Math 180A HW3

Neo Lee

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Problem 3. Obviously, $P(A) = \frac{1}{2}$ and $P(B) = \frac{1}{5}$. Then, $P(A \cap B) = P(A|B)P(B) = \frac{1}{2} \times \frac{1}{5} = P(A)P(B)$. Hence, A and B are indeed independent.

Problem 4.

(a) $E(X)$ will be larger. For Y, each outcome is weighted evenly with $p(x_i) = \frac{1}{4}$. Yet, for X, the outcome of more students are weighted more while the outcome of fewer students are weighted less with $p(x_i) = \frac{x_i}{90}$. In other words, there's a higher probability of choosing a student from a larger class than from a smaller class in X.

(b)

$$\begin{aligned} E(X) &= \sum x_i p(x_i) && (1) \\ &= 21 \times \frac{21}{90} + 24 \times \frac{24}{90} + 17 \times \frac{17}{90} + 28 \times \frac{28}{90} && (2) \\ &= \frac{209}{9} && (3) \\ E(Y) &= 21 \times \frac{1}{4} + 24 \times \frac{1}{4} + 17 \times \frac{1}{4} + 28 \times \frac{1}{4} && (4) \\ &= \frac{45}{2} && (5) \end{aligned}$$

Problem 5.

(a) Possible values of $Y = 0, 1$. $P(Y = 1) = \frac{1}{2} + \frac{1}{6} = \frac{2}{3}$ and $P(Y = 0) = \frac{1}{3}$.

$$\begin{aligned} E(|X|) &= E(Y) && (6) \\ &= 1 \times \frac{2}{3} + 0 \times \frac{1}{3} && (7) \\ &= \frac{2}{3} && (8) \end{aligned}$$

(b)

$$\begin{aligned} E(|X|) &= |-1| \times \frac{1}{2} + |1| \times \frac{1}{6} + 0 \times \frac{1}{6} && (9) \\ &= \frac{2}{3} && (10) \end{aligned}$$

Problem 6. We can express X in terms of a collection of indictors $I_i = I_{\{i\text{th card is a pair with } (i+1)\text{th card}\}}$, for which $X = I_1 + I_2 + \cdots + I_{51}$. Note that

$$\begin{aligned} E(I_1) = E(I_2) = \cdots = E(I_{51}) &= 1 \cdot P(\{\text{first card is a pair with second card}\}) && (11) \\ &= \frac{3}{51}. && (12) \end{aligned}$$

Therefore, by additivity, $E(X) = E(I_1) + E(I_2) + \cdots + E(I_51) = 51 \times \frac{3}{51} = 3$.

Side note: this counting method counts 3-of-a-kind as two of two consecutive cards and 4-of-a-kind as three of two consecutive cards.

Problem 7.

(a) Let A be the event of a die roll and B be the event of the other die roll. Note that A and B are independent because $P(A \cap B) = P(A)P(B)$.

Now let us we consider $P(X \leq s) : 1 \leq s \leq 20, s \in \mathbb{Z}+$, which is the probability that both the dice rolls are less than or equal to s. Therefore, it can be written as $P(X \leq s) = P((A \leq s) \cap (B \leq s)) = \frac{s}{20} \times \frac{s}{20} = \frac{s^2}{400}$.

Hence, CDF of X is

$$F_x(s) = \begin{cases} 0, & s \leq 0 \\ \frac{s^2}{400}, & 1 \leq s \leq 20 \\ 1, & 20 < s. \end{cases} \tag{13}$$

Now let us determine the pmf of X. Note that $P(X = k) = P(X \leq k) - P(X \leq k-1)$. Therefore,

$$p(k) = P(X = k) = \begin{cases} 0, & k < 1 \text{ or } 20 < k \\ \frac{k^2}{400} - \frac{(k-1)^2}{400} = \frac{2k-1}{400}, & 1 \leq k \leq 20. \end{cases} \tag{14}$$

(b) Let us first approach the question with basic combinatorics knowledge then we'll approach the question with CDF, which will converge to the same conclusion.

For $Y = k$, one die roll has to be equal to k, and another die roll can be between k and 20 (end point inclusice), which has a total of $(21-k)$ possibilities. Then, let's take ordering into account. Since there are two die roll, for every combination there are two orders and we times two for $21-k$. Note we have to minus 1 because there is only one ordering if the two roll die have the same outcome. Finally, we divide the event outcomes by all possible outcomes $20 \times 20 = 400$.

Hence,

$$P(Y = k) = \begin{cases} 0, & 1 < k \text{ or } 20 < k \\ \frac{2(21-k)-1}{20 \times 20} = \frac{41-2k}{400}, & 1 \leq k \leq 20. \end{cases} \tag{15}$$

Now let us find the pmf of Y from CDF just like how we did it in (a). Let A be the event of a die roll and B be the event of the other die roll. Note that A and B are independent because $P(A \cap B) = P(A)P(B)$. For $1 \leq s \leq 20 : s \in \mathbb{Z}$,

$$\begin{aligned} P(Y \leq s) &= 1 - P(Y > s) && (16) \\ &= 1 - P(A > s)P(B > s) && (17) \\ &= 1 - \left(\frac{20-s}{20}\right)\left(\frac{20-s}{20}\right) && (18) \\ &= 1 - \frac{s^2 - 40s + 400}{400} && (19) \\ &= \frac{40s - s^2}{400}. && (20) \end{aligned}$$

Hence, the CDF of Y is

$$F_y(s) = \begin{cases} 0, & s \leq 0 \\ \frac{40s-s^2}{400}, & 1 \leq s \leq 20 \\ 1, & 20 < s. \end{cases} \tag{21}$$

By the same reasoning from (a), $P(Y = k) = P(Y \leq k) - P(Y \leq k-1)$. Therefore,

$$P(Y = k) = \begin{cases} 0, & 1 < k \text{ or } 20 < k \\ \frac{40k-k^2-\left(40(k-1)-(k-1)^2\right)}{400} = \frac{40k-k^2-(40k-40-k^2+2k-1)}{400} = \frac{41-2k}{400}, & 1 \leq k \leq 20, \end{cases} \tag{22}$$

which is the same as the combinatorics approach.