

Math 180A HW7

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Problem 1.

(a) 0.3.

(b) $\frac{7}{30}$.

Problem 2.

(a)

$$f_X(x) = \int_0^1 \frac{12}{7}(xy + y^2)dy \quad (1)$$

$$= \frac{12}{7} \left[\frac{1}{2}xy^2 + \frac{1}{3}y^3 \right]_0^1 \quad (2)$$

$$= \frac{12}{7} \left(\frac{x}{2} + \frac{1}{3} \right) \quad (3)$$

$$= \frac{6x + 4}{7} \quad (4)$$

for $x \in [0, 1]$ and 0 otherwise.

(b)

$$f_Y(y) = \int_0^1 \frac{12}{7}(xy + y^2)dx \quad (5)$$

$$= \frac{12}{7} \left[\frac{1}{2}x^2y + xy^2 \right]_0^1 \quad (6)$$

$$= \frac{12}{7} \left(\frac{y}{2} + y^2 \right) \quad (7)$$

$$= \frac{6y + 12y^2}{7} \quad (8)$$

for $y \in [0, 1]$ and 0 otherwise.

(c)

$$P(X < Y) = \int_0^1 \int_0^y \frac{12}{7} (xy + y^2) dx dy \quad (9)$$

$$= \frac{12}{7} \int_0^1 \left[\frac{1}{2} x^2 y + xy^2 \right]_{x=0}^{x=y} dy \quad (10)$$

$$= \frac{12}{7} \int_0^1 \frac{1}{2} y^3 + y^3 dy \quad (11)$$

$$= \frac{12}{7} \left[\frac{1}{8} y^4 + \frac{1}{4} y^4 \right]_0^1 \quad (12)$$

$$= \frac{12}{7} \left(\frac{3}{8} \right) \quad (13)$$

$$= \frac{9}{14}. \quad (14)$$

Problem 3.

$$P(X = x, Y = y) = P(X = x)P(Y = y) \quad (15)$$

$$= (p(1-p)^{x-1}) (r(1-r)^{y-1}). \quad (16)$$

Then,

$$P(X < Y) = P(X \in [1, \infty))P(Y > X) \quad (17)$$

$$= \sum_{n=1}^{\infty} p(1-p)^{n-1}(1-r)^n. \quad (18)$$

Problem 4.

(a)

$$\int_0^{\pi} \int_0^{\pi} c(1 - \cos(x)\cos(y)) dx dy = 1 \quad (19)$$

$$c \int_0^{\pi} \int_0^{\pi} dx dy - c \int_0^{\pi} \int_0^{\pi} \cos(x)\cos(y) dx dy = 1 \quad (20)$$

$$c\pi^2 - c \int_0^{\pi} [\sin(x)\cos(y)]_{x=0}^{x=\pi} dy = 1 \quad (21)$$

$$c\pi^2 = 1 \quad (22)$$

$$c = \frac{1}{\pi^2}. \quad (23)$$

(b)

$$f_X(x) = \frac{1}{\pi^2} \int_0^{\pi} 1 - \cos(x)\cos(y) dy \quad (24)$$

$$= \frac{1}{\pi^2} \left(\pi - [\cos(x)\sin(y)]_{y=0}^{y=\pi} \right) \quad (25)$$

$$= \frac{1}{\pi} \quad (26)$$

for $x \in [0, \pi]$ and 0 otherwise. Similarly,

$$f_Y(y) = \frac{1}{\pi} \quad (27)$$

for $y \in [0, \pi]$ and 0 otherwise. The probability distribution is uniform.

(c)

$$f(0,0) = \frac{1}{\pi^2} (1 - \cos(0)\cos(0)) \quad (28)$$

$$= 0. \quad (29)$$

On the other hand,

$$f_X(0)f_Y(0) = \left(\frac{1}{\pi}\right) \left(\frac{1}{\pi}\right) \quad (30)$$

$$= \frac{1}{\pi^2} \quad (31)$$

$$\neq f(0,0) \quad (32)$$

Hence, X and Y are not independent.

Problem 5.

- (a) We need to compute $F_T(t) = P(T \leq t) = 1 - P(T > t) = 1 - P(\min(X, Y) > t) = 1 - P(X, Y > t)$. Assuming the unit to be 1 year, we have, for $x, y \geq 0$,

$$f_X(x) = e^{-x} \quad (33)$$

and

$$f_Y(y) = \frac{1}{2}e^{-\frac{y}{2}} \quad (34)$$

and the joint density is

$$f(x, y) = \frac{1}{2}e^{-x-\frac{y}{2}}. \quad (35)$$

Hence,

$$F_T(t) = 1 - \int_t^\infty \int_t^\infty f(x, y) dx dy = 1 - \int_t^\infty \int_t^\infty \frac{1}{2}e^{-x-\frac{y}{2}} dx dy \quad (36)$$

$$= 1 - \frac{1}{2} \int_t^\infty e^{-x} dx \int_t^\infty e^{-\frac{y}{2}} dy \quad (37)$$

$$= 1 - \frac{1}{2}e^{-t} \cdot 2e^{-\frac{t}{2}} \quad (38)$$

$$= 1 - e^{-\frac{3t}{2}}. \quad (39)$$

- (b) For $t \in [0, \infty)$,

$$f_T(t) = \frac{d}{dt} \left(1 - e^{-\frac{3t}{2}}\right) \quad (40)$$

$$= \frac{3}{2}e^{-\frac{3t}{2}} \quad (41)$$

and 0 otherwise.

(c)

$$E[T] = \int_0^\infty t \cdot \frac{3}{2}e^{-\frac{3t}{2}} dt \quad (42)$$

$$= - \left[te^{-\frac{3t}{2}} \right]_0^\infty + \int_0^\infty e^{-\frac{3t}{2}} dt \quad (43)$$

$$= 0 + \frac{-2}{3} \lim_{z \rightarrow \infty} \left[e^{-\frac{3t}{2}} \right]_0^z \quad (44)$$

$$= \frac{2}{3}. \quad (45)$$

Problem 6. The convolution formula tells us

$$P_{X+Y}(z) = \sum_X P_X(x)P_Y(z-x) \quad (46)$$

$$= P(X=0)P(Y=z-0) + P(X=1)P(Y=z-1) \quad (47)$$

$$= (1-p)P(Y=z) + pP(Y=z-1). \quad (48)$$

Hence,

$$P_{X+Y}(z) = \begin{cases} (1-p)(1-r) & z=0, \\ p(1-r) + r(1-p) & z=1, \\ pr & z=2, \\ 0 & \text{otherwise.} \end{cases} \quad (49)$$

Problem 7. *Convolution Approach*

We know $f_Y(y) = 1$ for $y \in (1, 2)$ and 0 otherwise. For $z \leq 1$, $f_{X+Y}(z) = 0$. For $z \in [1, 2]$,

$$f_{X+Y}(z) = \int_1^z f_Y(y)f_X(z-y)dy \quad (50)$$

$$= \int_1^z 2(z-y)dy \quad (51)$$

$$= \int_1^z 2z - 2ydy \quad (52)$$

$$= [2yz]_{y=1}^{y=z} - [y^2]_1^z \quad (53)$$

$$= 2z^2 - 2z - z^2 + 1 \quad (54)$$

$$= z^2 - 2z + 1. \quad (55)$$

For $z \in [2, 3]$,

$$f_{X+Y}(z) = \int_{z-2}^1 f_Y(z-x)f_X(x)dx \quad (56)$$

$$= \int_{z-2}^1 2xdx \quad (57)$$

$$= [x^2]_{z-2}^1 \quad (58)$$

$$= -z^2 + 4z - 3. \quad (59)$$

For $z \geq 3$, $f_{X+Y}(z) = 0$.

CDF Approach

Since X and Y are independent, $f(x, y) = f_X(x)f_Y(y) = 2x$. For $z \leq 1$, $f_{X+Y}(z) = 0$. For $z \in [1, 2]$,

$$P(X+Y \leq Z) = \int_1^z \int_0^{z-y} 2xdxdy \quad (60)$$

$$= \int_1^z [x^2]_0^{z-y} dy \quad (61)$$

$$= \int_1^z z^2 - 2zy + y^2 dy \quad (62)$$

$$= \left[z^2 y - zy^2 + \frac{1}{3}y^3 \right]_1^z \quad (63)$$

$$= z^3 - z^3 + \frac{1}{3}z^3 - z^2 + z - \frac{1}{3} \quad (64)$$

$$= \frac{1}{3}z^3 - z^2 + z - \frac{1}{3}. \quad (65)$$

Hence, for $z \in [1, 2]$,

$$f_{X+Y}(z) = \frac{d}{dz} \left(\frac{1}{3}z^3 - z^2 + z - \frac{1}{3} \right) \quad (66)$$

$$= z^2 - 2z + 1. \quad (67)$$

For $z \in [2, 3]$,

$$P(X + Y \leq Z) = \int_0^1 \int_1^2 2xdydx - \int_{z-2}^1 \int_{z-x}^2 2xdydx \quad (68)$$

$$= \int_0^1 [2xy]_{y=1}^{y=2} dx - \int_{z-2}^1 [2xy]_{y=z-x}^{y=2} dx \quad (69)$$

$$= \int_0^1 2xdx - \int_{z-2}^1 2x^2 + 4x - 2zx dx \quad (70)$$

$$= 1 - \left[\frac{2}{3}x^3 \right]_{z-2}^1 - [2x^2]_{z-2}^1 + [zx^2]_{z-2}^1 \quad (71)$$

$$= -\frac{1}{3}z^3 + 2z^2 - 3z + 1. \quad (72)$$

Hence, for $z \in [2, 3]$,

$$f_{X+Y}(z) = \frac{d}{dz} \left(-\frac{1}{3}z^3 + 2z^2 - 3z + 1 \right) \quad (73)$$

$$= -z^2 + 4z - 3. \quad (74)$$

For $z \geq 3$, $f_{X+Y}(z) = 0$.

Problem 8.

$$p_{x+y}(z) = \sum_{x \in X} P(X = x)P(Y = z - x) \quad (75)$$

$$= \sum_{i=0}^{\infty} \left(\frac{\lambda^i}{i!} e^{-\lambda} \right) \left(\frac{\mu^{z-i}}{(z-i)!} e^{-\mu} \right) \quad (76)$$

$$= e^{-(\lambda+\mu)} \sum_{i=0}^{\infty} \left(\frac{\lambda^i}{i!} \right) \left(\frac{\mu^{z-i}}{(z-i)!} \right) \quad (77)$$

$$= e^{-(\lambda+\mu)} \sum_{i=0}^{\infty} \frac{1}{i!(z-i)!} (\lambda^i)(\mu^{z-i}) \quad (78)$$

$$= e^{-(\lambda+\mu)} \frac{1}{z!} \sum_{i=0}^{\infty} \frac{z!}{i!(z-i)!} (\lambda^i)(\mu^{z-i}) \quad (79)$$

$$= e^{-(\lambda+\mu)} \frac{1}{z!} \sum_{i=0}^{\infty} \binom{z}{i} (\lambda^i)(\mu^{z-i}) \quad (80)$$

$$= \frac{(\lambda + \mu)^z}{z!} e^{-(\lambda+\mu)} \quad (81)$$

$$= \text{Poisson}(\lambda + \mu). \quad (82)$$

Problem 9. $X = \sigma_1 Z + \mu_1$ and $Y = \sigma_2 Z + \mu_2 \Rightarrow X + Y = (\sigma_1 + \sigma_2)Z + (\mu_1 + \mu_2) \Rightarrow N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$.

$$f_{X+Y}(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z-x) dx \quad (83)$$

$$= \int_{-\infty}^{\infty} \left(\frac{1}{\sigma_1 \sqrt{2\pi}} e^{-\frac{(x-\mu_1)^2}{2\sigma_1^2}} \right) \left(\frac{1}{\sigma_2 \sqrt{2\pi}} e^{-\frac{(z-x-\mu_2)^2}{2\sigma_2^2}} \right) dx \quad (84)$$

$$= \frac{1}{\sigma_1 \sigma_2 \sqrt{2\pi} \sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(x-\mu_1)^2}{2\sigma_1^2} - \frac{(z-x-\mu_2)^2}{2\sigma_2^2}} dx \quad (85)$$

$$= \frac{1}{\sigma_1 \sigma_2 \sqrt{2\pi} \sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{\sigma_2^2(x-\mu_1)^2 + \sigma_1^2(z-x-\mu_2)^2}{2\sigma_1^2 \sigma_2^2}} dx \quad (86)$$

$$= \frac{1}{\sigma_1 \sigma_2 \sqrt{2\pi} \sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{\sigma_2^2(x^2 - 2x\mu_1 + \mu_1^2) + \sigma_1^2(z^2 + x^2 + \mu_2^2 - 2zx + 2x\mu_2 - 2z\mu_2)}{2\sigma_1^2 \sigma_2^2}} dx \quad (87)$$

$$= \frac{1}{\sigma_1 \sigma_2 \sqrt{2\pi} \sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2(\sigma_1^2 + \sigma_2^2) - 2x(\sigma_1^2(z-\mu_2) + \sigma_2^2\mu_1) + \sigma_1^2(z^2 + \mu_2^2 - 2z\mu_2) + \sigma_2^2\mu_1^2}{2\sigma_1^2 \sigma_2^2}} dx. \quad (88)$$

Let $\sigma_z = \sqrt{\sigma_1^2 + \sigma_2^2}$,

$$f_{X+Y}(z) = \frac{1}{\sqrt{2\pi}\sigma_z} \frac{1}{\sqrt{2\pi}\frac{\sigma_1\sigma_2}{\sigma_z}} \int_{-\infty}^{\infty} e^{-\frac{x^2 - 2x\frac{\sigma_1^2(z-\mu_2) + \sigma_2^2\mu_1}{\sigma_z^2} + \frac{\sigma_1^2(z^2 + \mu_2^2 - 2z\mu_2) + \sigma_2^2\mu_1^2}{\sigma_z^2}}{2\frac{\sigma_1\sigma_2}{\sigma_z}^2}} dx. \quad (89)$$

Interlude:

$$- \frac{x^2 - 2x\frac{\sigma_1^2(z-\mu_2) + \sigma_2^2\mu_1}{\sigma_z^2} + \frac{\sigma_1^2(z^2 + \mu_2^2 - 2z\mu_2) + \sigma_2^2\mu_1^2}{\sigma_z^2}}{2\frac{\sigma_1\sigma_2}{\sigma_z}^2} \quad (90)$$

$$= - \frac{\left(x - \frac{\sigma_1^2(z-\mu_2) + \sigma_2^2\mu_1}{\sigma_z^2}\right)^2 - \left(\frac{\sigma_1^2(z-\mu_2) + \sigma_2^2\mu_1}{\sigma_z^2}\right)^2 + \frac{\sigma_1^2(z-\mu_2)^2 + \sigma_2^2\mu_1^2}{\sigma_z^2}}{2\frac{\sigma_1\sigma_2}{\sigma_z}^2} \quad (91)$$

$$= - \left[\frac{\sigma_z^2(\sigma_1^2(z-\mu_2)^2 + \sigma_2^2\mu_1^2) - (\sigma_1^2(z-\mu_2) + \sigma_2^2\mu_1)^2}{2\sigma_z^2(\sigma_1\sigma_2)^2} \right] - \left[\frac{\left(x - \frac{\sigma_1^2(z-\mu_2) + \sigma_2^2\mu_1}{\sigma_z^2}\right)^2}{2\left(\frac{\sigma_1\sigma_2}{\sigma_z}\right)^2} \right] \quad (92)$$

$$= - \left[\frac{(z - (\mu_1 + \mu_2))^2}{2\sigma_z^2} \right] - \left[\frac{\left(x - \frac{\sigma_1^2(z-\mu_2) + \sigma_2^2\mu_1}{\sigma_z^2}\right)^2}{2\left(\frac{\sigma_1\sigma_2}{\sigma_z}\right)^2} \right] \quad (93)$$

Hence,

$$f_{X+Y}(z) = \frac{1}{\sqrt{2\pi}\sigma_z} \frac{1}{\sqrt{2\pi}\frac{\sigma_1\sigma_2}{\sigma_z}} \int_{-\infty}^{\infty} e^{-\left[\frac{(z-(\mu_1+\mu_2))^2}{2\sigma_z^2}\right] - \left[\frac{\left(x - \frac{\sigma_1^2(z-\mu_2) + \sigma_2^2\mu_1}{\sigma_z^2}\right)^2}{2\left(\frac{\sigma_1\sigma_2}{\sigma_z}\right)^2}\right]} dx \quad (94)$$

$$= \frac{1}{\sqrt{2\pi}\sigma_z} e^{-\left[\frac{(z-(\mu_1+\mu_2))^2}{2\sigma_z^2}\right]} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\frac{\sigma_1\sigma_2}{\sigma_z}} e^{-\left[\frac{\left(x - \frac{\sigma_1^2(z-\mu_2) + \sigma_2^2\mu_1}{\sigma_z^2}\right)^2}{2\left(\frac{\sigma_1\sigma_2}{\sigma_z}\right)^2}\right]} dx \quad (95)$$

$$= \frac{1}{\sqrt{2\pi}\sigma_z} e^{-\left[\frac{(z-(\mu_1+\mu_2))^2}{2\sigma_z^2}\right]} \quad (\text{the right hand side term evaluates to 1}) \quad (96)$$

$$= N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2). \quad (97)$$