Math 110 HW7

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Problem 1.

Suppose V is finite-dimensional and U, W are its subpaces. Prove that

$$(U \cap W)^0 = U^0 + W^0.$$

Proof. First we need the *Lemma:* $U^0 \cap W^0 = (U+W)^0$. Consider $\phi \in U^0 \cap W^0$ and $v = u + w \in U + W$, then

$$\phi(v) = \phi(u+w) = \phi(u) + \phi(w) = 0 + 0 = 0.$$

Hence $\phi \in (U+W)^0$ and $U^0 \cap W^0 \subseteq (U+W)^0$. Now consider $\varphi \in (U+W)^0$, indeed for $u \in U$,

$$\varphi(u) = \varphi(u+0) = 0,$$

and for $w \in W$,

$$\varphi(w) = \varphi(0+w) = 0$$

[notice $u+0$ and $0+w$ are both in $U+W$]. Therefore, $\varphi \in U^0 \cap W^0$ and $(U+W)^0 \subseteq U^0 \cap W^0$. The *Lemma* is proved.

Now we back to the original problem. Consider $\phi \in U^0 + W^0$ and $v \in U \cap W$, then

$$\phi(v) = \varphi_u(v) + \varphi_w(v) = 0 + 0 = 0 \qquad (\varphi_u \in U^0, \varphi_w \in W^0).$$

Hence, $\phi \in (U+W)^0$ and $U^0 + W^0 \subseteq (U+W)^0$.

Now consider the dimension of $(U \cap W)^0$ and $U^0 + W^0$.

$$\begin{aligned}
\dim(U \cap W)^0 &= \dim V - \dim(U \cap W) \\
&= \dim V - \dim U - \dim W + \dim(U+W) \\
&= \dim V - (\dim V - \dim U^0) - (\dim V - \dim W^0) + (\dim V - \dim(U+W)^0) \\
&= \dim U^0 + \dim W^0 - \dim(U+W)^0 \\
&= \dim U^0 + \dim W^0 - \dim(U^0 \cap W^0) \\
&= \dim(U^0 + W^0).
\end{aligned}$$

Since $U^0 + W^0 \subseteq (U+W)^0$ and $\dim(U^0 + W^0) = \dim(U \cap W)^0$, we have $(U \cap W)^0 = U^0 + W^0$. $\square$

Problem 2.

Suppose V and W are finite-dimensional, $T \in \mathcal{L}(V, W)$, and $\operatorname{null} T' = \operatorname{span}(\varphi)$ for some $\varphi \in W'$. Prove that $\operatorname{range} T = \operatorname{null} \varphi$. Give an example of such a pair $T \neq 0$, $\varphi \neq 0$ for $V = \mathbb{R}^2$, $W = \mathbb{R}^3$.

Proof. Notice $\operatorname{null} T' = (\operatorname{range} T)^0 = \operatorname{span}(\varphi)$. Consider $w \in \operatorname{range} T$ and arbitrary $\lambda \in \mathbb{F}$, then

$$\lambda\varphi(w) = 0 \quad (\lambda\varphi \in (\operatorname{range} T)^0) \implies \varphi(w) = 0.$$

Hence, $\operatorname{range} T \subseteq \operatorname{null} \varphi$.

Then we show that $\dim \operatorname{range} T = \dim \operatorname{null} \varphi$. For $\varphi \neq 0$,

$$\begin{aligned} \dim \operatorname{range} T &= \dim V - \dim(\operatorname{range} T)^0 \\ &= \dim V - \dim \operatorname{span}(\varphi) \\ &= \dim V - 1 \\ &= \dim V - \dim \operatorname{range} \varphi \qquad (1) \\ &= \dim \operatorname{null} \varphi. \end{aligned}$$

(1) is true because $\varphi \neq 0$, so $\operatorname{range} \varphi = \mathbb{R}$. If $\varphi = 0$,

$$\begin{aligned} \dim \operatorname{range} T &= \dim V - \dim(\operatorname{range} T)^0 \\ &= \dim V - \dim \operatorname{span}(\varphi) \\ &= \dim V - 0 \\ &= \dim \operatorname{null} \varphi. \end{aligned}$$

Since $\operatorname{range} T \subseteq \operatorname{null} \varphi$ and $\dim \operatorname{range} T = \dim \operatorname{null} \varphi$, we have $\operatorname{range} T = \operatorname{null} \varphi$.

Example: Let $T : (x, y) \mapsto (x, y, 0)$ and $\varphi : (x, y, z) \mapsto z$. Then indeed, $\operatorname{null} T' = \operatorname{span}(\varphi)$ [very obviously because φ annihilates $\operatorname{range} T$], and

$$\operatorname{range} T = (x, y, 0) = \operatorname{null} \varphi.$$

□

Problem 3.

Let $p \in \mathcal{P}_n(\mathbb{C})$ for some n and suppose there exist distinct real numbers $x_0, x_1, \ldots, x_n$ such that $p(x_j) \in \mathbb{R}$ for all $j = 0, \ldots, n$. Prove that all coefficients of p are real.

Proof. **Lemma:** *given distinct data sites x_j and arbitrary data y_j, $j = 0, \ldots, n$, there is a unique polynomial $p \in \mathcal{P}_n(\mathbb{R})$ such that $p(x_j) = y_j$, for all $j = 0, \ldots, n$.*

We have shown in **Problem 4** that such unique polynomial takes the form

$$p(x) = \sum_{k=1}^{n} y_k L_k(x) \quad \text{for} \quad L_k(x) = \prod_{j=0, j \neq k}^{n} \frac{x - x_j}{x_k - x_j}.$$

Then notice all the terms in this unique polynomial p are real and expanding the product $\prod_{j=0,j\neq k}^{n} \frac{x-x_j}{x_k-x_j}$ gives us a polynomial with real coefficients [no imaginary parts can arise]. $\square$

Problem 4.

[Lagrange interpolation.] Prove *using linear algebra*: given distinct *data sites* x_j and arbitrary *data* y_j, $j = 0, \ldots, n$, there is a unique polynomial $p \in \mathcal{P}_n(\mathbb{R})$ such that $p(x_j) = y_j$, for all $j = 0, \ldots, n$.

Proof. Define the Lagrange basis polynomials

$$L_k(x) = \prod_{j=0, j\neq k}^{n} \frac{x - x_j}{x_k - x_j}.$$

Notice each L_k are of degree n and hence each $L_k \in \mathcal{P}_n(\mathbb{R})$. Then notice these basis polynomials have the property that $L_k(x_k) = 1$ and $L_k(x_j) = 0$ for $j \neq k$.

Then we can define

$$p(x) = \sum_{k=0}^{n} y_k L_k(x).$$

Indeed, $p(x_j) = y_j$ because $L_j(x_j) = 1$ and $y_j L_j(x_j) = y_j$ while $L_k(x_j) = 0$ for $k \neq j$. To show that it's unique, assume to the contrary that there exists another polynomial $q \in \mathcal{P}_n(\mathbb{R})$ such that $q(x_j) = y_j$ for all $j = 0, \ldots, n$. Then

$$p(x) - q(x) = \sum_{k=0}^{n} y_k L_k(x) - q(x) = 0.$$

Notice $p(x) - q(x)$ is a polynomial of degree n and has $n + 1$ roots, which means $p - q$ must be the zero polynomial because non-zero polynomial of degree n has at most n roots [Theorem 4.11].

Therefore, $p - q = 0 \implies p = q$. $\square$

Problem 5.

Prove that every polynomial of odd degree with real coefficients has a real zero.

Proof. Assume for the sake of contradiction that there exists a polynomial of odd degree with real coefficients that has no real zero. Then we take the minimal example $p(x)$ with the lowest degree. Notice p can be factorized as follow

$$p(x) = (x - \lambda)(x - \bar{\lambda})q(x),$$

where λ is a complex zero of p and $q(x)$ is a polynomial of degree $n - 2$. Since we assumed p to be the minimal example, $q(x)$ must have a real zero x_0. Then we have

$$q(x) = (x - x_0)g(x) \implies p(x) = (x - \lambda)(x - \bar{\lambda})(x - x_0)g(x),$$

which means p has a real zero x_0 [contradiction]. $\square$