# Math 104 HW5

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## Exercise 11.3

Consider the sequences

$$s_n = \cos\left(\frac{n\pi}{3}\right), \qquad t_n = \frac{3}{4n+1}, \qquad u_n = \left(-\frac{1}{2}\right)^n, \qquad v_n = (-1)^n + \frac{1}{n}.$$

(a) For each sequence, give an example of a monotone subsequence.

*Solution.*

($s_n$): Consider $n_k = 6k$ for $k \in \mathbb{N}$, then $s_{n_k} = \cos\left(\frac{6k\pi}{3}\right) = \cos(2k\pi) = 1$ for all $n_k$, which is indeed monotone [a constant sequence is monotone].

($t_n$): Consider $n_k = 2k$ for $k \in \mathbb{N}$, then $t_{n_k} = \frac{3}{8k+1}$ for all $n_k$, which is apparently monotonically decreasing.

($u_n$): Consider $n_k = 2k$ for $k \in \mathbb{N}$, then $u_{n_k} = \left(-\frac{1}{2}\right)^{2k} = \frac{1}{4^k}$ for all $n_k$, which is apparently monotonically decreasing.

($v_n$): Consider $n_k = 2k$ for $k \in \mathbb{N}$, then $v_{n_k} = (-1)^{2k} + \frac{1}{2k} = 1 + \frac{1}{2k}$ for all $n_k$, which is apparently monotonically decreasing.

□

(b) For each sequence, give its set of subsequential limits.

*Solution.*

($s_n$): $\{1, 0.5, -0.5, -1\}$. The values of $s_n$ oscillates among constant values $\{1, 0.5, -0.5, -1\}$. We can construct the constant subsequences, which have the limits $1, 0.5, -0.5, -1$ respectively. Then consider any $x \notin \{1, 0.5, -0.5, -1\}$, any subsequence will have a minimum non-zero distance from $x$, hence unable to converge to $x$. Therefore, the set of subsequential limits is $\{1, 0.5, -0.5, -1\}$.

($t_n$): $\{0\}$. $\lim t_n = 0$ by Theorem 9.3 - 9.6, hence the set of subsequential limits only contains $\lim t_n = 0$

($u_n$): $\{0\}$. $\lim v_n = 0$ by Theorem 9.7, hence the set of subsequential limits only contains $\lim v_n = 0$

($v_n$): $\{1, -1\}$. Take only even $n$, then the subsequence converges to 1. Take only odd $n$, then the subsequence converges to $-1$. Then consider any subsequence with finite odd $n$, it will converge to 1, just like the only even $n$ subsequence because we can take $N$ larger than the finite odd $n$. Similarly, any subsequence with finite even $n$ will converge to $-1$. Finally, any subsequence with infinite odd and even $n$ do not converge because it will always have elements within the neighborhood of 1 and $-1$.

$\square$

(c) For each sequence, give its lim sup and lim inf.

*Solution.* Notice $\limsup x_n = \sup S$ and $\liminf x_n = \inf S$, where $x_n$ is any arbitrary sequence.

$(s_n)$: $\limsup s_n = 1, \liminf s_n = -1$.

$(t_n)$: $\limsup t_n = \liminf t_n = 0$.

$(u_n)$: $\limsup u_n = \liminf u_n = 0$.

$(v_n)$: $\limsup v_n = 1, \liminf v_n = -1$.

$\square$

(d) Which of the sequences converges? diverges to $+\infty$? diverges to $-\infty$?

*Solution.*

$(s_n)$: Diverges. $\limsup s_n = 1, \liminf s_n = -1$, hence $s_n$ diverges.

$(t_n)$: Converges. $\lim t_n = 0$, hence $t_n$ converges.

$(u_n)$: Converges. $\lim u_n = 0$, hence $u_n$ converges.

$(v_n)$: Diverges. $\limsup v_n = 1, \liminf v_n = -1$, hence $v_n$ diverges.

$\square$

(e) Which of the sequences is bounded?

*Solution.*

$(s_n)$: Bounded. $|s_n| \leq 1$ for all $n$, hence $s_n$ is bounded.

$(t_n)$: Bounded. $|t_n| \leq \frac{3}{4}$ for all $n$, hence $t_n$ is bounded.

$(u_n)$: Bounded. $|u_n| \leq \frac{1}{2}$ for all $n$, hence $u_n$ is bounded.

$(v_n)$: Unbounded. $|v_n| \leq 2$ for all $n$, hence $v_n$ is bounded.

$\square$

## Exercise 11.6

**Proposition 1.** *Every subsequence of a subseuqnce of a given sequence is itself a subsequence of the given sequence.*

*Proof.* Let $(s_n)$ be the original sequence, $(t_k)$ be a subsequence of $(s_n)$, and $(u_m)$ be a subsequence of $(t_k)$. We define $\sigma(m)$ as a function that maps $m$ to some $k$, the index of $t_k$ in order, and $\tau(k)$ as a function that maps $k$ to some $n$, the index of $s_n$ in order. Then we have

$$u_m = t \circ \sigma(m) = s \circ \tau \circ \sigma(m) = s \circ \rho(m),$$

where $\rho$ is the composite of $\sigma$ and $\tau$ that maps from $m$ to $n$, which preserves the order of $s_n$. Hence $(u_m)$ is a subsequence of $(s_n)$. $\square$

## Exercise 11.10

Let $(s_n)$ be the sequence of numbers in Fig. 11.2 listed in the indicated order.

(a) Find the set $S$ of the subsequential limits of $s_n$.

*Solution.* Take each column of the matrix as a subsequence, they are all constant subsequences with values $\frac{1}{n}$ for some $n \in \mathbb{N}$. Also, take the first row as a subsequence, which converges to 0. Therefore, $\{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\} \subseteq S$. On the other hand, consider any $x$ not in $\{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$, notice any subsequence will have a minimum non-zero distance from $x$, hence unable to converge to $x$. Therefore, $S \subseteq \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$, and $S = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$. $\square$

(b) Determine $\limsup s_n$ and $\liminf s_n$.

*Solution.* $\limsup s_n = \max S = 1$ and $\liminf s_n = \min S = 0$. $\square$

## Exercise 12.4

**Proposition 2.** $\limsup(s_n + t_n) \le \limsup s_n + \limsup t_n$ *for bounded sequences* $(s_n)$ *and* $(t_n)$.

*Proof.* We first show that

$$\sup\{s_n + t_n : n > N\} \le \sup\{s_n : n > N\} + \sup\{t_n : n > N\}. \tag{1}$$

Let $N$ be an arbitrary natural number, then for any $n > N$, we have

$$s_n \le \sup\{s_n : n > N\} \qquad \text{and} \qquad t_n \le \sup\{t_n : n > N\},$$

hence

$$s_n + t_n \le \sup\{s_n : n > N\} + \sup\{t_n : n > N\},$$

so $\sup\{s_n : n > N\}+\sup\{t_n : n > N\}$ is an upper bound for $\{s_n+t_n : n > N\}$. Hence, the least upper bound of $\{s_n + t_n : n > N\}$, aka $\sup\{s_n + t_n : n > N\}$, is less than or equal to $\sup\{s_n : n > N\} + \sup\{t_n : n > N\}$.

Next we show that if $a_n \le b_n$, then $\lim a_n \le \lim b_n$. Assume for the sake of contradiction that $\lim a_n > \lim b_n$. Let $\lim a_n = a$ and $\lim b_n = b$, then we can write $a = b + 2\epsilon$ for some $\epsilon > 0$. By the definition of limit, we know there exists $N_1$ such that for all $n > N_1$, $|a_n - a| < \epsilon$ and also exists $N_2$ such that for $n > N_2$, $|b_n - b| < \epsilon$. Then for all $n > \max\{N_1, N_2\}$, $a_n$ is within $\epsilon$ of $a$ and $b_n$ is within $\epsilon$ of $b$, hence $a_n > a - \epsilon = b + \epsilon > b_n$, which contradicts the assumption that $a_n \le b_n$.

Combining the two results, (1) implies

$$\limsup(s_n + t_n) \le \lim(\sup s_n + \sup t_n) = \limsup s_n + \limsup t_n,$$

since $\limsup s_n$ and $\limsup t_n$ are finite [bounded and monotone]. $\square$