Math 180A HW7

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Problem 1.

(a) 0.3.

(b) $\frac{7}{30}$.

Problem 2.

(a)

$$\begin{aligned}
f_X(x) &= \int_0^1 \frac{12}{7}(xy + y^2)dy && (1)\\
&= \frac{12}{7}\left[\frac{1}{2}xy^2 + \frac{1}{3}y^3\right]_0^1 && (2)\\
&= \frac{12}{7}\left(\frac{x}{2} + \frac{1}{3}\right) && (3)\\
&= \frac{6x + 4}{7} && (4)
\end{aligned}$$

for $x \in [0, 1]$ and 0 otherwise.

(b)

$$\begin{aligned}
f_Y(y) &= \int_0^1 \frac{12}{7}(xy + y^2)dx && (5)\\
&= \frac{12}{7}\left[\frac{1}{2}x^2y + xy^2\right]_0^1 && (6)\\
&= \frac{12}{7}\left(\frac{y}{2} + y^2\right) && (7)\\
&= \frac{6y + 12y^2}{7} && (8)
\end{aligned}$$

for $y \in [0, 1]$ and 0 otherwise.

(c)

$$P(X<Y)=\int_0^1\int_0^y \frac{12}{7}(xy+y^2)dxdy \tag{9}$$
$$=\frac{12}{7}\int_0^1\left[\frac{1}{2}x^2y+xy^2\right]_{x=0}^{x=y}dy \tag{10}$$
$$=\frac{12}{7}\int_0^1\frac{1}{2}y^3+y^3dy \tag{11}$$
$$=\frac{12}{7}\left[\frac{1}{8}y^4+\frac{1}{4}y^4\right]_0^1 \tag{12}$$
$$=\frac{12}{6}\left(\frac{3}{8}\right) \tag{13}$$
$$=\frac{3}{4}. \tag{14}$$

Problem 3.

$$P(X=x,Y=y)=P(X=x)P(Y=y) \tag{15}$$
$$=\left(p(1-p)^{x-1}\right)\left(r(1-r)^{y-1}\right). \tag{16}$$

Then,

$$P(X<Y)=P(X\in[1,\infty))P(Y>X) \tag{17}$$
$$=\sum_{n=1}^{\infty}p(1-p)^{n-1}(1-r)^n. \tag{18}$$

Problem 4.

(a)

$$\int_0^\pi\int_0^\pi c(1-cos(x)cos(y))dxdy=1 \tag{19}$$
$$c\int_0^\pi\int_0^\pi dxdy-c\int_0^\pi\int_0^\pi cos(x)cos(y)dxdy=1 \tag{20}$$
$$c\pi^2-c\int_0^\pi[sin(x)cos(y)]_{x=0}^{x=\pi}dy=1 \tag{21}$$
$$c\pi^2=1 \tag{22}$$
$$c=\frac{1}{\pi^2}. \tag{23}$$

(b)

$$f_X(x)=\frac{1}{\pi^2}\int_0^\pi 1-cos(x)cos(y)dy \tag{24}$$
$$=\frac{1}{\pi^2}\left(\pi-[cos(x)sin(y)]_{y=0}^{y=\pi}\right) \tag{25}$$
$$=\frac{1}{\pi} \tag{26}$$

for $x\in[0,\pi]$ and 0 otherwise. Similarly,

$$f_Y(y)=\frac{1}{\pi} \tag{27}$$

for $y\in[0,\pi]$ and 0 otherwise. The probability distribution is uniform.

(c)

$$f(0,0) = \frac{1}{\pi^2}\left(1 - cos(0)cos(0)\right) \tag{28}$$
$$= 0. \tag{29}$$

On the other hand,

$$f_X(0)f_Y(0) = \left(\frac{1}{\pi}\right)\left(\frac{1}{\pi}\right) \tag{30}$$
$$= \frac{1}{\pi^2} \tag{31}$$
$$\neq f(0,0) \tag{32}$$

Hence, X and Y are not independent.

Problem 6. The convolution formula tells us

$$P_{X+Y}(z) = \sum_X P_X(x)P_Y(z-x) \tag{33}$$
$$= P(X=0)P(Y=z-0) + P(X=1)P(Y=z-1) \tag{34}$$
$$= (1-p)P(Y=z) + pP(Y=z-1). \tag{35}$$

Hence,

$$P_{X+Y}(z) = \begin{cases} (1-p)(1-r) & z=0, \\ p(1-r)+r(1-p) & z=1, \\ pr & z=2, \\ 0 & otherwise. \end{cases} \tag{36}$$

Problem 7. *Convolution Approach*
We know $f_Y(y) = 1$ for $y \in (1,2)$ and 0 otherwise. For $z \leq 1$, $f_{X+Y}(z) = 0$. For $z \in [1,2]$,

$$f_{X+Y}(z) = \int_1^z f_Y(y)f_X(z-y)dy \tag{37}$$
$$= \int_1^z 2(z-y)dy \tag{38}$$
$$= \int_1^z 2z - 2ydy \tag{39}$$
$$= [2yz]_{y=1}^{y=z} - \left[y^2\right]_1^z \tag{40}$$
$$= 2z^2 - 2z - z^2 + 1 \tag{41}$$
$$= z^2 - 2z + 1. \tag{42}$$

For $z \in [2,3]$,

$$f_{X+Y}(z) = \int_{z-2}^1 f_Y(z-x)f_X(x)dx \tag{43}$$
$$= \int_{z-2}^1 2xdx \tag{44}$$
$$= \left[x^2\right]_{z-2}^1 \tag{45}$$
$$= -z^2 + 4z - 3. \tag{46}$$

For $z \geq 3$, $f_{X+Y}(z) = 0$.

CDF Approach

Since X and Y are independent, $f(x,y) = f_X(x)f_Y(y) = 2x$. For $z \leq 1$, $f_{X+Y}(z) = 0$. For $z \in [1,2]$,

$$\begin{aligned}
P(X+Y \leq Z) &= \int_1^z \int_0^{z-y} 2x dx dy && (47)\\
&= \int_1^z \left[x^2\right]_0^{z-y} dy && (48)\\
&= \int_1^z z^2 - 2zy + y^2 dy && (49)\\
&= \left[z^2 y - zy^2 + \frac{1}{3}y^3\right]_1^z && (50)\\
&= z^3 - z^3 + \frac{1}{3}z^3 - z^2 + z - \frac{1}{3} && (51)\\
&= \frac{1}{3}z^3 - z^2 + z - \frac{1}{3}. && (52)
\end{aligned}$$

Hence, for $z \in [1,2]$,

$$\begin{aligned}
f_{X+Y}(z) &= \frac{d}{dz}\left(\frac{1}{3}z^3 - z^2 + z - \frac{1}{3}\right) && (53)\\
&= z^2 - 2z + 1. && (54)
\end{aligned}$$

For $z \in [2,3]$,

$$\begin{aligned}
P(X+Y \leq Z) &= \int_0^1 \int_1^2 2x dy dx - \int_{z-2}^1 \int_{z-x}^2 2x dy dx && (55)\\
&= \int_0^1 \left[2xy\right]_{y=1}^{y=2} dx - \int_{z-2}^1 \left[2xy\right]_{y=z-x}^{y=2} dx && (56)\\
&= \int_0^1 2x dx - \int_{z-2}^1 2x^2 + 4x - 2zx dx && (57)\\
&= 1 - \left[\frac{2}{3}x^3\right]_{z-2}^1 - \left[2x^2\right]_{z-2}^1 + \left[zx^2\right]_{z-2}^1 && (58)\\
&= -\frac{1}{3}z^3 + 2z^2 - 3z + 1. && (59)
\end{aligned}$$

Hence, for $z \in [2,3]$,

$$\begin{aligned}
f_{X+Y}(z) &= \frac{d}{dz}\left(-\frac{1}{3}z^3 + 2z^2 - 3z + 1\right) && (60)\\
&= -z^2 + 4z - 3. && (61)
\end{aligned}$$

For $z \geq 3$, $f_{X+Y}(z) = 0$.

Problem 8.

$$
\begin{aligned}
p_{x+y}(z) &= \sum_{x \in X} P(X = x) P(Y = z - x) && (62)\\
&= \sum_{i=0}^{\infty} \left(\frac{\lambda^i}{i!} e^{-\lambda} \right) \left(\frac{\mu^{z-i}}{(z-i)!} e^{-\mu} \right) && (63)\\
&= e^{-(\lambda+\mu)} \sum_{i=0}^{\infty} \left(\frac{\lambda^i}{i!} \right) \left(\frac{\mu^{z-i}}{(z-i)!} \right) && (64)\\
&= e^{-(\lambda+\mu)} \sum_{i=0}^{\infty} \frac{1}{i!(z-i)!} (\lambda^i)(\mu^{z-i}) && (65)\\
&= e^{-(\lambda+\mu)} \frac{1}{z!} \sum_{i=0}^{\infty} \frac{z!}{i!(z-i)!} (\lambda^i)(\mu^{z-i}) && (66)\\
&= e^{-(\lambda+\mu)} \frac{1}{z!} \sum_{i=0}^{\infty} \binom{z}{i} (\lambda^i)(\mu^{z-i}) && (67)\\
&= \frac{(\lambda+\mu)^z}{z!} e^{-(\lambda+\mu)} && (68)\\
&= Poisson(\lambda + \mu). && (69)
\end{aligned}
$$

Problem 9. $X = \sigma_1 Z + \mu_1$ and $Y = \sigma_2 Z + \mu_2 \Rightarrow X + Y = (\sigma_1 + \sigma_2) Z + (\mu_1 + \mu_2) \Rightarrow N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$.

$$
\begin{aligned}
f_{X+Y}(z) &= \int_{-\infty}^{\infty} f_X(x) f_Y(z-x) dx && (70)\\
&= \int_{-\infty}^{\infty} \left(\frac{1}{\sigma_1 \sqrt{2\pi}} e^{-\frac{(x-\mu_1)^2}{2\sigma_1^2}} \right) \left(\frac{1}{\sigma_2 \sqrt{2\pi}} e^{-\frac{(z-x-\mu_2)^2}{2\sigma_2^2}} \right) dx && (71)\\
&= \frac{1}{\sigma_1 \sigma_2 \sqrt{2\pi}\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(x-\mu_1)^2}{2\sigma_1^2} - \frac{(z-x-\mu_2)^2}{2\sigma_2^2}} dx && (72)\\
&= \frac{1}{\sigma_1 \sigma_2 \sqrt{2\pi}\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{\sigma_2^2 (x-\mu_1)^2 + \sigma_1^2 (z-x-\mu_2)^2}{2\sigma_1^2 \sigma_2^2}} dx && (73)\\
&= \frac{1}{\sigma_1 \sigma_2 \sqrt{2\pi}\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{\sigma_2^2 (x^2 - 2x\mu_1 + \mu_1^2) + \sigma_1^2 (z^2 + x^2 + \mu_2^2 - 2zx + 2x\mu_2 - 2z\mu_2)}{2\sigma_1^2 \sigma_2^2}} dx && (74)\\
&= \frac{1}{\sigma_1 \sigma_2 \sqrt{2\pi}\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2(\sigma_1^2 + \sigma_2^2) - 2x(\sigma_1^2 (z-\mu_2) + \sigma_2^2 \mu_1) + \sigma_1^2 (z^2 + \mu_2^2 - 2z\mu_2) + \sigma_2^2 \mu_1^2}{2\sigma_1^2 \sigma_2^2}} dx. && (75)
\end{aligned}
$$

Let $\sigma_z = \sqrt{\sigma_1^2 + \sigma_2^2}$,

$$
f_{X+Y}(z) = \frac{1}{\sqrt{2\pi}\sigma_z} \frac{1}{\sqrt{2\pi}\frac{\sigma_1 \sigma_2}{\sigma_z}} \int_{-\infty}^{\infty} e^{-\frac{x^2 - 2x \frac{\sigma_1^2 (z-\mu_2) + \sigma_2^2 \mu_1}{\sigma_z^2} + \frac{\sigma_1^2 (z^2 + \mu_2^2 - 2z\mu_2) + \sigma_2^2 \mu_1^2}{\sigma_z^2}}{2 \frac{\sigma_1 \sigma_2}{\sigma_z}^2}} dx. \tag{76}
$$

Interlude:

$$
-\frac{x^2 - 2x\frac{\sigma_1^2(z-\mu_2)+\sigma_2^2\mu_1}{\sigma_z^2} + \frac{\sigma_1^2(z^2+\mu_2^2-2z\mu_2)+\sigma_2^2\mu_1^2}{\sigma_z^2}}{2\frac{\sigma_1\sigma_2}{\sigma_z}^2} \tag{77}
$$

$$
= -\frac{\left(x - \frac{\sigma_1^2(z-\mu_2)+\sigma_2^2\mu_1}{\sigma_z^2}\right)^2 - \left(\frac{\sigma_1^2(z-\mu_2)+\sigma_2^2\mu_1}{\sigma_z^2}\right)^2 + \frac{\sigma_1^2(z-\mu_2)^2+\sigma_2^2\mu_1^2}{\sigma_z^2}}{2\frac{\sigma_1\sigma_2}{\sigma_z}^2} \tag{78}
$$

$$
= -\left[\frac{\sigma_z^2(\sigma_1^2(z-\mu_2)^2+\sigma_2^2\mu_1^2)-(\sigma_1^2(z-\mu_2)+\sigma_2^2\mu_1)^2}{2\sigma_z^2(\sigma_1\sigma_2)^2}\right] - \left[\frac{\left(x - \frac{\sigma_1^2(z-\mu_2)+\sigma_2^2\mu_1}{\sigma_z^2}\right)^2}{2\left(\frac{\sigma_1\sigma_2}{\sigma_z}\right)^2}\right] \tag{79}
$$

$$
= -\left[\frac{(z-(\mu_1+\mu_2))^2}{2\sigma_z^2}\right] - \left[\frac{\left(x - \frac{\sigma_1^2(z-\mu_2)+\sigma_2^2\mu_1}{\sigma_z^2}\right)^2}{2\left(\frac{\sigma_1\sigma_2}{\sigma_z}\right)^2}\right] \tag{80}
$$

Hence,

$$
f_{X+Y}(z) = \frac{1}{\sqrt{2\pi}\sigma_z}\frac{1}{\sqrt{2\pi}\frac{\sigma_1\sigma_2}{\sigma_z}}\int_{-\infty}^{\infty} e^{-\left[\frac{(z-(\mu_1+\mu_2))^2}{2\sigma_z^2}\right]-\left[\frac{\left(x-\frac{\sigma_1^2(z-\mu_2)+\sigma_2^2\mu_1}{\sigma_z^2}\right)^2}{2\left(\frac{\sigma_1\sigma_2}{\sigma_z}\right)^2}\right]} dx \tag{81}
$$

$$
= \frac{1}{\sqrt{2\pi}\sigma_z}e^{-\left[\frac{(z-(\mu_1+\mu_2))^2}{2\sigma_z^2}\right]}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}\frac{\sigma_1\sigma_2}{\sigma_z}}e^{\left[\frac{-\left(x-\frac{\sigma_1^2(z-\mu_2)+\sigma_2^2\mu_1}{\sigma_z^2}\right)^2}{2\left(\frac{\sigma_1\sigma_2}{\sigma_z}\right)^2}\right]} dx \tag{82}
$$

$$
= \frac{1}{\sqrt{2\pi}\sigma_z}e^{-\left[\frac{(z-(\mu_1+\mu_2))^2}{2\sigma_z^2}\right]} \quad \text{(the right had side term evaluates to 1)} \tag{83}
$$

$$
= N(\mu_1+\mu_2, \sigma_1^2+\sigma_2^2). \tag{84}
$$