigcup_{G \subset A} G$$

But

$$\begin{aligned} &x \in A^\circ \implies \exists G \subset X \text{ open} : x \in G \implies \exists \epsilon > 0 : B_\epsilon(x) \subset G \subset A \\ &\therefore A^\circ \subset I(A) \ni x, \; I(A) \subset A \text{ by definition}, \; \therefore I(A) = A^\circ \end{aligned}$$

$\boxed{2}$ For the second proposition, we have

$$\begin{aligned} &\overline{A} = \left[(A^c)^\circ \right]^c \implies x \in A \iff x \in (A^c)^\circ \implies \forall \epsilon > 0 \; B_\epsilon(x) \not\subset A^c \\ &\therefore \forall \epsilon > 0 \; B_\epsilon(x) \cap A \neq \{\} \end{aligned}$$

$\boxed{3}$ For the last one, we have, taking into account the first two proofs

$$\begin{aligned} &x \in \partial A \iff x \in \overline{A} \setminus A^\circ \implies x \in \overline{A} \wedge x \notin A^\circ \\ &\boxed{1} \wedge \boxed{2} \implies x \in \overline{A} \iff \forall \epsilon > 0 \; B_\epsilon(x) \cap A \neq \{\} \\ &\therefore x \notin A^\circ \iff \forall \epsilon > 0 \; B_\epsilon(x) \cap A^c \neq \{\} \end{aligned}$$

□

Definition 0.1.7 (Isometry)**.** Let $(X, d), (Y, \rho)$ be two metric spaces and f an application, defined as follows

$$f : (X, d) \to (Y, d)$$

f is said to be an *isometry* iff

$$\forall x_1, x_2 \in X, \; \rho(f(x_1), f(x_2)) = d(x_1, x_2)$$

Remark. If f is an isometry, then f is injective, but it's not necessarily surjective

Example 0.1.1. Let $X = [0, 1]$ and $Y = [0, 2]$, therefore

$$\begin{aligned} f : &[0, 1] \to [0, 2] \\ &x \to f(x) = x \end{aligned}$$

f is obviously an isometry, since, for $x, y \in [0, 1]$

$$d(f(x), f(y)) = d(x, y)$$

But it's obviously not surjective.

Definition 0.1.8 (Diameter of a Set)**.** Let A be a set and the couple (A, d) be a metric space. We define the *diameter* of A as follows

$$\operatorname{diam}(A) = \sup_{x, y \in A} (d(x, y))$$

§ 0.2 Convergence and Compactness

Definition 0.2.1 (Convergence)**.** Let (X,d) be a metric space and $x \in X$. A sequence $(x_k)_{k\geq 0}$ in X is said to converge in X and it's indicated as $x_k \to x \in X$, iff

$$\forall \epsilon > 0\ \exists N > 0 : \forall k \geq N,\ d(x_k, x) < \epsilon\ \therefore \lim_{k\to\infty} x_k = x$$

Theorem 0.1 (Unicity of the Limit)**.** *Let (X,d) be a metric space and $(x_k)_{k\geq 0}$ a sequence in X. If $x_k \to x \wedge x_k \to y$, then $x = y$*

Definition 0.2.2 (Adherent point)**.** Let (X,d) be a metric space and $A \subset X$. $x \in X$ is said to be an *adherent point* of A if $\exists (x_k)_{k\geq 0} \in A : x_k \to x \in X$. The set of all adherent points of A is called $\operatorname{ad}(A)$

Definition 0.2.3 (Accumulation point)**.** Let (X,d) be a metric space and $A \subset X$. $x \in X$ is an *accumulation point* of A, or also *limit point* of A if $\exists (x_k)_{k\geq 0} : x_k \neq x \wedge x_k \to x \in \operatorname{ad}(A)$

Proposition 5. Let (X,d) be a metric space and $A \subset X$, then $\overline{A} = \operatorname{ad}(A)$

Proof. Let $Y = \operatorname{ad}(A)$, then

$$\begin{aligned} &x \in \overline{A} \implies \forall \epsilon > 0\ B_\epsilon(x) \cap A \neq \{\} \\ &\therefore \forall n \in \mathbb{N}\ B_{\frac{1}{n}}(x) \cap A \neq \{\} \implies \forall n \in \mathbb{N}\ \exists x_n \in B_{\frac{1}{n}}(x) \end{aligned}$$

But $d(x, x_n) < n^{-1}$, therefore $x \in Y \implies x \in \operatorname{ad}(A)$, and by definition

$$\begin{aligned} &\exists (x_n)_{n\geq 0} : \forall \epsilon > 0\ \exists N \in \mathbb{N} : \forall k \geq N\ d(x_k, x) < \epsilon \implies x_N \in B_\epsilon(x)\ \therefore x_N \in A \\ &\therefore \forall \epsilon > 0\ x_N \in B_\epsilon(x) \cap A \neq \{\} \implies x \in \overline{A} \implies Y \subset \overline{A},\ \therefore Y = \operatorname{ad}(A) = \overline{A} \end{aligned}$$

□

Proposition 6. Let (X,d) be a metric space and $A \subset X$. Then A is closed iff $\exists (x_k)_{k\geq 0} \in A : x_k \to x \in \overline{A} \implies \operatorname{ad}(A) \subset A$

Definition 0.2.4 (Dense Set)**.** Let (X,d) be a metric space and $A, B \subset X$. A is said to be dense in B iff $B \subset \overline{A}$, therefore $\forall \epsilon > 0\ \exists y \in A : d(x,y) < 0$. One example for this is $\mathbb{Q} \subset \mathbb{R}$, with the usual euclidean distance defined through the modulus.

Definition 0.2.5. Let (X,d) be a metric space and $(x_k)_{k\geq 0} \in X$. The sequence x_k is said to be a *Cauchy sequence* iff

$$\forall \epsilon > 0\ \exists N > 0 : \forall k, n \geq N\ d(x_k, x_n) < \epsilon$$

Proposition 7. Let (X,d) be a metric space and $(x_k)_{k\geq 0} \in X$ a sequence. Then, if $x_k \to x$, x_k is a Cauchy sequence

Definition 0.2.6 (Complete Space)**.** Let (X,d) be a metric space. (X,d) is said to be *complete* iff $\forall (x_k)_{k\geq 0} \in X$ Cauchy sequences, we have $x_k \to x \in X$

Theorem 0.2 (Completeness)**.** *Let (X,d) be a metric space and $Y \subset X$. (Y,d) is complete iff $Y = \overline{Y}$ in X*

Proof. Let (Y, d) be a complete space, then

$$(x_k) \in Y \text{ Cauchy sequence } \implies \exists y \in Y \,:\, x_k \to y$$

Let $z \in \mathrm{ad}(A)$ and η_k a subsequence of x_k, then

$$\exists (\eta_k) \in Y \,:\, \eta_k \to z \implies \exists y \in Y \,:\, \eta_k \to y \;\therefore z = y \implies \mathrm{ad}(Y) \subset Y$$

Going the opposite way we have that $\mathrm{ad}(Y) = Y$ and therefore $Y = \overline{Y}$ □

Definition 0.2.7 (Compact Space)**.** A metric space (X, d) is said to be *compact* or *sequentially compact* if

$$\forall (x_k) \in X \;x_k \to x \in X, \exists (y_k) \text{ Subsequence } : \, y_k \to y \in X$$

Theorem 0.3. *Let (X, d) be a compact space. Then (X, d) is also complete*

Proof. (X, d) is compact, therefore

$$\forall (x_k) \in X \text{ Cauchy sequence } \implies x_k \to x \in X$$

Taken $(x_{n_k})_k \in X$ a subsequence, we have

$$x_k \to x \implies x_{n_k} \to x \in X$$

□

Definition 0.2.8 (Completely Bounded)**.** Let (X, d) be a metric space. X is *totally bounded* iff

$$\exists Y \subset X \,:\, \forall \epsilon > 0, \forall x \in Y \;X = \bigcup_{i=1}^{n} B_\epsilon(x)$$

Definition 0.2.9 (Poligonal Chain)**.** Let $z, w \in \mathbb{C}$. We define a *polygonal* $[z, w]$ as follows

$$[z, w] := \{\, z, w \in \mathbb{C} |\; z + t(w - z),\;\; t \in [0, 1] \subset \mathbb{R}\}$$

A *polygonal chain* will be indicated as follows $P_{z,w}$ and it's defined as follows

$$P_{z,w} = \bigcup_{k=1}^{n-1} [z_k, z_{k+1}] = [z, z_1, \cdots, z_{n-1}, w]$$

It can also be defined analoguously for every metric space $(X, d) \neq (\mathbb{C}, \|\cdot\|)$, where $\|\cdot\| : \mathbb{C} \to \mathbb{R}$ is the usual complex norm $\|z\| = \sqrt{z\overline{z}} = \sqrt{\mathrm{Re}(z)^2 + \mathrm{Im}(z)^2}$

Definition 0.2.10 (Connected Space)**.** Let (G, d) be a metric space, G is *connected* if

$$\forall z, w \in G \;\exists P_{z,w} \subset G$$

Definition 0.2.11 (Contraction Mapping)**.** Let (X, d) be a complete metric space. Let $T : X \longrightarrow X$. T is said to be a *contraction mapping* or *contractor* if

$$\forall x, y \in X \;\exists q \in [0, 1) \,:\, d(T(x), T(y)) \leq q d(x, y) \tag{6}$$

Note that a contractor is necessarily continuous.

Theorem 0.4 (Banach Fixed Point)**.** *Let (X, d) be a complete metric space, with $X \neq \{\}$ and equipped with a contractor $T : X \longrightarrow X$. Then*

$$\exists! x^\star \in X \,:\, T(x^\star) = x^\star \tag{7}$$

Proof. Take $x_0 \in X$ and a sequence $x_n : \mathbb{N} \longrightarrow X$, where

$$x_n = T(x_{n-1}), \quad \forall n \in \mathbb{N}$$

It's obvious that

$$d(x_{n+1}, x_n) = d(T(x_n), T(x_{n-1})) \leq q d(x_n, x_{n-1}) \leq q^n d(x_1, x_0)$$

We need to prove that x_n is a Cauchy sequence. Let $m, n \in \mathbb{N} \,:\, m > n$, then

$$d(x_m, x_n) \leq d(x_m, x_{m-1}) + \cdots + d(x_{n+1}, x_n) \leq q^{m-1} d(x_1, x_0) + \cdots + q^n d(x_1, x_0)$$

Regrouping, we have

$$d(x_m, x_n) \leq q^n d(x_1, x_0) \sum_{k=0}^{m-n-1} q^k \leq q^n d(x_1, x_0) \sum_{k=0}^{\infty} q^k = q^n d(x_1, x_0) \left(\frac{1}{1-q} \right)$$

By definition of convergence, we have then

$$\forall \epsilon > 0, \ \exists N \in \mathbb{N} \,:\, \forall n > N \ d(s_n, s) < \epsilon$$

Then

$$\frac{q^n d(x_1, x_0)}{1-q} < \epsilon \implies q^n < \frac{\epsilon(1-q)}{d(x_1, x_0)}, \quad \forall n > N$$

Therefore, after taking $m > n > N$, we have

$$d(x_m, x_n) < \epsilon$$

Therefore x_n is a Cauchy sequence. Since (X, d) is a complete metric space, this sequence must have a limit $x_n \to x^\star \in X$, but, by definition of convergence and limit, we have that by continuity

$$x^\star = \lim_{n\to\infty} x_n = \lim_{n\to\infty} T(x_{n-1}) = T\left(\lim_{n\to\infty} x_{n-1} \right) = T(x^\star)$$

This point is unique. Take $y^\star \in X$ such that $T(y^\star) = y^\star \neq x^\star$, then

$$0 < d(T(x^\star), T(y^\star)) = d(x^\star, y^\star) > q d(x^\star, y^\star) \quad \lightning$$

Therefore

$$\exists! x^\star \in X \,:\, T(x^\star) = x^\star$$

And $x^\star$ is the fixed point of the contractor T $\square$

§ 0.3 Vector Spaces

Definition 0.3.1 (Vector Space)**.** A vector space $\mathcal{V}$ over a field $\mathbb{F}$ is a set, where $\mathcal{V} \neq \{\}$ and it satisfies the following properties, $\forall u, v, w \in \mathcal{V}$ and $a, b \in \mathbb{F}$

1. $u + v \in \mathcal{V}$ sum closure
2. $av \in \mathcal{V}$ scalar closure
3. $u + v = v + u$
4. $(u + v) + w = u + (v + w)$
5. $\exists! 0 \in \mathcal{V} \, : \, u + 0 = 0 + u = u$
6. $\exists! v \in \mathcal{V} \, : \, u + v = 0 \implies v = -u$
7. $\exists! 1 \in \mathcal{V} \, : \, 1 \cdot u = u$
8. $(ab)u = a(bu) = b(au) = abu$
9. $(a + b)u = au + bu$
10. $a(u + v) = au + av$

Definition 0.3.2 (Norm)**.** Let $\mathcal{V}$ be a vector space over a field $\mathbb{F}$, then the *norm* is an application defined as follows

$$\|\cdot\| : \mathcal{V} \longrightarrow \mathbb{F}$$

Where it satisfies the following properties

1. $\|u\| \geq 0 \; \forall u \in \mathcal{V}$
2. $\|u\| = 0 \iff u = 0$
3. $\|cu\| = |c|\|u\| \; \forall u \in \mathcal{V} \; c \in \mathbb{F}$
4. $\|u + v\| \leq \|u\| + \|v\| \; \forall u, v \in \mathcal{V}$

Definition 0.3.3 (Normed Vector Space)**.** A *normed vector space* is defined as a couple $(\mathcal{V}, \|\cdot\|)$, where $\mathcal{V}$ is a vector space over a field $\mathbb{F}$.

Proposition 8. A normed vector space (NVS), is also a metric vector space (MVS) if we define our distance as follows

$$d(u, v) = \|u - v\| \; \forall u, v \in \mathcal{V}$$

Definition 0.3.4 (Vector Subspace)**.** Let $\mathcal{V}$ be a vector space and $\mathcal{U} \subset \mathcal{V}$. $\mathcal{U}$ is a *vector subspace* of $\mathcal{V}$ iff

1. $u, v \in \mathcal{U} \implies u + v \in \mathcal{U}$
2. $u \in \mathcal{U}, \; a \in \mathbb{F} \implies au \in \mathcal{U}$

Proposition 9. If $(\mathcal{V}, \|\cdot\|)$ is an normed vector space and $\mathcal{W} \subset \mathcal{V}$ is a subspace of $\mathcal{V}$, then $(\mathcal{W}, \|\cdot\|)$ is a normed vector space

Definition 0.3.5 (p-norm)**.** Let $(\mathcal{V}, \|\cdot\|_p)$ be a normed vector space. The norm $\|\cdot\|_p$ is said to be a *p-norm* if it's defined as follows

$$\|v\|_p := \left(\sum_{i=1}^{\dim(\mathcal{V})} (v_i)^p \right)^{\frac{1}{p}}, \ \forall v \in \mathcal{V}, \ \forall p \in \mathbb{N}^\star := \mathbb{N} \cup \{\pm\infty\} \tag{8}$$

Setting $p = \infty$ we have that

$$\|v\|_\infty = \max_{i \leq \dim(\mathcal{V})} |v_i| \tag{9}$$

Definition 0.3.6 (Dual Space)**.** Let $\mathcal{V}$ be a vector space over the field $\mathbb{F}$, we define a *linear functional* as an application $\varphi : \mathcal{V} \longrightarrow \mathbb{F}$ such that $\forall u, v \in \mathcal{V}$ and $c \in \mathbb{F}$

$$\begin{aligned} \varphi(u + v) &= \gamma(u) + \varphi(v) \\ \varphi(\lambda u) &= \lambda \varphi(u) \end{aligned} \tag{10}$$

Defining the sum of two linear functionals as $(\varphi_1 + \varphi_2)(v) = \varphi_1(v) + \varphi_2(v)$ we immediately see that the set of all linear functionals forms a vector space over $\mathcal{V}$, which will be called the *dual space* $\mathcal{V}^\star$.

§§ 0.3.1 Hölder and Minkowski Inequalities

Having defined p-norms, we can prove two inequalities that work with these norms, the *Minkowski inequality* and the *Hölder Inequality*

Theorem 0.5 (Hölder Inequality)**.** *Let* $p_q \in \mathbb{N}^\star$*, where*

$$\frac{1}{p} + \frac{1}{q} = 1$$

Then

$$\forall x, y \in \mathbb{R}^n \ \|x\|_p \|y\|_q \geq \sum_{k=1}^{n} |x_k y_k| \tag{11}$$

Proof. Taking $p = 1$, we have $q = \infty$, and the demonstration is obvious

$$\|x\|_p \|y\|_q = \|x\|_1 \|p\|_\infty = \max_{k \leq n} |y_k| \sum_{k=1}^{n} |x_k| \geq \sum_{k=1}^{n} |x_k y_k|$$

Else, if $p > 1$, we ave that

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q} \ \forall a, b \geq 0$$

Let

$$s = \frac{x}{\|x\|_p}, \ t = \frac{y}{\|y\|_q}$$

We have

$$\sum_{k=1}^{n} \|s\|^p = \frac{1}{\|x\|_p^p} \sum_{k=1}^{n} |x_k|^p = 1 = \sum_{k=1}^{n} |t|^q = \frac{1}{\|y\|_q^q} \sum_{k=1}^{n} |y|^p$$

Therefore

$$\sum_{k=1}^{n} |s_k t_k| \leq \frac{1}{p} \sum_{k=1}^{n} |s_k|^p + \frac{1}{q} \sum_{k=1}^{n} |t_k|^q$$

Substituting again the definitions of s, t we have

$$\sum_{i=1}^{n} |y_k x_k| = \|x\|_p \|y\|_q \sum_{k=1}^{n} |s_k t_k| \leq \|x\|_p \|y\|_q$$

□

Theorem 0.6 (Minkowski Inequality)**.** *Let* $p \geq 1$*, therefore* $\forall x, y \in \mathbb{R}^n$ *we have*

$$\|x + y\|_p \leq \|x\|_p + \|y\|_p \tag{12}$$

Proof. We begin by writing explicitly the p-norm

$$\|x + y\|_p^p = \sum_{k=1}^{n} (|x_k| + |y_k|)^p = \sum_{k=1}^{n} (|x_k| + |y|_k) (|x_k| + |y_k|)^{p-1}$$

Letting $u_k = (|x_k| + |y_k|)^{p-1}$ we have, after imposing the condition on q of the p-norm as $q(p+1) = p$ and using that the sum is Abelian, we have

$$\begin{cases} \sum_{k=1}^{n} |x_k| u_k \leq \|x\|_p \|u\|_q = \|x\|_p \left(\sum_{k=1}^{n} (|x_k| + |y_k|)^p \right)^{\frac{1}{q}} \\ \sum_{k=1}^{n} |y_k| u_k \leq \|y\|_p \|u\|_q = \|y\|_p \left(\sum_{k=1}^{n} (|x_k| + |y_k|)^p \right)^{\frac{1}{q}} \end{cases}$$

Therefore, summing and imposing that $1 - q^{-1} = p$ we have that

$$\|x + y\|_p \leq \|x\|_p + \|y\|_q$$

□