Math 110 HW10

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Problem 1.

Determine whether or not the function taking the pair $((x_1, x_2, x_3), (y_1, y_2, y_3)) \in \mathbb{R}^3 \times \mathbb{R}^3$ to $x_1y_1 + x_2y_2 - 3x_2y_3 + 3x_3y_2 + x_3y_3$ is an inner product.

Proof. **Conjugate symmetry:** Let $u = (1, 2, 3), v = (4, 5, 6)$,

$$\begin{aligned}\langle u, v\rangle &= 1 \times 4 + 2 \times 5 + 3 \times 6 - 3 \times 2 \times 6 + 3 \times 3 \times 5 \\ &\neq 1 \times 4 + 2 \times 5 + 3 \times 6 - 3 \times 3 \times 5 + 3 \times 2 \times 6 = \langle v, u\rangle.\end{aligned}$$

Hence, the function is not an inner product. □

Problem 2.

Consider a complex vector space $V = \operatorname{span}(1, \cos x, \sin x, \cos 2x, \sin 2x)$ with an inner product

$$\langle f, g \rangle := \int_{-\pi}^{\pi} f(t)\overline{g(t)}dt.$$

Let U be the subspace of odd functions in V. What is $U^\perp$? Find an orthonormal basis for both U and $U^\perp$.

Proof. We have proved previously in the midterm exam that

$$U = \operatorname{span}\{\sin x, \sin 2x\}.$$

Then, notice the set $\{1, \cos x, \cos 2x\} \perp U$ because let $f(t) \in U$ and $g(t) \in \{1, \cos x, \cos 2x\}$, then $f(t)\overline{g(t)}$ is an odd function because $f(t)$ is odd while $\overline{g(t)} = g(t)$ is even, hence $\int_{-\pi}^{\pi} f(t)\overline{g(t)}dt = 0$.

Notice $\operatorname{span}\{1, \cos x, \cos 2x\} \subseteq U^\perp$ because any $w \in U^\perp$ can be written as a linear combination of $\{1, \cos x, \cos 2x\}$, and by linearity of integration, the inner product with any $u \in U$ is 0.

In fact, $\operatorname{span}\{1, \cos x, \cos 2x\} = U^\perp$ because

$$\dim U^\perp = \dim V - \dim U = 5 - 2 = 3$$

and $\dim \operatorname{span}\{1, \cos x, \cos 2x\} = 3$. $\square$

Problem 3.

Consider the space $\mathcal{P}_3(\mathbb{R})$ with the inner product

$$\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)dx.$$

Use the Gram-Schmidt algorithm to orthonormalize the basis $1, x, x^2, x^3$.

Solution.

$$\begin{aligned}
e_1 &= \frac{1}{\sqrt{\langle 1, 1\rangle}} = \frac{1}{\sqrt{2}} \\
e_2 &= \frac{x - \langle x, e_1\rangle e_1}{\|x - \langle x, e_1\rangle e_1\|} \\
&= \frac{x - \frac{1}{\sqrt{2}}\int_{-1}^{1}\frac{x}{\sqrt{2}}dx}{\|x - \frac{1}{\sqrt{2}}\int_{-1}^{1}\frac{x}{\sqrt{2}}dx\|} \\
&= \frac{x - 0}{\|x - 0\|} \\
&= \frac{x}{\sqrt{\int_{-1}^{1} x^2 dx}} \\
&= \frac{x}{\sqrt{2/3}} = \sqrt{\frac{3}{2}}x \\
e_3 &= \frac{x^2 - \langle x^2, e_1\rangle e_1 - \langle x^2, e_2\rangle e_2}{\|x^2 - \langle x^2, e_1\rangle e_1 - \langle x^2, e_2\rangle e_2\|} \\
&= \frac{x^2 - \frac{1}{\sqrt{2}}\int_{-1}^{1}\frac{x^2}{\sqrt{2}}dx - \frac{3}{2}x\int_{-1}^{1} x^3 dx}{\|x^2 - \frac{1}{\sqrt{2}}\int_{-1}^{1}\frac{x^2}{\sqrt{2}}dx - \frac{3}{2}x\int_{-1}^{1} x^3 dx\|} \\
&= \frac{x^2 - 1/3}{\|x^2 - 1/3\|} \\
&= \frac{x^2 - 1/3}{\sqrt{8/45}} \\
e_4 &= \frac{x^3 - \langle x^3, e_1\rangle e_1 - \langle x^3, e_2\rangle e_2 - \langle x^3, e_3\rangle e_3}{\|x^3 - \langle x^3, e_1\rangle e_1 - \langle x^3, e_2\rangle e_2 - \langle x^3, e_3\rangle e_3\|} \\
&= \frac{x^3 - \frac{1}{\sqrt{2}}\int_{-1}^{1}\frac{x^3}{\sqrt{2}}dx - \frac{3}{2}x\int_{-1}^{1} x^4 dx - \frac{x^2-1/3}{\sqrt{8/45}}\int_{-1}^{1} x^3\left(\frac{x^2-1/3}{\sqrt{8/45}}\right)dx}{\|x^3 - \frac{1}{\sqrt{2}}\int_{-1}^{1}\frac{x^3}{\sqrt{2}}dx - \frac{3}{2}x\int_{-1}^{1} x^4 dx - \frac{x^2-1/3}{\sqrt{8/45}}\int_{-1}^{1} x^3\left(\frac{x^2-1/3}{\sqrt{8/45}}\right)dx\|} \\
&= \frac{x^3 - \frac{3}{2}x\cdot\frac{2}{5}}{\|x^3 - \frac{3}{2}x\cdot\frac{2}{5}\|} \\
&= \frac{x^3 - \frac{3}{5}x}{\sqrt{\int_{-1}^{1}(x^3 - \frac{3}{5}x)^2 dx}} \\
&= \frac{x^3 - \frac{3}{5}x}{\sqrt{8/175}}.
\end{aligned}$$

$\square$

Problem 4.

Let e_1, …, e_m be an orthonormal list of vectors. Prove that $v \in \operatorname{span}(e_1, \ldots, e_m)$ if and only if

$$\|v\|^2 = \sum_{j=1}^{m} |\langle v, e_j \rangle|^2.$$

Proof. The forward direction falls directly from *Theorem 6.30 (b)* where $V = \operatorname{span}\{e_1, \ldots, e_m\}$.

For backward direction, assume $\|v\|^2 = \sum_{j=1}^{m} |\langle v, e_j \rangle|^2$ but $v \notin \operatorname{span}(e_1, \ldots, e_m)$. Now, we take the smallest vector space containing v and write out its basis as $(v_1, \ldots, v_n)$, where there are some $v_j \notin (e_1, \ldots, e_m)$. We take those basis and append to $(e_1, \ldots, e_m)$ to get $(e_1, \ldots, e_m, v_j, \ldots)$. Then we apply Gram-Schmidt algorithm to $(e_1, \ldots, e_m, v_j, \ldots)$ to get a new orthonormal basis $(e_1, \ldots, e_m, w_j, \ldots)$, which has the same span as $(e_1, \ldots, e_m, v_j, \ldots)$. Therefore, v can be written as a linear combination of this new basis, i.e.

$$v = \underbrace{a_1 e_1 + \cdots + a_m e_m}_{u} + \underbrace{b_j w_j, \ldots}_{w},$$

where w is orthogonal to u and is non-zero. Then by the *Pythagorean Theorem*, we have

$$\|v\|^2 = \|u\|^2 + \|w\|^2 > \|u\|^2 = \sum_{j=1}^{m} |\langle v, e_j \rangle|^2,$$

which is a contradiction. Hence, $v \in \operatorname{span}(e_1, \ldots, e_m)$. □

Problem 5.

Suppose that $e_1, \ldots, e_n$ is a list of vectors in V of length 1 (i.e., $\|e_k\| = 1$ for all $k = 1, \ldots, n$) such that

$$\|v\|^2 = |\langle v, e_1\rangle|^2 + \cdots + |\langle v, e_n\rangle|^2 \quad \text{for all } \ v \in V.$$

Prove that $e_1, \ldots, e_n$ is an orthonormal basis of V.

Proof. First, we show that $(e_1, \ldots, e_n)$ is orthogonal. Take $v = e_j$ for some $j \in \{1, \ldots, n\}$, then put in the equation, we see that

$$\begin{aligned} \|e_j\|^2 &= |\langle e_j, e_1\rangle|^2 + \cdots + |\langle e_j, e_j\rangle|^2 + \cdots + |\langle e_j, e_n\rangle|^2 \\ 1 &= |\langle e_j, e_1\rangle|^2 + \cdots + 1 + \cdots + |\langle e_j, e_n\rangle|^2, \end{aligned}$$

which means $|\langle e_j, e_i\rangle|^2 = 0 \implies \langle e_j, e_i\rangle = 0$ for $i \neq j$. Since e_i all have unit length, they are orthonormal. Then from *Problem 4*, we know that $v \in \operatorname{span}(e_1, \ldots, v_n)$ for all $v \in V$. In other words, $(e_1, \ldots, e_n)$ span the entire vector space V. At the same time, they are orthogonal hence linearly independent. Therefore, $(e_1, \ldots, e_n)$ is a basis, in particular orthonormal basis, of V. □