Math 109 HW3

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(1)

Proposition 1. *$n^2 + n$ is even for all $n \in \mathbb{N}$.*

Proof. Note that all natural numbers n, n is either even or odd. If n is even, it can be written as $n = 2k$ for some positive integer k. If n is odd, it can be written as $n = 2k + 1$ for some whole number k.

Let n be even, $n^2 + n = (2k)^2 + 2k = 4k^2 + 2k = 2(2k^2 + k)$, which is divisible by 2. Therefore, $n^2 + n$ is even if n is even.

Let n be odd, $n^2 + n = (2k + 1)^2 + (2k + 1) = (4k^2 + 4k + 1) + (2k + 1) = 2(2k^2 + 3k + 1)$, which is also divisible by 2. Therefore, $n^2 + n$ is even if n is odd.

Hence, $n^2 + n$ is always even for $n \in \mathbb{N}$. $\square$

(2)

Proposition 2. *$\sqrt{xy} \leq \frac{x+y}{2}$ if $x, y \geq 0$ are real numbers.*

Proof.

$$\begin{aligned}
\sqrt{xy} \leq \frac{x+y}{2} &\Leftarrow 2\sqrt{xy} \leq x + y && (1)\\
&\Leftarrow 4xy \leq (x + y)^2 && (2)\\
&\Leftarrow 4xy \leq x^2 + 2xy + y^2 && (3)\\
&\Leftarrow 0 \leq x^2 - 2xy + y^2 && (4)\\
&\Leftarrow 0 \leq (x - y)^2 && (5)\\
&\Leftarrow 0 \leq x, y \ (x, y \in \mathbb{R}) && (6)
\end{aligned}$$

Hence, for all real numbers $x, y \geq 0 \Rightarrow (x - y)^2 \geq 0 \Rightarrow \sqrt{xy} \leq \frac{x+y}{2}$. $\square$

(3)

Proposition 3. *For all real numbers $x > 2$, $\frac{x+1}{x-1} < \frac{x+2}{x-2}$.*

Proof.

$$\begin{aligned}
\frac{x+1}{x-1} < \frac{x+2}{x-2} &\Leftarrow (x - 2)(x + 1) < (x - 1)(x + 2) && (7)\\
&\Leftarrow x^2 - x - 2 < x^2 + x - 2 && (8)\\
&\Leftarrow 0 < 2x && (9)\\
&\Leftarrow 4 < 2x && (10)\\
&\Leftarrow 2 < x \ (x \in \mathbb{R}) && (11)
\end{aligned}$$

$\square$

(4)

Proposition 4. *Let $n \geq 2$ be a natural number. Let k be the maximum integer such that $2^k \leq n$. Among the numbers $1, ..., n$, the number 2^k is the only one which is divisible by 2^k.*

Proof. Assume to the contrary that other than 2^k, there exists i such that $2^k | i$, for which $1 \leq i \leq n$ and $i \in \mathbb{N}$. Since $2^k | i$, i can be written as $i = 2^k \cdot b = 2^k \cdot (2 + b - 2) = 2^{k+1} + (b-2)2^k$ for some positive integer $b \geq 2$. Note that $b \geq 2 \Rightarrow b - 2 \geq 0 \Rightarrow i = 2^{k+1} + (b-2)2^k \geq 2^{k+1}$, which contradicts that k is the maximum integer such that $2^k \leq n$. Hence, 2^k is the only number that is divisible by 2^k within [1, n].

The claim would not be true if 2^k is replaced by 3^k. For example, for $n = 26$, the greatest k such that $3^k \leq n$ is 2. In this example, 18 is divisible by $3^2 = 9$. □

(5)

Proposition 5. $\sum_{k=1}^{2^n} \frac{1}{k} \geq 1 + \frac{n}{2}$ *for* $n \in \mathbb{N}$.

Proof. Proving $P(n) : \sum_{k=1}^{2^n} \frac{1}{k} \geq 1 + \frac{n}{2}$ for $n \in \mathbb{N}$ by induction.

Base case:

$$P(1) : \sum_{k=1}^{2^1} \frac{1}{k} = \frac{1}{1} + \frac{1}{2} \tag{12}$$

$$= \frac{3}{2} \tag{13}$$

$$\geq 1 + \frac{1}{2}. \tag{14}$$

Thus, $P(n)$ is true for $n = 1$.

Induction step: assuming $P(m)$ is true for $n = m$,

$$P(m+1) : \sum_{k=1}^{2^{m+1}} \frac{1}{k} = \sum_{k=1}^{2^m} \frac{1}{k} + \sum_{k=2^m+1}^{2^{m+1}} \frac{1}{k} \tag{15}$$

$$\geq 1 + \frac{m}{2} + (2^{m+1} - (2^m + 1) + 1)\frac{1}{2^{m+1}} \tag{16}$$

$$\geq 1 + \frac{m}{2} + (2^{m+1} - \frac{2^{m+1}}{2})\frac{1}{2^{m+1}} \tag{17}$$

$$\geq 1 + \frac{m}{2} + (\frac{1}{2} \cdot 2^{m+1})\frac{1}{2^{m+1}} \tag{18}$$

$$\geq 1 + \frac{m}{2} + \frac{1}{2} \tag{19}$$

$$\geq 1 + \frac{m+1}{2}. \tag{20}$$

Therefore, $P(m+1)$ is true.

By Mathematical Induction, $P(n)$ is true for $n \in \mathbb{N}$. □

(6)

Proposition 6. $3 | 4^n + 5$ *for* $n \in \mathbb{Z}^+$.

Proof. Proving $P(n) : 3|4^n + 5$ for $n \in \mathbb{Z}^+$ by induction.
Base case:

$$\begin{aligned} P(1) : 4^1 + 5 &= 9 && (21)\\ &= 3 \cdot 3. && (22) \end{aligned}$$

Hence, $4^1 + 5$ is divisible by 3 and $P(n)$ is true for $n = 1$.
Induction step: assuming $P(m)$ is true for $n = m$, which means $4^m + 5 = 3 \cdot b$ for $b \in \mathbb{Z}^+$,

$$\begin{aligned} P(m+1) : 4^{m+1} + 5 &= 4 \cdot 4^m + 5 && (23)\\ &= 3 \cdot 4^m + 4^m + 5 && (24)\\ &= 3 \cdot 4^m + 3 \cdot b && (25)\\ &= 3(4^m + b). && (26) \end{aligned}$$

Since $4^m + b$ is an integer, $4^{m+1} + 5$ is divisibleb by 3 and $P(m+1)$ is true.
By Mathematical Induction, $P(n)$ is true for $n \in \mathbb{Z}^+$. ☐

(7)

Proposition 7. $\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$ *for* $n \in \mathbb{Z}^+$.

Proof. Proving $P(n) : \sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$ for $n \in \mathbb{Z}^+$ by induction.
Base case:

$$\begin{aligned} P(1) : \sum_{i=1}^{1} \frac{1}{i(i+1)} &= \frac{1}{1(1+1)} && (27)\\ &= \frac{1}{2} && (28)\\ &= \frac{1}{1+1}. && (29) \end{aligned}$$

Hence, $P(n)$ is true for $n = 1$.
Induction step: assuming $P(m)$ is true for $n = m$,

$$\begin{aligned} P(m+1) : \sum_{i=1}^{m+1} \frac{1}{i(i+1)} &= \sum_{i=1}^{m} \frac{1}{i(i+1)} + \frac{1}{(m+1)(m+2)} && (30)\\ &= \frac{m}{m+1} + \frac{1}{(m+1)(m+2)} && (31)\\ &= \frac{m(m+2)+1}{(m+1)(m+2)} && (32)\\ &= \frac{m^2+2m+1}{(m+1)(m+2)} && (33)\\ &= \frac{(m+1)^2}{(m+1)(m+2)} && (34)\\ &= \frac{m+1}{m+2} && (35)\\ &= \frac{m+1}{(m+1)+1}. && (36) \end{aligned}$$

Thus, $P(m+1)$ is true.
By Mathematical Induction, $P(n)$ is true for $n \in \mathbb{Z}^+$. ☐

(8)

Proposition 8. $\Pi_{i=2}^{n}\left(1-\frac{1}{i^2}\right)=\frac{n+1}{2n}$ *for integers* $n \geq 2$.

Proof. Proving $P(n) : \Pi_{i=2}^{n}\left(1-\frac{1}{i^2}\right)$ for integers $n \geq 2$ by induction.
Base case:

$$P(2) : \Pi_{i=2}^{2}\left(1-\frac{1}{i^2}\right) = 1-\frac{1}{2^2} \tag{37}$$
$$= 1-\frac{1}{4} \tag{38}$$
$$= \frac{3}{4} \tag{39}$$
$$= \frac{2+1}{2\cdot 2}. \tag{40}$$

Hence, $P(n)$ is true for $n = 2$.
Induction step: assuming $P(m)$ is true for $n = m$,

$$P(m+1) : \Pi_{i=2}^{m+1}\left(1-\frac{1}{i^2}\right) = \Pi_{i=2}^{m}\left(1-\frac{1}{i^2}\right)\cdot\left(1-\frac{1}{(m+1)^2}\right) \tag{41}$$
$$= \frac{m+1}{2m}\cdot\left(1-\frac{1}{(m+1)^2}\right) \tag{42}$$
$$= \frac{m+1}{2m}-\frac{1}{(2m)(m+1)} \tag{43}$$
$$= \frac{(m+1)^2-1}{(2m)(m+1)} \tag{44}$$
$$= \frac{m^2+2m}{(2m)(m+1)} \tag{45}$$
$$= \frac{m(m+2)}{(2m)(m+1)} \tag{46}$$
$$= \frac{m+2}{2m+2} \tag{47}$$
$$= \frac{(m+1)+1}{2(m+1)}. \tag{48}$$

Thus, $P(m+1)$ is true.
By Mathematical Induction, $P(n)$ is true for interger $n \geq 2$. □