

Math 128A HW3

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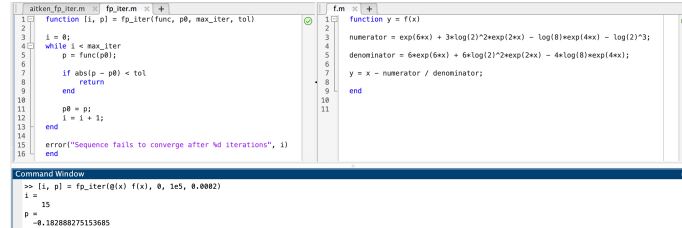
Section 2.5

Problem 2

Consider the function $f(x) = e^{6x} + 3(\ln 2)^2 e^{2x} - (\ln 8)e^{4x} - (\ln 2)^3$. Use Newton's method with $p_0 = 0$ to approximate a zero of f . Generate terms until $|p_{n+1} - p_n| < 0.0002$. Construct the sequence $\{\hat{p}_n\}$. Is the convergence improved?

Solution.

$$\begin{aligned} f'(x) &= 6e^{6x} + 6(\ln 2)^2 e^{2x} - 4(\ln 8)e^{4x} \\ g(x) &= x - \frac{f(x)}{f'(x)} \\ &= x - \frac{e^{6x} + 3(\ln 2)^2 e^{2x} - (\ln 8)e^{4x} - (\ln 2)^3}{6e^{6x} + 6(\ln 2)^2 e^{2x} - 4(\ln 8)e^{4x}}. \end{aligned}$$

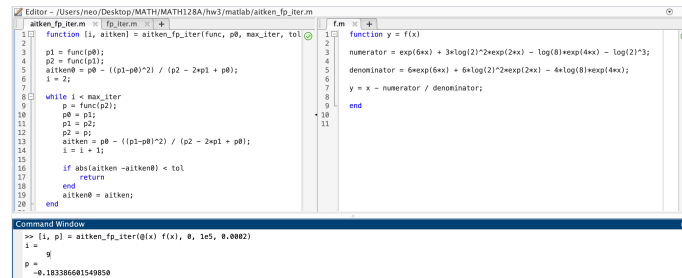


```
1 function fp_iterm = fp_iterm(func, p0, max_iter, tol)
2
3 l = 0;
4 while l < max_iter
5     p = func(p0);
6     if abs(p - p0) < tol
7         return
8     end
9     p0 = p;
10    l = l + 1;
11 end
12 error("Sequence fails to converge after %d iterations", l)
13 end
```

```
1 function y = f(x)
2
3 numerator = exp(6*x) + 3*log(2)^2*exp(2*x) - log(8)*exp(4*x) - log(2)^3;
4 denominator = 6*exp(6*x) + 6*log(2)^2*exp(2*x) - 4*log(8)*exp(4*x);
5 y = x - numerator / denominator;
6
7 end
```

```
>> [l, p] = fp_iterm(f, 0, 1e5, 0.0002)
l =
    15
p =
-0.182888275153685
```

Figure 1: Regular Newton's method: iterations = 15



```
1 function [l, aiken] = aiken_fp_iterm(func, p0, max_iter, tol)
2
3 p1 = func(p0);
4 p2 = func(p1);
5 aiken0 = p0 - ((p1-p0)^2) / (p2 - 2*p1 + p0);
6 l = 2;
7
8 while l < max_iter
9     p = func(p2);
10    p0 = p1;
11    p1 = p2;
12    p2 = p;
13    aiken = p0 - ((p1-p0)^2) / (p2 - 2*p1 + p0);
14    l = l + 1;
15
16    if abs(aiken - aiken0) < tol
17        return
18    end
19    aiken0 = aiken;
20 end
```

```
1 function y = f(x)
2
3 numerator = exp(6*x) + 3*log(2)^2*exp(2*x) - log(8)*exp(4*x) - log(2)^3;
4 denominator = 6*exp(6*x) + 6*log(2)^2*exp(2*x) - 4*log(8)*exp(4*x);
5 y = x - numerator / denominator;
6
7 end
```

```
>> [l, p] = aiken_fp_iterm(f, 0, 1e5, 0.0002)
l =
     9
p =
-0.183386681549858
```

Figure 2: Newton's method with Aitken's Δ^2 process: iterations = 9

The sequence converged with fewer iterations with Aitken's Δ^2 process.

□

Problem 4

Let $g(x) = 1 + (\sin x)^2$ and $p_0^{(0)} = 1$. Use Steffensen's method to find $p_0^{(1)}$ and $p_0^{(2)}$.

Solution.

```

function [i, p] = steffesen_fp_iter(func, p0, max_iter, tol)
1  i = 0;
2  while i < max_iter
3      p1 = func(p0);
4      p2 = func(p1);
5      p = p0 - (p1-p0)^2 / (p2 - 2*p1 + p0);
6      i = i + 1;
7      fprintf('p0^%d = %f\n', i, p);
8      if abs(p - p0) < tol
9          return
10     end
11     p0 = p;
12 end
13 % error('Sequence fails to converge after %d iterations', i)
14 end

```

```

>> [i, p] = steffesen_fp_iter(@g(x), 1, 2, 0.0002);
p0^1 = 1.708073
p0^2 = 1.981273

```

Figure 3: $p_0^{(1)} = 1.708, p_0^{(2)} = 1.981$

□

Problem 7

Use Steffensen's method to find, to an accuracy of 10^{-4} , the root of $x^3 - x - 1 = 0$ that lies in $[1, 2]$ and compare this to the results of Exercise 8 of Section 2.2.

Solution. Define $g(x) := (x + 1)^{1/3}$ as our fixed-point iteration function and use $p_0 = 1$.

```

function [i, p] = steffesen_fp_iter(func, p0, max_iter, tol)
1  i = 0;
2  while i < max_iter
3      p1 = func(p0);
4      p2 = func(p1);
5      p = p0 - (p1-p0)^2 / (p2 - 2*p1 + p0);
6      i = i + 1;
7      if abs(p - p0) < tol
8          return
9      end
10     p0 = p;
11 end
12 % error('Sequence fails to converge after %d iterations', i)
13 end

```

```

>> [i, p] = steffesen_fp_iter(@g(x), 1, 1e5, 1e-4)
i =
     6
p =
1.324781748518359

```

Figure 4: $p = 1.3247$, iterations = 6

```

function [i, p] = fp_iter(func, p0, max_iter, tol)
1  i = 0;
2  while i < max_iter
3      p = func(p0);
4      if abs(p - p0) < tol
5          return
6      end
7      p0 = p;
8      i = i + 1;
9  end
10 % error('Sequence fails to converge after %d iterations', i)
11 end

```

```

>> [i, p] = fp_iter(@g(x), 1, 1e5, 1e-4)
i =
     5
p =
1.324781748518359

```

Figure 5: $p = 1.3247$, iterations = 5

We re-did Exercise 8 of Section 2.2 up to the same accuracy, but it took only 5 iterations.

□

Problem 14

A sequence $\{p_n\}$ is said to be superlinearly convergent to p if

$$\lim_{n \rightarrow \infty} \frac{|p_{n+1} - p|}{|p_n - p|} = 0.$$

- a. Show that if $p_n \rightarrow p$ of order α for $\alpha > 1$, then $\{p_n\}$ is superlinearly convergent to p .

Proof.

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{|p_{n+1} - p|}{|p_n - p|} &= \lim_{n \rightarrow \infty} \frac{|p_{n+1} - p|}{|p_n - p|^\alpha} \cdot |p_n - p|^{\alpha-1} \\ &= \lim_{n \rightarrow \infty} \frac{|p_{n+1} - p|}{|p_n - p|^\alpha} \cdot \lim_{n \rightarrow \infty} |p_n - p|^{\alpha-1} \\ &= C \cdot 0 \\ &= 0. \end{aligned}$$

□

- b. Show that $p_n = \frac{1}{n^n}$ is superlinearly convergent to 0 but does not converge to 0 of order α for any $\alpha > 1$.

Proof. Superlinear convergence:

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{|p_{n+1} - p|}{|p_n - p|} &= \lim_{n \rightarrow \infty} \frac{\frac{1}{(n+1)^{n+1}}}{\frac{1}{n^n}} \\ &= \lim_{n \rightarrow \infty} \frac{n^n}{(n+1)^{n+1}} \\ &= \lim_{n \rightarrow \infty} \left(\frac{n}{n+1} \right)^n \cdot \frac{1}{n+1} \\ &= 0. \end{aligned}$$

Not of order α for any $\alpha > 1$: We want to evaluate

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{|p_{n+1} - p|}{|p_n - p|^\alpha} &= \lim_{n \rightarrow \infty} \frac{\frac{1}{(n+1)^{n+1}}}{\left(\frac{1}{n^n}\right)^\alpha} \\ &= \lim_{n \rightarrow \infty} \frac{n^{\alpha n}}{(n+1)^{n+1}} \\ &= \lim_{n \rightarrow \infty} \frac{n^{\alpha n}}{n^{n+1} + O(n^n)} \quad (\text{binomial expansion of denominator}) \\ &= \lim_{n \rightarrow \infty} \frac{n^{\alpha n - n - 1}}{1 + O\left(\frac{1}{n}\right)}. \end{aligned} \tag{1}$$

Now notice

$$\begin{aligned} \alpha n - n - 1 &= n(\alpha - 1) - 1 \\ \lim_{n \rightarrow \infty} \alpha n - n - 1 &= \lim_{n \rightarrow \infty} n(\alpha - 1) - 1 \\ &= \infty. \quad (\text{if } \alpha > 1) \end{aligned}$$

Therefore, the limit at (1) is ∞ and p_n is not of order α for any $\alpha > 1$. □

Problem 15

Suppose that $\{p_n\}$ is superlinearly convergent to p . Show that

$$\lim_{n \rightarrow \infty} \frac{|p_{n+1} - p_n|}{|p_n - p|} = 1.$$

Proof.

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{|p_{n+1} - p_n|}{|p_n - p|} &\leq \lim_{n \rightarrow \infty} \frac{|p_{n+1} - p| + |p - p_n|}{|p_n - p|} \\ &\leq \lim_{n \rightarrow \infty} \frac{|p_{n+1} - p|}{|p_n - p|} + \lim_{n \rightarrow \infty} \frac{|p - p_n|}{|p_n - p|} \\ &\leq 1. \end{aligned}$$

At the same time,

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{|p_{n+1} - p_n|}{|p_n - p|} &= \lim_{n \rightarrow \infty} \frac{|(p_{n+1} - p) - (p_n - p)|}{|p_n - p|} \\ &\geq \lim_{n \rightarrow \infty} \frac{||p_{n+1} - p| - |p_n - p||}{|p_n - p|} \\ &\geq \lim_{n \rightarrow \infty} \left| \frac{|p_{n+1} - p|}{|p_n - p|} - \frac{|p_n - p|}{|p_n - p|} \right| \\ &\geq \lim_{n \rightarrow \infty} \left| \frac{|p_{n+1} - p|}{|p_n - p|} - 1 \right| \\ &\geq \left| \lim_{n \rightarrow \infty} \frac{|p_{n+1} - p|}{|p_n - p|} - 1 \right| \\ &\geq 1. \end{aligned}$$

Therefore,

$$\begin{aligned} 1 &\leq \lim_{n \rightarrow \infty} \frac{|p_{n+1} - p_n|}{|p_n - p|} \leq 1 \\ \Rightarrow \lim_{n \rightarrow \infty} \frac{|p_{n+1} - p_n|}{|p_n - p|} &= 1. \end{aligned}$$

□

Section 2.6

Problem 2be

Find approximations to within 10^{-5} to all the zeros of **b**.

$$f(x) = x^4 - 2x^3 - 12x^2 + 16x - 40$$

and **e**.

$$f(x) = 16x^4 + 88x^3 + 159x^2 + 76x - 240$$

by first finding the real zeros using Newton's method and then reducing to polynomials of lower degree to determine any complex zeros.

b. Solution.

```

1 function p = fp_iter(func, p0, max_iter, tol)
2     i = 0;
3     while i < max_iter
4         p = func(p0);
5         if abs(p - p0) < tol
6             return p;
7         end
8         p0 = p;
9         i = i + 1;
10    end
11    error("Sequence fails to converge after %d iterations", i)
12 end

13 newtons_polynomial.m
14 function g = newtons_polynomial(coeff)
15     n = length(coeff) - 1;
16     deriv_coeff = zeros(1, n);
17     for i = 1:n
18         deriv_coeff(i) = (n-i+1)*coeff(i);
19     end
20     g = @(x) x - polyval(coeff, x) / polyval(deriv_coeff, x);
21 end

22 deflation.m
23 function b = deflation(a, x0)
24     b = zeros(1, length(a) - 1);
25     b(1) = a(1);
26     for i = 2:length(a)-1
27         b(i) = a(i) + b(i-1)*x0;
28     end
29 end

Command Window
>> P_coeff = [1, -2, -12, 16, -40];
>> x1 = fp_iter(newtons_polynomial(P_coeff), 4, 1e5, 1e-5)
x1 =
    4.381113440995944
>> Q1_coeff = deflation(P_coeff, x1)
Q1_coeff =
    1.000000000000000    2.381113440995944   -1.568071899116568    9.130899126332370
>> x2 = fp_iter(newtons_polynomial(Q1_coeff), -3, 1e5, 1e-5)
x2 =
   -3.548232897979703
>> x2 = fp_iter(newtons_polynomial(Q1_coeff), -3, 1e5, 1e-5)
x2 =
   -3.548232897979703
>> Q2_coeff = deflation(Q1_coeff, x2)
Q2_coeff =
    1.000000000000000   -1.167119456983759    2.573139754025412

```

Figure 6: $x_1 \approx 4.38111, x_2 \approx -3.54823$

```

>> Q2_coeff = deflation(Q1_coeff, x2)
Q2_coeff =
    1.000000000000000   -1.167119456983759    2.573139754025412
>> a = 1; b = -1.167119456983759; c = 2.573139754025412;
>> (-b + sqrt(b^2 - 4*a*c))/(2*a)
ans =
    0.583559728491880 + 1.494188006011257i
>> (-b - sqrt(b^2 - 4*a*c))/(2*a)
ans =
    0.583559728491880 - 1.494188006011257i

```

Figure 7: $x_3 \approx 0.58356 + 1.49419i, x_4 \approx 0.58356 - 1.49419i$

Figure 6: We first use Newton's method on $f(x)$ to find x_1 with starting point at $p_0 = 4$, and we get $x_1 \approx 4.38111$. Then we use Horner's method to deflate the polynomial to

$$Q_1(x) \approx x^3 + 2.381113x^2 - 1.56807x + 9.1301.$$

Then, we run Newton's method on $Q_1(x)$ to find x_2 with starting point at $p_0 = -3$, and we get

$x_2 \approx -3.54823$. Finally, we use Horner's method to deflate the polynomial to

$$Q_2(x) \approx x^2 - 1.16712x + 2.57314.$$

Figure 7: We solve $Q_2(x)$ by quadratic formula and get $x_3 \approx 0.58356 + 1.49419i$ and $x_4 \approx 0.58356 - 1.49419i$. □

e. *Solution.*

```

fp_iter.m
1 function p = fp_iter(func, p0, max_iter, tol)
2
3 i = 0;
4 while i < max_iter
5     p = func(p0);
6
7     if abs(p - p0) < tol
8         return
9     end
10
11     p0 = p;
12     i = i + 1;
13 end
14 error("Sequence fails to converge after %d iterations", i)
15
16
deflation.m
1 function b = deflation(a, x0)
2
3 b = zeros(1, length(a) - 1);
4 b(1) = a(1);
5
6 for i = 2:length(a)-1
7     b(i) = a(i) + b(i-1)*x0;
8 end
9
10
newtons_polynomial.m
1 function g = newtons_polynomial(coeff)
2
3 n = length(coeff) - 1;
4 deriv_coeff = zeros(1, n);
5
6 for i = 1:n
7     deriv_coeff(i) = (n-i+1)*coeff(i);
8 end
9
10 g = @(x) x - polyval(coeff, x) / polyval(deriv_coeff, x);
11 end

```

```

Command Window
>> P_coeff = [16, 88, 159, 76, -240];
x1 = fp_iter(newtons_polynomial(P_coeff), 4, 1e5, 1e-5)
Q1_coeff = deflation(P_coeff, x1)
x2 = fp_iter(newtons_polynomial(Q1_coeff), -3, 1e5, 1e-5)
Q2_coeff = deflation(Q1_coeff, x2)
x1 =
    0.846742571722201
Q1_coeff =
    1.0e+02 *
    0.160000000000000    1.015478811475552    2.449849140358213    2.834391561438335
x2 =
   -3.358044481406977
Q2_coeff =
   16.000000000000000   47.819169445043585   84.406015975427522
>> a = 16; b = 47.819169445043585; c = 84.406015975427522;
x3 = (-b + sqrt(b^2 - 4*a*c))/(2*a)
x4 = (-b - sqrt(b^2 - 4*a*c))/(2*a)
x3 =
   -1.494349845157612 + 1.744218142808047i
x4 =
   -1.494349845157612 - 1.744218142808047i

```

Figure 8: $x_1 \approx 0.84674$, $x_2 \approx -3.35804$, $x_3 \approx -1.49435 + 1.74422i$, $x_4 \approx -1.49435 - 1.74422i$

We followed the exact same procedure from **b.** to find the zeros of $f(x)$. □

Problem 4be

Repeat Exercise 2 using Muller's method.

b. *Solution.*

```

1 function p = mullers(p0, p1, p2, tol, max_iter, f)
2   h1 = p1 - p0;
3   h2 = p2 - p1;
4   del1 = (f(p1) - f(p0)) / h1;
5   del2 = (f(p2) - f(p1)) / h2;
6   d = (del2 - del1) / (h2 + h1);
7   i = 3;
8   while i < max_iter
9     b = del2 + h2 * d;
10    D = (b^2 - 4*f(p2)*d)^(1/2);
11
12    if abs(b - D) < abs(b + D)
13      E = b + D;
14    else
15      E = b - D;
16    end
17    h = -2*f(p2) / E;
18    p = p2 + h;
19    if abs(h) < tol
20      return
21    end
22  end
23
1 function b = deflation(a, x0)
2
3   b = zeros(1, length(a) - 1);
4   b(1) = a(1);
5
6   for i = 2:length(a)-1
7     b(i) = a(i) + b(i-1)*x0;
8   end
9
10  end
11
12

```

```

>> P_coeff = [1, -2, -12, 16, -40];
>> x1 = mullers(-5, -4, -3, 1e-5, 1e5, @(x) polyval(P_coeff, x))
x1 =
-3.548232897971227
>> Q1_coeff = deflation(P_coeff, x1)
Q1_coeff =
1.000000000000000 -5.548232897971227 7.686422494187745 -11.273217161583013
>> x2 = mullers(3, 4, 5, 1e-5, 1e5, @(x) polyval(Q1_coeff, x))
x2 =
4.38113448996176
>> Q2_coeff = deflation(Q1_coeff, x2)
Q2_coeff =
1.000000000000000 -1.167119456985050 2.573139753954054
>> x3 = mullers(-1, 0, 1, 1e-5, 1e5, @(x) polyval(Q2_coeff, x))
x3 =
0.583559728492525 -1.4941880059871261
>> Q3_coeff = deflation(Q2_coeff, x3)
Q3_coeff =
1.000000000000000 + 0.000000000000001i -0.583559728492525 - 1.4941880059871261i
>> x4 = mullers(-1, 0, 1, 1e-5, 1e5, @(x) polyval(Q3_coeff, x))
x4 =
0.583559728492525 + 1.4941880059871261i

```

Figure 9: Pretty much same answer as Exercise 2b.

We use Muller's method to find the first root, then deflate the polynomial. We repeat applying Muller's method on the deflated polynomial until we find all 4 roots. □

e. *Solution.*

```

1 function p = mullers(p0, p1, p2, tol, max_iter, f)
2   h1 = p1 - p0;
3   h2 = p2 - p1;
4   del1 = (f(p1) - f(p0)) / h1;
5   del2 = (f(p2) - f(p1)) / h2;
6   d = (del2 - del1) / (h2 + h1);
7   i = 3;
8   while i < max_iter
9     b = del2 + h2 * d;
10    D = (b^2 - 4*f(p2)*d)^(1/2);
11
12    if abs(b - D) < abs(b + D)
13      E = b + D;
14    else
15      E = b - D;
16    end
17    h = -2*f(p2) / E;
18    p = p2 + h;
19    if abs(h) < tol
20      return
21    end
22  end
23
1 function b = deflation(a, x0)
2
3   b = zeros(1, length(a) - 1);
4   b(1) = a(1);
5
6   for i = 2:length(a)-1
7     b(i) = a(i) + b(i-1)*x0;
8   end
9
10  end
11
12

```

```

>> P_coeff = [16, 88, 159, 76, -240];
>> x1 = mullers(-5, -4, -3, 1e-5, 1e5, @(x) polyval(P_coeff, x))
x1 =
-3.358044481406976
>> Q1_coeff = deflation(P_coeff, x1)
Q1_coeff =
16.000000000000000 34.271288297488383 43.915489461911662 -71.470167035858680
>> x2 = mullers(3, 4, 5, 1e-5, 1e5, @(x) polyval(Q1_coeff, x))
x2 =
-1.494349845157550 - 1.7442181428078921i
>> Q2_coeff = deflation(Q1_coeff, x2)
Q2_coeff =
16.000000000000000 + 0.000000000000001i 10.361703574967585 -27.9874982849261191-20.245263256749501i +23.6304600941671871i
>> x3 = mullers(-1, 0, 1, 1e-5, 1e5, @(x) polyval(Q2_coeff, x))
x3 =
0.846742571721998 - 0.000000000000001i
>> Q3_coeff = deflation(Q2_coeff, x3)
Q3_coeff =
16.000000000000000 + 0.000000000000001i 23.909584722519551 -27.9874982849270471i
>> x4 = mullers(-1, 0, 1, 1e-5, 1e5, @(x) polyval(Q3_coeff, x))
x4 =
-1.494349845157472 + 1.7442181428079401i

```

Figure 10: Pretty much same answer as Exercise 2e.

Same procedure as b. □

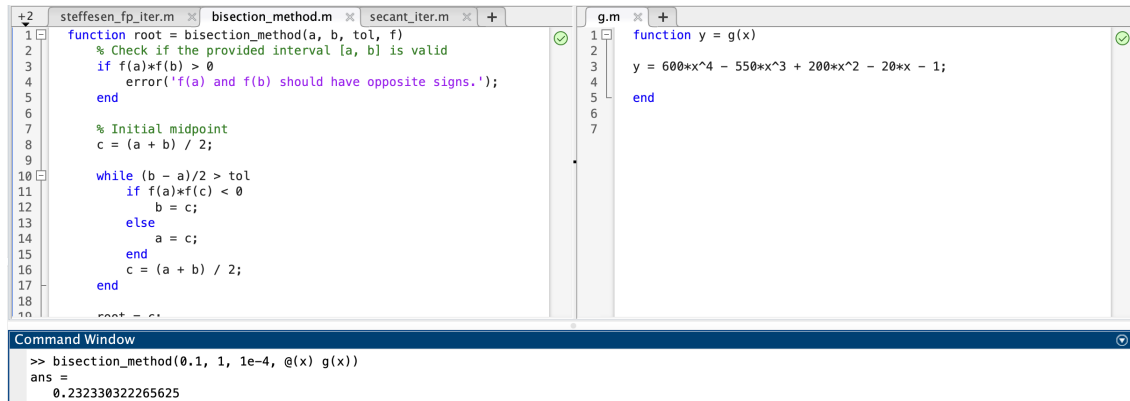
Problem 7abce

Use each of the following methods to find a solution in $[0.1, 1]$ accurate to within 10^{-4} for

$$600x^4 - 550x^3 + 200x^2 - 20x - 1 = 0.$$

a. Bisection method b. Newton's method c. Secant method e. Muller's method

a. *Solution.*



```

1 function root = bisection_method(a, b, tol, f)
2 % Check if the provided interval [a, b] is valid
3 if f(a)*f(b) > 0
4 error('f(a) and f(b) should have opposite signs.');
```

```

5 end
6
7 % Initial midpoint
8 c = (a + b) / 2;
9
10 while (b - a)/2 > tol
11 if f(a)*f(c) < 0
12 b = c;
13 else
14 a = c;
15 end
16 c = (a + b) / 2;
17 end
18 root = c;
19 end
```

```

1 function y = g(x)
2
3 y = 600*x^4 - 550*x^3 + 200*x^2 - 20*x - 1;
4
5 end
6
7
```

```

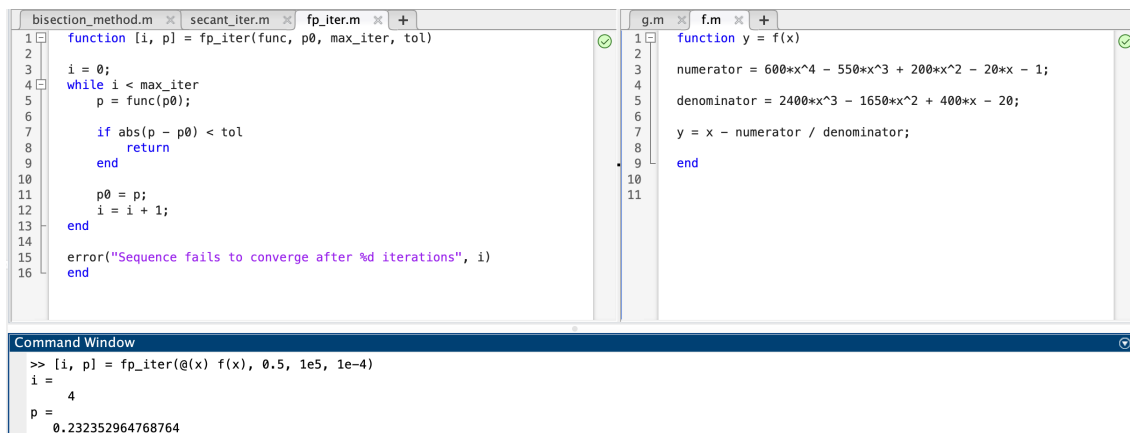
>> bisection_method(0.1, 1, 1e-4, @(x) g(x))
ans =
0.232330322265625
```

Figure 11: Bisection method

b. *Solution.*

$$f'(x) = 2400x^3 - 1650x^2 + 400x - 20$$

$$g(x) = x - \frac{600x^4 - 550x^3 + 200x^2 - 20x - 1}{2400x^3 - 1650x^2 + 400x - 20}.$$



```

1 function [i, p] = fp_iter(func, p0, max_iter, tol)
2
3 i = 0;
4 while i < max_iter
5 p = func(p0);
6
7 if abs(p - p0) < tol
8 return
9 end
10
11 p0 = p;
12 i = i + 1;
13 end
14
15 error('Sequence fails to converge after %d iterations', i)
16 end
```

```

1 function y = f(x)
2
3 numerator = 600*x^4 - 550*x^3 + 200*x^2 - 20*x - 1;
4 denominator = 2400*x^3 - 1650*x^2 + 400*x - 20;
5
6 y = x - numerator / denominator;
7
8 end
9
10
11
```

```

>> [i, p] = fp_iter(@(x) f(x), 0.5, 1e5, 1e-4)
i =
4
p =
0.232352964768764
```

Figure 12: Newton's method

c. Solution.

The figure shows a MATLAB editor with two files. The left file, `secant_iter.m`, contains the following code:

```

1 function [i, p] = secant_iter(func, p0, p1, max_iter, tol)
2
3     i = 0;
4
5     while i < max_iter
6         p = p1 - func(p1) * (p1 - p0) / (func(p1) - func(p0));
7
8         if abs(p - p1) < tol
9             return
10        end
11
12        p0 = p1;
13        p1 = p;
14        i = i + 1;
15    end
16
17    error("Sequence fails to converge after %d iterations", i)
18 end

```

The right file, `g.m`, contains the following code:

```

1 function y = g(x)
2
3     y = 600*x^4 - 550*x^3 + 200*x^2 - 20*x - 1;
4
5 end

```

The Command Window shows the execution of the secant method:

```

>> [i, p] = secant_iter(@(x) g(x), 0.5, 0.9, 1e5, 1e-4)
i =
    7
p =
    0.232352966799052

```

Figure 13: Secant method

e. Solution.

The figure shows a MATLAB editor with two files. The left file, `mullers.m`, contains the following code:

```

1 function [i, p] = mullers(p0, p1, p2, tol, max_iter, f)
2     h1 = p1 - p0;
3     h2 = p2 - p1;
4     del1 = (f(p1) - f(p0)) / h1;
5     del2 = (f(p2) - f(p1)) / h2;
6     d = (del2 - del1) / (h2 + h1);
7     i = 3;
8     while i < max_iter
9         b = del2 + h2 * d;
10        D = (b^2 - 4*f(p2)*d)^(1/2);
11
12        if abs(b - D) < abs(b + D)
13            E = b + D;
14        else
15            E = b - D;
16        end
17
18        h = -2*f(p2) / E;
19        p = p2 + h;
20
21        if abs(h) < tol
22            return
23        end
24
25        p0 = p1;
26        p1 = p2;
27        p2 = p;
28        h1 = p1 - p0;
29        h2 = p2 - p1;
30        del1 = (f(p1) - f(p0)) / h1;
31        del2 = (f(p2) - f(p1)) / h2;
32        d = (del2 - del1) / (h2 + h1);
33        i = i + 1;
34    end
35    error("Sequence fails to converge after %d iterations", i)
36 end

```

The right file, `f.m`, contains the following code:

```

1 function y = g(x)
2
3     y = 600*x^4 - 550*x^3 + 200*x^2 - 20*x - 1;
4
5 end

```

The Command Window shows the execution of Muller's method:

```

>> [i, p] = mullers(0.2, 0.3, 0.4, 1e-4, 1e5, @(x) g(x))
i =
    6
p =
    0.232352964752751

```

Figure 14: Muller's method

Problem 9

A can in the shape of a right circular cylinder is to be constructed to contain 1000cm^3 . The circular top and bottom of the can must have a radius of 0.25cm more than the radius of the can so that the excess can be used to form a seal with the side. The sheet of material being formed into the side of the can must also be 0.25cm longer than the circumference of the can so that a seal can be formed. Find, to within 10^{-4} , the minimal amount of material needed to construct the can.

Solution. We can formulate the problem as

$$\begin{aligned} \min_{r,h} \quad & f(r, h) = 2\pi(r + 0.25)^2 + (2\pi r + 0.25)h \\ \text{s.t.} \quad & \pi r^2 h = 1000. \end{aligned}$$

Notice we can rewrite $f(r, h)$ as

$$\begin{aligned} f(r) &= 2\pi(r + 0.25)^2 + (2\pi r + 0.25) \frac{1000}{\pi r^2} \\ &= 2\pi(r + 0.25)^2 + \frac{2000}{r} \\ &= 2\pi r^2 + \pi r + 0.125\pi + \frac{2000}{r} + \frac{250}{\pi r^2}. \end{aligned}$$

Then, we set $f'(r) = 0$ to find the critical points that will minimize the function.

$$\begin{aligned} f'(r) &= 4\pi r + \pi - \frac{2000}{r^2} - \frac{500}{\pi r^3} = 0 \\ 4\pi r^4 + \pi r^3 - 2000r - \frac{500}{\pi} &= 0. \end{aligned}$$

```

1 function p = fp_iter(func, p0, max_iter, tol)
2
3 i = 0;
4 while i < max_iter
5     p = func(p0);
6     if abs(p - p0) < tol
7         return
8     end
9     p0 = p;
10    i = i + 1;
11 end
12 error("Sequence fails to converge after %d iterations", i)
13 end
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