§Fresnel diffraction

Fresnel's approximation of the (KF) integral, written below

$$\psi_P = \frac{\psi_0 e^{-i\omega t}}{2i\lambda} \iint_\Sigma \frac{e^{ik(r+r')}}{r'r} \left(\cos(\hat{n}, \underline{r}) - \cos(\hat{n}, \underline{r}') \right) d^2\xi \qquad (KF)$$

Was developed by Fresnel expanding to 2nd order in δ the difference bw $r+r'$ @ Σ

I.e.:

$$\Delta_\xi \simeq \left(\frac{h'}{d'} + \frac{h}{d} \right) \delta + \frac{1}{2} \left(\frac{1}{d'} + \frac{1}{d} \right) \delta^2$$

This approximation explicitly considers a NON-PLANE wave (δ^2), and therefore it's known as CLOSE FIELD APPROXIMATION. Here, either the source S, the measuring point P are close to Σ, we cannot neglect wave curvature.

The close nature of this approximation makes it easy to see in labs

§§ Fresnel Zones

Consider a plane aperture Σ illuminated by a point source S, such that the plane of Σ is $\perp$ to the observing point P, as in fig.

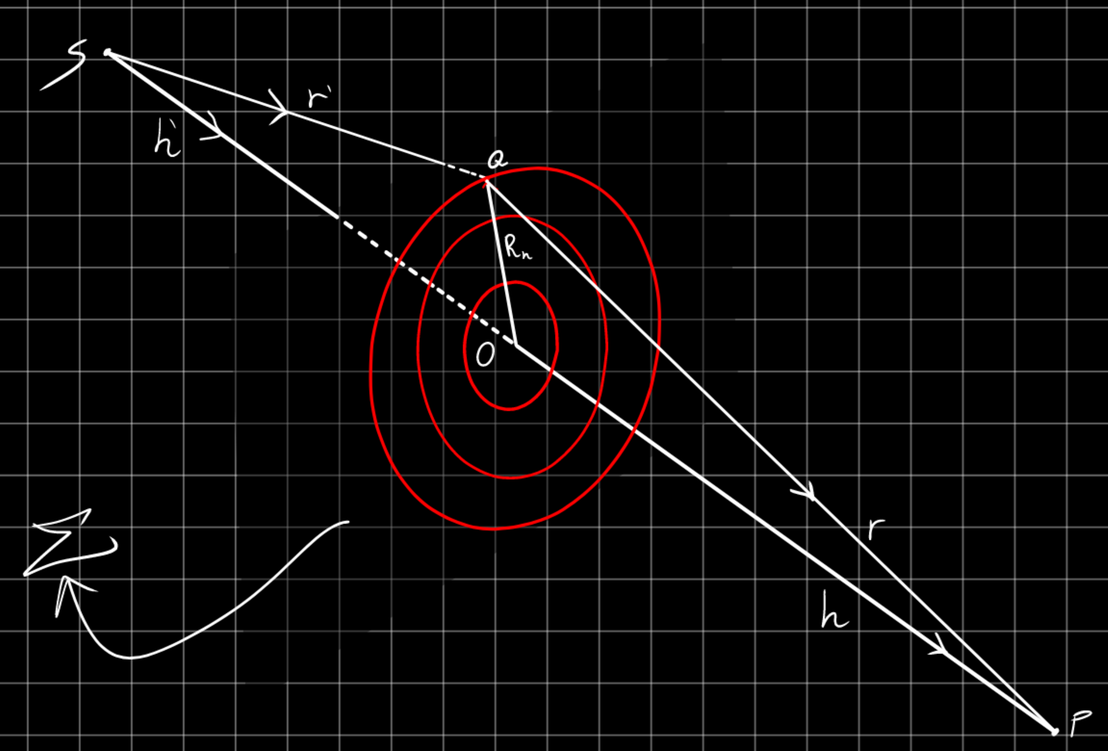

Fig 1) Source S close to a plane Σ, distant $h+h'$ from point P (measuring point). Fresnel zones in red

Said R the distance bw O & Q on Σ, we have that the distance $\overline{SPQ}$ can be written

$$\| \overline{SPQ} \| = r + r'$$

Q:

$$\boxed{\begin{aligned} r + r' &= \sqrt{h^2 + R^2} + \sqrt{h'^2 + R'^2} \\ r + r' &\simeq h + h' + \frac{1}{2}\left(\frac{1}{h} + \frac{1}{h'} \right) R^2 + O(R^3) \end{aligned}}$$

The approximation that we will use is then:

$$r + r' = h + h' + \frac{1}{2}\left(\frac{1}{h} + \frac{1}{h'}\right)R^2$$

Suppose now that Σ is divided in concentric circles around O as in Fig.(1), such that between the n-th circle & the (n+1)-th circle, the following constraint holds.

Said $\xi_{n,m}$ the difference of distance $\|\overline{SQ_iP}\|$ b/w a point Q_n & R_n from O & Q_m at R_m from O, then:

$$\xi_{n,n+1} = \frac{1}{2}\lambda$$

The zones created w/ this constraint are known as Fresnel Zones. From the formula for the approximation of $r + r'$ we get

$$R_1 = \sqrt{\lambda L}, \quad R_2 = \sqrt{2\lambda L}, \ldots$$

I.e.:

$$R_n = \sqrt{n\lambda L}$$

Where

$$L = \left(\frac{1}{h'} + \frac{1}{h}\right)^{-1} = \frac{hh'}{h+h'}$$

Note that, since two consecutive radii form a torus, then the area is

$$A_n = \pi R_{n+1}^2 - \pi R_n^2$$

$$A_n = \pi\left((n+1)\lambda L - n\lambda L\right) = \pi\lambda L = \pi R_1^2$$

I.e. the area of the Fresnel zone is constant, $A_n = A_{n+1} = A \quad \forall n \geq 1$

Typically this is very small, since for the 1ST zone in the optical range

$$\begin{cases} h = h' \simeq 60\ cm \\ \lambda \simeq 600\ nm \end{cases} \Rightarrow R_1 \simeq 2\ mm$$

Since $R_n \propto \sqrt{n}$ we can also say that the 100-th zone is approximately @ $R_{100} = 21\ mm$ (!)

The optical disturbance @ P $\{\Psi_P\}$ can be evaluated by summing of the contributions of each Fresnel zone $\{\Psi_1, \dots, \Psi_n\}$
Since we have for each zone a phase change of π, then

$$|\Psi_P| = \sum_{n=1}^{N} (-1)^{n+1} |\Psi_n|$$

Consider now Σ as a circular aperture centered in O.
If Σ contains exactly N zones, then

$$\Psi_P \begin{cases} = 0 & N \bmod 2 = 0 \\ \simeq \Psi_1 & N \bmod 2 = 1 \end{cases}$$

Considerations of the obliquity factor in the KF integral also show that

$$\Psi_n \leq \Psi_{n+1} \quad (\text{slowly})$$

Therefore, if $\Sigma \to \mathbb{R}^2$, we have, regrouping the sum as follows:

$$\Psi_P = \frac{1}{2}\Psi_1 + \left(\frac{1}{2}\Psi_1 - \Psi_2 - \frac{1}{2}\Psi_3\right) + \left(\frac{1}{2}\Psi_3 - \Psi_4 - \frac{1}{2}\Psi_5\right) + \dots$$

for $n \to \infty$ (∞ Fresnel zones)

$$\Psi_P = \frac{1}{2}\Psi_1$$

Note that this corresponds to the case that there is <u>NO APERTURE</u>
Consider now the case of having a circular obstacle instead of the aperture Σ, then, the Fresnel zones start @ the edge of the obstacle, then again

$$\Psi_P = \frac{1}{2}\Psi_1$$

And therefore @ the center of the shadow it's possible to observe a bright spot. If I is the irradiance w/o the obstacle, then

$$\Psi_P = \frac{1}{2}\Psi_1 \Rightarrow I_P \simeq I$$

This spot is known as <<Arago spot>>.
For an irregular object, depending on where it's positioned wrt the zones, we have
a) It's offset from the center $\Rightarrow$ Ψ_P hardly changes & higher contribs $\to 0$ quickly
b) It's over the center $\Rightarrow$ Terms diminish @ both ends & inside the shadow zone we get $I \simeq 0 \Rightarrow$ <u>SHADOW AS USUAL</u> (!) {check next drawing for visual}

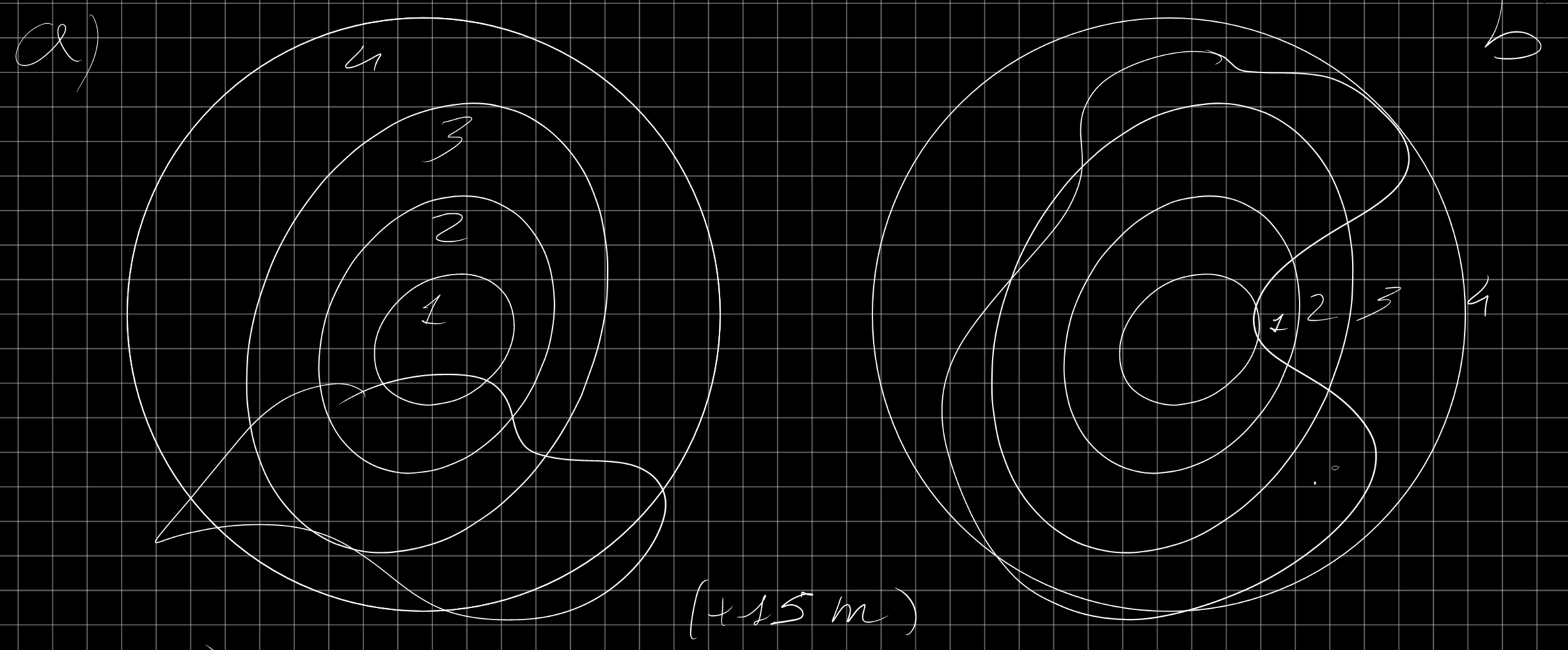

Fig 2) Cases a, b explained before

§§ Zone Plates

It's possible to construct plates whose job is to obstruct certain Fresnel zones. These plates are known as « ZONE PLATES ».

Suppose that we build one that obstructs only even zones, then:

$$\Psi_p = \sum_{n=1}^{N} \Psi_{2n}$$

Thus $I_p \gg I$ (!) and the plate acts like a lens

The focal length of a plate (L) can be calculated as

$$L = \frac{R_1^2}{\lambda} = \frac{hh'}{h+h'}$$

Note also that $L \propto \lambda^{-1}$, therefore this can be seen as a very chromatic lens.

§§ Rectangular Aperture, Fresnel diffraction

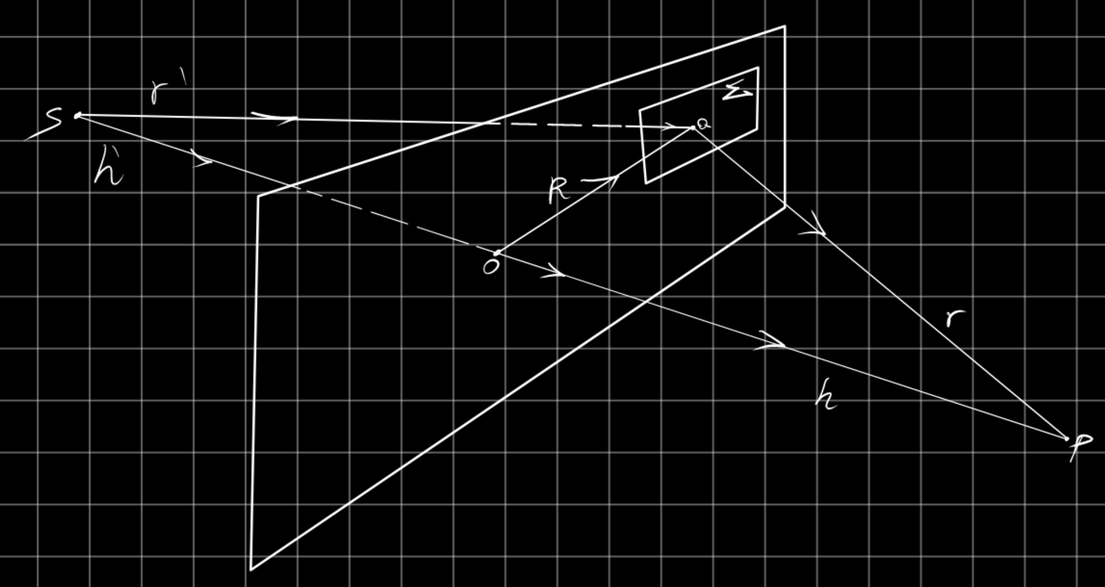

Fig. 3) Rectangular aperture in CFA (Fresnel diffraction)

For a rectangular aperture as in Fig. (3) we employ the KF integral together with cartesian coordinates $\{O; x, y\}$. Then, by definition

$$R^2 = x^2 + y^2$$

Therefore

$$r + r' = h + h' + \frac{1}{2L}(x^2 + y^2)$$

As for Fraunhofer diffraction, we shall assume that:

1) $\hat{O}(\underline{r}, \underline{r}', \hat{n}) \simeq 1$ {obliquity}

2) $1/rr' \simeq$ constant {radial factor}

Thus, the KF integral becomes:

$$\psi_P = C \iint_{\Sigma} e^{ik(x^2+y^2)/2L}\, d^2s = \underbrace{C \int_{x_1}^{x_2} e^{i\frac{kx^2}{2L}} dx \int_{y_1}^{y_2} e^{i\frac{ky^2}{2L}} dy}_{\text{Fubini}}$$

Introducing the dimensionless factors (u, v) as

$$\begin{cases} u_k = \sqrt{\dfrac{k}{\pi L}}\, x \\ v_k = \sqrt{\dfrac{k}{\pi L}}\, y \end{cases} \quad \text{or, using } k = \frac{2\pi}{\lambda} \quad u_\lambda = x\sqrt{\frac{2}{\lambda L}} \quad v_\lambda = y\sqrt{\frac{2}{\lambda L}}$$

We have

$$\psi_P = \psi_1 \int_{u_1}^{u_2} e^{\frac{i\pi u^2}{2}} du \int_{v_1}^{v_2} e^{\frac{i\pi v^2}{2}} dv \quad ; \quad \psi_1 = \frac{C\pi L}{k} = \frac{C\lambda L}{2}$$

This integral can be evaluated as the real and imaginary part of the following integral w/ $z_e \in \mathbb{C}$

$$\int_0^s e^{\frac{i\pi z^2}{2}} dz_e = \int_0^s \cos\left(\frac{\pi z^2}{2}\right) dz + i \int_0^s \sin\left(\frac{\pi z^2}{2}\right) dz = C(s) + i\, S(s)$$

The 2 integrals are known as <u>Fresnel integrals</u>. Both, taken as coordinates in $\mathbb{C}$, form what's known as the <u>Cornu Spiral</u>

Evaluating with the aid of numerical methods & tables, we have that for our Σ we only evaluate a piece of the spiral, namely Δs

$$\Delta s = s_2 - s_1 = u_2 - u_1 = (x_2 - x_1)\sqrt{\frac{2}{\lambda L}} \; ; \; \Delta s = v_2 - v_1 = (y_2 - y_1)\sqrt{\frac{2}{\lambda L}}$$

In the case of an infinite aperture ($\Sigma = \mathbb{R}^2$), we set $u_1 = v_1 = -\infty$ $u_2 = v_2 = \infty$. Since

$$\begin{cases} \lim_{s \to \infty} C(s) = \lim_{s \to \infty} S(s) = \frac{1}{2} \\ \lim_{s \to -\infty} C(s) = \lim_{s \to -\infty} S(s) = -\frac{1}{2} \end{cases}$$

We obtain the value for an unperturbed wave

$$\psi_P = \psi_1 (1+i)^2 = 2i\psi_1$$

Said $\psi_0 = \psi_1 (1+i)^2$, we have that in the general case of Cornu spirals for which $s \in I \subseteq \mathbb{R}$,

$$\psi_P = \frac{\psi_0}{(1+i)^2} \left(C(u) + iS(u) \right)_{u_1}^{u_2} \left(C(v) + iS(v) \right)_{v_1}^{v_2}$$

Which is simply the evaluation of the previous integrals in $[u_1, u_2] \times [v_1, v_2]$. In normal cases we tho only care about lower order Fresnel zones.

§§ SLIT & STRAIGHTEDGE

Fresnel diffraction from a long slit is treated as the limiting (1D) case of the rectangular aperture, namely

$$u \in \mathbb{R}, \quad v \in [v_1, v_2]$$

Thus:

$$\psi_P = \frac{\psi_0}{1+i} \left(C(v) + iS(v) \right)_{v_1}^{v_2}$$

The straightedge instead is the limiting case of the slit, when

$$u \in \mathbb{R}, \quad v \in (-\infty, v_2]$$

I.e.:

$$\psi_P = \frac{\psi_0}{1+i} \left(C(v_2) + iS(v_2) - \overset{-1/2}{C(-\infty)} - i\overset{-1/2}{S(-\infty)} \right)$$

$$\psi_P = \frac{\psi_0}{1+i} \left(C(v_2) + iS(v_2) \right) + \frac{1}{2} \frac{\psi_0}{\cancel{1+i}} \cancel{(1+i)}$$

$$\therefore \psi_P = \frac{\psi_0}{1+i} \left(C(v_2) + iS(v_2) \right) + \frac{1}{2} \psi_0$$

— 11/23/22 — 2h 4m 22.66s — NEW WEEKLY RECORD! —

§ FOURIER TRANSFORM & DIFFRACTION

Consider again the case of Fraunhofer diffraction {Far field}, it's possible to consider the most general case for diffraction, a generic aperture Σ with (also/either) general transmission properties, including phase retardation

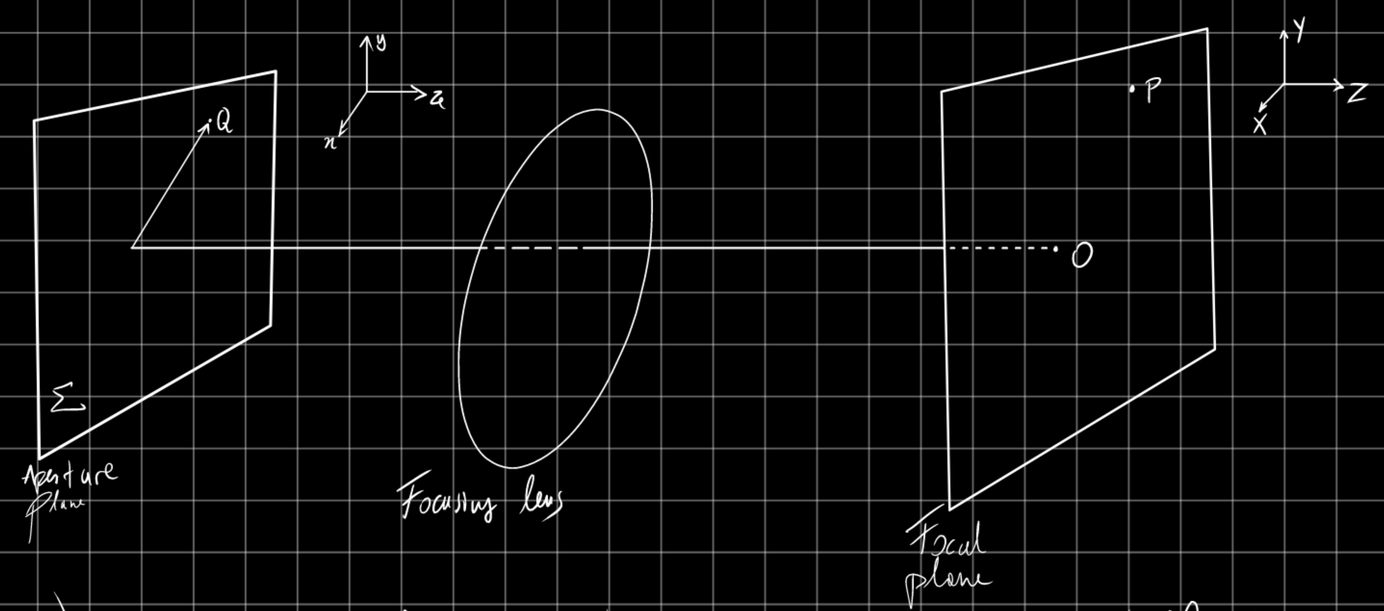

Fig. 1) scheme for evaluation of Fraunhofer diffraction in the general case

All rays leaving Σ in a generic direction, specified by the director cosines $\{\hat{\underline{r}} = (\alpha, \beta, \gamma)\}$ are brought to a common focus point (through the lens), which we'll call P.

by definition, also

$$P = (X, Y) \simeq (L\alpha, L\beta)$$

here we're assuming $\alpha, \beta << 1 \Rightarrow \alpha \simeq \tan\alpha$, $\beta \simeq \tan\beta$, also we assume $\gamma \simeq 1$.

The difference bw/ 2 parallel rays starting from Q & O is $\underline{R} = (x, y)$, when we choose $\hat{\underline{n}} = \frac{\underline{R}}{\|\underline{R}\|}$. Since, for what we said before $\hat{\underline{n}} = \hat{\underline{r}} = (\alpha, \beta, \gamma)$, we will have

$$\delta r = \underline{R} \cdot \hat{\underline{n}} = \alpha x + \beta y = x \frac{X}{L} + y \frac{Y}{L}$$

With X, Y, as in Fig (1) are the coordinates of P. It follows that the fundamental KF integral for diffraction is:

$$\psi(X, Y) = \iint_{\Sigma} e^{ik\delta r} d^2s = \iint_{\Sigma} e^{ik(xX + yY)/L} dx\,dy$$

This for an uniform aperture Σ. Inserting the apertures we treated before, we'll get the same result.

For a NON-UNIFORM aperture, we introduce the << APERTURE FUNCTION >> $g(x, y)$, defined such that $g(x,y)\,dx\,dy$ is the amplitude of the diffracted wave on the element d^2s. Thus:

$$\psi(X, Y) = \iint_{\Sigma} g(x, y)\, e^{ik\left(\frac{X}{L}x + \frac{Y}{L}y\right)} dx\,dy$$

Introducing the spatial frequencies μ, ν; defined as

$$\begin{cases} \mu = \dfrac{kX}{L} \\ \nu = \dfrac{kY}{L} \end{cases}$$

We can write the KF integral as a 2D Fourier transform

$$\psi(\mu, \nu) = \iint_{\Sigma} g(x, y)\, e^{i(\mu x + \nu y)} dx\,dy = \hat{\mathcal{F}}_2[g](\mu, \nu)$$

Therefore the diffraction pattern (Image) is the Fourier transform of the aperture function. Ψ, g are FOURIER PAIRS

Consider now a grating 1D for simplicity. Then $g(y)$ is a periodic step function

$$g(y) = \sum_{n=0}^{\infty} g_n \cos(n\nu_0 y) \quad ; \quad \nu_0 = \frac{2\pi}{h} \quad ; \; h \text{ Spacing}$$

The (1D) Fourier transform what we expect already for the grating.

The maximes of higher orders correspond to components w/ $n > 1$ of $g(y)$

§§ APODIZATION

Apodization (A-pod-ization / remove the feet): Process where the aperture function is modified in order to redistribute energy in the diff. pat.

It's employed for reducing the intensity of the 2ND-ary maxima of $\Psi(X, Y)$

Consider a single slit. Here:

$$g(y) = \mathbb{1}_{\left[-\frac{b}{2}, \frac{b}{2}\right]}(y)$$

Then, integrating the transform, or using $\hat{\mathcal{F}}[1](\nu) = \text{sinc}(\nu) = \frac{\sin(\nu)}{\nu}$, we get

$$\Psi(\nu) = b\,\text{sinc}\left(\tfrac{1}{2}\nu b\right)$$

This is exactly equivalent to what we found for the single slit before {Fraunhofer}

Suppose that we now apodize $g(y)$, with

$$g_A(y) = \cos\left(\frac{\pi y}{b}\right), \quad y \in \left[-\frac{b}{2}, \frac{b}{2}\right]$$

Then, integrating on the aperture, we get

$$\Psi_A(\nu) = \int_{-\frac{b}{2}}^{b/2} \cos\left(\frac{\pi y}{b}\right) e^{i\nu y}\, dy = \frac{1}{2}\int_{-\frac{b}{2}}^{b/2} \left(e^{\frac{i\pi y}{b}} + e^{-\frac{i\pi y}{b}}\right) e^{i\nu y}\, dy = \frac{1}{2}\int_{-\frac{b}{2}}^{\frac{b}{2}} e^{iy\left(\nu + \frac{\pi}{b}\right)} + e^{iy\left(\nu - \frac{\pi}{b}\right)}\, dy = \frac{1}{2}\left[\frac{1}{i\left(\nu + \frac{\pi}{b}\right)} e^{iy\left(\nu + \frac{\pi}{b}\right)} + \frac{1}{i\left(\nu - \frac{\pi}{b}\right)} e^{iy\left(\nu - \frac{\pi}{b}\right)}\right]_{-\frac{b}{2}}^{\frac{b}{2}}$$

$$\Psi(\nu) = \frac{1}{2i}\left[\frac{b}{b\nu - \pi} e^{\frac{ib}{2}\left(\nu - \frac{\pi}{b}\right)} + \frac{b}{b\nu - \pi} e^{-\frac{ib}{2}\left(\nu - \frac{\pi}{b}\right)} + \frac{b}{b\nu + \pi} e^{\frac{ib}{2}\left(\nu + \frac{\pi}{b}\right)} + \frac{b}{b\nu + \pi} e^{-\frac{ib}{2}\left(\nu + \frac{\pi}{b}\right)}\right] = \frac{b}{b\nu - \pi}\sin\left[\frac{1}{2}(\nu b - \pi)\right] + \frac{b}{b\nu + \pi}\sin\left[\frac{1}{2}(\nu b + \pi)\right]$$

Or:

$$\Psi_A(\nu) = \cos\left(\frac{\nu b}{2}\right)\left[\frac{b}{b\nu - \pi} + \frac{b}{b\nu + \pi}\right]$$

The apodized $\Psi_A(\nu)$ clearly respects the relation $\ulcorner \Psi_A(\nu) \leq \Psi(\nu) \lrcorner$ for higher frequencies. Apodization is typically used on telescope apertures, making sure that a dimmer object can be seen NEXT to a brighter one {see binary stars & exoplanets} ← 2022 black shit

§§ SPATIAL FILTERING

Consider Fig. (1). The xy plane (Σ) the location of a coherently illuminated object, imaged by some optical system (lens) & imaged again on the focal plane. Said $\mu\nu$ the plane of the optical system, then here we have $\Psi(\mu, \nu)$. $g(x, y)$ is simply our object.

The shadow casted on $X'Y'$ (call it x', y' for ease), which we will call $\tilde{g}(x, y)$ is again the Fourier transform of $\Psi(\mu, \nu)$

$$S \longrightarrow \underset{\text{object}}{g(x,y)} \xrightarrow{\hat{\mathcal{F}}_{(\mu,\nu)}} \underset{\text{lens (OS)}}{\Psi(\mu,\nu)} \xrightarrow{\hat{\mathcal{F}}_{(x',y')}} \underset{\text{image}}{\tilde{g}(x',y')}$$

If ALL $(\mu, \nu) \in \mathbb{R}^2$ were transmitted EQUALLY by the optical system, then $\tilde{g}(x', y') \propto g(x, y)$, i.e. the image is a true & precise representation! This is clearly not possible if $\Sigma \neq \mathbb{R}^2$ {finite aperture}, thus some frequencies are limited. Other optical fenomena like aberrations, defects etc.. result in a modified $\Psi(\mu, \nu)$. This mod can be incorporated by using a transfer function, such that $\Psi'(\mu, \nu) = T(\mu, \nu)\Psi(\mu, \nu)$ & $g'(x', y') = \hat{\mathcal{F}}[\Psi'](x', y')$

24/11/22 — 22:31 — 1h 37m 23.055s

Thus:

$$g'(x',y') = \iint_{\mathbb{R}^2} T(\mu,\nu)\,\Psi(\mu,\nu)\,e^{-i(\mu x'+\nu y')}\,d\mu\,d\nu$$

So, the image function $g'(x',y')$ is the Fourier transform of $T(\mu,\nu)\Psi(\mu,\nu)$. The limits of integration are $\mathbb{R}^2$ just formally, since they're actually determined by the transfer function $T(\mu,\nu)$. T can be modified by placing screens & apertures on the $(\mu\nu)$-plane. The process is known as << spatial filtering >> ($\sim$ electrical filters w/ passive components, accepts certain freq. (μ,ν) & rejects others)

Suppose that the object is a grating $\Rightarrow$ $\boxed{f(y) = \sum_{s\geq 0} \mathbb{1}_{[sh,\,sh+b]}(y)}$. If we $\hat{\mathcal{F}}[f](\nu) = \Psi(\nu)$, we get that $\lim_{\nu\to\pm\infty}\Psi(\nu)=0$, therefore it can be thought as a << low pass filter >>. Now let $\Sigma \subseteq \pi_{(\mu\nu)}$ transmit only freq. $\nu\in[-\nu_m,\nu_m]$. Since we chose $\nu = \frac{kY}{L}$

Thus $\nu = \frac{kb}{f}$, where $2b$ is the width of Σ in $\pi_{(\mu\nu)}$. The transfer function $T(\nu) = \mathbb{1}_{[-\nu_m,\nu_m]}(\nu)$, thus:

$$g'(y') = \int_{\mathbb{R}} \Psi(\nu)\,\mathbb{1}_{[-\nu_m,\nu_m]}(\nu)\,e^{-i\nu y'}\,d\nu = \int_{-\nu_m}^{\nu_m} \Psi(\nu)\,e^{-i\nu y'}\,d\nu$$

A high pass optical filter can be obtained by obstructing the central part of the diffraction pattern $\Psi(\nu)$, {with for e.g.: $T(\nu) = \mathbb{1}_{(-\infty,-\tilde{\nu}]}(\nu) + \mathbb{1}_{[\tilde{\nu},+\infty)}(\nu)$}

§§ Phase gratings

A method known as << phase contrast >> was invented by the Dutch physicist Zernike $\Longrightarrow$ making transparent obj w/ $n_{obj} \approx n_{amb}$ inside a transparent medium (amb).

The simplest method is given by the usage of << phase gratings >> i.e. alternate H-L index material (perfectly transparent). The object function $g(y)$ is then:

$$g(y) = e^{i\varphi(y)}$$

Where $\varphi(y)$ is a periodic step function, with height $\Delta\varphi = kZ_e\Delta n$ $\begin{cases} Z_e \to \text{thickness} \\ \Delta n \to n_H - n_L \end{cases}$

If $\Delta\varphi \ll 1$, we can write

$$g(y) \simeq 1 + i\varphi(y)$$

Thus

$$\Psi(\nu) = \int_{\mathbb{R}} (1+i\varphi(y))\,e^{i\nu y}\,dy = \int_{-b/2}^{b/2} e^{i\nu y}\,dy + i\int_{-b/2}^{b/2} \varphi(y)\,e^{i\nu y}\,dy = \mathrm{Re}\{\Psi(\nu)\} + i\,\mathrm{Im}\{\Psi(\nu)\} \quad \begin{cases} \mathrm{Re}\{\Psi\} = \Psi_1(\nu) \\ \mathrm{Im}\{\Psi\} = \Psi_2(\nu) \end{cases}$$

$\Psi_1(\nu)$ is the WHOLE OBJECT APERTURE DIFFRACTION PATTERN ($\Psi_1(\nu)\approx 0\ \forall\nu\neq 0$). $\Psi_2(\nu)$ is the $\varphi(y)$ - DIFFRACTION PATTERN. By definition, also Ψ_1, Ψ_2 are dephased by $\frac{\pi}{2}$ degrees $\{i = e^{i\frac{\pi}{2}}\}$. In the phase contrast method a phase plate is introduced, shifting Ψ_2 by an additional $\frac{\pi}{2}$ degrees

A PHASE PLATE is a transparent glass plate whose $Z_{e_{opt}} = Z_{e_{plate}} + \frac{1}{4}\lambda$. The thicker section is on $\pi_{(\mu\nu)}$. The phase plate applies the transformation $\Psi(\nu) = \Psi_1(\nu) + i\Psi_2(\nu) \longrightarrow \hat{P}_{\frac{\pi}{2}}[\Psi] = \Psi_1(\nu) + \Psi_2(\nu)$. The image function is then:

$$g'(y') = \hat{\mathcal{F}}[\Psi_1(\nu)](y') + \hat{\mathcal{F}}[\Psi_2(\nu)](y') = g_1'(y') + g_2'(y')$$

Phase contrast works like a phase-modulated signal getting transformed to an amplitude-modulated signal via a $\frac{\pi}{2}$ phase transformation.

Optics of Solids

§ Macroscopic Fields & Maxwell's Equations

The EM field @ any given point is described by 4 quantities:

1) $\frac{dQ}{d^3x} = \rho$ {Volumetric density of charge}

2) $\frac{dP}{d^3x} = \underline{P}$ {Polarization}

3) $\frac{dm}{d^3x} = \underline{M}$ {Magnetization}

4) $\frac{d}{d^3x}\left(\underline{J} + \frac{1}{c^2}\frac{\partial E}{\partial t}\right) = \underline{J}$ {current density}

ALL of these quantities are averaged in order to smooth the micro-variations due to the atomic (discrete) composition of matter

Considering the generic Maxwell's equations, considering magnetization currents $\underline{J}_M$ & polarization charges ρ_m, we have:

$$\begin{cases} \nabla\cdot\underline{E} = \frac{1}{\epsilon_0}(\rho + \rho_p) \\ \nabla\times\underline{E} = -\frac{\partial B}{\partial t} \\ \nabla\cdot\underline{B} = 0 \\ \nabla\times\underline{B} = \mu_0(\underline{J} + \underline{J}_M) + \epsilon_0\frac{\partial E}{\partial t} \end{cases} \qquad \begin{cases} \rho_p = -\nabla\cdot\underline{P} \\ \underline{J}_M = \nabla\times\underline{M} \end{cases}$$

Inserting the relations b/w Maxwell's equations & $(\rho_p, \underline{P})$; $(\underline{J}_M, \underline{M})$ we have:

$$(a)\begin{cases} \nabla\cdot(\epsilon_0\underline{E} + \underline{P}) = \rho \\ \nabla\times\underline{E} = -\frac{\partial B}{\partial t} \\ \nabla\cdot\underline{B} = 0 \\ \nabla\times\left(\frac{\underline{B}}{\mu_0} - \underline{M}\right) = \underline{J} + \frac{\partial}{\partial t}(\epsilon_0\underline{E} + \underline{P}) \end{cases} \qquad \begin{cases} \underline{D} = \epsilon_0\underline{E} + \underline{P} \\ \underline{B} = \mu_0\underline{H} + \underline{M} \end{cases}$$

Inserting the relations (b) -> (a), we get

$$\begin{cases} \nabla\cdot\underline{D} = \rho \\ \nabla\times\underline{E} = -\mu_0\frac{\partial H}{\partial t} - \mu_0\frac{\partial M}{\partial t} \\ \nabla\cdot\underline{B} = 0 \\ \nabla\times\underline{H} = \underline{J} + \frac{\partial D}{\partial t} + \epsilon_0\frac{\partial P}{\partial t} \end{cases} \qquad \begin{cases} \nabla\cdot\underline{D} = \rho \\ \nabla\times\underline{D} = \nabla\times\underline{P} - \frac{1}{c^2}\frac{\partial H}{\partial t} - \frac{1}{c^2}\frac{\partial M}{\partial t} \\ \nabla\cdot\underline{H} = -\nabla\cdot\underline{M} \\ \nabla\times\underline{H} = \underline{J} + \frac{\partial D}{\partial t} + \epsilon_0\frac{\partial P}{\partial t} \end{cases}$$

We note also <u>OHM'S LAW</u>, where $\underline{J} = \sigma\underline{E}$. Said $\underline{D} = \epsilon\underline{E}$, where $\epsilon = \epsilon_0(1 + \underline{\underline{\chi_e}})$ & $\mu\underline{H} = \underline{B}$, $\mu = \mu_0(1 + \underline{\underline{\chi_m}})$, where χ_e, χ_m are the electric & magnetic susceptibilities of the medium. Generally they're <u>rank 2 tensors</u>

For isotropic media, χ_e, χ_m are diagonal and can be considered scalar values

§ WAVE EQUATION IN SOLIDS

Going back to Maxwell's equations in media, we can write a wave equation as we did for the empty space. Considering the coupled $(\underline{E}, \underline{H})$ equations, we have

$$\begin{cases} \nabla \cdot \underline{E} = -\frac{1}{\epsilon_0} \nabla \cdot \underline{P} + \frac{\rho}{\epsilon_0} \\ \nabla \times \underline{E} = -\mu_0 \left(\frac{\partial H}{\partial t} + \frac{\partial M}{\partial t} \right) \\ \nabla \cdot \underline{H} = -\nabla \cdot \underline{M} \\ \nabla \times \underline{H} = \underline{J} + \epsilon_0 \frac{\partial E}{\partial t} + \frac{\partial P}{\partial t} \end{cases} \quad (\text{MW G})$$

Consider now the case of electrically neutral media, where $\rho = 0$, $\underline{M} = 0$, then:

$$\begin{cases} \nabla \cdot \underline{E} = -\frac{1}{\epsilon_0} \nabla \cdot \underline{P} & (1) \\ \nabla \times \underline{E} = -\mu_0 \frac{\partial H}{\partial t} & (2) \\ \nabla \cdot \underline{H} = 0 & (3) \\ \nabla \times H = \underline{J} + \epsilon_0 \frac{\partial E}{\partial t} + \frac{\partial P}{\partial t} & (4) \end{cases} \quad (\text{MW EN})$$

Using the operatorial relation $\hat{\nabla} \times \hat{\nabla} \times [\ ] = \hat{\nabla}(\hat{\nabla} \cdot [\ ]) - \hat{\nabla}^2 [\ ]$ we can use 2 to get:

$$\nabla \times \nabla \times \underline{E} = -\mu_0 \nabla \times \frac{\partial H}{\partial t} = -\mu_0 \frac{\partial}{\partial t} \left(\underline{J} + \epsilon_0 \frac{\partial E}{\partial t} + \frac{\partial P}{\partial t} \right) = -\mu_0 \frac{\partial J}{\partial t} - \frac{1}{c^2} \frac{\partial^2 E}{\partial t^2} - \mu_0 \frac{\partial^2 P}{\partial t^2}$$

$$\boxed{\nabla \times \nabla \times \underline{E} + \frac{1}{c^2} \frac{\partial E}{\partial t^2} = -\mu_0 \frac{\partial J}{\partial t} - \mu_0 \frac{\partial^2 P}{\partial t^2}} \quad (\text{EMWE})$$

Or, using $\Box \underline{E} = \frac{1}{c^2} \frac{\partial^2 E}{\partial t^2} - \nabla^2 \underline{E}$ (d'Alembertian), we can write the eqn in an explicitly covariant manner:

$$\nabla(\nabla \cdot \underline{E}) + \Box \underline{E} = -\mu_0 \frac{\partial J}{\partial t} - \mu_0 \frac{\partial^2 P}{\partial t^2} \quad \star$$

Also, further expanding, using (1), we can also compactify the equations in terms of $\underline{D}$

$$-\frac{1}{\epsilon_0} \nabla(\nabla \cdot \underline{P}) + \Box \underline{E} = -\mu_0 \frac{\partial J}{\partial t} - \mu_0 \frac{\partial^2 P}{\partial t^2}$$

Using $\mu_0 = (c^2 \epsilon_0)^{-1}$

$$-\frac{1}{\epsilon_0} \left(\nabla \times \nabla \times \underline{P} + \nabla^2 \underline{P} - \frac{1}{c^2} \frac{\partial^2 P}{\partial t^2} \right) + \Box \underline{E} = -\mu_0 \frac{\partial J}{\partial t}$$

$$\frac{1}{\epsilon_0} \Box \underline{P} - \frac{1}{\epsilon_0} \nabla \times \nabla \times \underline{P} + \Box \underline{E} = -\mu_0 \frac{\partial J}{\partial t}$$

$$\Box \underline{D} - \nabla \times \nabla \times \underline{P} = -\frac{1}{c^2} \frac{\partial J}{\partial t}$$

We will use the equation (Star) for our evaluations. Thus, our wave equation is

$$\nabla \times \nabla \times \underline{E} - \frac{1}{c^2} \frac{\partial E}{\partial t} = -\mu_0 \frac{\partial J}{\partial t} - \mu_0 \frac{\partial^2 P}{\partial t^2}$$

The RHS terms are known as SOURCE TERMS for the wave, & in our case they explicitly indicate oscillations in the polarization field + current density changes. We consider 2 new special cases:

1) DIELECTRIC MEDIA
2) CONDUCTING MEDIA

§§ NONCONDUCTING MEDIA / WAVES IN DIELECTRICS

For nonconducting media we have $\underline{J}=0$, thus our wave equation reduces to:

$$\nabla\times\nabla\times\underline{E} + \frac{\partial^2 \underline{E}}{\partial t^2} = -\mu_0 \frac{\partial^2 \underline{P}}{\partial t^2}$$

Here e^- are PERMANENTLY BOUND to the atom, and the only force exerted is the elastic rebound force of the e^- in the atomic field. Thus, for a lattice of N e^- w/ charge $q_e = -e$, we have

$$\begin{cases} \underline{P}_s = Q_T \underline{r} = -Ne\underline{r} \\ \underline{F} = q\underline{E} = -e\underline{E} = K\underline{r} \end{cases}$$

where K is the elastic force constant & $\underline{P}_s$ is the <u>STATIC</u> polarization

$$\begin{cases} \underline{r} = -\dfrac{e\underline{E}}{K} \\ \underline{P}_s = -Ne\underline{r} \end{cases} \Rightarrow \boxed{\underline{P}_s = -\frac{Ne^2}{K}\underline{E}}$$

<u>Note</u> if $\underline{E} \equiv \underline{E}(\underline{r},t)$ the above equation is <u>NOT</u> correct since we're not taking the e^- motion into account, since now $\underline{r} \equiv \underline{r}(t)$. This motion can be described as a bound harmonic oscillator w/ damping, then Newton's second law gives:

$$\underline{F} = m\frac{d^2\underline{r}}{dt^2} + \underbrace{m\gamma\frac{d\underline{r}}{dt}}_{\text{damping}} + K\underline{r} = -e\underline{E}$$

(Note that we neglected the magnetic force, since $\underline{F}_m \ll \underline{F}_e$ in our case.)

We now also suppose that $\underline{E}$ is harmonic (wave), i.e.

$$\underline{E}(\underline{r},t) = \underline{E}(\underline{r})e^{-i\omega t}$$

Assuming that e^- has harmonic motion w/ $\underline{r}(t) = \underline{r}e^{-i\omega t}$

$$\frac{d\underline{r}}{dt} = -i\omega\underline{r}e^{-i\omega t} \qquad \frac{d^2\underline{r}}{dt^2} = -\omega^2\underline{r}e^{-i\omega t}$$

Thus:

$$+m\omega^2\underline{r}\cancel{e^{-i\omega t}} + im\omega\gamma\underline{r}\cancel{e^{-i\omega t}} - K\underline{r}\cancel{e^{-i\omega t}} = +e\underline{E}\cancel{e^{-i\omega t}}$$

$$\begin{cases} \underline{r} = \dfrac{-e}{K - m\omega^2 - im\omega\gamma}\underline{E} \\ \underline{P} = -Ne\underline{r} \end{cases}$$

I.e., solving the system for $\underline{P}$

$$\underline{P} = \frac{Ne^2}{K - m\omega^2 - im\gamma\omega}\,\underline{E}$$

Using $\omega_0 = \sqrt{\frac{K}{m}}$, we can rewrite in a more significant way the previous equation

$$\underline{P} = \frac{Ne^2}{m}\left(\frac{1}{\omega_0^2 - \omega^2 - i\gamma\omega}\right)\underline{E}$$

The frequency ω_0 is the <<EFFECTIVE RESONANCE FREQUENCY>>, which depends <u>ONLY</u> on the medium analyzed (K varies with materials)
The result obtained is really similar to one given by a driven HO (DHO) as it should. We therefore <u>expect</u> optical resonance around the "natural frequency" ω_0.
Returning to the general equation, we have

$$\begin{cases} \nabla\times\nabla\times\underline{E} + \frac{1}{c^2}\frac{\partial^2 \underline{E}}{\partial t^2} = -\mu_0 \frac{\partial^2 \underline{P}}{\partial t^2} \\ \underline{P}(t) = \frac{Ne^2}{m}\left(\frac{1}{\omega_0^2 - \omega^2 - i\gamma\omega}\right)\underline{E} \end{cases}$$

Thus:

$$\nabla\times\nabla\times\underline{E} + \frac{1}{c^2}\frac{\partial^2 \underline{E}}{\partial t^2} + \frac{Ne^2}{c^2\epsilon_0 m}\left(\frac{1}{\omega_0^2 - \omega^2 - i\gamma\omega}\right)\frac{\partial^2 \underline{E}}{\partial t^2} = 0$$

Also, since in our case $\underline{P} \propto \underline{E}$, $\nabla\cdot\underline{E} = 0$, and $\nabla\times\nabla\times\underline{E} = -\nabla^2\underline{E}$, giving

$$\nabla^2\underline{E} = \left[1 + \frac{Ne^2}{m\epsilon_0}\left(\omega_0^2 - \omega^2 - i\gamma\omega\right)^{-1}\right]\frac{1}{c^2}\frac{\partial^2 \underline{E}}{\partial t^2}$$

If we search for a homogeneous plane wave solution (HPWs) of the form:

$$\underline{E} = \underline{E}_0 e^{-i(\mathcal{K}z - \omega t)} \qquad \mathcal{K} \in \mathbb{C}$$

we have

$$\nabla^2\underline{E} = -\mathcal{K}^2\underline{E}, \quad \frac{\partial^2 \underline{E}}{\partial t^2} = -\omega^2\underline{E}$$

Thus it's satisfied <u>IFF</u>

$$\mathcal{K}^2 = \frac{\omega^2}{c^2}\left[1 + \frac{Ne^2}{m\epsilon_0}(\omega_0 - \omega - i\gamma\omega)^{-1}\right]$$

We can rewrite the RHS remembering that for $z \in \mathbb{C}$, $z^{-1} = \frac{\bar{z}}{\|z\|^2}$, i.e.

$$(\omega_0 - \omega - i\gamma\omega)^{-1} = \left[(\omega_0 - \omega)^2 + \gamma^2\omega^2\right]^{-1}(\omega_0 - \omega + i\gamma\omega)$$

Giving, then:

$$\mathcal{K}^2 = \frac{\omega^2}{c^2} + \frac{Ne^2\omega^2}{m\epsilon_0 c^2}\,\frac{\omega_0 - \omega + i\gamma\omega}{(\omega_0 - \omega)^2 + \gamma^2\omega^2}$$

The fact that $\tilde{k} \in \mathbb{C}$ indicates that we can describe it as a real part + $i \cdot$ imaginary part, where

$$\left[\tilde{k} = K + i\alpha\right] \quad (K \text{ def})$$

with

$$\tilde{k} = \frac{\omega}{c} \tilde{N}, \quad \tilde{N} \in \mathbb{C}$$

our value $\tilde{N}$ is a <u>COMPLEX</u> REFRACTION INDEX. The solution, with $\tilde{k}$ as in (K def) becomes

$$\underline{E} = \underline{E}_0 e^{i(\tilde{k}z - \omega t)} = \underline{E}_0 e^{-\alpha z} e^{i(Kz - \omega t)}$$

The exponential damping of $\underline{E}$ indicates that there's absorption in the medium to be precise, we have that, inside the medium

$$I \propto e^{-2\alpha z} = e^{-az}$$

where $a = 2\alpha$ is the absorption coefficient. The refraction index, as we said is $\tilde{N} \propto \tilde{k} \in \mathbb{C}$, where

$$\left[\tilde{N}^2 = \frac{c^2}{\omega^2}\tilde{k} = 1 + \frac{Ne^2}{m\epsilon_0} \frac{\omega_0 - \omega + i\gamma\omega}{(\omega_0 - \omega)^2 + \gamma^2\omega^2}\right]$$

As before, $\tilde{N} = n + i\eta$, where

$$\begin{cases} \tilde{N} = \frac{c}{\omega}\tilde{k} = \frac{c}{\omega}(K + i\alpha) \\ \tilde{N} = n + i\eta \end{cases}$$

I.e., considering separately real & imaginary parts

$$\begin{cases} Re\{\tilde{N}\} = \frac{c}{\omega} K = n \\ Im\{\tilde{N}\} = \frac{c}{\omega}\alpha = \eta \end{cases}$$

Note that:

1) From $Re\{\tilde{N}\}$ we get the usual relation:

$$K = \frac{\omega}{c} n$$

2) From $Im\{\tilde{N}\}$ we get all absorption phenomena

$$\alpha = \frac{\omega}{c}\eta$$

Looking again @ the solution, we have a harmonic wave with <u>PHASE VELOCITY</u>

$$u = \frac{\omega}{k} = \frac{c}{n}$$

& from $\mathcal{N}^2$ we also find

$$\mathcal{N}^2 = n^2 - \eta^2 + i n\eta = 1 + \frac{Ne^2}{m\epsilon_0} \frac{\omega_0 - \omega + i\gamma\omega}{(\omega_0-\omega)^2 + \gamma^2\omega^2}$$

Note that:

$$\frac{\omega_0 - \omega + i\gamma\omega}{(\omega_0-\omega)^2+\gamma^2\omega^2} = \frac{\omega_0-\omega}{(\omega_0-\omega)^2+\gamma^2\omega^2} + i\frac{\gamma\omega}{(\omega_0-\omega)^2+\gamma^2\omega^2}$$

Therefore

$$\begin{cases} \mathrm{Re}\{\mathcal{N}^2\} = n^2 - \eta^2 = 1 + \dfrac{Ne^2}{m\epsilon_0}\dfrac{\omega_0-\omega}{(\omega_0-\omega)^2+\gamma^2\omega^2} \\ \mathrm{Im}\{\mathcal{N}^2\} = n\eta = \dfrac{\gamma\omega}{(\omega_0-\omega)^2+\gamma^2\omega^2}\dfrac{Ne^2}{m\epsilon_0} \end{cases}$$

From the previous equation it's possible to determine n, η. In general, their dependency on frequency is of the following kind:

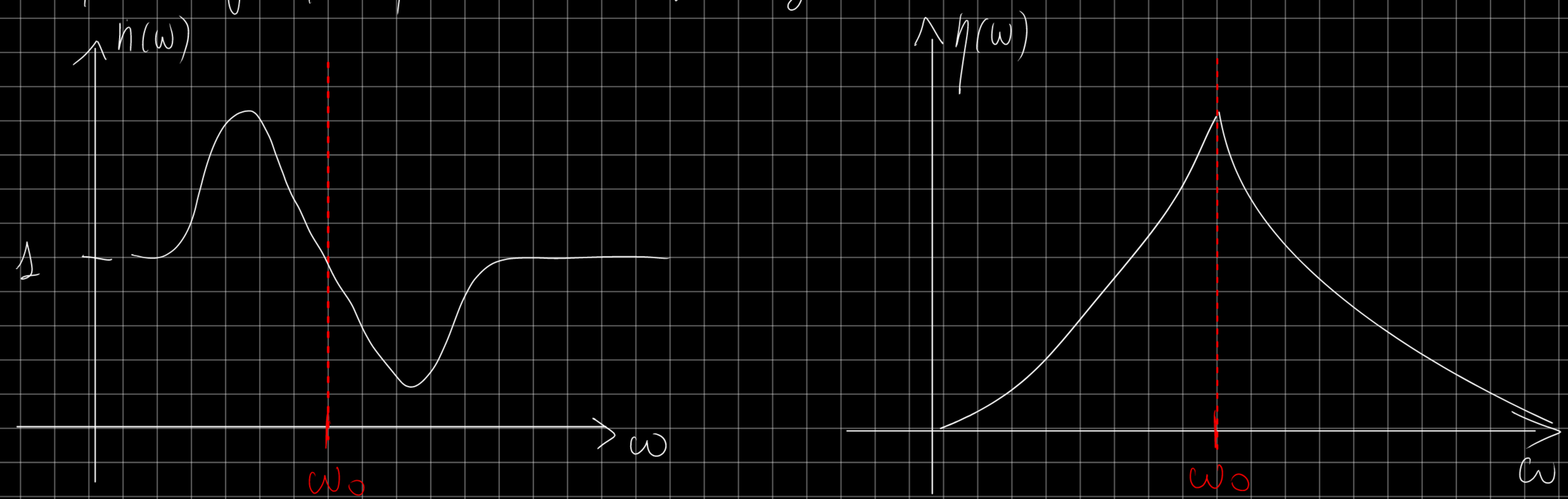

Fig 1.) Generic functional relations for $\mathcal{N}(\omega)$.

From fig. (1) it's clear that the absorption is MAX when $\omega \simeq \omega_0$, & $n > 1$ for frequencies $\omega < \omega_0$. The resonance frequencies of optical transport media are usually all in the UV region, therefore the absorption peak is not observed in VIS, & also we have $n>1$, giving our usual behaviour, known as <<NORMAL DISPERSION>>. After the peak for $\omega \lesssim \omega_0$, we enter the zone of <<ANOMALOUS DISPERSION>>, where instead of having n grow with ω, it decreases with increasing frequency ($n<1$!)

The previous remarks are valid only if e^- are bound equally to their respective atom

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This is not true in general, in fact, we may assume that each fraction f_j of electrons has resonant frequency ω_j, thus

$$\mathcal{N}^2 = 1 + \frac{Ne^2}{m\epsilon_0} \sum_{j \leq N} \frac{f_j}{\omega_j^2 - \omega^2 - i\gamma_j \omega}$$

The fractions f_j are known as << OSCILLATOR STRENGTHS >> {QHO}

Note that:

$$\begin{cases} \lim_{\omega \to 0} \mathcal{N}^2(\omega) = 1 + \frac{Ne^2}{m\epsilon_0} \sum_{j \leq N} \frac{f_j}{\omega_j} = 1 + \chi_e \\ \lim_{\omega \to \infty} \mathcal{N}^2(\omega) = \infty \end{cases}$$

While, if $\gamma_j << 1 \ \forall j \leq N$, then $\mathcal{N}^2 \in \mathbb{R}$, &

$$n^2 = 1 + \frac{Ne^2}{m\epsilon_0} \sum_{j \leq N} \frac{f_j}{\omega_j^2 - \omega^2}$$

Expressing the previous formula in terms of λ, we have what's known as <u>SELLMEIER'S FORMULA</u> (experimentally found by data fit at first)

§β PROPAGATION IN CONDUCTING MEDIA

As before, the equation analyzed is

$$\nabla \times \nabla \times \underline{E} + \frac{1}{c^2} \frac{\partial^2 E}{\partial t^2} = -\mu_0 \frac{\partial^2 P}{\partial t^2} - \mu_0 \frac{\partial J}{\partial t}$$

In the case of conducting media, we can consider e^- unbound from the atoms, thus $\underline{P} \simeq 0$. Due to this we can consider the force equation in terms only of m-e plus a dissipation term.

Remembering that $\underline{J} = -e\underline{v}$ & $\underline{J} = -Ne\underline{v}$, we have the following coupled equations for $\underline{v}$

$$\begin{cases} m\frac{dv}{dt} - \frac{m}{\tau}\underline{v} = -e\underline{E} \\ \underline{J} = -Ne\underline{v} \end{cases} \Longrightarrow \frac{m}{\tau Ne}\underline{J} - \frac{m}{Ne}\frac{dJ}{dt} = -e\underline{E}$$

$$\underline{v} = -\frac{1}{Ne}\underline{J}$$

From the previous equations, we have

$$\frac{d\underline{J}}{dt} + \tau^{-1}\underline{J} = \frac{Ne^2}{m}\underline{E} \qquad (A)$$

This is a 1st order non linear ODE, thus we find the solution the usual way

Firstly we find the <<TRANSIENT CURRENT>> by solving for the homogeneous ODE

$$\frac{d\underline{J}}{dt} - \tau^{-1}\underline{J} = 0 \Rightarrow \underline{J}(t) = \underline{J}_0 e^{-t/\tau}$$

The constant τ is known as <<RELAXATION TIME>> of the medium.

We consider another two cases tied to $\underline{E}$

1) STATIC FIELD

2) HARMONIC FIELD

(1) The field is static, thus (A) becomes

$$\tau^{-1}\underline{J} = \frac{Ne^2}{m}\underline{E}$$

From OHM'S LAW $\underline{J} = \sigma\underline{E}$, thus

$$\tau^{-1}\sigma\underline{E} = \frac{Ne^2}{m}\underline{E} \Rightarrow \sigma = \frac{Ne^2\tau}{m}$$

σ, as usual, is the conductivity of the medium.

(2) The field is harmonic $\Rightarrow E \propto e^{-i\omega t}$, thus, remembering OHM

$$(\tau^{-1} - i\omega)\underline{J} = \frac{Ne^2}{m}\underline{E} = \frac{\sigma}{\tau}\underline{E}$$

Therefore

$$\underline{J} = \frac{\sigma}{\tau(\tau^{-1} - i\omega)}\underline{E} = \frac{\sigma}{1 - i\omega\tau}\underline{E} \qquad (V)$$

Note how now Ohm's law depends explicitly on the frequency, and especially

$$\lim_{\omega \to 0} \underline{J} = \sigma\underline{E}$$

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For the complete treatment we get back the dynamic equation, where

$$\nabla\times\nabla\times\underline{E} + \frac{1}{c^2}\frac{\partial^2\underline{E}}{\partial t^2} = -\mu_0\frac{\partial^2\underline{P}}{\partial t^2} - \mu_0\frac{\partial\underline{J}}{\partial t}$$

Since $\underline{P} = 0$ in this case, & therefore $\nabla\cdot\underline{P} = -\epsilon_0\nabla\cdot\underline{E} = 0$, we have

$$\Box\underline{E} = -\mu_0\frac{\partial\underline{J}}{\partial t}$$

Since $\Box E = \left(-\frac{\omega^2}{c^2} - k^2\right) \underline{E}$, as before, & as we've seen

$$\underline{J} = \frac{\sigma}{1-i\omega\tau} \underline{E}$$, the equation becomes

$$\Box \underline{E} = -\frac{\mu_0 \sigma}{1-i\omega\tau} \frac{\partial \underline{E}}{\partial t} \Rightarrow \left(k^2 - \frac{\omega^2}{c^2}\right) \underline{E} = i\omega\mu_0\sigma \frac{1}{1-i\omega\tau} \underline{E}$$

Therefore

$$k^2 \underline{E} = \left(\frac{\omega^2}{c^2} + \frac{i\omega\mu_0\sigma}{1+i\omega\tau}\right) \underline{E}$$

$$k^2 = \frac{\omega^2}{c^2} + \frac{i\omega\mu_0\sigma}{1+i\omega\tau}$$

Note how in the regime of low frequencies ($\underline{E} = \frac{1}{\sigma}\underline{J}$), we have

$$k^2 \approx i\omega\mu_0\sigma$$

Which gives

$$k_{LF} = \sqrt{i\omega\mu_0\sigma} = \sqrt{\frac{\omega\mu_0\sigma}{2}}(1+i)$$

In this case, $k_{LF} = \kappa + i\alpha$ & $\kappa = \alpha$, where $k = \kappa + i\alpha$

$$\kappa = \alpha = \sqrt{\frac{\omega\mu_0\sigma}{2}}$$

Similarly, said $k = \frac{\omega}{c}\mathcal{N}$ we have $\mathcal{N}_{LF} = \frac{c}{\omega}\sqrt{\frac{\omega\mu_0\sigma}{2}}(1+i) = \sqrt{\frac{\sigma}{2\omega\epsilon_0}}(1+i)$, therefore, also for $\mathcal{N}_{LF} = n + i\eta$

$$n = \eta = \sqrt{\frac{\sigma}{2\omega\epsilon_0}}$$

The max depth of penetration of the wave is known as << SKIN DEPTH >> of the material, determined as the e-folding value of the amplitude, thus:

$$\delta = \frac{1}{\alpha} = \sqrt{\frac{2}{\omega\mu_0\sigma}}$$

Using $\lambda_0\omega = 2\pi c \Rightarrow \lambda_0 = \frac{2\pi c}{\omega}$, we can rewrite the skin-depth in terms of VACUUM WAVELENGTH

$$\boxed{\delta = \sqrt{\frac{\lambda_0}{\pi c \mu_0 \sigma}}}$$

This formula clearly explains why good conductors (big σ) are HIGHLY OPAQUE

Returning to the general definition for k, we have

$$k^2 = \frac{\omega^2}{c^2} + i\omega\mu_0\sigma \frac{1}{1+i\omega\tau}$$

Thus, the complex index of refraction

$$N^2 = \frac{c^2}{\omega^2}k^2 = 1 + \frac{ic^2\mu_0\sigma}{\omega}\frac{1}{1+i\omega\tau}$$

Defined the <<PLASMA FREQUENCY>> ω_p as

$$\omega_p = \underbrace{\sqrt{\frac{\mu_0\sigma c^2}{\tau}}} = \sqrt{\frac{\mu_0 c^2}{\tau}}\sqrt{\frac{Ne^2\tau}{m}} = \sqrt{\frac{Ne^2c^2\mu_0}{m}} = \underbrace{\sqrt{\frac{Ne^2}{m\epsilon_0}}}$$

We can rewrite

$$N^2 = 1 + \frac{1}{\omega}\frac{i\omega_p^2}{\tau^{-1}+i\omega} = 1 - \frac{\omega_p^2}{\omega^2+i\omega\tau^{-1}}$$

Rationalizing on the right, we have

$$N^2 = 1 - \omega_p^2\frac{1}{\omega^4+\omega^2\tau^{-2}}\left(\omega^2 - i\omega\tau^{-1}\right)$$

$$N^2 = 1 - \frac{\omega_p^2}{\omega^2+\tau^{-2}} + \frac{i\omega_p^2}{\omega^2+\tau^{-2}}\frac{1}{\omega\tau}$$

Thus:

$$\begin{cases} \mathrm{Re}\{N^2\} = n^2-\eta^2 = 1 - \dfrac{\omega_p^2}{\omega^2+\tau^{-2}} \\ \mathrm{Im}\{N^2\} = 2n\eta = \dfrac{\omega_p^2}{\omega^2+\tau^{-2}}\left(\dfrac{1}{\omega\tau}\right) \end{cases} \quad \checkmark$$

The optical parameters n, η can be then obtained (numerically). For what we wrote, we have that both the parameters depend on ω_p, τ & ω.

For <u>METALS</u> we have $\tau \propto 10^{-13}\,s$, which corresponds to IR frequencies; $\omega_p \propto 10^{15}\,s^{-1}$ corresponding to VIS & NUV.

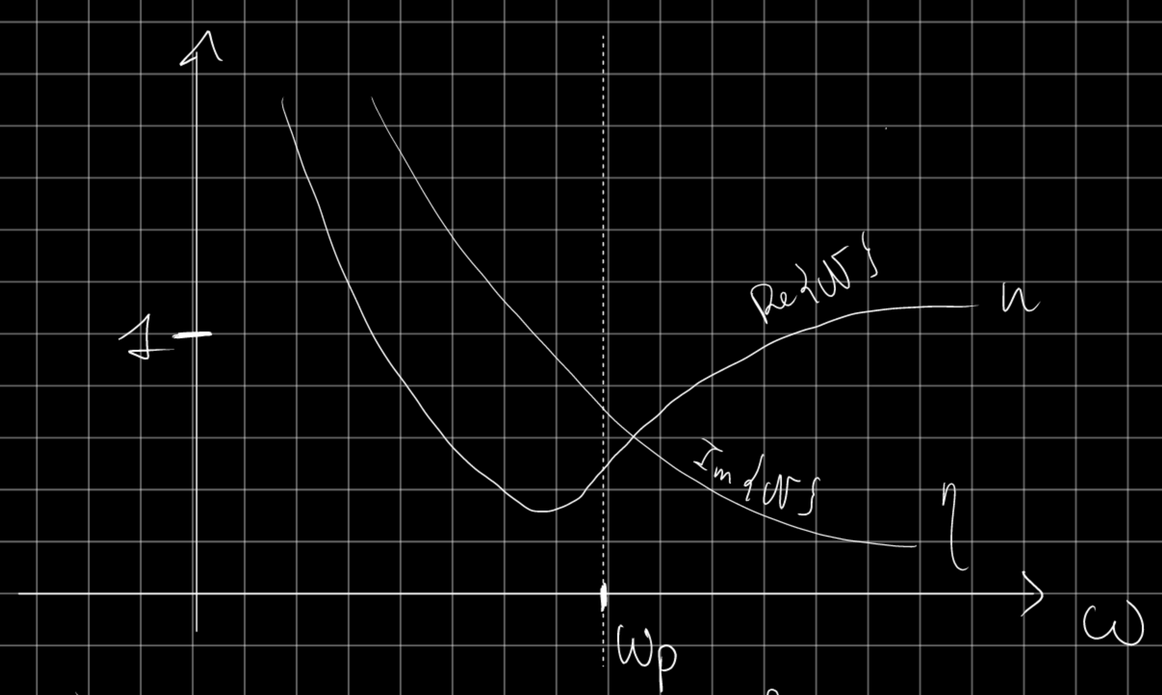

Fig 1) $n(\omega)$, $\eta(\omega)$. See how $n(\omega) < 1$ for many ω around ω_p

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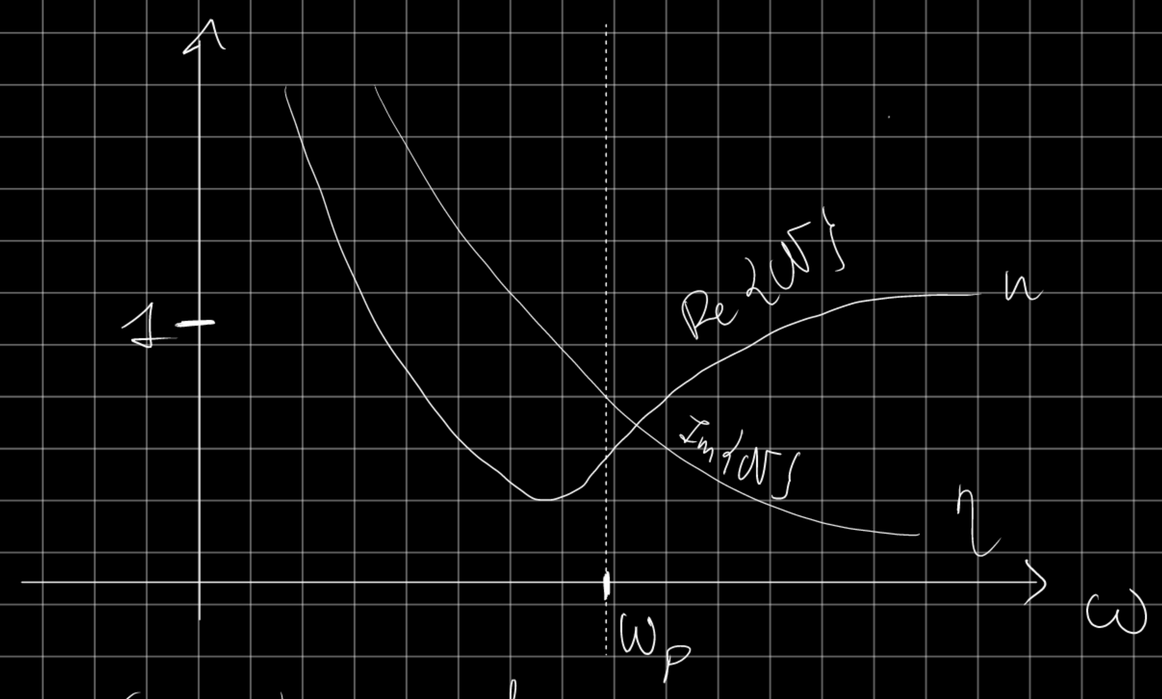

Fig 2.1c) Complex ref. index vs ω

As from the graph, the extinction coefficient η follows

$$\lim_{\omega \to \infty} \eta(\omega) = 0$$

Thus metals, become <u>TRANSPARENT</u> at high frequencies (low wavelengths)

For poor conductors & semiconductors we have

$$N_{SC}^2 = N_M^2 + N_D^2 = 1 - \frac{\omega_p^2}{\omega^2 + i\omega\tau^{-1}} + \frac{Ne^2}{m\epsilon_0} \sum_{j \geq 1} \left(\frac{f_j}{\omega_j^2 - \omega^2 - i\gamma_j \omega} \right)$$

QM gives a similar relation & can predict the value of the parameters

§ REFLECTION & REFRACTION @ BOUNDARY OF ABS. MEDIA

Let Ψ be a wave, incident at the boundary of a medium, with properties

$$\begin{cases} \mathcal{N} = n + i\eta \\ \underline{k} = \underline{K} + i\underline{\alpha} \end{cases}$$

Consider the first medium NONABSORBING ($\underline{\alpha}, \eta = 0$) w/ index n_1, as follows

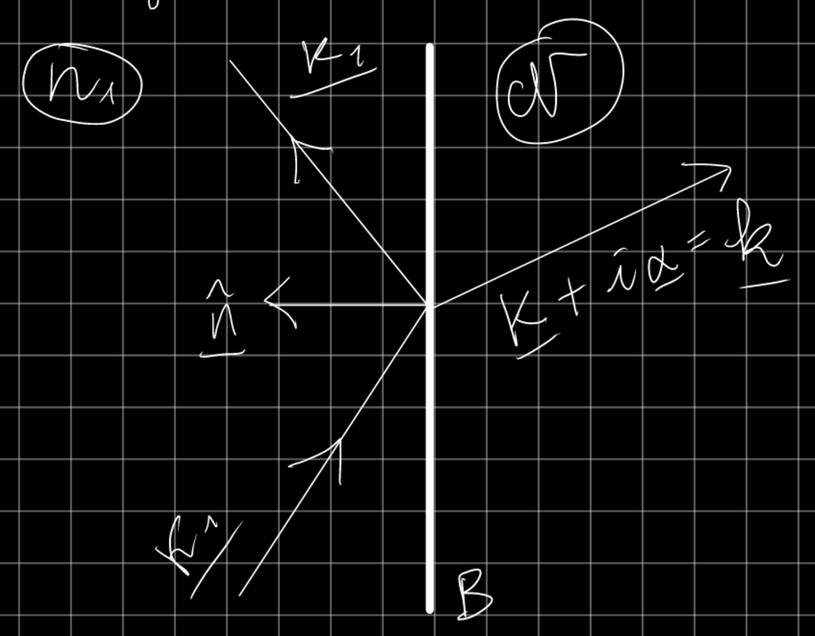

Fig 1.) Schema of the rt problem in absorbing media

As in the figure @ the left, at the boundary we get 3 waves

$$\begin{cases} e^{i\underline{k}_1 \underline{r} - i\omega t} = \Psi_1 & \text{INCIDENT WAVE} \\ e^{i\underline{k}_{1R} \underline{r} - i\omega t} = \Psi_{1R} & \text{REFLECTED WAVE} \\ e^{i(\underline{k}\,\underline{r} - i\omega t)} = \Psi_T & \text{TRANSMITTED WAVE} \end{cases}$$

As for the usual case, in the first region $\underline{K}_1 = \underline{K}_{1R}$ since they both lay on the region w/ index n_1. In the second region we have:

$$i\underline{k} = i\underline{K} - \underline{\alpha} \Rightarrow \Psi_T = e^{-\alpha r} e^{iKr - i\omega t}$$

Thus, as usual

$$\begin{cases} \underline{K}_1 \underline{r} = \underline{K}_{1R} \underline{r} \\ \underline{K}_1 \underline{r} = (\underline{K} + i\underline{\alpha})\underline{r} \end{cases} \qquad \begin{cases} \underline{K}_1 \underline{r} = \underline{K}\,\underline{r} \\ \underline{\alpha}\,\underline{r} = 0 \end{cases}$$

When the second system pops out equating Im & Re of both sides. For this reason, in general, $\underline{K}$ & $\underline{\alpha}$ DON'T HAVE THE SAME DIRECTION. The wave is known as <<INHOMOGENEOUS>>

In this case, $\underline{\alpha}\,\underline{r} = 0 \Rightarrow \underline{\alpha} \perp B$ (oundary)

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The waves travel in the direction of $\underline{K}$ but decay exponentially with $\underline{\alpha}$

Therefore, it's possible to define the planes w/ normal $\underline{K}$ define the phase planes, while, the planes with $\underline{\alpha}$ as normal define the planes of constant amplitude.

Denoting θ as the incidence angle & φ as the refraction angles we get

$$\underline{K}_1 \underline{r} = \underline{K}\,\underline{r} \Rightarrow K_1 \sin\theta = K \sin\varphi \quad \text{(by phase matching @ B)}$$

Note that $\underline{K}$ is not constant! $\underline{K} \equiv \underline{K}(\varphi)$

The relationship $\underline{k}(\varphi)$ can be determined starting from the wave equation.
For what we have said so far we can write the equation as follows

$$\nabla^2 \underline{E} = \frac{\mathcal{N}^2}{c^2}\frac{\partial^2 \underline{E}}{\partial t^2}$$

Therefore: $\hat{\nabla}\psi = i\underline{k}\psi$, $\partial_t \psi = -i\omega\psi$, thus:

$$(\underline{k}\cdot\underline{k})\underline{E} = \frac{\omega^2}{c^2}\mathcal{N}^2\underline{E} = k_0^2\mathcal{N}^2\underline{E}$$

where $k_0^2 = \frac{\omega^2}{c^2}$, thus:

$$(\underline{k}+i\underline{\alpha})(\underline{k}+i\underline{\alpha}) = k_0^2(n+i\eta)^2$$

$$k^2-\alpha^2+2i\underline{k}\cdot\underline{\alpha} = k_0^2(n^2-\eta^2+2in\eta)$$

Equating $Re\{\cdot\}$ & $Im\{\cdot\}$ we have

$$\begin{cases} k^2-\alpha^2 = k_0^2(n^2-\eta^2) \\ k\alpha\cos\varphi = k_0^2\eta n \end{cases}$$

(Can't work it out rn) The result is shown to be;

$$k\cos\varphi + i\alpha = k_0\sqrt{\mathcal{N}^2-\sin^2\theta} \quad (A)$$

Thus, for $\theta=0$ (normal incidence)

$$k\cos\varphi + i\alpha = k_0\mathcal{N} \Rightarrow \text{(same relation as for HOMOGENEOUS WAVES)}$$

In a <u>PURELY FORMAL WAY</u> the law of Refraction (SNELL) can be rewritten as

$$\mathcal{N} = \sin\theta\csc z_e \qquad \theta\in[0,2\pi) \quad z_e\in\mathbb{C}$$

From the above definition;

$$\sin z_e = \frac{\sin\theta}{\mathcal{N}} \Rightarrow \cos z_e = \sqrt{1-\frac{\sin^2\theta}{\mathcal{N}^2}} \quad (1)$$

<u>Therefore</u>, with (A) we have

$$k\cos\varphi + i\alpha = k_0\mathcal{N}\sqrt{1-\frac{\sin^2\theta}{\mathcal{N}^2}}$$

$$k\cos\varphi + i\alpha = k_0\mathcal{N}\cos z_e$$

Thus, the second equation is

$$\mathcal{N} = \frac{k\cos\varphi + i\alpha}{k_0}\sec z_e \quad (2)$$

We can now find the R of the absorbent medium.
The setup is as follows:

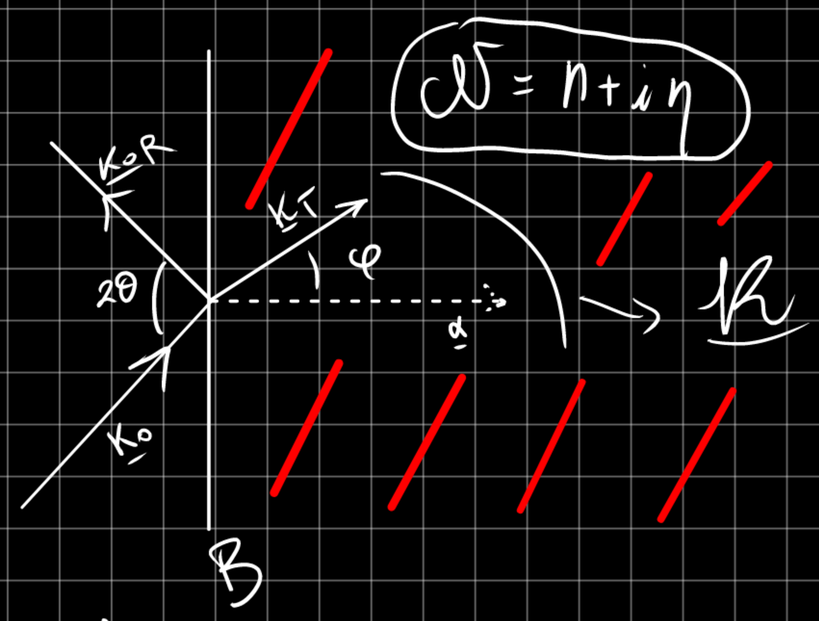


Fig a.) schema of the problem: $\underline{k}_0$ before, $\underline{k}$ after B

From Maxwell's equations, we know already that the relations between $\underline{E}$, $\underline{H}$ are

$$\begin{cases} \underline{H} = (\mu_0\omega)^{-1}\, \underline{k}_0 \times \underline{E} \\ \underline{H}_R = (\mu_0\omega)^{-1}\, \underline{k}_{0R} \times \underline{E} \\ \underline{H}_T = (\mu_0\omega)^{-1}\, \underline{k} \times \underline{E}_T \end{cases} \begin{cases} \| \\ \| \\ \underline{H}_T = (\mu_0\omega)^{-1}\left[\underline{k} \times \underline{E}_T + i\,\underline{\alpha} \times \underline{E}\right] \end{cases} \quad (E, H)$$

And, from the boundry continuity equations, we have for Senkrecht (s, TE) pol. & p, TM polarization:

$$\begin{cases} E + E_R = E_T \\ (H - H_R)\cos\theta = H_T \end{cases} \text{(S)} \qquad \begin{cases} H - H_R = H_T \\ (E + E_R)\cos\theta = E_T \cos(\tau_e) \end{cases} \text{(P)}$$

Which, can be rewritten as

$$\begin{cases} 1 + r_s = t_s \\ k_0(1 - r_s)\cos\theta = (k\cos\varphi + i\alpha)\, t_s \end{cases} \text{(S)} \qquad \begin{cases} k_0(1 - r_p) = k_0 \mathcal{N}\, t_p \\ (1 + r_p)\cos\theta = t_p \cos(\tau_e) \end{cases} \text{(P)}$$

Starting from Senkrecht polarization we have

$$(1 - r_s)\, k_0 \cos\theta = (1 + r_s)(k\cos\varphi + i\alpha)$$

Using $k\cos\varphi + i\alpha = k_0 \mathcal{N} \cos(\tau_e)$ we have

$$(1 - r_s)\, \cancel{k_0} \cos\theta = (1 + r_s)\, \cancel{k_0} \mathcal{N} \cos(\tau_e) \Rightarrow r_s\left(\mathcal{N}\cos(\tau_e) + \cos\theta\right) = \cos\theta - \mathcal{N}\cos(\tau_e)$$

Therefore

$$\boxed{r_s = \frac{\cos\theta - \mathcal{N}\cos(\tau_e)}{\cos\theta + \mathcal{N}\cos(\tau_e)}}$$

From the equation $1+r_s=t_s$, we can also get the transmission coefficient

$$\boxed{t_s = 1+\frac{\cos\theta-\tilde{n}\cos(\tilde{\theta})}{\cos\theta+\tilde{n}\cos(\tilde{\theta})} = \frac{2\cos\theta}{\cos\theta+\tilde{n}\cos(\tilde{\theta})}}$$

For (P) polarization, the evaluation is completely analogous

$$\begin{cases}(1-r_p)=\tilde{n}t_p\\(1+r_p)\cos\theta=t_p\cos(\tilde{\theta})\end{cases}\qquad\begin{cases}\tilde{n}^{-1}(1-r_p)=t_p\\(1+r_p)\cos\theta=\tilde{n}^{-1}(1-r_p)\cos(\tilde{\theta})\end{cases}$$

Thus

$$r_p\left(\cos\theta+\frac{1}{\tilde{n}}\cos(\tilde{\theta})\right)=\frac{1}{\tilde{n}}\cos(\tilde{\theta})-\cos\theta$$

which, multiplying both sides by $\tilde{n}$ gives

$$\boxed{r_p=\frac{\cos(\tilde{\theta})-\tilde{n}\cos\theta}{\cos(\tilde{\theta})+\tilde{n}\cos\theta}}$$

The transmission coefficient is

$$\boxed{t_p=\frac{1}{\tilde{n}}(1-r_p)=\frac{1}{\tilde{n}}\left(1-\frac{\cos(\tilde{\theta})-\tilde{n}\cos\theta}{\cos(\tilde{\theta})+\tilde{n}\cos\theta}\right)=\frac{2\cos\theta}{\tilde{n}\cos(\tilde{\theta})+\tilde{n}^2\cos\theta}}$$

As usual one can find R_p, R_s & T_p, T_s using $T=t\bar{t}$, $R=r\bar{r}$

Note that for P polarization, while $\mathrm{Im}\{\tilde{n}\}\neq 0$, $r_p\neq 0$! Thus, the Brewster angle is substituted by the << principal angle of incidence >> θ_1, for which we have $\min\{r_p\}=r_p(\theta_1)$.

As usual, mixes of S & p polarized light get transmitted in elliptical polarization $\tilde{n}$, can be evaluated by measuring I_T & the polarization of outgoing light with methods of ELLIPSOMETRY.

§ NORMAL INCIDENCE

In the special case of $\theta=0$ (normal incidence) we have that again $r_p=r_s=r$ &

$$r=\frac{1-\tilde{n}}{1+\tilde{n}}=\frac{1-n-i\eta}{1+n+i\eta}$$

Thus

$$R=r\bar{r}=\frac{(1-n)^2+\eta^2}{(1+n)^2+\eta^2}$$

which reduces to the dielectric case when $\eta=\mathrm{Im}\{\tilde{n}\}=0$.

Since we know that

$$\lim_{\omega \to 0} \mathrm{Re}\{\tilde{n}\} = \lim_{\omega \to 0} \mathrm{Im}\{\tilde{n}\} = \sqrt{\frac{\sigma}{2\omega\epsilon_0}}$$

& from the last equation when $R \equiv R(\omega)$ we have then:

$$\lim_{\omega \to 0} R(\omega) = 1 - \frac{2}{n} = 1 - \sqrt{\frac{8\omega\epsilon_0}{\sigma}} \quad (HR)$$

known as Hagen-Rubens formula. Therefore, when $\omega \to$ red (λ grows) we have that good conductors become better and better reflectors. As an example

$$\text{Cu, Ag, Au} \quad \begin{cases} R(\lambda) \approx 1 & \text{NIR } (\lambda \approx 1\,\mu m / 2\,\mu m) \\ R(\lambda) \cong 1 & \text{FIR } (\lambda > 20\,\mu m) \end{cases}$$

§ PROPAGATION OF LIGHT IN CRYSTALS

5/12/22 — 22.47 — 1h 54m 34.00s

The main property of matter in the cristalline state, in the case of optical properties, is its electrical anisotropy.
This means that the polarization produced by the application of a $\underline{E}$ field is dependent on direction.
There are 2 possible values of propagation velocity for a given direction in a crystal, tied to mutually orthogonal polarizations, thus crystals are said << BIREFRENGENT >>, i.e. doubly refracting.
Note tho that NOT ALL CRYSTALS EXHIBIT BIREFRENGENCE, it depends only on their symmetry. Cubic crystals like NaCl do NOT exhibit double refrengence, while others do.
One practical way to think physically of why such phenomenon arises is by thinking the lattice atom bonds as springs with different strength k, thus electron displacement is different in each direction, thus, the dependence $\underline{P}(\underline{E})$ must be expressed TENSORIALLY

$$P^i = \epsilon_0 \chi^i_j E_j$$

where χ^i_j is the ELECTRIC SUSCEPTIBILITY TENSOR. The corresponding $\underline{D}$ field is then:

$$D^i = \epsilon_0 (\delta^i_j + \chi^i_j) E^j = \epsilon^i_j E^j$$

where

$$\epsilon^i_j = \epsilon_0 (\delta^i_j + \chi^i_j)$$

Is the DIELECTRIC TENSOR.
For ordinary NON-ABSORBING crystals the tensor χ^i_j is diagonalizable into principal axes $(\hat{\underline{e}}_1, \hat{\underline{e}}_2, \hat{\underline{e}}_3)$, where

$$\chi^i_j = (\chi_{11} \hat{\underline{e}}_1 + \chi_{22} \hat{\underline{e}}_2 + \chi_{33} \hat{\underline{e}}_3) \delta^i_j$$

And the eigenvalues $\chi_{11}, \chi_{22}, \chi_{33}$ are known as the PRINCIPAL SUSCEPTIBILITIES of the crystal, corresponding each to dielectric constants

$$(\epsilon_r)^i_j = \delta^i_j + \chi^i_j$$

The general wave equation can then be rewritten as

$$\epsilon^i{}_{jk}\,\partial^j \epsilon^k{}_{lm}\,\partial^l E^m + \frac{1}{c^2}\frac{\partial^2 E^i}{\partial t^2} = -\frac{1}{c^2}\chi^i_j \frac{\partial^2 E^j}{\partial t^2}$$

Imposing $E^i = E_0^i \exp(ik^j r_j - i\omega t)$ as usual, & writing the operatorial relation

$$\begin{cases} \partial_j \to ik_j \\ \partial_t \to -i\omega \end{cases}$$

The previous equation becomes:

$$\epsilon^i{}_{jk}\,k^j \epsilon^k{}_{lm}\,k^l E^m + \frac{\omega^2}{c^2}E^i = -\frac{\omega^2}{c^2}\chi^i_j E^j$$

If χ^i_j is diagonal, in terms of components we can write

$$\begin{cases} \left(\frac{\omega^2}{c^2} - k_y^2 - k_z^2\right)E_x + k_x k_y E_y + k_x k_z E_z = -\frac{\omega^2}{c^2}\chi_{11}E_x \\ \left(\frac{\omega^2}{c^2} - k_x^2 - k_z^2\right)E_y + k_y k_z E_z + k_y k_x E_x = -\frac{\omega^2}{c^2}\chi_{22}E_y \\ \left(\frac{\omega^2}{c^2} - k_y^2 - k_x^2\right)E_z + k_z k_y E_y + k_z k_x E_x = -\frac{\omega^2}{c^2}\chi_{33}E_z \end{cases} \quad \text{(BRF)}$$

This system is quite complicated, therefore, for understanding it, suppose that the wave propagates along the x axis, thus

$$\underline{k} = (k, 0, 0)$$

The equations simplify to

$$\begin{cases} \frac{\omega^2}{c^2}E_x = -\frac{\omega^2}{c^2}\chi_{11}E_x \\ \left(\frac{\omega^2}{c^2} - k^2\right)E_y = -\frac{\omega^2}{c^2}\chi_{22}E_y \\ \left(\frac{\omega^2}{c^2} - k^2\right)E_z = -\frac{\omega^2}{c^2}\chi_{33}E_z \end{cases}$$

This system has clearly 2 different solutions. 1st we note that $\underline{E} \perp \underline{k}$ as it should, but, if

1) $E_y \neq 0 \Rightarrow k_1 = \sqrt{1+\chi_{22}} = \frac{\omega}{c}\sqrt{(\epsilon_r)_{22}}$

2) $E_z \neq 0 \Rightarrow k_2 = \sqrt{1+\chi_{33}} = \frac{\omega}{c}\sqrt{(\epsilon_r)_{33}}$

Since $\frac{\omega}{k}$ is the phase velocity of the wave, we have <u>TWO</u> different PV

$$u_1 = \frac{c}{\sqrt{1+\chi_{22}}}$$

$$u_2 = \frac{c}{\sqrt{1+\chi_{33}}}$$

More generally, in terms of refraction index $n = \sqrt{1+\chi}$, we have that for a generic diagonalizable χ^i_j tensor, we have $\underline{3}$ n_i indexes (PRINCIPAL REFRACTION INDEXES)

$$\begin{cases} n_1 = \sqrt{1+\chi_{11}} \\ n_2 = \sqrt{1+\chi_{22}} \\ n_3 = \sqrt{1+\chi_{33}} \end{cases}$$

These come in handy for simplifying eqn (BRF). For having NONZERO solutions we must have

$$\det_3 \begin{pmatrix} \frac{n_1^2\omega^2}{c^2} - k_y^2 - k_z^2 & k_y k_x & k_x k_z \\ k_y k_x & \frac{n_2^2\omega^2}{c^2} - k_x^2 - k_z^2 & k_y k_z \\ k_z k_x & k_z k_y & \frac{n_3^2\omega^2}{c^2} - k_x^2 - k_y^2 \end{pmatrix} \neq 0$$

$\Rightarrow$ This is a surface in $\underline{k}$ space!

Putting ourselves on the slice $k_z = 0$ the determinant reduces to a simple product

$$\left(\frac{n_3^2\omega^2}{c^2} - k_x^2 - k_y^2\right)\left\{\left[\frac{n_1^2\omega^2}{c^2} - k_y^2\right]\left[\frac{n_2^2\omega^2}{c^2} - k_x^2\right] - k_x^2 k_y^2\right\} = 0$$

We get thus $\underline{2}$ solutions

$$k_x^2 + k_y^2 = n_3^2 \frac{\omega^2}{c^2} \qquad \text{(SPHERE)}$$

$$\left(\frac{n_1\omega}{c}\right)^2\left(\frac{n_2\omega}{c}\right)^2 - k_y^2\left(\frac{n_2\omega}{c}\right)^2 - k_x\left(\frac{n_1\omega}{c}\right)^2 = 0$$

The second equation can be made clearer dividing by $\left(\frac{n_1\omega}{c}\right)^2\left(\frac{n_2\omega}{c}\right)^2$, giving

$$\frac{k_x^2}{(n_2\omega/c)^2} + \frac{k_y^2}{(n_1\omega/c)^2} = 1 \qquad \text{(ELLIPSE)}$$

Similar equations are found slicing the xz & yz planes (+10 m). The resulting were phase surface consists of an INNER (spherical) sheet & an EXTERNAL (ellipsoidal) sheet

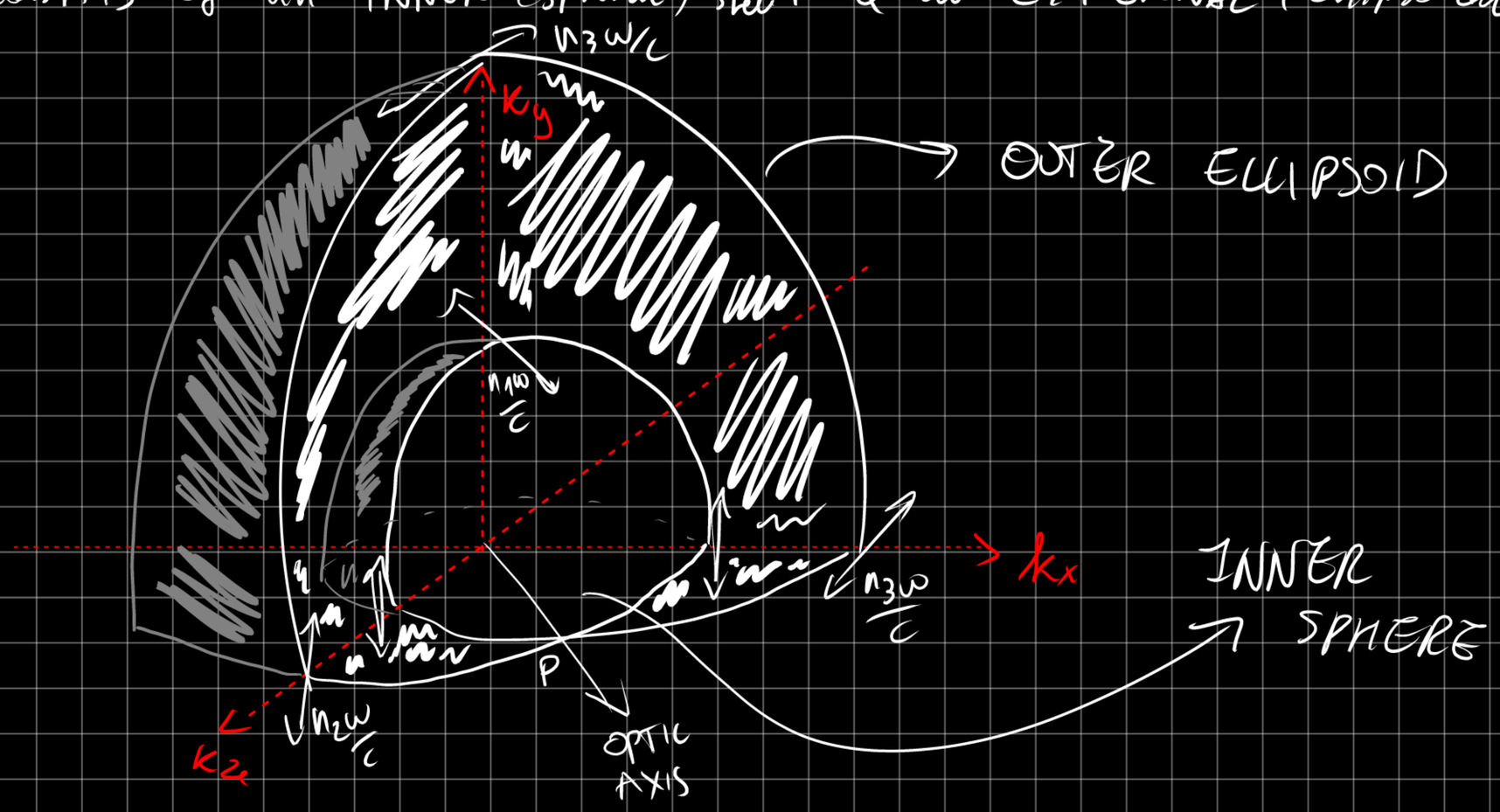


Fig A) SPHERO-ELLIPSOID FOR CRYSTAL PROPAGATION W/ PHASE VELOCITIES $u_i = \frac{n_i\omega}{c}$

09/11/22 22.57 — 4h 19m 32.35s

New record!

So, recap; the 2 equations solved by $\underline{k}$ are (on $k_x k_y$ plane)

$$\begin{cases} k_x^2 + k_y^2 = \left(\frac{n_3\omega}{c}\right)^2 & (\text{CIRCLE } R = \frac{n_3\omega}{c}) \\ \left(\frac{c}{n_3\omega}\right)^2 k_x^2 + \left(\frac{c}{n_1\omega}\right)^2 k_y^2 = 1 & (\text{ELLIPSE } a = \frac{n_3\omega}{c},\ b = \frac{n_1\omega}{c}) \end{cases}$$

Thus, the vector $\underline{k}$ lays both on an ellipsoid & a sphere {after solving for $k_z k_y$, $k_z k_x$ planes}, forming an INNER SPHERICAL SHEET & an OUTER ELLIPSOIDAL SHEET, giving the 2 possible solutions for $\|\underline{k}\| \Rightarrow$ 2 different orthogonal polarizations of $\underline{E}$. When an emw passes through a crystal can then be divided into 2 ORTHOGONAL polarizations traveling with 2 different phase velocities.

Said z the propagation direction of the wave & said $\hat{\underline{\chi}}_1, \hat{\underline{\chi}}_2$ the 2 principal directions of the crystal in the xy plane, we can see the generally polarized emw as:

$$\underline{E}(\underline{r},t) = (\underline{E}\cdot\hat{\underline{\chi}}_1)e^{i\underline{k}_1\underline{r} - i\omega t} + (\underline{E}\cdot\hat{\underline{\chi}}_2)e^{i\underline{k}_2\underline{r} - i\omega t}$$

where k_1 & k_2 are the 2 possible values of $\underline{k}$, corresponding to 2 refraction indexes

$$n_1 = \sqrt{1+\chi_{11}}, \quad n_2 = \sqrt{1+\chi_{22}}$$

where χ_{11}, χ_{22} are the eigenvalues of the χ^i_j tensor.

The two phase surfaces S_I, E_O have an intersection point $\{P\} = S_I \cap E_O$. This point defines the OPTICAL AXIS OF THE CRYSTAL. When the wave travels along this axis the 2 phase velocities ARE EQUAL $u_1 = u_2 = u$

From the different values of n_i we can distinguish 3 kinds of crystals

CRYSTALLOGRAPHIC SYSTEM

1) ISOTROPIC $\chi^i_j = \begin{pmatrix} \chi & 0 & 0 \\ 0 & \chi & 0 \\ 0 & 0 & \chi \end{pmatrix}$, $n = \sqrt{1+\chi}$ {MONO REFRACTING}

→ cubic

2) UNIAXIAL $\chi^i_j = \begin{pmatrix} \chi_1 & 0 & 0 \\ 0 & \chi_1 & 0 \\ 0 & 0 & \chi_2 \end{pmatrix}$, $n_O = \sqrt{1+\chi_1}$, $n_E = \sqrt{1+\chi_2}$ {BIREFRENGENT *}

→ {trigonal, tetragonal, hexagonal}

3) BIAXIAL $\chi^i_j = \begin{pmatrix} \chi_1 & 0 & 0 \\ 0 & \chi_2 & 0 \\ 0 & 0 & \chi_3 \end{pmatrix}$, $n_1 = \sqrt{1+\chi_1}$, $n_2 = \sqrt{1+\chi_2}$, $n_3 = \sqrt{1+\chi_3}$ {BIREFRENGENT}

→ {triclinic, monoclinic, orthorhombic}

In uniaxial crystals, when $\chi^i_j = (\chi_1, \chi_1, \chi_2)^i_j$ we distinguish 2 refraction indexes

- $n_E = \sqrt{1+\chi_2} \rightarrow$ EXTRAORDINARY
- $n_O = \sqrt{1+\chi_1} \rightarrow$ ORDINARY

And from these it's possible to distinguish 2 kinds of uniaxial crystals $\begin{cases} n_E > n_O & \text{POSITIVE} \\ n_O > n_E & \text{NEGATIVE} \end{cases}$

Thus, optically, crystals can be categorized in

ISOTROPIC $\rightarrow$ ∞ OPTICAL AXES {SPHERE}

UNIAXIAL $\begin{cases} \text{POSITIVE} \rightarrow SCE \\ \text{NEGATIVE} \rightarrow ECS \end{cases}$ 1 OPTICAL AXIS {SPHERE + REV ELLIPSOID}

BIAXIAL $\rightarrow$ 2 OPTICAL AXES {SPHERE + ELLIPSOID}

{ PHASE VELOCITY SURFACE $\begin{cases} v \rightarrow \text{Phase velocity} \\ u \rightarrow \text{ray velocity} \end{cases}$

By definition, $\underline{k} = \frac{v}{v^2}\omega$, therefore it's possible to rewrite the surface-defining determinant as

$$\det_3 \begin{pmatrix} n_1^2 \frac{v^4}{c^2} - v_y^2 - v_z^2 & v_x v_y & v_x v_z \\ v_y v_x & n_2^2 \frac{v^4}{c^2} - v_x^2 - v_z^2 & v_y v_z \\ v_z v_x & v_z v_y & n_3^2 \frac{v^4}{c^2} - v_y^2 - v_x^2 \end{pmatrix}\Bigg|_{xy} = \left(n_3^2 \frac{v_3^4}{c^2} - v_y^2 - v_x^2\right)\left\{\left(n_1^2 \frac{v^4}{c^2} - v_y^2\right)\left(\frac{n_2 v^4}{c^2} - v_x^2\right) - v_x^2 v_y^2\right\} = 0$$

which defines, as before, 2 surfaces

$$\begin{cases} G_1 = \left\{ v_x^2 + v_y^2 = \frac{n_3^2 v^4}{c^2} \right\} \\ G_2 = \left\{ \frac{c^2}{n_2^2 v^4} v_x^2 + \frac{c^2}{n_1^2 v^4} v_y^2 = 1 \right\} \end{cases}$$

These 2 surfaces are <u>RECIPROCAL</u> to the $\underline{k}$ surface, known as PHASE-VELOCITY SURFACES. They form circles + 4th degree ovals in all xy, xz, yz planes

{} POYNTING VECTOR & RAY VELOCITY

In the case of crystals, $\underline{k} \nparallel \underline{S} = \underline{E} \times \underline{H}$ generally, since, in general $\underline{k} \nperp \underline{E}$. This holds for $\underline{H}$ since

$$\underline{k} \times \underline{E} = \mu_0 \omega \underline{H} \Rightarrow \underline{H} \perp \underline{E}, \underline{H} \perp \underline{k}$$

By definition: $\underline{S} \perp \underline{H}$, $\underline{S} \perp \underline{E}$. Consider now a narrow beam of light in a generic crystal. $\underline{k}$ still defines the planes of constant phase, <u>BUT</u> $\underline{k} \nparallel \underline{S}$ thus $\underline{S} \perp \pi_{cp}$.

Let

$$\theta = \arccos\left(\frac{S \cdot k}{\|S k\|}\right)$$

then the planes of constant phase π_{cp} move w/ velocity u {RAY VELOCITY} along the RAY DIRECTION defined by $\underline{S}$. By definition, then

$$u = \frac{v}{\cos\theta}$$

So $u < v$ unless $\theta = 0$ (when the prop. direction is along a principal axis of the crystal {Here $S \parallel k$ (!)}

§ RAY VELOCITY SURFACE

For convenience in calculations later, we use the vector D^i, defined as before

$$D^i = \epsilon_0 \left(\delta^i_j + \chi^i_j\right) E^j$$

Then, the wave equation becomes

$$\epsilon^i{}_{jk} k^j \epsilon^k{}_{lm} k^l E^m = -\frac{\omega^2}{c^2 \epsilon_0} D^i$$

Using the identities of $\epsilon^i{}_{jk}$, we can rewrite the equation as

$$k^i (k^j E_j) - k^2 E^i = -\frac{\omega^2}{c^2 \epsilon_0} D^i$$

Since $k^i D_i = 0$, we have

$$k^2 D^i E_i = \frac{\omega^2}{c^2 \epsilon_0} D^2$$

Using $v = \frac{\omega}{k}$, we can rewrite it as

$$D^i E_i = E D \cos\theta = \frac{v^2}{c^2 \epsilon_0} D^2$$

If we put ourselves in the principal coordinate system of the crystal $(\hat{x}_1, \hat{x}_2, \hat{x}_3)$, then we know:

$$1)\ \epsilon^i_j = 1 + \chi^i_j = \begin{pmatrix} n_1^2 & 0 & 0 \\ 0 & n_2^2 & 0 \\ 0 & 0 & n_3^2 \end{pmatrix}$$

$$2)\ \epsilon_0 n^2_{(i)} E^i = D^i$$

Thus:

$$\begin{cases} \left(\frac{c^2}{n_1^2} - u_y^2 - u_z^2\right) D_x + u_x u_y D_y + u_x u_z D_z = 0 \\ u_y u_x D_x + \left(\frac{c^2}{n_2^2} - u_x^2 - u_z^2\right) D_y + u_y u_z D_z = 0 \\ u_z u_x D_x + u_z u_y D_y + \left(\frac{c^2}{n_3^2} - u_y^2 - u_x^2\right) D_z = 0 \end{cases}$$

or, in matrix form

$$\begin{pmatrix} \frac{c^2}{n_1^2} - u_y^2 - u_z^2 & u_x u_y & u_x u_z \\ u_y u_x & \frac{c^2}{n_2^2} - u_x^2 - u_z^2 & u_y u_z \\ u_z u_x & u_z u_y & \frac{c^2}{n_3^2} - u_x^2 - u_y^2 \end{pmatrix} \begin{pmatrix} D_x \\ D_y \\ D_z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Therefore, as before, putting ourselves in the xy plane $\{u_z = 0\}$ we get <u>2</u> equations that solve

+1 PG

the equations, as before, are found with:

$$\det_3 \begin{pmatrix} \frac{c^2}{n_1^2} - u_y^2 & u_x u_y & 0 \\ u_y u_x & \frac{c^2}{n_2^2} - u_x^2 & 0 \\ 0 & 0 & \frac{c^2}{n_3^2} - u_x^2 - u_y^2 \end{pmatrix} = \left(\frac{c^2}{n_3^2} - u_x^2 - u_y^2\right)\left\{\left[\frac{c^2}{n_1^2} - u_x^2\right]\left[\frac{c^2}{n_2^2} - u_y^2\right] - u_x^2 u_y^2\right\} = 0$$

with solutions

a) $u_x^2 + u_y^2 = \frac{c^2}{n_3^2}$ (circle w/ $R = \frac{c}{n_3}$)

b_1) $\frac{c^2}{n_2^2} u_x^2 + \frac{c^2}{n_1^2} u_y^2 = \frac{c^4}{n_1^2 n_2^2}$

b) $n_1^2 u_x^2 + n_2^2 u_y^2 = c^2$ (Ellipse)

The intercepts <u>are the SAME</u> of the phase-velocity surface & define the <u>RAY AXES</u>

§ DOUBLE REFRACTION @ BOUNDARY

Consider an incident wave to a crystal $(\underline{k}_0, \underline{E}_0, \underline{H}_0)$. Consider only transmitted wave(s) $(\underline{k}_1, \underline{E}_1, \underline{H}_1)$, $(\underline{k}_2, \underline{E}_2, \underline{H}_2)$. Said (θ, φ) the refraction angles for resp. $\underline{k}_1$ & $\underline{k}_2$, using

$$\underline{k}_0 \cdot \underline{r} = \underline{k} \cdot \underline{r} \quad (\text{at } B)$$

And noting the presence of 2 solutions $\underline{k}_1 \cdot \underline{r}$, $\underline{k}_2 \cdot \underline{r}$ <u>INSIDE</u> the crystal, we get from the following drawing

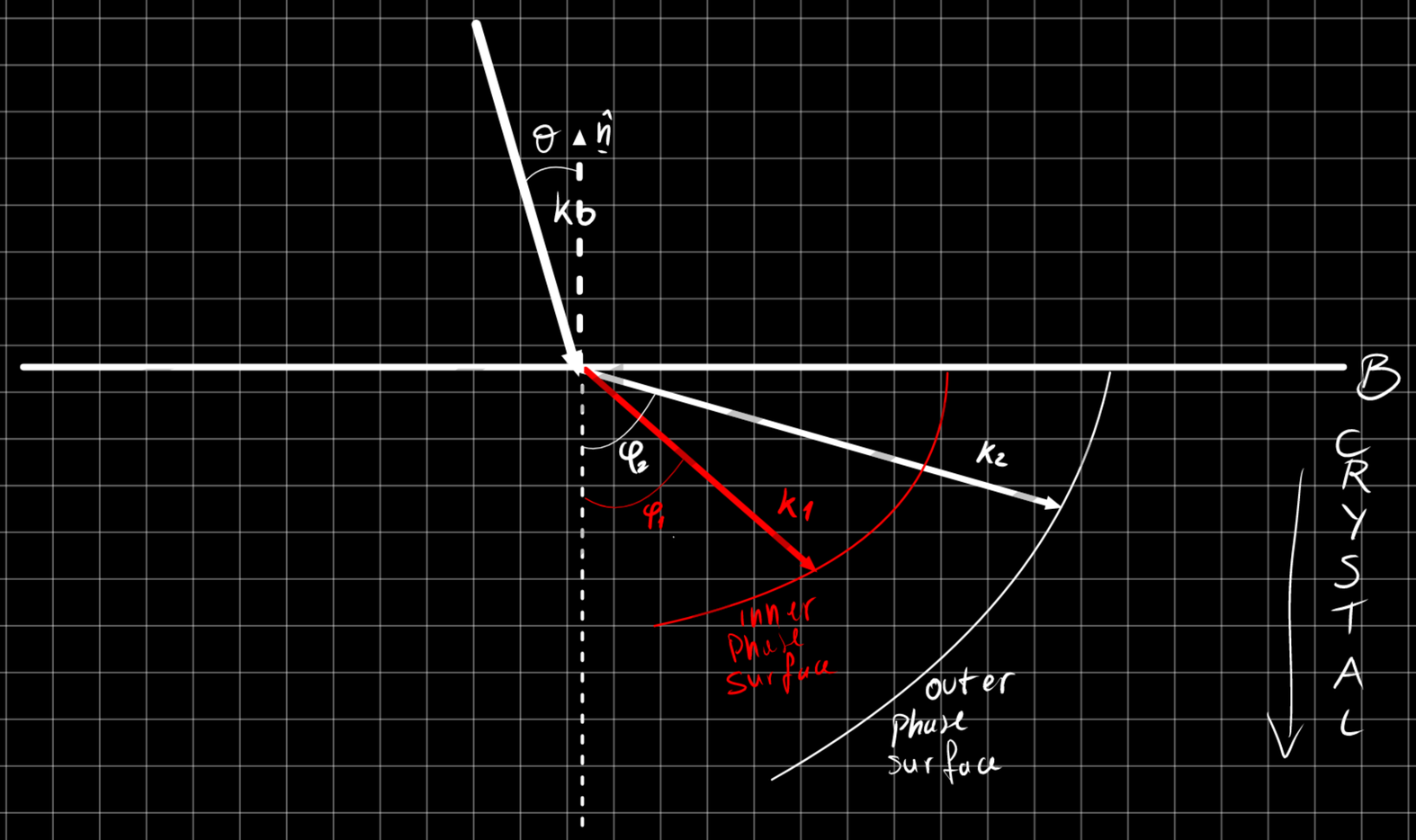


Fig BR.) Birefringence scheme for a generic crystal

The solutions are 2

$$\begin{cases} k_0 \sin\theta = k_1 \sin\varphi_1 \\ k_0 \sin\theta = k_2 \sin\varphi_2 \end{cases}$$

THIS IS NOT EQUIVALENT TO SNELL'S LAW. $k_1 \equiv k_1(\varphi_1)$, $k_2 \equiv k_2(\varphi_2)$!

$$\frac{\sin\theta}{\sin\varphi_i} \equiv f(\varphi_i) \qquad ((!))$$

In general it's not directly solvable without knowing the functional dependence on φ_1 & φ_2 of k_1, k_2.

This changes for uniaxial crystals. Here, in fact it's possible to say, considering the $\underline{k}_0$ $\{\underline{k}\text{-ORDINARY}, \|k\| = n_0^2 \frac{\omega^2}{c^2}\}$ the idea of an <u>ORDINARY</u> & <u>EXTRAORDINARY</u> wave.

Said φ_0 the ordinary ref. angle & φ_E the extraordinary angle we have, for φ_0

$$\frac{\sin\theta}{\sin\varphi_0} = n_0 \quad (! \text{ Just because } \|k\| = n_0^2 \frac{\omega^2}{c^2} \; !!)$$

But

$$\frac{\sin\theta}{\sin\varphi_E} \neq n_E \quad (!! \; \|k\| \text{ ON ELLIPSOID } !!)$$

In general, we can say that:

- FOR POSITIVE UNIAXIAL CRYSTALS ($n_O < n_E$)

$$\varphi_E \leq \varphi_O$$

- FOR NEGATIVE UNIAXIAL CRYSTALS ($n_O > n_E$)

$$\varphi_E \geq \varphi_O$$

SEE
Born M., Wolf E.
Principles of Optics
NY 1964 MacMillan
FOR PROOF

§ POLARIZING PRISMS

Consider an incident wave from the INSIDE of a crystal to a boundary $\mathcal{B}$, where the optical axis is $\perp$ plane of incidence

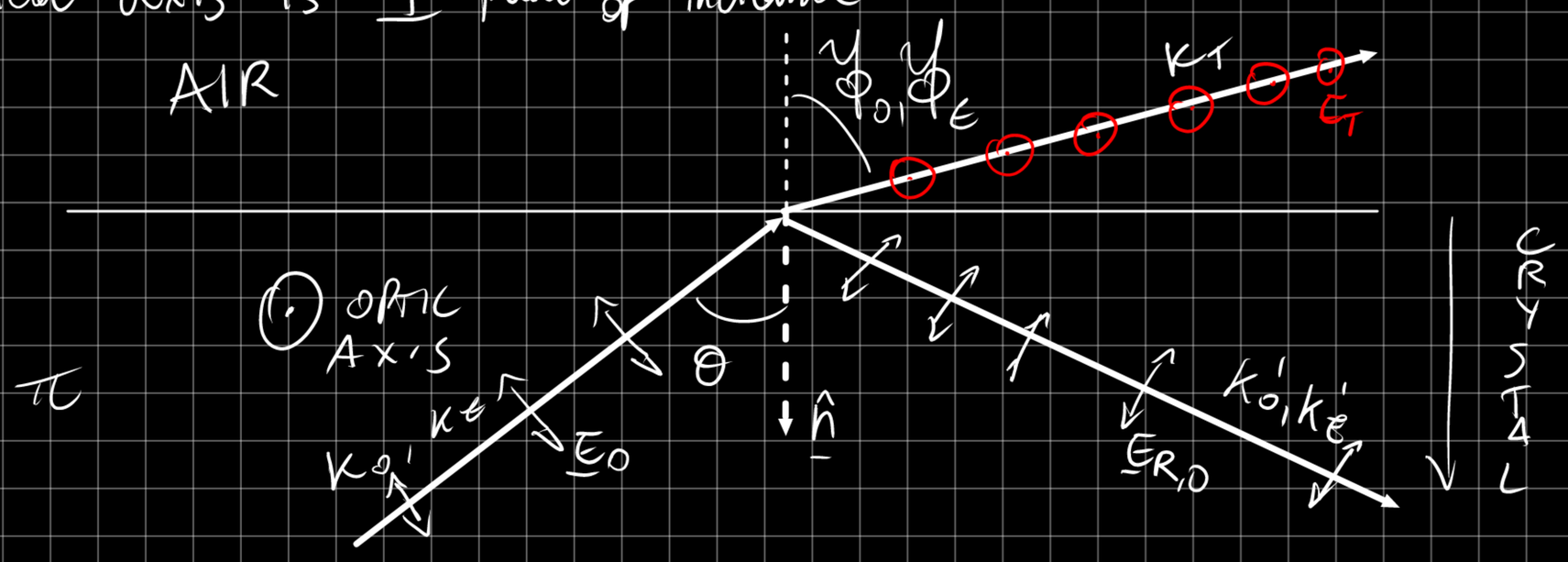


Fig PP1) incoming wave from INSIDE a crystal, OUTGOING into air

Said π the plane of incidence, since $OA \perp \pi$ {OA = OPTIC AXIS} Snell holds for both ordinary & extraordinary waves. Thus:

$$\begin{cases} k_O \sin\varphi_O = k\sin\theta \\ k_E \sin\varphi_E = k\sin\theta \end{cases} \Rightarrow \begin{cases} n_O \sin\varphi_O = \sin\theta \\ n_E \sin\varphi_E = \sin\theta \end{cases}$$

If the ordinary wave is (K_O, E_O, H_O) & the extraordinary is (K_E, E_E, H_E) then:

$$\begin{cases} \underline{E}_O \perp OA \\ \underline{E}_E \parallel OA \end{cases}$$

Suppose now that we have a NEGATIVE UNIAXIAL CRYSTAL $\{\varphi_E \geq \varphi_O, n_O > n_E\}$.
Then, given that $\varphi_O, \varphi_E, \theta \in (0, \frac{\pi}{2})$, we have for $\sin\theta$

$$\text{REF/TRA} \leftarrow \left(n_E < \frac{1}{\sin\theta} < n_O\right) \rightarrow \text{TIR}$$

or, for better clarity

$$\begin{cases} \sin\theta > n_O \Rightarrow \text{TIR for ORDINARY} \\ \sin\theta < n_E \Rightarrow \text{REF/TRA for EXTRAORDINARY} \end{cases}$$

the refracted (transmitted) wave is thus COMPLETELY POLARIZED ON THE DIRECTION OF THE OPTICAL AXIS (By means of Double Refraction)

§ OPTICAL ACTIVITY

An object is said <<OPTICALLY ACTIVE>> IFF it can rotate the polarization plane of a wave. The polarization gets rotated by an angle θ, where $\theta \propto l$, where l is the length of the path INSIDE the object.

The amount of rotation / length ($d\theta/dx$) is known as the SPECIFIC ROTATORY POWER

Objects can be DEXTROROTATORY or LEVOROTATORY depending on the handedness of the rotation

The amount of optical activity is wavelength dependent {Rotatory dispersion}.

The phenomena can be explained on the basis that LCP light has a different phase velocity than RCP light (left/right circularly polarized). Said n_R, n_L the indices of "right" & "left" refraction and noted $k_R = n_R \frac{\omega}{c}$, $k_L = n_L \frac{\omega}{c}$ we can write:

$$\begin{cases} \langle \underline{k}, t | RCP \rangle = \begin{pmatrix} 1 \\ -i \end{pmatrix} e^{ik_R z - i\omega t} \\ \langle \underline{k}, t | LCP \rangle = \begin{pmatrix} 1 \\ i \end{pmatrix} e^{ik_L z - i\omega t} \end{cases}$$

Suppose that our beam $|\underline{k}\rangle$ is LINEARLY POLARIZED. Then:

$$\langle \underline{r} | \underline{k} \rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

And

$$\left(\langle LCP| + \langle RCP| \right) |\underline{k}\rangle = \frac{1}{2}\begin{pmatrix} 1 \\ -i \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 \\ i \end{pmatrix} \rightarrow \frac{1}{2}\left(|RCP\rangle + |LCP\rangle \right)$$

After a distance l, said $\hat{T}_l$ the translation operator by l, we have

$$\hat{T}_l |\underline{k}\rangle = \frac{1}{2}\hat{T}_l |RCP\rangle + \frac{1}{2}\hat{T}_l |LCP\rangle = \frac{1}{2} e^{ik_R l} |RCP\rangle + \frac{1}{2} e^{ik_L l} |LCP\rangle$$

I.e.

$$\hat{T}_l |\underline{k}\rangle = \frac{1}{2} e^{\frac{1}{2} i (k_R + k_L) l} \left\{ e^{\frac{1}{2} i (k_R - k_L) l} |RCP\rangle + e^{-\frac{1}{2} i (k_R - k_L) l} |LCP\rangle \right\}$$

Introducing the quantities

$$\begin{cases} \psi = \frac{l}{2}(k_R + k_L) \\ \theta = \frac{l}{2}(k_R - k_L) \end{cases}$$

We have

$$\hat{T}_l |\underline{k}\rangle = \frac{1}{2} e^{i\psi} \left\{ e^{i\theta} |RCP\rangle + e^{-i\theta} |LCP\rangle \right\}$$

Expanding in terms of polarization basis:

$$\hat{T}_l \langle P | \underline{k}\rangle = \frac{1}{2} e^{i\psi} \left\{ \begin{bmatrix} e^{i\theta} \\ -ie^{i\theta} \end{bmatrix} + \begin{bmatrix} e^{-i\theta} \\ ie^{-i\theta} \end{bmatrix} \right\}$$

Summing, thus

$$\hat{T}_\ell \langle P | \underline{k} \rangle = e^{i\varphi} \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

Which is exactly a linearly polarized wave rotated by the angle θ:

$$\theta = \frac{1}{2}\ell(k_R - k_L) = \frac{\ell}{2}\left(n_R\frac{\omega}{c} - n_L\frac{\omega}{c}\right) = \frac{\omega\ell}{2c}(n_R - n_L) \quad \checkmark$$

In terms of wavelengths, then:

$$\theta(\lambda) = \frac{\pi\ell}{\lambda_0}(n_R - n_L)$$

Therefore, said $\ell(x) = x$

$$\frac{\partial\theta}{\partial x} = \delta = \frac{\pi}{\lambda_0}(n_R - n_L)$$

Remember that <u>OBVIOUSLY</u> $n_R \equiv n_R(\lambda)$, $n_L \equiv n_L(\lambda)$ (!)

§ Susceptibility tensor of an optically active medium

Given an optically active medium, it's clear that if E_x, E_y must be rotated, that the susceptibility tensor will have IMAGINARY OFF DIAGONALS (see Pauli Matrices & SO(2) symmetry) thus:

$$\chi^i_j = \begin{pmatrix} \chi^1_1 & i\chi^2_1 & 0 \\ -i\chi^1_2 & \chi^2_2 & 0 \\ 0 & 0 & \chi^3_3 \end{pmatrix}$$

When $\chi^1_2 = \chi^2_1 \in \mathbb{R}$, the medium is OPTICALLY ACTIVE. Then, for a wave propagating on the z axis $\{k^i = (0, 0, k_z)^i\}$

$$\begin{cases} \left(\frac{\omega^2}{c^2} - k^2\right)E_x = -\frac{\omega^2}{c^2}\left(\chi^1_1 E_x + i\chi^2_1 E_y\right) \\ \left(\frac{\omega^2}{c^2} - k^2\right)E_y = -\frac{\omega^2}{c^2}\left(\chi^2_2 E_y - i\chi^1_2 E_x\right) \\ \frac{\omega^2}{c^2}E_z = -\frac{\omega^2}{c^2}E_z \implies (E_z = 0) \end{cases}$$

The last equation is trivial, therefore we're left with:

$$\begin{pmatrix} \frac{\omega^2}{c^2}(1 + \chi^1_1) - k^2 & i\frac{\omega^2}{c^2}\chi^2_1 \\ -i\frac{\omega^2}{c^2}\chi^1_2 & \frac{\omega^2}{c^2}(1 + \chi^2_2) - k^2 \end{pmatrix} \begin{pmatrix} E_x \\ E_y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Which is solved as usual by taking the determinant & equating to 0

$$\left\{\frac{\omega^2}{c^2}(1 + \chi^1_1) - k^2\right\}\left\{\frac{\omega^2}{c^2}(1 + \chi^2_2) - k^2\right\} - \frac{\omega^4}{c^4}\chi^2_{12} = 0$$

Expanding the product, we get a biquadratic equation for k (quadratic for k^2)

$$\frac{\omega^4}{c^4}\left(1+\chi_1^1\right)\left(1+\chi_2^2\right)-\frac{\omega^2}{c^2}k^2\left(1+\chi_2^2\right)-\frac{\omega^2}{c^2}k^2\left(1+\chi_1^1\right)-\frac{\omega^4}{c^4}\chi_{12}^2+k^4=0$$

Therefore, reassembling

$$k^4-k^2\frac{\omega^2}{c^2}\left[1+\chi_2^2+\chi_1^1\right]-\frac{\omega^4}{c^4}\left[\chi_{12}^2-\left(1+\chi_1^1\right)\left(1+\chi_2^2\right)\right]=0$$

The only nontrivial solutions then are for

$$k=\frac{\omega}{c}\sqrt{1+\chi_1^1\pm\chi_{12}}$$

Reinserting into the matrix equation then we get for E_x, E_y

$$E_x=\pm i E_y$$

Thus, for RCP

$$E_x=iE_y$$

& for LCP

$$E_x=-iE_y$$

Accordingly, we get, remembering that $\frac{\omega}{c}k=n$

$$\begin{cases} n_R=\sqrt{1+\chi_1^1+\chi_{12}} \\ n_L=\sqrt{1+\chi_1^1-\chi_{12}} \end{cases}$$

It follows that, then

$$n_R-n_L\approx\frac{\chi_{12}}{\sqrt{1+\chi_1^1}}=\frac{\chi_{12}}{n_0}$$

Thus

$$\delta=\frac{\pi}{\lambda_0}\left(n_R-n_L\right)\approx\frac{\chi_{12}\pi}{n_0\lambda_0}$$

SPECIAL CASE: QUARTZE K-SURFACE

For quartz it's not as simple as before:

- Quartz is birefringent
- Quartz is optically active

The equation is then given by the generic 3x3 determinant

$$\det_3\begin{pmatrix} \left(\frac{n_1\omega}{c}\right)^2-k_y^2-k_z^2 & k_xk_y+i\frac{\omega^2}{c^2}\chi_{12} & k_xk_z \\ k_yk_x & \left(\frac{n_2\omega}{c}\right)^2-k_x^2-k_z^2 & k_yk_z \\ k_zk_x & k_zk_y & \left(\frac{n_3\omega}{c}\right)^2-k_x^2-k_y^2 \end{pmatrix}=0$$

The 2 possible surfaces of $\underline{K}$ now refer to ORTHOGONAL ELLIPTICAL POLARIZATIONS

Along the direction of the OA, the inner & outer surfaces do <u>NOT</u> touch & are separated by an amount $d(\chi_{12})$ & therefore is measure of δ

§ MAGNETO -/ ELECTRO-OPTICS

§§ FARADAY ROTATIONS

If an isotropic dielectric is placed in a magnetic field & a beam of NEARLY polarized light is sent through the material IN THE DIRECTION OF THE FIELD, a rotation of the polarization plane π_p is measured $\Rightarrow$ THE FIELD ACTIVATES THE DIELECTRIC

Discovery: 1845 - M. Faraday.

The amount of rotation $\theta \propto B$ & l, where l is the distance traveled in the medium. Thus:

$$\theta = VBl \quad (V \in \mathbb{R} \text{ prop. const.})$$

V is known as VERDET CONSTANT.

The explanation comes from the motion of bound electrons, when a static field $\underline{B}$ is added

Thus, since $\underline{F}_m = q\underline{v} \times \underline{B}$, $q = -e$

$$m\frac{d^2\underline{r}}{dt^2} + Kr = -e\left[\underline{E} + \overset{F_L/q}{\overset{\shortparallel}{\underline{v} \times \underline{B}}}\right]$$

where $\underline{r}$ is the displacement & K is the elastic force constant of the electrons.

If $\underline{r}$ is harmonic (it is) w/ freq ω, then:

$$\frac{d\underline{r}}{dt} = -i\omega\underline{r}, \quad \frac{d^2\underline{r}}{dt^2} = -\omega^2\underline{r}$$

Thus the equation becomes

$$K\underline{r} - m\omega^2\underline{r} = i\omega e\,\underline{r} \times \underline{B} - e\underline{E}$$

Since $\underline{P} = -Ne\underline{r}$, we can write

$$(K - m\omega^2)\underline{P} = Ne^2\underline{E} - i\omega e\,\underline{P} \times \underline{B}$$

The result will be of the form

$$P^i = \epsilon_0 \chi^i_j E_j$$

W/ χ^i_j being the "effective" susceptibility tensor. Its form is precisely one of an OA medium

$$\chi^i_j = \begin{pmatrix} \chi^1_1 & i\chi_{12} & 0 \\ -i\chi_{12} & \chi^2_2 & 0 \\ 0 & 0 & \chi^3_3 \end{pmatrix}^i_j$$

where:

$$\begin{cases} \chi^1_1 = \dfrac{Ne^2}{m\epsilon_0}\left[\dfrac{\omega_0^2 - \omega^2}{(\omega_0^2-\omega^2)^2 - \omega^2\omega_c^2}\right] \\ \chi^3_3 = \dfrac{Ne^2}{m\epsilon_0}\left[\dfrac{1}{\omega_0^2 - \omega^2}\right] \\ \chi_{12} = \dfrac{Ne^2}{m\epsilon_0}\left[\dfrac{\omega\omega_c}{(\omega_0^2-\omega^2)^2 - \omega^2\omega_c^2}\right] \end{cases}$$

Where $\omega_0^2 = \frac{K}{m}$ is the resonance frequency, $\omega_c = \frac{eB}{m}$ is the ciclotron frequency
+ 8m 30s

§ KERR EFFECT

Although an object becomes OPT. active w/ a field $\underline{B}$ since $\chi_1^1 \neq \chi_3^3$, the double refraction is small unless $\omega \simeq \omega_0$ {VOIGT EFFECT}.

Instead, when an OPTICALLY isotropic substance is placed in a STRONG $\underline{E}$ field, it becomes birefringent (1875 J. Kerr {not the new one}). It's observed in both GAS & LIQUIDS.

Idea of the effect: Molecular alignment makes it behave as an uniaxial crystal, where $\hat{OA} = \hat{E}$

The effect strength is propto E^2, where

$$n_{\parallel} - n_{\perp} = K E^2 \lambda_0 \quad (K \in \mathbb{R}, \text{ Kerr constant})$$

Where $n_{\parallel}$ is n along the direction of $\underline{E}$, $n_{\perp}$ analogously along the $\perp$ direction of $\underline{E}$

§ OTHER EFFECTS

- COTTON MOUTON: Magnetic analogue of Kerr effect, in liquids, for the lining up of molecules by $\underline{B}$, also $\mathcal{F}_{CM} \propto B^2$
- POCKELS: Some birefrengent crystals, their indexes of refraction n_i are altered by the field $\mathcal{F}_P \propto E, B$. Used for shutters & modulators. Usually made of ADP (NH_3PH_2) or KDP (KPH_2)

(4h. 08m 33.43s)

§ NONLINEAR OPTICS

When passing through an optical medium, the oscillating EM field exerts a LINEAR polarizing force. When $\omega \propto 10^8$ V/cm it's comparable to the atomic fields, thus it's NOT LINEAR.

The necessary requisites for having non linearity are reachable w/ lasers. Observed nonlinear effects are:

- OPTICAL HARMONIC GENERATION
- PRODUCTION OF COMBINATION FREQUENCIES
- OPTICAL RECTIFICATION (etc...)

In the most general case, we have

$$P^i = \epsilon_0 \left(\chi E^i + \chi^i_j E^j + \chi^i_{jk} E^j E^k + \chi^i_{jkl} E^j E^k E^l + \dots \right)$$

with growing orders of the χ^i_j tensor.

If $\underline{E}(t) \equiv E \exp(-i\omega t)$, then

$$P^i = \epsilon_0 \sum_{\alpha=1}^{\infty} \chi^i_{jk} (E^j)^k e^{-i\alpha\omega t}$$

This generates OPTICAL HARMONICS $P_n(\omega)$. Generally $P_n(\omega)$ decrease rapidly w/ n.

If the reversal of $\underline{E}$ induces a reversal of $\underline{P}$ ($P(E)$ is <u>ODD</u> $[P(-E) = -P(E)]$) then ALL EVEN TERMS ARE ZERO.

For crystalline media $\underline{P} \parallel \underline{E}$, thus:

$$P^i(E^j) = \epsilon_0 \left(\chi^i_j E^j + \chi^i_{jk} E^j E^k + \chi^i_{jkl} E^j E^k E^l + \dots \right)$$

Dividing in two parts we can say

$$P^i = P^i_L + P^i_{NL} \begin{cases} P^i_L = \epsilon_0 \chi^i_j E^j \\ P^i_{NL} = \epsilon_0 \left(\chi^i_{jk} E^j E^k + \chi^i_{jkl} E^j E^k E^l + \dots \right) \end{cases}$$

If $\underline{E}$ is a wave w/ freq. ω, the second polarization harmonic arises from χ^i_{jk} as:

$$P^i(2\omega) = \chi^i_{jk} E^j E^k$$

In order for $\chi^i_{jk} \neq 0$, the crystal <u>MUST NOT</u> have inversion symmetry. It's the same requirement of PIEZOELECTRICITY, therefore piezoelectric crystals can produce second optical harmonics {KDP, quartz, ...}.

Consider now a plane wave traveling w/ freq ω in a crystal which can produce second optical harmonics 2ω.

Then:

$$\underline{E}_1(t) \propto \exp(i k_1 r - i\omega t)$$
$$\underline{E}_2(t) \propto \exp(i k_2 r - 2i\omega t)$$

If the crystal is a slab thick ℓ, then we have that

$$\underline{E}_2(2\omega, \ell) \propto \int_0^{\ell} E^2(\omega, \ell)\, dz \propto \int_0^{\ell} e^{2i[k_1 z - \omega(t-\tau)]}\, dz$$

where τ is the time needed for the wave w/ freq 2ω to go from z to ℓ, thus equal to

$$\tau = \frac{k_2(\ell - z)}{2\omega}$$

Then:

$$\underline{E}_2(2\omega, \ell) \propto \int_0^{\ell} e^{2i[k_1 z - \frac{1}{2}(2\omega t - k_2(\ell - z))]}\, dz = e^{2i(k_1 z - \omega t)} \left. \frac{e^{-ik_2(\ell - z)}}{-ik_2(\ell - z)} \right|_0^{\ell}$$

Therefore

$$|E^2(2\omega)| \propto \ell^2 \operatorname{sinc}^2\left(k_1 - \frac{1}{2}k_2\right)$$

Note also, said $I_{2\omega} = |E^2(2\omega)|$, then, if $k_1 = k_2 \Rightarrow I_{2\omega} \propto \ell^2$.

The max irradiance can be obtained then w/ a slab of thickness:

$$\ell_c = \frac{\pi}{2k_1 - k_2}$$

Known as interaction length. In general $\ell_c \propto (10 \div 20)\lambda_0$ for typical crystals (dispersion)

It's maximable by velocity matching by making sure that

$$u = v$$

Via a suitable choice of ray direction (along OA)

← 10/12/22 ← 22.22 — 4h41m03.505 ——

(RP!)

§ LASERS

Light Amplification by means of Stimulated Emissions of Radiation

Vacuum tube emitting coherent EMR up to $\omega \propto 10^9$ Hz already known pre-1950 (MASER)

1st LASER - 1960 Hughes Lab. He-Ne LASER @ Bell Lab.

Hughes LASER: Ruby LASER (emits Red)

He-Ne LASER: He, Ne SER (emits Red, IR)

Since these 2 there are more lasers @ different emission frequencies

Work of a LASER: Optical oscillator; Amp medium inside a resonator / cavity (see LASER cavity).

The amp is via external excitation. The oscillation in the LASER cavity can be seen as a STANDING WAVE

The output is an intense beam of HIGHLY MONOCHROMATIC LIGHT.

Note that for common filaments $T_F \propto 10^4$ K, while $T_{LASER} \propto (10^{20} \div 10^{30})$ K therefore they're useful for: NONLINEAR PHENOMENA, OPTICAL BEATING, LONG DISTANCE INTERFEROMETRY, etc....

§ SE & Thermal Radiation

Einstein 1917. Consider a quantum system w/ levels $n = 1, 2, 3, \dots$ w/ energies $E_1, \dots$ & populations $N_1, \dots$ If the system is @ equilibrium for some temp T, then the pop. ratio can be described w/ Boltzmann statistics.

Said $\beta = \frac{1}{K_B T}$

$$\frac{N_2}{N_1} = \frac{\exp(-\beta E_2)}{\exp(-\beta E_1)} = e^{-\beta(E_2 - E_1)}$$

For logical reasons we assume $N_2 < N_1$, $E_2 > E_1$. As usual $2 \to 1$ decay by γ emission, where

$$E_\gamma = \hbar\omega = E_2 - E_1 \Longrightarrow \omega = \frac{E_2 - E_1}{\hbar}$$

Said $\rho(\omega)$ the radiation density @ given ω; B_{12}, B_{21} SE coefficients & A_{21} the E (spontaneous, (Stimulated Absorption) coefficient, we have that, per unit second:

$$\begin{cases} \frac{dN_1}{dt} = N_1 B_{12} \rho(\omega) \\ \frac{dN_2}{dt} = N_2 B_{21} \rho(\omega) \end{cases} \Rightarrow \text{Stimulated Absorption / Emission (SA, SE)}$$

While, adding spontaneous photon emissions (E)

$$\frac{dN_2}{dt} = N_2 A_{21} \Rightarrow \text{Spontaneous Emission (E)}$$

Therefore, in order to have equilibrium, we have that SA + SE & transitions must be:

$$\underbrace{N_2 A_{21}}_{\text{Spontaneous Emission}} + \underbrace{N_2 B_{21} \rho(\omega)}_{\text{Stimulated Emission}} = \underbrace{N_1 B_{12} \rho(\omega)}_{\text{Stimulated Absorption}}$$

E + SE = SA

Solving for the radiation density we have

$$\rho(\omega)\left[N_2 B_{21} - N_1 B_{12}\right] = -N_2 A_{21}$$

$$\rho(\omega) = \frac{N_2 A_{21}}{N_1 B_{12} - N_2 B_{21}}$$

which can be rewritten as follows

$$\begin{cases} \rho(\omega) = \dfrac{A_{21}}{B_{21}} \dfrac{1}{\dfrac{N_1}{N_2}\dfrac{B_{12}}{B_{21}} - 1} \\ \dfrac{N_1}{N_2} = e^{\beta(E_2 - E_1)} = e^{\beta\hbar\omega} \end{cases}$$

Noting also that $\rho(\omega)$, being the photon radiation density, it must follow Plank's Blackbody radiation formula, thus

$$\begin{cases} \rho(\omega) = \dfrac{\hbar\omega^3}{4\pi^3 c^3}\left(\dfrac{1}{e^{\beta\hbar\omega} - 1}\right) \\ \rho(\omega) = \dfrac{A_{21}}{B_{21}} \dfrac{1}{\dfrac{B_{12}}{B_{21}} e^{\beta\hbar\omega} - 1} \end{cases} \Rightarrow \begin{cases} \dfrac{A_{21}}{B_{21}} = \dfrac{\hbar\omega^3}{4\pi^3 c^3} \\ B_{12} = B_{21} \end{cases} \checkmark$$

Thus, at thermal equilibrium, the ratio SE/E is:

$$\frac{SE}{E} = \frac{B_{21}}{A_{21}}\rho(\omega) = \frac{4\pi^3 c^3}{\hbar\omega^3}\frac{\hbar\omega^3}{4\pi^3 c^3}\frac{1}{e^{\beta\hbar\omega} - 1} = \left(e^{\beta\hbar\omega} - 1\right)^{-1}$$

This is <u>EXACTLY</u> the number of γ per mode (OCCUPATION INDEX).

Thus, SE is very small @ VIS w/ ordinary sources $10^2 K \leq T \leq 10^3 K$, thus, most emission must be SPONTANEOUS $\Rightarrow$ RANDOM! {Incoherent}

In LASERS the radiation density for certain modes builds up s.t. $SE \gg E$.

Note that SE is EXTREMELY COHERENT. Also, the spectral radiance of LASERs at operating frequency is much higher than ordinary sources {remember definition of $T'(\tau)$}

§ AMPLIFICATION IN MEDIA

Consider a medium w/ radiation passing through, suppose that its atoms have random energy levels $E_1, E_2, E_3, \ldots$ and consider a two level approx w/ $E_1 < E_2$.

Since:

$$\frac{dN_2}{dt} \propto N_2 B_{21} \quad (SA) \ (*)$$

$$\frac{dN_1}{dt} \propto N_1 B_{12} \quad (SE)$$

Since $B_{21} = B_{12}$, then $SE > SA$ iff $N_2 > N_1$ (*)

(*) If the levels are degenerate $N_1 \to \frac{g_2}{g_1} N_1 < N_2$ {$g_i \to$ degeneration of the level}

The process described is known as POPULATION INVERSION (PI). If $\exists$ PI, the light beam will increase in intensity $\Rightarrow$ AMPLIFICATION, since $SE > SA$.

The emitted radiation has the same direction of the initial beam & definite phase relationship (coherent w/ beam). For a single atom, from SE you get $+1\gamma$ @ same mode of the stimulation

§ GAIN CONSTANT

Suppose a parallel beam of radiation propagates in media where there is PI. For a <u>COLLIMATED</u> beam, the spectral energy density $\rho(\omega)$ is related to the irradiance $I(\omega)$ in the interval $[\omega, \omega+\Delta\omega]$ as:

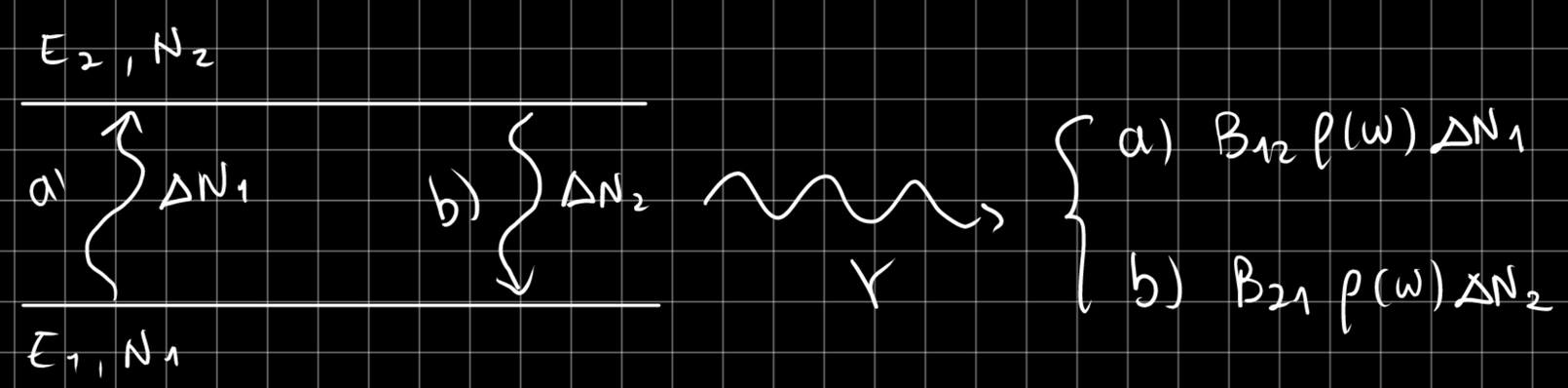
$$\rho(\omega)\,\Delta\omega = I(\omega)\,\frac{\Delta\omega}{c}$$

Note: due to Doppler & line broadening not all atoms @ given E_n will be available, thus, said n_1 the volumetric density of atoms available at level 1 of the N_1 possible, we have:

$$B_{12}\,\rho(\omega)\,\Delta N_1 = B_{12}\left(\frac{I(\omega)}{c}\right)\Delta N_1 \quad \text{(UPWARD transition)}$$

And

$$B_{21}\,\rho(\omega)\,\Delta N_2 = B_{21}\left(\frac{I(\omega)}{c}\right)\Delta N_2 \quad \text{(DOWNWARD transition)}$$



For each a) we get $-\hbar\omega$ energy, but we get $+\hbar\omega$ for each (b). Thus in an interval $\Delta\omega$, we have

$$\frac{\partial}{\partial t}\left(\rho(\omega)\,\Delta\omega\right) = \hbar\omega\left(B_{21}\Delta N_2 - B_{12}\Delta N_1\right)\rho(\omega)$$

But $\frac{dt}{c} = dx$, thus

$$\frac{d}{dx}I(\omega) = \frac{\hbar\omega}{c}\left(\frac{\Delta N_2}{\Delta\omega} - \frac{\Delta N_1}{\Delta\omega}\right)B_{21}\,I(\omega)$$

Thus, integrating the equation we get, said:

$$I(\omega) = I_0\,e^{\alpha_\omega x}$$

where α_ω is the gain constant

$$\alpha_\omega = \frac{\hbar\omega}{c}\left(\frac{\Delta N_2}{\Delta\omega} - \frac{\Delta N_1}{\Delta\omega}\right)B_{21}$$

At the center of the spectral line, we can find α_ω by approximating $\Gamma \approx \Delta\omega$. The result is correct except a numerical constant $C \propto 10^0$. Thus, $\Delta N_i \approx N_i$, and

we get

$$\alpha_{max} = \frac{\hbar\omega}{c\Delta\omega}(N_2 - N_1)B_{12}$$

Since we have that

$$\frac{A_{21}}{B_{12}} = \frac{A_{21}}{B_{21}} = \frac{\hbar\omega^3}{4\pi^3 c^3}$$

i.e

$$\frac{4\pi^3 c^3}{\hbar\omega^3} A_{12} = B_{12}$$

We get

$$\alpha_{max} \approx \frac{\hbar\omega}{c\Delta\omega}(N_2 - N_1)\frac{4\pi^3 c^3}{\hbar\,\omega^3} A_{21} = \frac{2\pi^3 c^2}{\omega^2\Delta\omega}(N_2 - N_1)A_{21} = 2\pi^3 c^2 \frac{\lambda^2}{\Delta\omega}(N_2 - N_1)A_{12}$$

Therefore, $\alpha > 0$ iff $N_2 > N_1$ (AMPLIFICATION), otherwise $\alpha < 0$ when $N_2 < N_1$ which is the usual condition for thermodynamical equilibrium.

§ GAIN CURVE

In order to determine the gain dependency on ω, it's necessary to consider line broadening. In case of thermal motion alone, we have that the fraction of atoms whose $V_x \in [V_x, V_x + \Delta V_x]$ is described by a Gaussian distribution

$$\sqrt{\frac{m\beta}{2\pi}}\, e^{-\frac{1}{2}m\beta V_x^2}\Delta V_x \quad , \quad \beta = \frac{1}{K_B T}$$

Due to Doppler effect, the atoms will absorb & emit γ in the x direction, of slightly different frequency ω wrt the resonant frequency ω_0, thus, applying Doppler's formula, we have

$$\frac{\omega - \omega_0}{\omega_0} = \frac{V_x}{c} \Rightarrow V_x = c\,\frac{\omega - \omega_0}{\omega_0}$$

Thus, the number of atoms in a given level $|i\rangle$ that can absorb/emit radiation are given from Boltzmann's distribution, where

$$\begin{cases} e^{-\frac{1}{2}\beta m V_x^2} = e^{-\frac{1}{2}\beta m c^2 \frac{(\omega-\omega_0)^2}{\omega_0^2}} \\ \Delta V_x = c\,\frac{\Delta\omega}{\omega_0} \end{cases}$$

Thus

$$\Delta N_i = \sqrt{\frac{\beta m c^2}{2\pi}}\, N_i\, e^{-\frac{1}{2\omega_0^2}\beta m c^2(\omega-\omega_0)^2}\frac{\Delta\omega}{\omega_0}$$

For the simpler 2-level problem then, substituting for $\Delta N_1, \Delta N_2$ in α_ω, we have

$$\begin{cases} \alpha_\omega = \frac{\hbar\omega}{c\Delta\omega}(\Delta N_2 - \Delta N_1)B_{12} \\ \Delta N_1 = \sqrt{\frac{\beta m c^2}{2\pi}}\, N_1 \exp\left(-\frac{1}{2\omega_0^2}\beta m c^2(\omega-\omega_0)^2\right)\frac{\Delta\omega}{\omega_0} \\ \Delta N_2 = \sqrt{\frac{\beta m c^2}{2\pi}}\, N_2 \exp\left(-\frac{1}{2\omega_0^2}\beta m c^2(\omega-\omega_0)^2\right)\frac{\Delta\omega}{\omega_0} \end{cases}$$

Thus
$$\alpha_\omega = \frac{\hbar\omega_0}{c\Delta\omega}\sqrt{\frac{\beta mc^2}{2\pi}}\left(N_2 - N_1\right)\exp\left(-\frac{1}{2\omega_0^2}\beta mc^2(\omega-\omega_0)^2\right)\frac{\Delta\omega}{\omega_0}B_{21}$$

which, simplified is
$$\alpha_\omega = \frac{\hbar B_{21}}{c}\sqrt{\frac{\beta mc^2}{2\pi}}\left(N_2 - N_1\right)e^{-\frac{1}{2\omega_0^2}\beta mc^2(\omega-\omega_0)^2}$$

The variation w/ ω of α is a Gaussian centered on ω_0! Also
$$\alpha_{max} = \frac{\hbar B_{21}}{c}\sqrt{\frac{\beta mc^2}{2\pi}}\left(N_2 - N_1\right) = \frac{\lambda_0^3}{8\pi c}\sqrt{\frac{\beta mc^2}{2\pi}}A_{21}$$

The half width at half maximum $(\Gamma_{1/2})$ is found by setting:
$$e^{-\frac{1}{2\omega_0^2}\beta mc^2(\omega-\omega_0)^2} = \frac{1}{2}$$

Thus
$$-\frac{1}{2\omega_0^2}\beta mc^2(\omega-\omega_0)^2 = \log\left(\frac{1}{2}\right)$$
$$(\omega-\omega_0)^2 = \frac{2\omega_0^2\log(2)}{\beta mc^2}$$

which finally gives for the half width at half maximum
$$\Gamma_{1/2} = \omega-\omega_0 = \omega_0\sqrt{\frac{2\log(2)}{\beta mc^2}}$$

Since the full width is $2\Gamma_{1/2} = \Gamma$, we have
$$\Gamma = 2\omega_0\sqrt{\frac{2\log(2)}{\beta mc^2}}$$

Since
$$\alpha_{max} = 2\pi^3c^2\frac{\lambda^2}{\Delta\omega}\left(N_2 - N_1\right)A_{21} = \frac{\lambda_0^3}{8\pi c}\sqrt{\frac{\beta mc^2}{2\pi}}\left(N_2 - N_1\right)A_{21}$$

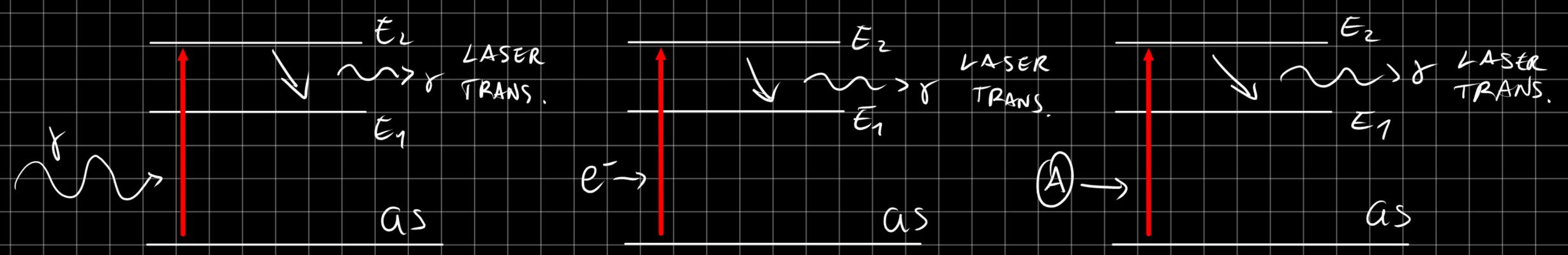
We have
$$\alpha_{max} = \frac{2\omega_0\lambda_0^3}{8\pi c\Delta\omega}\sqrt{\frac{\beta mc^2}{2\pi}}\sqrt{\frac{2\log(2)}{\beta mc^2}}\left(N_2 - N_1\right)A_{21}$$
$$\alpha_{max} = \frac{\omega_0}{2\pi c}\frac{\lambda_0^3}{2\Delta\omega}\sqrt{\frac{\log(2)}{\pi}}\left(N_2 - N_1\right)A_{21}$$

using $\lambda_0^{-1} = \frac{\omega_0}{2\pi c}$
$$\alpha_{max} = \frac{1}{2}\frac{\lambda_0^2}{\Delta\omega}\sqrt{\frac{\log(2)}{\pi}}\left(N_2 - N_1\right)A_{21}$$

§ EXPERIMENTAL PRODUCTION OF A POPULATION INVERSION

There are various methods for producing a population inversion:

- Photon excitation (optical pumping)
- Electron excitation
- Inelastic atomic collisions
- Chemical reactions



a) OPTICAL PUMPING (DIRECT PHOTON EXCITATION) b) DIRECT ELECTRON EXCITATION c) INELASTIC ATOMIC COLLISION

In the case of optical pumping, an external light source is employed for producing a high population at some energy level in the LASER medium via selective optical absorption. This method is used in solid state LASERs like the ruby prototypes.

DEE in gaseous discharge is used in gaseous ion LASERs like Ar LASERs. In this case the LASER medium itself carries the discharge current. It depends on e^- excitation CS & level lifetime.

For IAC an electric discharge is also employed such that two elements A, B both have excited states A^* & B^* that (nearly) coincide.

The transfer of excitation happens with the general reaction

$$A^* + B \longrightarrow A + B^*$$

If A^* or B^* is metastable, then the unexcited element will serve as outlet, getting a PI & subsequent LASER transition. examples of this kind of LASER is the HeNe LASER. In this LASER, Ne receives excitation from He^*.

Chemical LASERS instead has for the reaction a molecule / atom left in an excited state after the reaction, e.g. HF LASERs, which use the reaction

$$H_2 + F_2 \longrightarrow 2HF + \gamma$$

§ LASER OSCILLATIONS

The optical cavity of a LASER resonator is composed by a tube closed by 2 (plane / curved) mirrors with inside the amplifying medium. If a sufficient PI is reached, then the radiation gets amplified & establishes itself as a standing wave b/w the mirrors. The energy is coupled to the resonator via partially transmitting mirrors.

With plane mirrors, the cavity is similar to FP interferometers. The pass bands occur at ∞ equally

spaced bands $\{\dots, \omega_n, \omega_{n+1}, \dots\}$ that differ by a FSR of

$$\omega_{n+1} - \omega_n = \frac{\pi c}{d}$$

where $c = (\mu_0 \epsilon_0)^{-1/2}$ & d is the distance b/w the mirrors. These frequencies are known as: << LONGITUDINAL MODES >> of the resonator. TRANSVERSE MODES also exist. Oscillation can occur at one or more frequencies depending on mode spacing. Usually they oscillate at multiple frequencies. If extremely high spectral purity is needed, it's possible to fine-tune LASER parameters in order to get single mode oscillation. The inherent linewidth Γ is determined by the quality factor Q of a resonator. Usually $\Gamma \propto 10^0$ Hz, but in practice it's possible to obtain $\Gamma \propto 10^3$ Hz depending on thermal and mechanical stability of the system

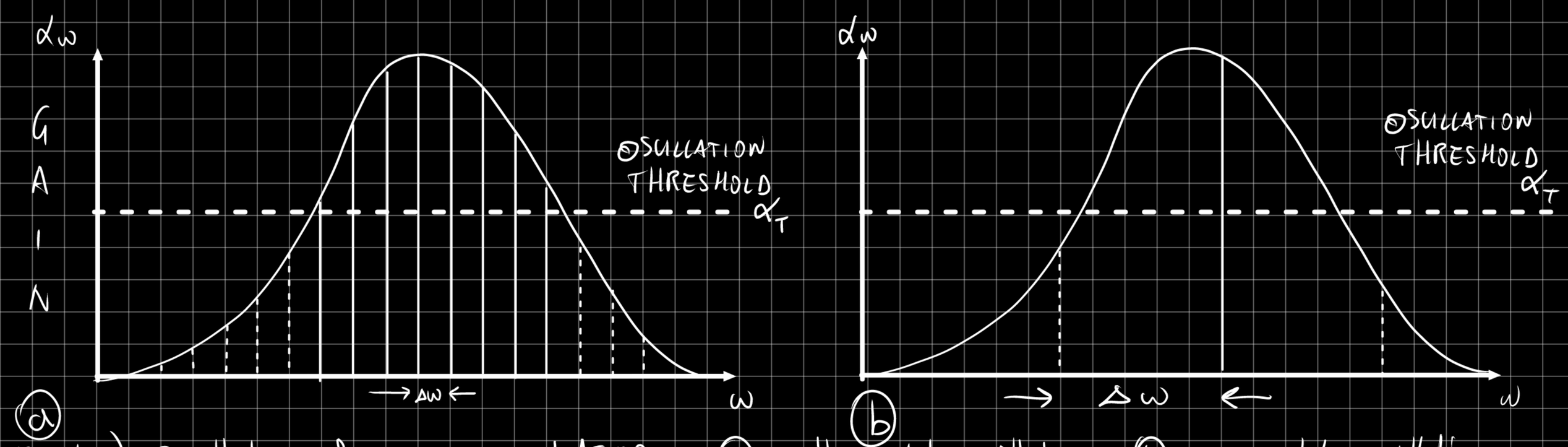


Fig 1.) Oscillation frequencies in LASERs, (a) multimodal oscillation, (b) monomodal oscillation. (a) 8 modes, (b) 1 mode {taken only ω_n st. $\alpha(\omega_n) > \alpha_T$}

§ THRESHOLD CONDITION FOR LASER OSCILLATION

As we saw, in an amplifying medium we have

$$I(\omega) = I_{0\omega} e^{\alpha_\omega x}$$

Suppose a wave starts at some point & travels back & forth in the cavity. At return, a certain amount of I will be lost due to scattering, reflection, etc... For getting oscillation, we need that the gain is higher than the loss, thus:

$$I(\omega) - I_{0\omega} \geq \delta I(\omega)$$

I.e., if the cavity is long ℓ (active part)

$$I_{0\omega}(e^{2\alpha_\omega \ell} - 1) \geq \delta I(\omega)$$

Thus

$$e^{2\alpha_\omega \ell} - 1 \geq \delta$$

If $2\alpha_\omega \ell \ll 1$, we can write by power series approximation

$$\underbrace{2\alpha_\omega \ell}_{\text{gain}} \geq \underbrace{\delta}_{\text{loss}}$$

When this is satisfied, the oscillation grows till an equilibrium condition is reached. Note that δ is constant & independent of $I(\omega)$. A depletion of the medium occurs until

$$2\alpha_\omega \ell = \delta$$

The depletion occurs in a band centered on the oscillation frequency, & it's known as << HOLE BURNING >>. It has the shape of an inverted resonance curve of a HO, known as << LORENTZ PROFILE >>.
For a radiative lifetime of an atom of τ_R, the linewidth of the profile is

$$\Gamma_L \propto \tau_R^{-1}$$

(Note that the evaluated τ_R is from the lasing atom)
If $\Gamma_L \geq \Gamma$ then all atoms are said to be "in communication with the oscillating laser mode". This is known as << HOMOGENEOUS BROADENING >> if $\Gamma_L < \Gamma$ then only a part of the atoms participate in a given laser mode, aka << INHOMOGENEOUS BROADENING >>

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§ OPTICAL RESONATOR THEORY

EMR spatial modes in a closed cavity can be described by 3 integers related to the standing wave pattern. For a LASER resonator the cavity is NOT closed. Such resonators can still support a 3D standing wave referred as << QUASI-MODE >>. Part of the energy of this mode spills around the mirrors & is lost. This loss is known as << DIFFRACTION LOSS >> of the resonator. It's very important in low frequency LASERs such as He-Ne LASERs w/ a gain per pass of only a few percent.
Taken the 2 mirrors w/ coordinates (x,y) & (x',y'). If $\psi(x,y)$ & $\psi'(x',y')$ are the complex amplitudes at both mirrors, using KF diffraction theory we have

$$\psi'(x',y') = -\frac{ik}{4\pi}\iint_{LC} \psi(x,y)\,\frac{e^{ikr}}{r}(1+\cos\theta)\,dx\,dy$$

where LC is the LASER cavity, & normal incidence is considered. If we take a point on the mirror wrt the first mirror, we can write using trigonometry

$$\begin{cases} r = \sqrt{d^2 + (x'-x)^2 + (y'-y)^2} \\ \cos\theta = \dfrac{d}{r} \end{cases}$$

with d being the distance b/w the centers of the mirrors.
If the mirrors are identical & we reach a steady state condition $\psi \propto \psi'$, Thus:

$$\gamma\,\psi(x',y') = \iint_{LC} \psi(x,y)\,K(x,y,x',y')\,dx\,dy$$

This equation is an integral equation for ψ, where K is the kernel and γ is the Eigenvalue. In general, by comparison, we have

$$K(x,y,x',y') = -\frac{ik}{4\pi r}e^{ikr}(1+\cos\theta) = \frac{1}{2i\lambda}(1+\cos\theta)\frac{e^{ikr}}{r}$$

Since $\gamma \in \mathbb{C}$ in general, for each mode ψ_n we have $\gamma_n = \|\gamma_n\| e^{i\varphi_n}$.

Therefore $\|\gamma_n\|$ is the amplitude ratio & φ_n is the phase shift of each mode. Also, since $\|\psi\|^2 \propto I$, we can write that the general energy loss ratio (for diffraction)

$$\frac{I_n - I_L}{I_n} = 1 - \|\gamma_n\|^2$$

Approximating $K(x, y, x', y')$ as we did w/ Fraunhofer diffraction, i.e.

$$K(x, y, x', y') \approx C e^{-ik_1(xx' + yy')}$$

The equation becomes

$$\gamma \psi(x', y') = C \iint_{L_c} \psi(x, y) e^{-ik_1(xx' + yy')} dx dy$$

Or, noting the 2D Fourier transform on the RHS

$$\gamma \psi(x', y') = C \hat{\mathcal{F}}_2 [\psi(x, y)](x, y)$$

Thus ψ is an eigenfunction of the Fourier transform operator. The simplest possible example is a Gaussian wave packet

$$\psi(x, y) = e^{-\frac{x^2+y^2}{w^2}}$$

where w is a scaling constant. Other more general solutions are products of Hermite polynomials & Gaussians, as

$$\psi_{pq}(x, y) = H_p\left(\frac{x\sqrt{2}}{w}\right) H_q\left(\frac{y\sqrt{2}}{w}\right) e^{-\frac{x^2+y^2}{w^2}}$$

Each couple (n_1, n_2) defines a mode of the resonator. Noted the modes as

$$TEM_{n,p,q} \begin{cases} n & \text{LONGITUDINAL MODE Nr} \\ \left.\begin{matrix} p \\ q \end{matrix}\right\} & \text{TRANSVERSE MODE Nr} \end{cases}$$

§ RESONATOR CONFIGS & STABILITY

There are various combinations of curved & plane mirrors, mainly plane parallel resonators, concave - plane resonators & confocal (concave - concave) resonators, as in the next drawing

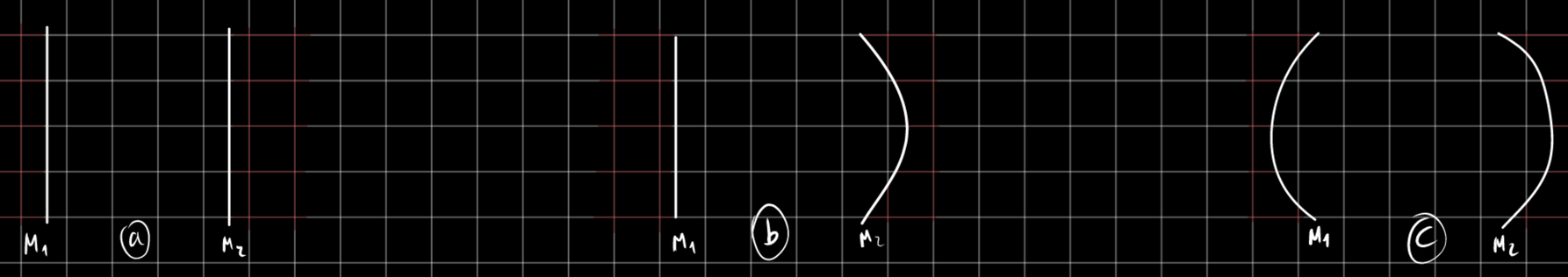

Fig R) Types of resonator, M_1 & M_2 indicate the two mirrors of the instrument.

a - Plane parallel resonator

b - Plane - concave resonator

c - Confocal resonator (convex - convex)

The most common kind of resonator is type c, confocal resonators, made by two concave spherical mirrors separated by a distance equal to the radius of curvature. The confocal cavity is much easier to collimate than the design a. Design c needs only $1/4^{\circ}$ of precision, while a needs an accuracy of collimation of the order of 1 arcsec.

The diffraction loss is measured in terms of the Fresnel number N_F, where

$$N_F = \frac{a^2}{\lambda d} \quad \begin{cases} a & \text{mirror radius} \\ d & \text{separation} \end{cases}$$

A system will be said "stable" when $N_F > 1$, i.e. after reflection the beam stays collimated with the optic axis of the mirror. Also, diffraction losses on confocal cavities are NEGLIGIBLY SMALL when $N_F > 1$, making this design the superior one.

SPOT SIZE

The parameter w introduced before is a measure of the lateral distribution of energy of the beam inside the LASER cavity. We have that $\exp\left(-\frac{x^2+y^2}{w^2}\right) = e^{-1}$ when $\sqrt{x^2+y^2} = w$. Since $I \propto \exp\left(-\frac{x^2+y^2}{w^2}\right) \Rightarrow I = e^{-2}$ when $\rho^2 = x^2+y^2 = w^2$.

Therefore w is known as the SPOT SIZE of the mode $TEM_{0,0}$ (dominant)

Given a resonator, said z_e the longitudinal distance from midpoint, then

$$w^2 \equiv w^2(z_e) = w_0^2 + \left(\frac{\lambda z_e}{\pi w_0}\right)^2$$

with λ the wavelength of the beam & w_0 a second parameter, the spot size at the center, depending from the curvature of the 2 mirrors. In general, for 2 mirrors w/ radii R separated by d, then

$$w_0^2 = \frac{\lambda}{\pi}\sqrt{\frac{d}{2}\left(R - \frac{d}{2}\right)}$$

The radius of curvature of the wave is then

$$r_w = z_e + \frac{d}{4z_e}(2R - d)$$

For a confocal resonator $R = d$, thus

$$\begin{cases} w_{0_c} = \sqrt{\frac{\lambda d}{2\pi}} \\ w = \sqrt{\frac{\lambda d}{\pi}} \end{cases}$$

Evaluating phase surfaces of the standing wave we have:

- at the mirrors ($z_e = \pm d/2$) the wave curvature is R (matches the mirrors), w is at the max
- at the center ($z_e = 0$) the wave are <u>planar</u> and w is at min.

Note that two wave surfaces (anywhere!) define a cavity if they are replaced by mirrors

§ GAS LASERS

In gas LASERs the optical cavity is provided by two external mirrors, coated by multilayer dielectric films in order to obtain high R at the desired λ. The mirrors are in confocal configuration

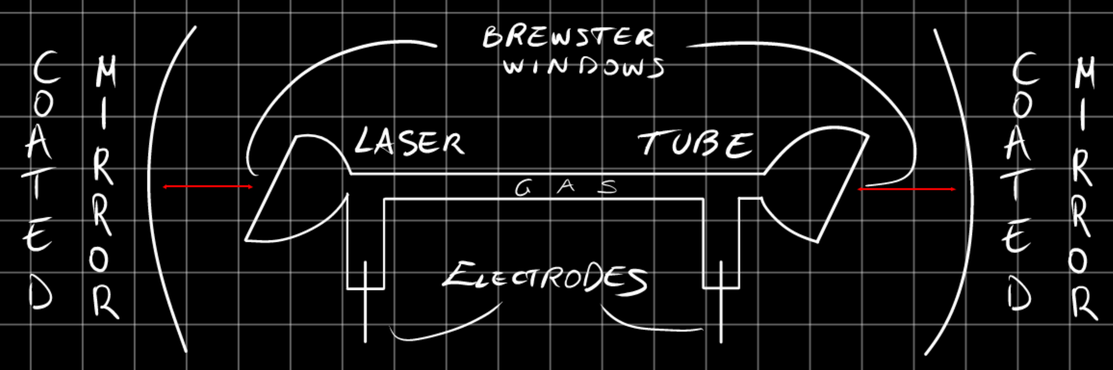

Fig GL) Schematics of a generic gas LASER

The LASER tube is fitted with Brewster windows in order to obtain the maximum transparency possible giving out linearly polarized light (p-polarized). External electrical excitation might be provided as follow

1) DC discharge
2) AC discharge } COMMONLY EMPLOYED IN COMMERCIAL LASERS
3) Electrodeless high-frequency discharge] FIRST GAS LASER SOLUTION (He-Ne)
4) High voltage pulses] High power pulsed lasers, steady PI not mantainable

1) is advantageous when the LASER is used for heterodyning, comms, etc...
2) is the simplest

§ He-Ne LASER

In He-Ne LASERs, He atoms are excited via e^- impact. The populations of the metastable $|^3S\rangle$ and $|^1S\rangle$ states of He build up due to selection rules (no allowed dipole/optical transition). Note that if E^{He}_{3S} and E^{He}_{1S} are the He energies & E^{Ne}_{nem} are the Ne energies then note that

$$E^{He}_{1S} \approx E^{Ne}_{3S}$$

$$E^{He}_{3S} \approx E^{Ne}_{2S}$$

Therefore, the probability of energy transfer is really high. These possible transfers are:

$$He^{3S} + Ne \longrightarrow He + Ne^{2S}$$
$$He^{1S} + Ne \longrightarrow He + Ne^{3S}$$

Since $E^{Ne}_{3S} > E^{Ne}_{3p}$, and $E^{Ne}_{2S} > E^{Ne}_{2p}$, a PI is possible. The optimal pressure is $p \approx 1$ torr $= 1$ atm, and the ideal concentration ratio of gases is $[He]/[Ne] = 7$.

The PI transitions are $\{ |30\rangle = |ne\rangle$

$|30\rangle \longrightarrow |21\rangle$ $\quad 3S^2 \longrightarrow 2p^4$ ($\lambda \simeq 632.8$ mm (RED)) $\quad \Delta n = -1 \quad \Delta l = +1$

$|20\rangle \longrightarrow |21\rangle$ $\quad 2S^2 \longrightarrow 2p^4$ ($\lambda \simeq 1.1523\ \mu m$) $\quad \Delta n = 0 \quad \Delta l = +1$

$|30\rangle \longrightarrow |31\rangle$ $\quad 3S^2 \longrightarrow 3p^4$ ($\lambda \simeq 3.39\ \mu m$) $\quad \Delta n = 0 \quad \Delta l = +1$

Other weaker transitions in Ne undergo LASER oscillation

OTHER GAS LASERS

Electric discharges in pure gases and various mixtures produce LASER action at different wavelengths from FIR to UV. All noble gases (He, Ne, Ar, Kr, Xe) exhibit LASER transitions.
Argon ion laser generates VIS light in the BLUE region. Pulse discharges have also been used to produce LASER action in Zn, Cd, Hg, Pb, Tn & other metals. Similarly, the halogens, like Cl, Br, I similarly show LASER behaviour in pulsed conditions.
Molecular gas LASERs have been built with N_2 (red, UV), CO_2 (10 μm)

§ OPTICALLY PUMPED SOLID STATE LASERS

In solid state LASERs (SSL), like ruby LASERs, the optically active atoms are embedded in a solid both crystals and glasses have been used.
The crystal / glass is usually made in a cylindrical rod with ground & polished in order to be highly parallel and flat.
The rods can be the cavity itself via edge coating, or by using external mirrors.
The optical pumping can be achieved by using an external light source (flashlamp) which might be pulsed or continuous. High intensity lamps, such as Xe lamp or high pressure Hg lamps are used usually

RUBY LASERS

The rod of a ruby laser is made of synthetic sapphire (Al_2O_3) doped by ~0.05% by Cr_2O_3. The resulting material is pink-colored due to the presence of Cr^{3+} ions that replace Al in the crystal lattice.
During the pumping the Cr^{3+} ions transition from the GS ^4A to excited states $^3T_1 / {}^4T_2$ from which RRLT (Rapid Radiationless Transition) can happen via decay to the level ^2E.
The decay from ^2E is relatively slow, therefore PI $^2E \rightarrow {}^4A$ can happen, with $\lambda_L = 6934$ Å

OTHER SSL

Other doped crystals exhibit LASER transition, like:

CaF_2, $CaWO_4$, Glass, YAG } Nd^{3+}	$CaF_2 \rightarrow Ho^{3+}$	BaF_2, CaF_2 } U^{3+}
	CaF_2, glass $\rightarrow Er^{3+}$	
	SrF_2, $CaWO_4 \rightarrow Tm^{3+}$	
	Glass $\rightarrow Yb^{3+}$	

N.b. YAG → Nd doped Yttrium Aluminum Garnet

§ DYE LASERS

Stimulated emission in liquids was firstly observed in 1966 at IBM labs. The dye solution was firstly pumped by a ruby LASER and then by a fast flash lamp.

The organic sources used were fluorescein & rhodamine. The fluorescence bands are broad, $\Delta\lambda \sim (50 \div 100)$ nm. The amp bands are similarly large, & the output frequency can be tuned w/ prisms, gratings and interferometers inside the cavity.
Thanks to this, we now have a high variety of LASERS emitting in different frequencies, permitting the span of the entire spectrum, from NUV to NIR. Using continuous wave gas LASERS like Ar or Kr LASERs as pump sources

§ SEMICONDUCTOR DIODE LASERS

The most compact LASER possible is the semiconductor diode laser, aka INJECTION LASER.
In the simplest form, the diode LASER has a P-N junction in a doped semiconductor crystal such as Gallium Arsenide. When a forward bias is applied to the diode, e^- are injected into the P side of the junction, & holes are injected into the N section. The e^--hole recombination results in RECOMBINATION RADIATION (see LEDs, SPONTANEOUS recombination radiation).
If the junction current is high enough, then PI can be obtained via e^-/hole transitions.
When the gain exceeds the loss at the junction we have LASER action. This layer is small ($d \sim 10^0 \mu m$), & crystal faces are treated in order to be partially reflective in order to form a resonator cavity. In Gallium Arsenide the threshold current density J_T is

$$J_T^{GaAs} \simeq 10^4 A/cm^2$$

The emitted radiation in is NIR, $\lambda \simeq (830 \div 850)$ nm

§ Q-SWITCHING & MODE LOCKING

High power lasers pumped by intense optical pulses, the LASER action begins when PI density reaches a threshold value when the gain is bigger than the loss.
The reciprocal of the fractional loss per cycle is known as RESONANT Q of the cavity.
Higher Q $\Rightarrow$ Lower PI density for LASER action.
In high power pulsed lasers the PI density is quickly used as soon as LASER oscillation commences.
It's possible to delay the oscillation via Q SWITCHING in the cavity, using a shutter (Q SWITCH)
The shutter is closed at the beginning of the pump cycle and opened when the PI is maximum. Examples of methods of Q switches are

- rotation of one of the mirrors about an axis $\perp$ OA (high rot freq.)
- Electrooptic shutter, Pockel cells
- Saturable absorber method: dye which bleaches or becomes transparent when strongly irradiated (PASSIVE Q SWITCHING)

With Q-switching it's possible to produce GW & TW LASER pulses which can be used for LASER fusion

MODE LOCKING

Given a NONLINEAR absorber (e.g. bleachable dye) inside the resonator cavity, it's possible to change the LASER output to short regular pulses. The pulse time-separation is a simple (Q) fraction of the cycle time, thus radiation is bunched into one or more narrow pulses that bounce back & forth.

A Fourier analysis shows that the freq. spectrum consists of discrete freqs separated by the rep. frequency of the pulses. For a round trip, the time needed for the radiation to travel is:

$$\tau = 2\frac{d}{u} = 2\frac{nd}{c}$$

Where d is the spacing b/w mirrors. The freq is then spaced by (or integral multiples of)

$$\nu_B = \frac{1}{\tau} = \frac{u}{2d} = \frac{c}{2nd}$$

where ν_B is the longitudinal mode spacing.

In this case, the oscillating longitudinal modes are not oscillating simultaneously incoherently, the resulting oscillating modes are coherent, thus the LASER is said to be MODE LOCKED / PHASE LOCKED. The pulse width $\propto \Delta\nu^{-1}$ with $\Delta\nu$ being the bandwidth; in a mode locked LASER this is the total bandwidth occupied by the coherent modes. Therefore, for N modes, the pulse time-width is $\frac{1}{N}$ the interval b/w pulses.

Therefore LASERS with big amp bandwidth can support various modes & therefore very short mode locked pulses.

In gas LASERS such as He-Ne or Ar LASERS, the mode locked pulses are limited to a few ns (small amp band). In dye LASERS or Nd/glass LASERS mode locked pulses can reach ps. 3ps pulses are thick only 1mm and can be considerated as c-traveling sheets of energy.

§ RING LASER (1963)

One example of LASER utilization in measurements is the ring LASER, that measures rotation using counterrotating coherent light beams.

The LASER cavity is made with 4 mirrors at the vertex of a square, and 1 or more tubes are inserted for amp. Oscillation happens then at

$$\nu_n = n\frac{c}{L}, \quad n \in \mathbb{N}$$

With L being the effective loop length. If the system is rotating about a perp axis to the LASER plane then there is a difference in effective path lengths b/w the CW & the CCW direction, which give a freq. difference of osc. frequency in the 2 beams, which is

$$\Delta\nu = \nu_0 \frac{4A}{c}\Omega$$

where ν_0 is the nonrotating osc. freq. A the loop area (L^2) & Ω the angular velocity. Optical heterodyning is used, where the 2 beams are sent together to a photodetector, giving then a beating resonant signal w/ freq $\nu_B = \Delta\nu \propto \Omega$

GEOMETRIC OPTICS

§ REFLECTION AND REFRACTION AT A SPHERICAL SURFACE

Virtually ALL optical instruments use two types of optical surfaces: plane or spherical.
The path of light is conveniently analyzed in terms of rays, and this path can be analyzed easily using Snell's law. Consider two main cases:

a) reflection by a spherical surface
b) refraction through a spherical surface separating 2 different media

Consider in the both cases the origin of the ray as a pointlike source P. Said O the point that intersects the origin of the ray and the mirror. Calling the point of return to the axis as Q we have the following schemes for the 2 cases

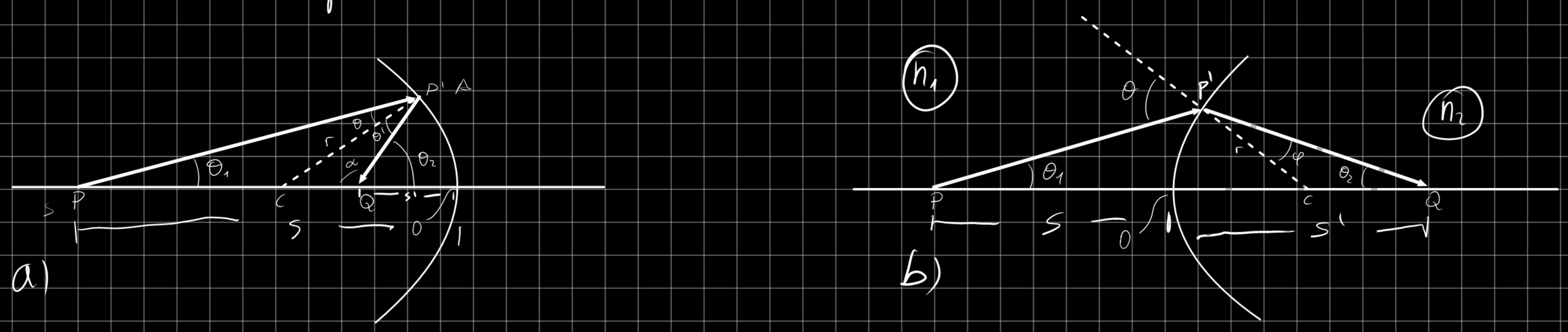

Fig (15) Drawing of the two cases described above

Called the distance $\overline{OP} = s$ (the OBJECT DISTANCE), and the distance $\overline{OQ} = s'$ (the IMAGE DISTANCE), the radius of curvature of the surface is $\overline{OC} = r$. In the case of a spherical mirror (a) we have for the law of sines, chosen a first triangle $\widehat{PP'C}$ that 1) $\overline{PC} = s - r$, thus

$$\frac{\sin\theta}{s-r} = \frac{\sin\theta_1}{r}$$

I.e.

$$r\sin\theta = (s-r)\sin\theta_1$$

Taken a second triangle $\widehat{CQP'}$ we have that $\theta_2 + \alpha = \pi$ & $\sin\alpha = \sin\theta_2$. Thus, said $\overline{CQ} = r - s'$, we get again

$$\frac{\sin\theta'}{r-s'} = \frac{\sin\theta_2}{r}$$

Therefore

$$r\sin\theta' = (r-s')\sin\theta_2$$

Since $\theta = \theta'$ we must have:

$$\boxed{\frac{r-s'}{s-r} = \frac{\sin\theta_1}{\sin\theta_2}} \quad (a)$$

Which gives the relationship bw the slopes of the incident & reflected rays.
Equivalently, for the refracting surface, we get

$$r\sin\theta = (r+s)\sin\theta_1 \quad , \quad r\sin\varphi = (s'-r)\sin\theta_2$$

where $n_1 \sin\theta = n_2 \sin\varphi$, thus

$$r\sin\theta = (r+s)\sin\theta_1 = \frac{n_2}{n_1} r\sin\varphi = \frac{n_2}{n_1}(s'-r)\sin\theta_2$$

Thus:

$$\boxed{\frac{\sin\theta_1}{\sin\theta_2} = \frac{s'-r}{r+s}\,\frac{n_2}{n_1} = n\,\frac{s'-r}{r+s}} \qquad \text{(b)}$$

PARAXIAL APPROXIMATION

When a bundle of rays originates from an axial point, the images are functions of the slope angles θ_1 at P, therefore rays DO NOT COME AT A SINGLE FOCUS. This characteristic of spherical optical surfaces is known as spherical aberration. But if the angles $\theta_i, \varphi \ll 1$, then we can approximate $\sin\theta \simeq \theta$. This approximation is known as the PARAXIAL APPROXIMATION, i.e.

$$\begin{cases} \text{a)} \quad \dfrac{\theta_1}{\theta_2} \simeq \dfrac{r-s'}{s-r} \\[2ex] \text{b)} \quad \dfrac{\theta_1}{\theta_2} \simeq n\,\dfrac{s'-r}{s+r} \end{cases}$$

DIDN'T UNDERSTAND HOW

IO QUANDO SONO RITARDATO STUPIDO PAZZO COGLIONE IMBECILLE XK NN CAPISCO DA GIOVEDÌ 15 (È 19 ORA)

Equation a) and b) yield then the following results:

$$\boxed{\frac{1}{s} + \frac{1}{s'} = \frac{2}{r} \quad ; \quad \frac{n_1}{s} + \frac{n_2}{s'} = \frac{n_2 - n_1}{r}}$$

PRESO PER BUONO (DIOCANE)

SIGN CONVENTION (FOWLES HA CAMBIATO DI NUOVO NOTAZIONE SA BACASSA SUA)

PER QUALCHE CAZZO DI MOTIVO QUEL ROTTO IN CULO HA DECISO SENZA NESSUN CAZZO DI MOTIVO CHE $s' = q$ PERCHÉ SI (ATTENTO)

It can happen, from the previous equation, that $s' = q < 0$, this means that rays diverge and appear to come from behind the lens. For a curved reflector w/ curvature radius r we have:

- $r > 0$ CONCAVE SURFACE $\rightarrow$)
- $r < 0$ CONVEX SURFACE $\rightarrow$ (

FOCAL LENGTH OF A SPHERICAL MIRROR

From the paraxial approximation equation, we see how if the rays come parallel to the OA, then the resulting image distance is $\frac{r}{2}$. Said $\frac{r}{2} = f$ {FOCAL LENGTH}, then the paraxial approx equation becomes

$$\frac{1}{s} + \frac{1}{s'} = \frac{1}{f}$$

Where again, $f = \frac{r}{2}$, with r being the radius of curvature of the mirror

§ LENSES

Consider a lens w/ curvature radii r_1, r_2 & refraction index n, we have that:

$$\begin{cases} \frac{1}{s} + \frac{1}{s'} = \frac{n_2 - n_1}{r} \\ n_2 = n, \; n_1 = 1 \end{cases}$$

For the first encounter (from $n_A = 1 \rightarrow n$ w/ radius of curvature r_1)

$$\frac{1}{s} + \frac{1}{s'} = \frac{n-1}{r_1}$$

For the second interface (from $n \rightarrow n_A = 1$ w/ radius r_2)

$$\frac{1}{s} + \frac{1}{s'} = \frac{1-n}{r_2} = -\frac{n-1}{r_2}$$

We get, for the full path

$$\frac{1}{s} + \frac{1}{s'} = (n-1)\left(\frac{1}{r_1} - \frac{1}{r_2}\right) \quad \checkmark$$

Since $s^{-1} + s'^{-1} = f^{-1}$, we have that the focal length of the lens is

$$\frac{1}{f} = (n-1)\left(\frac{1}{r_1} - \frac{1}{r_2}\right) \quad (LM)$$

Which is known as the << LENSMAKER'S FORMULA >>

If there are multiple lenses <u>in contact</u>, each w/ focal length f_i, we have that

$$\frac{1}{f_{eff}} = \sum_{i=1}^{N} \frac{1}{f_i} \qquad \left\{ f_{eff} \rightarrow \text{effective focal length} \right.$$

Considering two lenses separated by a distance d, instead:

$$\frac{1}{f_{eff}} = \frac{1}{f_1} + \frac{1}{f_2} - \frac{d}{f_1 f_2}$$

THICK LENSES

For thick lenses, the calculation is slightly different; said (t) the thickness at the OA of the lens, we define 2 principal planes H_1, H_2 which determine the focal length by evaluating from the planes to 2 appropriate points in the OA F_1, F_2. The planes depend on the radii of the 2 curved surfaces of the lens (r_1, r_2). Then, if the lens has refraction index n, we get

$$\frac{1}{f} = (n-1)\left[\frac{1}{r_1} + \frac{1}{r_2} - \frac{(n-1)^2}{n r_1 r_2} t\right]$$

If the principal planes are at distances d_1, d_2 we get

$$\begin{cases} d_1 = f t \left(\frac{1-n}{r_2}\right) \\ d_2 = f t \left(\frac{1-n}{r_1}\right) \end{cases}$$

CHROMATIC ABERRATION

Due to dispersion $\{n \equiv n(\lambda)\}$, we have that, from the previous formulae $f \equiv f(\lambda)$ implicitly w/ n. This aberration is known as chromatic aberration and it's reduced by using multiple lens combinations where each lens has different dispersion relations an achromatic combination is obtained if

$$\begin{cases} f_1 = f\left(1 - \frac{\delta_1}{\delta_2}\right) & f_2 = f\left(1 - \frac{\delta_2}{\delta_1}\right) \\ \delta_1 = \frac{1}{n_1 - 1}\frac{dn_1}{d\lambda} & \delta_2 = \frac{1}{n_2 - 1}\frac{dn_2}{d\lambda} \end{cases}$$

Note that the achromatization is possible ONLY in a determinate interval of wavelengths

SPHERICAL ABERRATION OF A SINGLE LENS

For a simple lens, f_{eff} varies w/ the distance from the OA of the lens to the incident ray's arrival point (h). For paraxial rays we have that the difference of focal length Δf can be expressed as

$$\Delta f = \frac{1}{2} h^2 \frac{f^2(n-1)}{n^2}\left[\frac{1}{r_1^3} + \left(\frac{1}{f} + \frac{1}{r_2}\right)\left(\frac{n+1}{f} + \frac{1}{r_2}\right)\right] = \frac{1}{2} k h^2$$

We also have

$$\min(\Delta f) \text{ when } \frac{r_1}{r_2} = \frac{n+4-2n^2}{n+2n^2}$$

This formula gives the shape of a simple lens for minimal spherical aberration iff the object is at $s = \infty$

RAY EQUATIONS

Consider a set of paraxial rays traveling in the general direction of the OA (z). The position & direction of ANY ray can be determined w/ 2 parameters:

1) distance from the OA (ρ)
2) angle b/w ray & OA (θ)

In particular, if the ray goes from $z_e = z_{e_1}$ to $z_e = z_{e_1} + d$, then, since $\sin\theta \approx \tan\theta \approx \theta$, $\cos\theta \approx 1$ we have that

$$\begin{pmatrix} \rho_2 \\ \theta_2 \end{pmatrix} = \begin{pmatrix} \rho_1 + \theta_1 d \\ \theta_1 \end{pmatrix}$$

Suppose now that it passes from one dielectric w/ n_1 & another w/ n_2, then

$$n_1\theta_1 = n_2\theta_2 \Rightarrow \theta_2 = \frac{n_1}{n_2}\theta_1$$

Therefore:

$$\begin{pmatrix} \rho_2 \\ \theta_2 \end{pmatrix} = \begin{pmatrix} \rho_1 \\ n_1/n_2\, \theta_1 \end{pmatrix} \quad \{\text{at boundary}\}$$

If the dielectric is curved instead

$$\begin{pmatrix} \rho_2 \\ \theta_2 \end{pmatrix} = \begin{pmatrix} \rho_1 \\ \frac{n_1}{n_2}\theta_1 - \left(1 - \frac{n_1}{n_2}\right)\frac{\rho_1}{r} \end{pmatrix}$$

For lenses or curved mirrors, it reduces to

$$\begin{pmatrix} \rho_2 \\ \theta_2 \end{pmatrix} = \begin{pmatrix} \rho_1 \\ \theta_1 - \rho_1/f \end{pmatrix}$$

All ray equations can then be written in terms of matrix transformations $\begin{pmatrix} A & B \\ C & D \end{pmatrix} = M$ known as ray matrices. Due to the matrix treatment we can indicate multiple interactions simply via matrix multiplication. In general we have

$\begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}$ free travel by d

$\begin{pmatrix} 1 & 0 \\ \frac{1}{r}\left(\frac{n_1}{n_2}-1\right) & \frac{n_1}{n_2} \end{pmatrix}$ curved dielectric
$r>0$ convex
$r<0$ concave

$\begin{pmatrix} 1 & 0 \\ 0 & n_1/n_2 \end{pmatrix}$ plane dielectric

$\begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix}$ thin lens
$\{f = r/2$ for a mirror$\}$

Thus, for two thin lenses w/ f_1, f_2, we have

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1/f_1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1/f_2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -(1/f_1 + 1/f_2) & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1/f_{eff} & 1 \end{pmatrix}$$

where we get again

$$\frac{1}{f_{eff}} = \frac{1}{f_1} + \frac{1}{f_2}$$

§ PERIODIC LENSES & RESONATORS

We can now analyze systems of multiple lenses with ray matrices: Consider multiple lenses with equal focal length f uniformly distant d from each other.

Consider a ray starting at the surface of one mirror, travels by a distance d & gets refl(r/l) by the mirror/lens $(2/r \,/\, 1/f)$. Thus

$$\begin{pmatrix} \rho_2 \\ \theta_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -f^{-1} & 1 \end{pmatrix}\begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \rho_1 \\ \theta_1 \end{pmatrix} = \begin{pmatrix} 1 & d \\ -f^{-1} & 1 - d/f \end{pmatrix}\begin{pmatrix} \rho_1 \\ \theta_1 \end{pmatrix}$$

We might ask: Is there an eigenray (obviously yes)

The secular equation is

$$\det_2\begin{pmatrix} 1-\lambda & d \\ -\frac{1}{f} & 1-\frac{d}{f}-\lambda \end{pmatrix} = (1-\lambda)\left(1-\frac{d}{f}-\lambda\right) + \frac{d}{f} = 0$$

Thus

$$1-\lambda-(1-\lambda)\frac{d}{f} - \lambda(1-\lambda) + \frac{d}{f} = 1-\lambda+\lambda\frac{d}{f} - \lambda + \lambda^2 = \lambda^2 + \lambda\left(2-\frac{d}{f}\right) + 1 = 0$$

said $\alpha = 1 - \frac{d}{2f}$ we have

$$\lambda^2 + 2\alpha\lambda + 1 = 0$$

Thus:

$$\lambda_{1,2} = -\alpha \pm \sqrt{\alpha^2 - 1}$$

Mmm, it depends on the sign of α. note that

$$\left.\begin{aligned} \lambda_{1,2} &= -\alpha \pm \sqrt{\alpha^2-1} = e^{\pm \vartheta} \quad |\alpha|>1 \\ \lambda_{1,2} &= -\alpha \pm i\sqrt{1-\alpha^2} = e^{\pm i\vartheta} \quad |\alpha|<1 \end{aligned}\right\} \vartheta \in \mathbb{R}$$

Returning back to our system, then, if our ray is an eigenray, then, after N passages

$$\begin{pmatrix} \rho_N \\ \theta_N \end{pmatrix} = \lambda^N \begin{pmatrix} \rho_1 \\ \theta_1 \end{pmatrix}$$

We need the ray to stay <u>CLOSE</u> to the OA. We prefer $|\alpha|<1$(!) thus, we have:

$$|\alpha|<1 \Rightarrow |\lambda|=1 \Rightarrow \lambda^N = e^{\pm iN\vartheta}$$

Since we need $|\alpha|<1$ for the stability criterion, we have

$$0 < 1-\frac{d}{2f} < 1 \Rightarrow \begin{cases} 0<d<4f & \text{(LENSES)} \\ 0<d<2r & \text{(MIRRORS \{RESONATOR\})} \end{cases}$$

We also get that $f>0 \Rightarrow$ converging lens/mirror. In the confocal configuration, we have

$$d = 2f = r$$

which for what we found also satisfies $|\alpha|<1$ {stability criterion}.

For the asymmetrical resonator / optical waveguide w/ 2 lenses w/ f_1, f_2 we get

$$\alpha_1 = 1-\frac{d}{2f_1} \qquad \alpha_2 = 1-\frac{d}{2f_2}$$

$$0<\alpha_1\alpha_2<1 \quad \text{\{stability\}}$$

{reached based 毛泽东 100(百)元 banknote!} END (?)

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