Math 170A HW3

Neo Lee

04/21/2023

Problem 1.

Gauss_solve.m

```matlab
function x = Gauss_solve(A,b)

if (size(A,1) ~= size(A,2))
    error('A is not square')
end

if (size(A,2) ~= size(b,1))
    error("A and b dimensions don't match")
end

n = size(A,1);
L = eye(n);

for i = 1:n

    index = max_entry(A,i);
    A = swap(A, i, index);
    b = swap(b, i, index);

    if A(i,i) == 0
        error('A is not invertible')
    end

    for j = i+1 : n
    L(j,i) = A(j,i)/A(i,i);

        for k = i+1: n
        A(j,k) = A(j,k) - L(j,i)*A(i,k);
        end

    A(j,i) = 0;
    b(j) = b(j) - L(j,i)*b(i);
    end
end

% this is the function you'll write in HW2
% that solves upper triangular systems
x = solve_upper_tri(A,b);
end
```

Figure 1: *Gauss_solve.m*

Problem 2.

$$\begin{bmatrix} 2 & 2 & -4 \\ 1 & 1 & 5 \\ 1 & 3 & 6 \end{bmatrix} \Rightarrow (\textit{row 2 - } 0.5\times \textit{ row 1, row 3 - } 0.5\times \textit{ row 1})$$

$$\Rightarrow \begin{bmatrix} 2 & 2 & -4 \\ 0 & 0 & 7 \\ 0 & 2 & 8 \end{bmatrix} \Rightarrow (\textit{row 2 and row 3 swap})$$

$$\Rightarrow \begin{bmatrix} 2 & 2 & -4 \\ 0 & 2 & 8 \\ 0 & 0 & 7 \end{bmatrix}$$

$$\Rightarrow P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}; \quad L = \begin{bmatrix} 1 & 0 & 0 \\ 0.5 & 1 & 0 \\ 0.5 & 0 & 1 \end{bmatrix}; \quad U = \begin{bmatrix} 2 & 2 & -4 \\ 0 & 2 & 8 \\ 0 & 0 & 7 \end{bmatrix}$$

Problem 3.

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 2 & -1 & 9 \end{bmatrix} = \begin{bmatrix} r_{1,1} & 0 & 0 \\ r_{1,2} & r_{2,2} & 0 \\ r_{1,3} & r_{2,3} & r_{3,3} \end{bmatrix} \begin{bmatrix} r_{1,1} & r_{1,2} & r_{1,3} \\ 0 & r_{2,2} & r_{2,3} \\ 0 & 0 & r_{3,3} \end{bmatrix}$$

$$\begin{aligned} &\Rightarrow r_{1,1} = \sqrt{1} = 1 \\ &\Rightarrow r_{1,1} \times r_{1,2} = 0 \Rightarrow r_{1,2} = 0 \\ &\Rightarrow r_{1,1} \times r_{1,3} = 2 \Rightarrow r_{1,3} = 2 \\ &\Rightarrow r_{1,2}^2 + r_{2,2}^2 = 1 \Rightarrow r_{2,2} = 1 \\ &\Rightarrow r_{1,2} \times r_{1,3} + r_{2,2} \times r_{2,3} = -1 \Rightarrow r_{2,3} = -1 \\ &\Rightarrow r_{1,3}^2 + r_{2,3}^2 + r_{3,3}^2 = 9 \Rightarrow r_{3,3} = 2 \\ &\Rightarrow R = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix} \end{aligned}$$

Since all the diagonal entires are greater than 0, R is positive definite.

Problem 4. $B = X^\top AX = X^\top R^\top RX = (RX)^\top (RX)$. Let $M = RX$, then $B = (RX)^\top (RX) \Leftrightarrow B = M^\top M$. Since R, X are both invertible with determinant $\neq 0$, M is also invertible with determinant $\neq 0$.

$B^\top = (M^\top M)^\top = M^\top M = B$. So B is symmetric.

Then let $\vec{x} \neq \vec{0}$. $\vec{x}^\top B\vec{x} = \vec{x}^\top M^\top M\vec{x} = (M\vec{x})^\top M\vec{x} = M\vec{x} \cdot M\vec{x}$. Let $y = M\vec{x}$. Since M is invertible and $\vec{x} \neq \vec{0}$, $\vec{y} \neq \vec{0}$. Hence, $M\vec{x} \cdot M\vec{x} = \vec{y} \cdot \vec{y} > 0$.

Therefore, B is positive definite.

Problem 5.

a)

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 2 & 3 & 3 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 2 & 3 & 1 \\ 0 & 0 & 0 & 5 & 2 \end{bmatrix} \Rightarrow L = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}; \quad U = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 2 & 3 & 1 \\ 0 & 0 & 0 & 5 & 2 \end{bmatrix}$$

$$\Rightarrow L = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}; \quad U = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 2 & 3 & 1 \\ 0 & 0 & 0 & 5 & 2 \end{bmatrix}$$

$$\Rightarrow L = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}; \quad U = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 5 & 2 \end{bmatrix}$$

$$\Rightarrow L = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}; \quad U = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

L is a lower triangular matrix with bandwidth $s = 1$ (only have non-zero entries on diagonal and subdiagonal), and U is an upper triangular matrix with bandwidth $t = 1$ (only have non-zero entries on diagonal and superdiagonal).

b) L would be a lower triangular matrix with bandwidth s and U would be an upper triangular matrix with bandwidth t.