Math 180A HW4

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Problem 1.

(a) $E[3X+2] = 5$

(b) $E[X^2] = Var(X) + (E[X])^2 = 4$

(c) $E[(2X+1)^2] = E[4X^2+4X+1] = 16+4+1 = 21$

(d) $Var(-2X+7) = E[(-2X+7)^2] - (E[-2X+7])^2 = E[4X^2-28X+49] - (-2+7)^2 = 16-28+49-25 = 12$

Problem 2.

(a)

$$p_x(k) = \begin{cases} \frac{1}{2} \times \frac{1}{4} + \frac{1}{2} \times \frac{1}{6} = \frac{5}{24}, & 1 \leq k \leq 4 \\ \frac{1}{12}, & 5 \leq k \leq 6 \\ 0, & otherwise \end{cases} ; \forall k \in \mathbb{Z}. \tag{1}$$

(b) $E[X] = E[X \leq 4] + E[5 \leq X \leq 6] = \frac{5}{24} \times (1+2+3+4) + \frac{1}{12} \times (5+6) = 3.$

(c) $Var(X) = E[X^2] - (E[X])^2 = \frac{5}{24} \times (1+4+9+16) + \frac{1}{12} \times (25+36) = \frac{34}{3} - 3^2 = \frac{7}{3}.$

Problem 3.

(a) Let X be random variable of number of correct questions.

$$\begin{aligned} P(X \geq 3) &= P(X=3) + P(x=4) & (2) \\ &= \binom{4}{3} \times 0.8^3 \times 0.2 + 0.8^4 & (3) \\ &= 0.8192. & (4) \end{aligned}$$

(b) $P = \binom{3}{2} \times 0.8^2 \times 0.2 + 0.8^3 = 0.896.$

Problem 4.

(a) Let X be the number of winning games. So the experiment is a $Binomial(20, p)$.

$$P(X \geq 12) = \sum_{k=12}^{20} \binom{20}{k} p^k (1-p)^{20-k} \tag{5}$$

(b) Let A be at leat one wins.

$$\begin{aligned} P(A) &= 1 - P(A^c) & (6) \\ &= 1 - \left(\sum_{k=0}^{11} \binom{20}{k} p^k (1-p)^{20-k} \right)^{10} & (7) \end{aligned}$$

Problem 5.

(a) It is $Binomial(9, \frac{3}{7})$.

$$
\begin{aligned}
P(X \geq 1) &= 1 - P(X = 0) && (8)\\
&= 1 - \left(\frac{4}{7}\right)^9 && (9)\\
&\approx 0.9935 && (10)
\end{aligned}
$$

$$
\begin{aligned}
P(X \leq 5) &= 1 - \sum_{k=6}^{9} \binom{9}{k} \left(\frac{3}{7}\right)^k \left(\frac{4}{7}\right)^{9-k} && (11)\\
&\approx 0.8653 && (12)
\end{aligned}
$$

(b) It is $Geometric(\frac{4}{7})$.

$$
\begin{aligned}
P(X \leq 9) &= \sum_{k=1}^{9} \left(\frac{3}{7}\right) \left(\frac{4}{7}\right)^{k-1} && (13)\\
&\approx 0.9935 && (14)
\end{aligned}
$$

(c) Yes, because $P(X \leq 9)$ can also be written as $1 - P(X > 9) = 1 - \left(\frac{4}{7}\right)^9$.

Problem 6.

(a) X is $Geometric(1/5)$. $p_x(k) = (1/5)(4/5)^{k-1}\ \forall k \in \mathbb{Z}^+$

(b) $E(X) = \frac{1}{\frac{1}{5}} = 5$.

$Var(X) = \frac{1-\frac{1}{5}}{\left(\frac{1}{5}\right)^2} = 20$.

$\sigma = \sqrt{Var(X)} = \sqrt{20}$.

Problem 7. $Binomial(n, p)$ can be represented by a collection of Indicator Random Variables for which $I_i = I_{\text{ith trial succeds}}$. Thus, $X = I_1 + \cdots + I_p$. Note $P(I_1 = 1) = \cdots = P(I_p = 1) = p$. Therefore,

$$
\begin{aligned}
E[X] &= np, && (15)\\
Var(X) &= \sum_{k=1}^{n} Var(I_k) && (16)\\
&= \sum_{k=1}^{n} \left(E[I_k^2] - (E[I_k])^2\right) && (17)\\
&= n(p - p^2) && (18)\\
&= np(1 - p) && (19)
\end{aligned}
$$