Math 109 HW5

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Problem 1 Let A, B, C be the set of students that like Reasoning, Algebra, and Calculus respectively. The question is asking for $|(A \cup B \cup C)^c|$.

$$\begin{aligned}
|(A \cup B \cup C)^c| &= 182 - |A \cup B \cup C| \\
&= 182 - [|A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|] \\
&= 182 - [129 + 129 + 129 - 85 - 89 - 86 + 53] \\
&= 2.
\end{aligned}$$

Problem 2. Let $X = (A \cap B); Y = (A \cap C); Z = (B \cap C); W = (A \cap B \cap C)$, and $x = |X|; y = |Y|; z = |Z|; w = |W|$. The question is asking for w. We know

$$A = (A \cap B^c \cap C^c) \cup (X \cup Y) \tag{1}$$
$$B = (A^c \cap B \cap C^c) \cup (X \cup Z) \tag{2}$$
$$C = (A^c \cap B^c \cap C) \cup (Y \cup Z). \tag{3}$$

and notice that the sets enclosed with parentheses are disjoint. Now, let us apply the inclusion-exclusion principle,

$$\begin{aligned}
170 - |A \cup B \cup C| = 2 &= 170 - [|A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|] \\
2 &= 170 - [124 + 124 + 124 - x - y - z + w] \\
204 &= x + y + z + -w, \qquad (4)
\end{aligned}$$

which is our first equation. Then we can apply the addition principle and inclusion-exclusion principle on (1), (2), and (3), and we get

$$\begin{aligned}
|A| = 124 &= 10 + (x + y - w) \\
114 &= x + y - w \qquad (5) \\
|B| = 124 &= x + z - w \qquad (6) \\
|C| = 124 &= 4 + y + z - w \\
120 &= y + z - w. \qquad (7)
\end{aligned}$$

Solving (4), (5), (6), and (7), we get $w = 50, x = 84, y = 80, z = 90$.

Problem 3.

(i) Let A, B, C be the set of people that speak English, Spanish, and Swahili respectively. Let

$$\begin{aligned}
a &= |(A \cap B^c \cap C^c)| \\
b &= |(A^c \cap B \cap C^c)| \\
c &= |(A^c \cap B^c \cap C)| \\
x &= |(A \cap B)| \\
y &= |(A \cap C)| \\
z &= |(B \cap C)| \\
w &= |(A \cap B \cap C)|.
\end{aligned}$$

Our goal is to maximize $a+b+c$. We know

$$\begin{aligned}|A \cup B \cup C| = 100 &= [|A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|] \\ 100 &= 75 + 60 + 45 - x - y - z + w \\ 80 &= x + y + z - w. \end{aligned} \tag{8}$$

Also,

$$|A| = 75 = a + x + y - w \tag{9}$$
$$|B| = 60 = b + x + z - w \tag{10}$$
$$|C| = 45 = c + y + z - w. \tag{11}$$

Then, from (9) to (11),

$$\begin{aligned}|A| + |B| + |C| = 75 + 60 + 45 &= a + b + c + 2x + 2y + 2z - 3w \\ 180 &= a + b + c + 2x + 2y + 2z - 3w.\end{aligned} \tag{12}$$

(12) $- 2 \times$ (8):

$$a + b + c = 20 + w. \tag{13}$$

Hence, we need to maximize w. Note that $x, y, z \geq w$. Thus, by (8), $w \leq 40$, and $a + b + c \leq 60$. Taking $w, x, y, z = 40$ and solving (9) to (11), $a = 35, b = 20, c = 5$.

(ii) (9) – (8):

$$\begin{aligned}-5 &= a - z \\ a &= z - 5.\end{aligned}$$

Hence, maximizing a is equivalent to maximizing z. $z = |B \cap C| \leq |C| = 45 \Rightarrow a \leq 40$. Thus, we take $a = 40, z = 45$. Notice that $c = |(A^c \cap B^c \cap C)|$ an $z = |B \cap C|$, so if $z = 45 = |C|$, then $c = 0$. Then, solving (9) to (11), we can conclude with the following equations equations:

$$\begin{aligned}&a = 40 \\ &z = 45 \\ &c = 0 \\ &x = 35 \\ &b = w - 20 = y - 20 \\ &0 \leq b \leq 15. \qquad (\because w = y \leq |A| - a = 35)\end{aligned}$$

Maximum number of people who speak only English = 40. In this case, the number of people who speak only Spanish = 0, and the number of people who speak only Swahili can range from 0 to 15.

(iii) Already proved in (i).