Math 109 HW3

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Problem 6.5

(i)

Proposition 1. $A \subseteq B \Leftrightarrow A \cup B = B$.

Proof. ($\Rightarrow; A \cup B \subseteq B$) $\forall x \in A \cup B, x \in B$ because $A \subseteq B$.

($\Rightarrow; B \subseteq A \cup B$) By definition, $\forall y \in B, y \in B \cup S$ for any arbitrary set S. Therefore, $B \subseteq A \cup B$.

Since $A \cup B \subseteq B$ and $B \subseteq A \cup B$, $A \cup B = B$, and ($\Rightarrow$) is proved.

($\Leftarrow$) By definition, $\forall z \in A, z \in A \cup S$ for any arbitrary set S, which means $A \subseteq A \cup S$. Hence, $A \subseteq A \cup B$, which is equivalent to $A \subseteq B$. $\square$

(ii)

Proposition 2. $A \subseteq B \Leftrightarrow A \cap B = A$.

Proof. ($\Rightarrow; A \cap B \subseteq A$) By definition, $\forall x \in A \cap B, x \in A$, thus $A \cap B \subseteq A$.

($\Rightarrow; A \subseteq A \cap B$) $\forall y \in A, x \in A \cap B$ because $A \subseteq B$.

Since $A \cap B \subseteq A$ and $A \subseteq A \cap B$, $A \cap B = A$, and ($\Rightarrow$) is proved.

($\Leftarrow$) By definition, $(B \cap S) \subseteq B$ for any arbitrary set S. Hence, $A = A \cap B \subseteq B$. $\square$

Problem 6.6

Proposition 3. *If $A \cap B \subseteq C$ and $x \in B$, then $x \notin A - C$.*

Proof. Assume to the contrary that if $A \cap B \subseteq C$ and $x \in B$, then $x \in A - C$. It means that $x \in A$ and $x \notin C$. Since $A \cap B \subseteq C$, $x \notin C \Rightarrow x \notin A \cap B$. We know $x \in A$ and $x \notin A \cap B$, therefore, $x \in A \cap B^c$. It means $x \in B^c \Rightarrow x \notin B$, which contradicts that $x \in B$. $\square$

Problem 6.7

Proposition 4. *For subsets of a universal set U, $A \subseteq B$ if and only if $B^c \subseteq A^c$.*

Proof. $A \subseteq B$ means that for an arbitrary x, if $x \in A$, then $x \in B$. Logically, it is equivalent to its contrapositive, which states for an arbitrary x, if $x \notin B$, then $x \notin A$. $x \notin B$ can be written as $x \in B^c$, and $x \notin A$ can be written as $x \in A^c$. Therefore, the entire statement can be rewritten as for an arbitrary x, if $x \in B^c$, then $x \in A^c$, which is the definition of $B^c \subseteq A^c$. Hence, $A \subseteq B \Leftrightarrow B^c \subseteq A^c$. $\square$

Problem 7.1

(i) $\mathbb{Z}^+$. Let $n = m$, $n, m \in \mathbb{Z}^+$ and $m \leq n$.

(ii) $\{1\}$. It is apparent that $\forall n \in \mathbb{Z}^+, 1 \leq n$. For $m \neq 1$, $n = 1$ is a counterexample to $\forall n \in \mathbb{Z}^+, m \leq n$.

(iii) $\mathbb{Z}^+$. Let $n = m$, $n, m \in \mathbb{Z}^+$ and $m \leq n$.

(iv) $\emptyset$. Let $m = n + 1$, $\forall m \in \mathbb{Z}^+, m \not\leq n$.

Problem 7.2

(i)

Proposition 5. *Disproving* $\forall m, n \in \mathbb{Z}^+, m \leq n$ *means proving* $\exists m, n \in \mathbb{Z}^+, m > n$.

Proof. Let $m = 3$ and $n = 2$, $m > n$. $\square$

(ii)

Proposition 6. $\exists m, n \in \mathbb{Z}^+, m \leq n$.

Proof. Let $m = 2$ and $n = 3$, $m \leq n$. $\square$

(iii)

Proposition 7. $\forall m \in \mathbb{Z}^+, \exists n \in \mathbb{Z}^+, m \leq n$.

Proof. Let $n = m$. $\forall m \in \mathbb{Z}^+, m = n \Rightarrow m \leq n$. $\square$

(iv)

Proposition 8. $\exists m \in \mathbb{Z}^+, \forall n \in \mathbb{Z}^+, m \leq n$.

Proof. Let $m = 1$. $\forall n \in \mathbb{Z}^+, m \leq n$. $\square$

(v)

Proposition 9. $\forall n \in \mathbb{Z}^+, \exists m \in \mathbb{Z}^+, m \leq n$.

Proof. Let $m = 1$. $\forall n \in \mathbb{Z}^+, m \leq n$. $\square$

(vi)

Proposition 10. *Disproving* $\exists n \in \mathbb{Z}^+, \forall m \in \mathbb{Z}^+, m \leq n$ *means proving* $\forall n \in \mathbb{Z}^+, \exists m \in \mathbb{Z}^+, m > n$.

Proof. Let $m = n + 1$. $\forall n \in \mathbb{Z}^+, m > n$. $\square$

Problem 7.4

(i)

Proposition 11. $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, x + y = 0$.

Proof. Let $y = -x$. $\forall x \in \mathbb{R}, x + y = x - x = 0$. $\square$

(ii)

Proposition 12. *Disproving* $\exists y \in \mathbb{R}, \forall x \in \mathbb{R}, x + y = 0$ *mean proving* $\forall y \in \mathbb{R}, \exists x \in \mathbb{R}, x + y \neq 0$.

Proof. Let $x = -y + 1$. $\forall y \in \mathbb{R}, y + x = y - y + 1 = 1 \neq 0$. $\square$

(iii)

Proposition 13. $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, xy = 0$.

Proof. Let $y = 0$. $\forall x \in \mathbb{R}, xy = x \cdot 0 = 0$. $\square$

(iv)

Proposition 14. $\exists y \in \mathbb{R}, \forall x \in \mathbb{R}, xy = 0.$

Proof. Let $y = 0$. $\forall x \in \mathbb{R}, xy = x \cdot 0 = 0$. ☐

(v)

Proposition 15. *Disproving* $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, xy = 1$ *means proving* $\exists x \in \mathbb{R}, \forall y \in \mathbb{R}, xy \neq 1$.

Proof. Let $x = 0$. $\forall y \in \mathbb{R}, xy = 0 \cdot y = 0 \neq 1$. ☐

(vi)

Proposition 16. *Disproving* $\exists y \in \mathbb{R}, \forall x \in \mathbb{R}, xy = 1$ *means proving* $\forall y \in \mathbb{R}, \exists x \in \mathbb{R}, xy \neq 1$.

Proof. Let $x = 0$. $\forall y \in \mathbb{R}, xy = 0 \cdot y = 0 \neq 1$. ☐

(vii)

Proposition 17. $\forall n \in \mathbb{Z}^+$, *($n$ is even or n is odd).*

Proof. $\forall n \in \mathbb{Z}^+$, n is either even or n is not even. By definition, if n is not even, then n is odd, which logically means n ie even or n is odd. ☐

(viii)

Proposition 18. *Disproving* ($\forall n \in \mathbb{Z}^+, n$ *is even*) *or* ($\forall n \in \mathbb{Z}^+, n$ *is odd*) *means proving* ($\exists n \in \mathbb{Z}^+, n$ *is odd*) *and* ($\exists n \in \mathbb{Z}^+, n$ *is even*).

Proof. For the first half of the statement, let $n = 1$, then n is odd, which proves ($\forall n \in \mathbb{Z}^+, n$ is odd). For the second half of the statement, let $n = 2$, then n is even, which proves ($\exists n \in \mathbb{Z}^+, n$ is even). ☐

Problem 7.7

Proposition 19. *For sets* A, B, C, D, $(A \times B) \cup (C \times D) \subseteq (A \cup C) \times (B \cup D)$.

Proof. Let $(x, y) \in (A \times B) \cup (C \times D)$. It means $(x, y) \in (A \times B)$ or $(x, y) \in (C \times D)$. If $(x, y) \in (A \times B)$, then indeed $x \in (A \cup C)$ and $y \in (B \cup D)$. If $(x, y) \in (C \times D)$, then again $x \in (A \cup C)$ and $y \in (B \cup D)$. Hence, $\forall (x, y) \in (A \times B) \cup (C \times D), (x, y) \in (A \cup C) \times (B \cup D)$.

For the counterexample, let $A = \{1\}, B = \{2\}, C = \{3\}, D = \{4\}$. $(A \times B) \cup (C \times D) = \{(1, 2), (3, 4)\}$ while $(A \cup C) \times (B \cup D) = \{(1, 2), (1, 4), (3, 2), (3, 4)\}$. ☐

Page 115 Problem 4

Proposition 20. $A \cap B = A \cap C$ *and* $A \cup B = A \cup C$ *if and only if* $B = C$.

Proof. ($\Rightarrow$)

$$\begin{aligned}
B &= B \cap (A \cup B) \quad (\because B \subseteq A \cup B) && (1)\\
&= B \cap (A \cup C) \quad (\because A \cup B = A \cup C) && (2)\\
&= (B \cap A) \cup (B \cap C) && (3)\\
&= (A \cap B) \cup (B \cap C) && (4)\\
&= (A \cap C) \cup (B \cap C) \quad (\because A \cap B = A \cap C) && (5)\\
&= (A \cup B) \cap C && (6)\\
&= (A \cup C) \cap C \quad (\because A \cup B = A \cup C) && (7)\\
&= C \quad (\because C \subseteq A \cup C) && (8)
\end{aligned}$$

Hence, $B = C$.

($\Leftarrow$) This is apparent because we only need to substitute B with C, then we will get $A \cap B = A \cap C$ and $A \cup B = A \cup C$. ☐

Page 117 Problem 13

(i)

Proposition 21. $A \times (B \cup C) = (A \times B) \cup (A \times C)$.

Proof.

$$
\begin{align}
(x, y) \in A \times (B \cup C) &\Leftrightarrow x \in A \text{ and } y \in (B \cup C) \tag{9}\\
&\Leftrightarrow x \in A \text{ and } (y \in B \text{ or } y \in C) \tag{10}\\
&\Leftrightarrow (x \in A \text{ and } y \in B) \text{ or } (x \in A \text{ and } y \in C) \tag{11}\\
&\Leftrightarrow (x, y) \in (A \times B) \cup (A \times C). \tag{12}
\end{align}
$$

□

(ii)

Proposition 22. $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$.

Proof.

$$
\begin{align}
(x, y) \in (A \times B) \cap (C \times D) &\Leftrightarrow (x, y) \in (A \times B) \text{ and } (x, y) \in (C \times D) \tag{13}\\
&\Leftrightarrow x \in A \text{ and } y \in B \text{ and } x \in C \text{ and } y \in D \tag{14}\\
&\Leftrightarrow (x \in A \text{ and } x \in C) \text{ and } (y \in B \text{ and } y \in D) \tag{15}\\
&\Leftrightarrow (x, y) \in (A \cap C) \times (B \cap D) \tag{16}
\end{align}
$$

□