Math 128A HW2

Neo Lee

09/13/2023

Section 2.2

Problem 1c

Use algebraic manipulation to show that

$$g_3(x) = \left(\frac{x+3}{x^2+2}\right)^{1/2}$$

has a fixed point at p precisely when $f(p) = 0$, where $f(x) = x^4 + 2x^2 - x - 3$.

Proof.

$$\begin{aligned} f(p) = 0 &= p^4 + 2p^2 - p - 3 \\ p^2(p^2+2) &= p + 3 \\ p &= \left(\frac{p+3}{p^2+2}\right)^{1/2} = g(p). \end{aligned}$$

□

Problem 8

Use a fixed-point iteration method to determine a solution accurate to within 10^{-2} for $x^3 - x - 1 = 0$ on $[1, 2]$. Use $p_0 = 1$.

Solution. Define $g(x) := (x+1)^{1/3}$. Then $g(x) = x$ when $x^3 - x - 1 = 0$. Indeed, $g(x) \in [1, 2]$ for $x \in [1, 2]$ and $|g'(x)| < 1$ for $x \in [1, 2]$. By the fixed-point theorem, we are sure that the fixed-point iteration will converge to the solution.

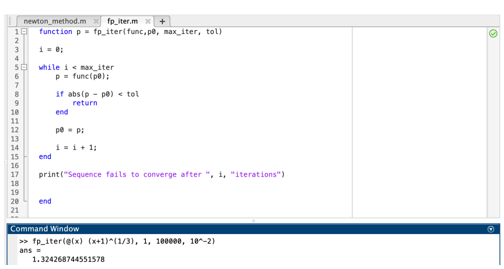

Figure 1: $x = 1.3243$

□

Problem 19

Let $g \in C^1[a,b]$ and p be in (a,b) with $g(p) = p$ and $|g'(p)| > 1$. Show that there exists a $\delta > 0$ such that if $0 < |p_0 - p| < \delta$, then $|p_0 - p| < |p_1 - p|$. Thus, no matter how close the initial approximation p_0 is to p, the next iteration p_1 is farther away, so the fixed-point iteration does not converge if $p_0 \neq p$.

Proof. $|g'(p)| > 1 \Rightarrow |g'(p)| = 1 + \epsilon$ for some $\epsilon > 0$. Since g' is continuous, there exists $\delta > 0$ such that for all $c \in (p - \delta, p + \delta)$, $|g'(c) - g'(p)| < \epsilon \Rightarrow |g'(c)| > 1$. Now consider such $p_0 \in (p - \delta, p + \delta) - \{p\}$,

$$
\begin{aligned}
|p_1 - p| &= |g(p_0) - p| \\
|p_1 - p| &= |g(p_0) - g(p)| \\
|p_1 - p| &= |g'(\xi)(p_0 - p)| \qquad (\textit{for } \xi \textit{ between } p_0, p) \\
\frac{|p_1 - p|}{|p_0 - p|} &= |g'(\xi)| \\
\frac{|p_1 - p|}{|p_0 - p|} &> 1. \qquad (\because \xi \in (p - \delta, p + \delta))
\end{aligned}
$$

□

Problem 20

Let A be a given positive constant and $g(x) = 2x - Ax^2$.

a. Show that if fixed-point iteration converges to a nonzero limit, then the limit is $p = 1/A$, so the inverse of a number can be found by using only multiplications and subtractions.

Solution. Let $g(p) = p = 2p - Ap^2$,

$$
\begin{aligned}
p &= 2p - Ap^2 \\
Ap^2 - p &= 0 \\
p(Ap - 1) &= 0.
\end{aligned}
$$

Hence, $p = 0$ or $Ap = 1 \Rightarrow \underline{p = 1/A}$. □

b. Find an interval about $1/A$ for which fixed-point iteration converges, provided p_0 is in that interval.

Solution. Define the interval $I = (\frac{1}{A} - |\frac{1}{2A}|, \frac{1}{A} + |\frac{1}{2A}|)$. Now consider $c \in I$, c can be written as $\frac{1}{A} + \epsilon$ such that $|\epsilon| < |\frac{1}{2A}|$.

$$
\begin{aligned}
|g'(c)| &= |2 - 2Ac| \\
&= |2 - 2A\left(\frac{1}{A} + \epsilon\right)| \\
&= |2A\epsilon| \\
&< |2A \cdot \frac{1}{2A}| \\
&< 1.
\end{aligned}
$$

Now consider $g(c) - g(\frac{1}{A})$,

$$\begin{aligned}
|g(c) - g(\frac{1}{A})| &= |g'(\xi)(c - \frac{1}{A})| \qquad (\xi \textit{ between } c, \frac{1}{A}) \\
|g(c) - \frac{1}{A}| &< 1 \cdot |c - \frac{1}{A}| \\
|g(c) - \frac{1}{A}| &< |\frac{1}{2A}|.
\end{aligned}$$

Hence, $g(c) \in I$.

We have verified that for all $c \in I$, $g'(c) < 1$ and $g(c) \in I$. Thus, by fixed point theorem, the fixed-point iteration with $p_0 \in I$ will converge. □

Section 2.3

Problem 6c

Use Newton's method to find solutions accurate to within 10^{-5} for

$$f(x) = 2xcos(2x) - (x-2)^2 = 0 \qquad \textit{for } 2 \le x \le 3 \textit{ and } 3 \le x \le 4.$$

Solution.

$$f'(x) = -4xsin(2x) + 2cos(2x) - 2(x-2).$$

Hence, for Newton's method,

$$g(x) = x - \frac{2xcos(2x) - (x-2)^2}{-4xsin(2x) + 2cos(2x) - 2(x-2)}.$$

With the fixed-point iteration function coded in Section 2.2 problem 8, we have □

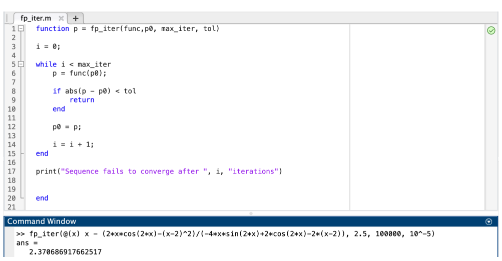

Figure 2: $p_0 = 2.5 \in [2, 3]$

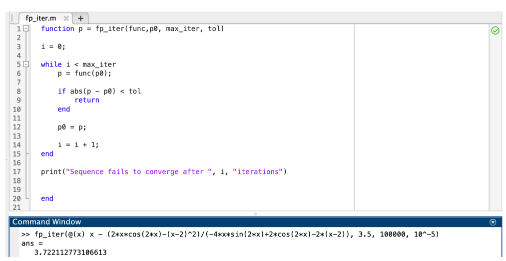

Figure 3: $p_0 = 3.5 \in [3, 4]$

Problem 8c

Repeat 6c using the Secant method.

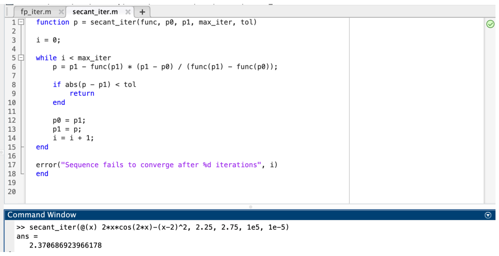

Figure 4: $p_0 = 2.25, p_1 = 2.75$

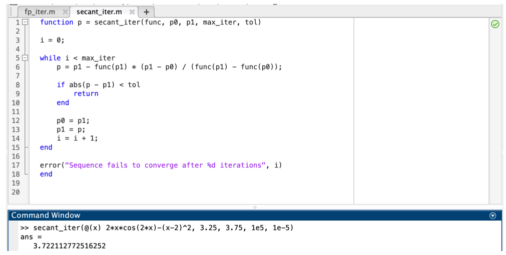

Figure 5: $p_0 = 3.25, p_1 = 3.75$

Solution. □

Problem 16

Use Newton's method to approximate the solution of $f(x) = x^2 - 10cos(x) = 0$ to within 10^{-5} with $\textbf{(a)} p_0 = -100$ and $\textbf{(d)} p_0 = 25$.

Solution.

$$f'(x) = 2x + 10sin(x)$$

$$g(x) = x - \frac{x^2 - 10cos(x)}{2x + 10sin(x)}.$$

□

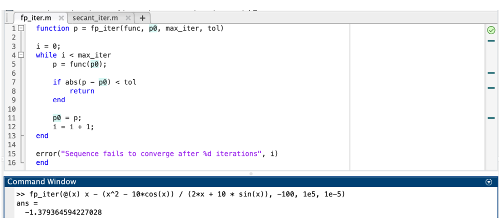

Figure 6: $\textbf{(a)} p_0 = -100$

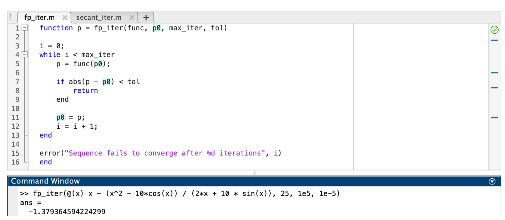

Figure 7: $\textbf{(d)} p_0 = 25$

Section 2.4

Problem 2c

Use Newton's method to find solutions accurate to within 10^{-5} to

$$f(x) = sin(3x) + 3e^{-2x}sin(x) - 3e^{-x}sin(2x) - e^{-3x} = 0 \qquad for\ 3 \le x \le 4.$$

Solution.

$$f'(x) = 3e^{-3x}\left(e^{3x}cos(3x) + e^{2x}sin(2x) - 2e^{2x}cos(2x) - 2e^{x}sin(x) + e^{x}cos(x) + 1\right)$$

$$g(x) = x - \frac{sin(3x) + 3e^{-2x}sin(x) - 3e^{-x}sin(2x) - e^{-3x}}{3e^{-3x}\left(e^{3x}cos(3x) + e^{2x}sin(2x) - 2e^{2x}cos(2x) - 2e^{x}sin(x) + e^{x}cos(x) + 1\right)}$$

□

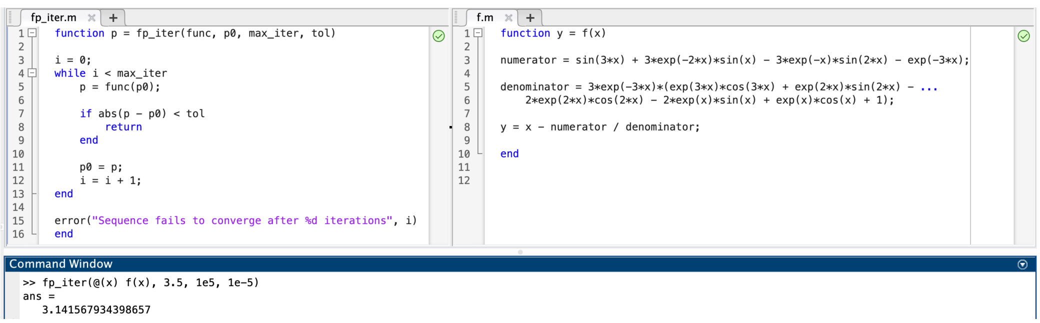

Figure 8: $x = 3.1415679$

Problem 8

a. Show that the sequence $p_n = 10^{-2^n}$ converges quadratically to 0.

Proof.

$$
\begin{aligned}
10^{-2^{n+1}} &= 10^{-2^n \cdot 2} \\
10^{-2^{n+1}} &= \left(10^{-2^n}\right)^2 \\
|10^{-2^{n+1}} - 0| &= |\left(10^{-2^n}\right) - 0|^2 \\
\frac{|10^{-2^{n+1}} - 0|}{|\left(10^{-2^n}\right) - 0|^2} &= 1 \\
\lim_{n\to\infty} \frac{|p_{n+1} - 0|}{|p_n - 0|^2} &= 1.
\end{aligned}
$$

□

b. Show that the sequence $p_n = 10^{-n^k}$ does not converge to 0 quadratically, regardless of the size of the exponent $k > 1$.

Proof. If the sequence converge quadratically, $\lim_{n\to\infty} \frac{|p_{n+1}|}{|p_n|^2} = C$ for some positive constant C. Consider

$$\begin{aligned}\lim_{n\to\infty} \frac{|10^{-(n+1)^k}|}{|10^{-n^k}|^2} &= \lim_{n\to\infty} \frac{10^{-(n+1)^k}}{10^{-2n^k}} \\ &= \lim_{n\to\infty} 10^{-(n+1)^k+2n^k}.\end{aligned}$$

Now we evaluate

$$\begin{aligned}\lim_{n\to\infty} -(n+1)^k + 2n^k &= \lim_{n\to\infty} \frac{-(n+1)^k + 2n^k}{n^k} \cdot n^k \\ &= \lim_{n\to\infty} \left(-\left(\frac{n+1}{n}\right)^k + 2\right) \cdot n^k \\ &= \infty.\end{aligned}$$

Hence, the limit

$$\lim_{n\to\infty} \frac{|10^{-(n+1)^k}|}{|10^{-n^k}|^2} = \infty \neq C,$$

and the sequence does not converge quadratically. ☐

Problem 9

a. Construct a sequence that converges to 0 of order 3.

Solution. Let $p_n = 10^{-3^n}$.

$$\begin{aligned}|10^{-3^{n+1}}| &= |10^{-3^n}|^3 \\ \lim_{n\to\infty} \frac{|10^{-3^{n+1}}|}{|10^{-3^n}|^3} &= 1.\end{aligned}$$

☐

b. Suppose $\alpha > 1$. Construct a sequence that converges to 0 of order α.

Solution. Let $p_n = \frac{1}{2^n} \cdot 10^{-\alpha^n}$.

$$\begin{aligned}|\frac{1}{2^{n+1}} \cdot 10^{-\alpha^{n+1}}| &= |\frac{1}{2^{n+1}}| \cdot |10^{-\alpha^n}|^\alpha \\ |\frac{1}{2^{n+1}} \cdot 10^{-\alpha^{n+1}}| &= \frac{1}{2} \cdot |\frac{1}{2^n}| \cdot |10^{-\alpha^n}|^\alpha \\ \lim_{n\to\infty} \frac{|(1/2^{n+1}) \cdot 10^{-\alpha^{n+1}}|}{|(1/2^n) \cdot 10^{-\alpha^n}|^\alpha} &= \frac{1}{2} \\ \lim_{n\to\infty} \frac{|p_{n+1} - 0|}{|p_n - 0|^\alpha} &= \frac{1}{2}.\end{aligned}$$

☐

Discussion question 4

What is the difference between the rate of convergence and the order of convergence? Have they any relationship to each other? Could two sequences have the same rates of convergence but different orders of convergence, or vice versa?

Solution. The order of convergence is defined as α for $\lim_{n\to\infty} \frac{|p_{n+1}-p|}{|p_n-p|^\alpha} = C$ for some finite constant C, while the rate of convergence is defined as $\lim_{n\to\infty} \frac{|p_{n+1}-p|}{|p_n-p|}$. Intuitively, the order of convergence can be thought of as a class of convergence, where the error is being raised to the order power, while the rate of convergence is the actual speed of convergence, which varies from sequence to sequence. They are both describing the speed of convergence, but the order of convergence is more general and significant. Two sequences cannot have same rates of convergence but different orders of convergence. However, two sequences can have same orders of convergence but different rates of convergence. □

Additional Problems

Question 1

The speed at which the sequence generated by an iterative method converges is called the methods ORDER of convergence. There are many types of orders of convergence: linear, superlinear, sublinear, quadratic, cubic, and so on. Discuss how a linearly convergent sequence could be accelerated.

Solution. It can be accelerate using Aitken's Δ^2 method. Let p_n be the sequence generated by an iterative method. Then the sequence q_n generated by Aitken's Δ^2 method is

$$\hat{p}_n = p_n - \frac{(p_{n+1}-p_n)^2}{p_{n+2}-2p_{n+1}+p_n}.$$

The convergence is faster in the sense that

$$\lim_{n\to\infty} \frac{|\hat{p}_n - p|}{|p_n - p|} = 0.$$

In particular for functional iteartion, Steffensen's method can be used to accelerate a linearly convergent sequence to quadratic convergence. □

Question 2

Show that the sequence $\{p_n\}$, for $p_n = 1/n^2$, converges sublinearly to $p = 0$, in that

$$\lim_{n\to\infty} \frac{|p_{n+1}-p|}{|p_n-p|^\alpha} = \infty \qquad (\textit{for } \alpha > 1)$$

and that there does not exist a $0 < \lambda < 1$ such that

$$\lim_{n\to\infty} \frac{|p_{n+1}-p|}{|p_n-p|} = \lambda.$$

How large must n be before $|p_n - p| \leq 5 \times 10^{-2}$?

Proof. To prove the first claim,

$$\begin{aligned}
\lim_{n\to\infty} \frac{|p_{n+1}-p|}{|p_n-p|^\alpha} &= \lim_{n\to\infty} \frac{|1/(n+1)^2|}{|1/n^2|^\alpha} \\
&= \lim_{n\to\infty} \frac{1/(n+1)^2}{1/n^{2\alpha}} \\
&= \lim_{n\to\infty} \frac{n^{2\alpha}}{(n+1)^2} \\
&= \lim_{n\to\infty} \frac{n^{2\alpha}}{n^2+2n+1} \\
&= \lim_{n\to\infty} \frac{n^{2\alpha-2}}{1+2/n+1/n^2} \\
&= \lim_{n\to\infty} \frac{n^{2\alpha-2}}{1+2/n+1/n^2} \qquad (2\alpha-2>0) \\
&= \infty.
\end{aligned}$$

To prove the second claim,

$$\begin{aligned}
\lim_{n\to\infty} \frac{|p_{n+1}-p|}{|p_n-p|} &= \lim_{n\to\infty} \frac{|1/(n+1)^2|}{|1/n^2|} \\
&= \lim_{n\to\infty} \frac{1/(n+1)^2}{1/n^2} \\
&= \lim_{n\to\infty} \frac{n^2}{(n+1)^2} \\
&= \lim_{n\to\infty} \frac{n^2}{n^2+2n+1} \\
&= \lim_{n\to\infty} \frac{1}{1+2/n+1/n^2} \\
&= \lim_{n\to\infty} \frac{1}{1+2/n+1/n^2} \\
&= 1.
\end{aligned}$$

For $|p_n - p| \leq 5 \times 10^{-2}$,

$$\begin{aligned}
|p_n-p| &\leq 5\times 10^{-2} \\
\frac{1}{n^2} &\leq 5\times 10^{-2} \\
n &\geq \sqrt{\frac{1}{5\times 10^{-2}}} \\
n &\geq 5.
\end{aligned}$$

$\square$