Math 181A HW10

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Problem 6.5.1 Let $k_1, k_2, \ldots, k_n$ be a random sample from the geometric probability function

$$p_X(k;p) = (1-p)^{k-1}p, k = 1, 2, \ldots$$

Find Λ, the generalized likelihood ratio for testing $H_0 : p = p_0$ versus $H_1 : p \neq p_0$.

Solution.

$$\Lambda = \frac{L(p_0)}{\underset{p\in\mathbb{R}}{max} L(p)}.$$

To find $\underset{p\in\mathbb{R}}{max} L(p)$, we take the derivative of $l(p) = \ln[L(p)]$ and set it to 0.

$$\begin{aligned}
L(p) &= \prod_{i=1}^{n}(1-p)^{k_i-1}p \\
&= p^n(1-p)^{\sum\limits_{i=1}^{n}(k_i)-n} \\
l(p) &= n\ln(p) + \left(\sum_{i=1}^{n}(k_i) - n\right)\ln(1-p) \\
l'(p) = 0 &= \frac{n}{p} - \frac{\sum\limits_{i=1}^{n}(k_i) - n}{1-p} \\
p\sum_{i=1}^{n}(k_i) - np &= n(1-p) \\
p &= \frac{n}{\sum_{i=1}^{n}(k_i)} \\
&= \frac{1}{\overline{K}}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\Lambda &= \frac{L(p_0)}{\underset{p\in\mathbb{R}}{max} L(p)} \\
&= \frac{\prod\limits_{i=1}^{n}(1-p_0)^{k_i-1}p_0}{\prod\limits_{i=1}^{n}(1-\frac{1}{\overline{K}})^{k_i-1}\frac{1}{\overline{K}}} \\
&= \frac{p_0^n(1-p_0)^{\sum_{i=1}^{n}(k_i)-n}}{(1/\overline{K})^n(1-(1/\overline{K}))^{\sum_{i=1}^{n}(k_i)-n}}
\end{aligned}$$

$\square$

Problem 6.5.2 Let $y_1, y_2, \ldots, y_{10}$ be a random sample from an exponential pdf with unknown parameter λ. Find the form of the GLRT for $H_0 : \lambda = \lambda_0$ versus $H_1 : \lambda \neq \lambda_0$. What integral would have to be evaluated to determine the critical value if α were equal to 0.05?

Solution.

$$\Lambda = \frac{L(\lambda_0)}{\underset{\lambda\in\mathbb{R}}{max} L(\lambda)}.$$

To find $\underset{\lambda\in\mathbb{R}}{max} L(\lambda)$, we first find the maximum likelihood estimator by taking the derivative of $l(\lambda) = \ln[L(\lambda)]$ and set it to 0.

$$\begin{aligned}
L(\lambda) &= \prod_{k=1}^{10} \lambda e^{-\lambda y_k} \\
&= \lambda^{10} e^{-\lambda \sum_{k=1}^{10} y_k} \\
l(\lambda) &= 10\ln(\lambda) - \lambda \sum_{k=1}^{10} y_k \\
l'(\lambda) = 0 &= \frac{10}{\lambda} - \sum_{k=1}^{10} y_k \\
\lambda &= \frac{10}{\sum_{k=1}^{10} y_k} \\
&= \frac{1}{\overline{Y}}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\Lambda &= \frac{L(\lambda_0)}{\underset{\lambda\in\mathbb{R}}{max} L(\lambda)} \\
&= \frac{\lambda_0^{10} \cdot e^{-\lambda_0 \sum_{k=1}^{10} y_k}}{(1/\overline{Y})^{10} \cdot e^{-(1/\overline{Y}) \sum_{k=1}^{10} y_k}} \\
&= \frac{\lambda_0^{10} \cdot e^{-\lambda_0 \sum_{k=1}^{10} y_k}}{(1/\overline{Y})^{10} \cdot e^{-10}} \\
&= (\overline{Y} \cdot \lambda_0)^{10} \cdot e^{10-\lambda_0 \sum_{k=1}^{10} y_k} \\
&= (\overline{Y} \cdot \lambda_0)^{10} \cdot e^{10-\lambda_0 \cdot 10\overline{Y}} \\
&= (\overline{Y} \cdot \lambda_0)^{10} \cdot e^{10(1-\lambda_0 \cdot \overline{Y})}.
\end{aligned}$$

To find the critical value, we need to find c such that

$$\begin{aligned}
\alpha &= P(\Lambda \leq c) \\
\alpha &= \int_0^c (\overline{Y} \cdot \lambda_0)^{10} \cdot e^{10(1-\lambda_0 \cdot \overline{Y})} d\overline{Y}.
\end{aligned}$$

□

Problem 6.5.3

Problem 6.5.4

Problem 6.5.5

Problem 6.5.6