Math 181A HW10

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Problem 6.5.1 Let $k_1, k_2, \ldots, k_n$ be a random sample from the geometric probability function

$$p_X(k; p) = (1-p)^{k-1}p, k = 1, 2, \ldots$$

Find Λ, the generalized likelihood ratio for testing $H_0 : p = p_0$ versus $H_1 : p \neq p_0$.

Solution.

$$\Lambda = \frac{L(p_0)}{\underset{p\in\mathbb{R}}{max} L(p)}.$$

To find $\underset{p\in\mathbb{R}}{max} L(p)$, we take the derivative of $l(p) = \ln[L(p)]$ and set it to 0.

$$\begin{aligned}
L(p) &= \prod_{i=1}^{n}(1-p)^{k_i-1}p \\
&= p^n(1-p)^{\sum\limits_{i=1}^{n}(k_i)-n} \\
l(p) &= n\ln(p) + \left(\sum_{i=1}^{n}(k_i) - n\right)\ln(1-p) \\
l'(\hat{p}) = 0 &= \frac{n}{\hat{p}} - \frac{\sum\limits_{i=1}^{n}(k_i) - n}{1-\hat{p}} \\
\hat{p}\sum_{i=1}^{n}(k_i) - n\hat{p} &= n(1-\hat{p}) \\
\hat{p} &= \frac{n}{\sum_{i=1}^{n}(k_i)} \\
&= \frac{1}{\overline{K}}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\Lambda &= \frac{L(p_0)}{\underset{p\in\mathbb{R}}{max} L(p)} \\
&= \frac{\prod\limits_{i=1}^{n}(1-p_0)^{k_i-1}p_0}{\prod\limits_{i=1}^{n}(1-\frac{1}{\overline{K}})^{k_i-1}\frac{1}{\overline{K}}} \\
&= \frac{p_0^n(1-p_0)^{\sum_{i=1}^{n}(k_i)-n}}{(1/\overline{K})^n(1-(1/\overline{K}))^{\sum_{i=1}^{n}(k_i)-n}}
\end{aligned}$$

□

Problem 6.5.2 Let $y_1, y_2, \dots, y_{10}$ be a random sample from an exponential pdf with unknown parameter λ. Find the form of the GLRT for $H_0 : \lambda = \lambda_0$ versus $H_1 : \lambda \neq \lambda_0$. What integral would have to be evaluated to determine the critical value if α were equal to 0.05?

Solution.

$$\Lambda = \frac{L(\lambda_0)}{\underset{\lambda \in \mathbb{R}}{max} L(\lambda)}.$$

To find $\underset{\lambda \in \mathbb{R}}{max} L(\lambda)$, we first find the maximum likelihood estimator by taking the derivative of $l(\lambda) = \ln[L(\lambda)]$ and set it to 0.

$$\begin{aligned}
L(\lambda) &= \prod_{k=1}^{10} \lambda e^{-\lambda y_k} \\
&= \lambda^{10} e^{-\lambda \sum_{k=1}^{10} y_k} \\
l(\lambda) &= 10 \ln(\lambda) - \lambda \sum_{k=1}^{10} y_k \\
l'(\hat{\lambda}) = 0 &= \frac{10}{\hat{\lambda}} - \sum_{k=1}^{10} y_k \\
\hat{\lambda} &= \frac{10}{\sum_{k=1}^{10} y_k} \\
&= \frac{1}{\overline{Y}}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\Lambda &= \frac{L(\lambda_0)}{\underset{\lambda \in \mathbb{R}}{max} L(\lambda)} \\
&= \frac{\lambda_0^{10} \cdot e^{-\lambda_0 \sum_{k=1}^{10} y_k}}{(1/\overline{Y})^{10} \cdot e^{-(1/\overline{Y}) \sum_{k=1}^{10} y_k}} \\
&= \frac{\lambda_0^{10} \cdot e^{-\lambda_0 \sum_{k=1}^{10} y_k}}{(1/\overline{Y})^{10} \cdot e^{-10}} \\
&= (\overline{Y} \cdot \lambda_0)^{10} \cdot e^{10 - \lambda_0 \sum_{k=1}^{10} y_k} \\
&= (\overline{Y} \cdot \lambda_0)^{10} \cdot e^{10 - \lambda_0 \cdot 10 \overline{Y}} \\
&= (\overline{Y} \cdot \lambda_0)^{10} \cdot e^{10(1 - \lambda_0 \cdot \overline{Y})}.
\end{aligned}$$

To find the critical value, we need to find c such that

$$\begin{aligned}
\alpha &= P(\Lambda \leq c | \lambda = \lambda_0) \\
\alpha &= \int_0^c f_\Lambda(z | \lambda = \lambda_0) dz.
\end{aligned}$$

$\square$

Problem 6.5.3 Let $y_1, y_2, ..., y_n$ be a random sample from a normal pdf with unknown mean μ and variance 1. Find the form of the GLRT for $H_0 : \mu = \mu_0$ versus $H_1 : \mu \neq \mu_0$.

Solution. Let's find the maximum likelihood estimator $\hat{\mu}$ first.

$$\begin{aligned}
L(\mu) &= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(y_i-\mu)^2} \\
&= \left(\frac{1}{\sqrt{2\pi}}\right)^n e^{-\frac{1}{2}\sum_{i=1}^{n}(y_i-\mu)^2} \\
l(\mu) &= -\frac{n}{2}\ln(2\pi) - \frac{1}{2}\sum_{i=1}^{n}(y_i-\mu)^2 \\
l'(\hat{\mu}) = 0 &= \sum_{i=1}^{n}(y_i - \hat{\mu}) \\
\hat{\mu} &= \frac{1}{n}\sum_{i=1}^{n} y_i \\
&= \overline{Y}.
\end{aligned}$$

Then, we can find the likelihood ratio by setting $\underset{\mu\in\mathbb{R}}{max} L(\mu) = L(\hat{\mu})$.

$$\begin{aligned}
\Lambda &= \frac{L(\mu_0)}{\underset{\mu\in\mathbb{R}}{max} L(\mu)} \\
&= \frac{\left(\frac{1}{\sqrt{2\pi}}\right)^n e^{-\frac{1}{2}\sum_{i=1}^{n}(y_i-\mu_0)^2}}{\left(\frac{1}{\sqrt{2\pi}}\right)^n e^{-\frac{1}{2}\sum_{i=1}^{n}(y_i-\overline{Y})^2}} \\
&= e^{-\frac{1}{2}\sum_{i=1}^{n}(y_i-\mu_0)^2+\frac{1}{2}\sum_{i=1}^{n}(y_i-\overline{Y})^2} \\
&= e^{-\frac{1}{2}\sum_{i=1}^{n}(y_i^2-2y_i\mu_0+\mu_0^2)+\frac{1}{2}\sum_{i=1}^{n}(y_i^2-2y_i\overline{Y}+\overline{Y}^2)} \\
&= e^{-\frac{1}{2}\sum_{i=1}^{n}(2y_i\overline{Y}-2y_i\mu_0-\overline{Y}^2+\mu_0^2)} \\
&= e^{-\frac{n}{2}\sum_{i=1}^{n}(2\frac{y_i}{n}\overline{Y}-2\frac{y_i}{n}\mu_0)+\frac{n}{2}\overline{Y}^2-\frac{n}{2}\mu_0^2} \\
&= e^{-\frac{n}{2}\left(2\overline{Y}^2-2\overline{Y}\mu_0\right)+\frac{n}{2}\overline{Y}^2-\frac{n}{2}\mu_0^2} \\
&= e^{-\frac{n}{2}\left(\overline{Y}^2-2\overline{Y}\mu_0+\mu_0^2\right)} \\
&= e^{-\frac{n}{2}\left(\overline{Y}-\mu_0\right)^2}.
\end{aligned}$$

To conduct a hypothesis test, we need to find c^* such that

$$\begin{aligned}
\alpha &= P(\Lambda \leq c|\mu = \mu_0) \qquad (\textit{notice } e^{-\frac{n}{2}\left(\overline{Y}-\mu_0\right)^2} \textit{ is always} \leq 1, \textit{so } c \leq 1) \\
\alpha &= P(e^{-\frac{n}{2}\left(\overline{Y}-\mu_0\right)^2} \leq c|\mu = \mu_0) \\
\alpha &= P\left(-\frac{n}{2}\left(\overline{Y}-\mu_0\right)^2 \leq \ln(c)|\mu = \mu_0\right) \\
\alpha &= P\left(\frac{\left(\overline{Y}-\mu_0\right)^2}{1/n} \geq -2\ln(c)|\mu = \mu_0\right) \\
\alpha &= P\left(\left(\frac{\overline{Y}-\mu_0}{1/\sqrt{n}}\right)^2 \geq -2\ln(c)|\mu = \mu_0\right) \\
\alpha &= P\left(Z^2 \geq -2\ln(c)|\mu = \mu_0\right) \\
\alpha &= P\left(Z \geq c^*|\mu = \mu_0\right) + P\left(Z \leq -c^*|\mu = \mu_0\right). \qquad (c^* = z_{\alpha/2})
\end{aligned}$$

□

Problem 6.5.4 In the scenario of Question 6.5.3, suppose the alternative hypothesis is $H_1 : \mu = \mu_1$, for some particular value of μ_1. How does the likelihood ratio test change in this case? In what way does the critical region depend on the particular value of μ_1?

Problem 6.5.5

Problem 6.5.6 Suppose a sufficient statistic exists for the parameter θ. Use Theorem 5.6.1 to show that the critical region of a likelihood ratio test will depend on the sufficient statistic.