Math 110 HW4

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Problem 1.

Let $a, b \in \mathbb{R}$. Define $T : \mathcal{P}(\mathbb{R}) \to \mathbb{R}^2$ by

$$Tp := (2p(1) + 5p'(2) + ap(-1)p(3), \int_{-1}^{1} x^3 p(x)\, dx + b \sin p(0)).$$

Under what conditions on a and b is the map T linear?

Solution. We will verify the linearity of T by checking the two properties of linear maps separately on the first and second components of Tp.

Additivity of first coordinate: Let $p, q \in \mathcal{P}(\mathbb{R})$. Then,

$$\begin{aligned}
T(p+q)^{(1)} &= 2(p+q)(1) + 5(p+q)'(2) + a(p+q)(-1)(p+q)(3) \\
&= 2(p(1)+q(1)) + 5(p'(2)+q'(2)) + a(p(-1)+q(-1))(p(3)+q(3)) \\
&= 2p(1) + 2q(1) + 5p'(2) + 5q'(2) + ap(-1)p(3) + ap(-1)q(3) + aq(-1)p(3) + aq(-1)q(3) \\
&= 2p(1) + 5p'(2) + ap(-1)p(3) + 2q(1) + 5q'(2) + aq(-1)q(3) + \underline{(ap(-1)q(3) + aq(-1)p(3))} \\
&= Tp^{(1)} + Tq^{(1)} + \underline{a\,(p(-1)q(3) + q(-1)p(3))}.
\end{aligned}$$

Hence, the linearity only holds if $a(p(-1)q(3) + q(-1)p(3)) = 0$ for all $p, q \in \mathcal{P}(\mathbb{R})$. This is only possible when $a = 0$ [we can show by considering p, q such that $p(-1)q(3) + q(-1)p(3) \neq 0$].

Additivity of second coordinate: Let $p, q \in \mathcal{P}(\mathbb{R})$. Then,

$$\begin{aligned}
T(p+q)^{(2)} &= \int_{-1}^{1} x^3 (p+q)(x)\, dx + b\sin(p+q)(0) \\
&= \int_{-1}^{1} x^3 p(x)\, dx + \int_{-1}^{1} x^3 q(x)\, dx + \underline{b\sin p(0) + b \sin q(0)} \\
&= Tp^{(2)} + Tq^{(2)} + \underline{b\,(\sin p(0) + \sin q(0))}.
\end{aligned}$$

Again, the linearity only holds if $b(\sin p(0) + \sin q(0)) = 0$ for all $p, q \in \mathcal{P}(\mathbb{R})$. This is only possible when $b = 0$ [we can show by considering p, q such that $\sin p(0) + \sin q(0) \neq 0$].

Finally, we check that homogeneity still holds when $a = b = 0$.

Homogeneity of first coordinate: Let $p \in \mathcal{P}(\mathbb{R})$ and $c \in \mathbb{R}$. Then,

$$\begin{aligned}
T(cp)^{(1)} &= 2(cp)(1) + 5(cp)'(2) + 0(cp)(-1)(cp)(3) \\
&= c \cdot 2p(1) + c \cdot 5p'(2) + c^2 \cdot 0p(-1)p(3) \\
&= c\left(2p(1) + 5p'(2)\right) \\
&= cTp^{(1)}.
\end{aligned}$$

Homogeneity of second coordinate: Let $p \in \mathcal{P}(\mathbb{R})$ and $c \in \mathbb{R}$. Then,

$$\begin{aligned}
T(cp)^{(2)} &= \int_{-1}^{1} x^3 (cp)(x)\,dx + 0\sin(cp(0)) \\
&= c\int_{-1}^{1} x^3 p(x)\,dx + 0 \\
&= cTp^{(2)}.
\end{aligned}$$

$\square$

Problem 2.

Suppose $T \in \mathcal{L}(V, W)$, $v_1, \dots, v_m \in V$ and the list $Tv_1, Tv_2, \dots Tv_m$ spans W. Prove or disprove that the list $v_1, \dots, v_m$ spans V.

Solution. I will provide a counterexample to show that the statement is false. Let $V = \mathbb{R}^2$, $W = \mathbb{R}$, and $T : V \to W$ be defined by $T : (x, y) \to (x)$. Let $v_1 = (1, 0)$. Then, $Tv_1 = (1)$, which spans W. However, v_1 does not span $V = \mathbb{R}^2$ obviously. $\square$

Problem 3.

Let $V = \mathcal{P}_2(\mathbb{R})$, $W = \mathbb{R}$. Are the maps

$$T_1 : f \mapsto f(0), \quad T_2 : f \mapsto f'(1), \quad T_3 : f \mapsto \int_0^1 f(x)dx$$

in $\mathcal{L}(V, W)$? Are they linearly independent?

Solution. To check if the maps are linear, we will check whether they satisfy additivity and homogeneity.

Additivity of T_1: Let $f, g \in \mathcal{P}_2(\mathbb{R})$. Then,

$$\begin{aligned} T_1(f+g) &= (f+g)(0) \\ &= f(0) + g(0) \\ &= T_1(f) + T_1(g). \end{aligned}$$

Homogeneity of T_1: Let $f \in \mathcal{P}_2(\mathbb{R})$ and $c \in \mathbb{R}$. Then,

$$\begin{aligned} T_1(cf) &= (cf)(0) \\ &= c \cdot f(0) \\ &= cT_1(f). \end{aligned}$$

Additivity of T_2: Let $f, g \in \mathcal{P}_2(\mathbb{R})$. Then,

$$\begin{aligned} T_2(f+g) &= (f+g)'(1) \\ &= f'(1) + g'(1) \\ &= T_2(f) + T_2(g). \end{aligned}$$

Homogeneity of T_2: Let $f \in \mathcal{P}_2(\mathbb{R})$ and $c \in \mathbb{R}$. Then,

$$\begin{aligned} T_2(cf) &= (cf)'(1) \\ &= cf'(1) \\ &= cT_2(f). \end{aligned}$$

Additivity of T_3: Let $f, g \in \mathcal{P}_2(\mathbb{R})$. Then,

$$\begin{aligned} T_3(f+g) &= \int_0^1 (f+g)(x)dx \\ &= \int_0^1 f(x) + g(x)dx \\ &= \int_0^1 f(x)dx + \int_0^1 g(x)dx \\ &= T_3(f) + T_3(g). \end{aligned}$$

Homogeneity of T_3: Let $f \in \mathcal{P}_2(\mathbb{R})$ and $c \in \mathbb{R}$. Then,

$$\begin{aligned} T_3(cf) &= \int_0^1 (cf)(x)dx \\ &= \int_0^1 cf(x)dx \\ &= c\int_0^1 f(x)dx \\ &= cT_3(f). \end{aligned}$$

To check linear independence of T_1, T_2, T_3, we will check if the only solution to the equation

$$\alpha T_1 + \beta T_2 + \gamma T_3 = 0$$

is $\alpha = \beta = \gamma = 0$.

Let $f(x) = ax^2 + bx + c$. Then,

$$\begin{aligned} \alpha T_1 + \beta T_2 + \gamma T_3 = \alpha f(0) + \beta f'(1) + \gamma \int_0^1 f(x)dx &= 0 \\ \alpha c + \beta(2a + b) + \gamma\left(\frac{a}{3} + \frac{b}{2} + c\right) &= 0 \\ 6\alpha c + 12\beta a + 6\beta b + 2\gamma a + 3\gamma b + 6\gamma c &= 0 \\ (12\beta + 2\gamma)a + (6\beta + 3\gamma)b + (6\alpha + 6\gamma)c &= 0. \end{aligned}$$

We claim this is only true for all $a, b, c \in \mathbb{R}$ if

$$\begin{cases} 12\beta + 2\gamma = 0 \\ 6\beta + 3\gamma = 0 \\ 6\alpha + 6\gamma = 0. \end{cases}$$

<u>Proof sketch of the claim:</u> Assume without loss of generality that $12\beta + 2\gamma \neq 0$ is part of the solution. Then, take $a' = 2a$, and we will reach a contradiction that the equation does not equal to 0.

Now we can solve the system of equations by gaussian elimination, and we will get $\alpha = \beta = \gamma = 0$. Hence, the maps are linearly independent. $\square$

Problem 4.

Suppose V is a vector space and S, $T \in \mathcal{L}(V,V)$ are such that

$$\operatorname{range} S \subset \operatorname{null} T.$$

Prove or disprove that $ST = TS = 0$.

Solution. We will provide a counter example.

Consider $V = \mathbb{R}^3$. Define

$$T : (x, y, z) \to (x, 0, 0)$$

$$S : (x, y, z) \to (0, 0, x).$$

It is trivial to see that S, T are indeed linear maps. Also,

$$\begin{aligned}\operatorname{range} S &= \{(0,0,x) : x \in \mathbb{R}\} \\ &\subset \{(0,y,z) : y, z \in \mathbb{R}\} = \operatorname{null} T.\end{aligned}$$

Now, take $v = (1, 0, 0)$,

$$ST(v) = S(1,0,0) = (0,0,1) \neq \vec{0}.$$

□

Problem 5.

Suppose V is a nonzero finite-dimensional vector space and $\mathcal{L}(V,W)$ is finite-dimensional for some vector space W. Prove or disprove that W is finite-dimensional.

Solution. Let $\dim V = n$ and $\dim \mathcal{L}(V,W) = k$.

Assume for the sake of contradiction that W is infinite-dimensional. We will show that $\dim \mathcal{L}(V,W) > k$ to reach the contradiction.

Let $m = \lceil \frac{k}{n} \rceil$. Then, we know there exists $m+1$ linearly independent vectors in W, denote $w_1, w_2, \ldots, w_{m+1}$, because W is infinite-dimensional. Now, consider the space $W' = span\{w_1, \cdots, w_{m+1}\}$, which is indeed a subspace of W. We can see that $\mathcal{L}(V,W')$ is a subspace of $\mathcal{L}(V,W)$ because every linear map from V to W' is also a linear map from V to W. Hence, $\dim \mathcal{L}(V,W') \leq \dim \mathcal{L}(V,W)$.

Now, we will show that $\dim \mathcal{L}(V,W') > k$. Since V and W' are both finite, $\dim \mathcal{L}(V,W') = \dim V \times \dim W' = n \times (m+1) > k$. Thus, we conclude that $\dim \mathcal{L}(V,W) \geq \dim \mathcal{L}(V,W') > k$, which is a contradiction. Hence, W must be finite-dimensional.

Extension of the proof: If we let $k \in \mathbb{N}$. Following the same argument as above, we can always construct a subspace W' of W such that $\dim \mathcal{L}(V,W) \geq \dim \mathcal{L}(V,W') > k$. Hence, $\mathcal{L}(V,W) > k$ for all $k \in \mathbb{N}$, and $\mathcal{L}(V,W)$ must be infinite-dimensional.

$\square$