Math 109 HW4

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Problem 8.1

Proposition 1. $g(x,y) = \begin{cases} x & if x \geq y \\ y & if x \leq y \end{cases}$ *is well defined for* $g : \mathbb{R}^2 \to \mathbb{R}$.

Proof. For all $(x,y) \in \mathbb{R}^2$, it is exclusively that $x > y$, $x < y$, or $x = y$. If $x > y$, $g(x,y)$ is uniquely defined as $x \in \mathbb{R}$. If $x < y$, $g(x,y)$ is uniquely defined as $y \in \mathbb{R}$. If $x = y$, $g(x,y)$ is uniquely defined as $x = y \in \mathbb{R}$. $\square$

Proposition 2. *Let* $f(x,y) = \frac{x+y}{2} + \frac{|x-y|}{2}$ *for* $f : \mathbb{R}^2 \to \mathbb{R}$, $f = g$.

Proof. If $x > y$, $f(x,y) = \frac{x+y}{2} + \frac{x-y}{2} = x$. If $x < y$, $f(x,y) = \frac{x+y}{2} + \frac{y-x}{2} = y$. If $x = y$, $f(x,y) = \frac{x+x}{2} + \frac{x-x}{2} = x = y$. Hence, $f(x,y) = g(x,y)$ for all $(x,y) \in \mathbb{R}^2$. $\square$

Problem 8.2

(i) $f \circ f = f(f(x)) = f(x^3) = x^{3^3} = x^9$ for $\mathbb{R} \to \mathbb{R}$.

(ii) $f \circ g = f(g(x)) = f(1-x) = (1-x)^3$ for $\mathbb{R} \to \mathbb{R}$.

(iii) $g \circ f = g(f(x)) = g(x^3) = 1 - x^3$ for $\mathbb{R} \to \mathbb{R}$.

(iv) $g \circ g = g(g(x)) = g(1-x) = 1 - (1-x) = x$ for $\mathbb{R} \to \mathbb{R}$.

$fg(x) = gf(x) \Leftrightarrow (1-x^3) = 1 - x^3 \Leftrightarrow 1 - 3x + 3x^2 - x^3 = 1 - x^3 \Leftrightarrow x(x-1) = 0 \Leftrightarrow x = 0$ or $x = 1$. Hence, $\{x \in \mathbb{R} | fg(x) = gf(x)\} = \{0, 1\}$.

Problem 8.3

(i) $f_1(x) = x$ for $\mathbb{R} \to \mathbb{R}$.

(ii) $f_2(x) = |x|$ for $\mathbb{R} \to \mathbb{R}$.

(iii) $f_3(x) = \begin{cases} x & if x \notin \mathbb{Z} \\ 0.1 & if x \in \mathbb{Z} \end{cases}$ for $\mathbb{R} \to \mathbb{R}$.

(iv) $f_4(x) = \lfloor x \rfloor$ for $\mathbb{R} \to \mathbb{R}$.

Problem 8.5 (i) and (iv) are graphs of a function $f : X \to Y$.

x	$f_i(x)$	$f_{iv}(x)$
a	z	y
b	y	z
c	z	w
d	x	x

For (ii), $\{c\} \times Y$ contains no elements, which means not every element in X is mapped to Y. For (iii), $\{b\} \times Y$ contains more than one element, which mean $f(x)$ is not uniquely defined in Y for $x = b$.

Problem 9.1

(i) Bijective. It is surjective because for every image y, there is a pre-image $x = \frac{y-5}{2}$. It is injective because if $y = f(x_1) = f(x_2)$, $x_1 = x_2 = \frac{y-5}{2}$.

(ii) Neither injective nor surjective. Let $f(x) = 1$, $x = -2$ or $x = 0$, thus it's not injective. Since there does not exists x for $f(x) = -1$, it's not surjective.

(iii) Neither injective nor surjective. Let $f(x) = 0$, $x = 0$ or $x = 2$, thus it's not injective. Since there does not exists x for $f(x) = -2$, it's not surjective.

(iv) Bijective. It is surjective because for every image $y \neq 0$, there is a pre-image $x = \frac{1}{y}$; for $y = 0$, $x = 0$. It is injective becasue if $0 \neq y = f(x_1) = f(x_2)$, $x_1 = x_2 = \frac{1}{y}$; if $y = 0$, $x = 0$.

Problem 9.2

(i) Injective only. It is injective because if $y = f(x_1) = f(x_2)$, $x_1 = x_2 = \frac{y-5}{2}$. It is not surjective because there does not exist x for $f(x) = 1$.

(ii) Injective only. It is injective because if $y = f(x_1) = f(x_2)$, $x_1 = x_2 = -1 + \frac{\sqrt{y}}{2}$. It is not surjective because there does not exist x for $f(x) = 0.1$.

(iii) Not a function. There does not exist $f(x) \in \mathbb{R}^+$ for $x = 0.1$.

(iv) Bijective. It is surjective because for every image y, there is a pre-image $x = \frac{1}{y}$. It is injective becasue if $y = f(x_1) = f(x_2)$, $x_1 = x_2 = \frac{1}{y}$.

Problem 9.3

(i) $f^{-1}(y) = \frac{y-2}{3}$.

(ii) $f^{-1}(y) = \sqrt[3]{y-1}$.

Problem 9.4

Proposition 3. *$g \circ f$ is injective if g and f are both injective.*

Proof.

$$\begin{aligned} z = g(f(x_1)) = g(f(x_1)) \Rightarrow f(x_1) = f(x_2) \qquad &\because g \text{ is injective} \qquad (1) \\ \Rightarrow x_1 = x_2 \qquad &\because f \text{ is injective} \qquad (2) \end{aligned}$$

Hence, $g \circ f(x_1) = g \circ f(x_2) \Rightarrow x_1 = x_2$. □

Problem 9.6

Proposition 4. *Let $f : X \to Y$ be a function with graph $G_f \subseteq X \times Y$. f is surjective if and only if $\forall y \in Y, (X \times \{y\} \cap G_f) \neq \emptyset$.*

Proof. ($\Rightarrow$) Since f is surjective, $\forall y \in Y$, $\exists x$ such that $f(x) = y$. Let $x_0 \in X$ such that $f(x_0) = y_0$ for arbitrary $y_0 \in Y$. Then $(x_0, y_0) \in (X \times \{y_0\} \cap G_f)$. Hence, for all $y \in Y$, $(X \times \{y\} \cap G_f) \neq \emptyset$.

($\Leftarrow$) Since $\forall y \in Y, (X \times \{y\} \cap G_f) \neq \emptyset$, we can take an arbitrary $y_1 \in Y$ and there must exist $(x_1, y_1) \in X \times \{y_1\}$. At the same time $(x_1, y_1) \in G_f$, so we know that $f(x_1) = y_1$. Hence, it satisfies that definition of surjection that $\forall y \in Y, \exists x$ such that $f(x) = y$. □

Problem 14 $f \circ f = x \mapsto x^4$. $f \circ g = x \mapsto x^4 - 2x^2 + 1$. $g \circ f = x \mapsto x^4 - 1$. $g \circ g = x \mapsto x^4 - 2x^2$. $\{x \in \mathbb{R} | fg(x) = gf(x)\} \Leftrightarrow x^4 - 2x^2 + 1 = x^4 - 1 \Leftrightarrow -2x^2 + 2 = 0 \Leftrightarrow -2(x^2 - 1) = 0 \Leftrightarrow x = -1$ or $x = 1 \Leftrightarrow \{-1, 1\}$.

Problem 15

(i) We can easily see that $\chi_A(x)\chi_B(x) \equiv \chi_{A\cap B}(x)$ by drawing a truth table.

$x \in A$	$x \in B$	$\chi_A(x)\chi_B(x)$	$\chi_{A\cap B}(x)$
T	T	1	1
T	F	0	0
F	T	0	0
F	F	0	0

(ii) Let $C = A \cup B$.

$x \in A$	$x \in B$	$\chi_A(x) + \chi_B(x) - \chi_A(x)\chi_B(x)$	$\chi_C(x)$
T	T	1	1
T	F	1	1
F	T	1	1
F	F	0	0

Problem 16

(i) Bijective. Surjective: $\forall y = f_1(x), \exists x = y+1 \in \mathbb{R}$. Injective: $y_0 = f_1(x_1) = f_1(x_2) \Rightarrow x_1 = x_2 = y_0+1$. $f_1^{-1}(y) = y+1$.

(ii) Bijective. Surjective: $\forall y = f_2(x), \exists x = \sqrt[3]{y} \in \mathbb{R}$. Injective: $y_0 = f_2(x_1) = f_2(x_2) \Rightarrow x_1 = x_2 = \sqrt[3]{y}$. $f_2^{-1}(y) = \sqrt[3]{y}$.

(iii) Surjective. Surjective: $\lim\limits_{x\to\infty} f_3(x) = \infty$ and $\lim\limits_{x\to-\infty} f_3(x) = -\infty$. Since $f_3(x)$ is a polynomial, it is a continuous function. By intermediate value theorem, $\forall y \in (-\infty,\infty) \equiv \mathbb{R}$, $\exists x$ such that $y = f_3(x)$. Not injective: let $f_3(x) = 0$, $x = -1$ or $x = 0$ or $x = 1$.

(iv) Bijective. Surjective: $\forall y = f_4(x), \exists x$ such that $x^3 - 3x^2 + 3x - 1 = y \Leftrightarrow (x-1)^3 = y \Leftrightarrow x = \sqrt[3]{y}+1$. Injective: $f_4(x)' = 3x^2 - 6x + 3 \geq 0$ for all $x \in \mathbb{R}$; so for every $y_0 = f_4(x)$, there exists only at most one x_0 such that $f_4(x_0) = y_0$. $f_4^{-1}(y) = \sqrt[3]{y}+1$.

(v) Injective. Not surjective: let $y = -1$, there does not exist $x \in \mathbb{R}$ such that $f_5(x) = e^x = y$. Injective: let $y_0 = f_5(x_1) = f_5(x_2) \Rightarrow x_1 = x_2 = ln(y_0)$.

(vi) Bijective. Surjective: $\forall y = f_6(x) \geq 0, \exists x = \sqrt{y}$; $\forall y = f_6(x) < 0, \exists x = \sqrt{-y}$. Injective: let $0 \geq y = f_6(x_1) = f_6(x_2) \Rightarrow x_1 = x_2 = \sqrt{y}$. Let $0 > y = f_6(x_3) = f_6(x_4) \Rightarrow x_3 = x_4 = \sqrt{-y}$. $f_6^{-1}(y) = \begin{cases} \sqrt{y} & \text{if } y \geq 0, \\ \sqrt{-y} & \text{if } y < 0. \end{cases}$