Math 128A HW3

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Section 2.5

Problem 2

Consider the function $f(x) = e^{6x} + 3(ln2)^2e^{2x} - (ln8)e^{4x} - (ln2)^3$. Use Newton's method with $p_0 = 0$ to approximate a zero of f. Generate terms until $|p_{n+1} - p_n| < 0.0002$. Construct the sequence $\{\hat{p}_n\}$. Is the convergence improved?

Solution.

$$\begin{aligned} f'(x) &= 6e^{6x} + 6(ln2)^2e^{2x} - 4(ln8)e^{4x} \\ g(x) &= x - \frac{f(x)}{f'(x)} \\ &= x - \frac{e^{6x} + 3(ln2)^2e^{2x} - (ln8)e^{4x} - (ln2)^3}{6e^{6x} + 6(ln2)^2e^{2x} - 4(ln8)e^{4x}}. \end{aligned}$$

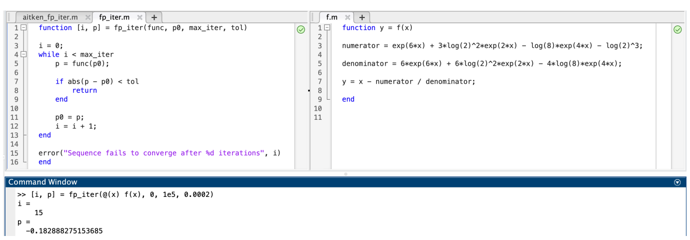


Figure 1: Regular Newton's method: iteartions = 15

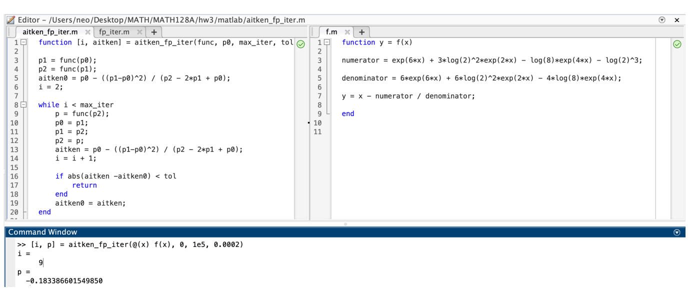


Figure 2: Newtons' method with Aitken's Δ^2 process: iterations = 9

The sequence converged with fewer iterations with Aitken's Δ^2 process. $\square$

Problem 4

Let $g(x) = 1 + (sinx)^2$ and $p_0^{(0)} = 1$. Use Steffensen's method to find $p_0^{(1)}$ and $p_0^{(2)}$.

Solution.

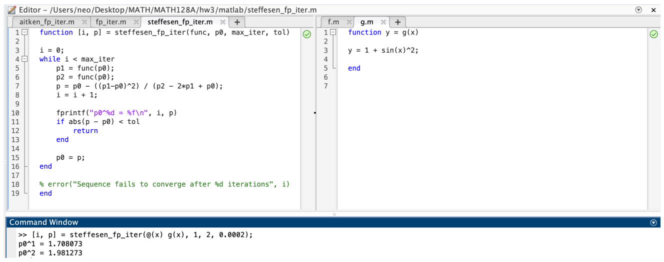


Figure 3: $p_0^{(1)} = 1.708, p_0^{(2)} = 1.981$

□

Problem 7

Use Steffensen's method to find, to an accuracy of 10^{-4}, the root of $x^3 - x - 1 = 0$ that lies in $[1, 2]$ and compare this to the results of Exercise 8 of Section 2.2.

Solution. Define $g(x) := (x + 1)^{1/3}$ as our fixed-point iteration function and use $p_0 = 1$.

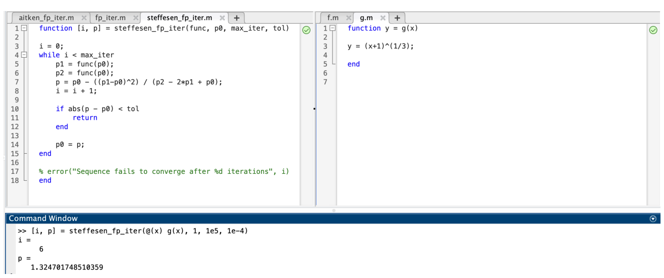


Figure 4: $p = 1.3247$, iterations $= 6$

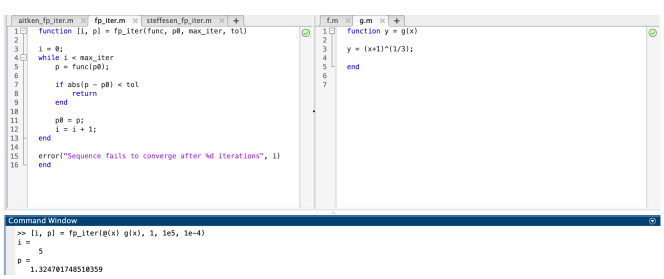


Figure 5: $p = 1.3247$, iterations $= 5$

We re-did Exercise 8 of Section 2.2 up to the same accuracy, but it took only 5 iterations. □

Problem 14

A sequence $\{p_n\}$ is said to be superlinearly convergent to p if

$$\lim_{n\to\infty} \frac{|p_{n+1}-p|}{|p_n-p|} = 0.$$

a. Show that if $p_n \to p$ of order α for $\alpha > 1$, then $\{p_n\}$ is superlinearly convergence to p.

Proof. We know

$$\begin{aligned}
\lim_{n\to\infty} \frac{|p_{n+1}-p|}{|p_n-p|^\alpha} &= C \\
\frac{\lim_{n\to\infty} |p_{n+1}-p|}{\lim_{n\to\infty} |p_n-p|^\alpha} &= C \\
\lim_{n\to\infty} |p_{n+1}-p| &= C \lim_{n\to\infty} |p_n-p|^\alpha.
\end{aligned}$$

Now, we are interested in evaluating

$$\begin{aligned}
\lim_{n\to\infty} \frac{|p_{n+1}-p|}{|p_n-p|} &= \frac{\lim_{n\to\infty} |p_{n+1}-p|}{\lim_{n\to\infty} |p_n-p|} \\
&= \frac{C \lim_{n\to\infty} |p_n-p|^\alpha}{\lim_{n\to\infty} |p_n-p|} \\
&= C \lim_{n\to\infty} |p_n-p|^{\alpha-1}.
\end{aligned}$$

Since $\lim_{n\to\infty} |p_n-p| = 0$ and $\alpha-1$ as a constant is larger than 0, $C \lim_{n\to\infty} |p_n-p|^{\alpha-1} = 0$. Hence, the limit is indeed 0 and $\{p_n\}$ is superlinearly convergent to p. □

b. Show that $p_n = \frac{1}{n^n}$ is superlinearly convergent to 0 but does not converge to 0 of order α for any $\alpha > 1$.

Proof. **Superlinear convergence:**

$$\begin{aligned}
\lim_{n\to\infty} \frac{|p_{n+1}-p|}{|p_n-p|} &= \lim_{n\to\infty} \frac{\frac{1}{(n+1)^{n+1}}}{\frac{1}{n^n}} \\
&= \lim_{n\to\infty} \frac{n^n}{(n+1)^{n+1}} \\
&= \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n \cdot \frac{1}{n+1} \\
&= 0.
\end{aligned}$$

Not of order α for any $\alpha > 1$: We want to evaluate

$$\begin{aligned}
\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|^\alpha} &= \lim_{n\to\infty} \frac{\frac{1}{(n+1)^{n+1}}}{\left(\frac{1}{n^n}\right)^\alpha} \\
&= \lim_{n\to\infty} \frac{n^{\alpha n}}{(n+1)^{n+1}} \\
&= \lim_{n\to\infty} \frac{n^{\alpha n}}{n^{n+1} + O(n^n)} \qquad (\textit{binomial expansion of denominator}) \\
&= \lim_{n\to\infty} \frac{n^{\alpha n - n - 1}}{1 + O\left(\frac{1}{n}\right)}. \qquad (1)
\end{aligned}$$

Now notice

$$\begin{aligned}
\alpha n - n - 1 &= n(\alpha - 1) - 1 \\
\lim_{n\to\infty} \alpha n - n - 1 &= \lim_{n\to\infty} n(\alpha - 1) - 1 \\
&= \infty. \qquad (\textit{if } \alpha > 1)
\end{aligned}$$

Therefore, the limit at (1) is ∞ and p_n is not of order α for any $\alpha > 1$. □

Problem 15

Suppose that $\{p_n\}$ is superlinearly convergent to p. Show that

$$\lim_{n\to\infty} \frac{|p_{n+1} - p_n|}{|p_n - p|} = 1.$$

Proof.

$$\begin{aligned}
\lim_{n\to\infty} \frac{|p_{n+1} - p_n|}{|p_n - p|} &\le \lim_{n\to\infty} \frac{|p_{n+1} - p| + |p - p_n|}{|p_n - p|} \\
&\le \lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|} + \lim_{n\to\infty} \frac{|p - p_n|}{|p_n - p|} \\
&\le 1.
\end{aligned}$$

At the same time,

$$\begin{aligned}
\lim_{n\to\infty} \frac{|p_{n+1} - p_n|}{|p_n - p|} &= \lim_{n\to\infty} \frac{|(p_{n+1} - p) - (p_n - p)|}{|p_n - p|} \\
&\ge \lim_{n\to\infty} \frac{||(p_{n+1} - p)| - |(p_n - p)||}{|p_n - p|} \\
&\ge \lim_{n\to\infty} \left|\frac{|(p_{n+1} - p)|}{|p_n - p|} - \frac{|(p_n - p)|}{|p_n - p|}\right| \\
&\ge \lim_{n\to\infty} \left|\frac{|(p_{n+1} - p)|}{|p_n - p|} - 1\right| \\
&\ge \left|\lim_{n\to\infty} \frac{|(p_{n+1} - p)|}{|p_n - p|} - 1\right| \\
&\ge 1.
\end{aligned}$$

Therefore,

$$
\begin{aligned}
1 \leq \lim_{n\to\infty} \frac{|p_{n+1} - p_n|}{|p_n - p|} \leq 1 \\
\Rightarrow \lim_{n\to\infty} \frac{|p_{n+1} - p_n|}{|p_n - p|} = 1.
\end{aligned}
$$

$\square$

Section 2.6

Problem 2be

Find approximations to within 10^{-5} to all the zeros of **b.**

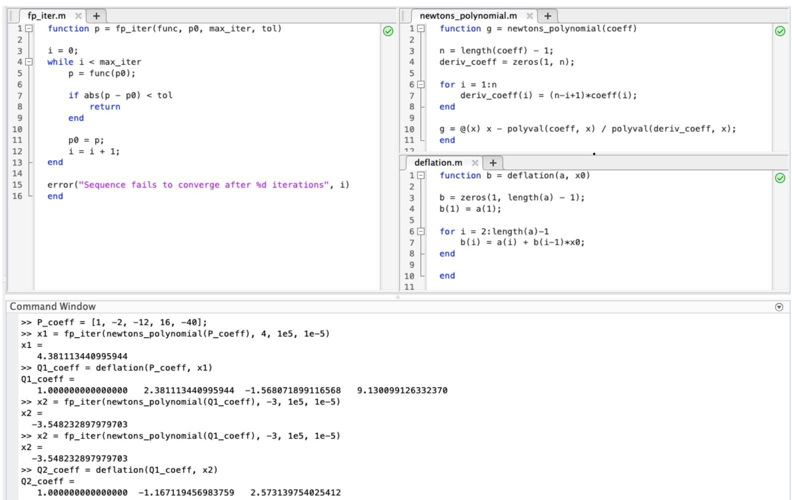
$$f(x) = x^4 - 2x^3 - 12x^2 + 16x - 40$$

and **e.**

$$f(x) = 16x^4 + 88x^3 + 159x^2 + 76x - 240$$

by first finding the real zeros using Newton's method and then reducing to polynomials of lower degree to determine any complex zeros.

b. *Solution.*



fp_iter.m

```
function p = fp_iter(func, p0, max_iter, tol)

i = 0;
while i < max_iter
    p = func(p0);

    if abs(p - p0) < tol
        return
    end

    p0 = p;
    i = i + 1;
end

error("Sequence fails to converge after %d iterations", i)
end
```

newtons_polynomial.m

```
function g = newtons_polynomial(coeff)

n = length(coeff) - 1;
deriv_coeff = zeros(1, n);

for i = 1:n
    deriv_coeff(i) = (n-i+1)*coeff(i);
end

g = @(x) x - polyval(coeff, x) / polyval(deriv_coeff, x);
end
```

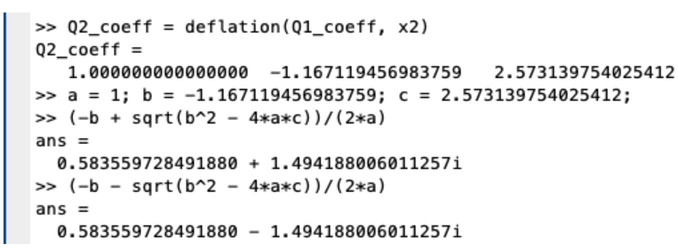
deflation.m

```
function b = deflation(a, x0)

b = zeros(1, length(a) - 1);
b(1) = a(1);

for i = 2:length(a)-1
    b(i) = a(i) + b(i-1)*x0;
end

end
```

Command Window

```
>> P_coeff = [1, -2, -12, 16, -40];
>> x1 = fp_iter(newtons_polynomial(P_coeff), 4, 1e5, 1e-5)
x1 =
   4.381113440995944
>> Q1_coeff = deflation(P_coeff, x1)
Q1_coeff =
   1.000000000000000   2.381113440995944  -1.568071899116568   9.130099126332370
>> x2 = fp_iter(newtons_polynomial(Q1_coeff), -3, 1e5, 1e-5)
x2 =
  -3.548232897979703
>> x2 = fp_iter(newtons_polynomial(Q1_coeff), -3, 1e5, 1e-5)
x2 =
  -3.548232897979703
>> Q2_coeff = deflation(Q1_coeff, x2)
Q2_coeff =
   1.000000000000000  -1.167119456983759   2.573139754025412
```

Figure 6: $x_1 \approx 4.38111, x_2 \approx -3.54823$



```
>> Q2_coeff = deflation(Q1_coeff, x2)
Q2_coeff =
   1.000000000000000  -1.167119456983759   2.573139754025412
>> a = 1; b = -1.167119456983759; c = 2.573139754025412;
>> (-b + sqrt(b^2 - 4*a*c))/(2*a)
ans =
  0.583559728491880 + 1.494188006011257i
>> (-b - sqrt(b^2 - 4*a*c))/(2*a)
ans =
  0.583559728491880 - 1.494188006011257i
```

Figure 7: $x_3 \approx 0.58356 + 1.49419i, x_4 \approx 0.58356 - 1.49419i$

Figure 6: We first use Newton's method on $f(x)$ to find x_1 with starting point at $p_0 = 4$, and we get $x_1 \approx 4.38111$. Then we use Horner's method to deflate the polynomial to

$$Q_1(x) \approx x^3 + 2.381113x^2 - 1.56807x + 9.1301.$$

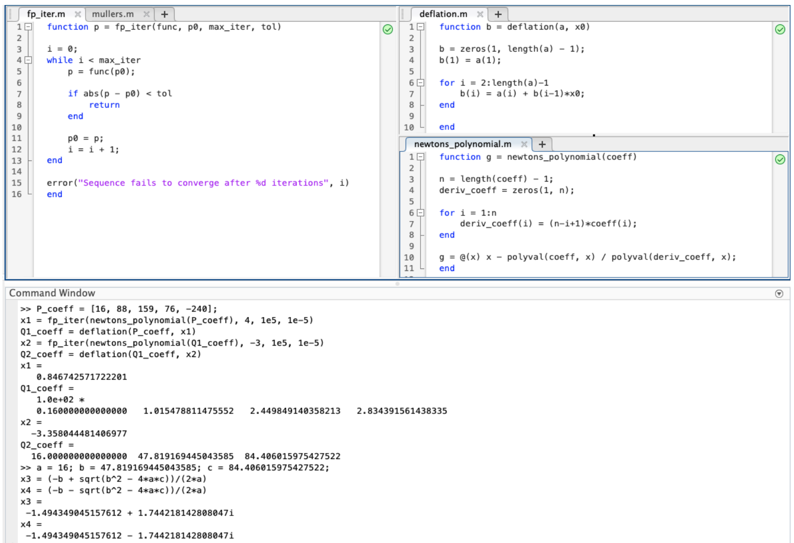
Then, we run Newton's method on $Q_1(x)$ to find x_2 with starting point at $p_0 = -3$, and we get

$x_2 \approx -3.54823$. Finally, we use Horner's method to deflate the polynomial to

$$Q_2(x) \approx x^2 - 1.16712x + 2.57314.$$

Figure 7: We solve $Q_2(x)$ by quadratic formula and get $x_3 \approx 0.58356 + 1.49419i$ and $x_4 \approx 0.58356 - 1.49419i$. ☐

e. *Solution.*



fp_iter.m | mullers.m

```
function p = fp_iter(func, p0, max_iter, tol)

i = 0;
while i < max_iter
    p = func(p0);

    if abs(p - p0) < tol
        return
    end

    p0 = p;
    i = i + 1;
end

error("Sequence fails to converge after %d iterations", i)
end
```

deflation.m

```
function b = deflation(a, x0)

b = zeros(1, length(a) - 1);
b(1) = a(1);

for i = 2:length(a)-1
    b(i) = a(i) + b(i-1)*x0;
end

end
```

newtons_polynomial.m

```
function g = newtons_polynomial(coeff)

n = length(coeff) - 1;
deriv_coeff = zeros(1, n);

for i = 1:n
    deriv_coeff(i) = (n-i+1)*coeff(i);
end

g = @(x) x - polyval(coeff, x) / polyval(deriv_coeff, x);
end
```

Command Window

```
>> P_coeff = [16, 88, 159, 76, -240];
x1 = fp_iter(newtons_polynomial(P_coeff), 4, 1e5, 1e-5)
Q1_coeff = deflation(P_coeff, x1)
x2 = fp_iter(newtons_polynomial(Q1_coeff), -3, 1e5, 1e-5)
Q2_coeff = deflation(Q1_coeff, x2)
x1 =
   0.846742571722201
Q1_coeff =
   1.0e+02 *
   0.160000000000000   1.015478811475552   2.449849140358213   2.834391561438335
x2 =
  -3.358044481406977
Q2_coeff =
  16.000000000000000  47.819169445043585  84.406015975427522
>> a = 16; b = 47.819169445043585; c = 84.406015975427522;
x3 = (-b + sqrt(b^2 - 4*a*c))/(2*a)
x4 = (-b - sqrt(b^2 - 4*a*c))/(2*a)
x3 =
 -1.494349045157612 + 1.744218142808047i
x4 =
 -1.494349045157612 - 1.744218142808047i
```

Figure 8: $x_1 \approx 0.84674, x_2 \approx -3.35804, x_3 \approx -1.49435 + 1.74422i, x_4 \approx -1.49435 - 1.74422i$

We followed the exact same procedure from **b.** to find the zeros of $f(x)$. ☐

Problem 4be

Repeat Exercise 2 using Muller's method.

b. *Solution.*

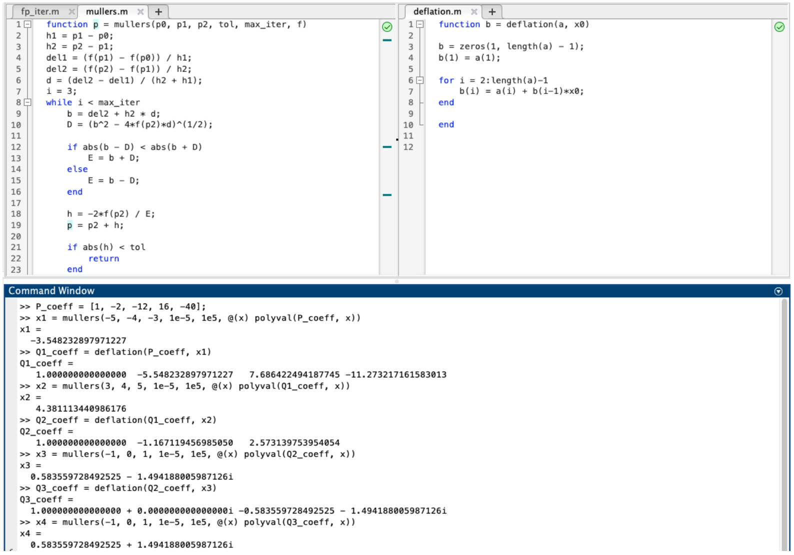


Figure 9: Pretty much same answer as Exercise 2b.

We use Muller's method to find the first root, then deflate the polynomial. We repeat applying Muller's method on the deflated polynomial until we find all 4 roots. □

e. *Solution.*

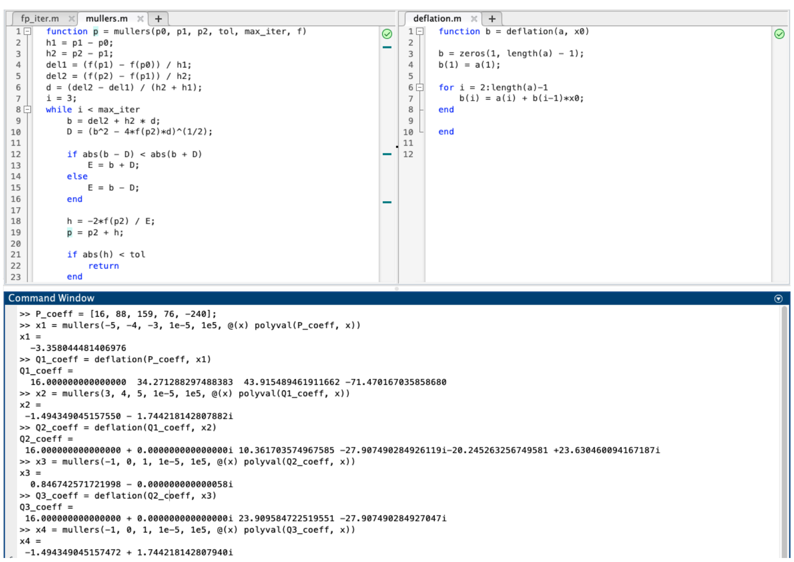


Figure 10: Pretty much same answer as Exercise 2e.

Same procedure as **b.** □

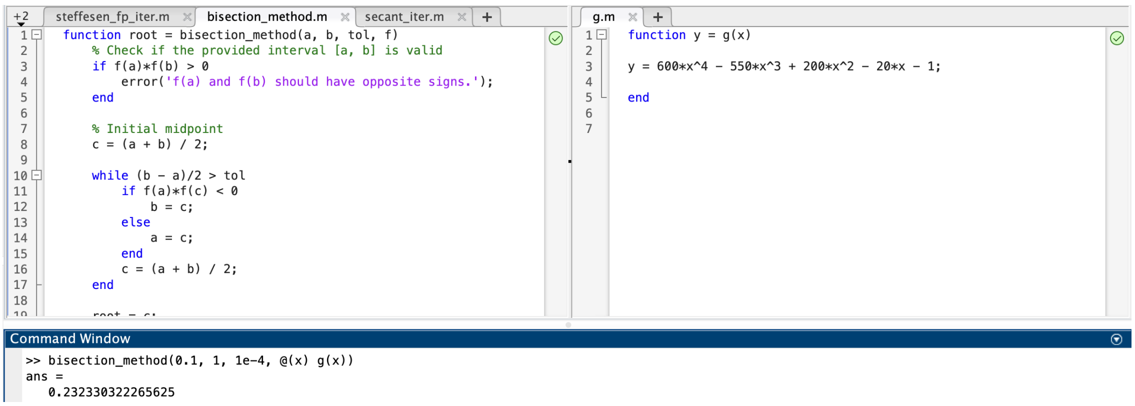
Problem 7abce

Use each of the following methods to find a solution in $[0.1, 1]$ accurate to within 10^{-4} for

$$600x^4 - 550x^3 + 200x^2 - 20x - 1 = 0.$$

a. Bisection method **b.** Newton's method **c.** Secant method **e.** Muller's method

a. *Solution.* □



```
function root = bisection_method(a, b, tol, f)
    % Check if the provided interval [a, b] is valid
    if f(a)*f(b) > 0
        error('f(a) and f(b) should have opposite signs.');
    end

    % Initial midpoint
    c = (a + b) / 2;

    while (b - a)/2 > tol
        if f(a)*f(c) < 0
            b = c;
        else
            a = c;
        end
        c = (a + b) / 2;
    end

    root = c;
```

```
function y = g(x)

y = 600*x^4 - 550*x^3 + 200*x^2 - 20*x - 1;

end
```

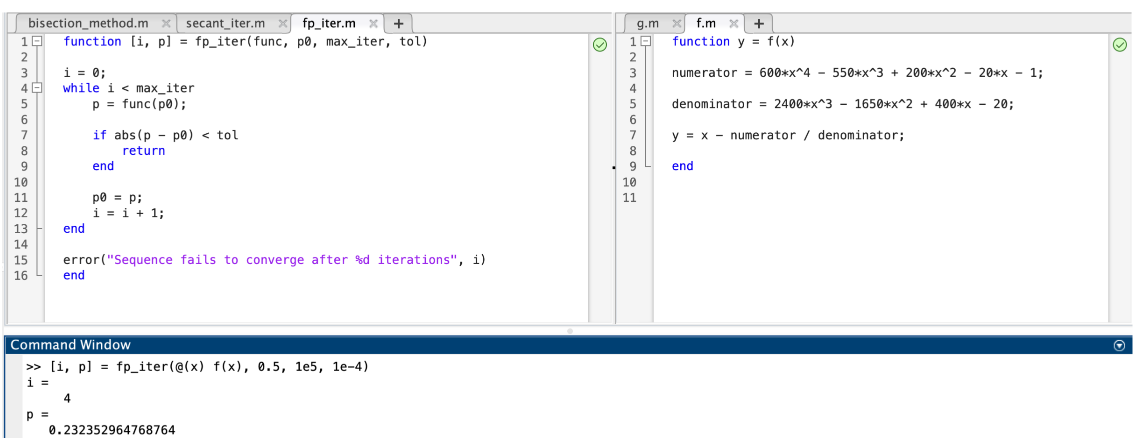
```
>> bisection_method(0.1, 1, 1e-4, @(x) g(x))
ans =
  0.232330322265625
```

Figure 11: Bisection method

b. *Solution.*

$$f'(x) = 2400x^3 - 1650x^2 + 400x - 20$$
$$g(x) = x - \frac{600x^4 - 550x^3 + 200x^2 - 20x - 1}{2400x^3 - 1650x^2 + 400x - 20}.$$

□



```
function [i, p] = fp_iter(func, p0, max_iter, tol)

i = 0;
while i < max_iter
    p = func(p0);

    if abs(p - p0) < tol
        return
    end

    p0 = p;
    i = i + 1;
end

error("Sequence fails to converge after %d iterations", i)
end
```

```
function y = f(x)

numerator = 600*x^4 - 550*x^3 + 200*x^2 - 20*x - 1;

denominator = 2400*x^3 - 1650*x^2 + 400*x - 20;

y = x - numerator / denominator;

end
```

```
>> [i, p] = fp_iter(@(x) f(x), 0.5, 1e5, 1e-4)
i =
     4
p =
  0.232352964768764
```

Figure 12: Newton's method

c. *Solution.* □

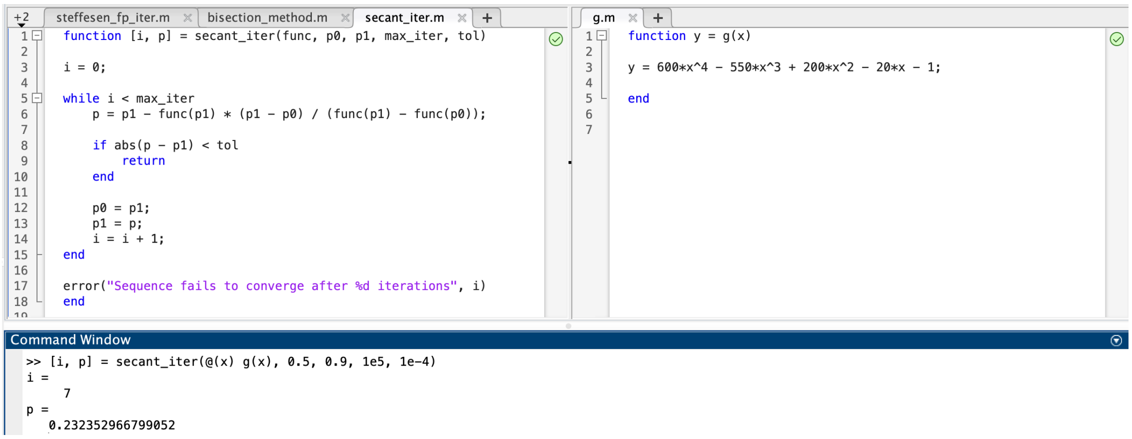


Figure 13: Secant method

e. *Solution.* □

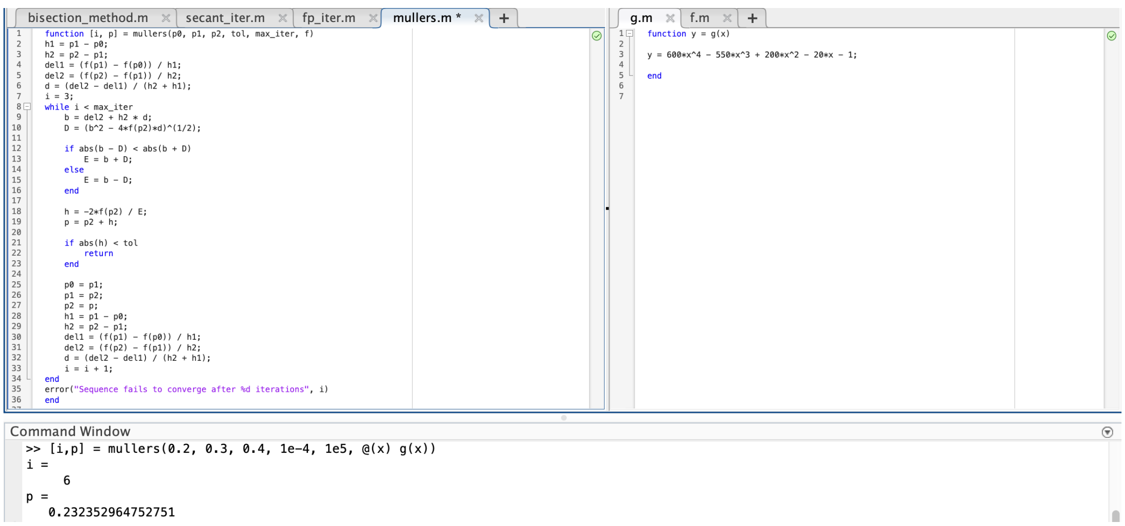


Figure 14: Muller's method

Problem 9

A can in the shape of a right circular cylinder is to be constructed to contain $1000cm^3$. The circular top and bottom of the can must have a radius of $0.25cm$ more than the radius of the can so that the excess can be used to form a seal with the side. The sheet of material being formed into the side of the can must also be $0.25cm$ longer than the circumference of the can so that a seal can be formed. Find, to within 10^{-4}, the minimal amount of material needed to construct the can.

Solution. We can formulate the problem as

$$\begin{aligned} \min_{r,h} \quad & f(r,h) = 2\pi(r+0.25)^2 + (2\pi r + 0.25)h \\ \text{s.t.} \quad & \pi r^2 h = 1000. \end{aligned}$$

Notice we can rewrite $f(r,h)$ as

$$\begin{aligned} f(r) &= 2\pi(r+0.25)^2 + (2\pi r + 0.25)\frac{1000}{\pi r^2} \\ &= 2\pi(r+0.25)^2 + \frac{2000}{r} \\ &= 2\pi r^2 + \pi r + 0.125\pi + \frac{2000}{r} + \frac{250}{\pi r^2}. \end{aligned}$$

Then, we set $f'(r) = 0$ to find the critical points that will minimize the function.

$$\begin{aligned} f'(r) = 4\pi r + \pi - \frac{2000}{r^2} - \frac{500}{\pi r^3} &= 0 \\ 4\pi r^4 + \pi r^3 - 2000r - \frac{500}{\pi} &= 0. \end{aligned}$$

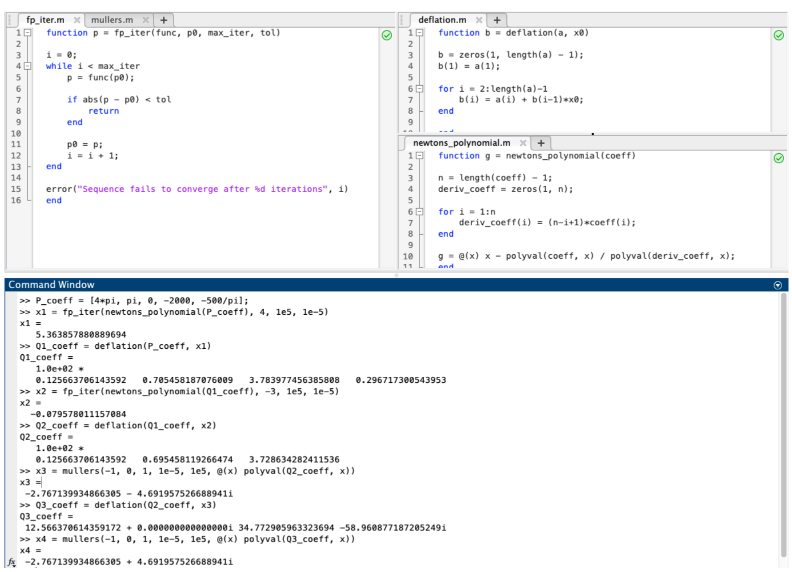


Figure 15: $r = x_1 \approx 5.36386$

The only reasonable root being a positive real numer is $r = x_1 \approx 5.36386$. With a few sanity check by pluggin in a few points near $r = 5.36386$ to $f(r)$, indeed $r = 5.36386$ is a local minimum.

$\square$