§ 0.1 Banach Spaces

§§ 0.1.1 Sequence Spaces

Definition 0.1.1 (Banach Space)**.** Given a space and a norm $(X, \|\cdot\|)$, the space is said to be a *Banach space* if it's complete with respect to the norm $\|\cdot\|$.
I.e. remembering the definition of completeness, we have that $\forall (x)_k \in X$ Cauchy sequence, $x_k \to x \in X$

Notation (The Field $\mathbb{F}$)**.** Here in this section, the field $\mathbb{F}$ should be intended as either the field of real numbers $\mathbb{R}$ or the field of complex numbers $\mathbb{C}$

Definition 0.1.2 (Sequence Space)**.** As a n-tuple in the field $\mathbb{F}^n$ can be seen as a sequence, as follows

$$x \in \mathbb{F}^n,\ x = (x_1, x_2, \cdots, x_n) = (x_k)_{k=1}^n$$

We can imagine a sequence as a point in a space. We will call this space $\mathbb{F}^{\mathbb{N}}$, and an element of this space will be indicated as follows

$$x \in \mathbb{F}^{\mathbb{N}},\ x = (x)_n = (x_1, x_2, \cdots, x_n, \cdots) = (x_k)_{k=1}^\infty$$

Therefore, every point in $\mathbb{F}^{\mathbb{N}}$ is a sequence. Note that the infinite sequence of 0s and 1s will be indicated as $0 = (0)_n,\ 1 = (1)_n$

Definition 0.1.3 (Sequence of Sequences)**.** We can see a sequence of sequences as a mapping from $\mathbb{N}$ to the space $\mathbb{F}^{\mathbb{N}}$, as follows

$$\begin{aligned} x :& \mathbb{N} \longrightarrow \mathbb{F}^{\mathbb{N}} \\ & n \to ((x)_k)_n \end{aligned}$$

It's important to note how there are two indexes, since every element of the sequence is a sequence in itself (i.e. $((x)_k)_n \in \mathbb{F}^{\mathbb{N}}$ for any fixed $n \in \mathbb{N}$)

Definition 0.1.4 (Convergence of a Sequence of Sequences)**.** A sequence of sequences is said to converge to a sequence in $\mathbb{F}^{\mathbb{N}}$ if and only if

$$\lim_{n\to\infty} \|(x)_k - ((x)_k)_n\| = 0 \tag{1}$$

For some norm $\|\cdot\|$

Definition 0.1.5 (Pointwise Convergence)**.** A sequence of sequence is said to converge *pointwise* to a sequence in $\mathbb{F}^{\mathbb{N}}$ if and only if

$$\forall k \in \mathbb{N} \ \lim_{n\to\infty} ((x)_k)_n = (x)_k \tag{2}$$

And it's indicated as $((x)_k)_n \to (x)_k$

Example 0.1.1. Take the following sequence of sequences in $\mathbb{F}^{\mathbb{N}}$

$$\left((x)_k\right)_n = \frac{k}{n}$$

This sequence converges pointwise to the null sequence, since

$$\lim_{n\to\infty} \left((x)_k\right)_n = \lim_{n\to\infty} \frac{k}{n} = (0)_k$$

Space of Bounded Sequences

Definition 0.1.6 (Limited Sequence Space)**.** Let $(x)_k \in \mathbb{F}^{\mathbb{N}}$. Calling the space of bounded sequences as $\ell^\infty(\mathbb{F})$, we have that $(x)_k \in \ell^\infty(\mathbb{F})$ if and only if

$$\sup_{n\in\mathbb{N}} |(x)_n| = M \in \mathbb{F} \tag{3}$$

Therefore, this space is defined as follows

$$\ell^\infty(\mathbb{F}) := \{ (x)_n \in \mathbb{F}^{\mathbb{N}} \big| \sup_{n\in\mathbb{N}} |(x)_n| < M,\ M \in \mathbb{F}\} \tag{4}$$

Theorem 0.1. *The application* $\|\cdot\|_\infty = \sup_{n\in\mathbb{N}} |\cdot|$ *is a norm in* $\ell^\infty(\mathbb{F})$

Proof. 1) $\|(x)_n\| \geq 0\ \forall (x)_n \in \mathbb{F}^{\mathbb{N}}$, $\|(x)_n\|_\infty = 0 \iff (x)_n = (0)_n$, by definition of sup the first statement is obvious, meanwhile for the second

$$0 \leq |(x)_n| \leq \sup_{n\in\mathbb{N}} |(x)_n| = 0 \implies |(x)_n| = 0 \ \therefore (x)_n = (0)_n$$

2) $\|c(x)_n\|_\infty = |c| \|(x)_n\|_\infty$

$$\|c(x)_n\|_\infty = \sup_{n\in\mathbb{N}} |c(x)_n| = \sup_{n\in\mathbb{N}} |c||(x)_n| = |c| \sup_{n\in\mathbb{N}} |(x)_n| = |c| \|(x)_n\|_\infty$$

3) $\|(x)_n + (y)_n\|_\infty \leq \|(x)_n\|_\infty + \|(y)_n\|_\infty$

$$\sup_{n\in\mathbb{N}} |(x)_n + (y)_n| \leq \sup_{n\in\mathbb{N}} (|(x)_n| + |(y)_n|) = \sup_{n\in\mathbb{N}} |(x)_n| + \sup_{n\in\mathbb{N}} |(y)_n| = \|(x)_n\|_\infty + \|(y)_n\|_\infty$$

Since $\ell^\infty(\mathbb{F})$ is a vector space, the couple $(\ell^\infty(\mathbb{F}), \|\cdot\|_\infty)$ is a normed vector space □

Remark. Let $\mathcal{V}$ be a vector space over some field $\mathbb{F}$. If $\dim(\mathcal{V}) = \infty$, a closed and bounded subset $\mathcal{W} \subset \mathcal{V}$ isn't necessarily compact, whereas, a compact subset $\mathcal{Z} \subset \mathcal{V}$ is necessarily closed and bounded.

Example 0.1.2. Take $\mathcal{V} = \ell^\infty(\mathbb{F})$ and $\mathcal{W} = \overline{B_1((0)_n)}$, where

$$\overline{B_1((0)_n)} := \{ (x)_n \in \mathbb{F}^\infty |\ \|(x)_n\|_\infty \leq 1\}$$

We have that $\mathrm{diam}(\overline{B_1}) = 2$, therefore this set is bounded and closed by definition.
Take the *canonical sequence of sequences* $((e)_k)_n$, defined as follows:

$$((e)_k)_n = ((0)_k, (0)_k, \cdots, (0)_k, (1)_k, (0)_k, \cdots), \text{ for some } k \in \mathbb{N}$$

Therefore, $\forall n \neq m$

$$\left\| ((e)_k)_n - ((e)_k)_m \right\|_\infty = \|(1)_k\|_\infty = 1$$

Therefore there aren't converging subsequences, and therefore $\overline{B_1}$ can't be compact.

Space of Sequences Converging to 0

Definition 0.1.7 (Space of Sequences Converging to 0)**.** The space of sequences converging to 0 is indicated as $\ell_0(\mathbb{F})$ and is defined as follows

$$\ell_0(\mathbb{F}) := \{\, (x)_n \in \mathbb{F}^{\mathbb{N}} \big| \, (x)_n \to 0\} \tag{5}$$

Proposition 1. $\ell_0(\mathbb{F}) \subset \ell^\infty(\mathbb{F})$, and the couple $(\ell_0(\mathbb{F}), \|\cdot\|_\infty)$ is a normed vector space, where the norm $\|\cdot\|_\infty$ gets induced from the space $\ell^\infty(\mathbb{F})$

Proof.

$$\lim_{k\to\infty} (x)_k = 0 \implies \forall \epsilon > 0 \; \exists N \in \mathbb{N} \, : \, |(x)_n| < \epsilon \; \forall n \geq N$$
$$\therefore \sup_{n\in\mathbb{N}} |(x)_n| = \epsilon \leq M \in \mathbb{F} \implies (x)_n \in \ell^\infty(\mathbb{F}), \; \therefore \ell_0(\mathbb{F}) \subset \ell^\infty(\mathbb{F})$$

□

$\ell^p(\mathbb{F})$ Spaces

Definition 0.1.8. The sequence space $\ell^p(\mathbb{F})$ is defined as follows

$$\ell^p(\mathbb{F}) := \{\, (x)_n \in \mathbb{F}^{\mathbb{N}} \big| \, \|(x)_n\|_p^p = M \in \mathbb{F}\} \tag{6}$$

Where $\|\cdot\|_p$ is the usual $p-$norm

Proposition 2. The application $\|\cdot\|_p : \ell^p(\mathbb{F}) \longrightarrow \mathbb{F}$ is a norm in $\ell^p(\mathbb{F})$, and the couple $(\ell^p(\mathbb{F}), \|\cdot\|_p)$ is a normed vector space

Proof. We begin by proving that $\ell^p(\mathbb{F})$ is actually a vector space, therefore 1) $\forall (x)_n, (y)_n \in \ell^p(\mathbb{F}), \; (x)_n + (y)_n = (x+y)_n \in \ell^p(\mathbb{F})$

$$(x+y)_n \in \ell^p(\mathbb{F}) \implies \sum_{n=0}^{\infty} |(x)_n + (y)_n|^p = \|(x)_n + (y)_n\|_p^p < M \in \mathbb{F}$$
$$\|(x)_n + (y)_n\|_p^p \leq \|(x)_n\|_p^p + \|(y)_n\|_p^p < M \in \mathbb{F}$$

2) $\forall (x)_n \in \ell^p(\mathbb{F}), \; c \in \mathbb{F}, \; c(x)_n \in \ell^p(\mathbb{F})$

$$c(x)_n \in \ell^p(\mathbb{F}) \implies \|c(x)_n\|_p^p < M \in \mathbb{F}$$
$$\|c(x)_n\|_p^p = \sum_{n=0}^{\infty} |c(x)_n|^p = |c|^p \sum_{n=0}^{\infty} |(x)_n|^p = |c|^p \|(x)_n\|_p^p < M \in \mathbb{F}$$

□

Remark. $(x)_n \in \ell^p(\mathbb{F}) \implies (x)_n \in \ell_0(\mathbb{F})$.

Proof. The proof is simple, taking $(y)_n = |(x)_n|^p$, we can see that $(y)_n \to 0$, therefore $(x)_n \to 0$ and $(x)_n \in \ell_0(\mathbb{F})$ □

Space of Finite Sequences

Definition 0.1.9 (Space of Finite Sequences)**.** The space of finite sequences is indicated as $\ell_f(\mathbb{F})$ and it's defined as follows

$$\ell_f(\mathbb{F}) := \{ (x)_n \in \mathbb{F}^{\mathbb{N}} \big| \, (x)_n = 0 \; \forall n > N \in \mathbb{N}\} \tag{7}$$

It's already obvious that $\ell_f(\mathbb{F}) \subset \ell^p(\mathbb{F}) \subset \ell^q(\mathbb{F}) \subset \ell_0(\mathbb{F}) \subset \ell^\infty(\mathbb{F})$, where $p < q \in \mathbb{R}^+ \setminus \{0\}$ where $p < q \in \mathbb{R}^+ \setminus \{0\}$

§§ 0.1.2 Function Spaces

Notation. In this case, when there will be written the field $\mathbb{F}$, we might either mean $\mathbb{R}$ only, i.e. functions $\mathbb{R} \longrightarrow \mathbb{R}$, or $\mathbb{R}; \mathbb{C}$, i.e. functions $\mathbb{R} \longrightarrow \mathbb{C}$.

Definition 0.1.10 (Some Function Spaces)**.** We are already familiar from the basic courses in one dimensional real analysis, about the space of continuous functions $C(A)$, where $A \subset \mathbb{R}$. We can define three other spaces directly, adding some restrictions.

1. $C_b(\mathbb{F}) := \{ f \in C(\mathbb{F}) | \sup_{x\in\mathbb{F}}(f(x)) \leq M \in \mathbb{F}\}$
2. $C_0(\mathbb{F}) := \{ f \in C(\mathbb{F}) | \lim_{x\to\infty}(f(x)) = 0\}$
3. $C_c(\mathbb{F}) := \{ f \in C(\mathbb{F}) | \, f(x) = 0 \quad \forall x \in A^{\mathrm{c}} \subset \mathbb{F}\}$ i.e. $C_c(\mathbb{F}) := \{ f \in C(\mathbb{F}) | \operatorname{supp}(f) \text{ is compact } \}$, where with supp we indicate the following set $\operatorname{supp}_{\mathbb{F}}(f) := \overline{\{ x \in \mathbb{F} | \, f(x) \neq 0\}}$

Due to the properties of continuous functions, these spaces are obviously vector spaces.

Proposition 3. We have $C_c(\mathbb{F}) \subset C_0(\mathbb{F}) \subset C_b(\mathbb{F}) \subset C(\mathbb{F})$, the application

$$\|f\|_u = \|f\|_\infty = \sup_{x\in A} |f(x)| \tag{8}$$

Is a norm in $C(A)$, whereas

$$\|f\|_u = \|f\|_\infty = \sup_{x\in \mathbb{F}} |f(x)| \tag{9}$$

Is a norm in the other three spaces

Proof. The inclusion of these spaces is obvious, due to the definition of these. For the proof that the application $\|\cdot\|_u$ is a norm, it's immediately given from the proof that the application $\|\cdot\|_\infty$ is a norm in $\ell^\infty(\mathbb{F})$, and that $\|\cdot\|_u = \|\cdot\|_\infty$ □

Remark. Take $f_n \in C_b(\mathbb{F})$ a sequence of functions. The uniform convergence of this sequence means that $f_n \to f$ in the norm $\|\cdot\|_u = \|\cdot\|_\infty$

Proposition 4. If $f \in C_0(\mathbb{F})$, then f is uniformly continuous

Proof. Let $f \in C_0(\mathbb{F})$, then

$$\forall \epsilon > 0 \; \exists l \, : \, |x| \geq l \implies |f(x)| < \frac{\epsilon}{2}$$

Since every continuous function is uniformly continuous in a closed set, then

$$\forall \epsilon > 0 \; \exists \delta \, : \, \forall x, y \in [-L-1, L+1] \wedge |x-y| < \delta \implies |f(x) - f(y)| < \epsilon$$

Hence we can have two cases. We either have $|x-y|<\delta$ or $x, y \in [-L-1, L+1]$. Hence we have, in the first case

$$|f(x)-f(y)| \leq |f(x)|+|f(y)|<\epsilon$$

Or, in the second case

$$\forall \epsilon>0 \ \exists \delta>0 : |x-y|<\delta \implies |f(x)-f(y)|<\epsilon$$

Demonstrating our assumption □

$C_p(\mathbb{F})$ spaces

Definition 0.1.11. We can define a set of function spaces analogous to the $\ell^p(\mathbb{F})$ spaces. These spaces are the $C_p(\mathbb{F})$ spaces. We define analogously the $p-$norm for functions as follows

$$\|f\|_p := \sqrt[p]{\int_{\mathbb{F}} |f(x)|^p \, \mathrm{d}x} \tag{10}$$

Thanks to what said about $\ell^p(\mathbb{F})$ spaces and $p-$norms, it's already obvious that these spaces are normed vector spaces

Remark. Watch out! $C_p(\mathbb{F}) \not\subset C_0(\mathbb{F})$, and $C_p(\mathbb{F}) \not\subset C_q(\mathbb{F})$ for $1 \leq p \leq q$. It's easy to find counterexamples

Proposition 5. If $1 \leq p \leq q$, then

$$C_p(\mathbb{F}) \cap C_b(\mathbb{F}) \subset C_q(\mathbb{F})$$

Proof. Let $f \in C_p(\mathbb{F}) \cap C_b(\mathbb{F})$. Therefore $\sup_{x\in\mathbb{F}} |f(x)| < M \in \mathbb{F}$, then

$$\int_{\mathbb{F}} |f(x)|^q \, \mathrm{d}x = \int_{\mathbb{F}} |f(x)|^p |f(x)|^{q-p} \, \mathrm{d}x \leq M^{q-p} \int_{\mathbb{F}} |f(x)|^p \, \mathrm{d}x \leq \infty$$

Therefore $f \in C_p(\mathbb{F}) \cap C_b(\mathbb{F}) \implies f \in C_q(\mathbb{F})$ □

§§ 0.1.3 Function Spaces in $\mathbb{R}^n$

Definition 0.1.12 (Seminorm)**.** A *seminorm* is an application $\|\cdot\|_{\alpha,\beta} : A \longrightarrow \mathbb{F}$ with A a function space and α, β multiindices, where

$$\|f\|_{\alpha,\beta} := \left\|x^\alpha \partial^\beta f\right\|_\infty = \sup_{x\in\mathbb{F}} \left|x^\alpha \partial^\beta f(x)\right| \tag{11}$$

Definition 0.1.13 (Schwartz Space)**.** The space $\mathcal{S}(\mathbb{R}^n)$ is called the *Schwartz space*, and it's defined as follows

$$\mathcal{S}(\mathbb{R}^n) := \left\{ f \in C^\infty(\mathbb{R}^n) | \ \|f\|_{\alpha,\beta} < \infty, \alpha, \beta \text{ multiindices} \right\} \tag{12}$$

Example 0.1.3. Taken $p(x) \in \mathbb{R}[x]$ a polynomial, a common example of functions $f(x) \in \mathcal{S}(\mathbb{R})$ is the following.

$$f(x) = p(x) e^{-a|x|^{2n}} \tag{13}$$

With $a>0, \ n \in \mathbb{N}$

Theorem 0.2. *A function $f \in C^\infty(\mathbb{R}^n)$ is in $\mathcal{S}(\mathbb{R}^n)$ if $\forall \beta$ multiindex, $\forall a > 0$ $\exists C_{\alpha,\beta}$ such that*

$$\left\|\partial^\beta f(x)\right\| \leq \frac{C_{\alpha,\beta}}{\left(1 + \|x\|^2\right)^{\frac{a}{2}}} \quad \forall x \in \mathbb{R}^n \tag{14}$$

Proof. Taken $n = 1$ and $f \in C^\infty(\mathbb{R})$, then

$$\left|x^j \partial^k f(x)\right| = |x|^j \left|\partial^k f(x)\right| \leq \frac{C_{j,k}|x|^j}{(1+x^2)^{\frac{j}{2}}} \leq C_{j,k} \quad \forall x \in \mathbb{R}$$

Therefore

$$\|f\|_{j,k} \leq C_{j,k} < \infty \implies f \in \mathcal{S}(\mathbb{R})$$

Taken $f \in S(\mathbb{R})$ we have that, if $|x| \geq 1$

$$(1+x^2)^{\frac{a}{2}} \leq 2^{\frac{a}{2}} |x|^a$$

Taken $j = \lceil a \rceil$

$$\left|\partial^k f(x)\right| = \frac{\left|x^j \partial^k f(x)\right|}{|x|^j} \leq \frac{\|f\|_{j,k}}{|x|^a} \leq \frac{2^{\frac{a}{2}} \|f\|_{j,k}}{(1+x^2)^{\frac{a}{2}}} \leq \frac{2^{\frac{a}{2}\|f\|_{0,k}}}{(1+x^2)^{\frac{a}{2}}} \quad |x| < 1$$

Taken $C_{j,k} = 2^{\frac{a}{2}} \max \|f\|_{\lceil a \rceil,k}, \|f\|_{0,k}$ the assert is proven ☐

Definition 0.1.14 (Space of Compact Support Function)**.** Given a function with compact support f, we define the space of compact functions $C_c^\infty(\mathbb{R}^n)$ as the space of all such functions.
We have obviously that $C_c^\infty(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n)$

Theorem 0.3. *Both $C_c^\infty(\mathbb{R}^n)$ and $\mathcal{S}(\mathbb{R}^n)$ are dense in $(C_p(\mathbb{R}^n), \|\cdot\|_p)$*

Theorem 0.4 (Other Function Spaces)**.**

1. *$C(\mathbb{R})$ Space of continuous functions*
2. *$\mathbb{R}[x]$ Space of real polynomials*
3. *$C^k(\mathbb{R})$ Space of continuous $k-$derivable functions*
4. *$C_c^k(\mathbb{R})$ Space of functions $f \in C^k(\mathbb{R})$ with compact support*
5. *$C^\infty(\mathbb{R})$ Space of infinitely differentiable (smooth) functions*
6. *$C_0(\mathbb{R})$ Space of smooth functions with* $\lim_{x\longrightarrow\pm\infty} f(x) = 0$
7. *$C_c^\infty(\mathbb{R})$ Space of smooth functions with compact support*
   We have the obvious inclusions

$$\begin{aligned}
\ell_f \subset \ell_p \subset \cdots \subset \ell_0 \subset \mathbb{R}^\infty \\
C_c(\mathbb{R}) \subset C_0(\mathbb{R}) \subset \cdots \subset C_p(\mathbb{R}) \subset C(\mathbb{R}) \\
\mathbb{R}[x] \subset C^\infty(\mathbb{R}) \subset \cdots \subset C(\mathbb{R}) \\
C_c^\infty(\mathbb{R}) \subset C_0^\infty(\mathbb{R}) \subset C^\infty(\mathbb{R})
\end{aligned}$$

§ 0.2 Hilbert Spaces

Definition 0.2.1 (Hermitian Product)**.** Given $\mathcal{V}$ a complex vector space, and an application $\langle \cdot, \cdot \rangle : \mathcal{V} \longrightarrow \mathbb{C}$ such that $\forall u, v, z \in \mathcal{V},\ c, d \in \mathbb{C}$

1. $\langle v, v \rangle \geq 0$
2. $\langle v, v \rangle = 0 \iff v = 0$
3. $\langle u, v \rangle = \overline{\langle v, u \rangle}$
4. $\langle u + v, z \rangle = \langle u, z \rangle + \langle v, z \rangle$
5. $\langle cu, dv \rangle = c\overline{d}\langle u, v \rangle$

The application $\langle \cdot, \cdot \rangle$ is called an *Hermitian product* in $\mathcal{V}$, and the couple $(\mathcal{V}, \langle \cdot, \cdot \rangle)$ is called an *Euclidean space*

Remark. It's usual in physics that for a Hermitian product, we have that

$$\langle cu, v \rangle = \overline{c}\langle u, v \rangle \tag{15}$$

Definition 0.2.2 (Hilbert Space)**.** Given $(\mathcal{V}, \langle \cdot, \cdot \rangle)$ an euclidean space. It's said to be a *Hilbert space* if it's complete

Theorem 0.5 (Cauchy-Schwartz Inequality)**.** *Given* $(\mathcal{V}, \langle \cdot, \cdot \rangle)$ *a complex euclidean space, then* $\forall u, v \in \mathcal{V}$

$$\|\langle u, v \rangle\|^2 \leq \langle u, u \rangle \langle v, v \rangle \tag{16}$$

Proof. Taken $t \in \mathbb{C}$, we define $p(t) = \langle tu + v, tu + v \rangle$. Then by definition of the Hermitian product, we have

$$p(t) = \|t\|^2 \langle u, u \rangle + t\langle u, v \rangle + \overline{t}\langle v, u \rangle + \langle v, v \rangle$$

Writing $\langle u, v \rangle = \rho e^{i\theta},\ t = se^{-i\theta}$ we have

$$p(se^{-i\theta}) = s^2 \langle u, u \rangle + 2s\rho + \langle v, v \rangle \geq 0 \quad \forall s \in \mathbb{R}$$

Then, by definition, we have

$$\rho^2 = \|\langle u, v \rangle\|^2 \leq \langle u, u \rangle \langle v, v \rangle$$

□

Theorem 0.6 (Induced Norm)**.** *Given a Hermitian product* $\langle \cdot, \cdot \rangle$ *we can define an induced norm* $\|\cdot\|$ *by the definition*

$$\|\cdot\| = \sqrt{\langle \cdot, \cdot \rangle} \tag{17}$$

Theorem 0.7. *Addition, multiplication by a scalar and the scalar product are continuous in an euclidean space* $\mathcal{V}$*, then, given two sequences* $u_n \longrightarrow u \in \mathcal{V},\ v_n \to v \in \mathcal{V}$ *and*

$$\begin{aligned} u_n + v_n &\longrightarrow u + v \\ c \in \mathbb{C} \implies cu_n &\longrightarrow u \\ \langle u_n, v_n \rangle &\longrightarrow \langle u, v \rangle \end{aligned}$$

Proof. Thanks to Cauchy-Schwartz we have

$$|\langle u,v\rangle - \langle u_n,v_n\rangle| = |\langle u,v\rangle - \langle u,v_n\rangle + \langle u,v_n\rangle - \langle u_n,v_n\rangle| \leq |\langle u,v\rangle - \langle u,v_n\rangle| + |\langle u,v_n\rangle - \langle u_n,v_n\rangle| = \\ = \|u\|\|v-v_n\| + \|v_n\|\|u-u_n\|$$

Since the successions are convergent, we have that $\exists M>0 : \|v_n\| \leq M\ \forall n \in \mathbb{N}$, therefore

$$|\langle u,v\rangle - \langle u_n,v_n\rangle| \leq \max\{\|u\|,M\}\left(\|v-v_n\| + \|u-u_n\|\right) \to 0$$

□

Example 0.2.1 (Some Euclidean Spaces). 1) $\ell^2(\mathbb{C})$
Given $x,y \in \ell^2(\mathbb{C})$ we define the scalar product as

$$\langle x,y\rangle = \sum_{i=1}^{\infty} x_i\overline{y_i}$$

2) $\ell^2(\mu)$, a weighted sequence space, where

$$\ell^2(\mu) := \left\{ x \in \mathbb{C}^{\mathbb{N}} \middle| \sum_{i=1}^{\infty} \mu_i|x_i|^2 < \infty,\ \mu_i \in \mathbb{R},\ \mu_i > 0\ \forall i \right\}$$

Given $x,y \in \ell^2(\mu)$ we define

$$\langle x,y\rangle = \sum_{i=1}^{\infty} \mu_i x_i\overline{y_i}$$

3) $C_2(\mathbb{C})$ Given $f,g \in C_2(\mathbb{C})$ we define

$$\langle f,g\rangle = \int_{\mathbb{R}} f(x)\overline{g(x)}\,\mathrm{d}x$$

4) $C_2(\mathbb{C}, p(x)\,\mathrm{d}x)$, weighted function spaces, where

$$C_2(\mathbb{C}, p(x)\,\mathrm{d}x) := \left\{ f \in C(\mathbb{C}) \middle| \int_{\mathbb{R}} f(x)\overline{f(x)}p(x)\,\mathrm{d}x < \infty,\ p(x) \in C(\mathbb{R};\mathbb{R}^+) \right\}$$

Given $f,g \in C_2(\mathbb{C}, p(x)\,\mathrm{d}x)$ we define

$$\langle f,g\rangle = \int_{\mathbb{R}} f(x)\overline{g(x)}p(x)\,\mathrm{d}x$$

The spaces C_2 *aren't complete* therefore they're not Hilbert spaces. The spaces $L^2(\mathbb{C})$ and the weighted alternative are the completion of such spaces and are therefore Hilbert spaces

Theorem 0.8 (Polarization Identity)**.** *Given a complex euclidean space* $(\mathcal{V}, \langle\cdot,\cdot\rangle)$ *we have,* $\forall u,v \in \mathcal{V}$

$$\langle u,v\rangle = \frac{1}{4}\left(\|u+v\|^2 - \|u-v\|^2 + i\left(\|u+iv\|^2\|u-iv\|^2\right)\right) \tag{18}$$

Theorem 0.9 (Parallelogram Rule)**.** *Let* $(\mathcal{V}, \|\cdot\|)$ *be a normed vector space. A necessary and sufficient condition that the norm is induced by a scalar product is that*

$$\|u+v\|^2 + \|u-v\|^2 = 2\left(\|u\|^2 + \|v\|^2\right) \quad \forall u,v \in \mathcal{V} \tag{19}$$

§ 0.3 Projections

§§ 0.3.1 Orthogonality

Definition 0.3.1 (Angle)**.** Given a real euclidean space $(\mathcal{V}, \langle\cdot,\cdot\rangle)$ we define the angle $\theta = u\angle v$ as follows

$$\theta = \arccos\left(\frac{\langle u, v\rangle}{\|u\|\|v\|}\right) \tag{20}$$

Definition 0.3.2 (Orthogonal Complement)**.** Given an euclidean vector space $\mathcal{V}$ and two vectors u, v, we say that the two vectors are orthogonal $u \perp v$ if

$$\langle u, v\rangle = 0 \tag{21}$$

If $X \subset \mathcal{V}$ and $\forall x \in X$ we have that

$$\langle u, x\rangle = 0$$

We say that $u \in X^\perp$ where this space is called the *Orthogonal Complement* of X, i.e.

$$X^\perp := \{\, v \in \mathcal{V} |\, \langle v, w\rangle = 0 \ \forall w \in X\} \tag{22}$$

Theorem 0.10. *Given $X \subset \mathcal{V}$ with $\mathcal{V}$ an euclidean space, the set $X^\perp$ is a closed subspace of $\mathcal{V}$*

Proof. $X^\perp$ is a subspace, hence $\forall v_1, v_2 \in X^\perp$ and $c_1, c_2 \in \mathbb{C}$ we have

$$\langle c_1 v_1 + c_2 v_2, w\rangle = c_1\langle v_1, w\rangle + c_2\langle v_2, w\rangle \quad \forall w \in X$$

Hence $c_1 v_1 + c_2 v_2 \in X^\perp$.
Given a sequence $(v)_n \in X^\perp : (v)_n \longrightarrow v \in \mathcal{V}$ we have, given $w \in X$

$$\langle v_n, w\rangle = 0 \quad \forall n \in \mathbb{N}$$

Due to the continuity of the scalar product we have that

$$\lim_{n\to\infty} \langle v_n, w\rangle = 0 = \langle v, w\rangle$$

Therefore $v \in X^\perp$ and the subspace $X^\perp$ is closed in $\mathcal{V}$ □

Theorem 0.11. *Given $X, Y \subset \mathcal{V}$ with $\mathcal{V}$ an euclidean space, we have*

$$X \subset Y \implies Y^\perp \subset X^\perp$$
$$X^\perp = (\overline{X})^\perp = \left(\overline{\operatorname{span}(X)}\right)^\perp$$

Proof. Taken $v \in Y^\perp$ we have by definition

$$\langle v, y\rangle = 0 \quad \forall y \in Y$$

Since $X \subset Y$ we have then

$$\langle v, x\rangle = 0 \quad \forall x \in X$$

Therefore $Y^\perp \subset X^\perp$.
By definition we have that $X \subset \overline{X} \subset \overline{\text{span}(X)}$, and thanks to the previous proof

$$X^\perp \supset (\overline{X})^\perp \supset (\overline{\text{span}(X)})^\perp$$

Taken $w \in \text{span}(X)$ we have

$$w = \sum_i c_i w_i \quad w_i \in X$$

And given $v \in X^\perp$, we get

$$\langle v, w \rangle = \sum_i c_i \langle v, w_i \rangle = 0$$

Now take $w \in \overline{\text{span}(X)}$. Take a sequence $(w)_n \in \overline{\text{span}(X)}$ such that $(w)_n \longrightarrow w$. Thanks to the continuity of the scalar product we have

$$\langle v, w \rangle = \langle v, \lim_{n\to\infty} w_n \rangle = \lim_{n\to\infty} \langle v, w_n \rangle = 0$$

Demonstrating that $X^\perp = (\overline{\text{span}(X)})^\perp$ $\square$

Lemma 0.3.1. Let $\mathcal{V}$ be a Hilbert space. Given $\mathcal{W} \subset \mathcal{V}$ a closed subspace. Given $v \in \mathcal{V}$

$$\exists! w_0 \in \mathcal{W} \,:\, \forall w \in \mathcal{W} \; d = \|v - w_0\| \leq \|v - w\|$$

Proof. Take $d = \inf_{w \in \mathcal{W}} \|v - w\|$. By definition of infimum we have that $\exists (w)_n \in \mathcal{W}$ such that

$$\lim_{n\to\infty} \|v - w_n\| = d$$

Using the parallelogram rule, we have that

$$\|w_n - w_k\|^2 = \|(w_n - v) + (v - w_k)\|^2 = 2\|v - w_n\|^2 + 2\|v - w_k\|^2 - 4\left\|\frac{1}{2}(w_n + w_k) - v\right\|^2$$

Since $1/2(w_n + w_k) \in \mathcal{W}$ we have by definition of d

$$\left\|\frac{1}{2}(w_n + w_k) - v\right\| \geq d$$

Therefore, we can rewrite

$$\|w_n - w_k\|^2 \leq 2\|v - w_n\|^2 + 2\|v - w_k\|^2 - 4d^2$$

Therefore, we have

$$\forall \epsilon > 0 \; \exists N \in \mathbb{N} \,:\, \forall n, k \geq N \; \|w_n - w_k\|^2 \leq 4(d + \epsilon)^2 - 4d^2$$

Hence $(w)_n$ is a Cauchy sequence. Since by definition $\mathcal{V}$ is complete and $\mathcal{W} \subset \mathcal{V}$ is closed, we have that $\mathcal{W}$ is also complete, therefore $(w)_n \to w_0 \in \mathcal{W}$ and we have

$$\|v - w_0\| = d$$

Now suppose that $\exists w_1, w_2 \in \mathcal{W}$ such that the previous is true, i.e.

$$\|v - w_1\| = \|v - w_2\| \leq \|v - w\| \quad \forall w \in \mathcal{W}$$

Taken $w_3 = 1/2(w_1 + w_2)$ we have

$$\|v - w_3\|^2 = \|v - w_1\|^2 - \frac{1}{4}\|w_2 - w_1\|^2$$

Taken $d = \|v - w_1\| = \|v - w_2\|$, $z_1 = v - w_3$ and $z_2 = 1/2(w_1 - w_2)$ we get

$$\|z_1 + z_2\|^2 + \|z_1 - z_2\|^2 = 2\left(\|z_1\|^2 + \|z_2\|^2\right)$$

And therefore

$$d^2 = \frac{1}{2}\left(\|v - w_1\|^2 + \|v - w_2\|^2\right) = \|v - w_3\|^2 + \frac{1}{4}\|w_1 - w_2\|^2$$

I.e. if $w_1 \neq w_2$, w_3 is the infimum between $v \in \mathcal{V}$ and $\mathcal{W}$ $\lightning$ □

§§ 0.3.2 Projections and Orthogonal Projections

Theorem 0.12 (Projection)**.** *Given $\mathcal{W} \subset \mathcal{V}$ closed subspace of a Hilbert space, we have*

$$v = w + z \quad \forall v \in \mathcal{V},\ w \in \mathcal{W},\ z \in \mathcal{W}^\perp$$

Proof. Given $v \in \mathcal{V}$, due to the previous lemma we have that $\exists! w \in \mathcal{W}$ such that

$$d = \|v - w\| \leq \|v - w'\| \quad \forall w' \in \mathcal{W}$$

Taken $z = v - w$, and an element $x \in \mathcal{W}$, define the vector $w + tx$ with $t \in \mathbb{C}$. Since $\mathcal{W}$ is a subspace $w + tx \in \mathcal{W}$ and $\forall t \in \mathbb{C}$ we have

$$d^2 \leq \|v - (w + tx)\|^2 = \|v - w\|^2 - \bar{t}\langle v - w, x, -\rangle t\langle x, v - w\rangle + \|t\|^2\|x\|^2$$

Writing $\langle x, v - w\rangle = \|\langle x, v - w\rangle\| e^{i\theta}$ and $t = se^{-i\theta}$ with $s \in \mathbb{R}$ we have

$$-2s\|\langle v - w, x\rangle\| + s^2\|x\|^2 \geq 0 \quad \forall s \in \mathbb{R}$$

Which implies

$$\langle v - w, x\rangle = 0 \implies z = v - w \in \mathcal{W}^\perp$$

Therefore there exists a representation $v = w + z$ with $w \in \mathcal{W},\ z \in \mathcal{W}^\perp$ Now, we suppose that $v = w' + z'$, then

$$0 = (w - w') + (z - z')$$

Therefore

$$0 = \|(w - w') + (z - z')\|^2 = \|w - w'\| + \|z - z'\|^2$$

Therefore the representation is unique. □

Theorem 0.13. *If $\mathcal{W} \subset \mathcal{V}$ with $\mathcal{V}$ a Hilbert space, we have*

$$\left(\mathcal{W}^{\perp}\right)^{\perp} = \overline{\mathcal{W}}$$

If $\mathcal{W}$ is closed

$$\left(\mathcal{W}^{\perp}\right)^{\perp} = \mathcal{W}$$

Proof. Taken $w \in \mathcal{W}$ we have that $w \perp v$ with $v \in \mathcal{W}^{\perp}$, therefore $w \in (\mathcal{W}^{\perp})^{\perp}$. Therefore

$$\mathcal{W} \subset \left(\mathcal{W}^{\perp}\right)^{\perp}$$

Since the space on the right is closed, we have

$$\overline{\mathcal{W}} = \overline{\left(\mathcal{W}^{\perp}\right)^{\perp}} = \left(\mathcal{W}^{\perp}\right)^{\perp}$$

Now taken $w \in \left(\mathcal{W}^{\perp}\right)^{\perp}$, since $\overline{\mathcal{W}}$ is a closed subspace by definition, we can write

$$w = v + z \quad v \in \overline{\mathcal{W}},\ z \in \overline{\mathcal{W}}^{\perp} = \mathcal{W}^{\perp}$$

We have $w \perp z$, and therefore

$$\|z\|^2 = \langle z, w - v \rangle = 0 \implies w = v \in \overline{\mathcal{W}}$$

□

Definition 0.3.3 (Orthogonal Projection)**.** Given a closed subspace $\mathcal{W} \subset \mathcal{V}$ we can define an operator $\hat{\pi}_{\mathcal{W}} : \mathcal{V} \longrightarrow \mathcal{W}$ such that

$$\hat{\pi}_{\mathcal{W}} v = w \iff w \in \mathcal{W},\ v - w \in \mathcal{W}^{\perp} \tag{23}$$

$\hat{\pi}_{\mathcal{W}}$ is linear and called a *orthogonal projection*

Theorem 0.14. *Given $\mathcal{W} \subset \mathcal{V}$ a closed subspace of the Hilbert space $\mathcal{V}$, then given an orthogonal projection $\hat{\pi}_{\mathcal{W}} : \mathcal{V} \longrightarrow \mathcal{W}$ we have, $\forall v, z \in \mathcal{V}$ and another closed subspace $\mathcal{Z} \subset \mathcal{V}$*

1. $\hat{\pi}_{\mathcal{W}}^2 = \hat{\pi}_{\mathcal{W}}$
2. $\langle \hat{\pi}_{\mathcal{W}} v, z \rangle = \langle v, \hat{\pi}_{\mathcal{W}} z \rangle \geq 0$
3. *If* $\mathcal{Z} \subseteq \mathcal{W}^{\perp} \quad \hat{\pi}_{\mathcal{W}} \circ \hat{\pi}_{\mathcal{Z}} = \hat{\pi}_{\mathcal{Z}} \circ \hat{\pi}_{\mathcal{W}} = 0$
4. *If* $\mathcal{Z} \subset \mathcal{W} \quad \hat{\pi}_{\mathcal{W}} \circ \hat{\pi}_{\mathcal{Z}} = \hat{\pi}_{\mathcal{Z}} \circ \hat{\pi}_{\mathcal{W}} = \hat{\pi}_{\mathcal{W}}$

Definition 0.3.4 (Direct Sum)**.** An euclidean space $\mathcal{V}$ is called the *direct sum* of closed subspaces $\mathcal{V}_i \subset \mathcal{V}$ and it's indicated as follows

$$\mathcal{V} = \mathcal{V}_1 \oplus \mathcal{V}_2 \oplus \cdots = \bigoplus_{k=1}^{\infty} \mathcal{V}_k \tag{24}$$

If

1. The spaces $\mathcal{V}_k$ are orthogonal in couples
2. $\forall v \in \mathcal{V}\ v = \sum_{k=1}^{\infty} v_k$ with $v_k \in \mathcal{V}_k$

Corollary 0.3.1. Given a Hilbert space $\mathcal{V}$ and a closed subspace $\mathcal{W}$, then

$$\mathcal{V} = \mathcal{W} \oplus \mathcal{W}^{\perp} \tag{25}$$

§§ 0.3.3 Orthogonal Systems and Bases

Definition 0.3.5 (Orthogonal System)**.** A set of vectors $X \subset \mathcal{V} \quad X \neq \{\}$ is said to be an *orthogonal system* if $\forall u, v \in X,\ u \neq v \quad u \perp v$.

Definition 0.3.6 (Orthonormal System)**.** Given an orthogonal system $X \subset \mathcal{V}$ such that $\forall u \in X$ we have $\|u\| = 1$, the system X is called an *orthonormal system*

Theorem 0.15. *Given an orthogonal system $X \subset \mathcal{V}$ with $(\mathcal{V}, \langle \cdot, \cdot \rangle)$ an euclidean space, then we have that X is a system of linearly independent vectors*

Definition 0.3.7 (Basis)**.** Given an orthogonal and complete set of vectors $(v)_\alpha$ in an euclidean space $\mathcal{V}$ it's said to be an *orthogonal basis* of $\mathcal{V}$. If it's an orthonormal and complete set of vectors it's said to be an *orthonormal basis* of $\mathcal{V}$

Lemma 0.3.2. Given an orthogonal system $(v)_{k=1}^n \in \mathcal{V}$ and let $u \in \mathcal{V}$ an arbitrary vector. Then given $z \in \mathcal{V}$ as follows

$$z := u - \sum_{k=1}^{n} \frac{\langle u, v_k \rangle}{\|v_k\|^2} v_k$$

We have that $z \perp v_i \ \forall 1 \leq i \leq n$ and therefore $z \perp \operatorname{span}\{v_1, \cdots, v_n\}$

Proof. $\forall i = 1, \cdots, n$ it's obvious that $\langle z, v_i \rangle = 0$, therefore

$$z \in \{v_1, \cdots, v_n\}^\perp$$

Therefore

$$z \in \{v_1, \cdots, v_n\}^\perp = \overline{\operatorname{span}\{v_1, \cdots, v_n\}}^\perp = \operatorname{span}\{v_1, \cdots, v_n\}^\perp$$

□

Theorem 0.16 (Gram-Schmidt Orthonormalization)**.** *Given $\mathcal{V}$ an euclidean space and $(v)_{n\in\mathbb{N}} \in \mathcal{V}$ a set of linearly independent vectors. Then $\exists (u)_{n\in\mathbb{N}} \in \mathcal{V}$ orthonormal system such that*

1. *u_n is a linear combination of $v_i \ \forall 0 \leq i \leq n$, i.e.*

$$u_n = \sum_{k=1}^{n} a_{nk} v_k \quad a_{nn} \neq 0$$

2. *v_n can be written as follows*

$$v_n = \sum_{k=1}^{n} b_{nk} u_k \quad b_{nn} \neq 0$$

Therefore, $\forall n \in \mathbb{N}$ we have that

$$\operatorname{span}\{v_1, \cdots, v_n\} = \operatorname{span}\{u_1, \cdots, u_n\}$$

Proof. Defining

$$w_n = v_n - \sum_{k=1}^{n-1} \frac{\langle v_n, v_k \rangle}{\|w_k\|^2} w_k \quad u_n = \frac{w_n}{\|w_n\|}$$

We can say immediately that $\forall n \geq 1 \; w_n \in \{w_1, \cdots, w_{n-1}\}^{\perp}$. By induction we can say that it holds $\forall (w)_{n\in\mathbb{N}}$, therefore $(w)_n$ is an orthogonal system and $(u)_{n\in\mathbb{N}}$ is an orthonormal system
We can also say that

$$v_n = w_n + \sum_{j=1}^{k-1} \beta_{nj} w_j$$

I.e. $\forall n \geq 1 \; w_n$ is a linear combination of $\{v_1, \cdots, v_n\}$, therefore, by definition

$$\operatorname{span}\{v_1, \cdots, v_n\} = \operatorname{span}\{w_1, \cdots, w_n\} = \operatorname{span}\{u_1, \cdots, u_n\}$$

$\square$

Example 0.3.1. 1) Legendre Polynomials
Using the Gram-Schmidt orthonormalization procedure, we can find an orthonormal system $\{p_0, \cdots, p_n\} \subset C_2[-1, 1]$ starting from the following system

$$(v)_n := \left\{1, x, x^2, x^3, x^4, \cdots, x^n\right\}$$

The final result will be called the *Legendre Polynomials* We begin by taking the canonical scalar product in $C_2[-1, 1]$ as follows

$$\langle f, g \rangle = \int_{-1}^{1} f(x)g(x) \, \mathrm{d}x$$

And using that

$$\int_{-1}^{1} x^n \, \mathrm{d}x = \begin{cases} \dfrac{2}{n+1} & n = 2k \in \mathbb{N} \\ 0 & n = 2k \in \mathbb{N} \end{cases}$$

Therefore, we have that

$$\begin{aligned}
&w_0 = 1 \quad \|w_0\|^2 = \int_{-1}^{1} \mathrm{d}x = 2 \\
&w_1 = x - \frac{1}{2}\int_{-1}^{1} x \, \mathrm{d}x = x \quad \|w_1\|^2 = \frac{-1}{1} x^2 \, \mathrm{d}x = \frac{2}{3} \\
&w_2 = x^2 - \frac{1}{3} \quad \|w_2\|^2 = \int_{-1}^{1} \left(x^2 \frac{1}{3}\right)^2 \mathrm{d}x = \frac{8}{45} \\
&\;\vdots
\end{aligned}$$

Normalizing, we find

$$\begin{aligned}
p_0 &= \frac{1}{\sqrt{2}} \\
p_1(x) &= \sqrt{\frac{3}{2}}x \\
p_2(x) &= \sqrt{\frac{45}{8}}\left(x^2 - \frac{1}{3}\right) = \frac{1}{2}\sqrt{\frac{5}{2}}\left(3x^2 - 1\right) \\
&\vdots
\end{aligned}$$

And so on. The set $(p)_n$ is called the set of *Legendre polynomials* 2) Hermite Polynomials
Using the same procedure, we can find the *Hermite polynomials* $H_n(x)$ in the space $C_2(\mathbb{R}, e^{-x^2}\,\mathrm{d}x)$. The first 5 are the following

$$\begin{aligned}
H_1 &= 1 \\
H_2(x) &= x \\
H_3(x) &= x^2 - \frac{1}{2} \\
H_4(x) &= x^3 - \frac{3}{2}x \\
H_5(x) &= x^4 - 3x^2 + \frac{3}{4}
\end{aligned}$$

Theorem 0.17 (Existence of an Orthonormal Basis)**.** *Given a separable or complete euclidean space $\mathcal{V}$, there always exists an orthonormal basis*

Proof. Taken $\mathcal{V}$ a separable euclidean space and $(v)_{n\in\mathbb{N}}$ a dense subset of $\mathcal{V}$.
Removing the linearly independent elements of this subset, we can call the new linearly independent set $(w)_{n\in\mathbb{N}}$. We have obviously

$$\begin{aligned}
\operatorname{span}\{(w)_{n=1}^{\infty}\} &= \operatorname{span}\{(v)_{n=1}^{\infty}\} \\
\overline{\operatorname{span}\{(w)_{n=1}^{\infty}\}} &= \overline{\operatorname{span}\{(v)_{n=1}^{\infty}\}} = \mathcal{V}
\end{aligned}$$

Orthonormalizing the system $(w)_{n\in\mathbb{N}}$ with the Gram-Schmidt procedure I obtain then a new set $(u)_{n\in\mathbb{N}}$ such that

$$\overline{\operatorname{span}\{(u)_{n\in\mathbb{N}}\}} = \overline{\operatorname{span}\{(w)_{n\in\mathbb{N}}\}} = \mathcal{V}$$

$(u)_{n\in\mathbb{N}}$ is complete and therefore a basis. □

Remark. If $\mathcal{V}$ is a complete euclidean space but not separable, we can find thanks to Zorn's lemma a maximal orthonormal basis, but

1. There isn't a standard procedure for finding this basis
2. Taken the basis $(u)_{\alpha\in I}$ it can't be numerable. If it was then $\mathcal{V} = \overline{\operatorname{span}\{(u)_{\alpha}\}}$. Taken $X = \operatorname{span}_{\mathbb{Q}}\{(u)_k\}$ as the set of finite linear combinations with rational coefficients, we have that $\overline{X} = \operatorname{span}\{(u)_k\}$ and therefore $\overline{X} = \mathcal{V}$ contradicting the fact that $\mathcal{V}$ is not separable