4. MEMORY COMPLEXITY

To simplify notation, in this section we assume there are N nodes in each frame of the lattice.

Although the matrices involved in (11) are large, only a small fraction of their values are non-zero. Matrices $\boldsymbol{A}^{\mathrm{I}}, \boldsymbol{A}^{\mathrm{O}} \in \mathbb{R}^{N(K-1)\times P}$, but each contains only P non-zero entries. Furthermore $\boldsymbol{A}^{\mathrm{B}} \in \mathbb{R}^{N(K-1)\times P}$ but contains only $2N^2(K-2)$ non-zero entries while $\boldsymbol{A}^{\mathrm{C}}_{K-2} \in \mathbb{R}^{P}$ contains merely N. The $\boldsymbol{x} \geq \boldsymbol{0}$ constraint requires a matrix with P non-zero entries. The total memory complexity including the entries in $\boldsymbol{c}_\rho$ and the right-hand-sides of (11) is $2N^2(K-2) + 4P + 2N(K-1) + N + 1$ non-zero floating-point numbers.

5. COMPLEXITY

Here we will compare the complexity of the LP formulation of the best L paths search to the greedy McAulay-Quatieri method as well a combinatorial algorithm proposed in [3].

Assuming the same number of nodes N in each frame of the lattice, the search for the lth best path in the generalized McAulay-Quatieri algorithm ($0 \leq l < L$) requires a search over $(N-l)^K$ possible paths.

The LP formulation of the L-best paths problem gives results equivalent to the solution to the L-best paths problem proposed in [3]. The complexity of the algorithm by Wolf in [3] is equivalent to the Viterbi algorithm for finding the single best path through a trellis whose kth frame has $\binom{N_k}{L}\binom{N_{k+1}}{L}L!$ connections where N_k and N_{k+1} are the number of nodes in two consecutive frames of the original lattice. Therefore, assuming a constant number N of nodes in each frame, its complexity is $O((\binom{N}{L}^2 L!)^2 K)$.

The complexity of the algorithm presented here is polynomial in the number of variables (the size of $\boldsymbol{x}$). Assuming we use the algorithm in [11] to solve the LP, our program has a complexity of $O(P^{3.5}B^2)$ where B is the number of bits used to represent each number in the input. However, this bound is conservative considering the reported complexity of modern algorithms.

For instance, the complexity of a log-barrier interior-point method is dominated by solving the system of equations

$$\begin{bmatrix} -\boldsymbol{D}\boldsymbol{G}^T\boldsymbol{G} & \boldsymbol{A}^T \\ \boldsymbol{A} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{u} \\ \boldsymbol{v} \end{bmatrix} = \begin{bmatrix} t\boldsymbol{c}_\rho + \boldsymbol{A}^T\boldsymbol{d} \\ \boldsymbol{0} \end{bmatrix} \tag{12}$$

some 10s of times [12, p. 590]. Each iteration then takes $\frac{2}{3}((K-1)N^2 + (K-2)N)^3$ flops (floating-point operations) to solve (12) using a standard LU-decomposition [13, p. 98]. As $\boldsymbol{D}$ is a diagonal matrix, if the number of nodes in each frame is N for all frames, then $\boldsymbol{D}\boldsymbol{G}^T\boldsymbol{G}$ will be a block-diagonal matrix made up of $K-1$ blocks $\boldsymbol{B}_k \in \mathbb{R}^{N^2\times N^2}$. The system can then be solved in

$$\frac{2}{3}(K-1)N^6 + 2(K-2)(K-1)N^5 + 2(K-2)^2(K-1)N^4 + \frac{2}{3}(K-2)^3N^3$$

flops [12, p. 675]; this complexity is without exploiting the sparsity of $\boldsymbol{A}$ nor the structure of $\boldsymbol{B}_k = \boldsymbol{D}_k\boldsymbol{C}$ — the product of some diagonal matrix $\boldsymbol{D}_k$ with an unchanging symmetric matrix $\boldsymbol{C}$.

6. PARTIAL PATHS ON AN EXAMPLE SIGNAL

We compare the greedy and LP based methods for peak matching on a synthetic signal. The signal is composed of $Q = 3$ chirps of constant amplitude, the qth chirp s at sample n described by the equation

$$s_q(n) = \exp(j(\phi_q + \omega_q n + \frac{1}{2}\psi_q n^2))$$

The parameters for the Q chirps are presented in Table 1.

A 1 second long signal is synthesized at a sampling rate of 16000 Hz, the chirps ramping from their initial to final frequency in that time. We add Gaussian distributed white noise at several SNRs to evaluate the technique in the presence of noise.

A spectrogram of each signal is computed with an analysis window length of 2048 samples and a hop-size H of 512 samples. Local maxima are searched in 100 Hz wide bands spaced 50 Hz apart. The bin corresponding to each local maximum and its two surrounding bins are used by the Distribution Derivative Method (DDM) [14] to estimate the local chirp parameters, the ith set of parameters in frame k denoted $\theta_i^k = \{\phi_i^k, \omega_i^k, \psi_i^k\}$ (the atoms used by the DDM are generated from 4-term once-differentiable Nuttall windows [15]). Partial tracking is performed on the resulting atomic decomposition.

We search for partial tracks using both the greedy and LP strategies. Both algorithms use the distance metric $\mathcal{D}_{\text{pr.}}$ between two parameters sets:

$$\mathcal{D}_{\text{pr.}}\left(\theta_i^k, \theta_j^{k+1}\right) = \left(\omega_i^k + \psi_i^k H - \omega_j^{k+1}\right) \tag{13}$$

which is the error in predicting jth frequency in frame $k+1$ from the ith parameters in frame k. For the greedy method, the search for partial paths is restricted to two frames ahead, i.e., paths of length $K_{\text{MQ}} = 3$ are sought, otherwise the computation becomes intractable. For the LP method the search is carried out over all frames ($K_{\text{LP}} = 28$). The cost thresholding values are $\Delta_{\text{MQ}} = \Delta_{\text{LP}} = 0.1$. For both methods, the search is restricted to nodes between frequencies 250 to 2000 Hz.

Figure 1 shows discovered partial trajectories for signals at various SNRs. It is seen that while the greedy method starts performing poorly at an SNR of -6 dB, the LP method still gives plausible partial trajectories. The LP method returns paths spanning all K frames, due to the constraints. The McAulay-Quatieri method in general does not, but longer paths can be formed in a straightforward way after the initial short path search step [1].

It is interesting to note that the paths are found by only considering the prediction error of the initial frequency of the atom. Other cost functions can be chosen depending on the nature of the signal: reasonable cost functions here might be similarity of the atoms's energies or frequency slopes.

Table 1**.** Parameters of qth chirp. ν_0 and ν_1 are the initial and final frequency of the chirp in Hz.

q	ϕ_q	ω_q	ψ_q	ν_0	ν_1
0	0	0.20	2.45×10^{-6}	500	600
1	0	0.39	4.91×10^{-6}	1000	1200
2	0	0.59	7.36×10^{-6}	1500	1800