Math 110 HW6

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Problem 1.

Suppose $q \in \mathcal{P}(\mathbb{R})$. Prove that there exists a polynomial $p \in \mathcal{P}(\mathbb{R})$ such that

$$q(x) = (x^2 - 3x)p''(x) + (2x - 3)p'(x) + p(0).$$

Proof. We want to show that this map, specifically a linear map, is surjective. Denote the map as T.

We first show that this is a linear map. Consider p_1 and p_2, then

$$\begin{aligned}
T(p_1 + p_2)(x) &= (x^2 - 3x)(p_1 + p_2)''(x) + (2x - 3)(p_1 + p_2)'(x) + (p_1 + p_2)(0) \\
&= (x^2 - 3x)(p_1'' + p_2'')(x) + (2x - 3)(p_1' + p_2')(x) + (p_1 + p_2)(0) \\
&= (x^2 - 3x)p_1''(x) + (2x - 3)p_1'(x) + p_1(0) + (x^2 - 3x)p_2''(x) + (2x - 3)p_2'(x) + p_2(0) \\
&= T(p_1)(x) + T(p_2)(x).
\end{aligned}$$

Consider $\lambda \in \mathbb{F}$, then

$$\begin{aligned}
T(\lambda p)(x) &= (x^2 - 3x)(\lambda p)''(x) + (2x - 3)(\lambda p)'(x) + (\lambda p)(0) \\
&= \lambda(x^2 - 3x)p''(x) + \lambda(2x - 3)p'(x) + \lambda p(0) \\
&= \lambda\left((x^2 - 3x)p''(x) + (2x - 3)p'(x) + p(0)\right) \\
&= \lambda T(p)(x).
\end{aligned}$$

Next for each $q \in \mathcal{P}(\mathbb{R})$, we restrict the domain and codomain of T to $\mathcal{P}_m(\mathbb{R})$ for $m = \text{degree}(q)$. Then we show that for each $q \in \mathcal{P}(\mathbb{R})$, $T : \mathcal{P}_m(\mathbb{R}) \to \mathcal{P}_m(\mathbb{R})$ is injective, hence surjective. Consider

$$\begin{aligned}
T(p) &= \vec{0} \\
(x^2 - 3x)p''(x) + (2x - 3)p'(x) + p(0) &= \vec{0}.
\end{aligned}$$

Notice if p is of degree ≥ 1, the left hand side is a degree ≥ 1 polynomial and cannot be the zero map. If p is of degree 0, which is a non-zero constant, then left hand side is a constant function, which again cannot be the zero map. Hence, in conclusion the only possible p is the zero polynomial function. Therefore, null $T = \{0\}$, and T is injective. Since we restricted the map $T : \mathcal{P}_m(\mathbb{R}) \to \mathcal{P}_m(\mathbb{R})$ for each q, the domain and codomain of T have the same finite dimension and injectivity implies surjectivity. $\square$

Problem 2.

Let V be a vector space over $\mathbb{F}$. Give a constructive proof that V and $\mathcal{L}(\mathbb{F}, V)$ are isomorphic, i.e., construct an explicit isomorphism between these spaces.

Solution. Define $T_v : x \to xv$, for $x \in \mathbb{F}, v \in V$.

$$\begin{aligned} T_v(x_1 + x_2) &= (x_1 + x_2)v \\ &= x_1 v + x_2 v \\ &= T_v(x_1) + T_v(x_2). \\ T_v(\lambda x) &= (\lambda x)v \\ &= \lambda(xv) \\ &= \lambda T_v(x). \end{aligned}$$

Therefore, T_v is a linear map and is indeed in $\mathcal{L}(\mathbb{F}, V)$.

Now define $\Phi : v \to T_v$.

$$\begin{aligned} \Phi(v_1 + v_2)(x) &= T_{v_1+v_2}(x) \\ &= x(v_1 + v_2) \\ &= xv_1 + xv_2 \\ &= T_{v_1}(x) + T_{v_2}(x) \\ &= \Phi(v_1)(x) + \Phi(v_2)(x). \\ \Phi(\lambda v)(x) &= T_{\lambda v}(x) \\ &= x(\lambda v) \\ &= \lambda(xv) \\ &= \lambda T_v(x) \\ &= \lambda \Phi(v)(x). \end{aligned}$$

Therefore, Φ is indeed a linear map.

Now we show that Φ is injective. Consider the zero map in $\mathcal{L}(\mathbb{F}, V)$, in order for Φ to map to the zero map, we need $T_v(x) = xv = 0$ for all $x \in \mathbb{F}$, which is only possible when $v = 0$. Hence, null $\Phi = \{0\}$, and Φ is injective.

Now we show that Φ is surjective. Notice

$$\dim \operatorname{range} T = \dim V - \dim \operatorname{null} T = \dim V = \dim \mathbb{F} \times \dim V = \dim \mathcal{L}(\mathbb{F}, V),$$

thus range$T = \mathcal{L}(\mathbb{F}, V)$. Hence, Φ is surjective.

Therefore, Φ is an isomorphism between V and $\mathcal{L}(\mathbb{F}, V)$. $\square$

Problem 3.

Which of the following maps on $\mathcal{P}(\mathbb{R})$ are linear functionals?

(a) $p(x) \mapsto \int_{-1}^{x} p(t)dt$

(b) $p(x) \mapsto \int_{0}^{1} p(4t^{10} + t^3 - 1)dt$

(c) $p(x) \mapsto p(0)p''(\pi)$

(d) $p(x) \mapsto 2p(1)$

Solution.

(a) This is not a linear functionals. Let $P(x) \in \mathcal{P}(\mathbb{R})$ such that $\frac{dP}{dx} = p$. Then

$$\int_{-1}^{x} p(t) = P(x) - P(1),$$

which is a polynomial function instead of a field.

(b) This is a linear functional. Indeed a definite integral will evaluate to a constant, which is a field. Consider $p_1, p_2 \in \mathcal{P}(\mathbb{R})$, then

$$\begin{aligned} T(p_1 + p_2) &= \int_0^1 (p_1 + p_2)(4t^{10} + t^3 - 1)dt \\ &= \int_0^1 p_1(4t^{10} + t^3 - 1)dt + \int_0^1 p_2(4t^{10} + t^3 - 1)dt \\ &= T(p_1) + T(p_2). \end{aligned}$$

Consider $\lambda \in \mathbb{F}$, then

$$\begin{aligned} T(\lambda p) &= \int_0^1 (\lambda p)(4t^{10} + t^3 - 1)dt \\ &= \lambda \int_0^1 p(4t^{10} + t^3 - 1)dt \\ &= \lambda T(p). \end{aligned}$$

Hence, it is linear.

(c) This is not a linear functional. Consider $p_1, p_2 \in \mathcal{P}(\mathbb{R})$ such that $p_1(x) = 1$ and $p_2(x) = x^2$, then

$$\begin{aligned} T(p_1 + p_2) &= (p_1 + p_2)(0)(p_1 + p_2)''(\pi) \\ &= (p_1(0) + p_2(0))\left(p_1''(\pi) + p_2''(\pi)\right) \\ &= (1 + 0)(0 + 2) \\ &= 2 \\ &\neq 0 + 0 = p_1(0)p''(\pi) + p_2(0)p_2''(\pi). \end{aligned}$$

(d) This is a linear functional. Indeed an evaluation of a polynomial function at a point will evaluate to a constant, which is a field. Consider $p_1, p_2 \in \mathcal{P}(\mathbb{R})$, then

$$\begin{aligned} T(p_1 + p_2) &= 2(p_1 + p_2)(1) \\ &= 2p_1(1) + 2p_2(1) \\ &= T(p_1) + T(p_2). \end{aligned}$$

Consider $\lambda \in \mathbb{F}$, then

$$\begin{aligned} T(\lambda p) &= 2(\lambda p)(1) \\ &= \lambda(2p(1)) \\ &= \lambda T(p). \end{aligned}$$

Hence, it is linear.

$\square$

Problem 4.

Let $V = \mathcal{P}_2(\mathbb{R})$ and suppose $\varphi_j(p) = p(j-1)$, $j = 1, 2, 3$. Prove that $(\varphi_1, \varphi_2, \varphi_3)$ is a basis for $\mathcal{P}_2(\mathbb{R})'$ and find a basis (p_1, p_2, p_3) of $\mathcal{P}_2(\mathbb{R})$ whose dual is $(\varphi_1, \varphi_2, \varphi_3)$.

Proof. $(\varphi_1, \varphi_2, \varphi_3)$ is of length 3 while $\dim \mathcal{P}_2(\mathbb{R})' = 3$, thus it suffices to show that $(\varphi_1, \varphi_2, \varphi_3)$ is linearly independent.

Consider $\alpha, \beta, \gamma \in \mathbb{R}$ and $p = ax^2 + bx + c \in \mathcal{P}(\mathbb{R})$, then

$$\begin{aligned}
\alpha\varphi_1(p) + \beta\varphi_2(p) + \gamma\varphi_3(p) &= 0 \\
\alpha p(0) + \beta p(1) + \gamma p(2) &= 0 \\
\alpha c + \beta(a + b + c) + \gamma(4a + 2b + c) &= 0 \\
a(4\gamma + \beta) + b(2\gamma + \beta) + c(\alpha + \beta + \gamma) &= 0.
\end{aligned}$$

Since a, b, c are arbitrary, we must have

$$\begin{cases} 4\gamma + \beta = 0 \\ 2\gamma + \beta = 0 \\ \alpha + \beta + \gamma = 0 \end{cases} \implies \begin{cases} \gamma = 0 \\ \beta = 0 \\ \alpha = 0 \end{cases}.$$

Hence, $(\varphi_1, \varphi_2, \varphi_3)$ is linearly independent and is a basis for $\mathcal{P}_2(\mathbb{R})'$.

To find the corresponding basis (p_1, p_2, p_3), we need

$$\begin{cases} p_1(0) = 1, p_1(1) = 0, p_1(2) = 0 \\ p_2(0) = 0, p_2(1) = 1, p_2(2) = 0 \\ p_3(0) = 0, p_3(1) = 0, p_3(2) = 1 \end{cases}$$

Define $p_1(x) = \frac{(x-1)(x-2)}{2}$, $p_2(x) = -x(x-2)$, and $p_3(x) = \frac{x(x-1)}{2}$. Indeed, these 3 functions satisfy the above conditions, so it suffices to show that they are linear independent. Consider $\alpha, \beta, \gamma \in \mathbb{R}$, then

$$\begin{aligned}
\alpha p_1(x) + \beta p_2(x) + \gamma p_3(x) &= 0 \\
\alpha\frac{(x-1)(x-2)}{2} + \beta(-x(x-2)) + \gamma\frac{x(x-1)}{2} &= 0 \\
\alpha(x^2 - 3x + 2) - 2\beta(x^2 - 2x) + \gamma(x^2 - x) &= 0 \\
(\alpha - 2\beta + \gamma)x^2 + (-3\alpha + 4\beta - \gamma)x + (2\alpha) &= 0.
\end{aligned}$$

We already know that $x^2, x, 1$ are linearly independent, thus we must have

$$\begin{cases} \alpha - 2\beta + \gamma = 0 \\ -3\alpha + 4\beta - \gamma = 0 \\ 2\alpha = 0 \end{cases} \implies \begin{cases} \alpha = 0 \\ \beta = 0 \\ \gamma = 0 \end{cases}.$$

Therefore, (p_1, p_2, p_3) is linearly independent and is a basis for $\mathcal{P}_2(\mathbb{R})$. $\square$

Problem 5.

Let V be a finite-dimensional vector space and let U be its proper subspace (i.e., $U \neq V$). Prove that there exists $\varphi \in V'$ such that $\varphi(u) = 0$ for all $u \in U$ but $\varphi \neq 0$.

Proof. It's equivalent to showing that the annihilator of U contains non-trivial vector. Since U is a subspace of V, we have

$$\dim U^0 = \dim V - \dim U > 0 \qquad \because \dim U < \dim V.$$

Therefore, the annihilator of U contains non-trivial vector. $\square$