Math 154 HW1

Neo Lee

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Problem 1.

(a) 15.

(b) $N(v_1) = \{v_4, v_7\}, N(v_5) = \{v_3, v_4, v_6\}, d(v_1) = 2, d(v_5) = 3, \delta(G) = 2, \Delta = 6$.

(c) 5.

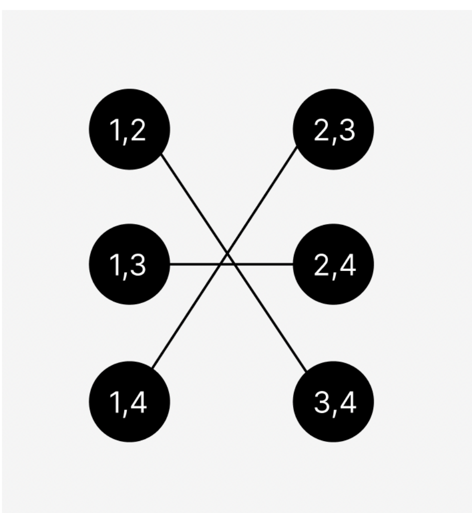
(d) 2.

(e) $K_{1,7}$ is not a subgraph of G. $\Delta(K) = 7 \Rightarrow \exists v \in V(K_{1,7})$ such that $d(v) = 7$. However, $\Delta(G) = 6 \Rightarrow \nexists w \in G$ such that $d(w) = 7 \Rightarrow v \notin V(G)$. Hence, $V(K_{1,7}) \nsubseteq V(G)$.

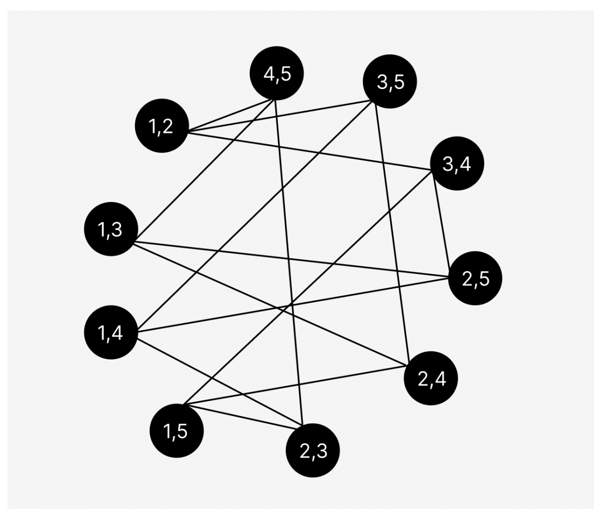
Problem 2.

(a) $K_{n:1}$ is just a clique K_n because there are $\binom{n}{1} = n$ number of vertices and all vertices are pairwise disjoint, so there will be an edge between every pair of vertices, which is the definition of a clique.

(b)



$K_{4:2}$



$K_{5:2}$

(c) By definition, there are a total of $\binom{n}{r}$ vertices. Then, for any arbitrary $v \in V(K_{n:r})$, there are $\binom{n-r}{r}$ disjoint vertex (basically just choosing r number of vertices that are not in v already), which would be the neighbors. Hence, for all $v \in V(K_{n:r})$, $d(v) = \binom{n-r}{r} \Rightarrow \sum_{v \in V} d(v) = \binom{n}{r} \times \binom{n-r}{r} \Rightarrow |E(K_{n:r})| = \frac{1}{2} \sum_{v \in V} d(v) = \frac{1}{2}\binom{n}{r} \times \binom{n-r}{r}$ [Handshake lemma].

Problem 3.

Proposition 1. *Every graph with at least two vertices contains two vertices with the same degree.*

Proof. Let G be a graph with $|V(G)| = n$, then for any $v \in V(G)$, $d(v) \leq n-1$. More specifically, $d(v) \in A = \{0 \leq m \leq n-1 : m \in \mathbb{Z}\}$.

However, we have to realize that there cannot exist any pair $v, w \in V(G)$ simultaneously such that $d(v) = 0$ and $d(w) = n-1$ because $\exists w \in V(G)$ such that $d(w) = n-1 \Rightarrow v \in N(w) \Rightarrow d(v) \neq 0$, and vice versa.

Hence, in fact $A = \{1 \leq m \leq n-1 : m \in \mathbb{Z}\}$ or $A = \{0 \leq m \leq n-2 : m \in \mathbb{Z}\}$. In either case, $|A| = n-1$. Yet, there are n vertices in $V(G)$, so by pigeonhole principle, there must be at least two vertices with the same degree. □

Problem 4.

Proposition 2. *Every tournament contains a Hamiltonian path.*

Proof. We can easily see that the proposition holds for $n = 1, 2$ vertices, so they will be omitted here. Now, let's prove by induction starting from $n = 3$.

Base case: $n = 3$.

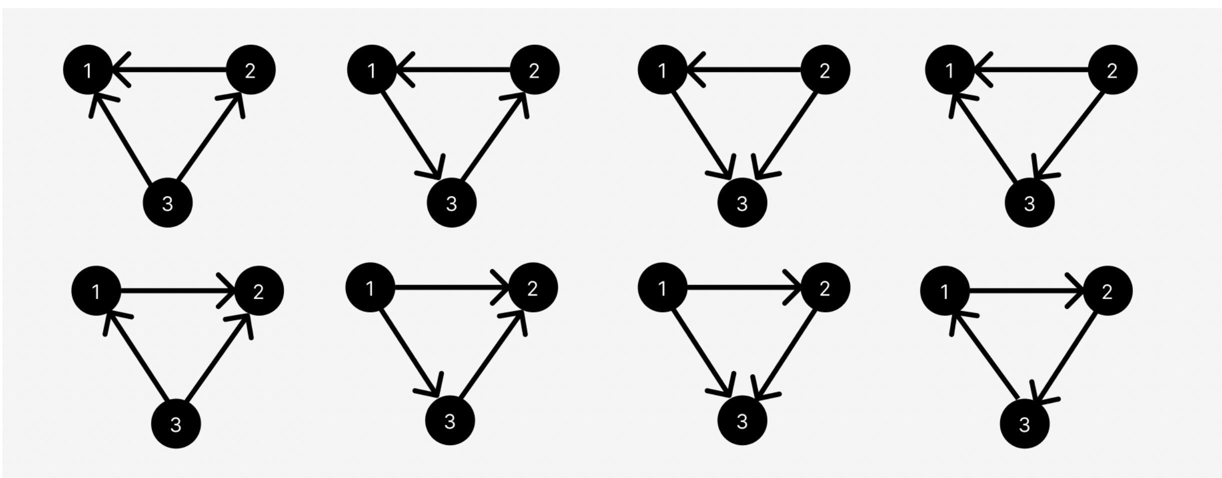


Figure 1: Orientations of K_3

We can see that every possible orientation of K_3 contains a Hamiltonian path. Hence, the proposition holds for $n = 3$.

Inductive step: assume every tournament with number of vertices k contains a Hamiltonian path.

Let G be a tournament with $|V(G)| = k+1$, we want to prove that G contains a Hamiltonian path. First, let's consider the subgraph $G' = G - w$ for which w is an arbitrary vertex in G. By induction hypothesis, G' contains a directed path containing all of its vertices. We can denote the directed path as $P' = (v_1, \cdots, v_i, \cdots, v_k)$. Then, there can be three cases for G:

Case 1: (w, v_1) is an arc of G. Then we know there is a Hamiltonian path in G by adding w to the beginning of P'.

Case 2: (v_k, w) is an arc of G. Then we also know there is a Hamiltonian path in G by appending w to the end of P'.

Case 3: (w, v_1) and (v_k, w) are both not arcs of $G \Rightarrow (v_1, w)$ and (w, v_k) must be arcs of G. In other words, v_1 is adjacent *to* w and v_k is adjacent *from* w. Then we know for $1 \leq i \leq k-1$, there much exist v_i such that v_i is adjacent *to* w and v_{i+1} is adjacent *from* w because the direction of the arcs between v_i and w must switch for $i \in [1, k]$.

Therefore, we can construct a Hamiltonian path in G by inserting w between v_i and v_{i+1}, giving $P = (v_1, \cdots, v_i, w, v_{i+1}, \cdots, v_k)$.

Hence, we have proved that every tournament with $k+1$ vertices contains a Hamiltonian path.

By Mathematical Induction, we have proved that every tournament with $|V| \geq 3$ contains a Hamiltonian path.

We know that the proposition is indeed also true for $n = 1, 2$, thus we have proved that every tournament contains a Hamiltonian path. $\square$