Math 180A HW7

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Problem 1.

(a) 0.3.

(b) $\frac{7}{30}$.

Problem 2.

(a)

$$\begin{aligned} f_X(x) &= \int_0^1 \frac{12}{7}(xy + y^2)dy & (1) \\ &= \frac{12}{7}\left[\frac{1}{2}xy^2 + \frac{1}{3}y^3\right]_0^1 & (2) \\ &= \frac{12}{7}\left(\frac{x}{2} + \frac{1}{3}\right) & (3) \\ &= \frac{6x+4}{7} & (4) \end{aligned}$$

for $x \in [0, 1]$ and 0 otherwise.

$$\begin{aligned} f_Y(y) &= \int_0^1 \frac{12}{7}(xy + y^2)dx & (5) \\ &= \frac{12}{7}\left[\frac{1}{2}x^2y + xy^2\right]_0^1 & (6) \\ &= \frac{12}{7}\left(\frac{y}{2} + y^2\right) & (7) \\ &= \frac{6y+12y^2}{7} & (8) \end{aligned}$$

for $y \in [0, 1]$ and 0 otherwise.

(b)

$$P(X < Y) = \int_0^1 \int_0^y \frac{12}{7}(xy + y^2)dxdy \tag{9}$$
$$= \frac{12}{7} \int_0^1 \left[\frac{1}{2}x^2 y + xy^2\right]_{x=0}^{x=y} dy \tag{10}$$
$$= \frac{12}{7} \int_0^1 \frac{1}{2}y^3 + y^3 dy \tag{11}$$
$$= \frac{12}{7} \left[\frac{1}{8}y^4 + \frac{1}{4}y^4\right]_0^1 \tag{12}$$
$$= \frac{12}{7}\left(\frac{3}{8}\right) \tag{13}$$
$$= \frac{9}{14}. \tag{14}$$

(c)

$$E[X^2Y] = \int_0^1 \int_0^1 x^2 y \cdot f(x,y)dxdy \tag{15}$$
$$= \frac{12}{7} \int_0^1 \int_0^1 x^2 y(xy + y^2)dxdy \tag{16}$$
$$= \frac{12}{7} \int_0^1 \int_0^1 x^3y^2 + x^2y^3 dxdy \tag{17}$$
$$= \frac{12 \times 2}{7} \int_0^1 \int_0^1 x^3 y^2 dxdy \tag{18}$$
$$= \frac{24}{7} \int_0^1 \left[\frac{1}{4}x^4 y^2\right]_{x=0}^{x=1} dy \tag{19}$$
$$= \frac{6}{7}\left[\frac{1}{3}y^3\right]_0^1 \tag{20}$$
$$= \frac{2}{7}. \tag{21}$$

Problem 3.

$$P(X = x, Y = y) = P(X = x)P(Y = y) \tag{22}$$
$$= \left(p(1-p)^{x-1}\right)\left(r(1-r)^{y-1}\right). \tag{23}$$

Then,

$$P(X < Y) = P(X \in [1, \infty))P(Y > X) \tag{24}$$
$$= \sum_{n=1}^{\infty} p(1-p)^{n-1}(1-r)^n. \tag{25}$$

Problem 4.

(a)

$$\int_0^\pi \int_0^\pi c(1-cos(x)cos(y))dxdy = 1 \tag{26}$$

$$c\int_0^\pi \int_0^\pi dxdy - c\int_0^\pi \int_0^\pi cos(x)cos(y)dxdy = 1 \tag{27}$$

$$c\pi^2 - c\int_0^\pi [sin(x)cos(y)]_{x=0}^{x=\pi} dy = 1 \tag{28}$$

$$c\pi^2 = 1 \tag{29}$$

$$c = \frac{1}{\pi^2}. \tag{30}$$

(b)

$$f_X(x) = \frac{1}{\pi^2}\int_0^\pi 1 - cos(x)cos(y)dy \tag{31}$$

$$= \frac{1}{\pi^2}\left(\pi - [cos(x)sin(y)]_{y=0}^{y=\pi}\right) \tag{32}$$

$$= \frac{1}{\pi} \tag{33}$$

for $x \in [0, \pi]$ and 0 otherwise. Similarly,

$$f_Y(y) = \frac{1}{\pi} \tag{34}$$

for $y \in [0, \pi]$ and 0 otherwise. The probability distribution is uniform.

(c)

$$f(0,0) = \frac{1}{\pi^2}\left(1 - cos(0)cos(0)\right) \tag{35}$$

$$= 0. \tag{36}$$

On the other hand,

$$f_X(0)f_Y(0) = \left(\frac{1}{\pi}\right)\left(\frac{1}{\pi}\right) \tag{37}$$

$$= \frac{1}{\pi^2} \tag{38}$$

$$\neq f(0,0) \tag{39}$$

Hence, X and Y are not independent.

Problem 5.

(a) We need to computer $F_T(t) = P(T \leq t) = 1 - P(T > t) = 1 - P(min(X,Y) > t) = 1 - P(X, Y > t)$. Assuming the unit to be 1 year, we have, for $x, y \geq 0$,

$$f_X(x) = e^{-x} \tag{40}$$

and

$$f_Y(y) = \frac{1}{2}e^{-\frac{y}{2}} \tag{41}$$

and the joint density is

$$f(x,y) = \frac{1}{2}e^{-x-\frac{y}{2}}. \tag{42}$$

Hence,

$$F_T(t) = 1 - \int_t^\infty \int_t^\infty f(x,y)dxdy = 1 - \int_t^\infty \int_t^\infty \frac{1}{2}e^{-x-\frac{y}{2}}dxdy \tag{43}$$
$$= 1 - \frac{1}{2}\int_t^\infty e^{-x}dx \int_t^\infty e^{-\frac{y}{2}}dy \tag{44}$$
$$= 1 - \frac{1}{2}e^{-t} \cdot 2e^{-\frac{t}{2}} \tag{45}$$
$$= 1 - e^{-\frac{3t}{2}} \tag{46}$$

for $t \geq 0$ and 0 otherwise.

(b) For $t \in [0, \infty)$,

$$f_T(t) = \frac{d}{dt}\left(1 - e^{-\frac{3t}{2}}\right) \tag{47}$$
$$= \frac{3}{2}e^{-\frac{3t}{2}} \tag{48}$$

and 0 otherwise.

(c)

$$E[T] = \int_0^\infty t \cdot \frac{3}{2}e^{-\frac{3t}{2}}dt \tag{49}$$
$$= -\left[te^{-\frac{3t}{2}}\right]_0^\infty + \int_0^\infty e^{-\frac{3t}{2}}dt \tag{50}$$
$$= 0 + \frac{-2}{3}\lim_{z\to\infty}\left[e^{-\frac{3t}{2}}\right]_0^z \tag{51}$$
$$= \frac{2}{3}. \tag{52}$$

Problem 6. The convolution formula tells us

$$P_{X+Y}(z) = \sum_X P_X(x)P_Y(z-x) \tag{53}$$
$$= P(X=0)P(Y=z-0) + P(X=1)P(Y=z-1) \tag{54}$$
$$= (1-p)P(Y=z) + pP(Y=z-1). \tag{55}$$

Hence,

$$P_{X+Y}(z) = \begin{cases} (1-p)(1-r) & z = 0, \\ p(1-r) + r(1-p) & z = 1, \\ pr & z = 2, \\ 0 & \textit{otherwise}. \end{cases} \tag{56}$$

Problem 7. *Convolution Approach*

We know $f_Y(y) = 1$ for $y \in (1,2)$ and 0 otherwise. For $z \leq 1$, $f_{X+Y}(z) = 0$. For $z \in [1,2]$,

$$
\begin{aligned}
f_{X+Y}(z) &= \int_1^z f_Y(y) f_X(z-y) dy && (57)\\
&= \int_1^z 2(z-y) dy && (58)\\
&= \int_1^z 2z - 2y dy && (59)\\
&= [2yz]_{y=1}^{y=z} - \left[y^2\right]_1^z && (60)\\
&= 2z^2 - 2z - z^2 + 1 && (61)\\
&= z^2 - 2z + 1. && (62)
\end{aligned}
$$

For $z \in [2,3]$,

$$
\begin{aligned}
f_{X+Y}(z) &= \int_{z-2}^1 f_Y(z-x) f_X(x) dx && (63)\\
&= \int_{z-2}^1 2x dx && (64)\\
&= \left[x^2\right]_{z-2}^1 && (65)\\
&= -z^2 + 4z - 3. && (66)
\end{aligned}
$$

For $z \geq 3$, $f_{X+Y}(z) = 0$.

CDF Approach
Since X and Y are independent, $f(x,y) = f_X(x) f_Y(y) = 2x$. For $z \leq 1$, $f_{X+Y}(z) = 0$. For $z \in [1,2]$,

$$
\begin{aligned}
P(X+Y \leq Z) &= \int_1^z \int_0^{z-y} 2x dx dy && (67)\\
&= \int_1^z \left[x^2\right]_0^{z-y} dy && (68)\\
&= \int_1^z z^2 - 2zy + y^2 dy && (69)\\
&= \left[z^2 y - zy^2 + \frac{1}{3} y^3\right]_1^z && (70)\\
&= z^3 - z^3 + \frac{1}{3} z^3 - z^2 + z - \frac{1}{3} && (71)\\
&= \frac{1}{3} z^3 - z^2 + z - \frac{1}{3}. && (72)
\end{aligned}
$$

Hence, for $z \in [1,2]$,

$$
\begin{aligned}
f_{X+Y}(z) &= \frac{d}{dz}\left(\frac{1}{3} z^3 - z^2 + z - \frac{1}{3}\right) && (73)\\
&= z^2 - 2z + 1. && (74)
\end{aligned}
$$

For $z \in [2,3]$,

$$
\begin{aligned}
P(X+Y \leq Z) &= \int_0^1 \int_1^2 2xdydx - \int_{z-2}^1 \int_{z-x}^2 2xdydx && (75)\\
&= \int_0^1 [2xy]_{y=1}^{y=2} dx - \int_{z-2}^1 [2xy]_{y=z-x}^{y=2} dx && (76)\\
&= \int_0^1 2xdx - \int_{z-2}^1 2x^2 + 4x - 2zxdx && (77)\\
&= 1 - \left[\frac{2}{3}x^3\right]_{z-2}^1 - \left[2x^2\right]_{z-2}^1 + \left[zx^2\right]_{z-2}^1 && (78)\\
&= -\frac{1}{3}z^3 + 2z^2 - 3z + 1. && (79)
\end{aligned}
$$

Hence, for $z \in [2,3]$,

$$
\begin{aligned}
f_{X+Y}(z) &= \frac{d}{dz}\left(-\frac{1}{3}z^3 + 2z^2 - 3z + 1\right) && (80)\\
&= -z^2 + 4z - 3. && (81)
\end{aligned}
$$

For $z \geq 3$, $f_{X+Y}(z) = 0$.

Problem 8.

$$
\begin{aligned}
p_{x+y}(z) &= \sum_{x \in X} P(X=x)P(Y=z-x) && (82)\\
&= \sum_{i=0}^{\infty} \left(\frac{\lambda^i}{i!}e^{-\lambda}\right)\left(\frac{\mu^{z-i}}{(z-i)!}e^{-\mu}\right) && (83)\\
&= e^{-(\lambda+\mu)} \sum_{i=0}^{\infty} \left(\frac{\lambda^i}{i!}\right)\left(\frac{\mu^{z-i}}{(z-i)!}\right) && (84)\\
&= e^{-(\lambda+\mu)} \sum_{i=0}^{\infty} \frac{1}{i!(z-i)!}(\lambda^i)(\mu^{z-i}) && (85)\\
&= e^{-(\lambda+\mu)} \frac{1}{z!} \sum_{i=0}^{\infty} \frac{z!}{i!(z-i)!}(\lambda^i)(\mu^{z-i}) && (86)\\
&= e^{-(\lambda+\mu)} \frac{1}{z!} \sum_{i=0}^{\infty} \binom{z}{i}(\lambda^i)(\mu^{z-i}) && (87)\\
&= \frac{(\lambda+\mu)^z}{z!}e^{-(\lambda+\mu)} && (88)\\
&= Poisson(\lambda+\mu). && (89)
\end{aligned}
$$

Problem 9. $X = \sigma_1 Z + \mu_1$ and $Y = \sigma_2 Z + \mu_2 \Rightarrow X + Y = (\sigma_1 + \sigma_2)Z + (\mu_1 + \mu_2) \Rightarrow N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$.

$$\begin{aligned}
f_{X+Y}(z) &= \int_{-\infty}^{\infty} f_X(x) f_Y(z-x)dx && (90)\\
&= \int_{-\infty}^{\infty} \left(\frac{1}{\sigma_1\sqrt{2\pi}} e^{-\frac{(x-\mu_1)^2}{2\sigma_1^2}}\right)\left(\frac{1}{\sigma_2\sqrt{2\pi}} e^{-\frac{(z-x-\mu_2)^2}{2\sigma_2^2}}\right) dx && (91)\\
&= \frac{1}{\sigma_1\sigma_2\sqrt{2\pi}\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(x-\mu_1)^2}{2\sigma_1^2} - \frac{(z-x-\mu_2)^2}{2\sigma_2^2}} dx && (92)\\
&= \frac{1}{\sigma_1\sigma_2\sqrt{2\pi}\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{\sigma_2^2(x-\mu_1)^2 + \sigma_1^2(z-x-\mu_2)^2}{2\sigma_1^2\sigma_2^2}} dx && (93)\\
&= \frac{1}{\sigma_1\sigma_2\sqrt{2\pi}\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{\sigma_2^2(x^2-2x\mu_1+\mu_1^2) + \sigma_1^2(z^2+x^2+\mu_2^2-2zx+2x\mu_2-2z\mu_2)}{2\sigma_1^2\sigma_2^2}} dx && (94)\\
&= \frac{1}{\sigma_1\sigma_2\sqrt{2\pi}\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2(\sigma_1^2+\sigma_2^2) - 2x(\sigma_1^2(z-\mu_2)+\sigma_2^2\mu_1) + \sigma_1^2(z^2+\mu_2^2-2z\mu_2) + \sigma_2^2\mu_1^2}{2\sigma_1^2\sigma_2^2}} dx. && (95)
\end{aligned}$$

Let $\sigma_z = \sqrt{\sigma_1^2 + \sigma_2^2}$,

$$f_{X+Y}(z) = \frac{1}{\sqrt{2\pi}\sigma_z} \frac{1}{\sqrt{2\pi}\frac{\sigma_1\sigma_2}{\sigma_z}} \int_{-\infty}^{\infty} e^{-\frac{x^2 - 2x\frac{\sigma_1^2(z-\mu_2)+\sigma_2^2\mu_1}{\sigma_z^2} + \frac{\sigma_1^2(z^2+\mu_2^2-2z\mu_2)+\sigma_2^2\mu_1^2}{\sigma_z^2}}{2\frac{\sigma_1\sigma_2}{\sigma_z}^2}} dx. \tag{96}$$

Interlude:

$$\begin{aligned}
& -\frac{x^2 - 2x\frac{\sigma_1^2(z-\mu_2)+\sigma_2^2\mu_1}{\sigma_z^2} + \frac{\sigma_1^2(z^2+\mu_2^2-2z\mu_2)+\sigma_2^2\mu_1^2}{\sigma_z^2}}{2\frac{\sigma_1\sigma_2}{\sigma_z}^2} && (97)\\
&= -\frac{\left(x - \frac{\sigma_1^2(z-\mu_2)+\sigma_2^2\mu_1}{\sigma_z^2}\right)^2 - \left(\frac{\sigma_1^2(z-\mu_2)+\sigma_2^2\mu_1}{\sigma_z^2}\right)^2 + \frac{\sigma_1^2(z-\mu_2)^2+\sigma_2^2\mu_1^2}{\sigma_z^2}}{2\frac{\sigma_1\sigma_2}{\sigma_z}^2} && (98)\\
&= -\left[\frac{\sigma_z^2(\sigma_1^2(z-\mu_2)^2 + \sigma_2^2\mu_1^2) - (\sigma_1^2(z-\mu_2) + \sigma_2^2\mu_1)^2}{2\sigma_z^2(\sigma_1\sigma_2)^2}\right] - \left[\frac{\left(x - \frac{\sigma_1^2(z-\mu_2)+\sigma_2^2\mu_1}{\sigma_z^2}\right)^2}{2\left(\frac{\sigma_1\sigma_2}{\sigma_z}\right)^2}\right] && (99)\\
&= -\left[\frac{(z-(\mu_1+\mu_2))^2}{2\sigma_z^2}\right] - \left[\frac{\left(x - \frac{\sigma_1^2(z-\mu_2)+\sigma_2^2\mu_1}{\sigma_z^2}\right)^2}{2\left(\frac{\sigma_1\sigma_2}{\sigma_z}\right)^2}\right] && (100)
\end{aligned}$$

Hence,

$$\begin{aligned}
f_{X+Y}(z) &= \frac{1}{\sqrt{2\pi}\sigma_z} \frac{1}{\sqrt{2\pi}\frac{\sigma_1\sigma_2}{\sigma_z}} \int_{-\infty}^{\infty} e^{-\left[\frac{(z-(\mu_1+\mu_2))^2}{2\sigma_z^2}\right] - \left[\frac{\left(x - \frac{\sigma_1^2(z-\mu_2)+\sigma_2^2\mu_1}{\sigma_z^2}\right)^2}{2\left(\frac{\sigma_1\sigma_2}{\sigma_z}\right)^2}\right]} dx && (101)\\
&= \frac{1}{\sqrt{2\pi}\sigma_z} e^{-\left[\frac{(z-(\mu_1+\mu_2))^2}{2\sigma_z^2}\right]} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\frac{\sigma_1\sigma_2}{\sigma_z}} e^{\left[\frac{-\left(x - \frac{\sigma_1^2(z-\mu_2)+\sigma_2^2\mu_1}{\sigma_z^2}\right)^2}{2\left(\frac{\sigma_1\sigma_2}{\sigma_z}\right)^2}\right]} dx && (102)\\
&= \frac{1}{\sqrt{2\pi}\sigma_z} e^{-\left[\frac{(z-(\mu_1+\mu_2))^2}{2\sigma_z^2}\right]} \quad \text{(the right hand side term evaluates to 1)} && (103)\\
&= N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2). && (104)
\end{aligned}$$