Math 110 HW8

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Problem 1.

Let $p \in \mathcal{P}(\mathbb{C})$. Define $q : \mathbb{C} \to \mathbb{C}$ by the formula

$$q(z) := p(z)\overline{p(\overline{z})}.$$

Prove that $q \in \mathcal{P}(\mathbb{R})$. If $\deg p = n$, then what is $\deg q$? Explain.

Proof. Write

$$p(z) = a_0 + \cdots + a_n z^n,$$

then

$$\begin{aligned} & \overline{p(z)} = \overline{a_0 + \cdots + a_n z^n} = \overline{a_0} + \cdots + \overline{a_n z^n} = \overline{a_0} + \cdots + \overline{a_n} \cdot \overline{z^n} \\ \implies & \overline{p(\overline{z})} = \overline{a_0} + \cdots + \overline{a_n} \cdot \overline{\overline{z}^n} = \overline{a_0} + \cdots + \overline{a_n} \cdot \overline{\overline{z^n}} = \overline{a_0} + \cdots + \overline{a_n} \cdot z^n = \sum_{k=0}^{n} \overline{a_k} \cdot z^k. \end{aligned}$$

Therefore, our desired result is

$$\begin{aligned} p(z)\overline{p(\overline{z})} &= \left(\sum_{j=0}^{n} a_j \cdot z^j\right)\left(\sum_{k=0}^{n} \overline{a_k} \cdot z^k\right) \\ &= \sum_{k=0}^{n} a_k \cdot \overline{a_k} \cdot z^{2k} + \sum_{j \neq k} a_j \cdot \overline{a_k} \cdot z^{j+k} \\ &= \sum_{k=0}^{n} |a_k|^2 \cdot z^{2k} + \sum_{j>k} (a_j \cdot \overline{a_k} + \overline{a_j} \cdot a_k) \cdot z^{j+k} \\ &= \sum_{k=0}^{n} |a_k|^2 \cdot z^{2k} + \sum_{j>k} (a_j \cdot \overline{a_k} + \overline{a_j \cdot \overline{a_k}}) \cdot z^{j+k} \\ &= \sum_{k=0}^{n} |a_k|^2 \cdot z^{2k} + \sum_{j>k} \left(2\,\mathrm{Re}\left(a_j \cdot \overline{a_k}\right)\right) \cdot z^{j+k}, \end{aligned}$$

where we can see all the coefficients are real numbers.

If $\deg p = n$, then from the polynomial expression above, we can see the highest degree term is z^{2n}, so $\deg q = 2n$. $\square$

Problem 2.

Let $V = \mathcal{P}_3(\mathbb{R})$ and let D denote the differentiation operator on V. Determine, with proof, all subspaces of V invariant under the action of D.

Proof. We note that $\mathcal{P}_0(\mathbb{R}), \mathcal{P}_1(\mathbb{R}), \mathcal{P}_2(\mathbb{R}), \mathcal{P}_3(\mathbb{R})$ are all invariant subspaces because for any vector $p \in \mathcal{P}_k(\mathbb{R})$, D acting on p will simply reduce the polynomial degree by 1, and hence fall back into $\mathcal{P}_k(\mathbb{R})$. Also, the zero subspace is invariant because $D(0) = 0$.

We claim that there are no more other invariant subspaces.

Assume for contradiction that there exists another invariant subspace W with $\dim W = m \leq 4$, then we take the polynomial with largest degree in W and call it p. Notice $\deg p \geq m - 1$, otherwise all the polynomials have degree $\leq m-2$ and hence W is a subspace of $\mathcal{P}_{m-2}(\mathbb{R})$ with $\dim W \leq m-1$. At the same time, $\deg p \leq m-1$ because otherwise $(p, \ldots, D^m p)$ will all be in W and they are linearly independent due to different degrees. Therefore, $\deg p = m - 1 \Longrightarrow W$ is a subspace of $\mathcal{P}_{m-1}(\mathbb{R})$ with the same dimension of $\mathcal{P}_{m-1}(\mathbb{R}) \Longrightarrow W = \mathcal{P}_{m-1}(\mathbb{R})$, which is already included in the invariant subspaces we found above. $\square$

Problem 3.

Suppose V is a finite-dimensional complex vector space and $T \in \mathcal{L}(V)$ satisfies the condition: For any $\varphi \in V'$ and any $v \in V$, $\lim_{n\to\infty} \varphi(T^n v) = 0$. What does this imply about the eigenvalues of T?

Proof. We claim that all eigenvalues of T must be less than 1. Consider two contradictory cases:

(1) $\exists \lambda = 1$: There exists eigenvector $v \neq 0$ such that $T^n v = \lambda^n v = v$ for $n \in \mathbb{N}$. Then we can construct ϕ that sends all vectors in $V \setminus \operatorname{span}(v)$ to 0 while $\phi(cv) = c$ for $c \in \mathbb{F}$. Therefore, $\lim_{n\to\infty} \phi(T^n v) = \lim_{n\to\infty} \phi(v) = 1 \neq 0$, a contradiction.

(2) $\exists \lambda > 1$: There exists eigenvector $v \neq 0$ such that $T^n v = \lambda^n v$ for $n \in \mathbb{N}$. Again, we construct the same ϕ. Then $\phi(T^n v) = \phi(\lambda^n v) = \lambda^n \Longrightarrow \lim_{n\to\infty} \phi(T^n v) = \lim_{n\to\infty} \lambda^n = \infty \neq 0$, a contradiction.

$\square$

Problem 4.

Suppose V is finite-dimensional, $T \in \mathcal{L}(V)$ has $\dim V$ distinct eigenvalues, and $S \in \mathcal{L}(V)$ has the same eigenvectors as T (not necessarily with with the same eigenvalues). **(a)** Prove that $TS = ST$. **(b)** Give an example of such operators T and S on $\mathbb{R}^2$, neither of which is a multiple of the identity operator.

Proof. Notation: $\dim V = n$.

(a) T has n distinct eigenvalues $\Longrightarrow T$ has n linearly independent eigenvectors with each eigenvector corresponding to a distinct eigenvalue. Therefore, this set of eigenvectors forms a basis of V; denote the set of eigenvectors as $\{v_1, \ldots, v_n\}$.

Now we evaluate the action of TS and ST on any arbitrary $v \in V$:

$$\begin{aligned}
TS(v) &= T(S(a_1v_1 + \cdots + a_nv_n)) \\
&= T(S(a_1v_1) + \cdots + S(a_nv_n)) \\
&= T(a_1S(v_1) + \cdots + a_nS(v_n)) \\
&= T(a_1\gamma_1v_1 + \cdots + a_n\gamma_nv_n) \\
&= T(a_1\gamma_1v_1) + \cdots + T(a_n\gamma_nv_n) \\
&= a_1\gamma_1T(v_1) + \cdots + a_n\gamma_nT(v_n) \\
&= a_1\gamma_1\lambda_1v_1 + \cdots + a_n\gamma_n\lambda_nv_n \\
&= a_1\lambda_1\gamma_1v_1 + \cdots + a_n\lambda_n\gamma_nv_n \\
&= a_1\lambda_1S(v_1) + \cdots + a_n\lambda_nS(v_n) \\
&= S(a_1\lambda_1v_1) + \cdots + S(a_n\lambda_nv_n) \\
&= S(a_1T(v_1)) + \cdots + S(a_nT(v_n)) \\
&= S(T(a_1v_1)) + \cdots + S(T(a_nv_n)) \\
&= S(T(a_1v_1) + \cdots T(a_nv_n)) \\
&= S(T(a_1v_1 + \cdots a_nv_n)) \\
&= S(T(v)),
\end{aligned}$$

where γ_k is the corresponding eiganvalues of S for v_k while λ_k is the corresponding eigenvalues of T for v_k.

(b) Let

$$T : (x, y) \mapsto (2x, 3y),$$

then T has eigenvalues 2 and 3 with eigenvectors $(1, 0)$ and $(0, 1)$ respectively. Let

$$S : (x, y) \mapsto (5x, 10y),$$

then S has eigenvalues 5 and 10 with eigenvectors $(1, 0)$ and $(0, 1)$ respectively. Therefore,

$$TS(x, y) = T(5x, 10y) = (10x, 30y) = 5(2x, 3y) = 5T(x, y) = ST(x, y).$$

$\square$

Problem 5.

Let $S, T \in \mathcal{L}(V)$ and suppose S is invertible. (a) Prove that, for any polynomial $p \in \mathcal{P}(\mathbb{F})$,

$$p(STS^{-1}) = S\,p(T)\,S^{-1}.$$

(b) How are the subspaces of V invariant under T related to the subspaces invariant under STS^{-1}?

Proof.

(a) **Lemma:** *For any linear map T and invertible linear map S, $(STS^{-1})^n = ST^nS^{-1}$.*

Proof. We show by induction on $\mathbb{N}$.
Base Case: $(STS^{-1})^1 = STS^{-1}$ trivially.
Inductive Step:

$$\begin{aligned}(STS^{-1})^{k+1} &= (STS^{-1})^k(STS^{-1}) \\ &= (ST^kS^{-1})(STS^{-1}) \qquad (\textit{applying inductive hypothesis}) \\ &= ST^k(S^{-1}S)TS^{-1} \\ &= ST^kTS^{-1} \\ &= ST^{k+1}S^{-1}.\end{aligned}$$

Hence, the lemme is shown by mathematical induction. □

Write

$$p(STS^{-1}) = a_0I + a_1STS^{-1} + \cdots + a_n(STS^{-1})^n,$$

then for $v \in V$,

$$\begin{aligned}p(STS^{-1})(v) &= (a_0I + a_1STS^{-1} + \cdots + a_n(STS^{-1})^n)(v) \\ &= a_0I(v) + a_1STS^{-1}(v) + \cdots + a_n(STS^{-1})^n(v) \\ &= a_0I(v) + a_1STS^{-1}(v) + \cdots + a_nST^nS^{-1}(v) \\ &= a_0SS^{-1}(v) + a_1STS^{-1}(v) + \cdots + a_nST^nS^{-1}(v) \\ &= Sa_0S^{-1}(v) + Sa_1TS^{-1}(v) + \cdots + Sa_nT^nS^{-1}(v) \\ &= S(a_0TS^{-1}(v) + a_1TS^{-1}(v) + \cdots + a_nT^nS^{-1}(v)) \\ &= S(a_0T + a_1T + \cdots + a_nT^n)\left(S^{-1}(v)\right) \\ &= S(p(T))S^{-1}(v)\end{aligned}$$

(b) If W is an invariant subspace of T, then $S(W)$ is an invariant subapsce of STS^{-1} because for any vector $v \in S(W)$, S^{-1} first maps the vector to W and applying T will map back into W, and eventually applying S will map it back to $S(W)$.

□