Math 109 Discussion 1

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Problem 1.

Proposition 1. $m, n \in \mathbb{Z}^+ \Rightarrow n^2 - m^2 \neq 1$.

Proof. Assume to the contrary that $n^2 - m^2 = 1$, then $(n+m)(n-m) = 1$. Thus, $n+m = 1$ and $n - m = 1$. Since $n, m \in \mathbb{Z}^+$, $n \geq 1$ and $m \geq 1$, $n + m \geq 2$, which contradicts $n + m = 1$. Hence, $m, n \in \mathbb{Z}^+ \Rightarrow n^2 - m^2 \neq 1$. $\square$

Problem 2.

Proposition 2. *Let n be an integer. If $n^2 - (n-2)^2$ is not divisible by 8, then n is even.*

Proof. The contrapositive of the statement is: if n is odd, $8|n^2 - (n-2)^2$. Since n is odd, it can be written as $n = 2k+1$ for some whole number k. Then $n^2 - (n-2)^2 \Rightarrow (2k+1)^2 - (2k+1-2)^2 \Rightarrow (4k^2+4k+1) - (4k^2-4k+1) \Rightarrow 8k^2 + 8k = 8(k^2+k)$, which is divisible by 8 for $k \in \mathbb{Z}^{\geq}$. Hence, if $n^2 - (n-2)^2$ is not divisible by 8, then n is even. $\square$

Problem 3.

Proposition 3. $\sqrt{2} + \sqrt{3}$ *is not rational.*

Proof. Primary proof: Assume to the contrary that $\sqrt{2}+\sqrt{3}$ is rational, $(\sqrt{2}+\sqrt{3})^2 = 2+2\sqrt{6}+3 = 5+2\sqrt{6}$ is rational $\Rightarrow 2\sqrt{6}$ is rational $\Rightarrow \sqrt{6}$ is rational.

Secondary proof: Now let's prove that $\sqrt{6}$ is irrational. Assume to the contrary that $\sqrt{6}$ is rational, which can be written as $\frac{a}{b}$ for $\{a, b \in \mathbb{Z} | gcd(a,b) = 1\}$.

$$b\sqrt{6} = a \tag{1}$$
$$6b^2 = a^2 \tag{2}$$

Thus a^2 is even, which means a is even and can be written as $a = 2k$ for $k \in \mathbb{Z}$. From (2), $6b^2 = a^2 \Rightarrow 6b^2 = (2k)^2 \Rightarrow 6b^2 = 4k^2 \Rightarrow 3b^2 = 2k^2 \Rightarrow b^2$ is even $\Rightarrow b$ is even, which contracts that $gcd(a,b) = 1$. Hence, $\sqrt{6}$ is irrational, which contradicts that $\sqrt{6}$ is rational is the primary proof. Thus, $\sqrt{2} + \sqrt{3}$ is not rational. $\square$