Math 104 HW1

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Exercise 1.3

Proposition 1. $1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2$ *for all positive integers* n.

Proof. We proceed by induction.

Base case: $n = 1$. We have $1^3 = 1^2$.

Inductive step: Assume that $1^3 + 2^3 + \cdots + k^3 = (1 + 2 + \cdots + k)^2$ for some $k \in \mathbb{N}$. Now consider $k + 1$,

$$\begin{aligned}
1^3 + 2^3 + \cdots + k^3 + (k+1)^3 &= (1 + 2 + \cdots + k)^2 + (k+1)^3 \\
&= \left(\frac{(k+1) \cdot k}{2}\right)^2 + (k+1)^3 \\
&= \frac{(k+1)^2 \cdot k^2 + (k+1)^2 \cdot 4(k+1)}{4} \\
&= \frac{(k+1)^2(k^2 + 4k + 4)}{4} \\
&= \frac{(k+1)^2(k+2)^2}{4} \\
&= \left(\frac{(k+1) \cdot (k+2)}{2}\right)^2 \\
&= (1 + 2 + \cdots + (k+1))^2.
\end{aligned}$$

Hence, by the principle of mathematical induction, $1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2$ for all positive integers n. $\square$

Exercise 1.5

Proposition 2. $1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n} = 2 - \frac{1}{2^n}$ *for all positive integers* n.

Proof. We again proceed by induction.

Base case: $n = 1$. We have $1 + \frac{1}{2} = 2 - \frac{1}{2}$.

Inductive step: Assume that $1+\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2^k}=2-\frac{1}{2^k}$ for some $k\in\mathbb{N}$. Now consider $k+1$,

$$\begin{aligned}
1+\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2^k}+\frac{1}{2^{k+1}} &= 2-\frac{1}{2^k}+\frac{1}{2^{k+1}}\\
&= 2-\frac{1}{2^k}+\frac{1}{2^k}\cdot\frac{1}{2}\\
&= 2-\frac{1}{2^k}\left(1-\frac{1}{2}\right)\\
&= 2-\frac{1}{2^k}\cdot\frac{1}{2}\\
&= 2-\frac{1}{2^{k+1}}.
\end{aligned}$$

Hence, by the principle of mathematical induction, $1+\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2^n}=2-\frac{1}{2^n}$ for all positive integers n. □

Exercise 1.11

(a)

Proposition 3. *If n^2+5n+1 is an even integer, then $(n+1)^2+5(n+1)+1$ is also an even integer for $n\in\mathbb{N}$.*

Consider

$$\begin{aligned}
(n+1)^2+5(n+1)+1 &= n^2+2n+1+5n+5+1\\
&= n^2+5n+1+2n+6\\
&= (n^2+5n+1)+2(n+3)\\
&= 2k+2(n+3) \qquad (\textit{for some } k\in\mathbb{Z}\because n^2+5n+1 \textit{ is an even integer})\\
&= 2(k+n+3).
\end{aligned}$$

Hence, $(n+1)^2+5(n+1)+1$ is an even integer.

(b) For which $n\in\mathbb{N}$ is n^2+5n+1 an even integer?

Solution. If n is even, then $n^2+5n+1=(2k)^2+5(2k)+1=2(2k^2+5k)+1$ for some $k\in\mathbb{Z}$, thus is an odd integer. If n is odd, then $n^2+5n+1=(2j+1)^2+5(2j+1)+1=2(2j^2+7j+3)+1$ for some $j\in\mathbb{Z}$, thus is also an odd integer. Hence, n^2+5n+1 is never an even integer.

The moral of the exercise is that even the inductive step is true, the proposition is not necessarily true without a proper and true base case. □

Exercise 2.7

Exercise 2.8

Exercise 3.1

Exercise 3.6a