EE6222 2022-2023-S1-4

(a) Since the sample mean is zero

$$\Sigma = \frac{1}{n} X X^T \qquad \Sigma \phi_i = \lambda_i \phi_i$$

let $X = [x_1 \ x_2 \cdots x_n]$

$$\Sigma = \frac{1}{n}\sum_{j=1}^{n}(x_j - 0)(x_j - 0)^T$$

$$= \frac{1}{n}\sum_{j=1}^{n} x_j x_j^T$$

the projected data: $y_j = \phi_i^T(x_j - 0) = \phi_i^T x_j$

$$\because \bar{y} = \frac{1}{n}\sum_{j=1}^{n} y_j = \frac{1}{n}\phi_i^T \sum_{j=1}^{n} x_j = 0$$

$$\therefore \sigma_i^2 = \frac{1}{n}\sum_{j=1}^{n}(y_j - \bar{y})(y_j - \bar{y})^T$$

$$= \frac{1}{n}\sum_{j=1}^{n} y_j y_j^T = \frac{1}{n}\sum_{j=1}^{n}(\phi_i^T x_j)(\phi_i^T(x_j))^T = \frac{1}{n}\phi_i^T\left(\sum x_j x_j\right)\phi_i$$

$$= \phi_i^T\left(\frac{1}{n}\sum x_j x_j^T\right)\phi_i = \phi_i^T \Sigma \phi_i = \lambda_i \phi_i^T \phi_i = \lambda_i$$

(b) Since we have

$$\Sigma \phi_i = \lambda_i \phi_i \text{ and } \Sigma \phi_j = \lambda_j \phi_j$$

$$\phi_j^T \Sigma \phi_i = \lambda_i \phi_j^T \phi_i \qquad (\Sigma \phi_j)^T \phi_i = (\lambda_j \phi_j)^T \phi_i$$

$$\phi_j^T \Sigma \phi_i = \lambda_i \phi_j^T \phi_i \qquad \phi_j^T \Sigma^T \phi_i = \lambda_j \phi_j^T \phi_i$$

$\because \Sigma = \Sigma^T$

$\therefore \lambda_i \phi_j^T \phi_i = \lambda_j \phi_j^T \phi_i$

$(\lambda_i - \lambda_j)\phi_j^T \phi_i = 0 \qquad \because \lambda_i \neq \lambda_j \Rightarrow \phi_j^T \phi_i = 0$

$\therefore \phi_i$ and ϕ_j are orthogonal

(c) $\sigma_{kj}^2 = \frac{1}{n}[\phi_k^T(X-0)][\phi_j^T(X-0)]^T$

$$= \frac{1}{n}\phi_k^T X X^T \phi_j$$

$$= \frac{1}{n}\phi_k^T \Sigma \phi_j$$