Math 180A HW5

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Problem 1. (b)

Problem 2. (a)

Problem 3. 4.

Problem 4.

$$\begin{aligned} E[X] &= \int_0^1 6x^2 - 6x^3 dx && (1) \\ &= \left[2x^3 - \frac{3}{2}x^4\right]_0^1 && (2) \\ &= \frac{1}{2} && (3) \end{aligned}$$

Problem 5.

(a)

$$\begin{aligned} E[X] &= \int_{-2}^0 \frac{x}{6} dx + \int_0^3 \frac{2x}{9} dx && (4) \\ &= \left[\frac{x^2}{12}\right]_{-2}^0 + \left[\frac{x^2}{9}\right]_0^3 && (5) \\ &= \frac{-1}{3} + 1 && (6) \\ &= \frac{2}{3} && (7) \end{aligned}$$

(b)

$$\begin{aligned} E[X^2] &= \int_{-2}^0 \frac{x^2}{6} dx + \int_0^3 \frac{2x^2}{9} dx && (8) \\ &= \left[\frac{x^3}{18}\right]_{-2}^0 + \left[\frac{2x^3}{27}\right]_0^3 && (9) \\ &= \frac{4}{9} + 2 && (10) \\ &= \frac{22}{9} && (11) \\ Var(X) &= E[X^2] - (E[X])^2 && (12) \\ &= \frac{22}{9} - \left(\frac{2}{3}\right)^2 && (13) \\ &= 2 && (14) \end{aligned}$$

(c)

$$E[(X-1)^2] = E[X^2 - 2X + 1] \tag{15}$$
$$= E[X^2] - 2E[X] + 1 \tag{16}$$
$$= \frac{22}{9} - \frac{4}{3} + 1 \tag{17}$$
$$= \frac{19}{9} \tag{18}$$

Problem 6.

(a) Consider $x \in [0, \infty)$,

$$f_X(x) = \frac{d}{dx}\frac{x}{1+x} \tag{19}$$
$$= \frac{(1+x) - x}{(1+x)^2} \tag{20}$$
$$= \frac{1}{(1+x)^2}. \tag{21}$$

Hence,

$$f_X(x) = \begin{cases} \frac{1}{(1+x)^2}, & x \geq 0 \\ 0, & x < 0. \end{cases} \tag{22}$$

(b)

$$P(2 < X < 3) = P(X < 3) - P(X < 2) \tag{23}$$
$$= F(3) - F(2) \tag{24}$$
$$= \frac{3}{4} - \frac{2}{3} \tag{25}$$
$$= \frac{1}{12} \tag{26}$$

(c)

$$E\left[(1+X)^2 e^{-2X}\right] = E\left[X^2 e^{-2X} + 2Xe^{-2X} + e^{-2X}\right] \tag{27}$$
$$= E\left[X^2 e^{-2X}\right] + E\left[2Xe^{-2X}\right] + E\left[e^{-2X}\right] \tag{28}$$
$$= \int_0^\infty \frac{x^2 e^{-2x}}{(1+x)^2} + \int_0^\infty \frac{2xe^{-2x}}{(1+x)^2} + \int_0^\infty \frac{e^{-2x}}{(1+x)^2} dx \tag{29}$$
$$= \int_0^\infty \frac{e^{-2x}(x^2 + 2x + 1)}{(1+x)^2} dx \tag{30}$$
$$= \int_0^\infty e^{-2x} dx \tag{31}$$
$$= \frac{1}{-2}\left[e^{-2x}\right]_0^\infty \tag{32}$$
$$= \frac{1}{2} \tag{33}$$

Problem 7. 150 meteros per hour is equivalent to 2.5 meteros per minute. We can divide the first minute into n inifinitely small intervals, and assume each interval is independent. Thus the probability of a metero appearing in one interval is $\frac{2.5}{n}$. Then we can model the situation with $Bionomial(n, p)$, and approximate the wanted probability with $Poisson(2.5)$. Hence,

$$\begin{align} P(X \geq 2) &= 1 - P(X = 0) - P(X = 1) \tag{34} \\ &= 1 - e^{-2.5} - 2.5e^{-2.5} \tag{35} \\ &\approx 0.7127. \tag{36} \end{align}$$

Problem 8.

(a) $f(x)$ is apparently non-negative because $\forall x \in \mathbb{R}, x^2 \geq 0 \Rightarrow \pi(1 + x^2) \geq 0 \Rightarrow \frac{1}{\pi(1+x^2)} \geq 0$.

$$\begin{align} \int_{-\infty}^{\infty} \frac{1}{\pi(1 + x^2)} dx &= \frac{1}{\pi} \left[tan^{-1}(y)\right]_{-\infty}^{\infty} \tag{37} \\ &= \frac{1}{\pi} \left(\frac{\pi}{2} - \frac{-\pi}{2}\right) \tag{38} \\ &= 1 \tag{39} \end{align}$$

(b)

$$\begin{align} E[|X|] &= \int_{-\infty}^{0} \frac{-x}{\pi(1 + x^2)} dx + \int_{0}^{\infty} \frac{x}{\pi(1 + x^2)} dx \tag{40} \\ &= \frac{-1}{\pi} \int_{\infty}^{1} \frac{1}{2u} du + \frac{1}{\pi} \int_{1}^{\infty} \frac{1}{2u} du \qquad \left(\text{let } u = 1 + x^2\right) \tag{41} \\ &= \frac{1}{2\pi} \int_{1}^{\infty} \frac{1}{u} du + \frac{1}{2\pi} \int_{1}^{\infty} \frac{1}{u} du \tag{42} \\ &= \lim_{t \to \infty} \frac{1}{\pi} \left[ln(u)\right]_1^t \tag{43} \\ &= \infty \tag{44} \end{align}$$

(c) Looks like a 0.

(d) Maybe start by finding the CDF, which should be $\frac{1}{\pi}\left[tan^{-1}(y)\right]$: $\frac{1}{\pi}$ represents the weight of the angle, and $tan^{-1}(y)$ converts the slope to the angle. Then differentiate it to get $\frac{1}{\pi(1+x^2)}$.

Problem 9. I think the number of commits to a GitHub Repo would be well-modeled by Poisson distribution. data-engineering-zoomcap has an average of 4.87 commits per day. Let X be the number of commits per day, it can be modeled by $Poisson(4.87)$. $P(X = 2) = \frac{4.87^2}{2} e^{-4.87}$, multiplied by the total of 210 days, we expect that there are 19 days that there are exactly 2 commits. In fact, there are a total of 17 days with total of 2 commits, which is pretty close in my opinion.

(https://github.com/DataTalksClub/data-engineering-zoomcamp/graphs/commit-activity)