Math 104 HW5

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Exercise 11.3

Consider the sequences

$$s_n = \cos\left(\frac{n\pi}{3}\right), \qquad t_n = \frac{3}{4n+1}, \qquad u_n = \left(-\frac{1}{2}\right)^n, \qquad v_n = (-1)^n + \frac{1}{n}.$$

(a) For each sequence, give an example of a monotone subsequence.

Solution.

(s_n): Consider $n_k = 6k$ for $k \in \mathbb{N}$, then $s_{n_k} = \cos\left(\frac{6k\pi}{3}\right) = \cos(2k\pi) = 1$ for all n_k, which is indeed monotone [a constant sequence is monotone].

(t_n): Consider $n_k = 2k$ for $k \in \mathbb{N}$, then $t_{n_k} = \frac{3}{8k+1}$ for all n_k, which is apparently monotonically decreasing.

(u_n): Consider $n_k = 2k$ for $k \in \mathbb{N}$, then $u_{n_k} = \left(-\frac{1}{2}\right)^{2k} = \frac{1}{4^k}$ for all n_k, which is apparently monotonically decreasing.

(v_n): Consider $n_k = 2k$ for $k \in \mathbb{N}$, then $v_{n_k} = (-1)^{2k} + \frac{1}{2k} = 1 + \frac{1}{2k}$ for all n_k, which is apparently monotonically decreasing.

$\square$

(b) For each sequence, give its set of subsequential limits.

Solution.

(s_n): $\{1, 0.5, -0.5, -1\}$. The values of s_n oscillates among constant values $\{1, 0.5, -0.5, -1\}$. We can construct the constant subsequences, which have the limits $1, 0.5, -0.5, -1$ respectively. Then consider any $x \notin \{1, 0.5, -0.5, -1\}$, any subsequence will have a minimum non-zero distance from x, hence unable to converge to x. Therefore, the set of subsequential limits is $\{1, 0.5, -0.5, -1\}$.

(t_n): $\{0\}$. $\lim t_n = 0$ by Theorem 9.3 - 9.6, hence the set of subsequential limits only contains $\lim t_n = 0$

(u_n): $\{0\}$. $\lim v_n = 0$ by Theorem 9.7, hence the set of subsequential limits only contains $\lim v_n = 0$

(v_n): $\{1, -1\}$. Take only even n, then the subsequence converges to 1. Take only odd n, then the subsequence converges to -1. Then consider any subsequence with finite odd n, it will converge to 1, just like the only even n subsequence because we can take N larger than the finite odd n. Similarly, any subsequence with finite even n will converge to -1. Finally, any subsequence with infinite odd and even n do not converge because it will always have elements within the neighborhood of 1 and -1.

□

(c) For each sequence, give its lim sup and lim inf.

Solution. Notice $\limsup x_n = \sup S$ and $\liminf x_n = \inf S$, where x_n is any arbitrary sequence.

(s_n): $\limsup s_n = 1, \liminf s_n = -1$.

(t_n): $\limsup t_n = \liminf t_n = 0$.

(u_n): $\limsup u_n = \liminf u_n = 0$.

(v_n): $\limsup v_n = 1, \liminf v_n = -1$.

□

(d) Which of the sequences converges? diverges to $+\infty$? diverges to $-\infty$?

Solution.

(s_n): Diverges. $\limsup s_n = 1, \liminf s_n = -1$, hence s_n diverges.

(t_n): Converges. $\lim t_n = 0$, hence t_n converges.

(u_n): Converges. $\lim u_n = 0$, hence u_n converges.

(v_n): Diverges. $\limsup v_n = 1, \liminf v_n = -1$, hence v_n diverges.

□

(e) Which of the sequences is bounded?

Solution.

(s_n): Bounded. $|s_n| \leq 1$ for all n, hence s_n is bounded.

(t_n): Bounded. $|t_n| \leq \frac{3}{4}$ for all n, hence t_n is bounded.

(u_n): Bounded. $|u_n| \leq \frac{1}{2}$ for all n, hence u_n is bounded.

(v_n): Bounded. $|v_n| \leq 2$ for all n, hence v_n is bounded.

□

Exercise 11.6

Proposition 1. *Every subsequence of a subseuqnce of a given sequence is itself a subsequence of the given sequence.*

Proof. Let (s_n) be the original sequence, (t_k) be a subsequence of (s_n), and (u_m) be a subsequence of (t_k). We define $\sigma(m)$ as a function that maps m to some k, the index of t_k in order, and $\tau(k)$ as a function that maps k to some n, the index of s_n in order. Then we have

$$u_m = t \circ \sigma(m) = s \circ \tau \circ \sigma(m) = s \circ \rho(m),$$

where ρ is the composite of σ and τ that maps from m to n, which preserves the order of s_n. Hence (u_m) is a subsequence of (s_n). □

Exercise 11.10

Let (s_n) be the sequence of numbers in Fig. 11.2 listed in the indicated order.

(a) Find the set S of the subsequential limits of s_n.

Solution. Take each column of the matrix as a subsequence, they are all constant subsequences with values $\frac{1}{n}$ for some $n \in \mathbb{N}$. Also, take the first row as a subsequence, which converges to 0. Therefore, $\{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\} \subseteq S$. On the other hand, consider any x not in $\{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$, notice any subsequence will have a minimum non-zero distance from x, hence unable to converge to x. Therefore, $S \subseteq \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$, and $S = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$. □

(b) Determine $\limsup s_n$ and $\liminf s_n$.

Solution. $\limsup s_n = \max S = 1$ and $\liminf s_n = \min S = 0$. □

Exercise 12.4

Proposition 2. $\limsup(s_n + t_n) \leq \limsup s_n + \limsup t_n$ *for bounded sequences* (s_n) *and* (t_n).

Proof. We first show that

$$\sup\{s_n + t_n : n > N\} \leq \sup\{s_n : n > N\} + \sup\{t_n : n > N\}. \tag{1}$$

Let N be an arbitrary natural number, then for any $n > N$, we have

$$s_n \leq \sup\{s_n : n > N\} \qquad \text{and} \qquad t_n \leq \sup\{t_n : n > N\},$$

hence

$$s_n + t_n \leq \sup\{s_n : n > N\} + \sup\{t_n : n > N\},$$

so $\sup\{s_n : n > N\} + \sup\{t_n : n > N\}$ is an upper bound for $\{s_n + t_n : n > N\}$. Hence, the least upper bound of $\{s_n + t_n : n > N\}$, aka $\sup\{s_n + t_n : n > N\}$, is less than or equal to $\sup\{s_n : n > N\} + \sup\{t_n : n > N\}$.

Next we show that if $a_n \leq b_n$, then $\lim a_n \leq \lim b_n$. Assume for the sake of contradiction that $\lim a_n > \lim b_n$. Let $\lim a_n = a$ and $\lim b_n = b$, then we can write $a = b + 2\epsilon$ for some $\epsilon > 0$. By the definition of limit, we know there exists N_1 such that for all $n > N_1$, $|a_n - a| < \epsilon$ and also exists N_2 such that for $n > N_2$, $|b_n - b| < \epsilon$. Then for all $n > \max\{N_1, N_2\}$, a_n is within ϵ of a and b_n is within ϵ of b, hence $a_n > a - \epsilon = b + \epsilon > b_n$, which contradicts the assumption that $a_n \leq b_n$.

Combining the two results, (1) implies

$$\limsup(s_n + t_n) \leq \lim(\sup s_n + \sup t_n) = \limsup s_n + \limsup t_n,$$

since $\limsup s_n$ and $\limsup t_n$ are finite [bounded and monotone]. □