

Math 180A HW6

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Problem 1.

1. (b)
2. (a)
- 3.

$$E[X^2] = \text{Var}(X) + (E[X])^2 \quad (1)$$

$$= \sigma^2 + \mu^2 \quad (2)$$

$$= 4 + 9 \quad (3)$$

$$= 13 \quad (4)$$

Problem 2. (c)

Problem 3.

(a)

$$P(X \geq 10 + 15 | X \geq 10) = P(X \geq 15) \quad (5)$$

$$= e^{-(\frac{1}{20} \times 15)} \quad (6)$$

$$\approx 0.4724 \quad (7)$$

(b)

$$P(X \geq 25 | X \geq 10) = \frac{P(X \geq 25)}{P(X \geq 10)} \quad (8)$$

$$= \frac{\int_{25}^{40} \frac{1}{40} dx}{\int_{10}^{40} \frac{1}{40} dx} \quad (9)$$

$$= \frac{40 - 25}{40 - 10} \quad (10)$$

$$= \frac{1}{2} \quad (11)$$

- (c) Since exponential random variable is memoryless, the probability of waiting for at least 15 additional minutes is always the same given the time before has already passed.

On the other hand, uniform random variable is not memoryless. In fact, the probability is eventually distributed throughout the interval $[0, 40]$. Therefore, the probability of waiting would be $\frac{P(X \geq n+15)}{P(X \geq n)}$ given that n has passed.

Problem 4.

(a)

$$E[Z^n] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z^n e^{-\frac{z^2}{2}} dz \quad (12)$$

$$= \frac{1}{\sqrt{2\pi}} \left[\left[-z^{n-1} e^{-\frac{z^2}{2}} \right]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} (n-1) z^{n-2} e^{-\frac{z^2}{2}} dz \right] \quad (13)$$

$$= \frac{n-1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z^{n-2} e^{-\frac{z^2}{2}} dz \quad (14)$$

$$= (n-1)E[Z^{n-2}] \quad (15)$$

Hence, $E[Z^3] = (3-1)E[Z] = 0$.

(b)

$$E[X^3] = E[(\sigma Z + \mu)^3] \quad (16)$$

$$= E[\sigma^2 Z^3 + 3\sigma^2 \mu Z^2 + 3\sigma \mu^2 Z + \mu^3] \quad (17)$$

$$= \sigma^2 E[Z^3] + 3\sigma^2 \mu E[Z^2] + 3\sigma \mu^2 E[Z] + E[\mu^3] \quad (18)$$

$$= 3\sigma^2 \mu + \mu^3 \quad (19)$$

Problem 5. For $a \in [0, 1]$,

$$P(X \in [0, 1] \cap X \in [a, 2]) = P(X \in [a, 1]) \quad (20)$$

$$= e^{-2a} - e^{-2}. \quad (21)$$

For the events to be independent,

$$P(X \in [0, 1] \cap X \in [a, 2]) = P(X \in [0, 1])P(X \in [a, 2]) \quad (22)$$

$$e^{-2a} - e^{-2} = (1 - e^{-2}) \times (e^{-2a} - e^{-4}) \quad (23)$$

$$e^{-2a} - e^{-2} = e^{-2a} - e^{-4} - e^{-2a-2} + e^{-6} \quad (24)$$

$$e^{-2a-2} = e^{-6} - e^{-4} + e^{-2} \quad (25)$$

$$e^{-2a-2} = e^{-2} (e^{-4} - e^{-2} + 1) \quad (26)$$

$$-2a - 2 = -2 + \ln(e^{-4} - e^{-2} + 1) \quad (27)$$

$$a = \frac{\ln(e^{-4} - e^{-2} + 1)}{-2} \quad (28)$$

$$\approx 0.0622 \quad (29)$$

Problem 6.

(a) There can be two scenarios: 1) if the stove breaks within r years, profit = $\$C - 600$; 2) if the stove lasts longer than r years, profit = $\$200 + C$.

Let $g(x)$ be a function that calculates the profit, then

$$E[g(x)] = (C - 600) \times (1 - P(X \geq r)) + (200 + C) \times P(X \geq r) \quad (30)$$

$$= (C - 600) \times (1 - e^{-\frac{r}{10}}) + (200 + C) \times e^{-\frac{r}{10}} \quad (31)$$

$$= C - 600 + 600e^{-\frac{r}{10}} + 200e^{-\frac{r}{10}} \quad (32)$$

$$= 800e^{-\frac{r}{10}} - 600 + C \quad (33)$$

(b)

$$800e^{\frac{-5}{10}} - 600 + C = 0 \quad (34)$$

$$C = 600 - 800e^{-0.5} \quad (35)$$

$$\approx 114.8 \quad (36)$$

Problem 7.

$$P((X, Y) \in [0, 1]) = 1 \quad (37)$$

$$c \int_0^1 \int_0^1 xy + y^2 dx dy = 1 \quad (38)$$

$$c \int_0^1 \left[\frac{1}{2}x^2y + xy^2 \right]_{x=0}^{x=1} = 1 \quad (39)$$

$$c \int_0^1 \frac{1}{2}y + y^2 = 1 \quad (40)$$

$$c \left[\frac{1}{4}y^2 + \frac{1}{3}y^3 \right]_0^1 = 1 \quad (41)$$

$$\frac{7}{12}c = 1 \quad (42)$$

$$c = \frac{12}{7} \quad (43)$$