Computer Modern

MATH 104 Notes
Book: Elementary Analysis - Ross

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Contents

Chapter 1

Metric Spaces

1.1 Some Topological Concepts in Metric Spaces

Definition 1.1: Metric and Metric Space

Let S be a set, and suppose d is a function defined for all paris (x, y) of elements from S satisfying

D1. $d(x, x) = 0$ for all $x \in S$ and $d(x, y) > 0$ for distinct x, y in S.

D2. $d(x, y) = d(y, x)$ for all $x, y \in S$.

D3. $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in S$.

We call such function d the *distance function* or a *metric* on S. A *metric space* S is a set S together with a metric on it. Properly speaking, the metric space is the pair S, d since a set S may have more than one metric on it.

Example 1.1

Let $\text{dist}(a, b) = |a - b|$ for $a, b \in \mathbb{R}$. Then dist is a metric on $\mathbb{R}$. Notice **D3** follows direcly from the triangular inequality for real numbers. Therefore, for any metric d, we call the property **D3** the *triangular inequality*.

Example 1.2

The space of all k-tuples

$$\mathbf{x} = (x_1, x_2, \ldots, x_k) \qquad \text{where} \qquad x_j \in \mathbb{R} \qquad \text{for } j = 1, 2, \ldots, k,$$

is called *k-dimensional Euclidean space* and written $\mathbb{R}^k$.

R^k has multiple metrics on it, and the most familiar metric is the *Euclidean norm*:

$$d(\mathbf{x}, \mathbf{y}) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_k - y_k)^2} \qquad \text{for } \mathbf{x}, \mathbf{y} \in \mathbb{R}^k.$$

Definition 1.2: Convergence in Metric Space

A sequence (s_n) in a metric space S, d *converges* to $s \in S$ if

$$\lim_{n \to \infty} d(s_n, s) = 0.$$

A sequence $(s_n) \in S$ is a *Cauchy sequence* if for each $\epsilon > 0$ there is an integer N such that

$$m, n > N \implies d(s_m, s_n) < \epsilon.$$

Note:
The metric space (S, d) is *complete* if every Cauchy sequence in S is convergent.

Lemma 1.1

A sequence $(\mathbf{x}^{(n)}) \in \mathbb{R}^k$ converges if and only if each of its component sequences $(x_j^{(n)})$ converges in $\mathbb{R}$. A sequence $(\mathbf{x}^{(n)}) \in \mathbb{R}^k$ is a Cauchy sequence if and only if each of its component sequences $(x_j^{(n)})$ is a Cauchy sequence in $\mathbb{R}$.

Theorem 1.1

Euclidean k-space $\mathbb{R}^k$ is complete.

Note:
In other words, every Cauchy sequence in $\mathbb{R}^k$ is convergent.

Definition 1.3: Boundedness of $\mathbb{R}^k$

A set $S \in \mathbb{R}^k$ is bounded if there exists $M > 0$ such that

$$\max\{|x_j| : j = 1, 2, \dots, k\} \leq M$$

for all $\mathbf{x} = (x_1, x_2, \dots, x_k) \in S$.

Theorem 1.2 Bolzano-Weierstrass Theorem for Euclidean k-space

Evеry bounded sequence in $\mathbb{R}^k$ has a convergent subsequence.

Definition 1.4: Open Set

Let (S, d) be a metric space. Let E be a subset of S. An element $s_0 \in E$ is interior to E if for some $r > 0$ we have

$$\{s \in S : d(s, s_0) < 0\} \subseteq E.$$

Note:
In other words, s_0 is interior to E if there exists an open ball with radius $r > 0$ centered at s_0 such that all points $\in S$ and in the ball are still in E.

We denote E° as the set of points in E that are interior to E. The set E is *open* on S if every point of E is interior to E, i.e., if $E = E^\circ$.

Theorem 1.3 Property of Open Set

(i) S is open in S.

(ii) The empty set $\emptyset$ is open in S.

(iii) The union of any collection of open sets in S is also open in S.

(iv) The intersection of finitely many open sets in S is also open in S.

Note:

Metric spaces are fairly general and useful objects, in the sense that when one is interested in convergence of certain objects (such as points or functions), there is often a metric that assists in the study of the convergence.

However, sometimes no metric will work but there is still some sort of convergence notion. Frequently, the toolbox we need for studying such behavior is what is called a *topology*.

Topologies are more general than metrics, and they are defined in terms of open sets rather than distance functions. In particular, the open sets defined by a metric form a topology. Because of this abstract theory of topology, concepts that can be defined in terms of open sets are called topological. Hence, we are slowly entering the realm of topology by studying open sets in metric spaces.

Definition 1.5: Closed Set

Let (S, d) be a metric space. A subset E of S is *closed* if its complement $S \setminus E$ is an open set in S. Because of *Theorem 1.3 (iii)*, the intersection of any collection of closed sets in S is also closed in S.

The closure E^- of a set E is the intersection of all closed sets in S that contain E.

The *boundary* of a set E is the set of $E^- \setminus E^\circ$; points in the boundary set are called *boundary points* of E.

Note:

For every bounary point s of E, all open balls centered at s must contain some point in E and some point in $S \setminus E$. Because if s has an open ball that does not contain any points in E, then s is not in a limit point of E and thus not in E^-. On the other hand, if s has an open ball that does not contain points in $S \setminus E$, then that open ball is contained in E and thus s is interior to E.

Theorem 1.4 Property of Closed Set

(i) The set E is closed if and only if $E^- = E$.

(ii) The set E is closed if and only if it contains the limits of all convergent sequences of points in E.

(iii) An element is in E^- if and only if it is the limit of some sequence of points in E.

(iv) A point s is in the boundary of E if and only if it belongs to both the closure of E and the closure of $S \setminus E$.

Example 1.3 (Open Intervals and Closed Intervals in $\mathbb{R}$)

In $\mathbb{R}$, open intervals (a, b) are open sets. Closed intervals $[a, b]$ are closed sets. The interior of $[a, b]$ is (a, b). The boundary of both (a, b) and $[a, b]$ is the two-element set $\{a, b\}$.

Every open set in $\mathbb{R}$ is the union of a disjoint sequence of open intervals. However, a closed set in $\mathbb{R}$ need not be the union of a disjoint sequence of closed intervals and points.

Example 1.4 (Open Balls and Closed Balls)

In $\mathbb{R}^k$, open balls $\{\mathbf{x} : d(\mathbf{x}, \mathbf{x}_0) < r\}$ are open sets, and closed balls $\{\mathbf{x} : d(\mathbf{x}, \mathbf{x}_0) \leq r\}$ are closed sets. The boundary of each of these sets is the sphere $\{\mathbf{x} : d(\mathbf{x}, \mathbf{x}_0) = r\}$.

In the plane $\mathbb{R}^2$, the sets

$$\{(x, y) : x > 0\} \qquad \text{and} \qquad \{(x, y) : x > 0, y > 0\}$$

are open. If $>$ is replaced by $\geq$, we obtain clsoed sets. Many sets are neither open nor closed. For example, $[0, 1)$ are neither open nor closed in $\mathbb{R}$, and $\{(x, y) : x > 0, y \geq 0\}$ is neither open nor closed in $\mathbb{R}^2$.

Theorem 1.5

Let F_n be a decreasing sequence $[i.e., F_1 \supseteqq F_2 \supseteq \cdots]$ of closed, bounded, and nonempty sets in $\mathbb{R}^k$. Then $\bigcap_{n=1}^{\infty} F_n$ is also closed, bounded, and nonempty.

Note:

- We can think of this theorem as a generalization of the Nested Interval Property in $\mathbb{R}$.
- Also, we can substitute the word "closed and bounded" with "compact" in the theorem.

Example 1.5 (Cantor Set)

Cantor set is a nonempty, closed, and bounded set in $\mathbb{R}$. It is constructed by removing the middle third of the interval $[0, 1]$, then removing the middle thirds of the two remaining pieces, and so on. The Cantor set is the set of all points that remain after this process is repeated infinitely many times.

The sum of the intervals removed at the nth stage is $\left(\frac{2}{3}\right)^n$, and this tends to zero as $n \to \infty$. Yet, the points remaining after the infinite process is so large that it cannot be written as a sequence; in set-theoretic terms it is "uncountable". The interior of the Cantor set is empty, and hence its boundary is the set itself.

Definition 1.6

Let (S, d) be a metric space. A family $\mathcal{U}$ of open sets is said to be an *open cover* for a set E if each point of E belongs to at least one set in $\mathcal{U}$, i.e.,

$$E \subseteq \{U : U \in \mathcal{U}\}.$$

A *subcover* of $\mathcal{U}$ is any subfamily of $\mathcal{U}$ that still covers E. A cover or subcover is finite if it contains only finitely many sets; yet the sets themselves can be infinite.

A set E is *compact* if every open cover of E has a finite subcover.

Note:

The word "family" here emphasizes it is a collections of sets, usually indexed loosely. So a family is a set of sets.

Theorem 1.6 Heine-Borel Theorem

A subset E of $\mathbb{R}^k$ is compact if and only if it is closed and bounded.

Definition 1.7: k-cell

A *k-cell* F in $\mathbb{R}^k$ is a set of the form

$$\{x \in \mathbb{R}^k : a_j \leq x_j \leq b_j \text{ for } j = 1, 2, \ldots, k\}$$

where $a_j, b_j \in \mathbb{R}$ and $a_j \leq b_j$ for $j = 1, 2, \ldots, k$.

The *diameter* of F is

$$\delta = \sqrt{(b_1 - a_1)^2 + (b_2 - a_2)^2 + \cdots + (b_k - a_k)^2},$$

that is, $\delta = \sup\{d(\mathbf{x}, \mathbf{y}) : \mathbf{x}, \mathbf{y} \in F\}$.

Note:

F is a union of 2^k k-cells each having diameter $\delta/2$.

Example 1.6

A 2-cell in $\mathbb{R}^2$ is a rectangle. A 3-cell in $\mathbb{R}^3$ is a rectangular box.

Theorem 1.7

Every k-cell in $\mathbb{R}^k$ is compact.

Example 1.7

Let E be a nonempty subset of a metric space (S, d). Consider the function $d(E, x) = \inf\{d(y, x) : y \in E\}$ for $x \in S$. This function satifies $|d(E, x_1) - d(E, x_2)| \leq d(x_1, x_2)$.

We want to show that if E is compact and if $E \subseteq U$ for some open subset $U \subseteq S$, then for some $\delta > 0$, we have

$$\{y \in S : d(E, y) < \delta\} \subseteq U,$$

in other words, any point in S that is within a distance δ of E is in U.

Proof: Since U is open, for each $x \in E$, which is also in U, there exists some r_x such that

$$\{y \in S : d(y, x) < r_x\} \subseteq U \text{ for some } r_x > 0.$$

The open balls $\{y \in S : d(y, x) < \frac{r_x}{2}\}$ cover E. Since E is compact, a finite subfamily also covers E. That is there are $x_1, \ldots, x_n \in E$ so that

$$E \subseteq \bigcup_{i=1}^{n} \{y \in S : d(y, x_i) < \frac{r_{x_i}}{2}\}.$$

Let $\delta = \frac{1}{2} \min\{r_{x_1}, \ldots, r_{x_n}\}$.

Now consider an arbitrary $y \in S$ and $d(E, y) < \delta$. Then for some $x \in E$, we have $d(y, x) < \delta$. Moreover, $d(x, x_k) < \frac{r_{x_k}}{2}$ for some $k \in \{1, 2, \ldots, n\}$. Therefore, by "passing through" this closest point x to reach x_k,

$$d(y, x_k) \leq d(x, y) + d(x, x_k) < \delta + \frac{r_k}{2} \leq \frac{r_k}{2} + \frac{r_k}{2} = r_k.$$

Hence, y is within the "safe" open ball centered at x_k, which is contained in U. 😊

1.2 Metric Spaces: Continuity

1.3 Metric Spaces: Connectedness

We generealize the notion of "connectedness" from $\mathbb{R}$ to metric spaces.

Definition 1.8: Disconnected

Let E be a subset of metric space (S, d). The set E is *disconnected* if one of the following two equivalent conditions holds:

(a) There are open subsets U_1 and U_2 of S such that

$$(E \cap U_1) \cap (E \cap U_2) = \emptyset \text{ and } E = (E \cap U_1) \cup (E \cap U_2) \tag{1.1}$$
$$E \cap U_1 \neq \emptyset \text{ and } E \cap U_2 \neq \emptyset. \tag{1.2}$$

(b) There are nonempty disjoint subsets A and B of E such that such that $E = A \cup B$ and neither set intersects the closure of the other set, i.e.,

$$A^- \cap B \neq \emptyset \text{ and } A \cap B^- \neq \emptyset. \tag{1.3}$$

We say that E is *connected* if it is not disconnected.