# Math 109 HW5

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**Problem 1** Let $A, B, C$ be the set of students that like Reasoning, Algebra, and Calculus respectively. The question is asking for $|(A \cup B \cup C)^c|$.

$$\begin{aligned}
|(A \cup B \cup C)^c| &= 182 - |A \cup B \cup C| \\
&= 182 - [|A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|] \\
&= 182 - [129 + 129 + 129 - 85 - 89 - 86 + 53] \\
&= 2.
\end{aligned}$$

**Problem 2.** Let $X = (A \cap B); Y = (A \cap C); Z = (B \cap C); W = (A \cap B \cap C)$, and $x = |X|; y = |Y|; z = |Z|; w = |W|$. The question is asking for $w$. We know

$$\begin{aligned}
A &= (A \cap B^c \cap C^c) \cup (X \cup Y) && (1) \\
B &= (A^c \cap B \cap C^c) \cup (X \cup Z) && (2) \\
C &= (A^c \cap B^c \cap C) \cup (Y \cup Z). && (3)
\end{aligned}$$

and notice that the sets enclosed with parentheses are disjoint. Now, let us apply the inclusion-exclusion principle,

$$\begin{aligned}
170 - |A \cup B \cup C| = 2 &= 170 - [|A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|] \\
2 &= 170 - [124 + 124 + 124 - x - y - z + w] \\
204 &= x + y + z + -w, && (4)
\end{aligned}$$

which is our first equation. Then we can apply the addition principle and inclusion-exclusion principle on (1), (2), and (3), and we get

$$\begin{aligned}
|A| = 124 &= 10 + (x + y - w) \\
114 &= x + y - w && (5) \\
|B| = 124 &= x + z - w && (6) \\
|C| = 124 &= 4 + y + z - w \\
120 &= y + z - w. && (7)
\end{aligned}$$

Solving (4), (5), (6), and (7), we get $w = 50, x = 84, y = 80, z = 90$.

**Problem 3.**

(i) Let $A, B, C$ be the set of people that speak English, Spanish, and Swahili respectively. Let

$$\begin{aligned}
a &= |(A \cap B^c \cap C^c)| \\
b &= |(A^c \cap B \cap C^c)| \\
c &= |(A^c \cap B^c \cap C)| \\
x &= |(A \cap B)| \\
y &= |(A \cap C)| \\
z &= |(B \cap C)| \\
w &= |(A \cap B \cap C)|.
\end{aligned}$$

Our goal is to maximize $a+b+c$. We know

$$
\begin{aligned}
|A\cup B\cup C| = 100 &= [|A|+|B|+|C|-|A\cap B|-|A\cap C|-|B\cap C|+|A\cap B\cap C|] \\
100 &= 75+60+45-x-y-z+w \\
80 &= x+y+z-w. \qquad (8)
\end{aligned}
$$

Also,

$$
\begin{aligned}
|A| &= 75 = a+x+y-w \qquad (9)\\
|B| &= 60 = b+x+z-w \qquad (10)\\
|C| &= 45 = c+y+z-w. \qquad (11)
\end{aligned}
$$

Then, from (9) to (11),

$$
\begin{aligned}
|A|+|B|+|C| = 75+60+45 &= a+b+c+2x+2y+2z-3w \\
180 &= a+b+c+2x+2y+2z-3w. \qquad (12)
\end{aligned}
$$

(12) $-\ 2 \times$ (8):

$$a+b+c = 20+w. \qquad (13)$$

Hence, we need to maximize $w$. Note that $x,y,z \geq w$. Thus, by (8), $w \leq 40$, and $a+b+c \leq 60$. Taking $w,x,y,z = 40$ and solving (9) to (11), $a=35, b=20, c=5$.

(ii) (9) – (8):

$$
\begin{aligned}
-5 &= a-z \\
a &= z-5
\end{aligned}
$$

Hence, maximizing $a$ is equivalent to maximizing $z$. $z = |B\cap C| \leq |C| = 45 \Rightarrow a \leq 40$. Thus, we take $a=40, z=45$. Notice that $c = |(A^c\cap B^c\cap C)|$ an $z = |B\cap C|$, so if $z = 45 = |C|$, then $c=0$. Then, solving (9) to (11), we can conclude with the following equations equations:

$$
\begin{aligned}
&a = 40\\
&z = 45\\
&c = 0\\
&x = 35\\
&b = w-20 = y-20\\
&0 \leq b \leq 15 \qquad (\because w = y \leq |A|-a = 35) \qquad (14)
\end{aligned}
$$

Maximum number of people who speak only English = 40. In this case, the number of people who speak only Spanish = 0, and the number of people who speak only Swahili can range from 0 to 15.

(iii) Already proved in (i).