STAT153 HW4 Sketch

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$B^k B^\ell x_t = B^k x_{t-\ell} = x_{t-\ell-k} = B^\ell x_{t-k} = B^\ell B^k x_t$

$$
\begin{aligned}
(1+\phi_1 B+\cdots+\phi_k B^k)(1+\varphi_1 B+\cdots+\varphi_\ell B^\ell) &= \left(\sum_{i=0}^{k}\phi_i B^i\right)\left(\sum_{j=0}^{\ell}\varphi_j B^j\right) \qquad (\text{let } \phi_0=\varphi_0=1)\\
&= \sum_{i=0}^{k}\sum_{j=0}^{\ell}\phi_i B^i\cdot\varphi_j B^j\\
&= \sum_{i=0}^{k}\sum_{j=0}^{\ell}\phi_i\varphi_j\cdot B^i B^j\\
&= \sum_{i=0}^{k}\sum_{j=0}^{\ell}\varphi_j\phi_i\cdot B^j B^i\\
&= \sum_{i=0}^{k}\sum_{j=0}^{\ell}\varphi_j B^j\cdot\phi_i B^i\\
&= \sum_{i=0}^{\ell}\sum_{j=0}^{k}\varphi_j B^j\cdot\phi_i B^i\\
&= (1+\varphi_1 B+\cdots+\varphi_\ell B^\ell)(1+\phi_1 B+\cdots+\phi_k B^k)
\end{aligned}
$$

Notice

$$\nabla^d\nabla_s^D = (1-B)^d(1-B^s)^D = (1-B^s)^D(1-B)^d$$

where we apply Q2 iteratively on $(1+\phi_1 B)$ where $\phi_1=-1$ and $(1+\varphi_1 B+\cdots+\phi_s B^s)$ where $\varphi_s=-1$ and 0 else. Therefore,

$$\nabla^d\nabla_s^D = \nabla_s^D\nabla^d.$$

Similarly,

$$
\begin{aligned}
\phi(B)\Phi(B^s) &= (1+\phi_1 B+\cdots+\phi_p B^p)(1+\Phi_s B^s+\cdots+\Phi_{Ps}B^{Ps})\\
&= (1+\Phi_s B^s+\cdots+\Phi_{Ps}B^{Ps})(1+\phi_1 B+\cdots+\phi_p B^p)
\end{aligned}
$$

by apply Q2 iteratively where $\varphi_i=\Phi_i$ for $i=s,2s,\ldots,Ps$ and 0 else. Therefore,

$$\phi(B)\Phi(B^s) = \Phi(B^s)\phi(B).$$

Similarly,

$$\begin{aligned}\theta(B)\Theta(B^s) &= (1-\theta_1 B - \cdots - \theta_q B^q)(1-\Theta_s B^s - \cdots - \Theta_{Qs} B^{Qs}) \\ &= (1-\Theta_s B^s - \cdots - \Theta_{Qs} B^{Qs})(1-\theta_1 B - \cdots - \theta_q B^q),\end{aligned}$$

by applying Q2 iteratively where $\phi_i = -\theta_i$ and $\varphi_i = -\Theta_i$ for $i = s, \ldots, Qs$ and 0 else. Therefore,

$$\theta(B)\Theta(B^s) = \Theta(B^s)\theta(B).$$

Therefore, we achieved the desired equivalence.

$$\forall t, \nabla x_t = u \Longrightarrow x_t = x_{t-1} + u = x_{t-2} + u + u = x_{t-3} + u + u + u = \cdots = x_0 + tu,$$

which is in the desired form of $x_t = a + bt$.

$$\begin{aligned}x_t &= u + vt + x_{t-1} \\ &= u + vt + u + v(t-1) + x_{t-2} \\ &= u + vt + u + v(t-1) + u + v(t-2) + x_{t-3} \\ &= u + vt + u + v(t-1) + u + v(t-2) + \cdots + u + v(t-(t-1)) + x_0 \\ &= ut + v\left(t^2 - (1 + \cdots + (t-1))\right) + x_0 \\ &= ut + v\left(t^2 - \frac{t(t-1)}{2}\right) + x_0 \\ &= ut + v\left(\frac{t^2+t}{2}\right) + x_0 \\ &= x_0 + \left(u + \frac{v}{2}\right)t + \frac{v}{2}t^2,\end{aligned}$$

which is in the desired form of $x_t = a + bt + ct^2$.

$$(1-\phi B)x_t = (1+\theta B)w_t \Longrightarrow x_t - \phi x_{t-1} = w_t + \theta w_{t-1} \Longrightarrow x_t = \phi x_{t-1} + w_t + \theta w_{t-1}.$$

Then,

$$\begin{aligned}\hat{x}_{t+1|t} &= \hat{\phi} x_t + w_{t+1} + \hat{\theta} w_t = \hat{\phi} x_t + \hat{\theta} w_t \qquad (w_{t+1} = 0) \\ \hat{x}_{t+2|t} &= \hat{\phi} x_{t+1|t} + w_{t+2} + \hat{\theta} w_{t+1} = \hat{\phi}\left(\hat{\phi} x_t + \hat{\theta} w_t\right) \qquad (w_{t+2} = w_{t+1} = 0) \\ \hat{x}_{t+3|t} &= \hat{\phi} x_{t+2|t} + w_{t+3} + \hat{\theta} w_{t+2} = \hat{\phi}\left(\hat{\phi}\left(\hat{\phi} x_t + \hat{\theta} w_t\right)\right) \qquad (w_{t+3} = w_{t+2} = 0) \\ &\vdots \\ \lim_{h\to\infty} \hat{x}_{t+h|t} &= \lim_{h\to\infty} \hat{\phi}^h x_t + \hat{\phi}^{h-1}\hat{\theta} w_t = 0 \qquad (\because \hat{\phi} < 1 \text{ and } x_t, \hat{\theta} w_t \in \mathbb{R})\end{aligned}$$

$$(1-\phi B)x_t = c + (1+\theta B)w_t \Longrightarrow x_t = \phi x_{t-1} + w_t + \theta w_{t-1} + c.$$

Then,

$$\begin{aligned}
\hat{x}_{t+1|t} &= \hat{\phi}x_t + w_{t+1} + \hat{\theta}\hat{w}_t + c = \hat{\phi}x_t + \hat{\theta}\hat{w}_t + c \qquad (w_{t+1} = 0) \\
\hat{x}_{t+2|t} &= \hat{\phi}x_{t+1|t} + w_{t+2} + \hat{\theta}w_{t+1} + c = \hat{\phi}\left(\hat{\phi}x_t + \hat{\theta}\hat{w}_t + c\right) + c \qquad (w_{t+2} = w_{t+1} = 0) \\
\hat{x}_{t+3|t} &= \hat{\phi}x_{t+2|t} + w_{t+3} + \hat{\theta}w_{t+2} + c = \hat{\phi}\left(\hat{\phi}\left(\hat{\phi}x_t + \hat{\theta}\hat{w}_t + c\right) + c\right) + c \qquad (w_{t+3} = w_{t+2} = 0) \\
&\vdots \\
\hat{x}_{t+h|t} &= \hat{\phi}^h x_t + \hat{\phi}^{h-1}\hat{\theta}\hat{w}_t + c \cdot \sum_{k=0}^{h-1} \hat{\phi}^k
\end{aligned}$$

$$\lim_{h\to\infty} \hat{x}_{t+h|t} = \lim_{h\to\infty} \hat{\phi}^h x_t + \hat{\phi}^{h-1}\hat{\theta}\hat{w}_t + c \cdot \sum_{k=0}^{h-1} \hat{\phi}^k = c \cdot \lim_{h\to\infty} \sum_{k=0}^{h-1} \hat{\phi}^k = C,$$

because $\hat{\phi} < 1$, $\hat{\theta}w_t \in \mathbb{R}$, and $\lim_{h\to\infty} \sum_{k=0}^{h-1} \hat{\phi}^k$ converges to some real number.

$$rows = t_0 + (t_0 + 1) + (t_0 + 2) + \cdots + n = \frac{(t_0 + n)(n - t_0 + 1)}{2}.$$

$\underline{c = 0, d = 1}$: Let $y_t = \nabla x_t$, then

$$(1 - \phi B)\nabla x_t = (1 + \theta B)w_t \Leftrightarrow (1 - \phi B)y_t = (1 + \theta B)w_t.$$

By Q9, $\hat{y}_{t+h|t} = 0$ as $h \to \infty$, thus $\nabla\hat{x}_{t+h|t} = 0$ as $h \to \infty$. By Q5, $\hat{x}_{t+h|t}$ approaches a constant sequence as $h \to \infty$.

$\underline{c = 0, d = 2}$: Let $y_t = \nabla^2 x_t$, then

$$(1 - \phi B)\nabla^2 x_t = (1 + \theta B)w_t \Leftrightarrow (1 - \phi B)y_t = (1 + \theta B)w_t.$$

By Q9, $\hat{y}_{t+h|t} = 0$ as $h \to \infty$, thus $\nabla^2\hat{x}_{t+h|t} = 0$ as $h \to \infty$. Notice $\nabla^2\hat{x}_{t+h|t} = \nabla(\nabla\hat{x}_{t+h|t})$, then by Q5, $\nabla\hat{x}_{t+h|t}$ approaches a constant sequence as $h \to \infty$. Further, by Q6, $\hat{x}_{t+h|t}$ approaches a linear function of t as $h \to \infty$.

$\underline{c \neq 0, d = 1}$: Let $y_t = \nabla x_t$, then

$$(1 - \phi B)\nabla x_t = c + (1 + \theta B)w_t \Leftrightarrow (1 - \phi B)y_t = c + (1 + \theta B)w_t.$$

By Q10, $\hat{y}_{t+h|t}$ approaches a non-zero constant as $h \to \infty$, thus $\nabla\hat{x}_{t+h|t}$ approaches a none-zero constant as $h \to \infty$. By Q6, $\hat{x}_{t+h|t}$ approaches a linear function of t as $h \to \infty$.

$\underline{c \neq 0, d = 2}$: Let $y_t = \nabla^2 x_t$, then

$$(1 - \phi B)\nabla x_t = c + (1 + \theta B)w_t \Leftrightarrow (1 - \phi B)y_t = c + (1 + \theta B)w_t.$$

By Q10, $\hat{y}_{t+h|t}$ approaches a non-zero constant as $h \to \infty$, thus $\nabla^2\hat{x}_{t+h|t}$ approaches a none-zero constant as $h \to \infty$. By Q8, $\hat{x}_{t+h|t}$ approaches a quadratic function of t as $h \to \infty$.