Math 154 HW4

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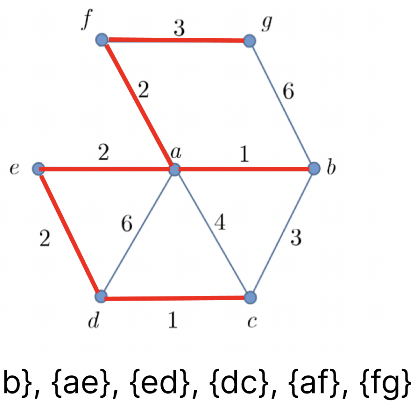
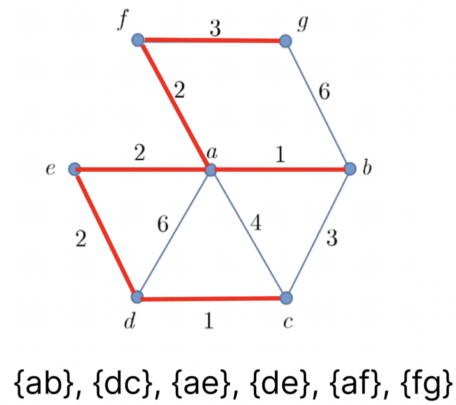
Problem 1.



Prim's



Kruskal's

Problem 2.

(a) Let v_1, v_2, v_3, v_4 be the only vertices in the graph G with edges $\{v_1, v_2\}$ and $\{v_3, v_4\}$ only. Then all vertices have degree one but the graph is disconnected.

(b) The proof using induction inherently assumed that all graphs are constructed algorithmically by adding one vertex each at a time. However, it overlooked the case where the graph can be constructed by adding multiple vertices at a time, and the condition of having at least degree one will still be true. Hence, the proof is not exhaustive.

More concretely saying, usually when we are doing induction on graph, we assume a graph G with something $n = k + 1$, *then* proceed to remove 1 edge / vertex or whatever to get G'. During the removal process, it's exhausting all possible cases, so we can say that the proof is exhaustive. Yet, the false proof misses this idea.

Problem 3.

Proposition 1. *Q_n is bipartite for all $n \geq 2$.*

Proof. We will prove by induction.

Base case: $n = 2$. $n = 2$ only have 4 vertices in total and can be checked to be true easily.

Inductive step: Assume Q_n is bipartite. We will show that Q_{n+1} is bipartite as well. Notice that all Q_{n+1} can be constructed by making a copy of Q_n, let Q_n', and connecting the corresponding vertices in Q_n and Q_n'.

To illustrate more clearly with examples, let all vertices of Q_n be $0x_1x_2 \ldots x_n$ and those of Q_n' be $1x_1x_2 \ldots x_n$ with x_i taking values 0 or 1. By inductive hypothesis, we know both Q_n and Q_n' are bipartite. Then, we just need to connect the corresponding vertices in Q_n and Q_n' to form Q_{n+1}.

Since Q'_n is a copy of Q_n, they have the same labeling of vertices. Then, we just need to flip the labeling of Q'_n and connect it to Q_n by adding the new edges. Since the vertices of Q_n and Q'_n are one-bit different, e.g. 010101 to 110101, the only new edges are the ones between the one-to-one corresponding vertices. Because we have flipped the labeling of Q'_n, the new edges are between vertices of different labels, and the entire Q_{n+1} are labeled with alternating labels.

Hence, Q_{n+1} is bipartite.

By Mathematical Induction, Q_n is bipartite for all $n \geq 2$. $\square$

Proposition 2. *Q_n contains at least $2 \cdot \left(2^{2^{n-1}}\right) - 1$ independent sets, for all $n \geq 2$.*

Proof. Proof goals: Q_n is bipartite $\Rightarrow$ parts A, B have same number of vertices $\Rightarrow$ parts A, B are independent sets $\Rightarrow$ subsets of A, B are also independent sets.

From *Proposition 1*, we have already proved that Q_n is bipartite. Then from the inductive step in *Prop 1*, we have flipped the labeling of the copy graph, so the new Hypercube Graph has $|A'| = |A| + |B| = |B'|$. Indeed, the parts A', B' have the same number of vertices, which is true for all Q_n.

Then, by definition of bipartite graph, all vertices exclusively in A or B are disjoint, which means A, B are themselves independent sets. Then, obviously all subsets of independent sets A, B are also independent sets.

Now, let's just count the number of independent sets in A. A has half the number of total vertices, 2^{n-1}, and the power set of A has $2^{2^{n-1}}$ elements. Now we do the same for B, so in total there are $2 \cdot \left(2^{2^{n-1}}\right)$ sets, since we double counted empty set, so it's just $2 \cdot \left(2^{2^{n-1}}\right) - 1$ in total. These are the sets that are guaranteed to be independent sets. Hence, we proved the weak inequality. $\square$

Problem 4.

Proposition 3. *A tree has at most one perfect matching.*

Proof. Suppose that a tree T has at least two perfect matchings M and M'. Since a leaf u only has one edge, it must be matched to it's parent v, and so $\{u, v\} \in M, M'$. We can then remove u, v from T, and the resulting graph is still a tree with perfect matching. We iterate the process above until we can not remove any more matchings. Since the resulting trees have even number of vertices and have at least one leaf, we can always remove an edge containing a leaf that is in both M and M' until T is empty, and so $M = M'$. $\square$