# STAT 153 sketch

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Proof of null$(X)$ contains at least one non-zero vector $\eta$:

Since $p > n$, the column vectors are linear dependent. Denote $(v_1, \cdots, v_p)$ as the column vectors of $X$. Then, there are non trivial coefficients $(c_1, \cdots, c_p)$ such that

$$\sum_{i=1}^{p} c_i v_i = 0.$$

Hence, $\eta = (c_1, \cdots, c_p)$ is a non-zero vector in null$(X)$.

Proof of $\hat{\beta} = \tilde{\beta} + \eta$ is also a least squares solution for $\eta \in \text{null}(X)$:

Denote the prediction from $\tilde{\beta}$ as $\tilde{y} = X\tilde{\beta}$ with MSE $= y - \tilde{y}$. Then the prediction from $\hat{\beta}$

$$\begin{aligned}
\hat{y} &= X\hat{\beta} && (1)\\
&= X(\tilde{\beta} + \eta) && (2)\\
&= X\tilde{\beta} + X\eta && (3)\\
&= \tilde{y} + X\eta && (4)\\
&= \tilde{y}. && (5)
\end{aligned}$$

Therefore, they have the same MSE. Since $\hat{\beta}$ is a least squares solution, $\tilde{\beta} + \eta$ is also a least squares solution.

Since null$(X) \not\perp e_j$, there exists some $v \in \text{null}(X)$ that has non-zero $j$-th coorindate. Denote the $j$-th coordinate of $v$ as a real number $c$. If $c > 0$, we can construct $\hat{\beta} = \tilde{\beta} - \left(\frac{\tilde{\beta}_j}{c}\right) v - v$, which has the $j$-th coordinate less than 0. If $c < 0$, we can construct $\hat{\beta} = \tilde{\beta} + \left(\frac{\tilde{\beta}_j}{c}\right) v - v$, which also has the $j$-th coordinate less than 0.

For either case, the prediction

$$\begin{aligned}
\hat{y} &= X\tilde{\beta}\\
&= X\left[\tilde{\beta} \pm \left(\frac{\tilde{\beta}_j}{c}\right) v - v\right]\\
&= X\tilde{\beta} \pm X\left(\frac{\tilde{\beta}_j}{c}\right) v - Xv\\
&= X\tilde{\beta} \pm \left(\frac{\tilde{\beta}_j}{c}\right) Xv - Xv\\
&= X\tilde{\beta} \qquad (\because v \in \text{null}(X))\\
&= \tilde{y}.
\end{aligned}$$

Hence, $\tilde{\beta}$ and $\hat{\beta}$ will have the same prediction under $X$, so as the MSE.