STAT 155 Notes

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Contents

Chapter 1

Combinatorial games

1.1 Lecture 1 (skipped)

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Definition 1.3.1: Combinatorial game

A two-player game of perfect information where the players alternate moves and the game ends in a finite number of moves. At each position $v \in V$, there is a set of moves E_v available to the player whose turn it is. The game ends when $E_v = \emptyset$.

Note:

We can think of a combinatorial game as a directed graph $G = (V, E)$ where V is the set of positions and E is the set of edges. The game starts at some vertex v_0 and ends when v_0 has no outgoing edges.

Definition 1.3.2: Impartial game

A game where the set of moves available to each player is the same at each position. The terminal reward to each player is the same. Players alternate moves.

Definition 1.3.3: Partisan game

The complement definition of an impartial game.

Example 1.3.1 (Impartial games)

1. Tic-tac-toe
2. Chomp
3. Subtraction games
4. Nim

Definition 1.3.4: Proressively bounded combinatorial game

A combinatorial game is progressively bounded if for every starting position v_0, there is a finite bound $B(v_0)$ on the number of moves in the game before the game ends.

Definition 1.3.5: Strategy

Let $V_{NT} \subseteq V$ is the set of non-terminal positions. A strategy for a player is a function that assigns a legal move to each non-terminal position $v \in V_{NT}$

$$\sigma : V_{NT} \to \cup_{v \in V_{NT}} E_v.$$

Definition 1.3.6: Winning strategy

A winning strategy for a player from position $v \in V$ is a strategy that is guaranteed to result in a win for that player.

Theorem 1.3.1 Existence of a winning strategy

In a progressively bounded, impartial combinatorial game, $V = F \sqcup S$. That is, from any initial position, one of the players has a winning strategy.

Note:

Denote F, S as the wining positions for first and second player respectively.

Proof: We prove this by induction on the finite bound $B(v)$ on the number of moves in the game before the game ends. If $B(v) = 0$, then the game ends immediately and v is a winning position either for the first or second player. Hence, for $B(v) = 0$, the theorem holds.

Now suppose that the theorem holds for all $B(v) < k$ and consider a game with $B(v) = k$. Let $v \in V$ be the initial position. Without loss of generality, suppose it is the first player's turn. The player can move along some edges $e \in E_v$ to v'. By inductive hypothesis, $V = F \sqcup S$ for $B(v) = k - 1$. If $v' \in F$, then the first player has a winning strategy at v, else if $v' \in S$, then the second player has a winning strategy at v. Hence, for all v, either $v \in F$ or $v \in S$. This proves the theorem. ☺

Theorem 1.3.2 Existence of winning strategy for first player in rectangular Chomp

In a non-terminal rectangular Chomp game with $m \times n$ for board, the first player always has a winning strategy.

Proof: First player eats the top right corner to move to v'. Then, by *Theorem 1.3.1*, the new position v' is either a winning position for the first player or the second player. Suppose $v' \in S$, and the next move by the second player moves to v''. Then, in fact, the first player can choose to not eat the right top corner in the first move, but instead directly move to v'', which would make v the winning position for the first player.

Note:

- v' is the rectangular board with the top right corner eaten.
- This is called a *strategy stealing*.

☺

1.4 Lecture 4

Proposition 1.4.1 Being first player is better

Assuming both players are perfect players, being the first player is usually better than being the second player in a progressively bounded impartial combinatorial game.

Proof: Let $v_0 \in V$ be the starting position. In order for v_0 to be in F, there only needs to be one neighbor of v_0 that is in S. However, for v_0 to be in S, all neighbors of v_0 must be in F.

Thinking in terms of edges, for v_0 to be in F, there only needs to be one good edge, but for v_0 to be in S, all edges must be bad. 😊

Lemma 1.4.1 Partitioning Nim columns

Given a position O in Nim, partition the board into two sets of columns X and Y such that X, Y are non-empty disjoint columns. Then X, Y are each individually smaller Nim boards.

1. If $X, Y \in S$, then $X \cup Y \in S$.
2. If $X \in S, Y \in F$ or $X \in F, Y \in S$, then $X \cup Y \in F$.

Note:

$X \cup Y$ is the union of the two sets of columns, which returns the original board O.

Proof: (1) : First, notice that $\emptyset \in S$ because if first player starts with an empty board, it means the previous move made by the second player removed the last column, which means the second player won.

(1) $\Longrightarrow$ (2) : Assume without loss of generality that $X \in F, Y \in S$, then we can move the position X to X', which is in S. Then, $X' \cup Y \in S$ by (1), which implies by reverting one move, $X \cup Y \in F$. 😊

Note:

This is only true for Nim because Nim guarantees that the two sets of columns are disjoint and making a move on one set of columns does not affect the other set of columns.

Chapter 2

Starting a new chapter

2.1 Demo of commands

Definition 2.1.1: Some defintion

yap

Question 1: Some question

yap

Solution

Some proof: yap ☺

Note:

Some note

Theorem 2.1.1 Some theorem

yap

Wrong Concept 2.1.1: Some wrong concept

yap

Lemma 2.1.1 Some lemma

yap

Proposition 2.1.1 Some proposition

yap

Example 2.1.1 (Some example)

yap

Claim 2.1.1 Some claim

yap

Corollary 2.1.1 Some corollary

yap

Some unlabeled theorem

This is a new paragraph

Algorithm 1: Some algorithm

```
Input: input
Output: output
/* This is a comment */
This is first line ;                                  // This is also a comment
if x > 5 then
    do nothing
else if x < 5 then
    do nothing
else
    do nothing
end
while x == 5 do
    still do nothing
end
foreach x = 1 : 5 do
    do nothing
end
return return nothing
```