Math 180B HW1

Neo Lee

04/14/2023

Problem 1.

(a)

$$\begin{aligned}
E[e^{tX}] &= \int_{-\infty}^{\infty} e^{tx} \cdot \frac{\lambda}{\Gamma(\alpha)}(\lambda x)^{\alpha-1} e^{-\lambda x} dx \\
&= \int_{0}^{\infty} e^{tx} \cdot \frac{\lambda}{\Gamma(\alpha)}(\lambda x)^{\alpha-1} e^{-\lambda x} dx \qquad (\because x > 0 \textit{ for Gamma Distribution}) \\
&= \int_{0}^{\infty} \frac{\lambda}{\Gamma(\alpha)}(\lambda x)^{\alpha-1} e^{-(\lambda - t)x} dx.
\end{aligned}$$

Let $u = (\lambda - t)x$ for u-substitution. Then $\frac{du}{\lambda - t} = dx$, and 1) $u \to \infty$ when $t < \lambda$ and $x \to \infty$; 2) $u \to -\infty$ when $t > \lambda$ and $x \to \infty$; 3) $u \to 0$ when $t = \lambda$.
Hence, when $t < \lambda$,

$$\begin{aligned}
E[e^{tX}] &= \int_{0}^{\infty} \frac{\lambda}{\Gamma(\alpha)}(\lambda \frac{u}{\lambda - t})^{\alpha-1} e^{-u} \frac{u}{\lambda - t} \\
&= \left(\frac{\lambda}{\lambda - t}\right)^{\alpha} \cdot \frac{1}{\Gamma(\alpha)} \int_{0}^{\infty} u^{\alpha-1} e^{-u} du \qquad (\textit{notice} \int_{0}^{\infty} u^{\alpha-1} e^{-u} du = \Gamma(\alpha)) \\
&= \left(\frac{\lambda}{\lambda - t}\right)^{\alpha}.
\end{aligned}$$

When $t = \lambda$, $\lambda - t = 0$, and

$$\begin{aligned}
E[e^{tX}] &= \left(\frac{\lambda}{\lambda - t}\right)^{\alpha} \cdot \frac{1}{\Gamma(\alpha)} \int_{0}^{0} u^{\alpha-1} e^{-u} du \\
&= \left(\frac{\lambda}{\lambda - t}\right)^{\alpha} \cdot \frac{1}{\Gamma(\alpha)} \\
&= \lim_{t \to \lambda} \left(\frac{\lambda}{\lambda - t}\right)^{\alpha} \cdot \frac{1}{\Gamma(\alpha)} \\
&= \infty.
\end{aligned}$$

When $t > \lambda$,

$$\begin{aligned}
E[e^{tX}] &= \left(\frac{\lambda}{\lambda - t}\right)^{\alpha} \cdot \frac{1}{\Gamma(\alpha)} \int_{0}^{-\infty} u^{\alpha-1} e^{-u} du \qquad (\textit{notice} \int_{0}^{-\infty} u^{\alpha-1} e^{-u} du \textit{ diverges}) \\
&= \infty.
\end{aligned}$$

Hence, we have reached the moment generating function of Gamma Distribution for all cases.

(b) Since we only care about $t = 0$ and $\lambda > 0$, we can consider $M_X(t) = \left(\frac{\lambda}{\lambda - t}\right)^\alpha$ when $t < \lambda$ only.

$$\begin{aligned}
M_X'(t) &= \alpha \left(\frac{\lambda}{\lambda - t}\right)^{\alpha-1} \left(\frac{-\lambda}{(\lambda - t)^2}\right)(-1) \\
&= \alpha \left(\frac{\lambda}{\lambda - t}\right)^{\alpha} \left(\frac{1}{\lambda - t}\right) \\
&= \alpha \cdot \lambda^\alpha \cdot (\lambda - t)^{-\alpha-1} \\
\mu &= M_X'(t = 0) = \frac{\alpha}{\lambda}. \\
M_X''(t) &= \alpha \cdot \lambda^\alpha \cdot (-\alpha - 1)(\lambda - t)^{-\alpha-2} \cdot (-1) \\
&= \alpha \cdot \lambda^\alpha \cdot (\alpha + 1)(\lambda - t)^{-\alpha-2} \\
\sigma^2 &= M_X''(t = 0) - M_X'(t = 0) \\
&= \frac{\alpha(\alpha + 1)}{\lambda^2} - \left(\frac{\alpha}{\lambda}\right)^2 \\
&= \frac{\alpha}{\lambda^2}.
\end{aligned}$$

Problem 2.

$$\vec{\mu} = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}; \quad \Sigma = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}; \quad \vec{A_1} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}; \quad \vec{A_2} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Then

$$\begin{aligned}
Y_1 &\sim N(\vec{A_1} \cdot \vec{\mu}, A_1^\top \sum A_1) \\
&= N(\mu_1 + 2\mu_2, \sigma_1^2 + 4\rho\sigma_1\sigma_2 + 4\sigma_2^2), \\
Y_2 &\sim N(\vec{A_2} \cdot \vec{\mu}, A_2^\top \sum A_2) \\
&= N(\mu_1 - \mu_2, \sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2).
\end{aligned}$$

The *pdf* of a normal random variable $N(m, s)$ for all $x \in \mathbb{R}$ is simply

$$f(x) = \frac{1}{\sqrt{2\pi s}} e^{-\frac{(x-m)^2}{2s}}.$$

Since we have already found the mean and variance of Y_1 and Y_2, and we have determined they are normal random variable, what's left is only plugging into the formula, which we will omit here for easier readability as a whole.

PK Problem 2.1.8 Let F be first ball, S be second ball.

$$\begin{aligned}
P(F = R|S = R) &= \frac{P(S = R|F = R)P(F = R)}{P(S = R|F = R)P(F = R) + P(S = R|F = G)P(F = G)} \\
&= \frac{\frac{2}{3} \cdot \frac{1}{2}}{\frac{2}{3} \cdot \frac{1}{2} + \frac{1}{3} \cdot \frac{1}{2}} \\
&= \frac{2}{3}.
\end{aligned}$$

PK Problem 2.1.9

$$p_N(n) = \begin{cases} \frac{e^{-1}}{n!}, & n \in \mathbb{Z}^{\geq} \\ 0, & \text{otherwise}, \end{cases}$$

$$p_{X|N=n}(x) = \begin{cases} \frac{1}{n+2}, & x \in [0, n+1] \\ 0, & \text{otherwise}. \end{cases}$$

Then,

$$
\begin{aligned}
p_X(x) &= \sum_{n=x-1}^{\infty} p_{X|N=n}(x) \cdot p_N(n) \\
&= \sum_{n=x-1}^{\infty} \frac{1}{n+2} \cdot \frac{e^{-1}}{n!} \\
&= \frac{1}{e} \sum_{n=x-1}^{\infty} \frac{n+1}{(n+2)!} \\
&= \frac{1}{e} \sum_{n=x}^{\infty} \frac{n}{(n+1)!} \\
&= \frac{1}{e} \sum_{n=x}^{\infty} \frac{n+1-1}{(n+1)!} \\
&= \frac{1}{e} \sum_{n=x}^{\infty} \left(\frac{n+1}{(n+1)!} - \frac{1}{(n+1)!} \right) \\
&= \frac{1}{e} \left[\sum_{n=x}^{\infty} \frac{n+1}{(n+1)!} - \sum_{n=x}^{\infty} \frac{1}{(n+1)!} \right] \\
&= \frac{1}{e} \left[\sum_{n=x}^{\infty} \frac{1}{n!} - \sum_{n=x}^{\infty} \frac{1}{(n+1)!} \right] \\
&= \frac{1}{e} \left[\left(\frac{1}{x!} + \frac{1}{(x+1)!} + \cdots \right) - \left(\frac{1}{(x+1)!} + \frac{1}{(x+2)!} + \cdots \right) \right] \\
&= \frac{1}{e} \cdot \frac{1}{x!} \\
&= \frac{e^{-1} \cdot 1^x}{x!} \sim \mathit{Poisson}(\lambda = 1).
\end{aligned}
$$

PK Exercise 2.3.1 Random Sum approach: Let $\xi_i \sim \mathit{Bernoulli}(\frac{1}{2})$, $Z = \xi_1 + \xi_2 + \cdots + \xi_n$, and $N \sim \mathit{Uniform}(1,6)$. Then

$$
\begin{aligned}
E[Z] &= E[\xi_i]E[N] \\
&= \frac{1}{2} \cdot \frac{7}{2} \\
&= \frac{7}{4}, \\
Var(Z) &= Var(\xi_i)E[N] + Var(N)(E[\xi_i])^2 \\
&= \frac{1}{4} \cdot \frac{7}{2} + \frac{6^2 - 1}{12} \cdot \left(\frac{1}{2}\right)^2 \\
&= \frac{77}{48}.
\end{aligned}
$$

<u>*pmf* approach</u>: Let $Z \sim \textit{Bionomial}(n, \frac{1}{2})$ and $N \sim \textit{Uniform}(1, 6)$. Then

$$
\begin{aligned}
p_Z(z) &= \sum_{n=z}^{6} p_{Z|N=n}(z) p_N(n) \\
&= \sum_{n=z}^{6} \binom{n}{z} \left(\frac{1}{2}\right)^n \cdot \frac{1}{6}, \\
E[Z] &= \sum_{z=0}^{6} z \cdot p_Z(z) \\
&= \sum_{z=0}^{6} z \cdot \left[\sum_{n=z}^{6} \binom{n}{z} \left(\frac{1}{2}\right)^n \cdot \frac{1}{6} \right] \\
&= \frac{7}{4}, \\
Var(Z) &= E[Z^2] - (E[Z])^2 = \frac{14}{3} - \left(\frac{7}{4}\right)^2 \\
&= \frac{77}{48}.
\end{aligned}
$$

```
In [19]: expectation = 0
         for z in range(7):
             sum = 0

             for n in range(z,7):
                 sum += math.comb(n, z)*(0.5**n)*1/6

             expectation += z*sum

         print(expectation)

1.7500000000000002
```

$E[Z]$

```
In [18]: expectation = 0
         for z in range(7):
             sum = 0

             for n in range(z,7):
                 sum += math.comb(n, z)*(0.5**n)*1/6

             expectation += z**2*sum

         print(expectation)

4.666666666666667
```

$E[Z^2]$

PK Exercise 2.3.5 Let N be a *Poisson* random variable representing number of accidents in a week, and ξ_i be a random variable representing the number of individuals injured in each accident (assume all ξ_i are independent and identically distributed). Then the number of individuals, Z, injued in a week can be written as $Z = \xi_1 + \xi_2 + \cdots + \xi_n$.

Hence,

$$
\begin{aligned}
E[Z] &= E[\xi_i]E[N] \\
&= 3 \times 2 \\
&= 6, \\
Var(Z) &= Var(\xi_i)E[N] + Var(N)(E[\xi_i])^2 \\
&= 4 \times 2 + 2 \times 9 \\
&= 26.
\end{aligned}
$$