Math 180B HW5

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PK Exercise 4.1.4

$$\begin{aligned} 0.3\pi_0 + 0.5\pi_1 + 0.5\pi_2 &= \pi_0 \\ 0.2\pi_0 + 0.1\pi_1 + 0.2\pi_2 &= \pi_1 \\ 0.5\pi_0 + 0.4\pi_1 + 0.3\pi_2 &= \pi_2 \\ \pi_0 + \pi_1 + \pi_2 &= 1 \end{aligned}$$

Then, solving the system of equations, we get $\pi_0 = \frac{5}{12}, \pi_1 = \frac{2}{11}, \pi_2 = \frac{53}{132}$. Therefore, the long run cost per period is $2 \times \frac{5}{12} + 5 \times \frac{2}{11} + 3 \times \frac{53}{132} \approx 2.95$.

PK Problem 4.1.3 *assume* $\alpha_i > 0$ *for* $i \in \{1, \dots, 6\}$

$$\begin{aligned} \pi_0\alpha_1 + \pi_1 = \pi_0 &\Rightarrow \pi_1 = (1-\alpha_1)\pi_0 \\ \pi_0\alpha_2 + \pi_2 = \pi_1 &\Rightarrow \pi_2 = (1-\alpha_1-\alpha_2)\pi_0 \\ \pi_0\alpha_3 + \pi_3 = \pi_2 &\Rightarrow \pi_3 = (1-\alpha_1-\alpha_2-\alpha_3)\pi_0 \\ \pi_0\alpha_4 + \pi_4 = \pi_3 &\Rightarrow \pi_4 = (1-\alpha_1-\alpha_2-\alpha_3-\alpha_4)\pi_0 \\ \pi_0\alpha_5 + \pi_5 = \pi_4 &\Rightarrow \pi_5 = (1-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5)\pi_0 \\ \pi_0 + \pi_1 + \pi_2 + \pi_3 + \pi_4 + \pi_5 = 1 &\Rightarrow \pi_0 = \frac{1}{\sum_{i=0}^{5}(1-\alpha_i)} \end{aligned}$$

PK Problem 4.1.5

$$P = \begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \left\Vert \begin{array}{cccc} 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \end{array} \right\Vert \end{array}.$$

$$\begin{aligned} \frac{1}{3}\pi_1 + \frac{1}{2}\pi_3 &= \pi_0 \\ \frac{1}{2}\pi_0 + \pi_2 + \frac{1}{2}\pi_3 &= \pi_1 \\ \frac{1}{3}\pi_1 &= \pi_2 \\ \pi_0 + \pi_1 + \pi_2 + \pi_3 &= 1 \end{aligned}$$

Solving the system of equations, we get $\pi_0 = \frac{1}{4}, \pi_1 = \frac{3}{8}, \pi_2 = \frac{1}{8}, \pi_3 = \frac{1}{4}$.

PK Problem 4.1.11 (a), (b)

(a)

$$\begin{aligned} 0.5\pi_0 + 0.2\pi_1 + 0.3\pi_2 + 0.2\pi_3 &= \pi_1 \\ 0.2\pi_1 + 0.4\pi_2 + 0.4\pi_3 &= \pi_2 \end{aligned}$$

Pluggin in $\pi_1 = \frac{119}{379}$ and $\pi_2 = \frac{81}{379}$ and solving the system of equations, we get $\underline{\pi_0 = \frac{117}{379}}, \pi_1 = \frac{119}{379}, \pi_2 = \frac{81}{379}, \underline{\pi_3 = \frac{62}{379}}$.

(b) $\mu = \pi_2 + \pi_3 = \frac{81}{379} + \frac{62}{379} \approx 0.377$.

PK Problem 4.2.4

(a)

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \left\| \begin{array}{cccc} 0.1 & 0.3 & 0.2 & 0.4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right\| \end{array}.$$

(b) P is regular because for $i = 0$, $P_{00} > 0$, and there is a path $k_1, \ldots, k_r$ for which $P_{ik_1} \cdots P_{k_r j} > 0$ for every state pair i, j.

$$\begin{aligned} 0.1\pi_0 + \pi_1 &= \pi_0 \\ 0.3\pi_0 + \pi_2 &= \pi_1 \\ 0.2\pi_0 + \pi_3 &= \pi_2 \\ 0.4\pi_0 &= \pi_3 \\ \pi_0 + \pi_1 + \pi_2 + \pi_3 &= 1 \end{aligned}$$

Then, solving the system of equations, we get $\underline{\pi_0 = \frac{10}{29}}, \pi_1 = \frac{9}{29}, \pi_2 = \frac{6}{29}, \pi_3 = \frac{4}{29}$.

(c)

$$\pi_0 = \frac{1}{E[\xi]} = \frac{1}{0.1 + 0.3 \times 2 + 0.2 \times 3 + 0.4 \times 4} = \frac{10}{29}.$$

PK Problem 4.2.6

On day n, if the computer is operating, which means at state 1, it has probability q of remaining "up" at state 1 and probability p of failing and going to state 0.

On the other hand, on day n, if the computer is down, which means at state 0, it has probability β of being repaired within a day and goes to state 1, and probability $1 - \beta = \alpha$ of remaining "down" at state 0.

$$\begin{aligned} \pi_0\beta + \pi_1 q &= \pi_1 \\ \pi_0 + \pi_1 &= 1. \end{aligned}$$

Then, by solving the system of equations, we get

$$\begin{aligned} (1 - \pi_1)\beta + \pi_1 q &= \pi_1 \\ \beta + \pi_1(q - \beta) &= \pi_1 \\ \beta &= \pi_1(1 - q + \beta) \\ \pi_1 &= \frac{\beta}{1 - q + \beta} \\ \pi_1 &= \frac{\beta}{p + \beta}. \end{aligned}$$