differentiating with respect to t we obtain

$$\frac{df}{dt}(t) = \frac{dx}{dt}(t)\overline{\psi}(t) + x(t)\frac{d\overline{\psi}}{dt}(t) = \left(\sum_{q=1}^{Q} q a_q t^{q-1}\right) x(t)\overline{\psi}(t) + x(t)\frac{d\overline{\psi}}{dt}(t) \quad (7)$$

Because ψ is zero outside of the interval $[-\frac{L_t}{2}, \frac{L_t}{2}]$, integrating $\frac{df}{dt}(t)$ we obtain

$$\int_{-\infty}^{\infty} \frac{df}{dt}(t)dt = \sum_{q=1}^{Q} q a_q \int_{-\frac{L_t}{2}}^{\frac{L_t}{2}} t^{q-1}x(t)\overline{\psi}(t)dt + \left\langle x, \frac{d\overline{\psi}}{dt}\right\rangle = 0 \quad (8)$$

or, using the operator $\mathcal{T}^{\alpha}$,

$$\sum_{q=1}^{Q} q a_q \left\langle \mathcal{T}^{q-1}x, \overline{\psi}\right\rangle = -\left\langle x, \frac{d\overline{\psi}}{dt}\right\rangle \quad (9)$$

Estimating coefficients a_q, $1 < q \leq Q$, simply requires R atoms ψ_r with $R \geq Q$ to solve the linear system of equations

$$\sum_{q=1}^{Q} q a_q \left\langle \mathcal{T}^{q-1}x, \overline{\psi}_r\right\rangle = -\left\langle x, \frac{d\overline{\psi}_r}{dt}\right\rangle \quad (10)$$

for $1 \leq r \leq R$.

To estimate a_0 we rewrite the signal we are analysing as

$$x(t) = \exp(a_0)\gamma(t) + \epsilon(t) \quad (11)$$

where $\epsilon(t)$ is the error signal, the part of the signal that is not explained by our model, and $\gamma(t)$ is the part of the signal whose coefficients have already been estimated, i.e.,

$$\gamma(t) = \exp\left(\sum_{q=1}^{Q} a_q t^q\right) \quad (12)$$

Computing the inner product $\langle x, \gamma\rangle$, we have

$$\langle x, \gamma\rangle = \langle \exp(a_0)\gamma, \gamma\rangle + \langle \epsilon, \gamma\rangle \quad (13)$$

The inner product between ϵ and γ is 0, by the orthogonality principle [13, ch. 12]. Furthermore, because $\exp(a_0)$ does not depend on t, we have

$$\langle x, \gamma\rangle = \exp(a_0)\langle \gamma, \gamma\rangle \quad (14)$$

so we can estimate a_0 as

$$a_0 = \log(\langle x, \gamma\rangle) - \log(\langle \gamma, \gamma\rangle) \quad (15)$$

As will be seen in subsequent sections, the DDM typically involves taking the discrete Fourier transform (DFT) of the signal windowed by both an everywhere once-differentiable function of finite support (e.g., the Hann window) and this function's derivative. A small subset of atoms corresponding to the peak bins in the DFT are used in Eq. 10 to solve for the parameters a_q.

3. ESTIMATING THE $a_{p,q}$ OF P COMPONENTS

We examine how the mixture model influences the estimation of the $a_{p,q}$ in Eq. 3. Consider a mixture of P components. If we define the weighted signal sum

$$g(t) = \sum_{p=1}^{P} x_p(t)\overline{\psi}(t) = \sum_{p=1}^{P} f_p(t) \quad (16)$$

and substitute g for f in Eq. 7 we obtain

$$\sum_{p=1}^{P}\int_{-\frac{L_t}{2}}^{\frac{L_t}{2}} \frac{df_p}{dt}(t)dt = 0 = \sum_{p=1}^{P}\left(\sum_{q=1}^{Q} q a_{p,q}\left\langle \mathcal{T}^{q-1}x_p, \overline{\psi}\right\rangle + \left\langle x_p, \frac{d\overline{\psi}}{dt}\right\rangle\right) \quad (17)$$

From this we see if $\left\langle \mathcal{T}^{q-1}x_p, \overline{\psi}_r\right\rangle$ and $\left\langle x_p, \frac{d\overline{\psi}_r}{dt}\right\rangle$ are small for all but $p = p^*$ and a subset of R atoms[1], we can simply estimate the parameters $a_{p^*,q}$ using

$$\sum_{q=1}^{Q} q a_{p^*,q}\left\langle \mathcal{T}^{q-1}x_{p^*}, \overline{\psi}_r\right\rangle = -\left\langle x_{p^*}, \frac{d\overline{\psi}_r}{dn}\right\rangle \quad (18)$$

for $1 \leq r \leq R$. To compute $a_{p^*,0}$ we simply use

$$\gamma_{p^*}(t) = \exp\left(\sum_{q=1}^{Q} a_{p^*,q}t^q\right) \quad (19)$$

in place of γ in Eq. 15.

4. DESIGNING THE ψ_R

In practice, an approximation of Eq. 4 is evaluated using the DFT on a signal x that is properly sampled and so can be evaluated at a finite number of times nT with $n \in [0, N-1]$ and T the sample period in seconds. In this way, the chosen atoms $\psi_\omega(t)$ are the products of the elements of the Fourier basis and an appropriately chosen window w that is once differentiable and finite, i.e.,

$$\psi_\omega(t) = w(t)\exp(-j\omega t) \quad (20)$$

Defining $N = \frac{L_t}{T}$ and angular frequency at bin r as $\omega_r = 2\pi\frac{r}{N}$, the approximate inner product is then

$$\langle x, \psi_\omega\rangle \approx \sum_{n=0}^{N-1} x(Tn)w(Tn)\exp(-2\pi jr\frac{n}{N}) \quad (21)$$

i.e., the definition of the DFT of a windowed signal[2]. The DFT is readily interpreted as a bank of bandpass filters centred at normalized frequencies $\frac{r}{N}$ and with frequency response described by

[1] The notation x^* will mean the value of the argument x maximizing or minimizing some function.

[2] Notice however that this is an approximation of the inner product and should not be interpreted as yielding the Fourier series coefficients of a properly sampled signal x periodic in L_t. This means that other evaluations of the inner product that yield more accurate results are possible. For example, the analytic solution is possible if x is assumed zero outside of $[-\frac{L_t}{2}, \frac{L_t}{2}]$ (the ψ are in general analytic). In this case the samples of x are convolved with the appropriate interpolating sinc functions and the integral of this function's product with ψ is evaluated.