Math 170A HW2

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Problem 1.

a) Let $U\vec{x} = \vec{b}$ be

$$\begin{bmatrix} u_{1,1} & u_{1,2} & \cdots & u_{1,n} \\ 0 & u_{2,2} & \cdots & u_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & u_{n,n} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

Then, we can perform backward substitution, starting from solving x_n with row n. The exact formula would be $x_i = \frac{1}{u_{i,i}} \left(b_i - \sum_{k=i+1}^{n} x_k \cdot u_{i,k} \right)$ for $i = n, n-1, \ldots, 2, 1$.

b)

solve_upper_tri.m

```
function x = solve_upper_tri(U,b)

x = b;
n = size(b,1);

x(end) = b(end) / U(end, end);

for i = (n-1):-1:1
    for k = n:-1:(i+1)
        x(i) = x(i) - x(k) * U(i, k);
    end

    x(i) = x(i) / U(i,i);
end

end
```

c) For each row $i \in [1, n]$, there are $n - i$ multiplications, $n - i$ subtractions, and 1 division. Therefore, there are a total of $\sum_{i=1}^{n} [2(n-i)+1] = 2n^2 + n - 2\sum_{i=1}^{n} i = 2n^2 + n - 2 \times \frac{(1+n)n}{2} = 2n^2 + n - n - n^2 = n^2$ flops.

Problem 2.

a) Let $LA = A'$. Then, for row i in A', $a'_{i,k} = m_{i,j} \times a_{j,k} + a_{i,k}$ for all k columns in A'. That means for each entry in $i - th$ row of A', it's the original entry in A plus m times the entry in $j - th$ row of the same column in A, which exactly is the elementary transformation.

b) $L^{-1} = I - L$. It means adding $-m$ times row j to row i (i.e. row $i - m\times$ row j).

Problem 3.

a) Let $L_1 \cdot L_2 = L_3$. Then, $l_3^{(i,j)} = \sum\limits_{k=1}^{n} l_1^{(i,k)} \times l_2^{(k,j)}$. Note that for $i < k$, $l_1^{(i,k)} = 0$ and for $k < j$, $l_2^{(k,j)} = 0$. Then, when $i < j$, for all $k \in [1, n]$, $i < k$ or $k < j$ must be true $\Rightarrow l_1^{(i,k)} \times l_2^{(k,j)} = 0 \Rightarrow l_3^{i,j} = 0 \Rightarrow$ all the entries above diagonal are $0 \Rightarrow L_3$ is lower triangular.

b) Let arbitrary lower triangular matrices L_1, L_2, L_3. $L_1 \cdot L_2 = L_3 \Leftrightarrow (L_1 \cdot L_2)^\top = L_3^\top \Leftrightarrow L_2^\top \cdot L_1^\top = L_3^\top \Leftrightarrow U_2 \cdot U_1 = U_3$. Hence, we get any arbitrary upper triangular matrix multiplication would get an upper triangular matrix.

Problem 4.

Proposition 1. *LU factorization is unique.*

Proof. Assume $A = LU = \tilde{L}\tilde{U}$. Then, $LU = \tilde{L}\tilde{U} \Leftrightarrow L = \tilde{L}\tilde{U}U^{-1} \Leftrightarrow \tilde{L}^{-1}L = \tilde{U}U^{-1}$. We can see the left hand side of the equation is a lower triangular matrix and right hand side is an upper triangular matrix. It is only possible when $\tilde{L}^{-1}L = \tilde{U}U^{-1} = nI$ for some constant n, which is a diagonal matrix.

Notice that $L, \tilde{L}$ are both lower triangular matrices with diagonal entries all equals 1 from the property of LU decomposition. Also notice that $(\tilde{L}^{-1})$ the inverse of a lower triangular matrix with diagonal entries all equals 1 is also a lower triangular matrix with diagonal entries all equals 1. Then, when we multiply (L) a lower triangular matrix to $(\tilde{L}^{-1})$ another lower triangular matrix both with diagonal entries equal 1, the result is also a lower triangular matrix with diagonal entries all equals 1. Hence, we can conclude that $\tilde{L}^{-1}L = \tilde{U}U^{-1} = I$ for $n = 1$.

Then, $\tilde{L}^{-1}L = I = L^{-1}L \Leftrightarrow \tilde{L}^{-1}LL^{-1} = L^{-1}LL^{-1} \Leftrightarrow \tilde{L}^{-1} = L^{-1}$. Since inverse matrix is unique, we can conclude that $\tilde{L} = L$. By similar reasoning, we can conclude that $\tilde{U} = U$. Therefore, we have proved LU decomposition is unique. □

Problem 5.

swap.m

```
function B = swap(A, i, j)

B = A;
B(i,:) = A(j,:);
B(j,:) = A(i,:);

end
```

swap.m

max_entry.m

```
function index = max_entry(A, i)

n = size(A,1);
maxi = abs(A(i,i));
index = i;
count = i;

for k = A(i:n, i)'
    if (abs(k) > maxi)
        maxi = abs(k);
        index = count;
    end
    count = count + 1;
end

end
```

max_entry.m