Math 110 HW2

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Problem 1.

Suppose $U := \{(x, x, 3x) : x \in \mathbb{R}\}$ and $W := \{(x, -x, -3x) : x \in \mathbb{R}\}$.

(a)

Proposition 1. *U and W are subspaces of $\mathbb{R}^3$.*

Proof. **Additive Identity:** Let $x = 0$ for both $u \in U$ and $w \in W$. Then $u = (0, 0, 0)$ and $w = (0, 0, 0)$, which are both in U and W and are additive identities.

Closed under addition: Let $u_1 = (x_1, x_1, 3x_1), u_2 = (x_2, x_2, 3x_2) \in U$. Then $u_1+u_2 = (x_1+x_2, x_1+x_2, 3x_1 + 3x_2) = ([x_1 + x_2], [x_1 + x_2], 3[x_1 + x_2])$, which is also in U.

Let $w_1 = (y_1, -y_1, -3y_1), w_2 = (y_2, -y_2, -3y_2) \in W$. Then $w_1+w_2 = (y_1+y_2, -y_1-y_2, -3y_1-3y_2) = ([y_1 + y_2], -[y_1 + y_2], -3[y_1 + y_2])$, which is also in W.

Closed under scalar multiplication: Let $u = (x, x, 3x) \in U$ and $c \in \mathbb{R}$. Then $cu = (cx, cx, 3cx) = ([cx], [cx], 3[cx])$, which is also in U.

Let $w = (y, -y, -3y) \in W$ and $c' \in \mathbb{R}$. Then $c'w = (c'y, -c'y, -3c'y) = ([c'y], -[c'y], -3[c'y])$, which is also in W. $\square$

(b) Describe $U + W$ using symbols.

Solution. For all $v \in (U + W)$, $v = u + w$ for $u \in U, w \in W$. $\square$

(c) Describe $U + W$ without symbols.

Solution. For every element in the set $U + W$, it is the sum of an element in U and an element in W. $\square$

Problem 2.

Suppose $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$ and let

$$U = \{(x, y, x + y, -y, -x) \in \mathbb{F}^5 : x, y \in \mathbb{F}\}.$$

Find three subspaces W_1, W_2, W_3 of $\mathbb{F}^5$, none of which equals $\{0\}$, such that $\mathbb{F}^5 = U \oplus W_1 \oplus W_2 \oplus W_3$.

Solution. Notice for all $u \in U$, $u = x(1,0,1,0,-1)+y(0,1,1,-1,0)$, which means U is a span of $(1,0,1,0,-1)$ and $(0,1,1,-1,0)$. Now we just need to define W_1, W_2, W_3 as the span of any three linearly independent vectors in $\mathbb{F}^5$ that are not in U. Let $W_1 = \mathit{span}(0,0,1,0,0), W_2 = \mathit{span}(0,0,0,1,0), W_3 = \mathit{span}(0,0,0,0,1)$. Then W_1, W_2, W_3 are subspaces of $\mathbb{F}^5$ and all the five vectors are linearly independent. We can check it by row reducing the matrix with these five vectors as column vectors, but we will omit here since it is trivial.

Now we have defined our subspaces U, W_1, W_2, W_3, we will show that $\mathbb{F}^5 = U \oplus W_1 \oplus W_2 \oplus W_3$. Let $v \in \mathbb{F}^5$, then we can write v as a linear combination of $u \in U, w_1 \in W_1, w_2 \in W_2, w_3 \in W_3$, which in turns is a linear combination of the five vectors we defined above. Since the five vectors are linearly independent, the linear combination is unique. Therefore, $\mathbb{F}^5 = U \oplus W_1 \oplus W_2 \oplus W_3$. □

Problem 3.

Proposition 2. *Let V be a vector space over $\mathbb{F}$. Suppose that $1+1 \neq 0$ in $\mathbb{F}$ and the list v_1, v_2, v_3, v_4 is linearly independent in V. Then the list $v_1 - v_2, v_1 + v_2, v_3 - v_2, v_4 - v_1$ is also linearly independent in $\mathbb{V}$.*

Proof. We proceed by showing that the zero vector can only be written as a trivial linear combination of $v_1 - v_2, v_1 + v_2, v_3 - v_2, v_4 - v_1$. Suppose there exists $a_1, a_2, a_3, a_4 \in \mathbb{F}$ such that $a_1(v_1 - v_2) + a_2(v_1 + v_2) + a_3(v_3 - v_2) + a_4(v_4 - v_1) = 0$. Then

$$
\begin{aligned}
a_1(v_1 - v_2) + a_2(v_1 + v_2) + a_3(v_3 - v_2) + a_4(v_4 - v_1) = 0 \\
(a_1 + a_2 - a_4)v_1 + (-a_1 + a_2 - a_3)v_2 + a_3 v_3 + a_4 v_4 = 0.
\end{aligned}
$$

Now since v_1, v_2, v_3, v_4 are linearly independent, $(a_1 + a_2 - a_4), (-a_1 + a_2 - a_3), a_3, a_4$ must all be zero. Now we solve the equations of

$$
\begin{aligned}
a_1 + a_2 - a_4 &= 0 \\
-a_1 + a_2 - a_3 &= 0 \\
a_3 &= 0 \\
a_4 &= 0.
\end{aligned}
$$

Solving the linear equations, which we omit here, we get $a_1 = a_2 = a_3 = a_4 = 0$. Therefore, the list $v_1 - v_2, v_1 + v_2, v_3 - v_2, v_4 - v_1$ is linearly independent. □

Problem 4.

Does the statement of Problem 3 still hold if we replace "linearly independent" by "a basis"?

Solution. We know $dimV = 4$ because the length of the basis v_1, v_2, v_3, v_4 is 4. Now, from the previous question, we know that $v_1 - v_2, v_1 + v_2, v_3 - v_2, v_4 - v_1$ is linearly independent in V. Since the length of the list is also 4, it must be a basis of V. □

Problem 5.

Proposition 3. *The space $\mathbb{R}^{[0,1]}$ is infinite-dimensional.*

Proof. Notice that a vector space is finite-dimensional if all its subspaces are finite-dimensional. Now we consider the subspace $\mathcal{P}(x)$, the set of polynomial functions that map from $[0,1]$ to $\mathbb{R}$. Assume for the sake of contradiction that $\mathcal{P}(x)$ is finite and let $dim\mathcal{P} = n$. Then we have $x, x^2, \ldots, x^n$, which are all linearly independent and with a total length of n, which form the basis of $\mathcal{P}$. Now consider the function $x \mapsto x^{n+1}$, which is still in the subspace $\mathcal{P}$ but not in the span of $x, \ldots, x^n$. Hence, by contradiction, $x, \ldots, x^n$ does not span the entire subspace $\mathcal{P}$, and $\mathcal{P}$ is infinite dimensional. Therefore, the vector space $\mathbb{R}^{[0,1]}$ is infinite-dimensional as well. $\square$