MATH 105 Notes
Professor: Khalilah Beal-Uribe

Neo Lee

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Chapter 1

First chapter

1.1 Lecture 1

Definition 1.1.1: Norm

Given a vector space V over a subfield $\mathbb{F}$ of $\mathbb{C}$, a norm of V is a real-valued function $p : V \to \mathbb{R}$ satisfying the following properties:

1. **Triangle inequality:** $p(v + w) \leq p(v) + p(w)$,
2. **Absolute homogeneity:** $p(\alpha v) = |\alpha| p(v)$,
3. **Positive definiteness:** $p(v) \geq 0$ and $p(v) = 0$ iff $v = 0$.

Note:

Usually, we denote the norm of v by $\|v\|$, and for clarity of the underlying vector space, we may write $\|v\|_V$.

Proposition 1.1.1 Normed space is a metric space

Let V be a normed space. Then the function $d : V \times V \to \mathbb{R}$ defined by $d(v, w) = p(v - w) = \|v - w\|$ is a metric on V.

Definition 1.1.2: Isomorphism in vector spaces

A function $f : V \to W$ between two vector spaces V and W over the same field $\mathbb{F}$ is called an isomorphism if it is bijective and linear. If such an isomorphism exists, we say that the two vector spaces are isomorphic.

Definition 1.1.3: Homeomorphism

A function $f : X \to Y$ between two topological spaces X and Y is called a homeomorphism if it satisfies the following properties:

1. f is bijective,
2. f is continuous,
3. f^{-1} is continuous.

If such a homeomorphism exists, we say that the two topological spaces are homeomorphic.

Note:

In general, isomorphism does not imply homeomorphism. However, in certain cases, they are equivalent, which will be discussed in details later.

Definition 1.1.4: Operator norm

Let $T : V \to W$ be a linear oepration between normed spaces. Denote $\|\cdot\|_V$ and $\|\cdot\|_W$ be the norms in V and W respectively. The operator norm of A is defined by

$$\begin{aligned} \|T\| &= \sup\left\{\frac{\|Tv\|_W}{\|v\|_V} : v \neq 0, v \in V\right\} \\ &= \inf\left\{c \geq 0 : \|Tv\|_W \leq c\|v\|_V, \forall v \in V\right\} \end{aligned}$$

Note:

We say that T is bounded if $\|T\| < \infty$.

1.2 Lecture 2

Theorem 1.2.1 Multiplication of matrices are composition of linear maps

$T_A \circ T_b = T_{AB}$.

Theorem 1.2.2 Bounded operator is equivalent to continuous

Let $T : V \to W$ be a linear transformation from one normed space to another. The following are equivalent:

1. $\|T\| < \infty$,
2. T is uniformly continuous,
3. T is continuous,
4. T is continuous at 0.

Proof: We show that (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) $\Longrightarrow$ (4) $\Longrightarrow$ (1).

- (1) $\implies$ (2): Let $M = \|T\| < \infty$, and let $\delta = \frac{\epsilon}{M}$. Then for any $x, y \in V$ such that $\|x - y\| < \delta$, we have

$$\begin{aligned}\|Tx - Ty\| &= \|T(x-y)\| \\ &\leq M\|x-y\| \\ &< M\delta \\ &= \epsilon.\end{aligned}$$

  Hence, T is uniformly continuous.

- (2) $\implies$ (3): Trivial. Uniformly continuous automatically implies continuous.

- (3) $\implies$ (4): Trivial. T is continuous over the whole domain implies that it is continuous at any point in the domain, including 0.

- (4) $\implies$ (1): Let $\epsilon = 1$, then there exists $\delta > 0$ such that $\|x\| < \delta$ implies $\|Tx\| < 1$. Then for any $v \neq 0$, define $v' = \frac{\delta}{2\|v\|}$, then $\|v'\| < \delta$ and hence $\|Tv'\| < 1$. Then we have

$$\begin{aligned}\|Tv'\| &< 1 \\ \left\|T\left(\frac{\delta}{2\|v\|}v\right)\right\| &< 1 \\ \frac{\delta}{2\|v\|}\|Tv\| &< 1 \\ \|Tv\| &< \frac{2}{\delta}\|v\|.\end{aligned}$$

  Then, from our *definition 1.1.4* of operator norm, we have $\|T\| < \frac{2}{\delta}$ and hence $\|T\| < \infty$.

😄

Theorem 1.2.3 Linear map from finite-dimensional Euclidean space to normed space is continuous

Let $T : \mathbb{R}^n \to W$, where T is linear and W is a normed space. Then

1. T is continuous,
2. if T is an isomorphism, then T is a homeomorphism.

Corollary 1.2.1 Linear maps from finite-dimensional normed space to normed space are continuous

All linear maps from finite-dimensional normed space to another normed space are continuous, and all isomorphisms from finite-dimensional space to normed space are homeomorphisms.

In particular, if a finite-dimensional vector spaces is equipped with two norms, then the identity map between them is a homeomorphism. For example, $T : \mathcal{M} \to \mathcal{L}$ is a homeomorphism.

Proof: Let V be a n-dimensional normed space and W be another normed space, and $T : V \to W$. Then, there exists an isoemorphism $S : V \to \mathbb{R}^n$. *Theorem 1.2.2* gaurentees that S and S^{-1} are homeomorphisms. Then, $T \circ S : \mathbb{R}^n \to W$ is also a continuous linear map guaranteed by *Theorem 1.2.2*. Then,

$$T = (T \circ S) \circ S^{-1}$$

is also a continuous linear because it is a composition of continuous linear maps. Hence, T is continuous.

Now, if $T : V \to W$ is an isomorphism where V is a finite-dimensional normed space. Then, W

is also a finite-dimensional normed space. Then, T is continuous by the above argument. Then, $T^{-1} : W \to V$ is a linear map from a finite-dimensional normed space, hence also continuous. Therefore, T is a homeomorphism.

Finally, let V be a finite-dimenisonal vector space equipped with two norms $\|\cdot\|_1$ and $\|\cdot\|_2$. Then, the identity map $I : V \to V$ is an isomorphism between the two finite-dimensional normed spaces. Then, I is a homeomorphism by the above argument. 😊

Chapter 2

Starting a new chapter

2.1 Demo of commands

Definition 2.1.1: Some defintion

yap

Question 1: Some question

yap

Solution

Some proof: yap ☺

Note:

Some note

Theorem 2.1.1 Some theorem

yap

Wrong Concept 2.1.1: Some wrong concept

yap

Lemma 2.1.1 Some lemma

yap

Proposition 2.1.1 Some proposition

yap

Example 2.1.1 (Some example)

yap

Claim 2.1.1 Some claim

yap

Corollary 2.1.1 Some corollary

yap

Some unlabeled theorem

This is a new paragraph

Algorithm 1: Some algorithm

```
Input: input
Output: output
/* This is a comment */
This is first line ;                    // This is also a comment
if x > 5 then
|   do nothing
else if x < 5 then
|   do nothing
else
|   do nothing
end
while x == 5 do
|   still do nothing
end
foreach x = 1 : 5 do
|   do nothing
end
return return nothing
```