EE7403
2022-2023-S2

3. (a) $\mu = \frac{1}{4}\left(\begin{bmatrix}1\\6\end{bmatrix}+\begin{bmatrix}4\\7\end{bmatrix}+\begin{bmatrix}2\\9\end{bmatrix}+\begin{bmatrix}5\\10\end{bmatrix}\right) = \begin{bmatrix}3\\8\end{bmatrix}$

$a_1 = \phi_1^T(x_1-\mu) = \begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}-2\\-2\end{bmatrix} = -\frac{4}{\sqrt{2}}$

$a_2 = \phi_1^T(x_2-\mu) = \begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}1\\-1\end{bmatrix} = 0$

$a_3 = \phi_1^T(x_3-\mu) = \begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}-1\\1\end{bmatrix} = 0$

$a_4 = \phi_1^T(x_4-\mu) = \begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}2\\2\end{bmatrix} = \frac{4}{\sqrt{2}}$

$b_1 = \phi_2^T(x_1-\mu) = \begin{bmatrix}-\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}-2\\-2\end{bmatrix} = 0$

$b_2 = \phi_2^T(x_2-\mu) = \begin{bmatrix}-\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}1\\-1\end{bmatrix} = -\frac{2}{\sqrt{2}}$

$b_3 = \phi_2^T(x_3-\mu) = \begin{bmatrix}-\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}-1\\1\end{bmatrix} = \frac{2}{\sqrt{2}}$

$b_4 = \phi_2^T(x_4-\mu) = 0$

(b) $\Sigma = \frac{1}{4}\sum_{i=1}^{4}(x_i-\mu)(x_i-\mu)^T = \frac{1}{2}\begin{bmatrix}5 & 3\\3 & 5\end{bmatrix}$.

$\begin{cases}\Sigma\phi_1 = \lambda_1\phi_1 \Rightarrow \lambda_1 = 4,\ \lambda_2 = 1\\ \Sigma\phi_2 = \lambda_2\phi_2\end{cases}$

(c) to minimize the error, let $\phi = \phi_1$ with greater eigenvalue.

$\phi_1\phi_1^T = I$

$\hat{x}_1 = \phi_1 a_1 + \mu = -\frac{4}{\sqrt{2}}\times\begin{bmatrix}\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\end{bmatrix}+\begin{bmatrix}3\\8\end{bmatrix} = \begin{bmatrix}1\\6\end{bmatrix}$

$\hat{x}_2 = \phi_1 a_2 + \mu = 0\times\begin{bmatrix}\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\end{bmatrix}+\begin{bmatrix}3\\8\end{bmatrix} = \begin{bmatrix}3\\8\end{bmatrix}$

$\hat{x}_3 = \phi_3 a_3 + \mu = 0\times\begin{bmatrix}\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\end{bmatrix}+\begin{bmatrix}3\\8\end{bmatrix} = \begin{bmatrix}3\\8\end{bmatrix}$

$\hat{x}_4 = \phi_4 a_4 + \mu = \frac{4}{\sqrt{2}}\times\begin{bmatrix}\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\end{bmatrix}+\begin{bmatrix}3\\8\end{bmatrix} = \begin{bmatrix}5\\10\end{bmatrix}$