Assignment 2, Part 1

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1 Complete the following problems. For all problems, <u>show your work</u>.

1.1 In one sentence, explain what the following homogeneous transformation accomplishes when applied to a point (x, y, z) in terms of yaw, pitch, roll, and translation.

$$T_1 = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 & -1 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

The transformation rolls about the z-axis by π/4 and then translates by (-1, 2, 0).

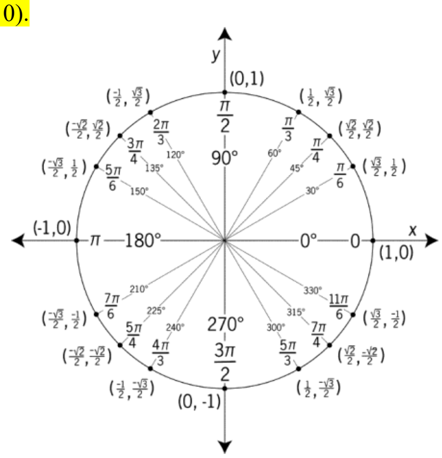

Figure 1: The unit circle.

$$R_z(\gamma) = \begin{pmatrix} \cos(\gamma) & -\sin(\gamma) & 0 \\ \sin(\gamma) & \cos(\gamma) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Figure 2: Example of roll from LaValle p. 76 example (3.16).

$$R_z(\pi/4) = \begin{pmatrix} \cos(\pi/4) & -\sin(\pi/4) & 0 \\ \sin(\pi/4) & \cos(\pi/4) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

γ is the rotation about the z-axis. If you look at the minor matrix of T_1, then you know that x and y are left alone and rotation or roll is about the z-axis by π/4 and then translates by (-1, 2, 0).

If $\gamma = \pi/4 \therefore \cos(\gamma) = (1/\sqrt{2})$
$(\cos \pi/4, \sin \pi/4) = (1/\sqrt{2}, 1/\sqrt{2})$

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1.2 Write out a 4x4 homogeneous transformation T_2, when applied to a point (x, y, z) in the global coordinate frame, translates the point $(3, 0, 2)^T$, then followed by a pitch of 45 degrees. Y our answer need not be simplified, and may be represented as a single matrix or the product of two or more matrices.

Pitch is the counterclockwise rotation of β about the x-axis.

$$R_x(\beta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\beta) & -\sin(\beta) \\ 0 & \sin(\beta) & \cos(\beta) \end{pmatrix}$$

Figure 3: Example of pitch from LaValle p. 77 example (3.17).

$$\text{Translation} = \begin{pmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\text{Pitch by } 45° \text{ or } \pi/4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\pi/4) & -\sin(\pi/4) & 0 \\ 0 & \sin(\pi/4) & \cos(\pi/4) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\text{which is equivalent to} \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

x-axis is left alone therefore we are rotating around it with respect to y and z. This is called pitch.

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{\text{Rotation}} \underbrace{\begin{pmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{\text{Translation}} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

First multiply translation then second multiply rotation because matrices go right to left in order.

1.3 We would like to reverse the transformation applied by T_2T_1 that is, write out $(T_2T_1)^{-1}$. Your answer need not be simplified, and may be represented as a single matrix or the product of two or more matrices.

$(T_2T_1)^{-1} = T_1^{-1}T_2^{-1}$

We need to find T_1^{-1}, T_2^{-1} separately then compute $T_1^{-1}\,T_2^{-1}$

T_2 = 1st translate then 2nd rotate (pitch) by $\pi/4$ or 45°.

T_2^{-1} = 1st rotate (pitch) by $\pi/4$ then 2nd translate by the negative.

$$\begin{pmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\pi/4) & -\sin(\pi/4) & 0 \\ 0 & \sin(\pi/4) & \cos(\pi/4) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

T_1 = 1st rotate (roll) by $\pi/4$ or 45° with respect to z, then 2nd translate by

$$\begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}$$

T_1^{-1} = 1st translate by

1
-2
0

then 2nd rotate (roll) by $\pi/4$ or 45° with respect to z.

$$\underbrace{\begin{pmatrix} \cos(-\pi/4) & -\sin(-\pi/4) & 0 & 0 \\ \sin(-\pi/4) & \cos(-\pi/4) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{\text{Rotate}}\underbrace{\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{\text{Translate}}$$

$(T_2T_1)^{-1} = T_1^{-1}T_2^{-1}$

$$\begin{pmatrix} \cos(-\pi/4) & -\sin(-\pi/4) & 0 & 0 \\ \sin(-\pi/4) & \cos(-\pi/4) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} * \begin{pmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\pi/4) & -\sin(\pi/4) & 0 \\ 0 & \sin(\pi/4) & \cos(\pi/4) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

1.4 Write out the quaternion equivalent to the rotations in T_1 and T_2 as q_1 and q_2. Then calculate the product, that is $q_1 * q_2$.

T_1 = rotate by $\pi/4$ with respect to z.

$(v, \theta) = ((0, 0, 1), \pi/4)$

$$q = (\cos(\theta/2), v_1\sin(\theta/2), v_2\sin(\theta/2), v_3\sin(\theta/2))$$

Figure 4: Quaternion representation of axis-angle representation of a 3D rotation from LaValle p. 82 example (3.30) [1].

$q_1 = (((\cos(\pi/4)) / 2), 0, 0, 1 * ((\sin(\pi/4)) / 2))$

$q_1 = (\cos(\pi/8), 0, 0, \sin(\pi/8))$

T_2 = rotate by $\pi/4$ with respect to x.

$(v, \theta) = ((1, 0, 0), \pi/4)$

$q_2 = (\cos(\pi/8), \sin(\pi/8), 0, 0)$

$w_1 = \cos(\pi/8)$
$w_2 = \cos(\pi/8)$
$x_1 = 0$
$x_2 = \sin(\pi/8)$
$y_1 = 0$
$y_2 = 0$
$z_1 = \sin(\pi/8)$
$z_2 = 0$

$w_3 = w_1*w_2 - x_1*x_2 - y_1*y_2 - z_1*z_2$
$w_3 = (\cos(\pi/8)\cos(\pi/8)) - (0*\sin(\pi/8)) - (0*0) - (\sin(\pi/8)*0)$
$w_3 = \cos^2(\pi/8)$

$x_3 = w_1*x_2 + w_2*x_1 + y_1*z_2 - y_2*z_1$
$x_3 = (\cos(\pi/8)*\sin(\pi/8)) + (\cos(\pi/8)*0) + (0*0) - (0*\sin(\pi/8))$
$x_3 = (\cos(\pi/8)*\sin(\pi/8))$

$y_3 = w_1*y_2 + w_2*y_1 + x_2*z_1 - x_1*z_2$
$y_3 = (\cos(\pi/8)*0) + (\cos(\pi/8)*0) + (\sin(\pi/8)*\sin(\pi/8)) - (0*0)$
$y_3 = \sin^2(\pi/8)$

$z_3 = w_1*z_2 + w_2*z_1 + x_1*y_2 - x_2*y_1$
$z_3 = (\cos(\pi/8)*0) + (\cos(\pi/8)*\sin(\pi/8)) + (0*0) - (\sin(\pi/8)*0)$
$z_3 = (\cos(\pi/8)*\sin(\pi/8))$

$q_3 = (w_3, x_3, y_3, z_3)$
$q_3 = (\cos^2(\pi/8), (\cos(\pi/8) * \sin(\pi/8)), \sin^2(\pi/8), (\cos(\pi/8)*\sin(\pi/8)))$

References

[1] S. LeValle. Chapter 3: The Geometry of Virtual Worlds. *Virtual Reality*, 2:65-94, Cambridge University Press, 2019.