Math 154 HW3

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Problem 1.

Proposition 1. *Any tree with $n \geq 2$ vertices has at least 2 leaves.*

Proof. Let $P = (v_1, \ldots, v_k)$ be the longest path in any arbitrary tree T with $n \geq 2$ vertices. Then, we know that all the neighbors of v_1 and v_k must be in P. Assume for the sake of contradiction that v_1 has more than one neighbor, let v_2, v_r be two of them, then $v_2, v_r \in P$, and there would be a cycle $(v_1, v_2, \ldots, v_r, v_1)$, which contradicts the definition of tree. Hence, v_1 has only one neighbor. Similarly, v_k has only one neighbor. Therefore, v_1 and v_k are the leaves of T. □

Problem 2.

Proposition 2. *If G is a graph in which there is a unique path between each pair of vertices, then G is a tree.*

Proof. We seek to prove that G is connected and acyclic.

Firstly, G is connected because there is a path between each pair of vertices.

Then, assume for the sake of contradiction that G has a cycle $C = (v_1, \ldots, v_k, \ldots, v_1)$. Then, there are two paths between v_1 and v_k with the one path being the first half of C and the other path being the other half of C, which contradicts the assumption that there is a unique path between each pair of vertices.

Therefore, G is a tree. □

Problem 3.

(a)

Proposition 3. *Any forest with n vertices and k components has exactly $n - k$ edges.*

Proof. Let $T_1, \ldots, T_k$ be the components of the forest. Then, each T_i is a tree, and has n_i vertices and $n_i - 1$ edges.

Therefore, the forest has $\sum_{i=1}^{k} n_i = n$ vertices and $\sum_{i=1}^{k} n_i - 1 = \left(\sum_{i=1}^{k} n_i\right) - k = n - k$ edges. □

(b)

Proposition 4. *Any n-vertex graph with at least n edges contains a cycle.*

Proof. Let an arbitrary graph G with n vertices.

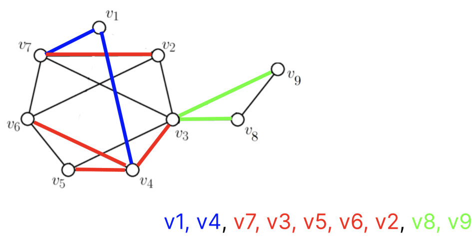
Case 1: If G is connected. Assume for the sake of contradiction that G has no cycle. Then, G is a tree, and has $n - 1$ edges, which contradicts the assumption that G has at least n edges. Hence, G must have a cycle.

Case 2: If G is not connected. Let $G_1, \ldots, G_k$ be the components of G.
Now assume for the sake of contradiction that all components G_i have $|E(G_i)| < |V(G_i)|$. However, notice that all components are disjoint, so $\sum_{i=1}^{k} |E(G_i)| \geq \sum_{i=1}^{k} |V_(G_i)|$, which is impossible under our contrary assumption. Hence, there must exist G_j such that $|E(G_j)| \geq |V(G_j)|$. Then, consider only that component G_j, and with the same argument from Case 1, there is a cycle in G_j.

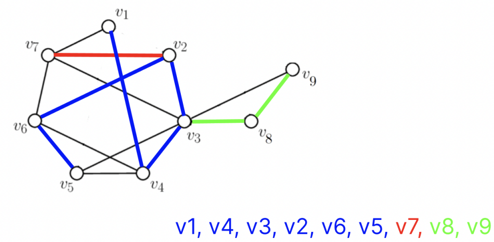
□

Problem 4.

(a)



BFS



DFS

(b) Height of the BFS tree is 3, and height of the DFS tree is 5.

(c) Radius of the graph is 2.

(d) Diameter of the graph is 3.