Math 180A HW7

Neo Lee

03/01/2023

Problem 1.

(a) 0.3.

(b) $\frac{7}{30}$.

Problem 2.

(a)

$$\begin{aligned} f_X(x) &= \int_0^1 \frac{12}{7}(xy+y^2)dy && (1) \\ &= \frac{12}{7}\left[\frac{1}{2}xy^2 + \frac{1}{3}y^3\right]_0^1 && (2) \\ &= \frac{12}{7}\left(\frac{x}{2}+\frac{1}{3}\right) && (3) \\ &= \frac{6x+4}{7} && (4) \end{aligned}$$

for $x \in [0,1]$ and 0 otherwise.

(b)

$$\begin{aligned} f_Y(y) &= \int_0^1 \frac{12}{7}(xy+y^2)dx && (5) \\ &= \frac{12}{7}\left[\frac{1}{2}x^2y + xy^2\right]_0^1 && (6) \\ &= \frac{12}{7}\left(\frac{y}{2}+y^2\right) && (7) \\ &= \frac{6y+12y^2}{7} && (8) \end{aligned}$$

for $y \in [0,1]$ and 0 otherwise.

(c)

$$
\begin{aligned}
P(X<Y) &= \int_0^1 \int_0^y \frac{12}{7}(xy+y^2)dxdy && (9)\\
&= \frac{12}{7}\int_0^1 \left[\frac{1}{2}x^2y+xy^2\right]_{x=0}^{x=y} dy && (10)\\
&= \frac{12}{7}\int_0^1 \frac{1}{2}y^3+y^3dy && (11)\\
&= \frac{12}{7}\left[\frac{1}{8}y^4+\frac{1}{4}y^4\right]_0^1 && (12)\\
&= \frac{12}{6}\left(\frac{3}{8}\right) && (13)\\
&= \frac{3}{4}. && (14)
\end{aligned}
$$

Problem 3.

$$
\begin{aligned}
P(X=x,Y=y) &= P(X=x)P(Y=y) && (15)\\
&= \left(p(1-p)^{x-1}\right)\left(r(1-r)^{y-1}\right). && (16)
\end{aligned}
$$

Then,

$$
\begin{aligned}
P(X<Y) &= P(X\in[1,\infty))P(Y>X) && (17)\\
&= \sum_{n=1}^{\infty} p(1-p)^{n-1}(1-r)^n. && (18)
\end{aligned}
$$

Problem 4.

(a)

$$
\begin{aligned}
\int_0^\pi \int_0^\pi c(1-cos(x)cos(y))dxdy &= 1 && (19)\\
c\int_0^\pi \int_0^\pi dxdy - c\int_0^\pi \int_0^\pi cos(x)cos(y)dxdy &= 1 && (20)\\
c\pi^2 - c\int_0^\pi \left[sin(x)cos(y)\right]_{x=0}^{x=\pi} dy &= 1 && (21)\\
c\pi^2 &= 1 && (22)\\
c &= \frac{1}{\pi^2}. && (23)
\end{aligned}
$$

(b)

$$
\begin{aligned}
f_X(x) &= \frac{1}{\pi^2}\int_0^\pi 1-cos(x)cos(y)dy && (24)\\
&= \frac{1}{\pi^2}\left(\pi - \left[cos(x)sin(y)\right]_{y=0}^{y=\pi}\right) && (25)\\
&= \frac{1}{\pi} && (26)
\end{aligned}
$$

for $x\in[0,\pi]$ and 0 otherwise. Similarly,

$$
f_Y(y) = \frac{1}{\pi} \qquad (27)
$$

for $y\in[0,\pi]$ and 0 otherwise. The probability distribution is uniform.

(c)

$$f(0,0) = \frac{1}{\pi^2}\left(1 - cos(0)cos(0)\right) \tag{28}$$
$$= 0. \tag{29}$$

On the other hand,

$$f_X(0)f_Y(0) = \left(\frac{1}{\pi}\right)\left(\frac{1}{\pi}\right) \tag{30}$$
$$= \frac{1}{\pi^2} \tag{31}$$
$$\neq f(0,0) \tag{32}$$

Hence, X and Y are not independent.

Problem 6. The convolution formula tells us

$$P_{X+Y}(z) = \sum_X P_X(x)P_Y(z-x) \tag{33}$$
$$= P(X=0)P(Y=z-0) + P(X=1)P(Y=z-1) \tag{34}$$
$$= (1-p)P(Y=z) + pP(Y=z-1). \tag{35}$$

Hence,

$$P_{X+Y}(z) = \begin{cases} (1-p)(1-r) & z=0 \\ p(1-r)+r(1-p) & z=1 \\ pr & z=2 \\ 0 & otherwise. \end{cases} \tag{36}$$

Problem 7. *Convolution Approach*
We know $f_Y(y) = 1$ for $y \in (1,2)$ and 0 otherwise. For $z \leq 1$, $f_{X+Y}(z) = 0$. For $z \in [1,2]$,

$$f_{X+Y}(z) = \int_1^z f_Y(y)f_X(z-y)dy \tag{37}$$
$$= \int_1^z 2(z-y)dy \tag{38}$$
$$= \int_1^z 2z - 2ydy \tag{39}$$
$$= [2yz]_{y=1}^{y=z} - \left[y^2\right]_1^z \tag{40}$$
$$= 2z^2 - 2z - z^2 + 1 \tag{41}$$
$$= z^2 - 2z + 1. \tag{42}$$

For $z \in [2,3]$,

$$f_{X+Y}(z) = \int_{z-2}^1 f_Y(z-x)f_X(x)dx \tag{43}$$
$$= \int_{z-2}^1 2xdx \tag{44}$$
$$= \left[x^2\right]_{z-2}^1 \tag{45}$$
$$= -z^2 + 4z - 3. \tag{46}$$

For $z \geq 3$, $f_{X+Y}(z) = 0$.

CDF Approach

Since X and Y are independent, $f(x,y) = f_X(x)f_Y(y) = 2x$. For $z \leq 1$, $f_{X+Y}(z) = 0$. For $z \in [1,2]$,

$$\begin{aligned}
P(X+Y \leq Z) &= \int_1^z \int_0^{z-y} 2x dx dy && (47)\\
&= \int_1^z \left[x^2\right]_0^{z-y} dy && (48)\\
&= \int_1^z z^2 - 2zy + y^2 dy && (49)\\
&= \left[z^2 y - zy^2 + \frac{1}{3}y^3\right]_1^z && (50)\\
&= z^3 - z^3 + \frac{1}{3}z^3 - z^2 + z - \frac{1}{3} && (51)\\
&= \frac{1}{3}z^3 - z^2 + z - \frac{1}{3}. && (52)
\end{aligned}$$

Hence, for $z \in [1,2]$,

$$\begin{aligned}
f_{X+Y}(z) &= \frac{d}{dz}\left(\frac{1}{3}z^3 - z^2 + z - \frac{1}{3}\right) && (53)\\
&= z^2 - 2z + 1. && (54)
\end{aligned}$$

For $z \in [2,3]$,

$$\begin{aligned}
P(X+Y \leq Z) &= \int_0^1 \int_1^2 2x dy dx - \int_{z-2}^1 \int_{z-x}^2 2x dy dx && (55)\\
&= \int_0^1 \left[2xy\right]_{y=1}^{y=2} dx - \int_{z-2}^1 \left[2xy\right]_{y=z-x}^{y=2} dx && (56)\\
&= \int_0^1 2x dx - \int_{z-2}^1 2x^2 + 4x - 2zx dx && (57)\\
&= 1 - \left[\frac{2}{3}x^3\right]_{z-2}^1 - \left[2x^2\right]_{z-2}^1 + \left[zx^2\right]_{z-2}^1 && (58)\\
&= -\frac{1}{3}z^3 + 2z^2 - 3z + 1. && (59)
\end{aligned}$$

Hence, for $z \in [2,3]$,

$$\begin{aligned}
f_{X+Y}(z) &= \frac{d}{dz}\left(-\frac{1}{3}z^3 + 2z^2 - 3z + 1\right) && (60)\\
&= -z^2 + 4z - 3. && (61)
\end{aligned}$$

For $z \geq 3$, $f_{X+Y}(z) = 0$.