Math 104 HW4

Neo Lee

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Exercise 9.12

Assume all $s_n \neq 0$ and that the limit $L = \lim \left|\frac{s_n+1}{s_n}\right|$ exists.

(a)

Proposition 1. *If $L < 1$, then $\lim s_n = 0$.*

Proof. Since $L < 1$, L can be written as $L + \delta = 1$ for some $\delta > 0$. We know there exists N such that for all $n > N$

$$\left|\frac{s_{n+1}}{s_n} - L\right| < \frac{\delta}{2}$$
$$\left|\frac{s_{n+1}}{s_n}\right| < L + \frac{\delta}{2} = a < 1.$$

Then, for $n > N$, we have

$$|s_n| < a^{n-N}|S_N|.$$

So, now we only have to show that there exists M such that for $n > M, \epsilon > 0$,

$$\begin{aligned}
a^{n-N}|S_N| &< \epsilon \\
a^n &< \frac{\epsilon}{|S_N|}a^N \\
|a^n| &< \frac{\epsilon}{|S_N|}a^N \\
|a^n| &< C \qquad (\textit{for some } C > 0).
\end{aligned}$$

Notice $\lim a^n = 0$ since $|a| < 1$ (Theorem 9.7). So, by definition of limit, indeed there exists such M. Therefore, for $n > \max\{M, N\}$,

$$|s_n - 0| = |s_n| < a^{n-N}|S_N| < \epsilon.$$

□

(b)

Proposition 2. *If $L > 1$, then* $\lim |s_n| = \infty$.

Proof. Define $t_n := \frac{1}{|s_n|}$. Then,

$$\begin{aligned}\lim \left|\frac{t_{n+1}}{t_n}\right| &= \lim \left|\frac{s_n}{s_{n+1}}\right| \\ &= \lim \frac{1}{\left|\frac{s_n+1}{s_n}\right|} \\ &= \frac{1}{L} < 1.\end{aligned}$$

Hence, from (a), $\lim t_n = 0$. Therefore, $\lim |s_n| = \infty$ by Theorem 9.10. $\square$

Exercise 9.15

Proposition 3. $\lim_{n\to\infty} \frac{a^n}{n!} = 0$ *for all* $a \in \mathbb{R}$.

Proof. If $a = 0$, then the limit is 0 trivially. If $a \neq 0$, denote $s_n = \frac{a^n}{n!}$, and

$$\begin{aligned}\left|\frac{s_{n+1}}{s_n}\right| &= \left|\frac{a^{n+1}}{(n+1)!}\frac{n!}{a^n}\right| \\ \left|\frac{s_{n+1}}{s_n}\right| &= \frac{|a|}{n+1} \\ \lim_{n\to\infty}\left|\frac{s_{n+1}}{s_n}\right| &= 0.\end{aligned}$$

Then, by (a) of Exercise 9.12, $\lim s_n = 0$. $\square$

Exercise 9.18

(a) Verify $1 + a + a^2 + \cdots + a^n = \frac{1-a^{n+1}}{1-a}$ for $a \neq 1$.

Solution. We prove by induction.

Base case: $n = 1$. LHS: $1 + a$. RHS: $\frac{1-a^2}{1-a} = \frac{(1-a)(1+a)}{1-a} = 1 + a$.

Inductive step: Assume the statement is true for $n = k$. Then,

$$\begin{aligned}1 + a + a^2 + \cdots + a^k + a^{k+1} &= \frac{1-a^{k+1}}{1-a} + a^{k+1} \\ &= \frac{1 - a^{k+1} + a^{k+1} - a^{k+2}}{1-a} \\ &= \frac{1-a^{k+2}}{1-a}.\end{aligned}$$

Therefore, the statement is true for all $n \in \mathbb{N}$. $\square$

(b) Find $\lim_{n\to\infty}(1 + a + a^2 + \cdots + a^n)$ for $|a| < 1$.

Solution.

$$\begin{aligned}\lim_{n\to\infty}(1 + a + a^2 + \cdots + a^n) &= \lim_{n\to\infty} \frac{1 - a^{n+1}}{1 - a} \\ &= \left(\lim 1 - a^{n+1}\right)\left(\lim \frac{1}{1-a}\right) \\ &= \frac{1}{1-a}. \qquad (\because \lim a^{n+1} = 0)\end{aligned}$$

Notice, this is just a gerometric series with $r = a$. $\square$

(c) Calculate $\lim_{n\to\infty} \frac{1}{n}\left(1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots + \frac{1}{3^n}\right)$.

Solution.

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{n}\left(1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots + \frac{1}{3^n}\right) &= \frac{1}{1 - 1/3} \\ &= \frac{3}{2}.\end{aligned}$$

$\square$

(d) What is $\lim_{n\to\infty}(1 + a + a^2 + \cdots + a^n)$ for $a \geq 1$.

Solution. For $a \geq 1$,

$$\begin{aligned}\lim_{n\to\infty}(1 + a + a^2 + \cdots + a^n) \geq \lim_{n\to\infty}(1 + 1 + \cdots + 1) &= \lim_{n\to\infty} n + 1 \\ &= \infty.\end{aligned}$$

$\square$

Exercise 10.6

(a)

Proposition 4. *Let* (s_n) *be a sequence such that*

$$|s_{n+1} - s_n| < 2^{-n} \qquad \textit{for all } n \in \mathbb{N},$$

then (s_n) *is a Cauchy sequence and hence a convergent sequence.*

Proof. We need to show the existence of N such that for all $n, m > N$, $|s_n - s_m| < \epsilon$. Without loss of generality, assume $n \geq m$. Also, $|s_n - s_m| < \epsilon$ is always true for $n = m$ so we consider $n > m$.

Notice

$$\begin{aligned}
|s_n - s_m| &\leq |s_n - s_{n-1}| + |s_{n-1} - s_{n-2}| + \cdots + |s_{m+1} - s_m| \\
&\leq \sum_{k=m}^{n-1} |s_{k+1} - s_k| \\
&\leq \sum_{k=m}^{n-1} 2^{-k} \\
&< \lim_{n\to\infty} \sum_{k=N+1}^{n} 2^{-k}.
\end{aligned}$$

So now we just need to find N such that $\sum_{k=N+1}^{\infty} 2^{-k} < \epsilon$.

$$\begin{aligned}
\lim_{n\to\infty} \sum_{k=N+1}^{n} 2^{-k} &< \epsilon \\
\frac{1}{2^{N+1}} + \frac{1}{2^{N+2}} + \cdots &< \epsilon \qquad (\textit{geometric series}) \\
\frac{1}{2^{N+1}} \left(\frac{1}{1 - \frac{1}{2}} \right) &< \epsilon \\
\frac{1}{2^N} &< \epsilon \\
2^N &> \frac{1}{\epsilon} \\
N &> \log_2 \frac{1}{\epsilon}.
\end{aligned}$$

Hence, we can take $N = \max\{\lceil \log_2 \frac{1}{\epsilon} \rceil + 1, 1\}$ and we will get

$$|s_n - s_m| < \lim_{n\to\infty} \sum_{k=N+1}^{n} 2^{-k} < \epsilon$$

for all $n, m > N, \epsilon > 0$. □

(b) Is the result in (a) true if we only assume $|s_{n+1} - s_n| < \frac{1}{n}$ for all $n \in \mathbb{N}$.

Solution. Not necessarily. Let $s_n = \sum_{k=1}^{n} \frac{1}{2k}$. Then $|s_{n+1} - s_n| = \frac{1}{2n} < \frac{1}{n}$ but there does not exists N such that for all $n, m > N$, $|s_n - s_m| < \epsilon$. Since $\lim_{n\to\infty} \sum_{N+1}^{n} \frac{1}{2k} = \lim_{n\to\infty} \frac{1}{2} \sum_{N+1}^{n} \frac{1}{k} > \frac{1}{2} \lim_{n\to\infty} \int_{N+1}^{n} \frac{1}{x} dx = \infty$, we can always set $m = N + 1$ and find n such that $|s_n - s_m| > \epsilon$. □

Exercise 10.9

Let $s_1 = 1$ and $s_{n+1} = \left(\frac{n}{n+1}\right) s_n^2$ for $n \geq 1$.

(a) Find s_2, s_3, s_4.

Solution.

$$\begin{aligned} s_2 &= \left(\frac{1}{2}\right) s_1^2 = \frac{1}{2} \\ s_3 &= \left(\frac{2}{3}\right) s_2^2 = \frac{1}{6} \\ s_4 &= \left(\frac{3}{4}\right) s_3^2 = \frac{1}{48}. \end{aligned}$$

$\square$

(b) Show $\lim s_n$ exists.

Proof. Observe that (s_n) is monotonically decreasing, which can be proved by induction. But in fact look at the equation $s_{n+1} = \left(\frac{n}{n+1}\right) s_n^2$ carefuly, $s_n < 1$ starting from $n = 2$ so a square of it will only get smaller while $\frac{n}{n+1}$ is always less than 1. So, $s_{n+1} < s_n$ for all $n \geq 2$ [also true for $n = 1$].

Also, (s_n) is bounded below by 0 because every s_{n+1} is defined by multiplication of positive numbers, so $s_{n+1} > 0$ for all $n \in \mathbb{N}$.

Hence, (s_n) is monotonically decreasing and bounded below, so it converges and $\lim s_n$ exists. $\square$

(c) Prove $\lim s_n = 0$.

Proof. Notice since $\lim s_n$ exists, we can let $\lim s_n = s = \lim s_{n+1}$. Then according to the recusive definition of (s_n),

$$\begin{aligned} s_{n+1} &= \left(\frac{n}{n+1}\right) s_n^2 \\ \lim s_{n+1} &= \lim \left(\frac{n}{n+1}\right) s_n^2 \\ \lim s_{n+1} &= \lim \left(\frac{n}{n+1}\right) \cdot \lim s_n^2 \\ s &= s^2 \\ s^2 - s &= 0. \end{aligned}$$

So, $s = 0$ or $s = 1$. But since (s_n) is monotonically decreasing and s_2 is already less than 1, $s = 1$ is not possible.

Alternatively, we can prove by using squeeze theorem. We prove by induction that $0 \leq s_n \leq \frac{1}{n}$.

Base case: $n = 1$. $s_1 = 1$ and $\frac{1}{1} = 1$.

Inductive step: Assume $0 \leq s_k \leq \frac{1}{k}$. Then,

$$0 \leq s_{k+1} = \left(\frac{k}{k+1}\right) s_k^2 \leq \left(\frac{k}{k+1}\right) \left(\frac{1}{k}\right)^2 = \frac{1}{k(k+1)} \leq \frac{1}{k+1}.$$

Therefore, $0 \leq s_n \leq \frac{1}{n}$ for all $n \in \mathbb{N}$. Then, by squeeze theorem, $\lim s_n = 0$ because $\lim 0 = 0$ and $\lim \frac{1}{n} = 0$. $\square$