Math 110 HW12

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Problem 1.

Let $T \in \mathcal{L}(V, W)$. Prove

(a) T is injective if and only if T^* is surjective;

(b) T^* is injective if and only if T is surjective.

Proof.

(a)

$$\begin{aligned}\operatorname{null} T = \{0\} &\iff (\operatorname{range} T^*)^\perp = \{0\}\\ &\iff \operatorname{range} T^* = V.\end{aligned}$$

(b)

$$\begin{aligned}\operatorname{null} T^* = \{0\} &\iff (\operatorname{range} T)^\perp = \{0\}\\ &\iff \operatorname{range} T = W.\end{aligned}$$

□

Problem 2.

Suppose S, $T \in \mathcal{L}(V)$ are self-adjoint. Prove that ST is self-adjoint if and only if $ST = TS$.

Proof. We will prove both direction in one go. Let $v, w \in V$,

$$\begin{aligned} ST = TS &\iff \overline{\langle w, STv\rangle} = \overline{\langle w, TSv\rangle} && (1)\\ &\iff \langle STv, w\rangle = \langle TSv, w\rangle && (2)\\ &\iff \langle v, (ST)^* w\rangle = \langle Sv, T^* w\rangle && (3)\\ &\iff \langle v, (ST)^* w\rangle = \langle v, S^* T^* w\rangle && (4)\\ &\iff \langle v, (ST)^* w\rangle = \langle v, STw\rangle && (5)\\ &\iff (ST)^* = ST. && (6) \end{aligned}$$

(1) is by the uniqueness of complex conjugate and the Riesz representation theorem. (6) is by the uniqueness of the Riesz representation theorem. $\square$

Problem 3.

Let $P \in \mathcal{L}(V)$ be such that $P^2 = P$. Prove that there is a subspace U of V such that $P_U = P$ if and only if P is self-adjoint.

Proof. **Forward direction:** Let $W = \operatorname{null} P$, then $U \oplus W = V$ because P is an orthogonal projection.

Now, let $x, y \in V$,

$$\begin{aligned}
\langle x, P^* y\rangle = \langle Px, y\rangle &= \langle P(x_u + x_w), y_u + y_w\rangle \\
&= \langle x_w, y_u + y_w\rangle \\
&= \langle x_u, y_u\rangle + \langle x_u, y_w\rangle \\
&= \langle x_u, y_u\rangle \\
&= \langle x_u + x_w, y_u\rangle \\
&= \langle x, Py\rangle.
\end{aligned}$$

Hence, by the uniqueness of the Riesz representation theorem, $P^* = P$.

Backward direction: P is self-adjoint and hence normal. Then by either the complex or the real spectral theorem, V can be decomposed into a direct sum of eigenspaces of P where all the eigenvectors are orthonormal. Since P is self-adjoint, all its eigenvalues are real. Also, $P^2 = P$ means the only eigenvalues can only be 0 or 1. Now, let U be the eiganespace of P with eigenvalue 1 and W be the eigenspace of P with eigenvalue 0. We know $U \perp W$ since P has orthonormal eigenvectors. Then for any $v \in V$,

$$Pv = P(\underbrace{u + w}_{u \in U, w \in W}) = u.$$

Hence, by definition of orthogonal projection, $P_U = P$. $\square$

Problem 4.

Let $n \in \mathbb{N}$ be fixed. Consider the real space $V := \operatorname{span}(1, \cos x, \sin x, \cos 2x, \sin 2x, \ldots, \cos nx, \sin nx)$ with inner product

$$\langle f, g \rangle := \int_{-\pi}^{\pi} f(x)g(x)dx.$$

Show that the differentiation operator $D \in \mathcal{L}(V)$ is *anti-Hermitian*, i.e., satisfies $D^* = -D$.

Proof. Let $f, g \in V$, then

$$\begin{aligned}
\langle Df, g \rangle &= \int_{-\pi}^{\pi} Df(x)g(x)dx \\
&= \int_{-\pi}^{\pi} f'(x)g(x)dx \\
&= f(x)g(x)|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} f(x)g'(x)dx \\
&= -\int_{-\pi}^{\pi} f(x)g'(x)dx \\
&= -\langle f, Dg \rangle.
\end{aligned}$$

Hence, by the uniqueness of the Riesz representation theorem, $D^* = -D$.

$$f(x)g(x)|_{-\pi}^{\pi} = 0$$

is true because f and g can be written as

$$\alpha + \sum_{k=1}^{n} a_k \sin(kx) + b_k \cos(kx)$$

but with different coefficients. Either way, $f(\pi) = f(-\pi)$ and $g(\pi) = g(-\pi)$ because all the sin functions evaluated at $\pi, -\pi$ are 0 and all the cos functions are even functions. Therefore,

$$f(\pi)g(\pi) = f(-\pi)g(-\pi) \iff f(\pi)g(\pi) - f(-\pi)g(-\pi) = 0.$$

$\square$

Problem 5.

Suppose T is normal. Prove that, for any $\lambda \in \mathbb{F}$ and any $k \in \mathbb{N}$,

$$\operatorname{null}(T-\lambda I)^k = \operatorname{null}(T-\lambda I).$$

Lemma: Let S be a self-adjoint operator, then for any $k \in \mathbb{N}$,

$$\operatorname{null} S^k = \operatorname{null} S.$$

Proof of Lemma. Assume this is not true, then we take the minimal counterexample. Let $n \geq 2$ be the minimal counterexample.

Clearly, $\operatorname{null} S \subseteq \operatorname{null} S^n$. Let $v \in \operatorname{null} S^n$, then

$$\begin{aligned}
\langle S^n v, S^{n-2} v\rangle = 0 &\iff \langle S^{n-1} v, S^{n-1} v\rangle = 0 \\
&\iff \|S^{n-1} v\|^2 = 0 \\
&\iff S^{n-1} v = 0 \\
&\iff v \in \operatorname{null} S^{n-1} \\
&\iff v \in \operatorname{null} S \qquad (\because S^n \text{ is the minimal counterexample}),
\end{aligned}$$

which is a contradiction to the minimality of n. Hence, $\operatorname{null} S^k = \operatorname{null} S$. □

Proof of Problem 5. Clearly, $\operatorname{null}(T-\lambda I) \subseteq \operatorname{null}(T-\lambda I)^k$. Let $v \in \operatorname{null}(T-\lambda I)^k$, then let

$$S = (T-\lambda I)^*(T-\lambda I),$$

where S is self-adjoint because

$$S^* = [(T-\lambda I)^*(T-\lambda I)]^* = (T-\lambda I)^*(T-\lambda I) = S.$$

Also,

$$\begin{aligned}
S^k &= (T-\lambda I)^*(T-\lambda I)\cdots(T-\lambda I)^*(T-\lambda I) \\
&= [(T-\lambda I)^*]^k (T-\lambda I)^k,
\end{aligned}$$

by repeatedly swapping the positions of $(T-\lambda I)^*$ and $(T-\lambda I)$ because $(T-\lambda I)$ is normal. Now, let v in $\operatorname{null}(T-\lambda I)^k$, clearly $v \in \operatorname{null} S^k$. Then by the lemma, $v \in \operatorname{null} S$. Hence,

$$\begin{aligned}
\langle (T-\lambda I)^*(T-\lambda I)v, v\rangle = 0 &\iff \langle (T-\lambda I)v, (T-\lambda I)v\rangle = 0 \\
&\iff \|(T-\lambda I)v\|^2 = 0 \\
&\iff (T-\lambda I)v = 0 \\
&\iff v \in \operatorname{null}(T-\lambda I).
\end{aligned}$$

Thus we proved the inclusion of $\operatorname{null}(T-\lambda I)^k \subseteq \operatorname{null}(T-\lambda I)$. □