§ 0.1 Tensors and k-forms

§§ 0.1.1 Basic Definitions, Tensor Product and Wedge Product

Definition 0.1.1 (Multilinear Functions, Tensors)**.** Let $\mathcal{V}$ be a real vector space, and take $\mathcal{V}^k = \mathcal{V}\times\cdots\times\mathcal{V}$ $k-$times. A function $T : \mathcal{V}^k \longrightarrow \mathbb{R}$ is called *multilinear* if $\forall i = 1, \cdots, k$, $\forall a \in \mathbb{R}$, $\forall v, w \in \mathcal{V}$

$$T(v_1, \cdots, av_i + w_i, \cdots, v_k) = aT(v_1, \cdots, v_i, \cdots, v_k) + T(v_1, \cdots, w_i, \cdots, v_k) \tag{1}$$

A multilinear function of this kind is called *k-tensor* on $\mathcal{V}$. The set of all $k-$tensors is denoted as $\mathcal{T}^k(\mathcal{V})$ and is a real vector space.
The tensor T is usually denoted as follows

$$T_{\mu_1\ldots\mu_k} \tag{2}$$

Where each index indicates a slot of the multilinear application $T(-, \cdots, -)$

Definition 0.1.2 (Tensor Product)**.** Let $S \in \mathcal{T}^k(V), T \in \mathcal{T}^l(\mathcal{V})$, we define the *tensor product* $S \otimes T \in \mathcal{T}^{k+l}(\mathcal{V})$ as follows

$$(S \otimes T)(v_1, \cdots, v_k, v_{k+1}, \cdots, v_{k+l}) = S(v_1, \cdots, v_k)T(v_{k+1}, \cdots, v_{k+l}) \tag{3}$$

This product has the following properties

$$\begin{aligned}(S_1 + S_2) \otimes T &= S_1 \otimes T + S_2 \otimes T\\ S \otimes (T_1 + T_2) &= S \otimes T_1 + S \otimes T_2\\ (aS) \otimes T &= S \otimes (aT) = a(S \otimes T)\\ (S \otimes T) \otimes U &= S \otimes (T \otimes U) = S \otimes T \otimes U\end{aligned} \tag{4}$$

If $S = S_{\mu_1\ldots\mu_k}$ and $T = T_{\mu_{k+1}\ldots\mu_{k+l}}$ we have

$$(S \otimes T)_{\mu_1\ldots\mu_k\mu_{k+1}\ldots\mu_{k+l}} = S_{\mu_1\ldots\mu_k} T_{\mu_{k+1}\ldots\nu_{k+l}} \tag{5}$$

Definition 0.1.3 (Dual Space)**.** We define the *dual space* of a real vector space $\mathcal{V}$ as the space of all *linear functionals* from the space to the field over it's defined, and it's indicated with $\mathcal{V}^\star$. I.e. let $\varphi^\mu \in \mathcal{V}^\star$, then $\varphi^\mu : \mathcal{V} \longrightarrow \mathbb{R}$.
It's easy to see how $\mathcal{V}^\star = \mathcal{T}^1(\mathcal{V})$.

Theorem 0.1. *Let $\mathcal{B} = \{v_{\mu_1}, \cdots, v_{\mu_n}\}$ be a basis for the space $\mathcal{V}$, and let $\mathcal{B}^\star := \{\varphi^{\mu_1}, \cdots, \varphi^{\mu_n}\}$ be the basis of the dual space, i.e. $\varphi^\mu v_\nu = \delta^\mu_\nu$ $\forall \varphi^\mu \in \mathcal{B}^\star$, $v_\mu \in \mathcal{B}$, then the set of all k-fold tensor products has basis $\mathcal{B}_\mathcal{T}$, where*

$$\mathcal{B}_\mathcal{T} := \{\varphi^{\mu_1} \otimes \cdots \otimes \varphi^{\mu_k},\ \forall i = 1, \cdots, n\} \tag{6}$$

Theorem 0.2 (Linear Transformations on Tensor Spaces)**.** *If $f^\mu_\nu : \mathcal{V} \longrightarrow \mathcal{W}$ is a linear transformation, $f^\nu_\mu \in \mathcal{L}(\mathcal{V}, \mathcal{W})$, one can define a linear transformation $f^\star : \mathcal{T}^k(W) \longrightarrow \mathcal{T}^k(V)$ as follows*

$$f^\star T(v_{\mu_1}, \cdots, v_{\mu_k}) = T(f^\mu_\nu v_{\mu_1}, \cdots, f^\mu_\nu v_{\mu_k})$$

Theorem 0.3. *If g is an inner product on $\mathcal{V}$ (i.e. $g : \mathcal{V} \times \mathcal{V} \longrightarrow \mathbb{R}$, with the properties of an inner product), there is a basis $v_{\mu_1}, \cdots, v_{\mu_n}$ of $\mathcal{V}$ such that $g(v_\mu, v_\nu) = g_{\mu\nu} = g_{\nu\mu} = g(v_\nu, v_\mu) = \delta_{\mu\nu}$. This basis is called orthonormal with respect to T. Consequently there exists an isomorphism $f^\mu_\nu : \mathbb{R}^n \xrightarrow{\sim} \mathcal{V}$ such that*

$$g(f^\mu_\nu x^\nu, f^\mu_\nu y^\nu) = x_\mu y^\mu = g_{\mu\nu} x^\mu y^\nu \tag{7}$$

I.e.

$$f^\star g(\cdot, \cdot) = g_{\mu\nu} \tag{8}$$

Definition 0.1.4 (Alternating Tensor)**.** Let $\mathcal{V}$ be a real vector space, and $\omega \in \mathcal{T}^k(\mathcal{V})$. ω is said to be *alternating* if

$$\begin{aligned} &\omega(v_{\mu_1}, \cdots, v_{\mu_i}, \cdots, v_{\mu_j}, \cdots, v_{\mu_k}) = -\omega(v_{\mu_1}, \cdots, v_{\mu_j}, \cdots, v_{\mu_i}, \cdots, v_{\mu_k}) \\ &\omega(v_{\mu_1}, \cdots, v_{\mu_i}, \cdots, v_{\mu_i}, \cdots, v_{\mu_k}) = 0 \end{aligned} \tag{9}$$

Or, compactly

$$\begin{aligned} \omega_{\mu\ldots\nu\ldots\gamma\ldots\sigma} &= -\omega_{\mu\ldots\gamma\ldots\nu\ldots\sigma} \\ \omega_{\mu\ldots\nu\ldots\nu\ldots\gamma} &= 0 \end{aligned} \tag{10}$$

The space of all alternating $k-$tensors on $\mathcal{V}$ is indicated as $\Lambda^k(\mathcal{V})$, and we obviously have that $\Lambda^k(\mathcal{V}) \subset \mathcal{T}^k(\mathcal{V})$.
We can define an application $\mathrm{Alt} : \mathcal{T}^k(\mathcal{V}) \longrightarrow \Lambda^k(\mathcal{V})$ as follows

$$\mathrm{Alt}(T)(v^\mu_1, \cdots, v^\mu_k) = \frac{1}{k!} \sum_{\sigma \in \Sigma_k} \mathrm{sgn}(\sigma) T(v^\mu_{\sigma(1)}, \cdots, v^\mu_{\sigma(k)}) \tag{11}$$

With $\sigma = (i, j)$ a permutation and Σ_k the set of all permutations of natural numbers $1, \cdots, k$ Compactly, we define an operation on the indices, indicated in square brackets, called the *antisymmetrization* of the indices inside the brackets.
This definition is much more general, since it lets us define a partially antisymmetric tensor, i.e. antisymmetric on only some indices.

$$\mathrm{Alt}(T_{\mu_1\ldots\mu_k}) = \frac{1}{k!} T_{[\mu_1\ldots\mu_k]} \tag{12}$$

As an example, for a $2-$tensor $a_{\mu\nu}$ we can write

$$a_{[\mu\nu]} = \frac{1}{2}\left(a_{\mu\nu} - a_{\nu\mu}\right) = \tilde{a}_{\mu\nu} \in \Lambda^2(\mathcal{V}) \tag{13}$$

This is valid for general tensors. If we define a $k-$tensor over the product repeated k times for $\mathcal{V}$ and k for its dual space $\mathcal{V} \times \cdots \mathcal{V} \times \mathcal{V}^\star \times \cdots \times \mathcal{V}^\star$, we can define the space $\mathcal{T}^k(\mathcal{V} \times \mathcal{V}^\star) = \mathcal{W}$. Let the basis for this space be the following

$$\mathcal{B}_\mathcal{W} := \{v_{\mu_1} \otimes \cdots \otimes v_{\mu_k} \otimes \varphi^{\nu_1} \otimes \cdots \otimes \varphi^{\nu_k}\}$$

Then an element $\mathcal{Y}$ of the space $\mathcal{W}$ can be written as follows

$$\mathcal{Y}(v_{\mu_1}, \cdots, v_{\mu_k}, \varphi^{\nu_1}, \cdots, \varphi^{\nu_k}) = \mathcal{Y}^{\nu_1\ldots\nu_k}_{\mu_1\ldots\mu_k}$$

We can define a new element $Y \in \Lambda^k(\mathcal{V} \times \mathcal{V}^\star)$ using the antisymmetrization brackets

$$Y^{\nu_1 \dots \nu_k}_{\mu_1 \dots \mu_k} = \mathcal{Y}^{[\nu_1 \dots \nu_k]}_{[\mu_1 \dots \mu_k]}$$

We can define also partially antisymmetric parts as follows

$$R^{\nu_1 \dots \nu_k}_{\mu_1 \dots \mu_k} = \mathcal{Y}^{\nu_1 \dots [\nu_i \nu_{i+1}] \dots \nu_k}_{\mu_1 \dots [\mu_l \mu_{l+1}] \dots \mu_k} = \frac{1}{4!}\left(\mathcal{Y}^{\nu_1 \dots \nu_i \nu_{i+1} \dots \nu_k}_{\mu_1 \dots \mu_l \mu_{l+1} \dots \mu_k} - \mathcal{Y}^{\nu_1 \dots \nu_{i+1} \nu_i \dots \nu_k}_{\mu_1 \dots \mu_l \mu_{l+1} \dots \mu_k} + \mathcal{Y}^{\nu_1 \dots \nu_i \nu_{i+1} \dots \nu_k}_{\mu_1 \dots \mu_l \mu_{l+1} \dots \mu_k} - \mathcal{Y}^{\nu_1 \dots \nu_i \nu_{i+1} \dots \nu_k}_{\mu_1 \dots \mu_{l+1} \mu_l \dots \mu_k} \right)$$

Note how the indexes in the expressions with the label i and l simply got switched, and in the new definition, the tensor R is antisymmetric in both the *covariant* (lower) indexes μ_l, μ_{l+1} and in the *contravariant* (upper) indexes ν_i, ν_{i+1}, where obviously $i, l \leq k$

Theorem 0.4. *Let $T \in \mathcal{T}^k(\mathcal{V})$ and $\omega \in \Lambda^k(\mathcal{V})$. Then*

$$\begin{aligned} T_{[\mu_1 \dots \mu_k]} &\in \Lambda^k(\mathcal{V}) \\ \omega_{[\mu_1 \dots \mu_k]} &= \omega_{\mu_1 \dots \mu_k} \\ T_{[[\mu_1 \dots \mu_k]]} &= T_{[\mu_1 \dots \mu_k]} \end{aligned} \tag{14}$$

Definition 0.1.5 (Wedge Product)**.** Let $\omega \in \Lambda^k(\mathcal{V}),\ \eta \in \Lambda^l(\mathcal{V})$. In general $\omega \otimes \eta \notin \Lambda^{k+l}(\mathcal{V})$, hence we define a new product, called the *wedge product*, such that $\omega \wedge \eta \in \Lambda^{k+l}(\mathcal{V})$

$$\omega_{\mu_1 \dots \mu_k} \wedge \eta_{\nu_1 \dots \nu_k} = \frac{(k+l)!}{k!l!} \omega_{[\mu_1 \dots \mu_k} \eta_{\nu_1 \dots \nu_l]} \tag{15}$$

With the following properties
$\forall \omega, \omega_1, \omega_2 \in \Lambda^k(\mathcal{V}),\ \forall \eta, \eta_1, \eta_2 \in \Lambda^l(\mathcal{V}),\ \forall a \in \mathbb{R}, \forall f^\star \in \mathcal{L} : \mathcal{T}^k(\mathcal{V}) \longrightarrow \mathcal{T}^l(\mathcal{V})\ \forall \theta \in \Lambda^m(\mathcal{V})$

$$\begin{aligned} (\omega_1 + \omega_2) \wedge \eta &= \omega_1 \wedge \eta + \omega_2 \wedge \eta \\ \omega \wedge (\eta_1 + \eta_2) &= \omega \wedge \eta_1 + \omega \wedge \eta_2 \\ (\omega \wedge \eta) \wedge \theta &= \omega \wedge (\eta \wedge \theta) \\ a\omega \wedge \eta &= \omega \wedge a\eta = a(\omega \wedge \eta) \\ \omega \wedge \eta &= (-1)^{kl} \eta \wedge \omega \\ f^\star(\omega \wedge \eta) &= f^\star(\omega) \wedge f^\star(\eta) \end{aligned} \tag{16}$$

Theorem 0.5. *The set*

$$\{\varphi^{\mu_1} \wedge \dots \wedge \varphi^{\mu_k},\ k < n\} \subset \Lambda^k(\mathcal{V}) \tag{17}$$

Is a basis for the space $\Lambda^k(\mathcal{V})$, and therefore

$$\dim(\Lambda^k(\mathcal{V})) = \binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Where $\dim(\mathcal{V}) = n$.
Therefore, $\dim(\Lambda^n(\mathcal{V})) = 1$

Theorem 0.6. *Let $v_{\mu_1}, \cdots, v_{\mu_n}$ be a basis for $\mathcal{V}$, and take $\omega \in \Lambda^n(\mathcal{V})$, then, if $w_\mu = a^\nu_\mu v_\nu$*

$$\omega(w_{\mu_1} \cdots w_{\mu_n}) = \det_{\mu\nu}(a^\mu_\nu) \omega(v_{\mu_1}, \dots, v_{\mu_n}) \tag{18}$$

Or using the basis representation of a vector $t^\mu = t^\mu w_\mu = t^\mu a^\nu_\mu v_\nu$ we have

$$\omega_{\mu_1 \dots \mu_n} t^{\mu_1} \cdots t^{\mu_n} = \det_{\mu\nu}(a^\mu_\nu) \omega_{\nu_1 \dots \nu_n} t^{\nu_1} \cdots t^{\nu_n} \tag{19}$$

Proof. Define $\eta_{\mu_1\ldots\mu_n} \in \mathcal{T}^n(\mathbb{R}^n)$ as

$$\eta_{\mu_1\ldots\mu_n} a^{\mu_1}_{\nu_1} a^{\mu_2}_{\nu_2} \cdots a^{\mu_n}_{v_n} = \omega_{\mu_1\ldots\mu_n} a^{\mu_1}_{\nu_1} \cdots a^{\mu_n}_{\nu_n}$$

Hence $\eta \in \Lambda^n(\mathbb{R}^n)$ so $\eta = \lambda \det(\cdot)$ for some λ, and

$$\lambda = \eta_{\mu_1\ldots\mu_n} e^{\mu_1} \cdots e^{\mu_n} = \omega_{\mu_1\ldots\mu_n} v^{\mu_1} \cdots v^{\mu_n}$$

□

§§ 0.1.2 Volume Elements and Orientation

Definition 0.1.6 (Orientation)**.** The previous theorem shows that a $\omega \in \Lambda^n(\mathcal{V}),\ \omega \neq 0$ splits the bases of $\mathcal{V}$ in two disjoint sets.
Bases for which $\omega(\mathcal{B}_v) > 0$ and for which $\omega(\mathcal{B}_w) < 0$. Defining $w^\mu = a^\mu_\nu v^\nu$ we have that the two bases belong to the same group iff $\det_{\mu\nu}(a^\mu_\nu) > 0$. We call this the *orientation* of the basis of the space. The *usual orientation* of $\mathbb{R}^n$ is

$$[e_\mu]$$

Given another two basis of $\mathbb{R}^n$ we can define (taking the first two examples)

$$\begin{gathered}[v_\mu] \\ -[w_\mu]\end{gathered}$$

Definition 0.1.7 (Volume Element)**.** Take a vector space $\mathcal{V}$ such that $\dim(\mathcal{V}) = n$ and it's equipped with an inner product g, such that there are two bases $(v^{\mu_1}, \cdots, v^{\mu_n}),\ (w^{\mu_1}, \cdots, w^{\mu_n})$ that satisfy the *orthonormality condition* with respect to this scalar product

$$g_{\mu\nu} v^{\mu_i} v^{\nu_j} = g_{\sigma\gamma} w^{\sigma_i} w^{\gamma_j} = \delta_{ij} \tag{20}$$

Then

$$\omega_{\mu_1\ldots\mu_n} v^{\mu_1} \cdots v^{\mu_n} = \omega_{\mu_1\ldots\mu_n} w^{\mu_1} \cdots w^{\mu_n} = \det_{\mu\nu}(a^\mu_\nu) = \pm 1$$

Where

$$w^\mu = a^\mu_\nu v^\nu$$

Therefore

$$\exists! \omega \in \Lambda^n(\mathcal{V}) \ :\ \exists! [w^{\mu_1}, \cdots, w^{\mu_n}] = O$$

Where O is the *orientation* of the vector space.

Definition 0.1.8 (Cross Product)**.** Let $v^\mu_1, \cdots, v^\mu_n \in \mathbb{R}^{n+1}$ and define $\varphi_\nu w^\nu$ as follows

$$\varphi_\nu w^\nu = \det \begin{pmatrix} v^{\mu_1} \\ \vdots \\ v^{\mu_n} \\ w^\nu \end{pmatrix}$$

Then $\varphi \in \Lambda^1(\mathbb{R}^{n+1})$, and

$$\exists! z^\mu \in \mathbb{R}^{n+1} \ :\ z^\mu w_\mu = \varphi_\nu w^\nu$$

z^μ is called the *cross product*, and it's indicated as

$$z^\mu = v^{\nu_1} \times \cdots \times v^{\nu_n} = \epsilon^\mu_{\ \nu_1\ldots\nu_n} v^{\nu_1} \cdots v^{\nu_n}$$

§ 0.2 Tangent Space and Differential Forms

Definition 0.2.1 (Tangent Space)**.** Let $p \in \mathbb{R}^n$, then the set of all pairs $\{(p, v^\mu)|\, v^\mu \in \mathbb{R}^n\}$ is denoted as $T_p\mathbb{R}^n$ and it's called the *tangent space* of $\mathbb{R}^n$ (at the point. This is a vector space defining the following operations

$$(p, av^\mu) + (p, aw^\mu) = (p, a(v^\mu + w^\mu)) = a(p, v^\mu + w^\mu) \quad \forall v^\mu, w^\mu \in \mathbb{R}^n,\ a \in \mathbb{R}$$

Remark. If a vector $v^\mu \in \mathbb{R}^n$ can be seen as an arrow from 0 to the point v, a vector $(p, v^\mu) \in T_p\mathbb{R}^n$ can be seen as an arrow from the point p to the point $p + v$. In concordance with the usual notation for vectors in physics, we will write $(p, v^\mu) = v^\mu$ directly, or v_p^μ when necessary to specify that we're referring to the vector $v^\mu \in T_p\mathbb{R}^n$. The point $p + v$ is called the *end point* of the vector v_p^μ.

Definition 0.2.2 (Inner Product in $T_p\mathbb{R}^n$)**.** The *usual inner product* of two vectors $v_p^\mu, w_p^\mu \in T_p\mathbb{R}^n$ is defined as follows

$$\begin{aligned} \langle\cdot,\cdot\rangle_p :& T_p\mathbb{R}^n \times T_p\mathbb{R}^n \longrightarrow \mathbb{R} \\ & v_p^\mu w_\mu^p = v^\mu w_\mu = k \end{aligned} \tag{21}$$

Analogously, one can define the usual orientation of $T_p\mathbb{R}^n$ as follows

$$[(e^{\mu_1})_p, \cdots, (e^{\mu_n})_p]$$

Definition 0.2.3 (Vector Fields, Again)**.** Although we already stated a definition for a vector field, we're gonna now state the actual precise definition of vector field
Let $p \in \mathbb{R}^n$ be a point, then a function $f^\mu(p) : \mathbb{R}^n \longrightarrow T_p\mathbb{R}^n$ is called a vector field, if $\forall p \in A \subseteq \mathbb{R}^n$ we can define

$$f^\mu(p) = f^\mu(p)(e_\mu)_p \tag{22}$$

Where $(e_\mu)_p$ is the canonical basis of $T_p\mathbb{R}^n$
All the previous *(and already stated)* considerations on vector fields hold with this definition.

Definition 0.2.4 (Differential Form)**.** Analogously to vector fields, one can define $k-$forms on the tangent space. These are called *differential (k-)forms* and "live" on the space $\Lambda^k(T_p\mathbb{R}^n)$.
Let $\varphi_p^{\mu_1}, \cdots, \varphi_p^{\mu_k} \in (T_p\mathbb{R}^n)^\star$ be a basis on such space, then the differential form $\omega \in \Lambda^k(T_p\mathbb{R}^n)$ is defined as follows

$$\omega_{\mu_1\dots\mu_k}(p) = \omega_{\mu_1\dots\mu_k}\varphi_p^{[\mu_1}\cdots\varphi_p^{\mu_k]} \to \sum_{i_1<\dots<i_k} \omega_{i_1\dots i_k}(p)\varphi_{i_1}(p) \wedge \cdots \wedge \varphi_{i_k}(p) \tag{23}$$

A function $f : T_p\mathbb{R}^n \longrightarrow \mathbb{R}$ is defined as $f \in \Lambda^0(T_p\mathbb{R}^n)$, or a $0-$form. In general, so, we can write without incurring in errors

$$f(p)\omega = f(p) \wedge \omega = f(p)\omega_{\mu_1\dots\mu_k} \tag{24}$$

§§ 0.2.1 External Differentiation, Closed and Exact Forms

Definition 0.2.5 (Differential)**.** Now we will omit that we're working on a point $p \in \mathbb{R}^n$ and we'll use the usual notation.
Let $f : T_p\mathbb{R}^n \longrightarrow \mathbb{R}$ be a smooth (i.e. continuously differentiable) function, where $f \in C^\infty$, then,

using operatorial notation we have that $\partial_\mu f(v) \in \Lambda^1(\mathbb{R}^n)$, therefore, with a small modification, we can define

$$\mathrm{d}f(v_p^\nu) = \partial_\mu f(v^\nu) \tag{25}$$

It's obvious how $\mathrm{d}x^\mu(v_p^\nu) = \partial_\nu x^\mu(v^\nu) = v^\mu$, therefore $\mathrm{d}x^\mu$ is a basis for $\Lambda^1(T_p\mathbb{R}^n)$, which we will indicate as $\mathrm{d}x^\mu$, therefore $\forall\omega \in \Lambda^k(T_p\mathbb{R}^n)$

$$\omega_{\mu_1\ldots\mu_k} = \omega_{\mu_1\ldots\mu_k}\,\mathrm{d}x^{[\mu_1}\cdots\mathrm{d}x^{\mu_k]} \to \sum_{i_1<\ldots<i_k} \omega_{i_1\ldots i_k}(p)\,\mathrm{d}x^{i_1}\wedge\cdots\wedge\mathrm{d}x^{i_k} \tag{26}$$

Basically, the vectors $\mathrm{d}x^\mu$ are the *dual basis* with respect to the canonical basis $(e_\mu)_p$

Theorem 0.7. *Since $\mathrm{d}f(v_p^\nu) = \partial_\nu f(v^\nu)$ we have, expressing the differential of a function with the basis vectors,*

$$\mathrm{d}f = \frac{\partial f}{\partial x^\mu}\,\mathrm{d}x^\mu = \partial_\mu f\,\mathrm{d}x^\mu \tag{27}$$

Definition 0.2.6. Having defined a smooth linear transformation $f^\mu_\nu : \mathbb{R}^n \longrightarrow \mathbb{R}^m$, it induces another linear transformation $\partial_\gamma f^\mu_\nu : \mathbb{R}^n \longrightarrow \mathbb{R}^m$, which with some modifications becomes the application $(f_\star)^\mu_\nu : T_p\mathbb{R}^n \longrightarrow T_{f(p)}\mathbb{R}^m$ defined such that

$$(f_\star)^\mu_\nu(v^\nu) = \left(\mathrm{d}f|_{f(p)}\right)^\mu_\nu(v^\nu) \tag{28}$$

Which, in turn, also induces a linear transformation $f^\star : \Lambda^k(T_{f(p)}\mathbb{R}^m) \longrightarrow \Lambda^k(T_p\mathbb{R}^n)$, defined as follows. Let $\omega_p \in \Lambda^k(\mathbb{R}^m)$, then we can define $f^\star\omega \in \Lambda^k(T_{f(p)}\mathbb{R}^n)$ as follows

$$(f^\star\omega_p)(v_{\mu_1},\ldots,v_{\mu_k}) = \omega_{f(p)}\left((f_\star)^{\mu_1}_{\nu_1}v_{\mu_1},\cdots,(f_\star)^{\mu_k}_{\nu_k}v_{\mu_k}\right) \tag{29}$$

(Just remember that in this way we are writing explicitly the chosen base, watch out for the indexes!)

Theorem 0.8. *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a smooth function, then*

1. $(f^\star)^\mu_\nu(\mathrm{d}x^\nu) = \mathrm{d}f = \partial_\nu f^\mu\,\mathrm{d}x^\nu$
2. $f^\star(\omega_1+\omega_2) = f^\star\omega_1 + f^\star\omega_2$
3. $f^\star(g\omega) = (g\circ f)f^\star\omega$
4. $f^\star(\omega\wedge\eta) = f^\star\omega\wedge f^\star\eta$
5. $f^\star\left(h\,\mathrm{d}x^{[\mu_1}\cdots\mathrm{d}x^{\mu_n]}\right) = h\circ f\,\det_{\mu\nu}(\partial_\nu f^\mu)\,\mathrm{d}x^{[\mu_1}\cdots\mathrm{d}x^{\mu_n]}$

Definition 0.2.7 (Exterior Derivative)**.** We define the operator d as an operator $\Lambda^k(T_p\mathcal{V}) \xrightarrow{\mathrm{d}} \Lambda^{k+1}(T_p\mathcal{V})$ for some vector space $\mathcal{V}$. For a differential form ω it's defined as follows

$$(\mathrm{d}\omega)_{\nu\mu_1\ldots\mu_k} = \partial_{[\nu}\omega_{\mu_1\ldots\mu_k]} \tag{30}$$

This, using the classical mathematical notation can be written as follows

$$\begin{aligned}\mathrm{d}\omega &= \sum_{i_1<\ldots<i_k}\mathrm{d}\omega_{i_1,\ldots,i_k}\wedge\mathrm{d}x^{i_1}\wedge\cdots\wedge\mathrm{d}x^{i_k}\\ \mathrm{d}\omega &= \sum_{i_1<\ldots<i_k}\sum_{j=1}^{n}\frac{\partial}{\partial x^j}\omega_{i_1,\ldots,i_k}\,\mathrm{d}x^j\wedge\mathrm{d}x^{i_1}\wedge\cdots\wedge\mathrm{d}x^{i_k}\end{aligned} \tag{31}$$

Theorem 0.9 (Properties of d). *1.* $\mathrm{d}(\omega+\eta)=\mathrm{d}\omega+\mathrm{d}\eta$

2. $\mathrm{d}(\omega\wedge\eta)=\mathrm{d}\omega\wedge\eta+(-1)^k\omega\wedge\mathrm{d}\eta$ *for* $\omega\in\Lambda^k(\mathcal{V}),\ \eta\in\Lambda^l(\mathcal{V})$

3. $\mathrm{d}\mathrm{d}\omega=\mathrm{d}^2\omega=0$

4. $f^\star(\mathrm{d}\omega)=\mathrm{d}(f^\star\omega)$

Definition 0.2.8 (Closed and Exact Forms)**.** A form ω is called *closed* iff

$$\mathrm{d}\omega=0 \tag{32}$$

It's called exact iff

$$\omega=\mathrm{d}\eta \tag{33}$$

Theorem 0.10. *Let ω be an exact differential form. Then it's closed*

Proof. The proof is quite straightforward. Since ω is exact we can write $\omega=\mathrm{d}\rho$ for some differential form ρ, therefore

$$\mathrm{d}\omega=\mathrm{d}\mathrm{d}\rho=\mathrm{d}^2\rho=0$$

Hence $\mathrm{d}\omega=0$ and ω is closed. □

Example 0.2.1. Take $\omega\in\Lambda^1(\mathbb{R}^2)$, where it's defined as follows

$$\omega_\mu=p\,\mathrm{d}x+q\,\mathrm{d}y \tag{34}$$

The external derivative will be of easy calculus by remembering the mnemonic rule $\mathrm{d}\to\partial_\mu\wedge\mathrm{d}x^\mu$, or also as $\partial_{[\nu}$ then we have

$$\mathrm{d}\omega_{\mu\nu}=\partial_{[\nu}\omega_{\mu]}$$

But

$$\partial_\nu\omega_\mu=\begin{pmatrix}\partial_1\omega_1 & \partial_1\omega_2\\ \partial_2\omega_1 & \partial_2\omega_2\end{pmatrix}_{\mu\nu}$$

And

$$\partial_{[\nu}\omega_{\mu]}=\frac{1}{2}(\partial_\nu\omega_\mu-\partial_\mu\omega_\nu)=\frac{1}{2}(\partial\omega-\partial\omega^{\mathrm{T}})$$

Therefore

$$\mathrm{d}\omega_{\mu\nu}=\frac{1}{2}\begin{pmatrix}0 & \partial_x q-\partial_y p\\ \partial_y p-\partial_x q & 0\end{pmatrix}_{\mu\nu}$$

Which, expressed in terms of the basis vectors of $\Lambda^2(\mathbb{R}^2)$, $\mathrm{d}x\wedge\mathrm{d}y$, we get

$$\mathrm{d}\omega=\frac{1}{2}(\partial_x q-\partial_y p)\,\mathrm{d}x\wedge\mathrm{d}y+\frac{1}{2}(\partial_y p-\partial_x q)\,\mathrm{d}y\wedge\mathrm{d}x=(\partial_x q-\partial_y p)\,\mathrm{d}x\wedge\mathrm{d}y \tag{35}$$

Therefore

$$\mathrm{d}\omega=0\iff\partial_x q-\partial_y p=0 \tag{36}$$

Definition 0.2.9 (Star Shaped Set)**.** A set A is said to be *star shaped with respect to a point a* iff $\forall x\in A$ the line segment $[a,x]\subset A$

Lemma 0.2.1 (Poincaré's)**.** Let $A\subset\mathbb{R}^n$ be an open star shaped set, with respect to 0. Then every closed form on A is exact

§ 0.3 Chain Complexes and Manifolds

§§ 0.3.1 Singular $n-$cubes and Chains

Definition 0.3.1 (Singular $n-$cube)**.** A *singular n-cube* is an application $c : [0,1]^n \longrightarrow A \subset \mathbb{R}^n$. In general. A singular 0-cube is a function $f : \{0\} \longrightarrow A$ and a singular $1-$cube is a curve.

Definition 0.3.2 (Standard $n-$cube)**.** We define a *standard n-cube* as a function $I^n : [0,1]^n \longrightarrow \mathbb{R}^n$ such that $I^n(x^\mu) = x^\mu$.

Definition 0.3.3 (Face)**.** Given a standard $n-$cube I^n we define the $(i,\alpha)-$face of the cube as

$$I^n_{(i,\alpha)} = (x^1, \cdots, x^{i-1}, \alpha, x^i, \cdots, x^{n-1}) \quad \alpha = 0, 1 \tag{37}$$

Definition 0.3.4 (Chain)**.** Given n $k-$cubes c_i, we define a *n-chain* s as follows

$$s = \sum_{i=1}^{n} a_i c_i \quad a_i \in \mathbb{R} \tag{38}$$

Definition 0.3.5 (Boundary)**.** Given an $n-$cube c_i we define the *boundary* as ∂c_i. For a standard $n-$cube we have

$$\partial I^n = \sum_{i=1}^{n} \sum_{\alpha=0,1} (-1)^{i+\alpha} I^n_{(i,\alpha)} \tag{39}$$

For a $k-$chain s we define

$$\partial s = \partial(\sum_i a_i c_i) = \sum_i a_i \partial c_i \tag{40}$$

Where ∂s is a $(k-1)$-chain

Theorem 0.11. *For a chain c, we have that $\partial\partial c = \partial^2 c = 0$*

§§ 0.3.2 Manifolds

Definition 0.3.6 (Manifold)**.** Given a set $M \subset \mathbb{R}^n$, it is said to be a *k-dimensional manifold* if $\forall x^\mu \in M$ we have that

1. $\exists U \subset \mathbb{R}^k$ open set $x^\mu \in U$ and $V \subset \mathbb{R}^n$ and φ a diffeomorphism such that $U \simeq V$ and $\varphi\,(U \cap M) = V \cap \left(\mathbb{R}^k \times \{0\}\right)$, i.e. $U \cap M \simeq \mathbb{R}^k \cap \{0\}$
2. $\exists U \subset \mathbb{R}^k$ open and $W \subset \mathbb{R}^k$ open, $x^\mu \in U$ and $f : W \longrightarrow \mathbb{R}^n$ a diffeomorphism
   (a) $f(W) = M \cap U$
   (b) $\operatorname{rank}(f) = k\ \forall x^\mu \in W$
   (c) $f^{-1} \in C(f(W))$

The function f is said to be a *coordinate system in M*

Definition 0.3.7 (Half Space)**.** We define the *k-dimensional half space* $\mathbb{H}^k \subset \mathbb{R}^k$ as

$$\mathbb{H}^k := \left\{ x^\mu \in \mathbb{R}^k \middle|\, x^i \geq 0 \right\} \tag{41}$$

Definition 0.3.8 (Manifold with Boundary)**.** A *manifold with boundary* (MWB) is a manifold M such that, given a diffeomorphism h, an open set $U \supset M$ and an open set $V \subset \mathbb{R}^n$

$$h\left(U \cap V\right) = V \cap \left(\mathbb{H}^k \cap \{0\}\right) \tag{42}$$

The set of all points that satisfy this forms the set ∂M called the *boundary of M*

Definition 0.3.9 (Tangent Space)**.** Given a manifold M and a coordinate set f around $x^\mu \in M$, we define the *tangent space of M at $x^\mu \in M$* as follows

$$f : W \subset \mathbb{R}^k \longrightarrow \mathbb{R}^n \implies f_\star\left(T_x\mathbb{R}^k\right) = T_x M \tag{43}$$

Definition 0.3.10 (Vector Field on a Manifold)**.** Given a vector field f^μ we identify it as a vector field on a manifold M if $f^\mu(x^\nu) \in T_x M$. Analogously we define a $k-$differential form