Math 110 Final

Neo Lee

TOTAL POINTS

100 / 100

QUESTION 1

1 Problem 1 10 / 10

✓ **+ 10 pts** *Correct*

+ 6 pts Correctly show it is a basis

+ 4 pts correctly compute dual basis

+ 3 pts minor computation error/partial progress

+ 2 pts partial progress

+ 1 pts partial progress

+ 0 pts Click here to replace this description.

QUESTION 2

2 Problem 2 10 / 10

✓ **- 0 pts** *Correct*

a

- 2 pts Minor errors

- 4 pts Major errors

- 5 pts Negligible progress (including only discussing 0 without any other mathematical content)

b

- 2 pts Correct result but minor errors in proof

- 4 pts Incorrect result or major errors in proof

- 5 pts Negligible progress (including stating a value for dim S_D with no attempt at a proof)

- 10 pts Blank / negligible progress

- 10 pts Copied or close-to-copied solution to problem 2 on practice final without seriously engaging with problem

QUESTION 3

3 Problem 3 10 / 10

✓ **+ 3 pts** *(a) All correct*

+ 2 pts (a): Linearity checked

+ 1 pts (a): Containment of outputs in co-domain checked

✓ **+ 3 pts** *(b) All correct or at least consistent with misunderstanding of the action of T*

+ 2 pts (b) Partially correct / partially justified

+ 1 pts Minor progress / Major gap(s) in (b)

✓ **+ 4 pts** *(c) Correct*

+ 3 pts (c) Almost correct (such as a minor misstatement or correct solution re unitary operator, not isometry)

+ 2 pts Correct idea, major gap in (c)

+ 0 pts No progress / All wrong / No justification for answers

QUESTION 4

4 Problem 4 10 / 10

✓ **- 0 pts** *Correct. There are two possible Jordan canonical forms. For $$\lambda = 1$$ there are two Jordan blocks of size one or one Jordan block of size two. For $$\lambda = -1$$, there is one Jordan block of size 4. The corresponding minimal polynomials* *are $$p(z)=(z-1)(z+1)^4$$ and $$p(z)=(z-1)^2(z+1)^4$$, respectively.*

- 4 pts Gave one correct Jordan normal form and the corresponding minimal polynomial with justification, but missed the other possibility, or included Jordan normal forms that are not possible.

- 2 pts Incorrect minimal polynomial or the correspondence between Jordan normal forms and the minimal polynomial is incorrect.

- 2 pts Point adjustment: virtually no reasonings given.

- 1 pts Point adjustment: minor error, or included arguments that do not make sense.

- 8 pts Partial credit: some correct ideas. For example, pointed out the relation between the dimensions of null spaces and the sizes of Jordan blocks.

- 10 pts Blank, incorrect, or irrelevant.

- 0 pts flag

QUESTION 5

5 Problem 5 **15 / 15**

✓ **+ 15 pts** *Click here to replace this description.*

+ 6 pts Correct image of a basis: correct image of each base vector = 1 point

+ 6 pts Comupted the desired basis with correct approach: Each correct base vector worth 1 point

+ 3 pts Correct diagonal matrix

- 2 pts Click here to replace this description.

- 4 pts Click here to replace this description.

- 6 pts Click here to replace this description.

+ 0 pts Click here to replace this description.

QUESTION 6

6 Problem 6 **10 / 10**

✓ **+ 10 pts** *Correct (points not deducted for the opposite sign at the end, ie approximation of -cos(x) is ok)*

+ 9 pts Correct except for minor calculation error. Solution must be of the form f(x) = ax^2 + c to earn this.

+ 5 pts Set up system with <e, cos> = <e, p> or equivalent and computed solution, but made significant errors. Solution of f(x)=0 do not score higher than this.

+ 3 pts Computed an ONB of P2 with respect to the correct inner product (but -1 if basis is not actually ON or is incomplete)

+ 0 pts No meaningful progress toward successful solution. Assertions that cosine is orthogonal to P2 without (attempted) proof do not earn points. Solutions that aren't in P2 do not earn points

QUESTION 7

7 Problem 7 **10 / 10**

✓ **+ 10 pts** *Correct*

+ 4 pts Correct spectral theorem: **explicitly** stated that you are picking an orthonormal basis $$(e_i)$$ under which $$M(T)$$ is a diagonal matrix, **before** using $$e_i$$'s. Mentioning '$$e_i$$ being orthonormal' and '$$M(T)$$ being diagonal' each worth 2 points.

Alternatively, showed that the singular values of $$T$$ are $$|\lambda_i|$$'s.
If directly cited theorem 7.85, then this proof

worth 6 points.

+ 4 pts If $$(e_i)$$ is an orthonormal basis, then $$|\sum_i a_ie_i|^2 = \sum |a_i|^2$$.
Note that to get a correct proof, you (likely) have to compare $$|Tv|^2$$ to $$R^2|v|^2$$. All proofs I saw comparing $$|Tv|$$ to $$R|v|$$ with spectral theorem are fundamentally incorrect.

+ 2 pts Did some correct and useful estimation using $$|\lambda_i|\leq R$$.
In particular, simply multiplying something to both sides of the inequality worth no credit, but showing that for an eigenvector $$v$$ of $$T$$, $$|Tv| \leq R|v|$$ worth 2 points.

+ 0 pts Click here to replace this description.

- 1 pts Click here to replace this description.

- 2 pts Click here to replace this description.

- 3 pts Click here to replace this description.

QUESTION 8

8 Problem 8 **15 / 15**

✓ **- 0 pts** *all correct*

- 5 pts claimed T is self adjoint or did not answer part a

- 5 pts wrong singular values or singular values not listed

- 5 pts SVD is wrong by more than a sign, or is missing, or uses a formula that does not work in general without simplification. Follow through points are given only if your value of M(T) is given and your svd is completely correct for that matrix

- 2 pts sign of SVD is wrong

- 1 pts singular value of 0 is not recorded

QUESTION 9

9 Problem 9a **2 / 2**

✓ **- 0 pts** *Correct*

- 1 pts Blank

- 2 pts Incorrect

QUESTION 10

10 Problem 9b **2 / 2**

✓ **- 0 pts** *Correct*

- 1 pts Blank

- 2 pts Incorrect

QUESTION 11

11 Problem 9c **2 / 2**

✓ **- 0 pts** *Correct*

- 1 pts Blank

- 2 pts Incorrect

QUESTION 12

12 Problem 9d **2 / 2**

✓ **- 0 pts** *Correct*

- 1 pts Blank

- 2 pts Incorrect

QUESTION 13

13 Problem 9e **2 / 2**

✓ **- 0 pts** *Correct*

- 1 pts Blank

- 2 pts Incorrect

note: using LADR 3rd. ed.

MATH 110, Fall 2023, final test.

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All necessary work to **justify an answer** and all necessary steps of a proof must be shown clearly to obtain full credit. Partial credit **may** be given but only for significant progress towards a solution. Show all relevant work **in logical sequence** and **indicate all answers clearly**. Cross out all work you do not wish considered.

1. (10pts.) Let $V = \mathcal{P}_2(\mathbb{R})$ and let $\varphi_j \in V'$, $j = 0, 1, 2$, be defined as follows: $\varphi_0(f) := f(0)$, $\varphi_1(f) := f(1)$, $\varphi_2(f) := \int_0^1 f(t)\,dt$. Prove that $(\varphi_0, \varphi_1, \varphi_2)$ is a basis of V' and find a basis (f_0, f_1, f_2) of V such that $(\varphi_0, \varphi_1, \varphi_2)$ is dual to (f_0, f_1, f_2).

f_0 is of the form $a_0x^2 + b_0x + 1$, and

$$\begin{cases} f_0(1) = 0 \Rightarrow a_0 + b_0 + 1 = 0 \\ \int_0^1 f_0(x)\,dx = 0 \Rightarrow \frac{a_0}{3} + \frac{b_0}{2} + 1 = 0 \end{cases} \Rightarrow \begin{cases} a_0 = 3 \\ b_0 = -4 \end{cases} \Rightarrow f_0 = 3x^2 - 4x + 1$$

f_1 is of form $a_1x^2 + b_1x$, and

$$\begin{cases} f_1(1) = 1 \Rightarrow a_1 + b_1 = 1 \\ \int_0^1 f_1(x)\,dx = 0 \Rightarrow \frac{a_1}{3} + \frac{b_1}{2} = 0 \end{cases} \Rightarrow \begin{cases} a_1 = 3 \\ b_1 = -2 \end{cases} \Rightarrow f_1 = 3x^2 - 2x$$

f_2 is of form $a_2x^2 + b_2x$, and

$$\begin{cases} f_2(1) = 0 \Rightarrow a_2 + b_2 = 0 \\ \int_0^1 f_2(x)\,dx = 1 \Rightarrow \frac{a_2}{3} + \frac{b_2}{2} = 1 \end{cases} \Rightarrow \begin{matrix} a_2 = -6 \\ b_2 = 6 \end{matrix} \Rightarrow f_2 = -6x^2 + 6x$$

---

now, check

$$af_0 + bf_1 + cf_2 = 0$$

$$a(3x^2 - 4x + 1) + b(3x^2 - 2x) + c(-6x^2 + 6x) = 0$$

$$x^2(3a + 3b - 6c) + x(-4a - 2b + 6c) + a = 0$$

$$\Rightarrow a = 0 \Rightarrow \begin{cases} 3b - 6c = 0 \\ -2b + 6c = 0 \end{cases} \Rightarrow \begin{cases} b = 2c \\ b = 3c \end{cases} \Rightarrow \begin{cases} b = 0 \\ c = 0 \end{cases}$$

$\Rightarrow$ f_0, f_1, f_2 are linearly independent and of length $3 = \dim P_2(\mathbb{R})$

so f_0, f_1, f_2 are basis of V

$\Rightarrow$ indeed $\varphi_0, \varphi_1, \varphi_2$ are dual basis of f_0, f_1, f_2, by Thm 3.98 (3rd ed.)

$\varphi_0, \varphi_1, \varphi_2$ are basis of V'

2. (10pts.) Let $T \in \mathcal{L}(V)$ for some vector space V and let T' denote, as usual, its dual map. Define

$$S_T := \cap_{k=1}^{\infty} \text{range}\,((T')^k).$$

(a) Prove that S_T is a subspace of V'.
(b) Let $n \in \mathbb{N}$. Determine $\dim S_D$ for the differentiation operator D on $\mathcal{P}_n(\mathbb{R})$.

(a) $S_T = \text{range } T' \cap \text{range}(T')^2 \cap \text{range}(T')^3 \cap \cdots$

notice $T' \in \mathcal{L}(V')$ and $(T')^k \in \mathcal{L}(V')$ for all k, so

$\text{range}(T')^k$ is a subspace of V' for all k.

Clearly, $\vec{0} \in \text{range}(T')^k\ \forall k$ because $\text{range}(T')^k$ are subspaces.

let $s \in S_T$, then $s \in \text{range}(T')^k\ \forall k$, since all $\text{range}(T')^k$ are subspaces,

$\lambda s \in \text{range}(T')^k\ \forall k$, hence in S_T

Similarly, let $s_1, s_2 \in S_T \iff s_1, s_2 \in \text{range}(T')^k\ \forall k$

$\iff s_1 + s_2 \in \text{range}(T')^k\ \forall k$

$\iff s_1 + s_2 \in S_T$

$\square$

(b) $S_D = \text{range } D' \cap \text{range}(D')^2 \cap \text{range}(D')^3 \cap \cdots$

notice $(D')^k = (D^k)'\ \forall k$, which can be check from its matrix representation (since $P_n(\mathbb{R})$ is f-d) and following Thm 3.114.

From Thm 3.109, $\dim \text{range}(D^k)' = \dim \text{range } D^k$.

stop when codomain $= \{0\}$

Notice $D: P_n(\mathbb{R}) \to P_{n-1}(\mathbb{R})$, in particular $D^k: P_n(\mathbb{R}) \to P_{n-k}(\mathbb{R})$

That means $\lim_{k\to\infty} \dim \text{range } D^k = 0$, in other words $\text{range}(D^m) = \{0\}$

after certain m, then $\dim \text{range } D^m = 0 = \dim \text{range}(D')^m$.

Clearly, this $\text{range}(D')^m$ is in the chain of the intersection.

So S_D is a bunch of intersection with some $\{0\}$ in there.

$\Rightarrow \dim S_D = 0$

$e^{ix} = \cos x + i\sin x$ $\qquad$ $\cos(2x) = \cos^2 x - \sin^2 x$

$\sin(2x) = 2\sin x\cos x$ $\qquad$ $\cos^2 x = \frac{1}{2}\cos 2x + \frac{1}{2}$

3. (10pts.) Consider the complex vector spaces $V = \mathrm{span}(1, \cos x, \sin x)$ and $W = \mathrm{span}(1, \cos x, \sin x, \cos 2x, \sin 2x)$, both equipped with the inner product $\langle f, g\rangle = \int_{-\pi}^{\pi} f(t)\overline{g(t)}\,dt$. Let $(Tf)(x) := e^{ix} f(x)$.

(a) Prove that T so defined is a linear map from V to W.
(b) Determine, with proof, $\dim \mathrm{null}\, T$ and $\dim \mathrm{range}\, T$.
(c) Determine, with proof, whether T is an isometry.

(a) Let $v \in V$, $v = \alpha + \beta\cos x + \gamma\sin x$, then

$$
\begin{aligned}
Tv = (e^{ix})v &= (\cos x + i\sin x)(\alpha + \beta\cos x + \gamma\sin x)\\
&= \alpha\cos x + \alpha i\sin x + \beta\cos^2 x + \gamma i\sin^2 x + (\gamma + \beta i)\sin x\cos x\\
&= \alpha\cos x + \alpha i\sin x + \tfrac{1}{2}(\gamma+\beta i)\sin(2x) + \beta\cos^2 x + \gamma i(1-\cos^2 x)\\
&= \alpha\cos x + \alpha i\sin x + \tfrac{1}{2}(\gamma+\beta i)\sin 2x + \gamma i + (\beta - \gamma i)\cos^2 x\\
&= \alpha\cos x + \alpha i\sin x + \tfrac{1}{2}(\gamma+\beta i)\sin 2x + \gamma i + (\beta-\gamma i)(\tfrac{1}{2}\cos 2x + \tfrac{1}{2})\\
&= \alpha\cos x + \alpha i\sin x + \tfrac{1}{2}(\gamma+\beta i)\sin 2x + \tfrac{1}{2}(\beta-\gamma i)\cos 2x + \tfrac{1}{2}(\beta+\gamma i),
\end{aligned}
$$

which is indeed in W.

Now, if we look at $v_1 = \alpha_1 + \beta_1\cos x + \gamma_1\sin x$ and $v_2 = \alpha_2 + \beta_2\cos x + \gamma_2\sin x$, indeed we can track the coefficients and $T(v_1+v_2) = T(v_1) + T(v_2)$.
(also since RHS is all linear terms)

Similarly, $T(\lambda v) = \lambda T(v)$. $\square$

(b) Let $v = \alpha + \beta\cos x + \gamma\sin x$ s.t. $Tv = 0$, then

$$
\begin{cases}
\alpha = 0\\
\frac{1}{2}(\gamma+\beta i) = 0\\
\frac{1}{2}(\beta-\gamma i) = 0\\
\frac{1}{2}(\beta+\gamma i) = 0
\end{cases}
\Rightarrow \beta - \gamma i = \beta + \gamma i \Rightarrow \gamma i = -\gamma i \Rightarrow \gamma = 0 \Rightarrow \beta = 0
$$

Hence $v = \vec{0}$.

$\Rightarrow \dim \mathrm{null}\, T = 0$

$\Rightarrow \dim \mathrm{range}\, T = \dim V - \dim\mathrm{null}\, T = 3$ since V is f-d. $\square$

(c) Per Defn 7.37 in 3rd ed., no since S is not an operator, but take $v = \alpha + \beta\cos x + \gamma\sin x$, $\|v\|^2 = |\alpha|^2\cdot 2\pi + |\beta|^2\cdot\pi + |\gamma|^2\cdot\pi$

$$
\begin{aligned}
\|Tv\|^2 &= (|\alpha|^2 + |\alpha i|^2)\pi + \left(\left|\tfrac{1}{2}(\gamma+\beta i)\right|^2 + \left|\tfrac{1}{2}(\beta-\gamma i)\right|^2 + \left|\tfrac{1}{2}(\beta+\gamma i)\right|^2\cdot 2\right)\cdot\pi\\
&= |\alpha|^2\cdot 2\pi + \left(\tfrac{1}{4}|\gamma+\beta i|^2 + \tfrac{1}{4}|\beta-\gamma i|^2 + \tfrac{1}{2}|\beta+\gamma i|^2\right)\cdot\pi\\
&= \|v\|^2 \Leftrightarrow \|Tv\| = \|v\|.
\end{aligned}
$$

So per the defn of preserving norm, yes.

4. (10pts.) Let V be a complex vector space of dimension 6 and let $T \in \mathcal{L}(V)$ be such that 1 and -1 are its eigenvalues, $\dim \operatorname{null}(T-I)^2 = \dim \operatorname{null}(T+I)^2 = 2$, and $\dim \operatorname{null}(T+I)^4 = 4$. What possible Jordan Normal Form(s) and corresponding minimal polynomial(s) can T have? Justify your answer.

$\dim \operatorname{null}(T+I)$ is not 0 nor 2, otherwise the null space chain would have stopped and $\dim \operatorname{null}(T+I)^4 \neq 4$, so $\dim \operatorname{null}(T+I) = 1$

$\dim \operatorname{null}(T+I)^3$ is not 2 otherwise the chain would have stopped and not 4 because $\dim \operatorname{null}(T+I)^2 - \dim \operatorname{null}(T+I) = 1$, which is no. of jordan block size ≥ 2, and $\dim \operatorname{null}(T+I)^3 - \dim \operatorname{null}(T+I)^2$, which is no. of jordan block ≥ 3 cannot be more than 1, hence $\dim \operatorname{null}(T+I)^3 = 3$.

Hence,

$$\dim \operatorname{null}(T+I)^k = k \quad \text{for } k = 1, 2, 3, 4$$

Now, $\dim \operatorname{null}(T-I)^2 = 2$, which means the dim of generalized eigenspace wrt $\lambda = 1$ is at least 2. But the dim of generalized eigenspace wrt $\lambda = -1$ is at least 4, and they must add up to $\dim V = 6$. So $\dim G(1, T)$ is exactly 2.

Indeed, $\dim \operatorname{null}(T-I)$ can be either 1 or 2. so there can be a 2×2 or two 1×1 jordan block for $\lambda = 1$.

So jordan form may look like

$$\begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & -1 & 1 & & \\ & & & -1 & 1 & \\ & & & & -1 & 1 \\ & & & & & -1 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1 & 1 & & & & \\ & 1 & & & & \\ & & -1 & 1 & & \\ & & & -1 & 1 & \\ & & & & -1 & 1 \\ & & & & & -1 \end{bmatrix}$$

$p(z) = (z-1)(z+1)^4$

$p(z) = (z-1)^2(z+1)^4$

note: the power corresponds to the maximal size of a jordan block corresponding to that eigenvalue.

5. (15pts.) Let V be the real vector space of polynomials in x and y of (total) degree at most 2, and let $T \in \mathcal{L}(V)$ be defined as follows (you do not need to verify that $T \in \mathcal{L}(V)$; it is so):

$$(Tf)(x,y) := y^2 \frac{\partial^2}{\partial x^2} f(x,y) + x^2 \frac{\partial^2}{\partial y^2} f(x,y).$$

Find a basis of V that diagonalizes T (over $\mathbb{R}$) and the resulting diagonal matrix representation $\mathcal{M}(T)$ or prove that T is not diagonalizable over $\mathbb{R}$.

$V = \operatorname{span}(1, x, x^2, y, y^2, xy)$

$T(1) = 0$

$T(x) = 0$

$T(x^2) = y^2 \frac{\partial^2}{\partial x^2} x^2 + 0 = 2y^2$

$T(y) = 0$

$T(y^2) = 0 + x^2 \frac{\partial^2}{\partial y^2} y^2 = 2x^2$

$T(xy) = 0$

now, $(1, x, y, xy) \in E(0,T)$ and they are the basis for that subspace clearly.

now, look at the invariant subspace $\operatorname{span}(x^2, y^2)$

notice $T(x^2+y^2) = 2y^2 + 2x^2 = 2(x^2+y^2)$

$T(x^2-y^2) = 2y^2 - 2x^2 = -2(x^2-y^2)$

so x^2+y^2, x^2-y^2 are eigenvector corresponding to $\lambda = 2$, $\lambda = -2$,

and they are independent since they correspond to distinct eigenvalue.

$\Rightarrow$ they are the basis of that invariant subspace

Hence, $\mathcal{M}(T, (1, x, y, xy, x^2+y^2, x^2-y^2))$

$$= \begin{bmatrix} 0 & & & & & \\ & 0 & & & 0 & \\ & & 0 & & & \\ & & & 0 & & \\ & 0 & & & 2 & \\ & & & & & -2 \end{bmatrix}$$

6. (10pts.) Find a polynomial $f \in \mathcal{P}_2(\mathbb{R})$ which minimizes the integral

$$\int_{-\pi}^{\pi} |\cos(x) - f(x)|^2 dx.$$

Define the inner product space

$V = P_2(\mathbb{R}) + \text{span}(\cos x)$, which can be checked is indeed a vector space.

with inner product $\langle f, g \rangle = \int_{-\pi}^{\pi} f(x)\overline{g(x)}\,dx$.

Now, it becomes finding the orthogonal projection of $\cos(x)$ onto the subspace $P_2(\mathbb{R})$, which would minimize the integral.

From doing Gram-Schmidt, the ON basis of $P_2(\mathbb{R})$ is

$$\left(\frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\frac{2}{3}\pi^3}} x, \frac{1}{\sqrt{\frac{8}{45}\pi^5}}\left(x^2 - \frac{1}{3}\pi^2\right)\right) =: (e_1, e_2, e_3)$$

$\cos x$ is orthogonal to span(1).

$$\langle \cos x, x \rangle = \int_{-\pi}^{\pi} x \cos x\,dx = x\sin x\Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} \sin x\,dx = 0 - (-\cos x)_{-\pi}^{\pi} = 0$$

$$\langle \cos x, x^2 \rangle = \int_{-\pi}^{\pi} x^2 \cos x\,dx = x^2 \sin x\Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} 2x\sin x\,dx$$

$$= 0 - 2\int_{-\pi}^{\pi} x\sin x\,dx = 2\left[x\cos x\Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi}\cos x\,dx\right]$$

$$= 2\left[\pi(-1) - (-\pi)(-1) - \sin x\Big|_{-\pi}^{\pi}\right]$$

$$= -4\pi$$

Hence,

$$f = \langle \cos x, e_1 \rangle e_1 + \langle \cos x, e_2 \rangle e_2 + \langle \cos x, e_3 \rangle e_3$$

$$= 0 + 0 + \left(\langle \cos x, \frac{1}{\sqrt{\frac{8}{45}\pi^5}} x^2 \rangle - \underbrace{\langle \cos x, \frac{1}{\sqrt{\frac{8}{45}\pi^5}} \cdot \left(\frac{1}{3}\pi^2\right) \rangle}_{=0}\right) e_3$$

$$= \left(\frac{1}{\sqrt{\frac{8}{45}\pi^5}} \cdot (-4\pi)\right) \cdot e_3$$

7. (10pts.) Let T be a normal operator on a complex finite-dimensional inner product space V whose eigenvalues are $\lambda_1, \ldots, \lambda_k$. Define

$$R := \max\{|\lambda_1|, |\lambda_2|, \ldots, |\lambda_k|\}.$$

Prove that $\|Tv\| \leq R\|v\|$ for any vector $v \in V$.

By the complex spectral theorem, there is an ON basis consists of eigenvectors of T, denote $e_1, \ldots, e_n$.

Then all $v \in V$ can be written as $\sum_{k=1}^{n} a_k e_n$, and

$$Tv = \sum_{k=1}^{n} \lambda_k a_k e_k \text{ , where } \lambda_k \text{ is the eigenvalue wrt } e_k$$

$$\|Tv\|^2 = \langle \sum \lambda_k a_k e_k, \sum \lambda_k a_k e_k \rangle = \sum |\lambda_k a_k|^2 \text{ since } e_k\text{'s are orthogonal (Thm 6.25)}$$

$$= \sum (|\lambda_k||a_k|)^2$$

$$= \sum |\lambda_k|^2 |a_k|^2 \quad \text{from thm 4.5}$$

From thm 6.10,

$$R\|v\| = |R|\|v\| = \|Rv\|$$

Now, $\|Rv\|^2 = \langle R\sum a_k e_k, R\sum a_k e_k \rangle$

$$= \langle \sum R a_k e_k, \sum R a_k e_k \rangle$$

$$= \sum |R a_k|^2 = \sum |R|^2 |a_k|^2$$

$$\geq \sum |\lambda_k|^2 |a_k|^2 \text{ since } |R| \geq |\lambda_k| \ \forall k$$

$$= \|Tv\|^2 \qquad \Rightarrow |R|^2 \geq |\lambda_k|^2$$

Hence,

$$\|Rv\|^2 \geq \|Tv\|^2$$

$$\Rightarrow \|Rv\| \geq \|Tv\|$$

$$\Rightarrow R\|v\| \geq \|Tv\|$$

8. (15pts.) Consider the complex inner product space $V = \operatorname{span}(1, \cos x, \sin x)$ with the inner product

$$\langle f, g\rangle = \int_{-\pi}^{\pi} f(x)\overline{g(x)}dx$$

and the operator $T = D + 2D^3 : f(x) \mapsto f'(x) + 2f'''(x)$.
(a) Is T self-adjoint? Explain.
(b) Find the singular values of T.
(c) Determine the singular value decomposition of T.

We know from hw, the ON basis is

$$\left(\frac{1}{\sqrt{2\pi}}, \frac{\cos(x)}{\sqrt{\pi}}, \frac{\sin(x)}{\sqrt{\pi}}\right) =: (e_1, e_2, e_3)$$

$T(e_1) = 0$, $T(e_2) = \frac{-\sin(x)}{\sqrt{\pi}} + 2\frac{\sin(x)}{\sqrt{\pi}} = e_3$ $\quad T(e_3) = \frac{\cos(x)}{\sqrt{\pi}} - 2\frac{\cos(x)}{\sqrt{\pi}} = -e_2$

(a) $\mathcal{M}(T, (e_1, e_2, e_3)) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$ while $\mathcal{M}(T^*, (e_1, e_2, e_3))$

$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \neq \mathcal{M}(T)$

So T is not self-adjoint. note V is f-d, so $\mathcal{M}(T)$ makes sense.

(b) They are the positive sq. root of eigenvalues of T^*T.

$$\mathcal{M}(T^*)\mathcal{M}(T) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The eigenvalues are $0, 1, 1$ (diagonal of upper-triang. mat.)

So the singular values are $1, 1, 0$ (listed in desc. order)

(c) Recall $T(e_2) = e_3$ and $T(e_3) = -e_2$, so $f_2 = e_3$, $f_3 = -e_2$

and $\quad Tv = \langle v, e_2\rangle f_2 + \langle v, e_3\rangle f_3$

9. (10pts.) Decide if the following implications hold in the settings below. No need to justify your answers. You will receive 2pts for each correct answer, 1pt for each blank answer, 0pts for each incorrect answer. Please circle or underline the best answer.

(a) The annihilator of a subset U of a vector space V is a subspace of V'.

ALWAYS TRUE
TRUE ONLY IN FINITE DIMENSION
TRUE ONLY IF U IS A SUBSPACE OF V

(b) $S, T \in \mathcal{L}(V)$ ($\dim V < \infty$) satisfy $S = T^*$ if and only if their matrix representations are conjugate transposes of each other.

ALWAYS TRUE
TRUE ONLY IF USING THE SAME BASIS
TRUE ONLY IF USING THE SAME ORTHONORMAL BASIS

(c) If $\dim V$, $\dim W < \infty$ and $T \in \mathcal{L}(V, W)$ is injective, then $\dim V \leq \dim W$.

TRUE OVER $\mathbb{C}$ AND $\mathbb{R}$
TRUE OVER $\mathbb{C}$ BUT NOT $\mathbb{R}$
FALSE

(d) Any diagonalizable operator on a finite-dimensional inner product space is self-adjoint.

TRUE OVER $\mathbb{C}$ AND $\mathbb{R}$
TRUE OVER $\mathbb{R}$ BUT NOT $\mathbb{C}$
FALSE

(e) $T' \in \mathcal{L}(V')$ is injective whenever $T \in \mathcal{L}(V)$ is surjective.

ALWAYS TRUE
TRUE ONLY IN FINITE DIMENSION
FALSE