

Math 180A HW1

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Problem 3.

- (a) $\Omega = \{(1, 2, 3, 4, 5), (1, 2, 3, 4, 6), (1, 2, 3, 4, 7), \dots\}$ with no repetition of numbers in each element and orders do not matter, which means $(1, 2, 3, 4, 5)$ is the same as $(5, 4, 3, 2, 1)$. There are a total of $\binom{39}{5} = 575757$ elements in Ω .

Since each number are picked randomly and uniformly, the distribution of each element is uniform with $P = \frac{1}{|\Omega|} = \frac{1}{\binom{39}{5}} = \frac{1}{575757}$.

(b) $P = \frac{\binom{5}{3} \times \binom{39-5}{2}}{\binom{39}{5}} = \frac{10 \times 561}{575757} = \frac{5610}{575757}$

We first choose 3 numbers out of the 5 numbers we have in our ticket. Then we choose 2 numbers out of the $(39 - 5) = 34$ numbers that are not on our ticket.

Problem 4.

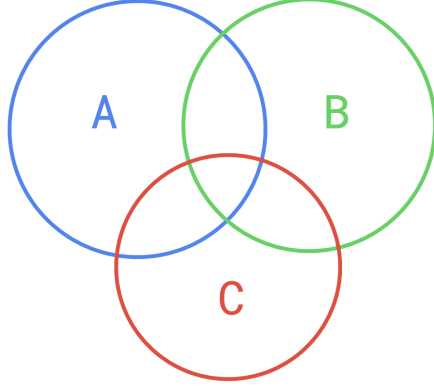
- (a) $\Omega = \{1C, 2C, 3C, \dots, 1D, 2D, \dots, 1H, 2H, \dots, 1S, 2S, \dots, 13S\}$ with C, D, H, S representing club, diamond, heart, and spade respectively. Therefore, there are a total of 52 elements in Ω .

Since one card is picked randomly and uniformly, the probability measure is uniform for each event with $P = \frac{1}{|\Omega|} = \frac{1}{52}$.

- (b) We can define event A as $(1S \cup 2S \cup 3S)$.

Since $1S, 2S, 3S$ are disjoint, $P(A) = P(1S \cup 2S \cup 3S) = P(1S) + P(2S) + P(3S) = \frac{1}{52} + \frac{1}{52} + \frac{1}{52} = \frac{3}{52}$.

Problem 5.



(a)

$$(b) \quad P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$

Let $p_1 = P(A \cap B^c \cap C^c)$, $p_2 = P(A^c \cap B \cap C^c)$, $p_3 = P(A^c \cap B^c \cap C)$, $p_{12} = P(A \cap B \cap C^c)$, $p_{13} = P(A \cap B^c \cap C)$, $p_{23} = P(A^c \cap B \cap C)$, $p_{123} = P(A \cap B \cap C)$. Note that all sets are pairwise disjoint.

$$\begin{aligned} & P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C) \\ &= (p_1 + p_{12} + p_{13} + p_{123}) + (p_2 + p_{12} + p_{23} + p_{123}) + (p_3 + p_{13} + p_{23} + p_{123}) \\ &\quad - (p_{12} + p_{123}) - (p_{13} + p_{123}) - (p_{23} + p_{123}) + p_{123} \\ &= p_1 + p_2 + p_3 + p_{12} + p_{13} + p_{23} + p_{123} \\ &= P(A \cup B \cup C) \end{aligned}$$

Problem 6.

$$1 \geq P(A \cup B) \tag{1}$$

$$1 \geq P(A) + P(B) - P(AB) \tag{2}$$

$$1 \geq 0.4 + 0.7 - P(AB) \tag{3}$$

$$-0.1 \geq -P(AB) \tag{4}$$

$$0.1 \leq P(AB) \tag{5}$$

We have proved that $0.1 \leq P(AB)$.

Since $(A \cap B) \subseteq A$ and $(A \cap B) \subseteq B$, $P(A \cap B) \leq P(A)$ and $P(A \cap B) \leq P(B)$. Therefore, $P(A \cap B) \leq 0.4$.

In conclusion, $0.1 \leq P(AB) \leq 0.4$.

Problem 7. Let $G = \{\text{exactly two balls are green}\}$, $R = \{\text{exactly two balls are red}\}$, $Y = \{\text{exactly two balls are yellow}\}$, $W = \{\text{exactly two balls are white}\}$.

$$P(G \cup R \cup Y \cup W) = P(G) + P(R) + P(Y) + P(W) \quad (6)$$

$$- P(G \cap R) - P(G \cap Y) - \dots \quad (7)$$

$$+ P(G \cap R \cap Y) + P(G \cap R \cap W) + \dots \quad (8)$$

$$- P(G \cap R \cap Y \cap W) \quad (9)$$

Now let's calculate the individual terms:

$$P(G) = P(R) = \dots = \frac{\binom{4}{2} \times 3 \times 3}{4^4} = \frac{54}{256} = \frac{27}{128} \quad (10)$$

$$P(G \cap R) = P(G \cap Y) = \dots = \frac{\binom{4}{2}}{4^4} = \frac{3}{128} \quad (11)$$

Note that it's impossible for more than two of the specified events to happen at the same time, therefore:

$$P(G \cap R \cap Y) = P(G \cap R \cap W) = \dots = P(G \cap R \cap Y \cap W) = 0 \quad (12)$$

Therefore,

$$P(G \cup R \cup Y \cup W) = 4 \times \frac{27}{128} - \binom{4}{2} \times \frac{3}{128} \quad (13)$$

$$= \frac{45}{64} \quad (14)$$

Problem 8.

By inclusion-exclusion,

$$P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2). \quad (15)$$

Since $P(A_1 \cap A_2) \geq 0$,

$$P(A_1 \cup A_2) \leq P(A_1) + P(A_2). \quad (16)$$

Therefore, $P(A_1 \cup \dots \cup A_k) \leq P(A_1) + \dots + P(A_k)$ is true for $k = 2$.

Suppose $P(A_1 \cup \dots \cup A_k) \leq P(A_1) + \dots + P(A_k)$ is true for some natural numbers $k \geq 2$. We get:

$$P(A_1 \cup \dots \cup A_k \cup A_{k+1}) = P(A_1 \cup \dots \cup A_k) + P(A_{k+1}) - P((A_1 \cup \dots \cup A_k) \cap A_{k+1}) \quad (17)$$

Since $P((A_1 \cup \dots \cup A_k) \cap A_{k+1}) \geq 0$,

$$P(A_1 \cup \dots \cup A_k \cup A_{k+1}) \leq P(A_1 \cup \dots \cup A_k) + P(A_{k+1}) \quad (18)$$

Since $P(A_1 \cup \dots \cup A_k) \leq P(A_1) + \dots + P(A_k)$,

$$P(A_1 \cup \dots \cup A_k) + P(A_{k+1}) \leq P(A_1) + \dots + P(A_k) + P(A_{k+1}) \quad (19)$$

and

$$P(A_1 \cup \dots \cup A_k \cup A_{k+1}) \leq P(A_1) + \dots + P(A_k) + P(A_{k+1}). \quad (20)$$

Thus, $P(A_1 \cup \dots \cup A_k \cup A_{k+1}) \leq P(A_1) + \dots + P(A_k) + P(A_{k+1})$ if the Induction Hypothesis is true.

Therefore, by Mathematical Induction, $P(A_1 \cup \dots \cup A_n) \leq P(A_1) + \dots + P(A_n) = \sum_{k=1}^n P(A_k)$.