A LINEAR PROGRAMMING APPROACH TO THE TRACKING OF PARTIALS

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A new approach to the tracking of sinusoidal chirps using linear programming is proposed. It is demonstrated that the classical algorithm of [1] is greedy and exhibits exponential complexity for long searches, while approaches based on the Viterbi algorithm exhibit factorial complexity [2] [3]. A linear programming (LP) formulation to find the best L paths in a lattice is described and its complexity is shown to be less than previous approaches. Finally it is demonstrated that the new LP formulation outperforms the classical algorithm in the tracking of sinusoidal chirps in high levels of noise.

ABSTRACT

Index Terms— partial tracking, linear programming, optimization, additive synthesis, atomic decomposition, regularized approximation

1. INTRODUCTION

Atomic decompositions of audio allow for the discovery of meaningful underlying structures such as musical notes [4] or sparse representations [5]. A classical structure sought in decompositions of speech and music signals is the sum-of-sinusoids model: windowed sinusoidal atoms in the decomposition of sufficient energy and in close proximity in both time and frequency are considered as *connected*. The progressions of these connected atoms in time form paths or *partial trajectories*.

Many authors have considered the partial tracking problem, beginning with [1]. Their technique is improved upon in [6] with the use of linear prediction to improve the plausibility of partial tracks. Rather than seeking individual paths, in [2] the most plausible sequence of connections and detachments between atoms is determined via an extension of the Viterbi algorithm proposed in [3]. Improvements to this technique are made in [7] by incorporating the frequency slope into atom proximity evaluations.

The latter techniques seeking globally optimal sets of paths incur great computational cost due to the large number of possible solutions. For this reason, in this paper we propose a linear programming *relaxation* formulation of the optimal path-set problem based on an algorithm for the tracking of multiple objects in video [8]. It will be shown that this algorithm has favourable asymptotic complexity and performs well on the tracking of chirps in high levels of noise.

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1.1. Note on notation

The atomic decompositions used in this paper consider blocks of of contiguous samples, called *frames* and these frames are computed every H samples, H being the *hop-size*. We will denote the N_k sets of parameters for atoms found in the decomposition in frame k as $\theta_0^k, \ldots, \theta_{N_k-1}^k$ and the N_{k+1} in frame k+1 as $\theta_0^{k+1}, \ldots, \theta_{N_{k+1}-1}^{k+1}$ where k and k+1 refer to adjacent frames.

where k and k+1 refer to adjacent frames. The total number of nodes is $M = \sum_{k=0}^{K-1} N_k$. θ_i^j is the *i*th node of the *j*th frame, θ^j the set of all the nodes in the *j*th frame and θ_m the *m*th node out of all M nodes $(0 \le m < M)$.

We are interested in paths that extend across K frames where each path touches only one parameter set and each parameter set is either exclusive to a single path or is not on a path.

In this paper, indexing starts at 0. If we have a vector \boldsymbol{x} then \boldsymbol{x}_i is the ith row or column of that vector depending on the orientation. The same notation is used for Cartesian products, e.g., if α and β are sets and $A=\alpha\times\beta$ then for the pair $a\in A$ a_0 is the first item in the pair and a_1 the second.

2. A GREEDY METHOD

In this section, we present the McAulay-Quatieri method of peak matching. It is conceptually simple and a set of short paths can be computed quickly, but it can be sensitive to spurious peaks and its complexity becomes unwieldly for long searches.

In [1, p. 748] the peak matching algorithm is described in a number of steps; we summarize them here in a way comparable with the linear programming formulation to be presented shortly. In that paper, the parameters of each data point are the instantaneous amplitude, phase, and frequency but here we allow for arbitrary parameter sets θ . Define a distance function $\mathcal{D}\left(\theta_i,\theta_j\right)$ that computes the similarity between 2 sets of parameters. We will now consider a method that finds L tuples of parameters that are closest.

We compute the cost tensor $C = \theta^k \otimes_{\mathcal{D}} \ldots \otimes_{\mathcal{D}} \theta^{k+K-1}$. For each $l \in [0 \ldots L-1]$, find the indices i_0, \ldots, i_{K-1} corresponding to the shortest distance, then remove the i_0, \ldots, i_{K-1} th rows (lines of table entries) in the their respective dimensions from consideration and continue until L tuples have been determined or a distance between a pair of nodes on the path exceeds some threshold Δ_{MQ} . This is summarized in Algorithm 1.

This is a greedy algorithm because on every iteration the smallest cost is identified and its indices are removed from consideration. Perhaps choosing a slightly higher cost in one iteration would allow

Algorithm 1: A generalized McAulay-Quatieri peakmatching algorithm.

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Input: the cost matrix C

Output: L tuples of indices \Gamma, or fewer if \Delta_{MQ} exceeded \Gamma \leftarrow \varnothing;

for l \leftarrow 0 to L-1 do

 \Gamma_l = \underset{[0,\dots,M_0-1]\times\dots\times[0,\dots,M_{K-1}-1]\setminus\Gamma}{\arg\min} C;

if \exists i,j\in\Gamma_l:\mathcal{D}\left(\theta_i,\theta_j\right)>\Delta_{MQ} then
 \mid \text{ return }\Gamma
end
 \Gamma\leftarrow\Gamma\cup C_{\Gamma_l};
end
return \Gamma
```

smaller costs to be chosen in successive iterations. This algorithm does not allow for that. In other terms, the algorithm does not find a set of pairs that represent a globally minimal sum of costs. Furthermore, the algorithm does not scale well: assuming equal numbers of parameter sets in all frames, the search space grows exponentially with K. Nevertheless, the method is simple to implement, computationally negligible when K is small, and works well with a variety of audio signals such as speech [1] and music [9].

3. L BEST PATHS THROUGH A LATTICE VIA LINEAR PROGRAMMING (LP)

In this section we show how to find L paths through a lattice of K frames such that the sets of nodes on each path are disjoint. The kth frame of the lattice contains N_k nodes for a total of $M = \sum_{k=0}^{K-1} N_k$ nodes.

Similar to the McAulay-Quatieri method we define the cost Δ_{LP} as the limiting cost under which the connection between two nodes will be considered in the LP method.

The solution vector \boldsymbol{x} to the linear program shall indicate the presence of a connection between a pair of nodes by having an entry equal to 1 and otherwise have entries equal to 0. To enumerate the set of possible connection-pairs we define

$$\rho = \{(i,j) : \mathcal{D}(\theta_i, \theta_j) \le \Delta_{LP}, 0 \le i < M, 0 \le j < M, i \ne j\}$$

The cost vector of the objective function is then

$$\mathbf{c}_{\rho} = \{ D(\theta_i, \theta_j) \forall (i, j) \in \rho \}$$
 (2)

and the length of c_{ρ} is $\#\rho = \#c_{\rho} = P$, in other words, P pairs of nodes. For convenience we define a bijective mapping $\mathcal{B}: \rho \to [0,\dots,M-1]$ giving the index in x of the pair $p \in \rho$. For the implementation considered in this paper, $\mathcal{D}(\theta_i,\theta_j) = \infty$ for all i,j not in adjacent frames and so P will be no larger than $(K-1)N^2$ (assuming the same number of nodes N in each frame).

The total cost of the paths in the solution is then calculated through the inner product $c_{\rho}^T x$. To obtain x^* that represents L disjoint paths we must place constraints on the structure of the solution. Some of the constraints presented in the following are redundant but the redundancies are kept for clarity; later we will show which constraints can be removed without changing the optimal solution x^* .

All nodes in \boldsymbol{x}^* will have at most one incoming connection or otherwise no connections, a constraint that can be enforced through the following linear inequality: define $\boldsymbol{A}^{\mathrm{I}} \in \mathbb{R}^{R_{\mathrm{I}} \times P}$ with $R_{\mathrm{I}} =$

 $\sum_{k=1}^{K-1} N_k$, the number of nodes in all the frames excluding the first. We sum all the connections into the node $r_{\rm I} + N_0$ represented by the respective entry in \boldsymbol{x} through an inner product with the $r_{\rm I}$ th row in $\boldsymbol{A}^{\rm I}$ and require that this sum be between 0 and 1, i.e.,

$$\boldsymbol{A}_{r_{1},\mathcal{B}(p)}^{I} = \begin{cases} 1 & \text{if } p_{1} = r_{I} + N_{0} \\ 0 & \text{otherwise} \end{cases}, 0 \leq r_{I} < R_{I}, p \in \rho$$
 (3)

and

$$\mathbf{0} \le \mathbf{A}^{\mathrm{I}} \mathbf{x} \le \mathbf{1} \tag{4}$$

Similarly, to constrain the number of outgoing connections into each node, we define $R_0 = \sum_{k=0}^{K-2} N_k$ and $\mathbf{A}^0 \in \mathbb{R}^{R_0 \times P}$ with

$$\mathbf{A}_{r_0, \mathcal{B}(p)}^{\mathcal{O}} = \begin{cases} 1 & \text{if } p_0 = r_0 \\ 0 & \text{otherwise} \end{cases}, 0 \le r_0 < R_0, p \in \rho$$
 (5)

and

$$0 < A^{\mathcal{O}}x < 1 \tag{6}$$

To forbid breaks in the paths it is required that the number of incoming connections into a given node equal the number of outgoing connections for the $R_{\rm B} = \sum_{k=1}^{K-2} N_k$ nodes potentially having both incoming and outgoing connections.

$$A_{r_{\rm B}}^{\rm B} = A_{r_{\rm B}}^{\rm B} - A_{r_{\rm B}+N_0}^{\rm B} \text{ for rows } 0 \le r_{\rm B} < R_{\rm B}$$
 (7)

and

$$A^{\mathrm{B}}x = 0 \tag{8}$$

Finally we ensure that there are L paths by counting the number of connections in each frame and constraining this sum to be L. We choose arbitrarily to count the number of outgoing connections by summing rows of \mathbf{A}^{O} into rows of $\mathbf{A}^{\mathrm{C}} \in \mathbb{R}^{(K-1) \times P}$

$$\boldsymbol{A}_{r_{\mathrm{C}}}^{\mathrm{C}} = \sum_{k=-1}^{b} \boldsymbol{A}_{k}^{\mathrm{O}} \tag{9}$$

with $a = \sum_{j=0}^{r_{\rm C}} N_j$ and $b = \sum_{j=0}^{r_{\rm C}+1} N_j$ and

$$\boldsymbol{A}^{\mathrm{C}}\boldsymbol{x} = L\boldsymbol{1} \tag{10}$$

As stated above, some of these constraints are redundant and can be removed. Indeed, we have $0 \le x \le 1$, therefore we will always have $A^{\rm I}x \ge 0$ and $A^{\rm O}x \ge 0$. Furthermore, all but the last row of (10) can be seen as constructed from linear combinations of rows of (8) and the last row of (10) so we only require $A_{K-2}^{\rm C}x = L$. Finally we always have $x \le 1$ because of the constraint that there be a maximum of 1 incoming and outgoing connection from each node.

The complete LP to find the L best disjoint paths through a lattice described by node connections ρ is then

$$\min_{m{x}} m{c}_{
ho}^T m{x}$$

subject to

$$Gx = \begin{bmatrix} A^{I} \\ A^{O} \\ -I \end{bmatrix} x \le \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$
$$Ax = \begin{bmatrix} A^{B} \\ A^{C}_{K-2} \end{bmatrix} x = \begin{bmatrix} 0 \\ L \end{bmatrix}$$
(11)

where I is the identity matrix. A proof that the solution x^* will have entries equal to either 0 or 1 can be found in [10, p. 167].

4. MEMORY COMPLEXITY

To simplify notation, in this section we assume there are N nodes in each frame of the lattice.

Although the matrices involved in (11) are large, only a small fraction of their values are non-zero. Matrices $\mathbf{A}^{\mathrm{I}}, \mathbf{A}^{\mathrm{O}} \in \mathbb{R}^{N(K-1) \times P}$, but each contains only P non-zero entries. Furthermore $\mathbf{A}^{\mathrm{B}} \in \mathbb{R}^{N(K-1) \times P}$ but contains only $2N^2(K-2)$ non-zero entries while $\mathbf{A}_{K-2}^C \in \mathbb{R}^P$ contains merely N. The $\mathbf{x} \geq \mathbf{0}$ constraint requires a matrix with P non-zero entries. The total memory complexity including the entries in \mathbf{c}_{ρ} and the right-hand-sides of (11) is $2N^2(K-2) + 4P + 2N(K-1) + N + 1$ non-zero floating-point numbers.

5. COMPLEXITY

Here we will compare the complexity of the LP formulation of the best L paths search to the greedy McAulay-Quatieri method as well a combinatorial algorithm proposed in [3].

Assuming the same number of nodes N in each frame of the lattice, the search for the lth best path in the generalized McAulay-Quatieri algorithm $(0 \le l < L)$ requires a search over $(N-l)^K$ possible paths.

The LP formulation of the L-best paths problem gives results equivalent to the solution to the L-best paths problem proposed in [3]. The complexity of the algorithm by Wolf in [3] is equivalent to the Viterbi algorithm for finding the single best path through a trellis whose kth frame has $\binom{N_k}{L}\binom{N_{k+1}}{L}L!$ connections where N_k and N_{k+1} are the number of nodes in two consecutive frames of the original lattice. Therefore, assuming a constant number N of nodes in each frame, its complexity is $O((\binom{N}{L}^2 L!)^2 K)$.

The complexity of the algorithm presented here is polynomial in the number of variables (the size of \boldsymbol{x}). Assuming we use the algorithm in [11] to solve the LP, our program has a complexity of $O(P^{3.5}B^2)$ where B is the number of bits used to represent each number in the input. However, this bound is conservative considering the reported complexity of modern algorithms.

For instance, the complexity of a log-barrier interior-point method is dominated by solving the system of equations

$$\begin{bmatrix} -DG^{T}G & A^{T} \\ A & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} tc_{\rho} + A^{T}d \\ 0 \end{bmatrix}$$
(12)

some 10s of times [12, p. 590]. Each iteration then takes $\frac{2}{3}((K-1)N^2+(K-2)N)^3$ flops (floating-point operations) to solve (12) using a standard LU-decomposition [13, p. 98]. As \boldsymbol{D} is a diagonal matrix, if the number of nodes in each frame is N for all frames, then $\boldsymbol{D}\boldsymbol{G}^T\boldsymbol{G}$ will be a block-diagonal matrix made up of K-1 blocks $\boldsymbol{B}_k \in \mathbb{R}^{N^2 \times N^2}$. The system can then be solved in

$$\frac{2}{3}(K-1)N^6 + 2(K-2)(K-1)N^5 + 2(K-2)^2(K-1)N^4 + \frac{2}{3}(K-2)^3N^3$$

flops [12, p. 675]; this complexity is without exploiting the sparsity of A nor the structure of $B_k = D_k C$ — the product of some diagonal matrix D_k with an unchanging symmetric matrix C.

6. PARTIAL PATHS ON AN EXAMPLE SIGNAL

We compare the greedy and LP based methods for peak matching on a synthetic signal. The signal is composed of Q=3 chirps of constant amplitude, the qth chirp s at sample n described by the equation

$$s_q(n) = \exp(j(\phi_q + \omega_q n + \frac{1}{2}\psi_q n^2))$$

The parameters for the Q chirps are presented in Table 1.

A 1 second long signal is synthesized at a sampling rate of 16000 Hz, the chirps ramping from their initial to final frequency in that time. We add Gaussian distributed white noise at several SNRs to evaluate the technique in the presence of noise.

A spectrogram of each signal is computed with an analysis window length of 2048 samples and a hop-size H of 512 samples. Local maxima are searched in 100 Hz wide bands spaced 50 Hz apart. The bin corresponding to each local maximum and its two surrounding bins are used by the Distribution Derivative Method (DDM) [14] to estimate the local chirp parameters, the ith set of parameters in frame k denoted $\theta_i^k = \left\{ \phi_i^k, \omega_i^k, \psi_i^k \right\}$ (the atoms used by the DDM are generated from 4-term once-differentiable Nuttall windows [15]). Partial tracking is performed on the resulting atomic decomposition.

We search for partial tracks using both the greedy and LP strategies. Both algorithms use the distance metric \mathcal{D}_{pr} between two parameters sets:

$$\mathcal{D}_{\text{pr.}}\left(\theta_i^k, \theta_j^{k+1}\right) = \left(\omega_i^k + \psi_i^k H - \omega_j^{k+1}\right) \tag{13}$$

which is the error in predicting jth frequency in frame k+1 from the ith parameters in frame k. For the greedy method, the search for partial paths is restricted to two frames ahead, i.e., paths of length $K_{\rm MQ}=3$ are sought, otherwise the computation becomes intractable. For the LP method the search is carried out over all frames ($K_{\rm LP}=28$). The cost thresholding values are $\Delta_{\rm MQ}=\Delta_{\rm LP}=0.1$. For both methods, the search is restricted to nodes between frequencies 250 to 2000 Hz.

Figure 1 shows discovered partial trajectories for signals at various SNRs. It is seen that while the greedy method starts performing poorly at an SNR of -6 dB, the LP method still gives plausible partial trajectories. The LP method returns paths spanning all K frames, due to the constraints. The McAulay-Quatieri method in general does not, but longer paths can be formed in a straightforward way after the initial short path search step [1].

It is interesting to note that the paths are found by only considering the prediction error of the initial frequency of the atom. Other cost functions can be chosen depending on the nature of the signal: reasonable cost functions here might be similarity of the atoms's energies or frequency slopes.

Table 1. Parameters of qth chirp. ν_0 and ν_1 are the initial and final frequency of the chirp in Hz.

q	ϕ_q	ω_q	ψ_q	ν_0	ν_1
0	0	0.20	2.45×10^{-6}	500	600
1	0	0.39	4.91×10^{-6}	1000	1200
2	0	0.59	7.36×10^{-6}	1500	1800

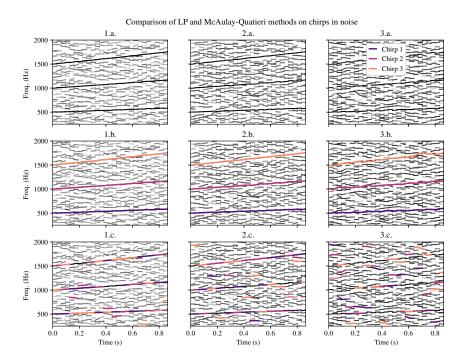


Figure 1. Line-segments representing the frequency and frequency-slope at local spectrogram maxima. The power of each atom is represented by shades of grey: black atoms have the highest power and white the lowest. The coloured segments correspond to connected paths returned by the search algorithms. Plots 1–3.a. show the atomic decomposition of the signal before partial tracking for SNRs of 0, -6 and -12 dB, respectively. Plots 1–3.b. show the partial paths discovered by the LP method and plots 1–3.c. show the paths discovered by the McAulay-Quatieri method. See Table 1 for the chirp parameters.

7. CONCLUSION

In this paper we reformulated the classical partial tracking technique of McAulay and Quatieri and showed that it can be seen as a greedy algorithm for finding the L shortest paths in a lattice. An algorithm was then proposed minimizing the sum of the L paths, using a linear programming approach. The complexity of the new algorithm was shown to be generally less than the Viterbi-based methods and the McAulay-Quatieri algorithm for large K. It was shown on synthetic signals that the new approach finds plausible paths in lattices with a large number of spurious nodes.

The proposed approach has some drawbacks. There are situations where it is undesirable to have paths extend throughout the entire lattice. Acoustic signals produced by striking media, such as strings or bars, exhibit a spectrum where the upper partials decay more quickly than the lower ones; it would be desirable in these situations to have shorter paths for these partials. This could be addressed as in [2] where the signal is divided into overlapping sequences of frames and partial paths are connected between sequences.

In its current form, the path search may choose undesirable paths if a convenient node is missing from the following frame. An extension could consider nodes some number of frames ahead.

The proposed algorithm, while asymptotically faster than other partial tracking algorithms, is still not fast. In situations where computational resources are limited, a McAulay-Quatieri method search over many sets of small K works sufficiently well. However in high amounts of noise the algorithm proposed here is robust while still

of tractable complexity.

It may be possible to improve the performance of the algorithm by the use of different cost functions and regularization. The cost function (13) could be extended to encourage similarity between frequency slopes or amplitude information. Each metric should be scaled according to its desired contribution to the cost.

There may also be a way to extract individual paths through the use of auxiliary variables in (11). If so, path specific costs such as overall smoothness or fit to a particular model could be incorporated.

In any case, it would be interesting to further investigate programming relaxations encouraging underlying discrete structures plausible for audio in the framework of regularized approximation. These structures are closer to ground-truth structures for speech (text) and music (the musical score).

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