§ 0.1 Canonical Variables

The idea behind the reformulation of mechanics comes directly from the theory of differential equations. Take a second order ODE as the one following

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} = f(y, y', t)$$

This equation can be reduced of order by imposing the transformation $u(t) = \dot{y}(t)$, which reduces the previous problem to a system of 2 ODEs of the first order

$$\begin{cases} \dot{u}(t) = f(y, u, t) \\ \dot{y}(t) = u(t) \end{cases}$$

This process can also be applied to Euler-Lagrange equations, where the N differential equations of the second order can be reduced to a system of $2N$ differential equations of the first order.
Since det $\dot{\partial}_{\mu\nu}\mathcal{L} \neq 0$ we know for sure that the following differential equation can be solved

$$\dot{q}^\mu = f^\mu(q^\nu, \dot{q}^\nu, t) \tag{1}$$

The space of dinamical configurations of the system can be described by the couple $(q^\mu, \dot{q}^\mu)$, or using $\dot{\partial}_\mu \mathcal{L} = p_\mu$ and its independece with respect to q^μ, we can define a new space, called the «phase space» , spanned by the couple (q^μ, p_μ). This space is of dimension $2n$, and it's denoted here as Γ^{2n}. The two variables q^μ, p_μ are known as the «canonical variables» of the system, and will describe a motion in this phase space via a curve $\gamma^\mu(t)$ which will be determined by the solution of the appropriate equations of motion.

§ 0.2 Canonical Equations of Motion

In order to solve the previous problem and actually reduce the Euler-Lagrange equations to a lower order, we begin by differentiating the Lagrangian

$$\mathrm{d}\mathcal{L} = \frac{\partial \mathcal{L}}{\partial \dot{q}^\mu}\mathrm{d}\dot{q}^\mu + \frac{\partial \mathcal{L}}{\partial q^\mu}\mathrm{d}q^\mu = \dot{p}_\mu \mathrm{d}q^\mu + p_\mu \mathrm{d}\dot{q}^\mu \tag{2}$$

Rewriting $p_\mu \mathrm{d}\dot{q}^\mu = \mathrm{d}\left(p_\mu \dot{q}^\mu\right) - \dot{q}^\mu \mathrm{d}p_\mu$, where we treat p_μ as an independent variable, we have

$$\mathrm{d}\left(p_\mu \dot{q}^\mu - \mathcal{L}\right) = \dot{q}^\mu \mathrm{d}p_\mu - \dot{p}_\mu \mathrm{d}q^\mu \tag{3}$$

The function on the left is known as «Hamiltonian» of the system, and corresponds to the generalized energy in canonical coordinates. It's indicated as $\mathcal{H}$, and differentiating we get ***Hamilton's equations of motion*** also known as the ***canonical equations of motion***

$$\begin{cases} \dfrac{\partial \mathcal{H}}{\partial p_\mu} = \dot{q}^\mu \\ \dfrac{\partial \mathcal{H}}{\partial q^\mu} = -\dot{p}_\mu \end{cases} \tag{4}$$

Integrating the differential on the left we can write

$$\mathcal{H}(p_\mu, q^\mu, t) = p_\mu \dot{q}^\mu - \mathcal{L}(q^\mu, q^\mu, t) \tag{5}$$

Where $\dot{q}^\mu = f^\mu(p_\mu, q^\mu, t)$. This process is called the «Legendre transformation» of the Lagrangian with respect to $\dot{q}^\mu$.
The previous equations (4) define the motion of the system in the phase space and are the searched reduction of the Euler-Lagrange equation from n ODEs of the second order to $2n$ ODEs of the first order.
Note that since in the phase space the Hamiltonian corresponds to the mechanical energy of the system, we can rewrite some theorems in a different way

Theorem 0.1 (Conservation of Energy)**.** ***The mechanical energy of the system E is conserved if the Hamiltonian function is independent from time, i.e.***

$$\frac{\partial \mathcal{H}}{\partial t} = 0 \implies \frac{\mathrm{d}E}{\mathrm{d}t} = 0$$

Proof. By definition, the Hamiltonian function of the system corresponds to the energy in the phase space, so we can immediately write its total derivative with respect to time

$$\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t} = \frac{\partial \mathcal{H}}{\partial t} + \frac{\partial \mathcal{H}}{\partial p_\mu}\dot{p}_\mu + \frac{\partial \mathcal{H}}{\partial q^\mu}\dot{q}^\mu \tag{6}$$

Substituting the canonical equations inside the expression we get

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t} = \frac{\partial \mathcal{H}}{\partial t}$$

Therefore

$$\frac{\partial \mathcal{H}}{\partial t} = 0 = \frac{\mathrm{d}E}{\mathrm{d}t}$$

□

Exercise 0.2.1 (Hamiltonians)**.** Find the Hamiltonian of a particle in

1. Cartesian coordinates
2. Cylindrical coordinates
3. Spherical coordinates

1) We begin by writing explicitly the Lagrangian for a particle in Cartesian coordinates.

$$\mathcal{L}(x^\mu, \dot{x}^\mu, t) = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - \mathcal{U}(x, y, z) \tag{7}$$

The canonical coordinates will be defined by taking the derivative with respect to the dotted coordinates, giving

$$\dot{\partial}_\mu \mathcal{L} = m\dot{x}_\mu \implies \dot{x}_\mu(p_\mu) = \frac{p_\mu}{m} \tag{8}$$

The kinetic counterpart transforms as $\dot{x}^\mu(p_\mu)\dot{x}_\mu(p_\mu)$, getting

$$\dot{x}^\mu \dot{x}_\mu = \frac{1}{m^2} p^\mu p_\mu$$

And the Hamiltonian will be

$$\mathcal{H}(p_\mu, q^\mu, t) = p_\mu \frac{p^\mu}{m} - \frac{m}{2}\frac{1}{m^2} p^\mu p_\mu + \mathcal{U}(x^\mu) \tag{9}$$

Which, simplified becomes the searched Hamiltonian

$$\mathcal{H}(p_\mu, q^\mu, t) = \frac{1}{2m} p^\mu p_\mu + \mathcal{U}(x^\mu) = \frac{1}{2m}(p_x^2 + p_y^2 + p_z^2) + \mathcal{U}(x, y, z) \tag{10}$$

2) Analogously, for cylindrical coordinates we have

$$\mathcal{L} = \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\theta}^2 + \dot{z}^2 \right) - \mathcal{U}(r, \theta, z) \tag{11}$$

The conjugated coordinates will therefore be

$$\dot{\partial}_\mu \mathcal{L} = \begin{pmatrix} m\dot{r} & mr^2\dot{\theta} & m\dot{z} \end{pmatrix} \implies \dot{x}_\mu = \frac{1}{m} \begin{pmatrix} p_r & \frac{p_\theta}{r^2} & p_z \end{pmatrix} \tag{12}$$

The Hamiltonian will be

$$\mathcal{H} = \frac{1}{m}\left(p_r^2 + \frac{p_\theta^2}{r^2} + p_z^2 \right) - \frac{m}{2}\left(\frac{p_r^2}{m^2} + \frac{r^2 p_\theta^2}{m^2 r^4} + \frac{p_z^2}{m^2} \right) + \mathcal{U}(r, \theta, z)$$

I.e.

$$\mathcal{H} = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2} + p_z^2 \right) + \mathcal{U}(r, \theta, z) \tag{13}$$

3) The Lagrangian is

$$\mathcal{L} = \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2\theta \dot{\varphi}^2 \right) - \mathcal{U}(r, \theta, \varphi) \tag{14}$$

The canonical coordinates are

$$p_\mu = \begin{pmatrix} m\dot{r} & mr^2\dot{\theta} & mr^2\sin^2\theta\dot{\varphi} \end{pmatrix}, \implies \dot{x}_\mu = \frac{1}{m} \begin{pmatrix} p_r & \frac{p_\theta}{r^2} & \frac{p_\varphi}{r^2\sin^2\theta} \end{pmatrix} \tag{15}$$

Substituting into the Legendre transform we have

$$\frac{1}{m}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\varphi^2}{r^2\sin^2\theta} \right) - \frac{1}{2} m \left(\frac{p_r^2}{m^2} + \frac{r^2 p_\theta^2}{m^2 r^4} + \frac{r^2 \sin^2\theta p_\varphi^2}{m^2 r^4 \sin^4\theta} \right) + \mathcal{U}(r, \theta, \varphi) \tag{16}$$

And therefore the Hamiltonian is

$$\mathcal{H} = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\varphi^2}{r^2\sin^2\theta} \right) + \mathcal{U}(r, \theta, \varphi) \tag{17}$$

§ 0.3 Hamilton-Jacobi Equation and Hamilton's Principle in Γ^{2n}

The principle of least action can be reformulated in Hamiltonian mechanics in a particular manner changing the boundary conditions for the variational principle, and considering the action as a function of coordinates.
Begin by considering that the path $q^\mu(t)$ will start from a fixed point $q^\mu(t_1) = q_1^\mu$ and ends in some unknown point $q^\mu(t_2)$. The boundary conditions for the variational principle will therefore be

$$\begin{cases} \delta q^\mu(t_1) = 0 \\ \delta q^\mu(t_2) = \delta q^\mu \end{cases}$$

Where $\delta q^\mu(t_1) = 0$ since q_1^μ is a constant vector.
The variation of the action integral will be, as usual

$$\delta \mathcal{S} = \left[\frac{\partial \mathcal{L}}{\partial \dot{q}^\mu} \delta q^\mu \right]_{t_2}^{t_1} + \int_{t_1}^{t_2} \left(\frac{\partial \mathcal{L}}{\partial q^\mu} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial \mathcal{L}}{\partial \dot{q}^\mu} \right) \delta q^\mu \mathrm{d}t$$

Imposing the obvious condition that $q^\mu(t)$ must represent a physical motion, the integral must be 0, since the Euler-Lagrange equations are automatically solved. Evaluating the term on the left we obtain the variation of the action as

$$\delta \mathcal{S} = \frac{\partial \mathcal{L}}{\partial \dot{q}^\mu} \delta q^\mu = p_\mu \delta q^\mu \tag{18}$$

This implies immediately that

$$\frac{\partial \mathcal{S}}{\partial q^\mu} = p_\mu$$

Now, considering $\mathcal{S} = \mathcal{S}(q^\mu, t)$ we also must have

$$\mathrm{d}\mathcal{S} = \frac{\partial \mathcal{S}}{\partial q^\mu} \mathrm{d}q^\mu + \frac{\partial \mathcal{S}}{\partial t} \mathrm{d}t = \mathcal{L} \mathrm{d}t$$

Or, substituting, we have

$$\mathrm{d}\mathcal{S} = p_\mu \mathrm{d}q^\mu + \frac{\partial \mathcal{S}}{\partial t} = \mathcal{L} \mathrm{d}t$$

Dividing by $\mathrm{d}t$, we have

$$\frac{\mathrm{d}\mathcal{S}}{\mathrm{d}t} = p_\mu \dot{q}^\mu + \frac{\partial \mathcal{S}}{\partial t} = \mathcal{L}$$

And rearranging in terms of $\partial_t \mathcal{S}$

$$\frac{\partial \mathcal{S}}{\partial t} = \mathcal{L} - p_\mu \dot{q}^\mu$$

Substituting inside the definition of the Hamiltonian function, we have

$$\frac{\partial \mathcal{S}}{\partial t} = -\mathcal{H} \tag{19}$$

This equation is called the ***Hamilton-Jacobi equation***. Rewriting the differential of the action, we have

$$\mathrm{d}\mathcal{S}(q^\mu, t) = p_\mu \mathrm{d}q^\mu - \mathcal{H} \mathrm{d}t$$

The previous Hamilton-Jacobi equation, if solved, imposes that the action as a function of (q^μ, t) must be a total differential. With this consideration, one can reformulate the principle of least action in Hamiltonian mechanics in a new and elegant way, where now the variation is made on a path in Γ^{2n}, the phase space.

$$\mathcal{S}[q^\mu(t)] = \int_{t_1}^{t_2} \left(p_\mu \mathrm{d}q^\mu - \mathcal{H}(p_\mu, q^\mu, t)\mathrm{d}t\right) \tag{20}$$

Imposing the usual conditions on the variation of the coordinates q^μ that were used already in the chapter on Lagrangian mechanics, we have

$$\delta\mathcal{S} = \int_{t_1}^{t_2} \left(\delta p_\mu \mathrm{d}q^\mu + p_\mu \mathrm{d}\delta q^\mu - \delta\mathcal{H}\mathrm{d}t\right)$$
$$\delta\mathcal{S} = \left[p_\mu \delta q^\mu\right]_{t_1}^{t_2} + \int_{t_1}^{t_2} \left(\delta p_\mu \mathrm{d}q^\mu + \delta q^\mu \mathrm{d}p_\mu - \frac{\partial\mathcal{H}}{\partial p_\mu}\delta p_\mu - \frac{\partial\mathcal{H}}{\partial q^\mu}\delta q^\mu\right)$$

Rearranging the terms and noting that the first term goes to zero we have

$$\delta\mathcal{S} = \int_{t_1}^{t_2} \delta p_\mu \left(\mathrm{d}q^\mu - \frac{\partial\mathcal{H}}{\partial p_\mu}\mathrm{d}t\right) - \int_{t_1}^{t_2} \delta q^\mu \left(\mathrm{d}p_\mu + \frac{\partial\mathcal{H}}{\partial q^\mu}\right) \tag{21}$$

The condition $\delta\mathcal{S} = 0$ imposes that both the integrals must be zero simultaneously, and since $\delta p_\mu, \delta q^\mu \neq 0$ in general, we must have

$$\begin{cases} \mathrm{d}q^\mu - \dfrac{\partial\mathcal{H}}{\partial p_\mu}\mathrm{d}t = 0 \\ \mathrm{d}p_\mu + \dfrac{\partial\mathcal{H}}{\partial q^\mu}\mathrm{d}t = 0 \end{cases} \tag{22}$$

Which are Hamilton's equations of motion. Note that dividing by $\mathrm{d}t$ and rearranging, we obtain the usual form of the equations

$$\begin{cases} \dfrac{\partial\mathcal{H}}{\partial p_\mu} = \dot{q}^\mu \\ \dfrac{\partial\mathcal{H}}{\partial q^\mu} = -\dot{p}_\mu \end{cases}$$

§§ 0.3.1 Maupertuis' Principle

A particular case of the previous variation was given by Maupertuis, where he stated the following theorem

Theorem 0.2 (Maupertuis Principle)**.** *Defined the «abbreviated action» of a system $\mathcal{S}_0$ as*

$$\mathcal{S}_0 = \int_{t_1}^{t_2} p_\mu \mathrm{d}q^\mu$$

Then, the equations of motion can be derived by finding an extremal of $\mathcal{S}_0$ if and only if energy is conserved.

Proof. The proof is similar to the previous derivation and quick. Since energy is conserved we have $\partial_t \mathcal{H} = 0$ and $\mathcal{H} = E$. Integrating directly the action $\mathcal{S}$ we have

$$\mathcal{S} = \int_{t_1}^{t_2} p_\mu \mathrm{d}q^\mu - E(t_2 - t_1)$$

Therefore

$$\mathcal{S} = \mathcal{S}_0 + E(t_2 - t_1)$$

Variating and imposing the least action principle, we have Maupertuis' principle

$$\delta\mathcal{S} = \delta\mathcal{S}_0 = 0$$

□