

# Math 181A HW2

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## Problem 1-1

(3.4.13) For  $y \in [0, 2]$ ,

$$\begin{aligned}f_Y(y) &= \frac{d}{dy} F_Y(y) \\f_Y(y) &= \frac{d}{dy} \frac{1}{12} (y^2 + y^3) \\f_Y(y) &= \frac{1}{12} \left[ \frac{y^3}{3} + \frac{y^4}{4} \right] \\f_Y(y) &= \frac{y^3}{36} + \frac{y^4}{48}.\end{aligned}$$

(3.4.17)

$$\begin{aligned}P(-a \leq Y \leq a) &= 1 - P(Y \leq -a) - P(Y \geq a) \\P(-a \leq Y \leq a) &= 1 - 2P(Y \geq a) \quad (\because P(Y \geq a) = P(Y \leq -a)) \\P(-a \leq Y \leq a) &= 1 - 2(1 - F_Y(a)) \\P(-a \leq Y \leq a) &= 1 - 2F_Y(a) - 1.\end{aligned}$$

(3.6.16)

$$\begin{aligned}E\left(\frac{W - \mu}{\sigma}\right) &= E\left(\frac{W}{\sigma}\right) - E\left(\frac{\mu}{\sigma}\right) \\&= \frac{1}{\sigma} E(W) - \frac{\mu}{\sigma} \\&= \frac{\mu}{\sigma} - \frac{\mu}{\sigma} \\&= 0.\end{aligned}$$

$$\begin{aligned}Var\left(\frac{W - \mu}{\sigma}\right) &= \frac{1}{\sigma^2} Var(W - \mu) \\&= \frac{Var(W)}{\sigma^2} \\&= \frac{\sigma^2}{\sigma^2} \\&= 1.\end{aligned}$$

**Problem 2-1** For discrete case,

$$\begin{aligned}
E[X] &= \sum_{x \geq a} xP(X = x) + \sum_{x < a} xP(X = x) \\
E[X] &\geq \sum_{x \geq a} aP(X = x) + 0 \\
E[X] &\geq a \sum_{x \geq a} P(X = x) \\
E[X] &\geq aP(X \geq a) \\
P(X \geq a) &\leq \frac{E[X]}{a}.
\end{aligned}$$

It is analogous for continuous case, but just replacing the summation with integral.

**Problem 2-2** Let  $\bar{S}_n = \frac{1}{n} \sum_{i=1}^n X_i$ , and all  $X_i$  are i.i.d. Then,

$$\begin{aligned}
E[\bar{S}_n] &= E\left[\frac{1}{n} \sum_{i=1}^n X_i\right] \\
E[\bar{S}_n] &= \frac{1}{n} \sum_{i=1}^n E[X_i] \\
E[\bar{S}_n] &= \frac{1}{n} nE[X_i] \\
E[\bar{S}_n] &= \mu,
\end{aligned}$$

and

$$\begin{aligned}
Var(\bar{S}_n) &= Var\left(\frac{1}{n} \sum_{i=1}^n X_i\right) \\
Var(\bar{S}_n) &= \frac{1}{n^2} Var\left(\sum_{i=1}^n X_i\right) \\
Var(\bar{S}_n) &= \frac{1}{n^2} \sum_{i=1}^n Var(X_i) \\
Var(\bar{S}_n) &= \frac{1}{n^2} nVar(X_i) \\
Var(\bar{S}_n) &= \frac{\sigma^2}{n}.
\end{aligned}$$

Then,

$$\begin{aligned}
P(|\bar{S}_n - E[\bar{S}_n]| \geq a) &\leq \frac{Var(\bar{S}_n)}{a^2} \\
P(|\bar{S}_n - \mu| \geq a) &\leq \frac{\sigma^2}{na^2}.
\end{aligned}$$

Since  $\sigma$  and  $a$  are constants,

$$\begin{aligned}
\lim_{n \rightarrow \infty} P(|\bar{S}_n - \mu| \geq a) &\leq 0 \quad (= 0 \because P \geq 0) \\
\lim_{n \rightarrow \infty} P(|\bar{S}_n - \mu| \leq a) &= 1.
\end{aligned}$$

### Problem 3-1

(5.2.23)

$$\begin{aligned}
 E[X] = \mu &= \sum_{k \in K} k P_X(k; \theta) \\
 \mu &= 0 \times [\theta^0(1-\theta)^{1-0}] + 1 \times [\theta^1(1-\theta)^{1-1}] \\
 \mu &= \theta.
 \end{aligned}$$

Then, from the sample, we can estimate

$$\begin{aligned}
 \hat{\mu} &= \frac{1+1}{5} \\
 \hat{\mu} &= \frac{2}{5} \\
 \theta = \mu &\approx \hat{\mu} = \frac{2}{5}.
 \end{aligned}
 \tag{1}$$

### Problem R

(1)

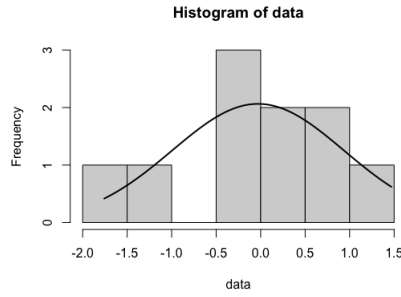
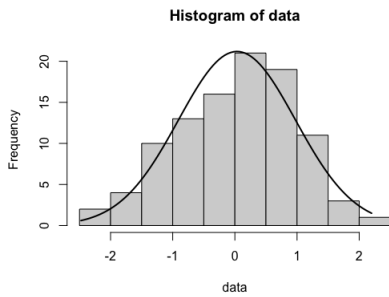
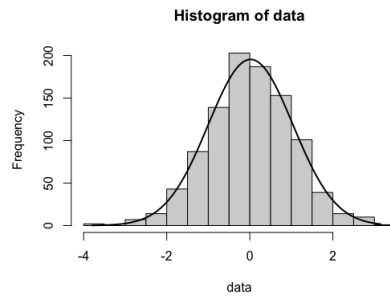


Figure 1:  $n = 10$

(2) I expect the distribution of the samples would trace closer and closer to a normal distribution, a result directly from Central Limit Theorem. We can see from the plot, the result is what we expected.



$n = 100$



$n = 1000$