Math 104 HW6

Neo Lee

10/13/2023

Exercise 12.10

Proposition 1. *(s_n) is bounded if and only if* $\limsup |s_n| < +\infty$.

Proof. Forward direction: Suppose (s_n) is bounded. Then there exists $M > 0$ such that $|s_n| \leq M$ for all $n \in \mathbb{N}$. Then $\sup |s_n| \leq M$, so $\limsup |s_n| \leq M < +\infty$ [recall we have proved previously that a sequence $(t_n) \leq M$ cannot have $\lim t_n > M$].

Backward direction: Suppose $\limsup |s_n| < +\infty$. Then let $\limsup |s_n| = L$. By definition of limit, there exists N such that for all $n > N$, $|s_n| \leq \sup |s_n| < L+1$. Then let $M = \max\{|s_1|, |s_2|, \ldots, |s_{N-1}|, |s_N|, L+1\}$. Then for all $n \in \mathbb{N}$, $|s_n| \leq M$, so (s_n) is bounded. $\square$

Exercise 14.3

Determine which of the following series converge:
(a) $\sum \frac{1}{\sqrt{n!}}$ **(b)** $\sum \frac{2+\cos n}{3^n}$ **(c)** $\sum \frac{1}{2^n+n}$ **(d)** $\sum \left(\frac{1}{2}\right)^n \left(50 + \frac{2}{n}\right)$ **(e)** $\sum \sin\left(\frac{n\pi}{9}\right)$ **(f)** $\sum \frac{(100)^n}{n!}$.

Solution.

(a) Ratio Test:

$$\begin{aligned}
\limsup \left| \frac{1}{\sqrt{(n+1)!}} \cdot \frac{\sqrt{n!}}{1} \right| &= \limsup \left| \frac{1}{\sqrt{n}} \right| \\
&= \lim \left| \frac{1}{\sqrt{n}} \right| \\
&= 0 < 1.
\end{aligned}$$

So the series converges.

(b) Root Test:

$$\begin{aligned}
\limsup \left| \frac{2+\cos n}{3^n} \right|^{\frac{1}{n}} &= \limsup \left| \frac{(2+\cos n)^{1/n}}{3} \right| \\
&\leq \limsup \left| \frac{(2+1)^{1/n}}{3} \right| \\
&= \lim \left| \frac{3^{1/n}}{3} \right| \\
&= \frac{1}{3} < 1.
\end{aligned}$$

So the series converges.

(c) Comparison Test: Consider the series $\sum \frac{1}{2^n}$. Then for all $n \in \mathbb{N}$,

$$\left|\frac{1}{2^n+n}\right| \leq \frac{1}{2^n}.$$

We know that $\sum \frac{1}{2^n}$ converges by Root Test with $\lim \left|\frac{1}{2^n}\right|^{1/n} = \frac{1}{2}$, so by comparison test, $\sum \frac{1}{2^n+n}$ converges.

(d) Comparison Test: Consider the series $\sum \left(\frac{1}{2}\right)^n \cdot (52)$. Then for all $n \in \mathbb{N}$,

$$\left|\left(\frac{1}{2}\right)^n \left(50+\frac{2}{n}\right)\right| \leq \left(\frac{1}{2}\right)^n \cdot (52).$$

We know the series $\sum \left(\frac{1}{2}\right)^n \cdot (52)$ converges by Ratio Test with $\lim \left|\frac{\left(\frac{1}{2}\right)^{n+1} \cdot (52)}{\left(\frac{1}{2}\right)^n \cdot (52)}\right| = \frac{1}{2} < 1$, so by comparison test, $\sum \left(\frac{1}{2}\right)^n \left(50+\frac{2}{n}\right)$ converges.

(e) Consider $n_k = 9k+1$ and $n_j = 9k+2$ for all $k \in \mathbb{N}$. Then $\sin\left(\frac{n_k \pi}{9}\right)$ and $\sin\left(\frac{n_j \pi}{9}\right)$ are two different constant subsequences of $\sin\left(\frac{n\pi}{9}\right)$ with different limits, so $\sin\left(\frac{n\pi}{9}\right)$ diverges. In particular, it does not converge to 0, so the series $\sum \sin\left(\frac{n\pi}{9}\right)$ diverges.

(f) Ratio Test:

$$\begin{aligned}\limsup \left|\frac{(100)^{n+1}}{(n+1)!} \cdot \frac{n!}{(100)^n}\right| &= \limsup \left|\frac{100}{n+1}\right| \\ &= \lim \left|\frac{100}{n+1}\right| \\ &= 0 < 1.\end{aligned}$$

□

Exercise 14.6

(a)

Proposition 2. *If $\sum |a_n|$ converges and (b_n) is a bounded sequence, then $\sum a_n b_n$ converges.*

Proof. We show that $\sum a_n b_n$ converges absolutely and hence converges. Since (b_n) is bounded, there exists $M > 0$ such that $|b_n| \leq M$ for all $n \in \mathbb{N}$. Now notice

$$\begin{aligned}|a_n b_n| &= |a_n||b_n| \\ &\leq |a_n| M.\end{aligned}$$

Further, since $\sum |a_n|$ converges, $\sum |a_n| M = M \sum |a_n|$ also converges. Then by comparison test, $\sum |a_n b_n|$ converges, so $\sum a_n b_n$ converges absolutely.

Alternatively, since (b_n) is bounded, we know there exists a M' such that $M' \geq |b_n|$ for all $n \in \mathbb{N}$. Now, denote $M = max\{M', 1\}$. Then, we know there exists $N \in \mathbb{N}$ such that for $n \geq m > N$,

$\sum_{k=m}^{n} |a_k| < \frac{\epsilon}{M}$ for all $\epsilon > 0$. Now, take such N and

$$\sum_{k=m}^{n} |a_k| < \frac{\epsilon}{M}$$

$$M \sum_{k=m}^{n} |a_k| < \epsilon$$

$$\left| \sum_{k=m}^{n} a_k b_k \right| \le \sum_{k=m}^{n} |a_k||b_k| \le \sum_{k=m}^{n} |a_k| M < \epsilon.$$

Hence, $\sum a_n b_n$ satifies the Cauchy criterion and thus converges. □

(b) Observe that *Corollary 14.7*: absolute convergent series are convergent, is a special case of (a).

Solution. *Corrollary 14.7* is a special case of (a) by taking $(b_n) = 1$. □

Exercise 14.8

Proposition 3. *If $\sum a_n$ and $\sum b_n$ are convergent series of nonnegative numbers, then $\sum \sqrt{a_n b_n}$ converges. Hint: show $\sqrt{a_n b_n} \le a_n + b_n$ for all n.*

Notice for all n

$$a_n^2 + b_n^2 + 2a_n b_n \ge a_n b_n$$

$$(a_n + b_n)^2 \ge a_n b_n$$

$$a_n + b_n \ge \sqrt{a_n b_n}.$$

Also, we know there exists N_1 such that for $n \ge m > N_1$, $|\sum_{k=m}^{n} a_k| < \epsilon/2$ and N_2 such that for $n \ge m > N_2$, $|\sum_{k=m}^{n} b_k| < \epsilon/2$. Now we take $N = \max\{N_1, N_2\}$ for some $\epsilon > 0$. Then, for all $n \ge m > N$

$$\begin{aligned}
\left| \sum_{k=m}^{n} \sqrt{a_n b_n} \right| &\le \left| \sum_{k=m}^{n} a_k + b_k \right| \\
&\le \left| \sum_{k=m}^{n} a_k + \sum_{k=m}^{n} b_k \right| \\
&\le \left| \sum_{k=m}^{n} a_k \right| + \left| \sum_{k=m}^{n} b_k \right| \\
&< \epsilon.
\end{aligned}$$

Hence, $\sum \sqrt{a_n b_n}$ satisfied Cauchy criterion and thus converges.

Exercise 14.9

Proposition 4. *The convergence of a series does not depend on any finite number of terms, though the value of the limit does. More precisely, consider series $\sum a_n$ and $\sum b_n$ and suppose that the set $\{n \in \mathbb{N} : a_n \neq b_n\}$ is finite. Then the series both converge or else the both diverge.*

Proof. Without loss of generality, we will focus on $\sum a_n$ and conclude the convergence of $\sum b_n$ based on $\sum a_n$. Also, denote $M = \max\{n \in \mathbb{N} : a_n \neq b_n\}$.

Case 1: $\sum a_n$ converges. We know $\sum a_n$ satisfies Cauchy criterion, thus we know there exists N_1 such that for all $n \geq m > N_1$, $|\sum_{k=m}^{n} a_k| < \epsilon$ for some $\epsilon > 0$.

Then, let $N_2 = \max\{N_1, M\}$. Since we have set N_2 to be at least M, any terms after N_2 for b_n is the same as a_n. Thus, any statement that holds true for a_n is also true for b_n after N_2 and we can conclude for all $n \geq m > N_2$ $|\sum_{k=m}^{n} b_k| < \epsilon$ for some $\epsilon > 0$.

Therefore, $\sum b_n$ satifies Cauchy criterion too and thus converges.

Case 2: $\sum a_n$ diverges. Assume for the sake of contradiction that $\sum b_n$ converges. Then there exists N_2 for all $\epsilon > 0$ such that for $n \geq m > N_2$, $|\sum b_n| < \epsilon$. Thus, we can take $N_1 = \max\{N_2, M\}$, which will make sure that for $n \geq m > N_1$, $|\sum_{k=m}^{n} a_n| < \epsilon$. But that contradicts that fact that $\sum a_n$ diverges. Hence, $\sum b_n$ must diverge. □

Exercise 14.14

Proposition 5. *Show that* $\sum_{n=1}^{\infty} \frac{1}{n}$ *diverges by comparing with the series* $\sum_{n=2}^{\infty} a_n$ *where* (a_n) *is the sequence*

$$(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{16}, \frac{1}{16}, \frac{1}{16}, \frac{1}{16}, \frac{1}{16}, \frac{1}{16}, \frac{1}{16}, \frac{1}{16}, \frac{1}{32}, \frac{1}{32}, \dots).$$

Note: this is also known as the Cauchy Condensation Test.

Proof. We will show that $\sum_{n=2}^{\infty} \frac{1}{n}$ diverges and thus $\sum_{n=1}^{\infty} \frac{1}{n}$, which differs only by $n = 1$.

Notice for all $2^k < n \leq 2^{k+1}$, $a_n = \frac{1}{2^{k+1}} \leq \frac{1}{n}$. This is true for all $k \in \mathbb{N}$. Hence, $\frac{1}{n} \geq a_n$ for all n. Now observe within each interval $(2^k, 2^{k+1}]$, there are 2^k terms. Therefore, $\sum_{n=2^k}^{2^{k+1}} a_n = \frac{1}{2}$ and $\sum_{n=2}^{\infty} a_n = \lim_{k\to\infty} k\left(\frac{1}{2}\right) = \infty$.

Hence, by Comparison Test, $\sum_{n=1}^{\infty} \frac{1}{n}$ also diverges. □