Math 104 HW3

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Exercise 7.4

Give examples of

(a) A sequence (x_n) of irrational numbers having a limit $\lim x_n$ that is rational.

Solution. Consider $(x_n) = \frac{1}{n} \cdot \sqrt{2}$. Clearly, $\lim x_n = 0$ and x_n is irrational for all n. $\square$

(b) A sequence (r_n) of rational numbers having a limit $\lim x_n$ that is irrational.

Solution. A simple one would be $(r_n) = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}$. Certianly, $\lim r_n = e$ and e is irrational, while r_n is rational for all n. $\square$

Exercise 7.5

Determine the following limits. No proofs are required, but show any relevant algebra.

(a) $\lim s_n$ where $s_n = \sqrt{n^2+1} - n$. Hint: first show $s_n = \frac{1}{\sqrt{n^2+1}+n}$.

Solution.

$$\begin{aligned} s_n &= \sqrt{n^2+1} - n \\ &= \left(\sqrt{n^2+1} - n\right) \times \frac{\sqrt{n^2+1}+n}{\sqrt{n^2+1}+n} \\ &= \frac{1}{\sqrt{n^2+1}+n} \\ \lim s_n &= 0. \end{aligned}$$

$\square$

(b) $\lim(\sqrt{n^2+n} - n)$.

Solution.

$$\begin{aligned} \sqrt{n^2+n} - n &= \frac{n}{\sqrt{n^2+n}+n} \\ &= \frac{1}{\sqrt{1+\frac{1}{n}}+1} \\ \lim\left(\sqrt{n^2+n} - n\right) &= \frac{1}{2}. \end{aligned}$$

$\square$

(c) $\lim(\sqrt{4n^2+n}-2n)$.

Solution.

$$\begin{aligned}\sqrt{4n^2+n}-2n &= \frac{n}{\sqrt{4n^2+n}+2n} \\ &= \frac{1}{\sqrt{4+\frac{1}{n}}+2} \\ \lim\left(\sqrt{4n^2+n}-2n\right) &= \frac{1}{4}.\end{aligned}$$

$\square$

Exercise 8.5

(a)

Proposition 1. *Consider three sequences (a_n), (s_n), and (c_n) such that $a_n \leq s_n \leq c_n$ for all n and $\lim a_n = \lim c_n = s$. Then, $\lim s_n = s$. This is called the squeeze lemma.*

Proof. For an arbitrary $\epsilon > 0$, we know $\exists N_c$ such that for $n > N_c$,

$$|c_n - s| < \epsilon \Rightarrow c_n < s + \epsilon$$

and $\exists N_a$ such that for $m > N_a$,

$$|a_m - s| < \epsilon \Rightarrow a_m > s - \epsilon.$$

Now take for $N = \max\{N_c, N_a\}$, we have for $k > N$,

$$s_k < c_k < s + \epsilon.$$

At the same time,

$$s - \epsilon < a_k < s_k.$$

Hence,

$$s - \epsilon < s_k < s + \epsilon$$

and $|s_k - s| < \epsilon$. $\square$

(b)

Proposition 2. *Suppose (s_n) and (t_n) are sequences such that $|s_n| \leq t_n$ for all n and $\lim t_n = 0$. Then $\lim s_n = 0$.*

Proof. Notice $-t_n \leq s_n \leq t_n$. If we can show that $\lim(-t_n) = 0$, then by the squeeze lemma, $\lim s_n = 0$. We know $\exists N$ such that for $\epsilon > 0$, take $n > N$,

$$|t_n - 0| < \epsilon \Rightarrow |-t_n - 0| = |t_n| = |t_n - 0| < \epsilon.$$

Hence, $\lim(-t_n) = 0$. $\square$

Exercise 8.6

Let (s_n) be a sequence in $\mathbb{R}$.

(a)

Proposition 3. $\lim s_n = 0$ *if and only if* $\lim |s_n| = 0$.

Proof. For any $\epsilon > 0$, we know $\exists N$ such that for $n > N$,

$$\begin{aligned} |s_n - 0| < \epsilon &\Leftrightarrow |s_n| < \epsilon \\ &\Leftrightarrow |(|s_n|)| < \epsilon \\ &\Leftrightarrow |(|s_n|) - 0| < \epsilon. \end{aligned}$$

□

Read the equivalence form in both forward and backward directions, we can see that $\lim s_n = 0$ if and only if $\lim |s_n| = 0$.

(b) Observe that if $s_n = (-1)^n$, then $\lim |s_n|$ exists, but $\lim s_n$ does not exist.

Solution. The first claim is trivial, since $|s_n| = 1$ for all n, so $\lim |s_n| = 1$.

Now assume for contradiction that $\lim s_n = s \in \mathbb{R}$ exists. Then, $\exists N$ such that for $n > N$ implies for any $\epsilon > 0$,

$$|(-1)^n - s| < \epsilon.$$

Consider $\epsilon = 1$, then $|(-1)^{N+1} - s| < 1$ and $|(-1)^{N+2} - s| < 1$. This means $|-1-s| < 1$ and $|1-s| < 1 \Rightarrow s \in (-2, 0)$ and $s \in (0, 2)$. This is a contradiction.

Or another way to arrive at contradiction is using the triangle inequality such that

$$\begin{aligned} 2 > |1-s| + |-1-s| &\geq |1-s-(-1-s)| \\ 2 > |1-s| + |-1-s| &\geq 2 \\ 2 &> 2. \end{aligned}$$

□

Exercise 8.9

Let (s_n) be a sequence that converges.

(a)

Proposition 4. *If* $s_n \geq a$ *for all but finitely many* n, *then* $\lim s_n \geq a$.

Proof. Assume for contradiction that $\lim s_n = s < a$, which can be written as $a = s + 2\epsilon$ for some $\epsilon > 0$. Now take $N = max\{n : s_n < a\}$, for all $n > N$,

$$s_n \geq a = s + 2\epsilon \Rightarrow |s_n - s| > \epsilon.$$

This contradicts the definition of limit $\lim s_n = s$. Hence, $\lim s_n \geq a$. □

(b)

Proposition 5. *If $s_n \leq b$ for all but finitely many n, then $\lim s_n \leq b$.*

Proof. Similary, assume for contradiction that $\lim s_n = s > b$, which can be written as $b = s - 2\epsilon$ for some $\epsilon > 0$. Now take $N = max\{n : s_n > b\}$, for all $n > N$,

$$s_n \leq b = s - 2\epsilon \Rightarrow |s_n - s| > \epsilon.$$

This contradicts the definition of limit $\lim s_n = s$. Hence, $\lim s_n \leq b$. □

(c)

Proposition 6. *If all but finitely many s_n belong to $[a, b]$, then $\lim s_n$ belongs to $[a, b]$.*

Proof. It means for all but finitely many n, $s_n \leq b$. Also, for all but finitely many m, and $s_m \geq a$. Following from (a) and (b), then $\lim s_n \geq a$ and $\lim s_n \leq b$. Hence $\lim s_n \in [a, b]$. □

Exercise 9.1a

Proposition 7. $\lim \frac{n+1}{n} = 1$.

Proof.

$$\begin{aligned}
\lim \frac{n+1}{n} &= \lim \frac{1 + 1/n}{1} \\
&= \lim(1 + 1/n) \cdot \lim 1 \qquad (\because \lim s_n/t_n = \lim s_n \cdot \lim 1/t_n) \\
&= (\lim 1 + \lim 1/n) \cdot \lim 1 \qquad (\because \lim s_n + t_n = \lim s_n + \lim t_n) \\
&= 1. \qquad (\because \lim 1 = 1, \lim 1/n = 0)
\end{aligned}$$

□

Exercise 9.4

Let $s_1 = 1$ and for $n \geq 1$ let $s_{n+1} = \sqrt{s_n + 1}$.

(a) List the first four terms of (s_n).

Solution.

1. 1
2. $\sqrt{2}$
3. $\sqrt{\sqrt{2}+1}$
4. $\sqrt{\sqrt{\sqrt{2}+1}+1}$

□

(b)

Proposition 8. *Assume* (s_n) *converges, then* $\lim(s_n) = \frac{1}{2}\left(1+\sqrt{5}\right)$.

Proof. Notice $\lim_{n\to\infty} s_{n+1} = s = \lim_{n\to\infty} s_n$. Hence,

$$\begin{aligned}
\lim_{n\to\infty} s_{n+1} &= \lim_{n\to\infty} \sqrt{s_n+1} \\
\left(\lim_{n\to\infty} s_{n+1}\right)^2 &= \lim_{n\to\infty} \sqrt{s_n+1} \cdot \lim_{n\to\infty} \sqrt{s_n+1} \\
\left(\lim_{n\to\infty} s_{n+1}\right)^2 &= \lim_{n\to\infty} (s_n+1) \\
s^2 &= s+1 \\
s^2 - s - 1 &= 0.
\end{aligned}$$

Solving the quadratic equation, we get $s = \frac{1}{2}\left(1 \pm \sqrt{5}\right)$. Notice $s_n > 0$ for all n, so $\lim s_n \geq 0$ [check *proposition 4*]. Thus, $\lim(s_n) = \frac{1}{2}\left(1+\sqrt{5}\right)$.

Attempt to prove (s_n) *converges:*

We first show that s_n is monotonic increasing in the interval $I = \left(\frac{(1-\sqrt{5})}{2}, \frac{(1+\sqrt{5})}{2}\right)$. Indeed, for $s_n \in I$,

$$\begin{aligned}
s_n^2 - s_n - 1 &< 0 \\
s_n^2 &< s_n + 1 \\
s_n &< \sqrt{s_n+1} \\
s_n &< s_{n+1}.
\end{aligned}$$

Then, we show that (s_n) is bounded by $\frac{(1+\sqrt{5})}{2}$. We proceed with induction to show that $s_n < \frac{(1+\sqrt{5})}{2}$ for all $n \in \mathbb{N}$. The base case $s_1 = 1$ is trivial. Now assume $s_k < \frac{(1+\sqrt{5})}{2}$ for some $k \in \mathbb{N}$. To show $s_{k+1} < \frac{(1+\sqrt{5})}{2}$, we need

$$\begin{aligned}
\sqrt{s_k+1} &< \frac{\left(1+\sqrt{5}\right)}{2} \\
s_k + 1 &< \frac{6+2\sqrt{5}}{4} \\
s_k &< \frac{6+2\sqrt{5}}{4} - 1 \\
s_k &< \frac{\left(1+\sqrt{5}\right)}{2},
\end{aligned}$$

which is indeed true by our inductive hypothesis.

Hence, by mathematical induction, $s_n = |s_n| < \frac{(1+\sqrt{5})}{2}$ for all $n \in \mathbb{N}$. Now, since s_n is a bounded monotone, it converges. $\square$