Math 181A HW4

Neo Lee

04/25/2023

Problem 5.4.2

(a)

$$P(2.8 \le \hat{\theta} \le 3.2) = P(\hat{\theta} \ge 2.8) = 1 - P(\{Y_1, \dots, Y_6\} \le 2.8)$$

$$= 1 - P(Y_1 \le 2.8)^6$$

$$= 1 - \left(\frac{2.8}{3}\right)^6$$

$$\approx 0.339.$$

(b)

$$P(2.8 \le \hat{\theta} \le 3.2) = P(\hat{\theta} \le 2.8) = 1 - P(\{Y_1, \dots, Y_3\} \le 2.8)$$

$$= 1 - P(Y_1 \le 2.8)^3$$

$$= 1 - \left(\frac{2.8}{3}\right)^3$$

$$\approx 0.187.$$

Problem 5.4.15

$$\begin{split} E[\overline{W}^2] &= Var(\overline{W}) + (E[\overline{W}])^2 \\ &= Var(\frac{1}{n}\sum_{i=1}^n W_i) + \left(E[\frac{1}{n}\sum_{i=1}^n W_i]\right)^2 \\ &= \frac{1}{n^2}\sum_{i=1}^n Var(W_i) + \left(\frac{1}{n}\sum_{i=1}^n E[W_i]\right)^2 \\ &= \frac{n\sigma^2}{n^2} + \left(\frac{n\mu}{n}\right)^2 \\ &= \frac{\sigma^2}{n} + \mu^2, \\ \lim_{n \to \infty} E[\overline{W}^2] &= \lim_{n \to \infty} \frac{\sigma^2}{n} + \mu^2 \\ &= \mu^2. \end{split}$$

Problem 5.4.18 By symmetry, $Var(Y_{min}) = Var(Y_{max})$, let σ^2 . Hence,

$$\begin{split} \mathit{MSE}(\hat{\theta}_1) &= \mathit{Var}(\hat{\theta}_1) + \mathit{Bias}(\hat{\theta}_1) \\ &= \mathit{Var}(\hat{\theta}_1) \\ &= \frac{36}{25}\sigma^2. \\ \mathit{MSE}(\hat{\theta}_2) &= \mathit{Var}(\hat{\theta}_2) + \mathit{Bias}(\hat{\theta}_2) \\ &= \mathit{Var}(\hat{\theta}_2) \\ &= 36\sigma^2 \\ &> \mathit{MSE}(\hat{\theta}_1). \end{split}$$

Therefore, $\hat{\theta}_1$ is better than $\hat{\theta}_2$.

Problem 5.4.21 With the same notation from last question, then

$$\frac{Var(\hat{\theta}_1)}{Var(\hat{\theta}_2)} = \frac{(n+1)^2 \sigma^2}{\left(\frac{n+1}{n}\right)^2 \sigma^2}$$
$$= n^2.$$

Problem 5.5.2

$$l(\lambda) = \log(\frac{e^{-\lambda}\lambda^k}{k!})$$

$$= -\lambda + k \log(\lambda) - \log(k!),$$

$$l'(\lambda) = -1 + \frac{k}{\lambda},$$

$$l''(\lambda) = -\frac{k}{\lambda^2}.$$

$$I(\lambda) = -E[l''(\lambda)]$$

$$= -E[\frac{-k}{\lambda^2}]$$

$$= \sum_{k=0}^{\infty} \frac{k}{\lambda^2} \cdot e^{-\lambda} \frac{\lambda^k}{k!}$$

$$= \frac{e^{-\lambda}}{\lambda} \sum_{k=0}^{\infty} \frac{\lambda^{k-1}}{(k-1)!}$$

$$= \frac{e^{-\lambda}}{\lambda} \left(\sum_{k=0}^{\infty} \frac{\lambda^k}{k!}\right)$$

$$= \frac{e^{-\lambda}}{\lambda} \left(\sum_{k=0}^{\infty} \frac{\lambda^k}{k!}\right)$$

$$= \frac{e^{-\lambda}}{\lambda} \cdot e^{\lambda}$$

$$= \frac{1}{\lambda}.$$

$$Var(\hat{\lambda}) = Var\left(\frac{1}{n} \sum_{i=1}^{n} X_i\right)$$

$$= \frac{1}{n^2} Var\left(\sum_{i=1}^{n} X_i\right)$$

$$= \frac{1}{n} Var(X_i)$$

$$= \frac{1}{n} \lambda$$

$$= \frac{1}{n(1/\lambda)}$$

$$= \frac{1}{nI(\lambda)}.$$