Probability Distribution Properties

1. Discrete Distributions

1.1 Uniform random variable

$$E[X] = \frac{x_1 + \cdots + x_n}{n}; \quad Var(X) = \frac{x_1^2 + \cdots + x_n^2}{n} - \left(\frac{x_1 + \cdots + x_n}{n}\right)^2.$$

$$E[X] = \frac{n+1}{2}; \quad Var(X) = \frac{n^2 - 1}{12}; \quad \textit{only when } x_i = [1, 2, \ldots, n].$$

1.2 Bernoulli random variable

$$I = \begin{cases} 1 & \text{if } X = 1, \\ 0 & \text{if } X = 0. \end{cases}$$

$$E[I] = p; \quad Var(I) = p(1-p).$$

1.3 Binomial random variable

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k} \quad \textit{for } k \in [0, n].$$

$$E[X] = np; \quad Var(X) = np(1-p).$$

1.4 Poisson random variable

$$P(X = k) = \frac{\lambda^k e^{-\lambda}}{k!} \quad \textit{for } k \in \mathbb{Z}^{\geq}.$$

$$E[X] = \lambda; \quad Var(X) = \lambda.$$

1.5 Geometric random variable

$$P(X = n) = (1-p)^{n-1} p \quad \textit{for } n \in \mathbb{N}.$$

$$E[X] = \frac{1}{p}; \quad Var(X) = \frac{1-p}{p^2}; \quad P(X > n) = (1-p)^n; \quad P(X > n + k | X > k) = P(X > n).$$

2. Continuous Distributions

2.1 Uniform random variable

$$f(x) = \begin{cases} \frac{1}{\beta-\alpha} & \text{if } x \in [\alpha, \beta], \\ 0 & \text{otherwise.} \end{cases}$$
$$E[X] = \frac{\alpha+\beta}{2}; \quad Var(X) = \frac{(\beta-\alpha)^2}{12}.$$

2.2 Exponential random variable

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$
$$E[X] = \frac{1}{\lambda}; \quad Var(X) = \frac{1}{\lambda^2}; \quad P(X \geq x) = e^{-\lambda x}; \quad P(X \geq x+y | X \geq y) = e^{-\lambda x}.$$

2.3 Normal random variable

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \quad \mathit{for}\ x \in \mathbb{R}.$$
$$E[X] = \mu; \quad Var(X) = \sigma^2; \quad Z \sim \mathit{N(0,1)} = \frac{X-\mu}{\sigma}.$$

2.4 Gamma random variable

$$f(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} \quad \mathit{for}\ \alpha, \beta, x \in \mathbb{R}^+; \quad \Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt.$$
$$E[X] = \frac{\alpha}{\lambda}; \quad Var(X) = \frac{\alpha}{\lambda^2}.$$

3. Multivariate Distributions

3.1 Multivariate Normal Distribution

$$f(\vec{x}) = \frac{1}{(2\pi)^{n/2}|\Sigma|^{1/2}} e^{-\frac{1}{2}(\vec{x}-\vec{\mu})^{\top}\Sigma^{-1}(\vec{x}-\vec{\mu})} \quad \textit{for } x \in \mathbb{R}^n.$$
$$E[X] = \vec{\mu}; \quad Var(X) = \Sigma.$$

Let random vector $\begin{bmatrix} X_1 \\ \vdots \\ X_n \end{bmatrix}$ have a multivariate normal distribution and Z be the linear combination of the random vector with coefficients $\vec{A} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$. Then, $Z \sim$ *N(*$\vec{A} \cdot \vec{\mu}, \vec{A}^{\top} \sum \vec{A}$*)*. For bivariate random vector $\begin{bmatrix} X \\ Y \end{bmatrix}$ with $\vec{A} = \begin{bmatrix} a \\ b \end{bmatrix}$,

$$Var(Z) = a^2\sigma_X^2 + 2ab\rho\sigma_X\sigma_Y + b^2\sigma_Y^2.$$

3.2 Multinomial Distribution

$$P(\vec{x}) = \frac{n!}{x_1! \cdots x_k!} p_1^{x_1} \cdots p_k^{x_k} \quad \textit{for } \vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_k \end{bmatrix},\ x_i \in \mathbb{N},\ \sum_{i=1}^{k} x_i = n,\ p_i \in [0,1].$$
$$E[\vec{x}] = n\vec{p}; \quad Var(\vec{x}) = n\overrightarrow{p(1-p)}.$$

4. Theorems

4.1 Chebyshev's Inequality

$$P(|X - \mu| \geq k\sigma) \leq \frac{1}{k^2}.$$

4.2 Markov's Inequality

$$P(X \geq k) \leq \frac{E[X]}{k}.$$

4.3 Central Limit Theorem

Let $S_n = X_1 + \cdots + X_n$. When n is large, S_n is approximately normally distributed with mean $n\mu$ and variance $n\sigma^2$.

$$\frac{S_n - n\mu}{\sqrt{n}\sigma} = \frac{\bar{S_n} - \mu}{\sigma/\sqrt{n}} \sim N(0,1) \quad \text{as } n \to \infty.$$

4.4 Weak Law of Large Numbers

Let $S_n = X_1 + \cdots + X_n$.

$$\lim_{n\to\infty} P(|\bar{S_n} - \mu| < \epsilon) = 1.$$

4.5 Strong Law of Large Numbers

Let $S_n = X_1 + \cdots + X_n$.

$$P(\lim_{n\to\infty} \bar{S_n} = \mu) = 1.$$

4.6 Moment Generating Function

$$M_X(t) = E[e^{tX}] = \int_{-\infty}^{\infty} e^{tx} f(x)dx.$$