An Investigation Into the Parameter Estimation of Sinusoidal Signals Using the Derivative Method

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1 Introduction

It has been shown that a sum of frequency and amplitude modulated sinusoids plus a noise component are a useful representation of speech[1] and musical[2] signals. In this paper we restrict ourselves to a second-order polynomial representation of the phase argument function and a first-order polynomial representation of the amplitude modulation function. Although there exist a variety of methods for estimating the coefficients for these functions, notable techniques being the quadratic interpolation of the spectrum[3] and the use of the reassignment operators for frequency and time[4], we will focus on a more recent technique that involves the use of signal derivatives[5].

2 Motivation

Considered as a sum of amplitude and frequency modulated sinusoids, an arbitrary signal has the following form

$$\sum_k a_k(t)e^{j\phi_k(t)}$$

where a_k and ϕ_k are the functions of amplitude and phase respectively.

The well-known Fast Fourier Transform (FFT) gives a Fourier Series representation of a signal whose summands have the form

$$e^{\alpha+j(\phi+\mu t)}$$

that is, a constant amplitude coefficient and a first-order polynomial representation of phase. Let us consider polynomials whose orders have been increased by one through integration, giving a sinusoidal representation

$$ae^{\alpha t+j(\phi+\mu t+\psi/2t^2)} \tag{1}$$

As the angular velocity is the derivate of the phase function, it is easy to see that this is a parameterization of a sinusoid with linearly varying amplitude and frequency.

$$\frac{1}{2\pi}\phi'(t) = \mu + \psi t = f(t)$$

3 Extraction of parameters

From now on we will only consider a single summand and therefor drop the k subscript notation.

The extraction of these parameters is motivated by the property of the exponential function that its derivatives and integrals are equivalent, barring adherence to the chain rule. Differentiating (1) we get

$$s'(t) = (\alpha + j(\mu + \psi t))e^{\alpha t+j(\phi+\mu t+\frac{\psi}{2}t^2)} \tag{2}$$

The quotient of (1) over (2) gives

$$\frac{s'(t)}{s(t)} = \alpha + j(\mu + \psi t)$$

As it is more convenient in practice to find the quotient between two values rather than two functions of time t, we find the Short-Time Fourier Transform of s with $\mathcal{F}\{s\} = S$ and evaluate S and S' at $\omega = [\omega : max\{S(\omega)\}]$.

If we neglect the contribution of ψt we find we have a simple and excellent estimation of the amplitude modulation and frequency

$$\Im\left\{\frac{S'(\omega)}{S(\omega)}\right\} = \hat{\mu}$$

$$\Re\left\{\frac{S'(\omega)}{S(\omega)}\right\} = \hat{\alpha}$$

Finding an estimation for ψ proves more difficult. In the original publication of the technique [5] S was simply differentiated a second time, giving

$$\frac{s''(t)}{s(t)} = \alpha^2 - \mu^2 - 2\mu\psi t - \psi^2 t^2 + j(2\alpha\mu + 2\alpha\psi t + \psi)$$

The contribution of ψt was again neglected and the contribution of other coefficients removed, giving

$$\hat{\psi} = \Im\left\{\frac{S''(\hat{\mu})}{S(\hat{\mu})}\right\} - 2\hat{\mu}\hat{\alpha}$$

This estimator was shown to perform poorly in practice, at least when compared to the reassignment method. In [6] it was shown that the derivatives of the signal multiplied by time t are neccessary to render the derivative method equivalent to the reassignment method. Given

$$u(\omega) = \Re\left\{\frac{S_t'(\omega)}{S(\omega)}\right\} - \Re\left\{\frac{S_t(\omega)S'(\omega)}{(S(\omega))^2}\right\}$$

where S_t is the Fourier Transform of the signal multiplied by time, it was shown that an improvement to the estimation of ψ using the derivative method was then

$$\hat{\psi} = \frac{\Im\left\{\frac{S''(\hat{\mu})}{S(\hat{\mu})}\right\} - 2\hat{\mu}\hat{\alpha}}{1 + u(\hat{\mu})}$$

4 Finding signal derivatives

It is worth mentioning the technique of finding signal derivatives used in [5]. The signal s was convolved with the impulse response

$$h[n] = \begin{cases} w[n]F_s\frac{(-1)^n}{n} & \text{if } n \neq 0 \\ 0 & \text{if } n = 0 \end{cases}$$

$$n = \begin{cases} -N/2, ..., N/2-1 & \text{if } N \text{ even} \\ \lfloor -N/2 \rfloor, ..., \lfloor N/2 \rfloor & \text{if } N \text{ odd} \end{cases}$$

where F_s is the sample rate and w a windowing function of length N. This consequently delays the signal s by $N/2$ samples, something which must be accounted for in the implementation.

5 Trials and implementation

We performed a series of trials, assessing the accuracy of the proposed method. The algorithms were implemented in Octave® (but should run in MATLAB® as well). For each trial, a sinusoid was synthesized using (1) and the $\log_{10}$ precentage error was plotted for various pairs of parameters. Finally, a sinusoid was synthesized using a known but arbitrary polynomial phase function and the analysis was compared to the theoretical frequency and frequency slope parameters. See the figures included at the end of the document for the results.

6 Conclusion

In this project the derivative method was succesfully implemented and evaluated. In terms of a real implementation, it seems the reassignment method is more practical as it requires the derivatives of the window rather than the derivatives of the signal [4]. The former can be precomputed. Both algorithms require five FFTs [6]. An advantage to the derivative method, however, is that arbitrary windows may be used in the analysis, even those without known derivatives. Nevertheless, the algorithm is an interesting contribution to the techniques of sinusoidal parameter frequency estimation, as it uses a basic property of the Fourier Transform

$$\mathcal{F}\{s'(t)\} = i2\pi f G(f)$$

Perhaps proceeding in this direction by considering higher order derivatives will yield useful results when attempting to recover the parameters of higher order functions of amplitude and frequency modulation.

References

[1] R.J. McAulay and T.F. Quatieri, "Speech Analysis/Synthesis Based on a Sinusoidal Representation," in *IEEE Trans. Acoust., Speech, and Signal Processing*, Vol. ASSP-34, No. 4, p. 744, 1986.

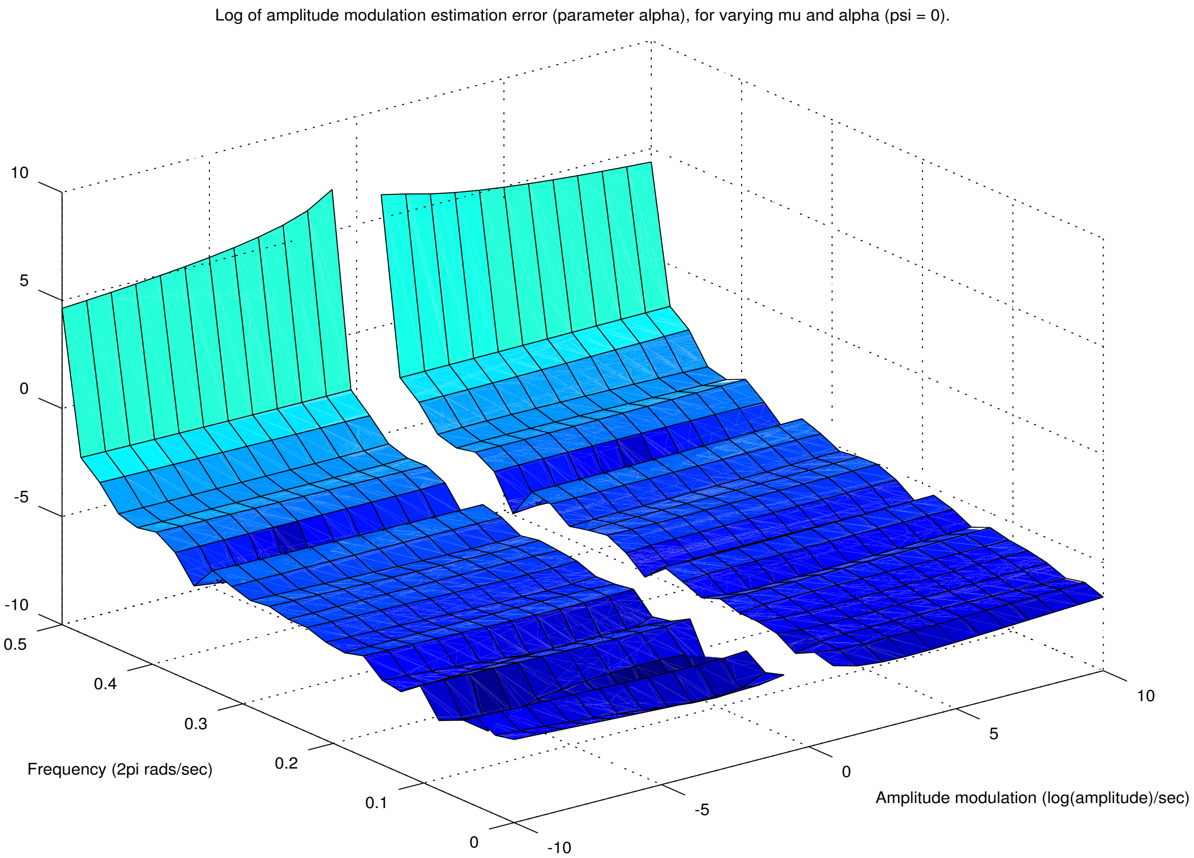
[2] X. Serra and J.O. Smith, "Spectral modeling synthesis: A sound analysis/synthesis system based on a deterministic plus stochastic decomposition," in *Computer Music Journal*, Vol. 14, No. 4, p. 12, 1990.

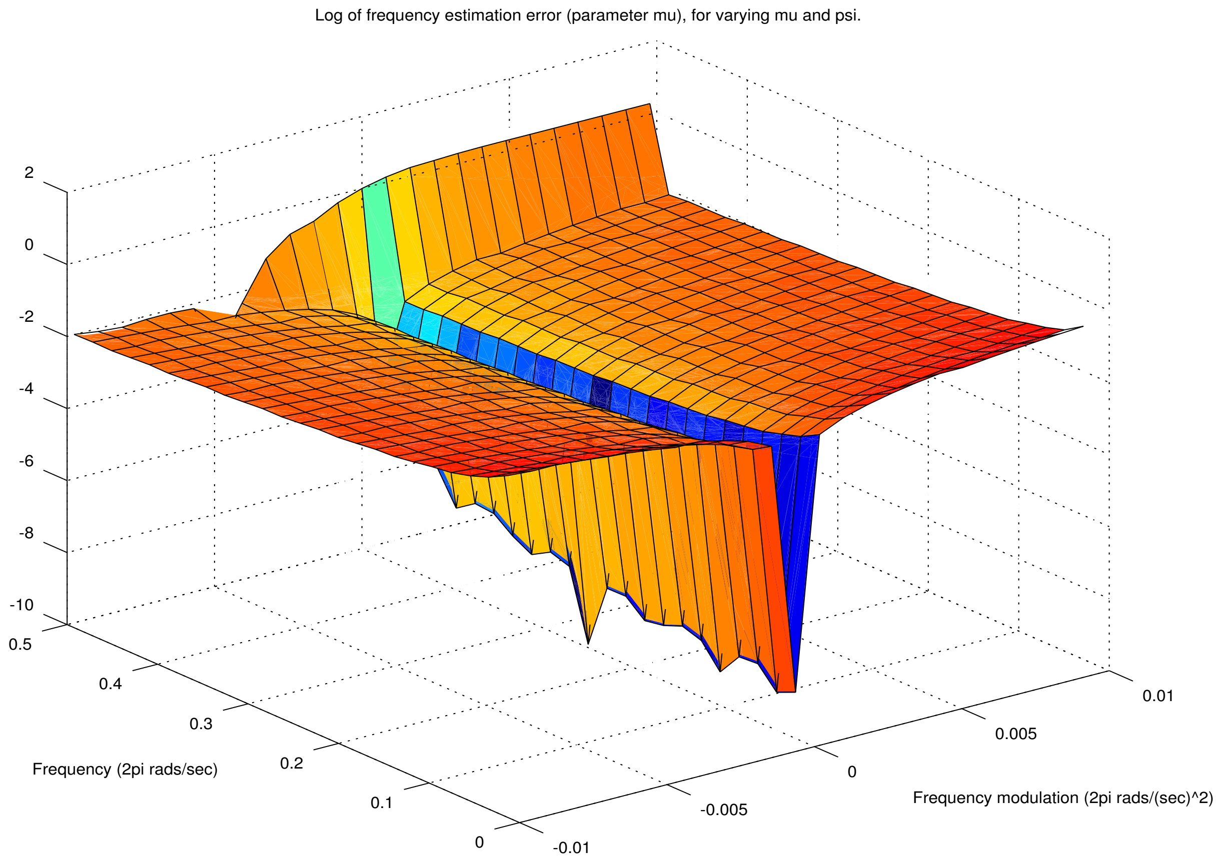
[3] M. Abe and J.O. Smith, "AM/FM Rate Estimation for Time-Vaying Sinusoidal Modeling," in *Proc. IEEE Int. Conf. Acoust., Speech, and Signal Processing*, Philadelphia, NY, USA, vol. III, pp. 201, March 2005.

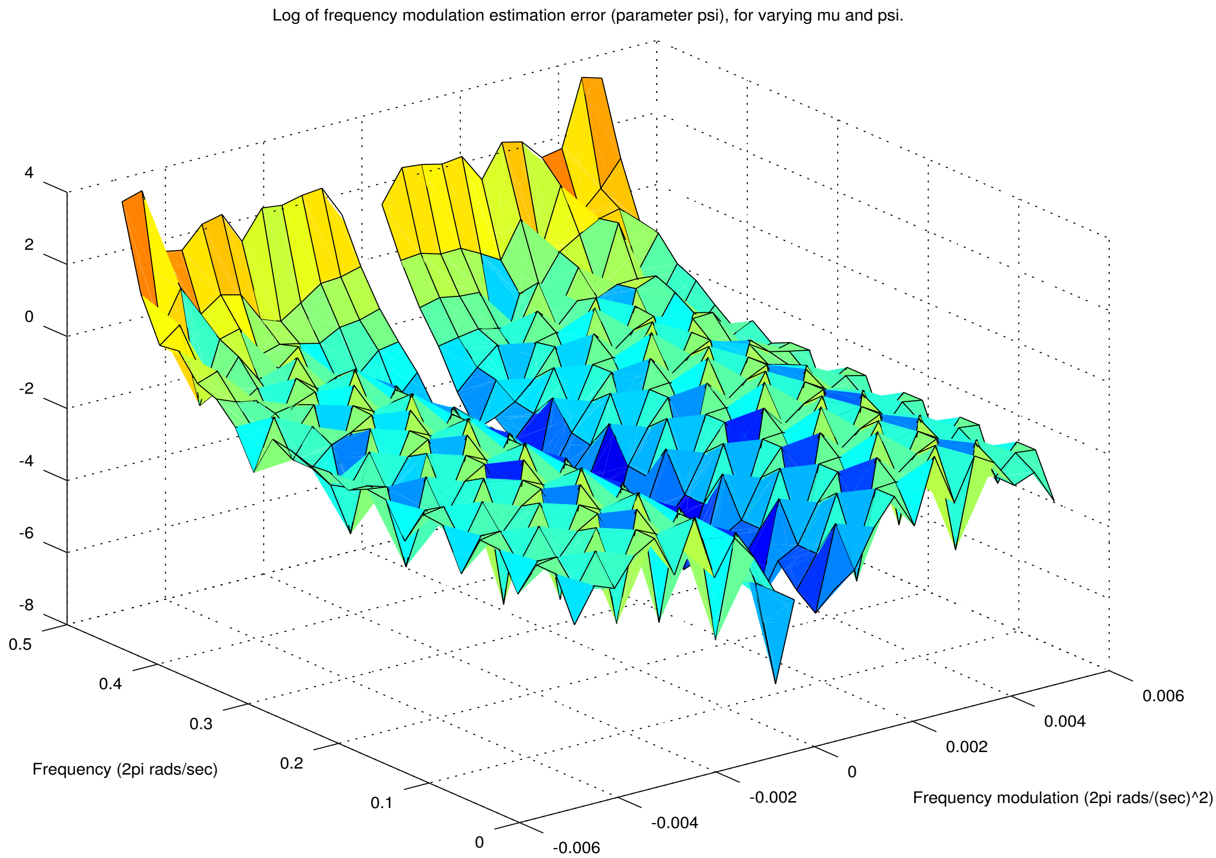
[4] A. Röbel, "Estimating Partial Frequency and Frequency Slope Using Reassignment Operators," in *Proc. Int. Computer Music Conference*, Göteborg, Sweden, pp. 122, September 2002.

[5] S. Marchand and P. Depalle, “Generalization of the derivative analysis method to non-stationary sinusoidal modeling.” in *Proc. DAFx Conf.*, Espoo, Finland, 2008.

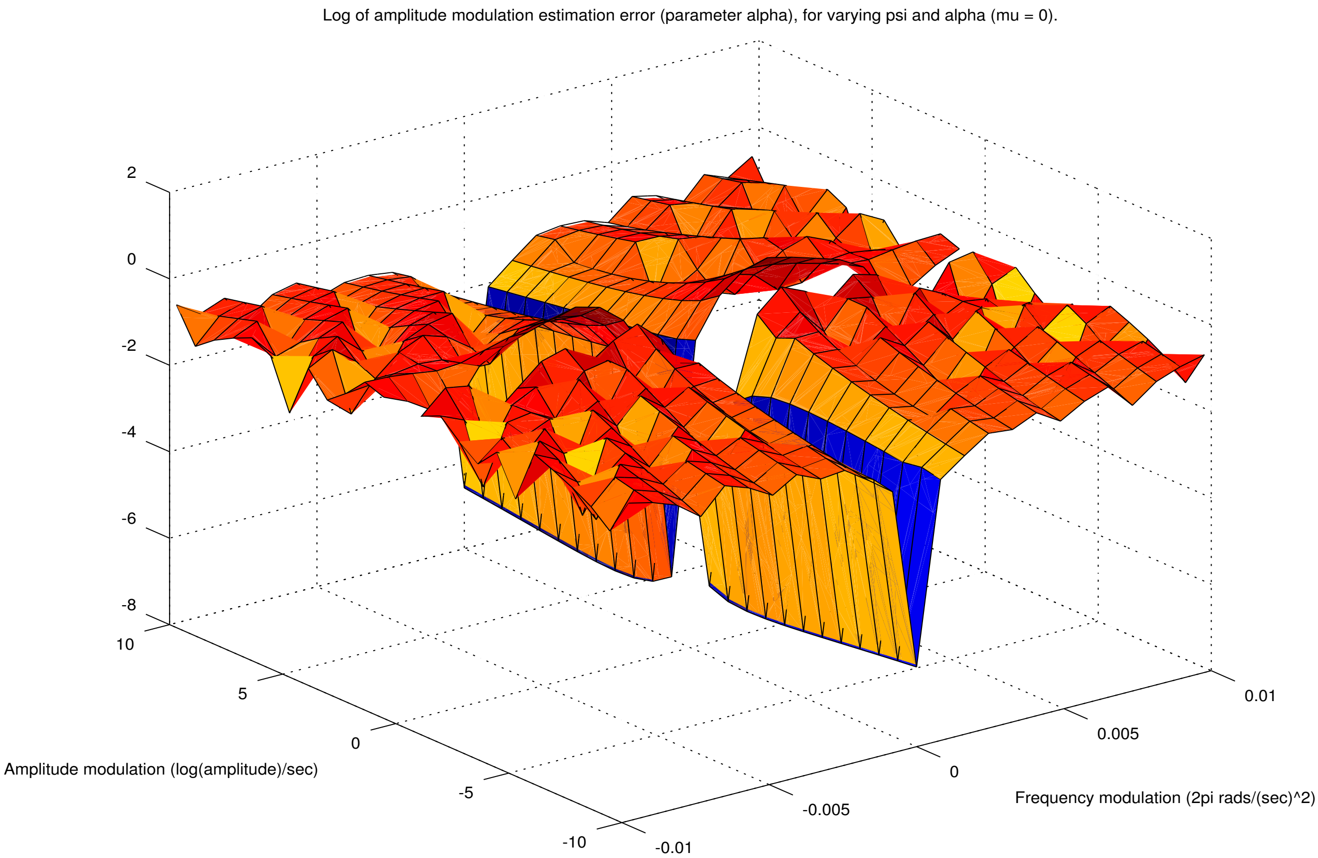
[6] B. Hamilton, P. Depalle, and S. Marchand, “Theoretical and practical comparisons of the reassignment method and the derivative method for the estimation of the frequency slope,” in *Proc. IEEE Workshop on Applications of Signal Processing to Audio and Acoustics*, New Paltz, NY, USA, pp. 345, October 2009.

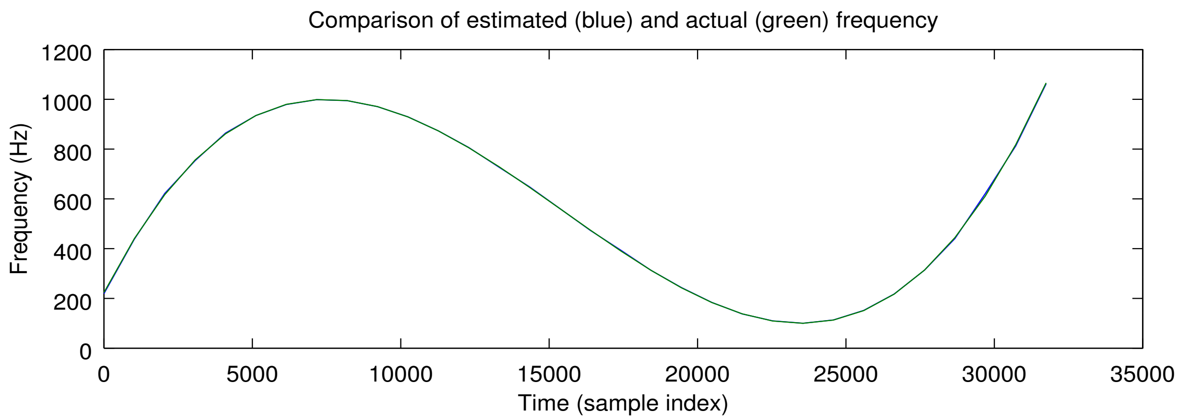

Log of amplitude modulation estimation error (parameter alpha), for varying mu and alpha (psi = 0).
10
5
0
-5
-10
0.5
0.4
0.3
0.2
0.1
0
Frequency (2pi rads/sec)
-10
-5
0
5
10
Amplitude modulation (log(amplitude)/sec)

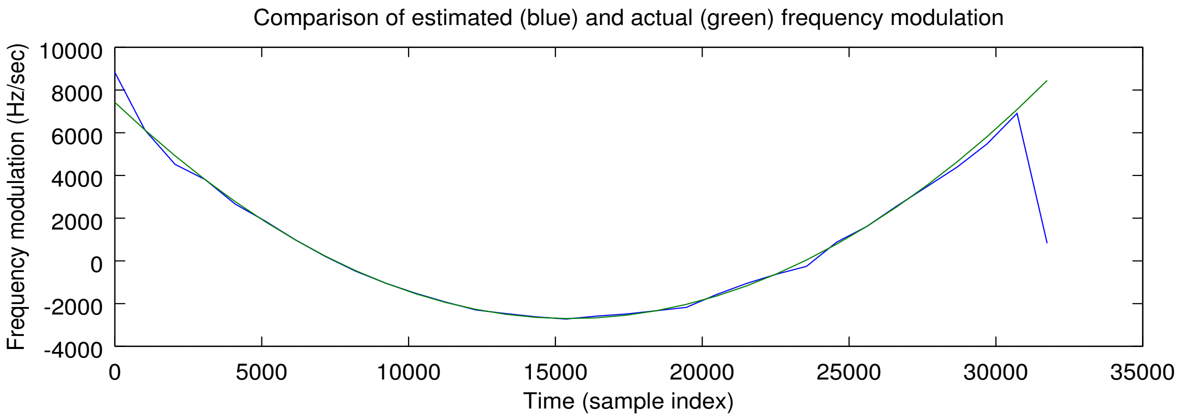

Log of frequency estimation error (parameter mu), for varying mu and psi.
2
0
-2
-4
-6
-8
-10
0.5
0.4
0.3
0.2
0.1
0
Frequency (2pi rads/sec)
-0.01
-0.005
0
0.005
0.01
Frequency modulation (2pi rads/(sec)^2)



Log of frequency modulation estimation error (parameter psi), for varying mu and psi.
4
2
0
-2
-4
-6
-8
0.5
0.4
0.3
0.2
0.1
0
Frequency (2pi rads/sec)
-0.006
-0.004
-0.002
0
0.002
0.004
0.006
Frequency modulation (2pi rads/(sec)^2)


Log of amplitude modulation estimation error (parameter alpha), for varying psi and alpha (mu = 0).
2
0
-2
-4
-6
-8
10
5
0
-5
-10
Amplitude modulation (log(amplitude)/sec)
-0.01
-0.005
0
0.005
0.01
Frequency modulation (2pi rads/(sec)^2)


Comparison of estimated (blue) and actual (green) frequency
Frequency (Hz)
Time (sample index)
0
200
400
600
800
1000
1200
0
5000
10000
15000
20000
25000
30000
35000


Comparison of estimated (blue) and actual (green) frequency modulation
Frequency modulation (Hz/sec)
Time (sample index)
-4000
-2000
0
2000
4000
6000
8000
10000
0
5000
10000
15000
20000
25000
30000
35000