(Incomplete)
Notes on Optics

From Maxwell to Crystals

§ Fresnel diffraction

Fresnel's approximation of the (KF) integral, written below

$$\psi_P = \frac{\psi_0 e^{-i\omega t}}{2i\lambda} \iint_\Sigma \frac{e^{ik(r+r')}}{r'r} \left(\cos(\hat{n}, \underline{r}) - \cos(\hat{n}, \underline{r}') \right) d_S^2 \qquad (KF)$$

Was developed by Fresnel expanding to 2nd order in δ the difference b/w $r+r'$ @ Σ

I.e.:

$$\Delta_\Sigma \simeq \left(\frac{h'}{d'} + \frac{h}{d} \right) \delta + \frac{1}{2} \left(\frac{1}{d'} + \frac{1}{d} \right) \delta^2$$

This approximation explicitly considers a NON-PLANE wave (δ^2), and therefore it's known as CLOSE FIELD APPROXIMATION. Here, either the source S, the measuring point P are close to Σ, we cannot neglect wave curvature.
The close nature of this approximation makes it easy to see
in later

§§ Fresnel Zones

Consider a plane aperture Σ illuminated by a point source S, such that the plane of Σ is $\perp$ to the observing point P, as in fig.

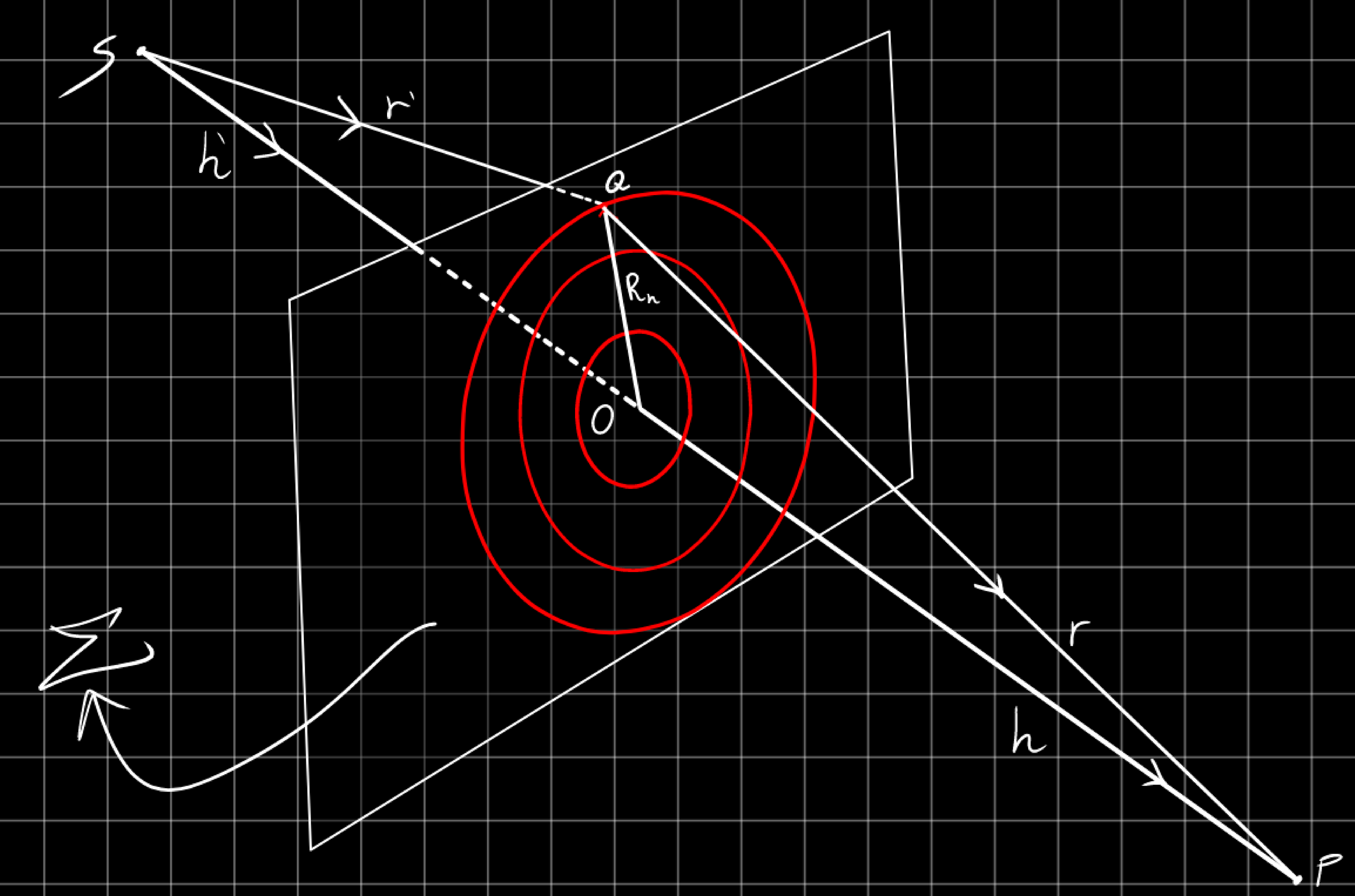


Fig 1) Source S close to a plane Σ, distant $h+h'$ from point P (measuring point). Fresnel zones in red

Said R the distance b/w O & Q on Σ, we have that the distance $\overline{SPQ}$ can be written

$$\| \overline{SPQ} \| = r + r'$$

Or:

$$\boxed{\begin{aligned} r + r' &= \sqrt{h^2 + R^2} + \sqrt{h'^2 + R'^2} \\ r + r' &\simeq h + h' + \frac{1}{2}\left(\frac{1}{h} + \frac{1}{h'} \right) R^2 + O(R^3) \end{aligned}}$$

The approximation that we will use is then:

$$r + r' = h + h' + \frac{1}{2}\left(\frac{1}{h} + \frac{1}{h'}\right)R^2$$

Suppose now that Σ is divided in concentric circles around O as in Fig.(1), such that between the n-th circle & the $(n+1)$-th circle, the following constraint holds.

Said $\xi_{n,m}$ the difference of distance $\|\overline{SQ_iP}\|$ b/w a point Q_n & R_n from O & Q_m at R_m from O, then:

$$\xi_{n,n+1} = \frac{1}{2}\lambda$$

The zones created w/ this constraint are known as Fresnel Zones. From the formula for the approximation of $r + r'$ we get

$$R_1 = \sqrt{\lambda L}\ ,\ R_2 = \sqrt{2\lambda L}\ ,\ \ldots$$

I.e.:

$$R_n = \sqrt{n\lambda L}$$

Where

$$L = \left(\frac{1}{h'} + \frac{1}{h}\right)^{-1} = \frac{h h'}{h + h'}$$

Note that, since two consecutive radii form a torus, then the area is

$$A_n = \pi R_{n+1}^2 - \pi R_n^2$$

$$A_n = \pi\left((n+1)\lambda L - n\lambda L\right) = \pi\lambda L = \pi R_1^2$$

I.e. the area of the Fresnel zone is constant, $A_n = A_{n+1} = A \quad \forall n \geq 1$

Typically this is very small, since for the 1st zone in the optical range

$$\begin{cases} h = h' \simeq 60\,\text{cm} \\ \lambda \simeq 600\,\text{nm} \end{cases} \Longrightarrow R_1 \simeq 0.4\,\text{mm}$$

Since $R_n \propto \sqrt{n}$ we can also say that the 100-th zone is approximately @ $R_{100} = 4\,\text{mm}$ (!)

The optical disturbance @ P $\{\psi_P\}$ can be evaluated by summation of the contributions of each Fresnel zone $\{\psi_1, \dots, \psi_n\}$

Since we have for each zone a phase change of π, then

$$|\psi_P| = \sum_{n=1}^{N} (-1)^{n+1} |\psi_n|$$

Consider now Σ as a circular aperture centered in O.

If Σ contains exactly N zones, then

$$\psi_P \begin{cases} = 0 & N \bmod 2 = 0 \\ \simeq \psi_1 & N \bmod 2 = 1 \end{cases}$$

Considerations of the obliquity factor in the KF integral also show that

$$\psi_n \leq \psi_{n+1} \quad (\text{slowly})$$

Therefore, if $\Sigma \to \mathbb{R}^2$, we have, regrouping the sum as follows:

$$\psi_P = \frac{1}{2}\psi_1 + \left(\frac{1}{2}\psi_1 - \psi_2 - \frac{1}{2}\psi_3\right) + \left(\frac{1}{2}\psi_3 - \psi_4 - \frac{1}{2}\psi_5\right) + \dots$$

for $n \to \infty$ (∞ Fresnel Zones)

$$\psi_P = \frac{1}{2}\psi_1$$

Note that this corresponds to the case that there is NO APERTURE

Consider now the case of having a circular obstacle instead of the aperture Σ, then, the Fresnel zones start @ the edge of the obstacle, then again

$$\psi_P = \frac{1}{2}\psi_1$$

And therefore @ the center of the shadow it's possible to observe a bright spot. If I is the irradiance w/o the obstacle, then

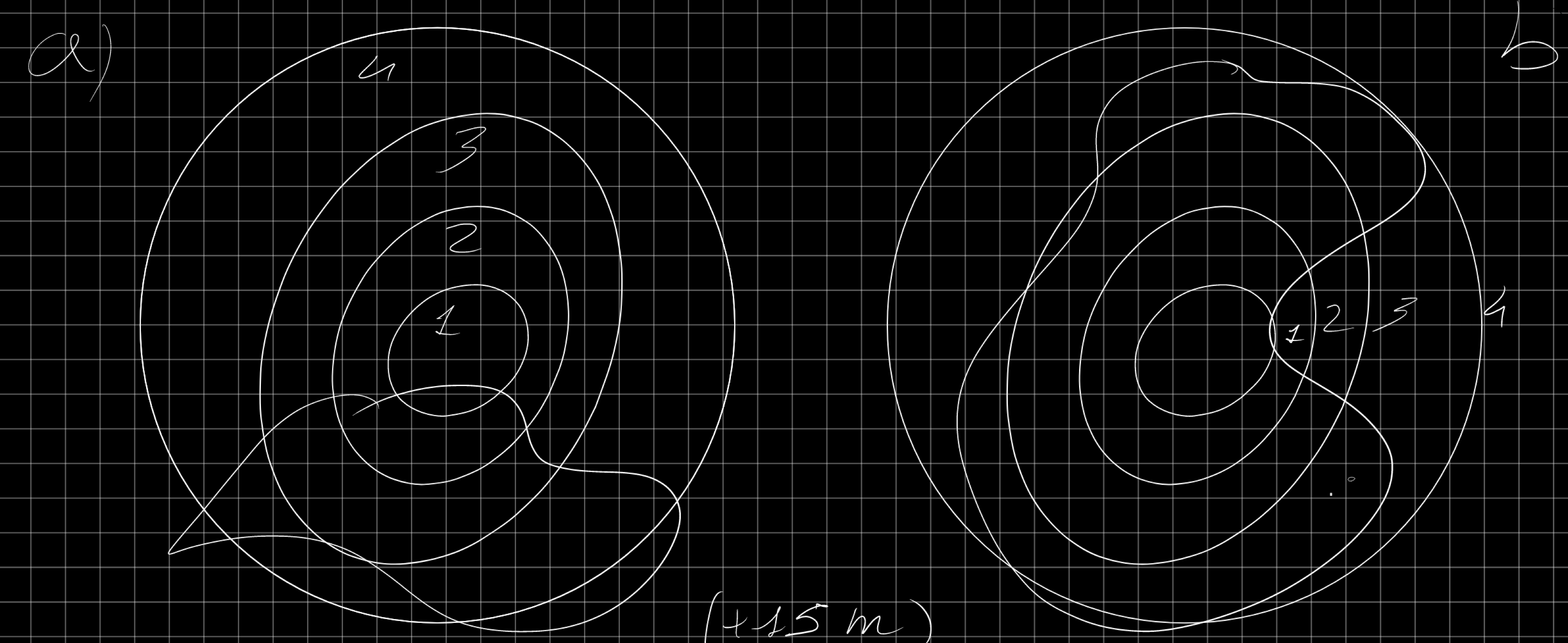
$$\psi_P = \frac{1}{2}\psi_1 \implies I_P \simeq I$$

This spot is known as << Arago spot >>.

For an irregular object, depending on where it's positioned wrt the zones, we have

a) It's offset from the center $\implies$ ψ_P hardly changes & higher contribs $\to 0$ quickly

b) It's over the center $\implies$ Terms diminish @ both ends & inside the shadow zone we get $I \simeq 0 \implies$ SHADOW AS USUAL (!) {check next drawing for visual}



(+15 m)

Fig 2) Cases a, b explained before

§§ Zone Plates

It's possible to construct plates whose job is to obstruct certain Fresnel zones. These plates are known as «ZONE PLATES».

Suppose that we build one that obstructs only even zones, then:

$$\psi_p = \sum_{n=1}^{N} \psi_{2n}$$

Thus $I_p \gg I$ (!) and the plate acts like a lens

The focal length of a plate (L) can be calculated as

$$L = \frac{R_1^2}{\lambda} = \frac{hh'}{h+h'}$$

Note also that $L \propto \lambda^{-1}$, therefore this can be seen as a very chromatic lens.

§§ Rectangular Aperture, Fresnel diffraction

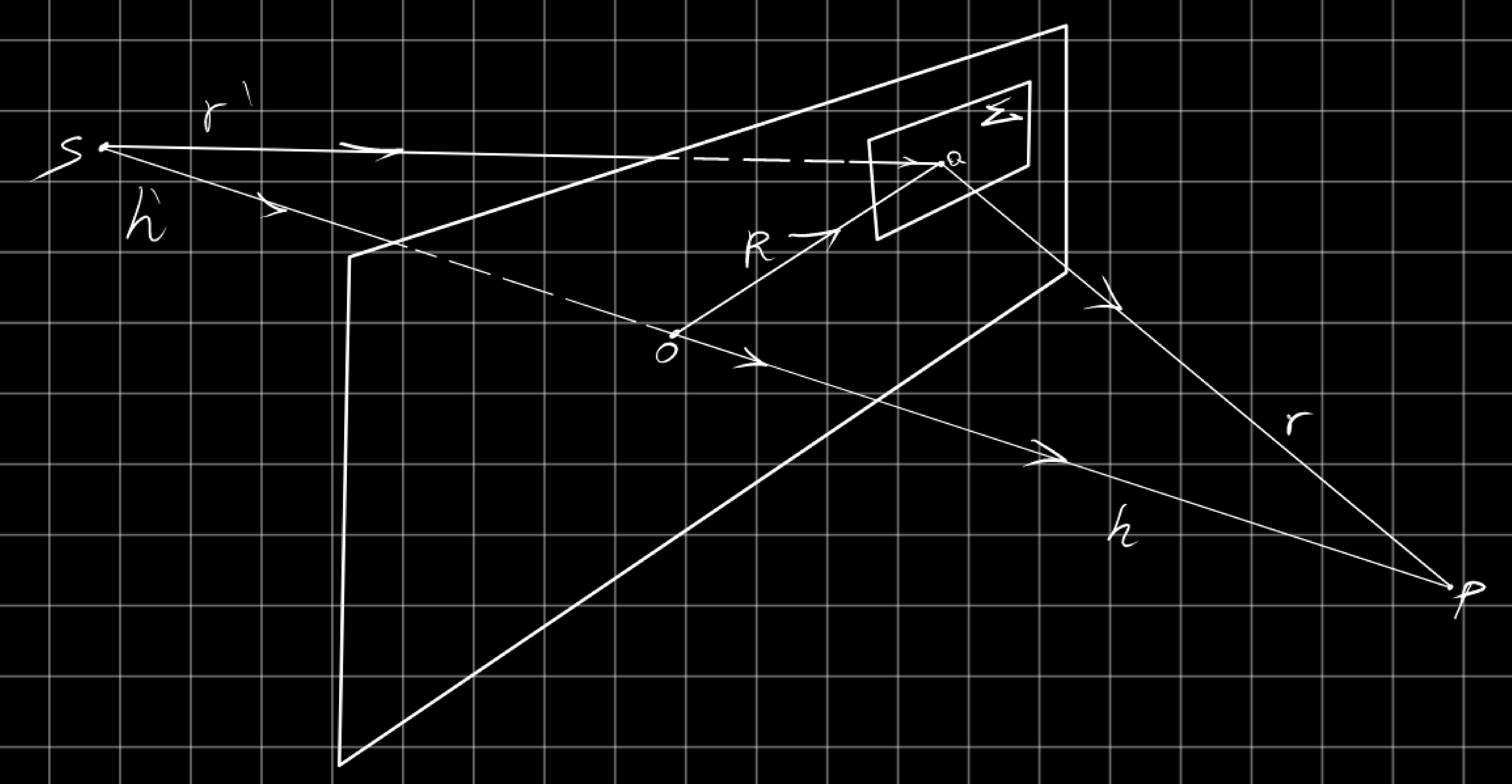


Fig 3) Rectangular aperture in CFA (Fresnel diffraction)

For a rectangular aperture as in Fig (3) we employ the KF integral together with cartesian coordinates $\{O; x, y\}$. Then, by definition

$$R^2 = x^2 + y^2$$

Therefore

$$r + r' = h + h' + \frac{1}{2L}(x^2 + y^2)$$

As for Fraunhofer diffraction, we shall assume that:

1) $\hat{O}(\underline{r}, \underline{r}', \underline{h}) \simeq 1$ {obliquity}

2) $1/rr' \simeq$ constant {radial factor}

Thus, the KF integral becomes:

$$\Psi_P = C \iint_{\Sigma} e^{ik(x^2+y^2)/2L} \, d^2s = \underbrace{C \int_{x_1}^{x_2} e^{\frac{ikx^2}{2L}} dx \int_{y_1}^{y_2} e^{\frac{iky^2}{2L}} dy}_{\text{Fubini}}$$

Introducing the dimensionless factors (u, v) as

$$\begin{cases} u_k = \sqrt{\dfrac{k}{\pi L}}\, x \\ v_k = \sqrt{\dfrac{k}{\pi L}}\, y \end{cases} \quad \text{or, using } k = \frac{2\pi}{\lambda} \qquad u_\lambda = x\sqrt{\frac{2}{\lambda L}} \qquad v_\lambda = y\sqrt{\frac{2}{\lambda L}}$$

We have

$$\Psi_P = \Psi_1 \int_{u_1}^{u_2} e^{\frac{i\pi u^2}{2}} du \int_{v_1}^{v_2} e^{\frac{i\pi v^2}{2}} dv \quad ; \quad \Psi_1 = \frac{C\pi L}{k} = \frac{C\lambda L}{2}$$

This integral can be evaluated as the real and imaginary parts of the following integral w/ $z \in \mathbb{C}$

$$\int_0^s e^{\frac{i\pi z^2}{2}} dz = \int_0^s \cos\left(\frac{\pi z^2}{2}\right) dz + i \int_0^s \sin\left(\frac{\pi z^2}{2}\right) dz = C(s) + i\, S(s)$$

The 2 integrals are known as Fresnel integrals. Both, taken as coordinates in $\mathbb{C}$, form what's known as the Cornu Spiral

Evaluating with the aid of numerical methods & tables, we have that for our Σ we only evaluate a piece of the spiral, namely Δs

$$\Delta s = s_2 - s_1 = u_2 - u_1 = (x_2 - x_1)\sqrt{\frac{2}{\lambda L}} \; ; \; \Delta s = v_2 - v_1 = (y_2 - y_1)\sqrt{\frac{2}{\lambda L}}$$

In the case of an infinite aperture ($\Sigma = \mathbb{R}^2$), we set $u_1 = v_1 = -\infty$ $u_2 = v_2 = \infty$. Since

$$\begin{cases} \lim_{s\to\infty} C(s) = \lim_{s\to\infty} S(s) = \frac{1}{2} \\ \lim_{s\to-\infty} C(s) = \lim_{s\to-\infty} S(s) = -\frac{1}{2} \end{cases}$$

We obtain the value for an unperturbed wave

$$\psi_P = \psi_1 (1+i)^2 = 2i\psi_1$$

Said $\psi_0 = \psi_1 (1+i)^2$, we have that in the general case of Cornu spirals for which $s \in I \subseteq \mathbb{R}$,

$$\psi_P = \frac{\psi_0}{(1+i)^2} \Big(C(u) + iS(u) \Big)_{u_1}^{u_2} \Big(C(v) + iS(v) \Big)_{v_1}^{v_2}$$

Which is simply the evaluation of the previous integrals in $[u_1, u_2] \times [v_1, v_2]$. In normal cases we then only care about lower order Fresnel zones.

§§ SLIT & STRAIGHTEDGE

Fresnel diffraction from a long slit is treated as the limiting (1D) case of the rectangular aperture, namely

$$u \in \mathbb{R}, \quad v \in [v_1, v_2]$$

Thus:

$$\psi_P = \frac{\psi_0}{1+i} \Big(C(v) + iS(v) \Big)_{v_1}^{v_2}$$

The straightedge instead is the limiting case of the slit, where

$$u \in \mathbb{R}, \quad v \in (-\infty, v_2]$$

I.e.:

$$\psi_P = \frac{\psi_0}{1+i} \Big(C(v_2) + iS(v_2) - \underbrace{C(-\infty)}_{-1/2} - i\underbrace{S(-\infty)}_{-1/2} \Big)$$

$$\psi_P = \frac{\psi_0}{1+i} \Big(C(v_2) + iS(v_2) \Big) + \frac{1}{2} \frac{\psi_0}{\cancel{1+i}} \cancel{(1+i)}$$

$$\therefore \psi_P = \frac{\psi_0}{1+i} \Big(C(v_2) + iS(v_2) \Big) + \frac{1}{2} \psi_0$$

— 11/23/22 — 2h 4m 22.66s — NEW WEEKLY RECORD! —

§ FOURIER TRANSFORM & DIFFRACTION

Consider again the case of Fraunhofer diffraction {Far field}, it's possible to consider the most general case for diffraction, a generic aperture Σ with (also/either) generic transmission properties, including phase retardation

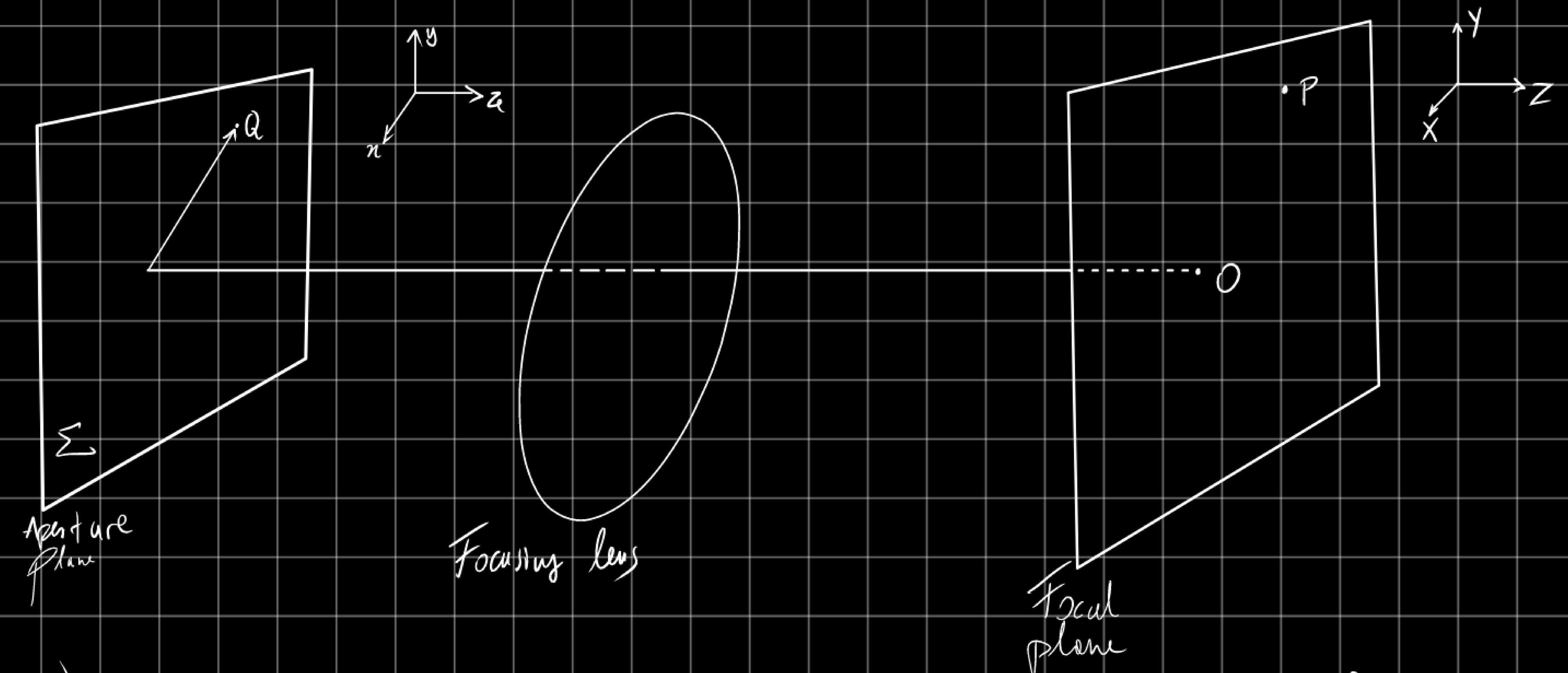


Fig. 1) scheme for evaluation of Fraunhofer diffraction in the general case

All rays leaving Σ in a generic direction, specified by the director cosines $\{\hat{\underline{r}} = (\alpha, \beta, \gamma)\}$ are brought to a common focus point (through the lens), which we'll call P.

by definition, also

$$P = (X, Y) \simeq (L\alpha, L\beta)$$

here we're assuming $\alpha, \beta \ll 1 \implies \alpha \simeq \tan\alpha$, $\beta \simeq \tan\beta$, also we assume $\gamma \simeq 1$.

The difference b/w 2 parallel rays starting from Q & O is $\underline{R} = (x, y)$, when we choose $\hat{\underline{n}} = \frac{\underline{R}}{\|R\|}$. Since, for what we said before $\hat{\underline{n}} = \hat{\underline{r}} = (\alpha, \beta, \gamma)$, we will have

$$\delta r = \underline{R} \cdot \hat{\underline{n}} = \alpha x + \beta y = x\frac{X}{L} + y\frac{Y}{L}$$

With X, Y, as in Fig (1) are the coordinates of P. It follows that the fundamental KF integral for diffraction is:

$$\psi(X, Y) = \iint_{\Sigma} e^{ik\delta r} d^2 s = \iint_{\Sigma} e^{ik(xX + yY)/L} \, dx\,dy$$

This for an uniform aperture Σ. Inserting the apertures we treated before, we'll get the same result.

For a NON-UNIFORM aperture, we introduce the << APERTURE FUNCTION >> $g(x, y)$, defined such that $g(x, y)\,dx\,dy$ is the amplitude of the diffracted wave on the element $d^2 s$. Thus,

$$\psi(X, Y) = \iint_{\Sigma} g(x, y)\, e^{ik\left(\frac{X}{L}x + \frac{Y}{L}y\right)} \, dx\,dy$$

Introducing the spatial frequencies μ, ν; defined as

$$\begin{cases} \mu = \dfrac{kX}{L} \\ \nu = \dfrac{kY}{L} \end{cases}$$

We can write the KF integral as a 2D Fourier transform

$$\psi(\mu, \nu) = \iint_{\Sigma} g(x, y)\, e^{i(\mu x + \nu y)} \, dx\,dy = \hat{\mathcal{F}}_2[g](\mu, \nu)$$

Therefore the diffraction pattern (Image) is the Fourier transform of the aperture function. ψ, g are FOURIER PAIRS

Consider now a grating 1D for simplicity. Then $g(y)$ is a periodic step function

$$g(y) = \sum_{n=0}^{\infty} g_n \cos(n\nu_0 y) \quad ; \quad \nu_0 = \frac{2\pi}{h} \; ; \; h \text{ Spacing}$$

The (1D) Fourier transform what we expect already for the grating.

The maximas of higher orders correspond to components w/ $n > 1$ of $g(y)$

§§ APODIZATION

Apodization (A-pod-ization / remove the feet): Process where the aperture function is modified in order to redistribute energy in the dif. pattern

It's employed for reducing the intensity of the 2ND-ary maxima of $\psi(x, y)$

Consider a single slit. Here:

$$g(y) = \mathbb{1}_{\left[-\frac{b}{2}, \frac{b}{2}\right]}(y)$$

Then, integrating the transform, or using $\hat{\mathcal{F}}[1](\nu) = \operatorname{sinc}(\nu) = \frac{\sin(\nu)}{\nu}$, we get

$$\psi(\nu) = b \operatorname{sinc}\left(\frac{1}{2}\nu b\right)$$

This is exactly equivalent to what we found for the single slit before {Fraunhofer}

Suppose that we now apodize $g(y)$, with

$$g_A(y) = \cos\left(\frac{\pi y}{b}\right) \quad , \; y \in \left[-\frac{b}{2}, \frac{b}{2}\right]$$

Then, integrating on the aperture, we get

$$\psi_A(\nu) = \int_{-\frac{b}{2}}^{\frac{b}{2}} \cos\left(\frac{\pi y}{b}\right) e^{i\nu y}\, dy = \frac{1}{2}\int_{-\frac{b}{2}}^{\frac{b}{2}} \left(e^{\frac{i\pi y}{b}} + e^{\frac{-i\pi y}{b}}\right) e^{i\nu y}\, dy = \frac{1}{2}\int_{-\frac{b}{2}}^{\frac{b}{2}} e^{iy\left(\nu + \frac{\pi}{b}\right)} + e^{iy\left(\nu - \frac{\pi}{b}\right)}\, dy = \frac{1}{2}\left[\frac{1}{i\left(\nu + \frac{\pi}{b}\right)} e^{iy\left(\nu + \frac{\pi}{b}\right)} + \frac{1}{i\left(\nu - \frac{\pi}{b}\right)} e^{iy\left(\nu - \frac{\pi}{b}\right)}\right]_{-\frac{b}{2}}^{\frac{b}{2}}$$

$$\psi(\nu) = \frac{1}{2i}\left[\frac{b}{b\nu - \pi} e^{i\frac{b}{2}\left(\nu - \frac{\pi}{b}\right)} + \frac{b}{b\nu - \pi} e^{-i\frac{b}{2}\left(\nu - \frac{\pi}{b}\right)} + \frac{b}{b\nu + \pi} e^{i\frac{b}{2}\left(\nu + \frac{\pi}{b}\right)} + \frac{b}{b\nu + \pi} e^{-i\frac{b}{2}\left(\nu + \frac{\pi}{b}\right)}\right] = \frac{b}{b\nu - \pi} \sin\left[\frac{1}{2}(\nu b - \pi)\right] + \frac{b}{b\nu + \pi} \sin\left[\frac{1}{2}(\nu b + \pi)\right]$$

Or:

$$\psi_A(\nu) = \cos\left(\frac{\nu b}{2}\right)\left[\frac{b}{b\nu - \pi} + \frac{b}{b\nu + \pi}\right]$$

The apodized $\psi_A(\nu)$ clearly respects the relation $\ulcorner \psi_A(\nu) \leq \psi(\nu) \lrcorner$ for higher frequencies. Apodization is typically used on telescope apertures, making sure that a dimmer object can be seen NEXT to a brighter one {see binary stars & exoplanets} [2022 big shit]

§§ SPATIAL FILTERING

Consider Fig. (1). The xy plane (Σ) the location of a coherently illuminated object, imaged by some optical system (lens) & imaged again on the focal plane. Said $\mu\nu$ the plane of the optical system, then here we have $\psi(\mu, \nu)$. $g(x, y)$ is simply our object.

The shadow casted on XY (call it x', y' for ease), which we will call $\tilde{g}(x, y)$ is again the Fourier transform of $\psi(\mu, \nu)$

$$\underset{}{S} \longrightarrow \underset{\text{object}}{g(x,y)} \xrightarrow{\hat{\mathcal{F}}_{(\mu,\nu)}} \underset{\text{lens (OS)}}{\psi(\mu,\nu)} \xrightarrow{\hat{\mathcal{F}}_{(x',y')}} \underset{\text{image}}{\tilde{g}(x', y')}$$

If ALL $(\mu, \nu) \in \mathbb{R}^2$ were transmitted EQUALLY by the optical system, then $\tilde{g}(x', y') \propto g(x, y)$, i.e. the image is a true & precise representation! This is clearly not possible if $\Sigma \neq \mathbb{R}^2$ {finite aperture}, thus some frequencies are limited. Other optical fenomena like aberrations, defects etc.. result in a modified $\psi(\mu, \nu)$. This mod can be incorporated by using a Transfer function, such that $\psi'(\mu, \nu) = T(\mu, \nu)\psi(\mu, \nu)$ & $g'(x', y') = \hat{\mathcal{F}}[\psi'](x', y')$

24/11/22 — 22:31 — 1h 37m 23.05s

Thus:

$$g'(x',y') = \iint_{\mathbb{R}^2} T(\mu,\nu)\,\psi(\mu,\nu)\,e^{-i(\mu x' + \nu y')}\,d\mu\,d\nu$$

So, the image function $g'(x',y')$ is the Fourier transform of $T(\mu,\nu)\psi(\mu,\nu)$. The limits of integration are $\mathbb{R}^2$ just formally, since they're actually determined by the transfer function $T(\mu,\nu)$. T can be modified by placing screens & apertures on the (μ,ν)-plane. The process is known as << spatial filtering >> ($\sim$ electrical filters w/ passive components, accepts certain freq. (μ,ν) & rejects others)

Suppose that the object is a grating $\Longrightarrow$ $\boxed{f(y) = \sum_{J\geq 0} \mathbb{1}_{[Jh,\,Jh+b]}(y)}$. If we $\hat{\mathcal{F}}[f](\nu) = \psi(\nu)$, we get that $\lim_{\nu\to\pm\infty}\psi(\nu) = 0$, therefore it can be thought as a << low pass filter >>. Now let $\Sigma \subseteq \pi_{(\mu,\nu)}$ transmit only freq. $\nu \in [-\nu_M, \nu_M]$. Since we chose $\nu = \frac{kY}{L}$

Thus $\nu = \frac{kb}{f}$, where $2b$ is the width of Σ in $\pi_{(\mu,\nu)}$. The transfer function $T(\nu) = \mathbb{1}_{[-\nu_M,\nu_M]}(\nu)$, thus:

$$g'(y') = \int_{\mathbb{R}} \psi(\nu)\,\mathbb{1}_{[-\nu_M,\nu_M]}(\nu)\,e^{-i\nu y'}\,d\nu = \int_{-\nu_M}^{\nu_M} \psi(\nu)\,e^{-i\nu y'}\,d\nu$$

A high pass optical filter can be obtained by obstructing the central part of the diffraction pattern $\psi(\nu)$. {with for s.a.: $T(\nu) = \mathbb{1}_{(-\infty,-\tilde{\nu}]}(\nu) + \mathbb{1}_{[\tilde{\nu},+\infty)}(\nu)$ }

Phase gratings

A method known as << phase contrast >> was invented by the Dutch physicist Zernike $\Longrightarrow$ making transparent obj w/ $n_{obj} \not\approx n_{amb}$ inside a transparent medium (amb).

The simplest method is given by the usage of << phase gratings >> i.e. alternate H-L index material (perfectly transparent). The object function $g(y)$ is then:

$$g(y) = e^{i\phi(y)}$$

Where $\phi(y)$ is a periodic step function, with height $\Delta\phi = k z_e \Delta n$ $\begin{cases} z_e \to \text{thickness} \\ \Delta n \to n_H - n_L \end{cases}$

If $\Delta\phi \ll 1$, we can write

$$g(y) \simeq 1 + i\phi(y)$$

Thus

$$\psi(\nu) = \int_{\mathbb{R}} (1 + i\phi(y))\,e^{i\nu y}\,dy = \int_{-b/2}^{b/2} e^{i\nu y}\,dy + i\int_{-b/2}^{b/2} \phi(y)\,e^{i\nu y}\,dy = \mathrm{Re}\{\psi(\nu)\} + i\,\mathrm{Im}\{\psi(\nu)\} \quad \begin{cases} \mathrm{Re}\{\psi\} = \psi_1(\nu) \\ \mathrm{Im}\{\psi\} = \psi_2(\nu) \end{cases}$$

$\psi_1(\nu)$ is the WHOLE OBJECT APERTURE DIFFRACTION PATTERN ($\psi_1(\nu) \approx 0\ \forall \nu \neq 0$). $\psi_2(\nu)$ is the $\phi(y)$ - DIFFRACTION PATTERN. By definition, also ψ_1, ψ_2 are dephased by $\frac{\pi}{2}$ degrees $\{ i = e^{i\frac{\pi}{2}} \}$. In the phase contrast method a phase plate is introduced, shifting ψ_2 by an additional $\frac{\pi}{2}$ degrees

A PHASE PLATE is a transparent glass plate whose $z_{e_{opt}} = z_{e_{plate}} + \frac{1}{4}\lambda$. The thicker section is on $\pi_{(\mu,\nu)}$. The phase plate applies the transformation $\psi(\nu) = \psi_1(\nu) + i\psi_2(\nu) \longrightarrow \hat{P}_{\frac{\pi}{2}}[\psi] = \psi_1(\nu) + \psi_2(\nu)$. The image function is then:

$$g'(y') = \hat{\mathcal{F}}[\psi_1(\nu)](y') + \hat{\mathcal{F}}[\psi_2(\nu)](y') = g_1'(y') + g_2'(y')$$

Phase contrast works like a phase-modulated signal getting transformed to an amplitude-modulated signal via a $\frac{\pi}{2}$ phase transformation.

Optics of Solids

§ Macroscopic Fields & Maxwell's Equations

The EM field @ any given point is described by 4 quantities:

1) $\frac{dQ}{d^3x} = \rho$ {Volumetric density of charge}

2) $\frac{dp}{d^3x} = \underline{P}$ {Polarization}

3) $\frac{dm}{d^3x} = \underline{M}$ {Magnetization}

4) $\frac{d}{d^3x}\left(\underline{J} + \frac{1}{c^2}\frac{\partial \varepsilon}{\partial t}\right) = \underline{J}$ {current density}

ALL of these quantities are averaged in order to smooth the micro-variations due to the atomic (discrete) composition of matter

Considering the generic Maxwell's equations, considering magnetization currents $\underline{J}_M$ & Polarization charges ρ_m, we have:

$$\begin{cases} \nabla\cdot\underline{E} = \frac{1}{\epsilon_0}(\rho + \rho_P) \\ \nabla\times\underline{E} = -\frac{\partial B}{\partial t} \\ \nabla\cdot\underline{B} = 0 \\ \nabla\times\underline{B} = \mu_0(\underline{J} + \underline{J}_M) + \epsilon_0\frac{\partial E}{\partial t} \end{cases} \qquad \begin{cases} \rho_P = -\nabla\cdot\underline{P} \\ \underline{J}_M = \nabla\times\underline{M} \end{cases}$$

Inserting the relations b/w Maxwell's equations & $(\rho_P, \underline{P})$; $(\underline{J}_M, \underline{M})$ we have:

$$(a)\begin{cases} \nabla\cdot(\epsilon_0 E + P) = \rho \\ \nabla\times E = -\frac{\partial B}{\partial t} \\ \nabla\cdot B = 0 \\ \nabla\times\left(\frac{\underline{B}}{\mu_0} - \underline{M}\right) = \underline{J} + \frac{\partial}{\partial t}(\epsilon_0\underline{E} + \underline{P}) \end{cases} \qquad \begin{cases} \underline{D} = \epsilon_0\underline{E} + \underline{P} \\ \underline{B} = \mu_0\underline{H} + \underline{M} \end{cases}$$

Inserting the relations (b) → (a), we get

$$\begin{cases} \nabla\cdot\underline{D} = \rho \\ \nabla\times\underline{E} = -\mu_0\frac{\partial H}{\partial t} - \mu_0\frac{\partial M}{\partial t} \\ \nabla\cdot\underline{B} = 0 \\ \nabla\times\underline{H} = \underline{J} + \frac{\partial D}{\partial t} + \epsilon_0\frac{\partial P}{\partial t} \end{cases} \qquad \begin{cases} \nabla\cdot\underline{D} = \rho \\ \nabla\times\underline{D} = \nabla\times\underline{P} - \frac{1}{c^2}\frac{\partial H}{\partial t} - \frac{1}{c^2}\frac{\partial M}{\partial t} \\ \nabla\cdot\underline{H} = -\nabla\cdot\underline{M} \\ \nabla\times\underline{H} = \underline{J} + \frac{\partial D}{\partial t} + \epsilon_0\frac{\partial P}{\partial t} \end{cases}$$

We note also OHM'S LAW, where $\underline{J} = \sigma\underline{E}$. Said $\underline{D} = \epsilon\underline{E}$, where $\epsilon = \epsilon_0(1 + \underline{\underline{\chi_e}})$ & $\mu\underline{H} = \underline{B}$, $\mu = \mu_0(1 + \underline{\underline{\chi_m}})$, where χ_e, χ_m are the electric & magnetic susceptibilities of the medium. Generally they're rank 2 tensors

For isotropic media, χ_e, χ_m are diagonal and can be considered scalar values

WAVE EQUATION IN SOLIDS

Going back to Maxwell's equations in media, we can write a wave equation as we did for the empty space. Considering the coupled $(\underline{E}, \underline{H})$ equations, we have

$$\begin{cases} \nabla \cdot \underline{E} = -\frac{1}{\epsilon_0} \nabla \cdot \underline{P} + \frac{\rho}{\epsilon_0} \\ \nabla \times \underline{E} = -\mu_0 \left(\frac{\partial H}{\partial t} + \frac{\partial M}{\partial t} \right) \\ \nabla \cdot \underline{H} = -\nabla \cdot \underline{M} \\ \nabla \times \underline{H} = \underline{J} + \epsilon_0 \frac{\partial E}{\partial t} + \frac{\partial P}{\partial t} \end{cases} \qquad (MW\ G)$$

Consider now the case of electrically neutral media, where $\rho = 0$, $\underline{M} = 0$, then:

$$\begin{cases} \nabla \cdot \underline{E} = -\frac{1}{\epsilon_0} \nabla \cdot \underline{P} & (1) \\ \nabla \times \underline{E} = -\mu_0 \frac{\partial H}{\partial t} & (2) \\ \nabla \cdot \underline{H} = 0 & (3) \\ \nabla \times H = \underline{J} + \epsilon_0 \frac{\partial E}{\partial t} + \frac{\partial P}{\partial t} & (4) \end{cases} \qquad (MW\ EN)$$

Using the operatorial relation $\hat{\nabla} \times \hat{\nabla} \times [\ ] = \hat{\nabla}(\hat{\nabla} \cdot [\ ]) - \hat{\nabla}^2 [\ ]$ we can use 2 to get:

$$\nabla \times \nabla \times \underline{E} = -\mu_0 \nabla \times \frac{\partial H}{\partial t} = -\mu_0 \frac{\partial}{\partial t} \left(\underline{J} + \epsilon_0 \frac{\partial E}{\partial t} + \frac{\partial P}{\partial t} \right) = -\mu_0 \frac{\partial J}{\partial t} - \frac{1}{c^2} \frac{\partial^2 E}{\partial t^2} - \mu_0 \frac{\partial^2 P}{\partial t^2}$$

$$\boxed{\nabla \times \nabla \times \underline{E} + \frac{1}{c^2} \frac{\partial E}{\partial t^2} = -\mu_0 \frac{\partial J}{\partial t} - \mu_0 \frac{\partial^2 P}{\partial t^2}} \qquad (EMWE)$$

Or, using $\Box \underline{E} = \frac{1}{c^2} \frac{\partial^2 E}{\partial t^2} - \nabla^2 \underline{E}$ (d'Alembertian), we can write the eqn in an explicitly covariant manner:

$$\nabla(\nabla \cdot \underline{E}) + \Box \underline{E} = -\mu_0 \frac{\partial J}{\partial t} - \mu_0 \frac{\partial^2 P}{\partial t^2} \qquad \star$$

Also, further expanding, using (1), we can also compactify the equations in terms of $\underline{D}$

$$-\frac{1}{\epsilon_0} \nabla(\nabla \cdot \underline{P}) + \Box \underline{E} = -\mu_0 \frac{\partial J}{\partial t} - \mu_0 \frac{\partial^2 P}{\partial t^2}$$

Using $\mu_0 = (c^2 \epsilon_0)^{-1}$

$$-\frac{1}{\epsilon_0} \left(\nabla \times \nabla \times \underline{P} + \nabla^2 \underline{P} - \frac{1}{c^2} \frac{\partial^2 P}{\partial t^2} \right) + \Box \underline{E} = -\mu_0 \frac{\partial J}{\partial t}$$

$$\frac{1}{\epsilon_0} \Box \underline{P} - \frac{1}{\epsilon_0} \nabla \times \nabla \times \underline{P} + \Box \underline{E} = -\mu_0 \frac{\partial J}{\partial t}$$

$$\Box \underline{D} - \nabla \times \nabla \times \underline{P} = -\frac{1}{c^2} \frac{\partial J}{\partial t}$$

We will use the equation (star) for our evaluations. Thus, our wave equation is

$$\nabla \times \nabla \times \underline{E} - \frac{1}{c^2} \frac{\partial E}{\partial t} = -\mu_0 \frac{\partial J}{\partial t} - \mu_0 \frac{\partial^2 P}{\partial t^2}$$

The RHS terms are known as SOURCE TERMS for the wave, & in our case they explicitly indicate oscillations in the polarization field + current density changes. We consider 2 new special cases

1) DIELECTRIC MEDIA

2) CONDUCTING MEDIA

❡❡ NONCONDUCTING MEDIA / WAVES IN DIELECTRICS

For nonconducting media we have $\underline{J}=0$, thus our wave equation reduces to:

$$\nabla\times\nabla\times\underline{E} + \frac{\partial^2 \underline{E}}{\partial t^2} = -\mu_0 \frac{\partial^2 \underline{P}}{\partial t^2}$$

Here e^- are PERMANENTLY BOUND to the atom, and the only force exerted is the elastic rebound force of the e^- in the atomic field. Thus, for a lattice of N e^- w/ charge $q_e = -e$, we have

$$\begin{cases} \underline{P_s} = Q_T \underline{r} = -Ne\underline{r} \\ \underline{F} = q\underline{E} = -e\underline{E} = K\underline{r} \end{cases}$$

Where K is the elastic force constant & $\underline{P_s}$ is the STATIC polarization

$$\begin{cases} \underline{r} = -\dfrac{e\underline{E}}{K} \\ \underline{P_s} = -Ne\underline{r} \end{cases} \Rightarrow \boxed{\underline{P_s} = -\frac{Ne^2}{K}\underline{E}}$$

Note if $\underline{E} \equiv \underline{E}(\underline{r},t)$ the above equation is NOT correct since we're not taking the e^- motion into account, since now $\underline{r} \equiv \underline{r}(t)$. This motion can be described as a bound harmonic oscillator w/ damping, then Newton's second law gives:

$$\underline{F} = m\frac{d^2\underline{r}}{dt^2} + \underbrace{m\gamma\frac{d\underline{r}}{dt}}_{\text{damping}} + K\underline{r} = -e\underline{E}$$

(Note that we neglected the magnetic force, since $F_m << F_e$ in our case.)
We now also suppose that $\underline{E}$ is harmonic (wave), i.e.

$$\underline{E}(\underline{r},t) = \underline{E}(\underline{r})e^{-i\omega t}$$

Assuming that e^- has harmonic motion w/ $\underline{r}(t) = \underline{r}e^{-i\omega t}$

$$\frac{d\underline{r}}{dt} = -i\omega\underline{r}e^{-i\omega t} \qquad \frac{d^2\underline{r}}{dt^2} = -\omega^2\underline{r}e^{-i\omega t}$$

Thus:

$$+m\omega^2\underline{r}\cancel{e^{-i\omega t}} + im\omega\gamma\underline{r}\cancel{e^{-i\omega t}} - K\underline{r}\cancel{e^{-i\omega t}} = +e\underline{E}\cancel{e^{-i\omega t}}$$

$$\begin{cases} \underline{r} = \dfrac{-e}{K - m\omega^2 - im\omega\gamma}\underline{E} \\ \underline{P} = -Ne\underline{r} \end{cases}$$

I.e., solving the system for $\underline{P}$

$$\underline{P} = \frac{Ne^2}{k - m\omega^2 - im\gamma\omega}\underline{E}$$

Using $\omega_0 = \sqrt{\frac{K}{m}}$, we can rewrite in a more significant way the previous equation

$$\underline{P} = \frac{Ne^2}{m}\left(\frac{1}{\omega_0^2 - \omega^2 - i\gamma\omega}\right)\underline{E}$$

The frequency ω_0 is the «EFFECTIVE RESONANCE FREQUENCY», which depends ONLY on the medium analyzed (K varies with materials)

The result obtained is really similar to one given by a driven HO (DHO) as it should. We therefore expect optical resonance around the "natural frequency" ω_0.

Returning to the general equation, we have

$$\begin{cases} \nabla\times\nabla\times\underline{E} + \frac{1}{c^2}\frac{\partial^2\underline{E}}{\partial t^2} = -\mu_0\frac{\partial^2 \underline{P}}{\partial t^2} \\ \underline{P}(t) = \frac{Ne^2}{m}\left(\frac{1}{\omega_0^2 - \omega^2 - i\gamma\omega}\right)\underline{E} \end{cases}$$

Thus:

$$\nabla\times\nabla\times\underline{E} + \frac{1}{c^2}\frac{\partial^2\underline{E}}{\partial t^2} + \frac{Ne^2}{c^2\epsilon_0 m}\left(\frac{1}{\omega_0^2 - \omega^2 - i\gamma\omega}\right)\frac{\partial^2\underline{E}}{\partial t^2} = 0$$

Also, since in our case $\underline{P} \propto \underline{E}$, $\nabla\cdot\underline{E} = 0$, and $\nabla\times\nabla\times E = -\nabla^2\underline{E}$, giving

$$\nabla^2\underline{E} = \left[1 + \frac{Ne^2}{m\epsilon_0}\left(\omega_0^2 - \omega^2 - i\gamma\omega\right)^{-1}\right]\frac{1}{c^2}\frac{\partial^2\underline{E}}{\partial t^2}$$

If we search for a homogeneous plane wave solution (HPWs) of the form:

$$\underline{E} = \underline{E}_0 e^{-i(\mathcal{k}z - \omega t)} \quad \mathcal{k} \in \mathbb{C}$$

we have

$$\nabla^2\underline{E} = -\mathcal{k}^2\underline{E}, \quad \frac{\partial^2\underline{E}}{\partial t^2} = -\omega^2\underline{E}$$

Thus it's satisfied IFF

$$\mathcal{k}^2 = \frac{\omega^2}{c^2}\left[1 + \frac{Ne^2}{m\epsilon_0}(\omega_0 - \omega - i\gamma\omega)^{-1}\right]$$

We can rewrite the RHS remembering that for $z \in \mathbb{C}$, $z^{-1} = \frac{\overline{z}}{\|z\|^2}$, i.e.

$$(\omega_0 - \omega - i\gamma\omega)^{-1} = \left[(\omega_0 - \omega)^2 + \gamma^2\omega^2\right]^{-1}(\omega_0 - \omega + i\gamma\omega)$$

Giving, then:

$$\mathcal{k}^2 = \frac{\omega^2}{c^2} + \frac{Ne^2\omega^2}{m\epsilon_0 c^2}\frac{\omega_0 - \omega + i\gamma\omega}{(\omega_0 - \omega)^2 + \gamma^2\omega^2}$$

The fact that $\tilde{k} \in \mathbb{C}$ indicates that we can describe it as a real part + i · imaginary part, where

$$\left[\tilde{k} = k + i\alpha\right] \quad (k \text{ def})$$

with

$$\tilde{k} = \frac{\omega}{c}\mathcal{N}, \quad \mathcal{N} \in \mathbb{C}$$

our value $\mathcal{N}$ is a COMPLEX REFRACTION INDEX. The solution, with $\tilde{k}$ as in (k def) becomes

$$E = E_0 e^{i(\tilde{k}z - \omega t)} = E_0 e^{-\alpha z} e^{i(kz - \omega t)}$$

The exponential damping of E indicates that there's absorption in the medium to be precise, we have that, inside the medium

$$I \propto e^{-2\alpha z} = e^{-az}$$

where $a = 2\alpha$ is the absorption coefficient. The refraction index, as we said is $\mathcal{N} \propto \tilde{k} \in \mathbb{C}$, where

$$\left[\mathcal{N}^2 = \frac{c^2}{\omega^2}\tilde{k} = 1 + \frac{Ne^2}{m\epsilon_0}\frac{\omega_0 - \omega + i\gamma\omega}{(\omega_0-\omega)^2 + \gamma^2\omega^2}\right]$$

As before, $\mathcal{N} = n + i\eta$, where

$$\begin{cases} \mathcal{N} = \frac{c}{\omega}\tilde{k} = \frac{c}{\omega}(k + i\alpha) \\ \mathcal{N} = n + i\eta \end{cases}$$

I.e., considering separately real & imaginary parts

$$\begin{cases} \mathrm{Re}\{\mathcal{N}\} = \frac{c}{\omega}k = n \\ \mathrm{Im}\{\mathcal{N}\} = \frac{c}{\omega}\alpha = \eta \end{cases}$$

Note that:

1) From $\mathrm{Re}\{\mathcal{N}\}$ we get the usual relation:

$$k = \frac{\omega}{c}n$$

2) From $\mathrm{Im}\{\mathcal{N}\}$ we get all absorption phenomena

$$\alpha = \frac{\omega}{c}\eta$$

Looking again @ the solution, we have a harmonic wave with PHASE VELOCITY

$$u = \frac{\omega}{k} = \frac{c}{n}$$

& from $\mathcal{N}^2$ we also find

$$\mathcal{N}^2 = n^2 - \eta^2 + i n \eta = 1 + \frac{Ne^2}{m\epsilon_0} \frac{\omega_0 - \omega + i\gamma\omega}{(\omega_0-\omega)^2 + \gamma^2\omega^2}$$

Note that:

$$\frac{\omega_0 - \omega + i\gamma\omega}{(\omega_0-\omega)^2+\gamma^2\omega^2} = \frac{\omega_0-\omega}{(\omega_0-\omega)^2+\gamma^2\omega^2} + i\frac{\gamma\omega}{(\omega_0-\omega)^2+\gamma^2\omega^2}$$

Therefore

$$\begin{cases} \mathrm{Re}\{\mathcal{N}^2\} = n^2 - \eta^2 = 1 + \frac{Ne^2}{m\epsilon_0}\frac{\omega_0-\omega}{(\omega_0-\omega)^2+\gamma^2\omega^2} \\ \mathrm{Im}\{\mathcal{N}^2\} = n\eta = \frac{\gamma\omega}{(\omega_0-\omega)^2+\gamma^2\omega^2}\frac{Ne^2}{m\epsilon_0} \end{cases}$$

From the previous equation it's possible to determine n, η. In general, their dependency on frequency is of the following kind:

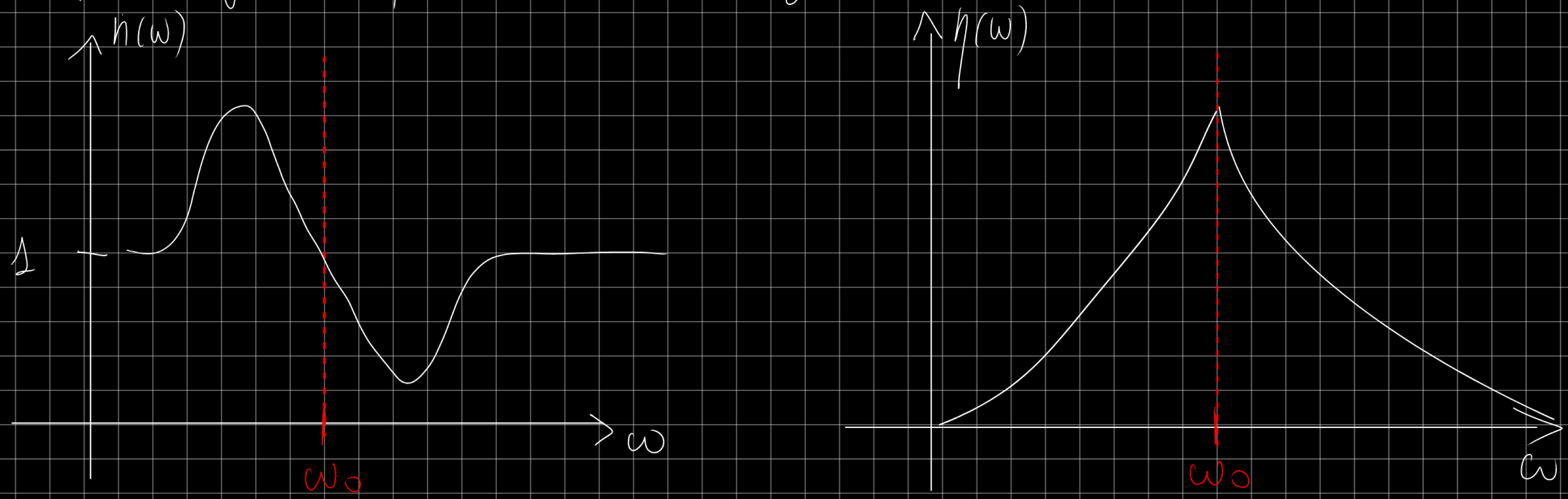


Fig 1.) Generic functional relations for $\mathcal{N}(\omega)$.

From fig. (1) it's clear that the absorption is MAX when $\omega \simeq \omega_0$, & $n > 1$ for frequencies $\omega < \omega_0$. The resonance frequencies of optical transparent media are usually all in the UV region, therefore the absorption peak is not observed in VIS, & also we have $n>1$, giving our usual behaviour, known as << NORMAL DISPERSION >>. After the peak for $\omega \gtrsim \omega_0$, we enter the zone of << ANOMALOUS DISPERSION >>, where instead of having n grow with ω, it decreases with increasing frequency ($n<1$!)

The previous remarks are valid only if e^- are bound equally to their respective atom

27/11/22 - 20.06 —— 4h 0m 1,8 s

This is not true in general, in fact, we may assume that each fraction f_j of electrons has resonant frequency ω_j, thus

$$\mathcal{N}^2 = 1 + \frac{Ne^2}{m\epsilon_0}\sum_{j\le N}\frac{f_j}{\omega_j^2-\omega^2-i\gamma_j\omega}$$

The fractions f_j are known as «OSCILLATOR STRENGTHS» {QM}

Note that:

$$\begin{cases}\lim_{\omega\to 0}\mathcal{N}^2(\omega) = 1 + \frac{Ne^2}{m\epsilon_0}\sum_{j\le N}\frac{f_j}{\omega_j} = 1+\chi_e\\ \lim_{\omega\to\infty}\mathcal{N}^2(\omega) = \infty\end{cases}$$

While, if $\gamma_j \ll 1\ \forall j\le N$, then $\mathcal{N}^2\in\mathbb{R}$, &

$$n^2 = 1 + \frac{Ne^2}{m\epsilon_0}\sum_{j\le N}\frac{f_j}{\omega_j^2-\omega^2}$$

Expressing the previous formula in terms of λ, we have what's known as SELLMEIER'S FORMULA (experimentally found by data fit at first)

§§ PROPAGATION IN CONDUCTING MEDIA

As before, the equation analyzed is

$$\nabla\times\nabla\times\underline{E} + \frac{1}{c^2}\frac{\partial^2\underline{E}}{\partial t^2} = -\mu_0\frac{\partial^2\underline{P}}{\partial t^2} - \mu_0\frac{\partial\underline{J}}{\partial t}$$

In the case of conducting media, we can consider e^- unbound from the atoms, thus $\underline{P}\approx 0$. Due to this we can consider the force equation in terms only of $m\cdot\underline{a}$ plus a dissipation term.

Remembering that $\underline{j} = -e\underline{v}$ & $\underline{J} = -Ne\underline{v}$, we have the following coupled equations for $\underline{v}$

$$\begin{cases} m\frac{d\underline{v}}{dt} - \frac{m}{\tau}\underline{v} = -e\underline{E}\\ \underline{J} = -Ne\underline{v}\end{cases} \Longrightarrow \frac{m}{\tau Ne}\underline{J} - \frac{m}{Ne}\frac{d\underline{J}}{dt} = -e\underline{E}$$

$$\underline{v} = -\frac{1}{Ne}\underline{J}$$

From the previous equations, we have

$$\frac{d\underline{J}}{dt} + \tau^{-1}\underline{J} = \frac{Ne^2}{m}\underline{E} \qquad \text{(A)}$$

This is a 1st order nonlinear ODE, thus we find the solution the usual way

Firstly we find the <<TRANSIENT CURRENT>> by solving for the homogeneous ODE

$$\frac{d\underline{J}}{dt} - \tau^{-1}\underline{J} = 0 \Rightarrow \underline{J}(t) = \underline{J}_0 e^{-t/\tau}$$

The constant τ is known as <<RELAXATION TIME>> of the medium.

We consider another two cases tied to $\underline{E}$

1) STATIC FIELD

2) HARMONIC FIELD

(1) The field is static, thus (A) becomes

$$\tau^{-1}\underline{J} = \frac{Ne^2}{m}\underline{E}$$

From OHM'S LAW $\underline{J} = \sigma\underline{E}$, thus

$$\tau^{-1}\sigma\underline{E} = \frac{Ne^2}{m}\underline{E} \Rightarrow \sigma = \frac{Ne^2\tau}{m}$$

σ, as usual, is the conductivity of the medium.

(2) The field is harmonic $\Rightarrow E \propto e^{-i\omega t}$, thus, remembering <u>OHM</u>

$$(\tau^{-1} - i\omega)\underline{J} = \frac{Ne^2}{m}\underline{E} = \frac{\sigma}{\tau}\underline{E}$$

Therefore

$$\underline{J} = \frac{\sigma}{\tau(\tau^{-1} - i\omega)}\underline{E} = \frac{\sigma}{1 - i\omega\tau}\underline{E}$$

Note how now Ohm's law depends explicitly on the frequency, and especially

$$\lim_{\omega \to 0} \underline{J} = \sigma\underline{E}$$

28/11/22 – 22.21 – 1h. 21m 21.95s

For the complete treatment we get back the dynamic equation, where

$$\nabla\times\nabla\times E + \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} = -\mu_0\frac{\partial^2 P}{\partial t^2} - \mu_0\frac{\partial J}{\partial t}$$

Since $\underline{P} = 0$ in this case, & therefore $\nabla\cdot P = -\varepsilon_0\nabla\cdot\underline{E} = 0$, we have

$$\Box E = -\mu_0\frac{\partial J}{\partial t}$$

Since $\Box \underline{E} = \left(-\frac{\omega^2}{c^2} - k^2\right) \underline{E}$, as before, & as we've seen

$\underline{J} = \frac{\sigma}{1-i\omega\tau} \underline{E}$, the equation becomes

$$\Box \underline{E} = -\frac{\mu_0 \sigma}{1-i\omega\tau} \frac{\partial E}{\partial t} \Rightarrow \left(k^2 - \frac{\omega^2}{c^2}\right) \underline{E} = i\omega\mu_0\sigma \frac{1}{1-i\omega\tau} \underline{E}$$

Therefore

$$k^2 \underline{E} = \left(\frac{\omega^2}{c^2} + \frac{i\omega\mu_0\sigma}{1+i\omega\tau}\right) \underline{E}$$

$$k^2 = \frac{\omega^2}{c^2} + \frac{i\omega\mu_0\sigma}{1+i\omega\tau}$$

Note how in the regime of low frequencies ($\underline{E} = \frac{1}{\sigma} \underline{J}$), we have

$$k^2 \approx i\omega\mu_0\sigma$$

Which gives

$$k_{LF} = \sqrt{i\omega\mu_0\sigma} = \sqrt{\frac{\omega\mu_0\sigma}{2}}(1+i)$$

In this case, $k_{LF} = K + i\alpha$ & $K = \alpha$, where $k = K + i\alpha$

$$K = \alpha = \sqrt{\frac{\omega\mu_0\sigma}{2}}$$

Similarly, said $k = \frac{\omega}{c}\mathcal{N}$ we have $\mathcal{N}_{LF} = \frac{c}{\omega}\sqrt{\frac{\omega\mu_0\sigma}{2}}(1+i) = \sqrt{\frac{\sigma}{2\omega\varepsilon_0}}(1+i)$, therefore, also for $\mathcal{N}_{LF} = n + i\eta$

$$n = \eta = \sqrt{\frac{\sigma}{2\omega\varepsilon_0}}$$

The max depth of penetration of the wave is known as « SKIN DEPTH » of the material, determined as the e-folding value of the amplitude, thus:

$$\delta = \frac{1}{\alpha} = \sqrt{\frac{2}{\omega\mu_0\sigma}}$$

Using $\lambda_0\omega = 2\pi c \Rightarrow \lambda_0 = \frac{2\pi c}{\omega}$, we can rewrite the skin-depth in terms of VACUUM WAVELENGTH

$$\boxed{\delta = \sqrt{\frac{\lambda_0}{\pi c\mu_0\sigma}}}$$

This formula clearly explains why good conductors (big σ) are HIGHLY OPAQUE

Returning to the general definition for k, we have

$$k^2 = \frac{\omega^2}{c^2} + i\omega\mu_0\sigma \frac{1}{1+i\omega\tau}$$

Thus, the complex index of refraction

$$\mathcal{N}^2 = \frac{c^2}{\omega^2}k^2 = 1 + \frac{ic^2\mu_0\sigma}{\omega}\frac{1}{1+i\omega\tau}$$

Defined the «PLASMA FREQUENCY» ω_p as

$$\omega_p = \underline{\sqrt{\frac{\mu_0\sigma c^2}{\tau}}} = \sqrt{\frac{\mu_0 c^2}{\tau}}\sqrt{\frac{Ne^2\tau}{m}} = \sqrt{\frac{Ne^2c^2\mu_0}{m}} = \underline{\sqrt{\frac{Ne^2}{m\epsilon_0}}}$$

We can rewrite

$$\mathcal{N}^2 = 1 + \frac{1}{\omega}\frac{i\omega_p^2}{\tau^{-1}+i\omega} = 1 - \frac{\omega_p^2}{\omega^2 + i\omega\tau^{-1}}$$

Rationalizing on the right, we have

$$\mathcal{N}^2 = 1 - \omega_p^2 \frac{1}{\omega^4+\omega^2\tau^{-2}}(\omega^2 - i\omega\tau^{-1})$$

$$\mathcal{N}^2 = 1 - \frac{\omega_p^2}{\omega^2+\tau^{-2}} + \frac{i\omega_p^2}{\omega^2+\tau^{-2}}\frac{1}{\omega\tau}$$

Thus:

$$\begin{cases} \mathrm{Re}\{\mathcal{N}^2\} = n^2 - \eta^2 = 1 - \dfrac{\omega_p^2}{\omega^2+\tau^{-2}} \\ \mathrm{Im}\{\mathcal{N}^2\} = 2n\eta = \dfrac{\omega_p^2}{\omega^2+\tau^{-2}}\left(\dfrac{1}{\omega\tau}\right) \end{cases}$$

The optical parameters n, η can be then obtained (numerically). For what we wrote, we have that both the parameters depend on ω_p, τ & ω.

For <u>METALS</u> we have $\tau \propto 10^{-13}\,s$, which corresponds to IR frequencies; $\omega_p \propto 10^{15}\,s^{-1}$ corresponding to VIS & NUV.

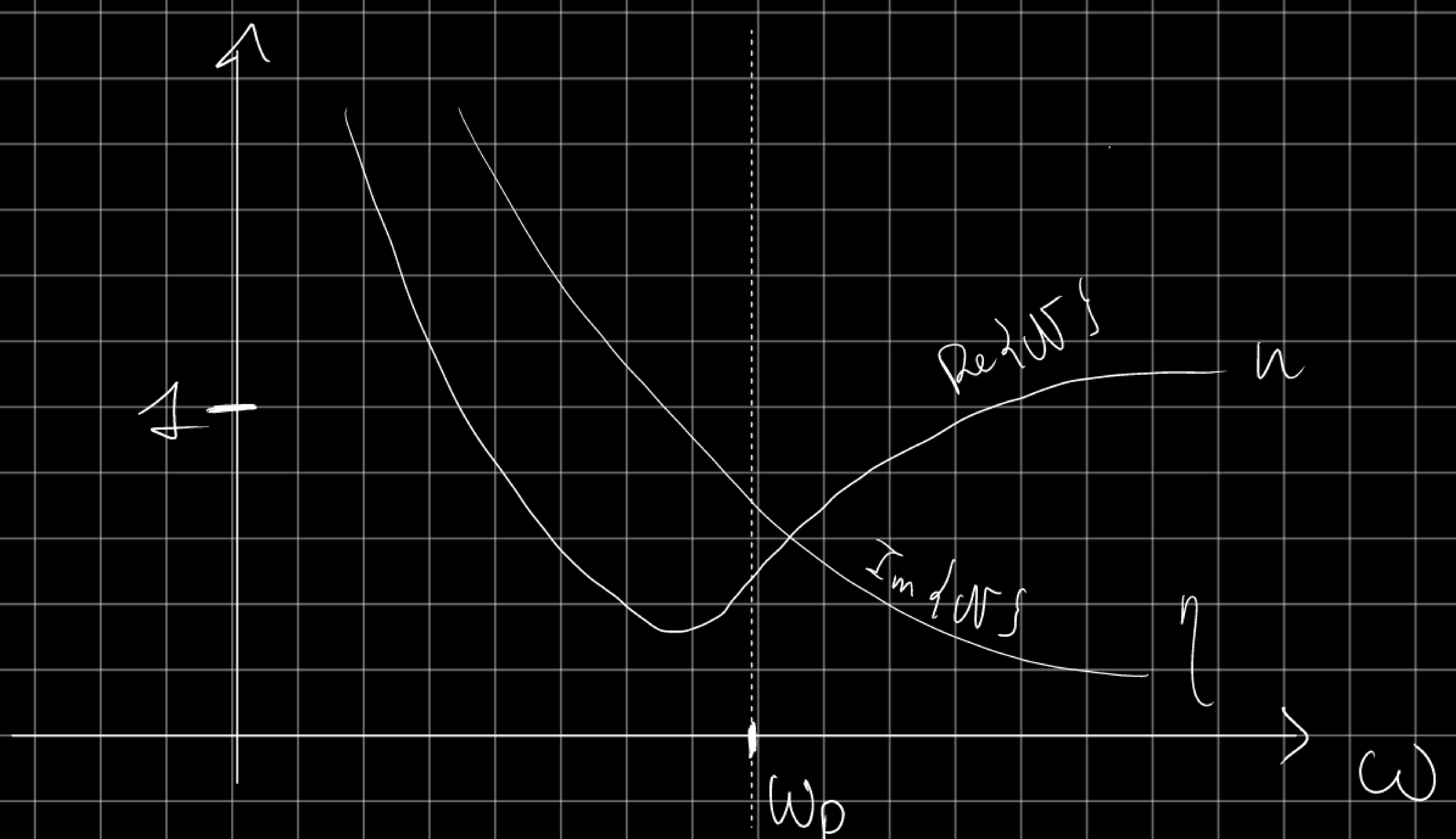


Fig 1) $n(\omega)$, $\eta(\omega)$. See how $n(\omega) < 1$ for many ω around ω_p

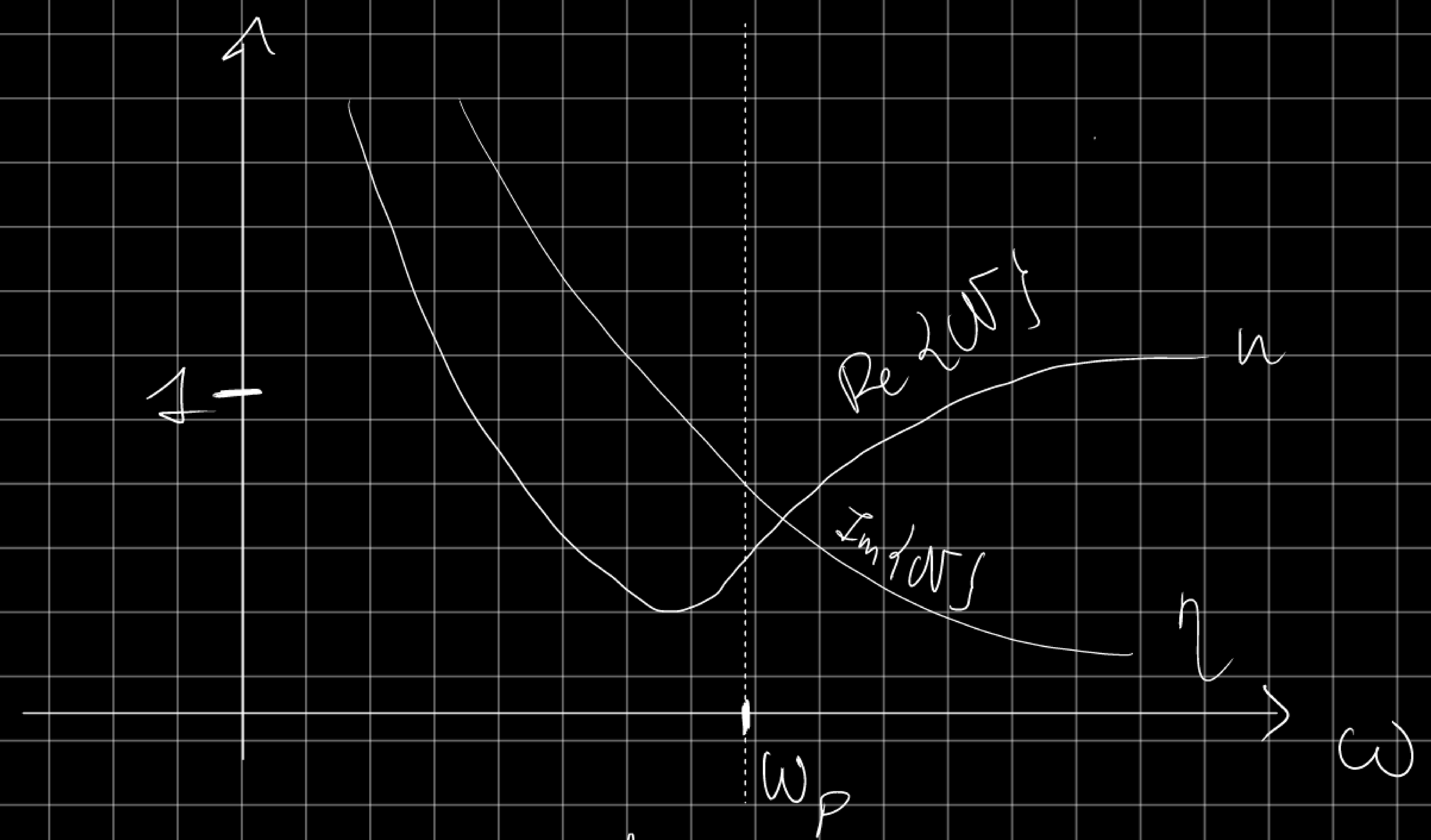


Fig 2.1c) Complex ref. index vs ω

As from the graph, the extinction coefficient η follows

$$\lim_{\omega \to \infty} \eta(\omega) = 0$$

Thus metals, become TRANSPARENT at high frequencies (long wavelengths)
For poor conductors & semiconductors we have

$$\mathcal{N}_{SC}^2 = \mathcal{N}_M^2 + \mathcal{N}_D^2 = 1 - \frac{\omega_p^2}{\omega^2 + i\omega\tau^{-1}} + \frac{Ne^2}{m\epsilon_0} \sum_{j \geq 1} \left(\frac{f_j}{\omega_j^2 - \omega^2 - i\gamma_j \omega} \right)$$

QM gives a similar relation & can predict the value of the parameters

§ REFLECTION & REFRACTION @ BOUNDARY OF ABS. MEDIA

Let ψ be a wave, incident at the boundary of a medium, with properties

$$\begin{cases} \mathcal{N} = n + i\eta \\ \underline{k} = \underline{K} + i\underline{\alpha} \end{cases}$$

Consider the first medium NONABSORBING ($\underline{\alpha}, \eta = 0$) w/ index n_1, as follows

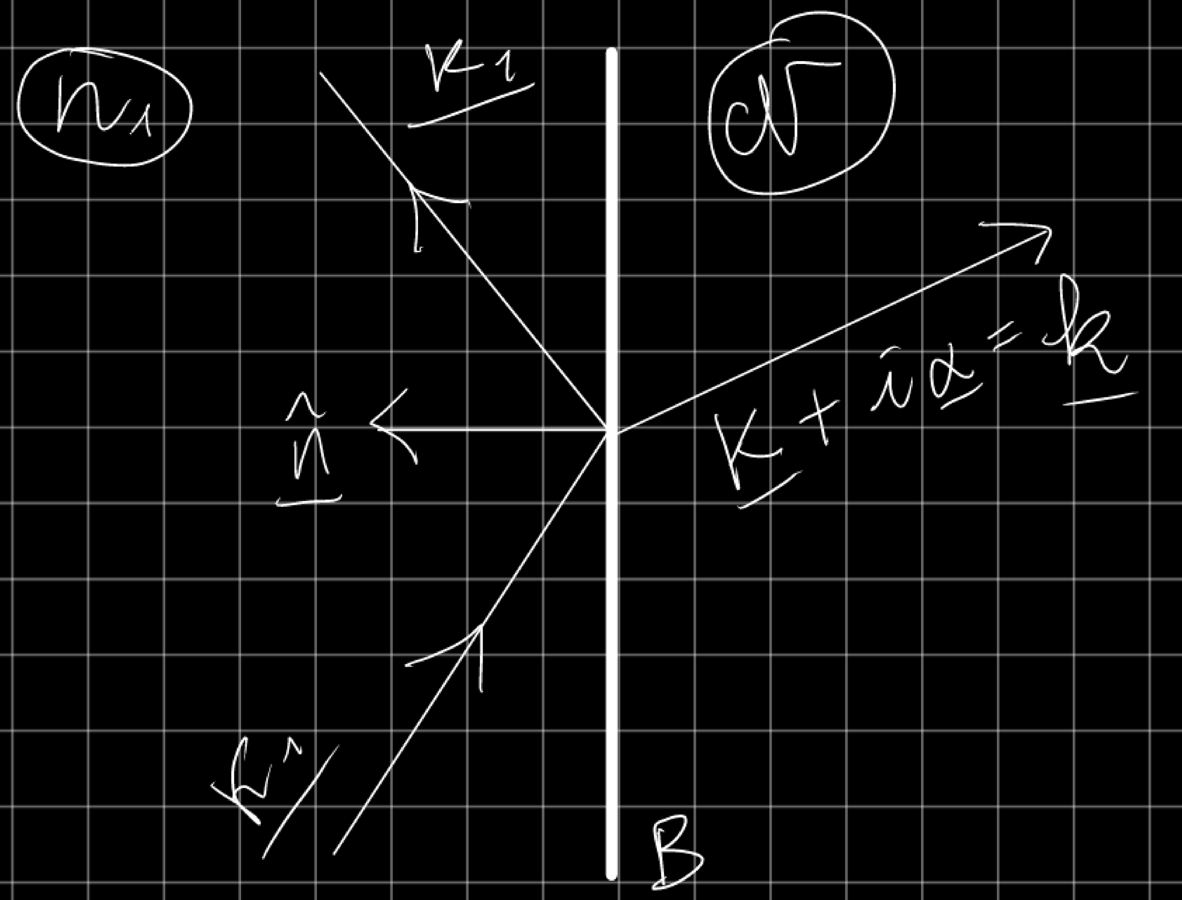


Fig 1.) Schema of the r t problem in absorbing media

As in the figure @ the left, at the boundary we get 3 waves

$$\begin{cases} e^{i\underline{k}_1\underline{r} - i\omega t} = \psi_1 & \text{INCIDENT WAVE} \\ e^{i\underline{k}_{1R}\underline{r} - i\omega t} = \psi_{1R} & \text{REFLECTED WAVE} \\ e^{i(\underline{k}\,\underline{r} - i\omega t)} = \psi_T & \text{TRANSMITTED WAVE} \end{cases}$$

As for the usual case, in the first region $\underline{K}_1 = \underline{K}_{1R}$ since they both lay on the region w/ index n_1. In the second region we have:

$$i\underline{k} = i\underline{K} - \underline{\alpha} \Rightarrow \psi_T = e^{-\underline{\alpha}\underline{r}} e^{i\underline{K}\underline{r} - i\omega t}$$

Thus, as usual

$$\begin{cases} \underline{K}_1\underline{r} = \underline{K}_{1R}\underline{r} \\ \underline{k}_1\underline{r} = (\underline{K} + i\underline{\alpha})\underline{r} \end{cases} \qquad \begin{cases} \underline{K}_1\underline{r} = \underline{K}\,\underline{r} \\ \underline{\alpha}\,\underline{r} = 0 \end{cases}$$

where the second system pops out equating Im & Re of both sides. For this reason, in general, $\underline{K}$ & $\underline{\alpha}$ DON'T HAVE THE SAME DIRECTION. The wave is known as <<INHOMOGENEOUS>>

In this case, $\underline{\alpha}\underline{r} = 0 \Rightarrow \underline{\alpha} \perp B$ (boundary)

1/12/22 - 39m 5.45

The waves travel in the direction of $\underline{K}$ but decay exponentially with $\underline{\alpha}$

Therefore, it's possible to define the planes w/ normal $\underline{K}$ define the phase planes, while, the planes with $\underline{\alpha}$ as normal define the planes of constant amplitude.

Denoting θ as the incidence angle & φ as the refraction angles we get

$$\underline{K}_1\underline{r} = \underline{K}\,\underline{r} \Rightarrow K_1 \sin\theta = K \sin\varphi \quad \text{(by phase matching @ } B\text{)}$$

Note that $\underline{K}$ is not constant! $\underline{K} \equiv \underline{K}(\varphi)$

The relationship $\underline{k}(\varphi)$ can be determined starting from the wave equation. For what we have said so far we can write the equation as follows

$$\nabla^2 \underline{E} = \frac{\mathcal{N}^2}{c^2}\frac{\partial^2 E}{\partial t^2}$$

Therefore: $\hat{\nabla}\psi = i\underline{k}\psi$, $\partial_t \psi = -i\omega\psi$, thus;

$$(\underline{k}\cdot\underline{k})\underline{E} = \frac{\omega^2}{c^2}\mathcal{N}^2\underline{E} = k_0^2\mathcal{N}^2\underline{E}$$

where $k_0^2 = \frac{\omega^2}{c^2}$, thus:

$$(\underline{k}+i\underline{\alpha})(\underline{k}+i\underline{\alpha}) = k_0^2(n+i\eta)^2$$

$$k^2-\alpha^2+2i\underline{k}\cdot\underline{\alpha} = k_0^2(n^2-\eta^2+2in\eta)$$

Equating $Re\{\cdot\}$ & $Im\{\cdot\}$ we have

$$\begin{cases} k^2-\alpha^2 = k_0^2(n^2-\eta^2) \\ k\alpha\cos\varphi = k_0^2\eta n \end{cases}$$

(Can't work it out rn) The result is shown to be;

$$k\cos\varphi + i\alpha = k_0\sqrt{\mathcal{N}^2-\sin^2\theta} \quad (A)$$

Thus, for $\theta=0$ (normal incidence)

$k\cos\varphi + i\alpha = k_0\mathcal{N} \Rightarrow$ (same relation as for HOMOGENEOUS WAVES)

In a <u>PURELY FORMAL WAY</u> the law of Refraction (SNELL) can be rewritten as

$$\mathcal{N} = \sin\theta\csc Z_e \qquad \begin{matrix}\theta\in[0,2\pi) \\ Z_e\in\mathbb{C}\end{matrix}$$

From the above definition;

$$\sin Z_e = \frac{\sin\theta}{\mathcal{N}} \Rightarrow \cos Z_e = \sqrt{1-\frac{\sin^2\theta}{\mathcal{N}^2}} \quad (1)$$

<u>Therefore</u>, with (A) we have

$$k\cos\varphi + i\alpha = k_0\mathcal{N}\sqrt{1-\frac{\sin^2\theta}{\mathcal{N}^2}}$$

$$k\cos\varphi + i\alpha = k_0\mathcal{N}\cos Z_e$$

Thus, the second equation is

$$\mathcal{N} = \frac{k\cos\varphi + i\alpha}{k_0}\sec Z_e \quad (2)$$

We can now find the R of the absorbent medium.

The setup is as follows:

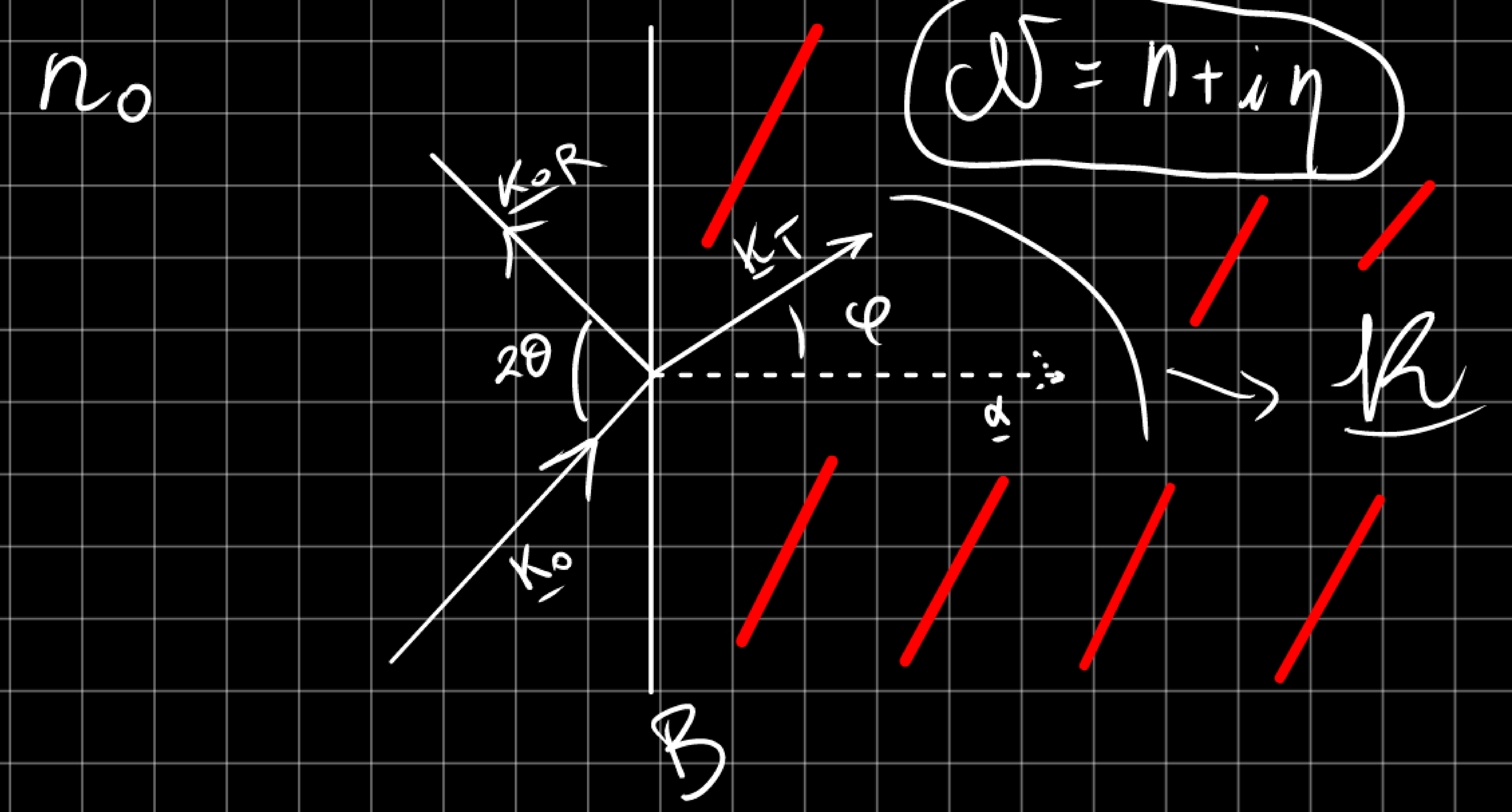


Fig a.) scheme of the problem: $\underline{k_0}$ before, $\underline{k}$ after B

From Maxwell's equations, we know already that the relations between $\underline{E}, \underline{H}$ are

$$\begin{cases} \underline{H} = (\mu_0 \omega)^{-1} \, \underline{k_0} \times \underline{E} \\ \underline{H_R} = (\mu_0 \omega)^{-1} \, \underline{k_{0R}} \times \underline{E} \\ \underline{H_T} = (\mu_0 \omega)^{-1} \, \underline{k} \times \underline{E_T} \end{cases} \begin{cases} \parallel \\ \parallel \\ \underline{H_T} = (\mu_0 \omega)^{-1} \left[\underline{k} \times \underline{E_T} + i \underline{\alpha} \times \underline{E} \right] \end{cases} \qquad (E, H)$$

And, from the boundary continuity equations, we have for Senkrecht (s, TE) pol. & p, TM polarization:

$$\begin{cases} E + E_R = E_T \\ (H - H_R) \cos\theta = H_T \end{cases} \quad (S) \qquad \begin{cases} H - H_R = H_T \\ (E + E_R) \cos\theta = E_T \cos(z_e) \end{cases} \quad (P)$$

Which, can be rewritten as

$$\begin{cases} 1 + r_s = t_s \\ k_0 (1 - r_s) \cos\theta = (k \cos\varphi + i\alpha) t_s \end{cases} \quad (S) \qquad \begin{cases} k_0 (1 - r_p) = k_0 \mathcal{N} t_p \\ (1 + r_p) \cos\theta = t_p \cos(z_e) \end{cases} \quad (P)$$

Starting from Senkrecht polarization we have

$$(1 - r_s) k_0 \cos\theta = (1 + r_s)(k \cos\varphi + i\alpha)$$

Using $k \cos\varphi + i\alpha = k_0 \mathcal{N} \cos(z_e)$ we have

$$(1 - r_s) \cancel{k_0} \cos\theta = (1 + r_s) \cancel{k_0} \mathcal{N} \cos(z_e) \Rightarrow r_s (\mathcal{N} \cos(z_e) + \cos\theta) = \cos\theta - \mathcal{N} \cos(z_e)$$

Therefore

$$\boxed{r_s = \frac{\cos\theta - \mathcal{N} \cos(z_e)}{\cos\theta + \mathcal{N} \cos(z_e)}}$$

From the equation $1+r_s = t_s$, we can also get the transmission coefficient

$$\boxed{t_s = 1 + \frac{\cos\theta - \tilde{\omega}\cos(\tilde{a})}{\cos\theta + \tilde{\omega}\cos(\tilde{a})} = \frac{2\cos\theta}{\cos\theta + \tilde{\omega}\cos(\tilde{a})}}$$

For (P) polarization, the evaluation is completely analogous

$$\begin{cases}(1-r_p) = \tilde{\omega}\, t_p \\ (1+r_p)\cos\theta = t_p\cos(\tilde{a})\end{cases} \qquad \begin{cases}\tilde{\omega}^{-1}(1-r_p) = t_p \\ (1+r_p)\cos\theta = \tilde{\omega}^{-1}(1-r_p)\cos(\tilde{a})\end{cases}$$

Thus

$$r_p\left(\cos\theta + \frac{1}{\tilde{\omega}}\cos(\tilde{a})\right) = \frac{1}{\tilde{\omega}}\cos(\tilde{a}) - \cos\theta$$

which, multiplying both sides by $\tilde{\omega}$ gives

$$\boxed{r_p = \frac{\cos(\tilde{a}) - \tilde{\omega}\cos\theta}{\cos(\tilde{a}) + \tilde{\omega}\cos\theta}}$$

The transmission coefficient is

$$\boxed{t_p = \frac{1}{\tilde{\omega}}(1-r_p) = \frac{1}{\tilde{\omega}}\left(1 - \frac{\cos(\tilde{a}) - \tilde{\omega}\cos\theta}{\cos(\tilde{a}) + \tilde{\omega}\cos\theta}\right) = \frac{2\cos\theta}{\tilde{\omega}\cos(\tilde{a}) + \tilde{\omega}^2\cos\theta}}$$

As usual one can find R_p, R_s & T_p, T_s using $T = t\bar{t}$, $R = r\bar{r}$

Note that for p polarization, while $\mathrm{Im}\{\tilde{\omega}\} \neq 0$, $r_p \neq 0$! Thus, the Brewster angle is substituted by the « principal angle of incidence » θ_1, for which we have $\min\{r_p\} = r_p(\theta_1)$.

As usual, mixes of s & p polarized light get transmitted in elliptical polarization

$\tilde{\omega}$, can be evaluated by measuring I_T & the polarization of outgoing light with methods of ELLIPSOMETRY.

§ NORMAL INCIDENCE

In the special case of $\theta = 0$ (normal incidence) we have that again $r_p = r_s = r$ &

$$r = \frac{1-\tilde{\omega}}{1+\tilde{\omega}} = \frac{1-n-i\eta}{1+n+i\eta}$$

Thus

$$R = r\bar{r} = \frac{(1-n)^2+\eta^2}{(1+n)^2+\eta^2}$$

which reduces to the dielectric case when $\eta = \mathrm{Im}\{\tilde{\omega}\} = 0$.

Since we know that

$$\lim_{\omega\to 0} \mathrm{Re}\{\tilde{n}\} = \lim_{\omega\to 0} \mathrm{Im}\{\tilde{n}\} = \sqrt{\frac{\sigma}{2\omega\epsilon_0}}$$

& from the last equation when $R \equiv R(\tilde{n})$ we have then:

$$\lim_{\omega\to 0} R(\tilde{n}) = 1 - \frac{2}{n} = 1 - \sqrt{\frac{8\omega\epsilon_0}{\sigma}} \quad (HR)$$

known as Hagen-Rubens formula. Therefore, when $\omega \to$ red (λ grows) we have that good conductors become better and better reflectors. As an example

$$Cu, Ag, Au \quad \begin{cases} R(\lambda) \approx 1 & \text{NIR} \quad (\lambda \approx 1\,\mu m / 2\,\mu m) \\ R(\lambda) \cong 1 & \text{FIR} \quad (\lambda > 20\,\mu m) \end{cases}$$

$\oint$ PROPAGATION OF LIGHT IN CRYSTALS

5/12/22 – 22.47 – 1h 54m 34.00 s

The main property of matter in the cristalline state, in the case of optical properties, is its electrical anisotropy.
This means that the polarization produced by the application of a $\underline{E}$ field is dependent on direction.
There are 2 possible values of propagation velocity for a given direction in a crystal, tied to mutually orthogonal polarizations, therefore crystals are said «BIREFRENGENT», i.e. doubly refracting.
Note tho that NOT ALL CRYSTALS EXHIBIT BIREFRENGENCE, it depends only on their symmetry. Cubic crystals like NaCl do NOT exhibit double refrengence, while others do.
One practical way to think physically of why such phenomenon arises is by thinking the lattice atom bonds as springs with different strength k, thus electron displacement is different in each direction, thus, the dependence $\underline{P}(\underline{E})$ must be expressed TENSORIALLY

$$P^i = \epsilon_0 \chi^i_j E_j$$

Where χ^i_j is the ELECTRIC SUSCEPTIBILITY TENSOR. The corresponding $\underline{D}$ field is then:

$$D^i = \epsilon_0(\delta^i_j + \chi^i_j)E^j = \epsilon^i_j E^j$$

Where

$$\epsilon^i_j = \epsilon_0(\delta^i_j + \chi^i_j)$$

Is the DIELECTRIC TENSOR.
For ordinary NON-ABSORBING crystals the tensor χ^i_j is diagonalizable into principal axes $(\hat{\underline{e}}_1, \hat{\underline{e}}_2, \hat{\underline{e}}_3)$, where

$$\chi^i_j = (\chi_{11}\hat{\underline{e}}_1 + \chi_{22}\hat{\underline{e}}_2 + \chi_{33}\hat{\underline{e}}_3)\delta^i_j$$

And the eigenvalues $\chi_{11}, \chi_{22}, \chi_{33}$ are known as the PRINCIPAL SUSCEPTIBILITIES of the crystal, corresponding each to dielectric constants

$$(\epsilon_r)^i_j = \delta^i_j + \chi^i_j$$

The general wave equation can then be rewritten as

$$\epsilon^i{}_{jk}\, \partial^j \epsilon^k{}_{em}\, \partial^e E^m + \frac{1}{c^2}\frac{\partial^2 E^i}{\partial t^2} = -\frac{1}{c^2}\chi^i_j \frac{\partial^2 E^j}{\partial t^2}$$

Imposing $E^i = E_0^i \exp(ik^j r_j - i\omega t)$ as usual, & writing the operatorial relation

$$\begin{cases} \partial_j \to ik_j \\ \partial_t \to -i\omega \end{cases}$$

The previous equation becomes:

$$\epsilon^i{}_{jk}\, k^j \epsilon^k{}_{em} k^e E^m + \frac{\omega^2}{c^2} E^i = -\frac{\omega^2}{c^2}\chi^i_j E^j$$

If χ^i_j is diagonal, in terms of components we can write

$$\begin{cases} \left(\frac{\omega^2}{c^2} - k_y^2 - k_z^2\right) E_x + k_x k_y E_y + k_x k_z E_z = -\frac{\omega^2}{c^2}\chi_{11} E_x \\ \left(\frac{\omega^2}{c^2} - k_x^2 - k_z^2\right) E_y + k_y k_z E_z + k_y k_x E_x = -\frac{\omega^2}{c^2}\chi_{22} E_y \\ \left(\frac{\omega^2}{c^2} - k_y^2 - k_x^2\right) E_z + k_z k_y E_y + k_z k_x E_x = -\frac{\omega^2}{c^2}\chi_{33} E_z \end{cases} \quad (BRF)$$

This system is quite complicated, therefore, for understanding it, suppose that the wave propagates along the x axis, thus

$$\underline{k} = (k, 0, 0)$$

The equations simplify to

$$\begin{cases} \frac{\omega^2}{c^2} E_x = -\frac{\omega^2}{c^2}\chi_{11} E_x \\ \left(\frac{\omega^2}{c^2} - k^2\right) E_y = -\frac{\omega^2}{c^2}\chi_{22} E_y \\ \left(\frac{\omega^2}{c^2} - k^2\right) E_z = -\frac{\omega^2}{c^2}\chi_{33} E_z \end{cases}$$

This system has clearly 2 different solutions. 1st we note that $\underline{E} \perp \underline{k}$ as it should, but, if

1) $E_y \neq 0 \Rightarrow k_1 = \sqrt{1+\chi_{22}} = \frac{\omega}{c}\sqrt{(\epsilon_r)_{22}}$

2) $E_z \neq 0 \Rightarrow k_2 = \sqrt{1+\chi_{33}} = \frac{\omega}{c}\sqrt{(\epsilon_r)_{33}}$

Since $\frac{\omega}{k}$ is the phase velocity of the wave, we have <u>TWO</u> different PV

$$u_1 = \frac{c}{\sqrt{1+\chi_{22}}}$$

$$u_2 = \frac{c}{\sqrt{1+\chi_{33}}}$$

More generally, in terms of refraction index $n = \sqrt{1+\chi}$, we have that for a generic diagonalizable χ^i_j tensor, we have $\underline{3}$ n_i indexes (PRINCIPAL REFRACTION INDEXES)

$$\begin{cases} n_1 = \sqrt{1+\chi_{11}} \\ n_2 = \sqrt{1+\chi_{22}} \\ n_3 = \sqrt{1+\chi_{33}} \end{cases}$$

These come in handy for simplifying eqn (BRF). For having NONZERO solutions we must have

$$\det_3 \begin{pmatrix} \frac{n_1^2\omega^2}{c^2} - k_y^2 - k_z^2 & k_y k_x & k_x k_z \\ k_y k_x & \frac{n_2^2\omega^2}{c^2} - k_x^2 - k_z^2 & k_y k_z \\ k_z k_x & k_z k_y & \frac{n_3^2\omega^2}{c^2} - k_x^2 - k_y^2 \end{pmatrix} \neq 0$$

=> This is a surface in $\underline{k}$ space!

Putting ourselves on the slice $k_z = 0$ the determinant reduces to a simple product

$$\left(\frac{n_3^2\omega^2}{c^2} - k_x^2 - k_y^2\right)\left\{\left[\frac{n_1^2\omega^2}{c^2} - k_y^2\right]\left[\frac{n_2^2\omega^2}{c^2} - k_x^2\right] - k_x^2 k_y^2\right\} = 0$$

We get thus $\underline{2}$ solutions

$$k_x^2 + k_y^2 = n_3^2 \frac{\omega^2}{c^2} \qquad \text{(SPHERE)}$$

$$\left(\frac{n_1\omega}{c}\right)^2\left(\frac{n_2\omega}{c}\right)^2 - k_y^2\left(\frac{n_2\omega}{c}\right)^2 - k_x\left(\frac{n_1\omega}{c}\right)^2 = 0$$

The second equation can be made clearer dividing by $\left(\frac{n_1\omega}{c}\right)^2\left(\frac{n_2\omega}{c}\right)^2$, giving

$$\frac{k_x^2}{(n_2\omega/c)^2} + \frac{k_y^2}{(n_1\omega/c)^2} = 1 \qquad \text{(ELLIPSE)}$$

Similar equations are found slicing the xz & yz planes (+10 m). The resulting wave phase surface consists of an INNER (spherical) sheet & an EXTERNAL (ellipsoidal) sheet

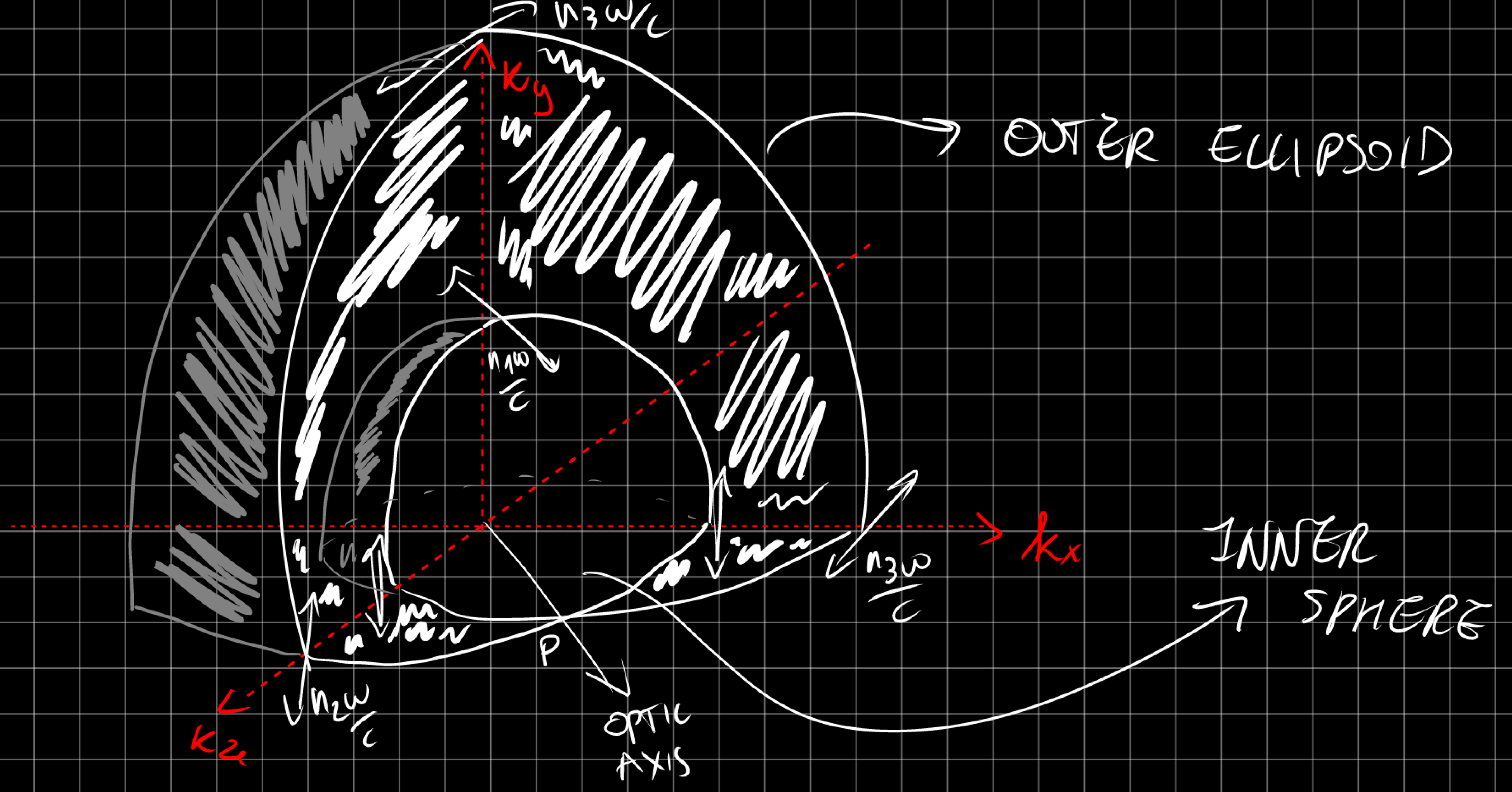


Fig A) SPHERO-ELLIPSOID FOR CRYSTAL PROPAGATION W/ PHASE VELOCITIES $u_i = \frac{n_i\omega}{c}$

09/11/22 22.57 4h 19m 32.35s

New record!