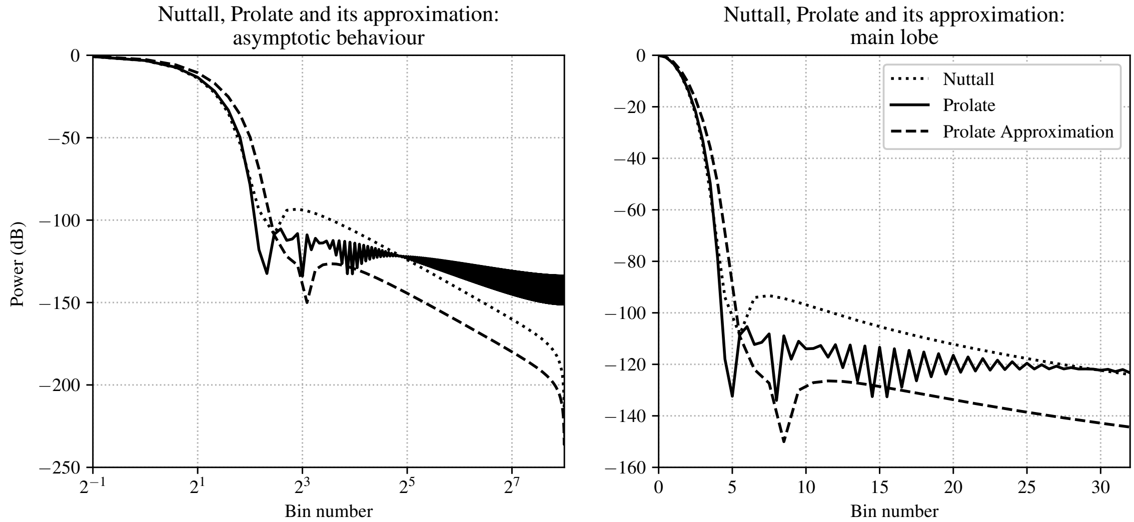

Figure 1: Comparing the main-lobe and asymptotic power spectrum characteristics of the continuous 4-term Nuttall window, the digital prolate window with $W = 0.008$, and the continuous approximation to the digital prolate window.

the Fourier transform of modulated w [14]. Therefore choosing ψ amounts to a filter design problem under the constraints that the impulse response of the filter be differentiable in t and finite. To minimize the influence of all but one component, granted the components's energy concentrations are sufficiently separated in frequency, we desire impulse responses whose magnitude response gives maximum out-of-band rejection or equivalently, windows whose Fourier transform exhibits the lowest sidelobes.

In all the publications reviewed on the DDM for this paper, the window used was the Hann window which is once-differentiable everywhere in the time-domain. In [11], a publication on the reassignment method, other windows than the Hann are considered but these windows must be twice-differentiable. Nuttall [15] has designed windows with lower sidelobes than the canonical Hann window which are everywhere at least once-differentiable. It is also possible to design approximations to arbitrary symmetrical window functions using harmonically related cosines, as is discussed in the following section.

5. DIFFERENTIABLE APPROXIMATIONS TO WINDOWS

A differentiable approximation to a symmetrical window can be designed in a straightforward way. In [16] and [17] it is shown how to design optimal windows of length N samples using a linear combination of M harmonically related cosines

$$\tilde{w}(n) = \sum_{m=0}^{M-1} b_m \cos(2\pi m \frac{n}{N}) \mathcal{R}(\frac{n}{N}) \tag{22}$$

where $\mathcal{R}$ is the *rectangle function*. This function is discontinuous at $n = \frac{\pm N}{2}$, and therefore not differentiable there, unless

$$\sum_{m=0}^{M-1} b_m \cos(\pm\pi m) = 0 \tag{23}$$

Rather than design based on an optimality criterion, such as the height of the highest sidelobe [17], a once-differentiable approximation to an existing window w is desired. To do this, we choose the b_m so that the window $\tilde{w}$'s squared approximation error to w is minimized while having $\tilde{w}(\frac{\pm N}{2}) = 0$, i.e. we find the solution $\{b_m^*\}$ to the mathematical program

$$\text{minimize} \sum_{n=0}^{N-1} (w(n) - \sum_{m=0}^{M-1} b_m \cos(2\pi m \frac{n}{N}))^2 \tag{24}$$

$$\text{subject to} \sum_{m=0}^{M-1} b_m \cos(\pi m) = 0 \tag{25}$$

which can be solved using constrained least-squares; a standard numerical linear algebra routine [18, p. 585].

6. A CONTINUOUS WINDOW DESIGN EXAMPLE

As a design example we show how to create a continuous approximation of a digital prolate spheroidal window.

Digital prolate spheroidal windows are a parametric approximation to functions whose Fourier transform's energy is maximized in a given bandwidth [19]. These can be tuned to have extremely low sidelobes, at the expense of main-lobe width. Differentiation of these window functions may be possible but is not as straightforward as differentiation of the sum-of-cosine windows above. Furthermore, the windows do not generally have end-points equal to 0. In the following we will demonstrate how to approximate a digital prolate spheroidal window with one that is everywhere at least once-differentiable.

In [20] it was shown how to construct digital prolate spheroidal windows under parameters N, the window length in samples, and a parameter W choosing the (normalized) frequency range in which the proportion of the main lobe's energy is to be maximized. We chose $N = 512$ based on the window length chosen in [1] for ease of comparison. Its W parameter's value was chosen by synthesizing windows with W ranging between 0.005 and 0.010 at a