

Math 104 Practice

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Chapter 14

Proposition 1. $\sum \frac{n^4}{2^n}$ converges.

Proof. We proceed with Ratio Test.

$$\begin{aligned}\lim \left| \frac{(n+1)^4}{2^{n+1}} \cdot \frac{2^n}{n^4} \right| &= \lim \frac{(n+1)^4}{2n^4} \\ &= \lim \frac{n^4 + O(n^3)}{2n^4} \\ &= \frac{1}{2} < 1.\end{aligned}$$

□

Proposition 2. $\sum \frac{2^n}{n!}$ converges.

Proof. We proceed with Ratio Test.

$$\begin{aligned}\lim \left| \frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n} \right| &= \lim \frac{2}{n+1} \\ &= 0 < 1.\end{aligned}$$

□

Proposition 3. $\sum \frac{n!}{n^4+3}$ diverges.

Proof. We proceed with Ratio Test.

$$\begin{aligned}\lim \left| \frac{(n+1)!}{(n+1)^4+3} \cdot \frac{n^4+3}{n!} \right| &= \lim \frac{n(n^4+3)}{(n+1)^4+3} \\ &= \lim \frac{n^5+3n}{n^4+O(n^3)} \\ &= \infty > 1.\end{aligned}$$

Hence,

$$\liminf \left| \frac{a_{n+1}}{a_n} \right| = \lim \left| \frac{a_{n+1}}{a_n} \right| = \infty > 1.$$

□

Proposition 4. $\sum \frac{\cos^2 n}{n^2}$ converges.

Proof. We proceed with Comparison Test.

$$\left| \frac{\cos^2 n}{n^2} \right| \leq \frac{1}{n^2}.$$

We know $\sum \frac{1}{n^2}$ converges. Hence, $\sum \frac{\cos^2 n}{n^2}$ converges. □

Proposition 5. $\sum_{n=2}^{\infty} \frac{1}{\log n}$ diverges.

Proof. We proceed with Comparison Test.

$$\frac{1}{\log n} \geq \frac{1}{n}.$$

We know $\sum_{n=2}^{\infty} \frac{1}{n}$ diverges to $+\infty$. Hence, $\sum_{n=2}^{\infty} \frac{1}{\log n}$ diverges to $+\infty$. □

Proposition 6. Suppose $\sum a_n = A, \sum b_n = B$ where $A, B \in \mathbb{R}$. Then, $\sum (a_n + b_n) = A + B$.

Proof. Define (a'_n) as the partial sums of (a_n) , (b'_n) as the partial sums of (b_n) , and (c'_n) as the partial sums of $(a_n + b_n)$. Then

$$\begin{aligned} \sum (a_n + b_n) &= \lim c'_n \\ &= \lim (a'_n + b'_n) \\ &= \lim a'_n + \lim b'_n \\ &= A + B. \end{aligned}$$

□

Proposition 7. Suppose $\sum a_n = A$ for $A \in \mathbb{R}$. Then, $\sum ka_n = kA$ for $k \in \mathbb{R}$.

Proof. Define (a'_n) as the partial sums of (a_n) and (c'_n) as the partial sums of (ka_n) . Then

$$\begin{aligned} \sum (ka_n) &= \lim c'_n \\ &= \lim (ka'_n) \\ &= k \lim a'_n \\ &= kA. \end{aligned}$$

□

Proposition 8. Suppose $\sum a_n = A, \sum b_n = B$ where $A, B \in \mathbb{R}$. Then, $\sum (a_n \cdot b_n) = AB$ is not true in general.

Proof. Define $(a_n) = (1, 0, 0, 0, \dots)$, $(b_n) = (1/2)^n$. Then $A = 1, B = 2$ and $AB = 2$. But notice $a_n \cdot b_n = 0$ for all $n \neq 0$ and $\sum (a_n \cdot b_n) = a_0 \cdot b_0 = 1 \neq AB = 2$. □

Proposition 9. *If $\sum |a_n|$ converges and (b_n) is a bounded sequence, then $\sum a_n b_n$ converges. Note: Corollary 14.7 that absolutely convergent series are convergent is a special case when (b_n) is taken to be 1 for all n .*

Proof. Since (b_n) is bounded, we know there exists a supremum for $(|b_n|)$, denote $M = \max\{\sup(|b_n|), 1\}$. Then, we know there exists $N \in \mathbb{N}$ such that for $n \geq m > N$, $\sum_{k=m}^n |a_k| < \frac{\epsilon}{M}$ for all $\epsilon > 0$. Now, take such N and

$$\begin{aligned} \sum_{k=m}^n |a_k| &< \frac{\epsilon}{M} \\ M \sum_{k=m}^n |a_k| &< \epsilon \\ \left| \sum_{k=m}^n a_k b_k \right| &\leq \sum_{k=m}^n |a_k| |b_k| \leq \sum_{k=m}^n |a_k| M < \epsilon. \end{aligned}$$

Hence, $\sum a_n b_n$ satisfies the Cauchy criterion and thus converges. \square

Proposition 10. *If $\sum a_n$ is a convergent series of nonnegative numbers and $p > 1$, then $\sum a_n^p$ converges.*

Proof. We know there exists N such that for $n \geq m > N$, $|\sum_{k=m}^n a_k| < \sqrt[p]{\epsilon}$ for all $\epsilon > 0$. Take some $\epsilon > 0$ and such N , then

$$\begin{aligned} \left| \sum_{k=m}^n a_k \right| &< \sqrt[p]{\epsilon} \\ \left| \sum_{k=m}^n a_k \right|^p &< \epsilon \\ \left| \sum_{k=m}^n a_k^p \right| &\leq \left| \left(\sum_{k=m}^n a_k \right)^p \right| < \epsilon. \end{aligned} \tag{1}$$

Hence, $\sum a_n^p$ satisfies Cauchy criterion and thus converges.

Note: the left inequality in (1) is true because $a_k \geq 0$ for all k so there are simply extra nonnegative terms in $\left| \left(\sum_{k=m}^n a_k \right)^p \right|$. \square

Proposition 11. If $\sum a_n$ and $\sum b_n$ are convergent series of nonnegative numbers, then $\sum \sqrt{a_n b_n}$ converges. Hint: show $\sqrt{a_n b_n} \leq a_n + b_n$ for all n .

Proof. Notice for all n

$$\begin{aligned} a_n^2 + b_n^2 + 2a_n b_n &\geq a_n b_n \\ (a_n + b_n)^2 &\geq a_n b_n \\ a_n + b_n &\geq \sqrt{a_n b_n}. \end{aligned}$$

Also, we know there exists N_1 such that for $n \geq m > N_1$, $|\sum_{k=m}^n a_k| < \epsilon/2$ and N_2 such that for $n \geq m > N_2$, $|\sum_{k=m}^n b_k| < \epsilon/2$. Now we take $N = \max\{N_1, N_2\}$ for some $\epsilon > 0$. Then, for all $n \geq m > N$

$$\begin{aligned} \left| \sum_{k=m}^n \sqrt{a_k b_k} \right| &\leq \left| \sum_{k=m}^n a_k + b_k \right| \\ &\leq \left| \sum_{k=m}^n a_k \right| + \left| \sum_{k=m}^n b_k \right| \\ &\leq \left| \sum_{k=m}^n a_k \right| + \left| \sum_{k=m}^n b_k \right| \\ &< \epsilon. \end{aligned}$$

Hence, $\sum \sqrt{a_n b_n}$ satisfied Cauchy criterion and thus converges. \square

Proposition 12. The convergence of a series does not depend on any finite number of terms, though the value of the limit does. More precisely, consider series $\sum a_n$ and $\sum b_n$ and suppose that the set $\{n \in \mathbb{N} : a_n \neq b_n\}$ is finite. Then the series both converge or else the both diverge.

Proof. Without loss of generality, we will focus on $\sum a_n$ and conclude the convergence of $\sum b_n$ based on $\sum a_n$. Also, denote $M = \max\{n \in \mathbb{N} : a_n \neq b_n\}$.

Case 1: $\sum a_n$ converges. We know $\sum a_n$ satisfies Cauchy criterion, thus we know there exists N_1 such that for all $n \geq m > N_1$, $|\sum_{k=m}^n a_k| < \epsilon$ for some $\epsilon > 0$.

Then, let $N_2 = \max\{N_1, M\}$. Since we have set N_2 to be at least M , any terms after N_2 for b_n is the same as a_n . Thus, any statement that holds true for a_n is also true for b_n after N_2 and we can conclude for all $n \geq m > N_2$ $|\sum_{k=m}^n b_k| < \epsilon$ for some $\epsilon > 0$.

Therefore, $\sum b_n$ satisfies Cauchy criterion too and thus converges.

Case 2: $\sum a_n$ diverges. Assume for the sake of contradiction that $\sum b_n$ converges. Then there exists N_2 for all $\epsilon > 0$ such that for $n \geq m > N_2$, $|\sum_{k=m}^n b_k| < \epsilon$. Thus, we can take $N_1 = \max\{N_2, M\}$, which will make sure that for $n \geq m > N_1$, $|\sum_{k=m}^n a_k| < \epsilon$ for each ϵ . But that contradicts that fact that $\sum a_n$ diverges. Hence, $\sum b_n$ must diverge. \square

Proposition 13. Let (a_n) be a sequence of nonzero real numbers such that the sequence $\left(\frac{a_{n+1}}{a_n}\right)$ of ratios is a constance sequence, then $\sum a_n$ is a geometric series.

Proof. Let $r = \frac{a_{n+1}}{a_n}$ for all n . Then we can define (a_n) recursively such that $a_{n+1} = a_n \cdot r$. Hence, $a_n = a_0 \cdot r^n$. Indeed,

$$\sum a_n = \sum_{k=0}^n a_0 \cdot r^k,$$

which is a geometric series. \square

Proposition 14. Let $(a_n)_{n \in \mathbb{N}}$ be a sequence such that $\liminf |a_n| = 0$, then there is a subsequence such that $\sum_{k=1}^{\infty} a_{n_k}$ converges.

Proof. Since $\liminf |a_n| = 0$, we know there exists a subsequence of $(|a_n|)$ that converges to 0. Hence, for each ϵ , the set $\{n : \mathbb{N} : |a_n| < \epsilon\}$ is infinite. Then we can construct a subsequence such that $\sum_{k=1}^{\infty} a_{n_k}$ converges.

For each $k + 1$, choose $n_{k+1} > n_k$ such that $|a_{n_{k+1}}| < \frac{1}{2^{k+1}} = b_{k+1}$. Then, for each k , $|a_{n_k}| \leq b_k$. Apparently, $\sum b_k$ is a convergent geometric series, thus by comparison test, $\sum_{k=1}^{\infty} a_{n_k}$ converges. \square

Proposition 15. $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$. *Hint:* $\sum_{k=1}^n \frac{1}{k(k+1)} = \sum_{k=1}^n \left(\frac{1}{k} - \frac{1}{k+1} \right)$.

Proof. Notice

$$\begin{aligned} \sum_{k=1}^n \frac{1}{k(k+1)} &= \sum_{k=1}^n \left(\frac{1}{k} - \frac{1}{k+1} \right) \\ &= \left(1 - \frac{1}{2} \right) + \left(\frac{1}{2} - \frac{1}{3} \right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1} \right) \\ &= 1 - \frac{1}{n+1}. \end{aligned}$$

Hence,

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{n(n+1)} &= \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1}{k(k+1)} \\ &= \lim_{n \rightarrow \infty} \left(1 - \frac{1}{n+1} \right) \\ &= 1. \end{aligned}$$

\square

Proposition 16. $\sum_{n=1}^{\infty} \frac{n-1}{2^{n+1}} = \frac{1}{2}$. *Hint:* $\frac{k-1}{2^{k+1}} = \frac{k}{2^k} - \frac{k+1}{2^{k+1}}$.

Proof. Notice

$$\begin{aligned} \sum_{k=1}^n \frac{k-1}{2^{k+1}} &= \sum_{k=1}^n \left(\frac{k}{2^k} - \frac{k+1}{2^{k+1}} \right) \\ &= \left(\frac{1}{2} - \frac{2}{2^2} \right) + \left(\frac{2}{2^2} - \frac{3}{2^3} \right) + \cdots + \left(\frac{n}{2^n} - \frac{n+1}{2^{n+1}} \right) \\ &= \frac{1}{2} - \frac{n+1}{2^{n+1}}. \end{aligned}$$

Hence,

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{n-1}{2^{n+1}} &= \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{k-1}{2^{k+1}} \\
&= \lim_{n \rightarrow \infty} \left(\frac{1}{2} - \frac{n+1}{2^{n+1}} \right) \\
&= \frac{1}{2} - \lim_{k \rightarrow \infty} \frac{k}{2^k} \\
&= \frac{1}{2} - \lim_{k \rightarrow \infty} \left(\frac{\sqrt[k]{k}}{2} \right)^k \\
&= \frac{1}{2}.
\end{aligned}$$

□

Proposition 17. Show that $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges by comparing with the series $\sum_{n=2}^{\infty} a_n$ where (a_n) is the sequence

$$\left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{16}, \frac{1}{16}, \frac{1}{16}, \frac{1}{16}, \frac{1}{16}, \frac{1}{16}, \frac{1}{16}, \frac{1}{16}, \frac{1}{32}, \frac{1}{32}, \dots \right).$$

Note: this is also known as the Cauchy Condensation Test.

Proof. We will show that $\sum_{n=2}^{\infty} \frac{1}{n}$ diverges and thus $\sum_{n=1}^{\infty} \frac{1}{n}$, which differs only by the first term.

Notice for all $2^k < n \leq 2^{k+1}$, $a_n = \frac{1}{2^{k+1}} \leq \frac{1}{n}$. This is true for all $k \in \mathbb{N}$. Hence, $\frac{1}{n} \leq a_n$ for all n . Now observe within each interval $(2^k, 2^{k+1}]$, there are 2^k terms. Therefore, $\sum_{n=2^k}^{2^{k+1}} a_n = \frac{1}{2}$ and $\sum_{n=2}^{\infty} a_n = \lim_{k \rightarrow \infty} k \left(\frac{1}{2} \right) = \infty$.

Hence, by Comparison Test, $\sum_{n=1}^{\infty} \frac{1}{n}$ also diverges.

□

Chapter 15

Proposition 18. $\sum \left[\sin \left(\frac{n\pi}{6} \right) \right]^n$ diverges.

Proof. Notice that when $n = 12k + 3$, $\left[\sin \left(\frac{n\pi}{6} \right) \right]^n = 1$. Hence, the summation never converges. \square

Proposition 19. $\sum \left[\sin \left(\frac{n\pi}{7} \right) \right]^n$ converges.

Proof. We will show that the summation converges absolutely, hence converges.

Notice $\left| \sin \left(\frac{n\pi}{7} \right) \right|$ is always between 0 and 1. In fact, it is bounded by above by some $r < 1$ such that $\left| \sin \left(\frac{n\pi}{7} \right) \right| \leq r < 1$ and $\left| \sin \left(\frac{n\pi}{7} \right) \right|^n \leq r^n < 1$. Then by Comparison Test, $\sum \left| \sin \left(\frac{n\pi}{7} \right) \right|^n$ converges because $\sum r^n$ converges, which can be shown easily by Ratio Test or Root Test. \square

Proposition 20. $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$ converges if and only if $p > 1$.

Proof. We proceed with Integral Test with $f(x) = \frac{1}{x(\log x)^p}$. Notice $f(x)$ is continuous, positive, and decreasing for $x \geq 2$. Also, $f(n) = a_n$. Then for $p \neq 1$

$$\lim_{n \rightarrow \infty} \int_2^n \frac{1}{x(\log x)^p} dx = \lim_{n \rightarrow \infty} \left[\frac{(\log x)^{1-p}}{1-p} \right]_2^n. \quad (2)$$

For $p = 1$, we have

$$\lim_{n \rightarrow \infty} \int_2^n \frac{1}{x(\log x)} dx = \lim_{n \rightarrow \infty} [\log(\log x)]_2^n = \infty. \quad (3)$$

Then for (\Rightarrow) direction, we know that if $p = 1$, (2) goes to infinity, thus the summation diverges. If $p < 1$, (1) goes to infinity, thus the summation diverges again. Hence, forward direction is shown by contrapositive.

For (\Leftarrow) direction, we know that if $p > 1$, (1) converges, thus the summation converges. \square

Proposition 21. $\sum_{n=4}^{\infty} \frac{1}{n(\log n)(\log \log n)}$ diverges.

Proof. We proceed with Integral Test with $f(x) = \frac{1}{x(\log x)(\log \log x)}$. Notice $f(x)$ is decreasing, $f(n) = a_n$, and all a_n are nonnegative. Then

$$\lim_{n \rightarrow \infty} \int_4^n \frac{1}{x(\log x)(\log \log x)} dx = \lim_{n \rightarrow \infty} [\log(\log(\log x))]_4^n = \infty.$$

\square

Proposition 22. $\sum_{n=2}^{\infty} \frac{\log n}{n^2}$ converges.

Proof. Integral Test:

We can integrate $f(x) = \frac{\log x}{x^2}$ to get

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_2^n \frac{\log x}{x^2} dx &= \lim_{n \rightarrow \infty} \int_2^n -(\log x) d\left(\frac{1}{x}\right) \\ &= \lim_{n \rightarrow \infty} \left[-\frac{\log x}{x} \right]_2^n + \lim_{n \rightarrow \infty} \int_2^n \frac{1}{x^2} dx \\ &= \lim_{n \rightarrow \infty} \left[-\frac{\log x}{x} \right]_2^n - \lim_{n \rightarrow \infty} \left[\frac{1}{x} \right]_2^n \\ &= \frac{1}{2}, \end{aligned}$$

and conclude that $\sum_{n=2}^{\infty} \frac{\log n}{n^2}$ converges.

Comparison Test:

We know that for $n > N$ where N is some constant, $\sqrt{n} > \log n$. This can be proved by observing that $\sqrt{n} > \log n$ when $n = 100$, and we see by first derivative that \sqrt{n} has a higher increasing rate than $\log n$ for all n . Hence, we can conclude that $\sqrt{n} > \log n$ for all $n \geq 100$.

Then, we see $\frac{\log n}{n^2} < \frac{1}{n^{3/2}}$ for $n \geq 100$. We know that $\sum \frac{1}{n^{3/2}}$ converges for $p > 1 \Rightarrow \sum_{100}^{\infty} \frac{1}{n^{3/2}}$ converges $\Rightarrow \sum_{100}^{\infty} \frac{\log n}{n^2}$ converges $\Rightarrow \sum_2^{\infty} \frac{\log n}{n^2}$ converges. \square

Proposition 23. If (a_n) is a decreasing sequence of real numbers and if $\sum a_n$ converges, then $\lim na_n = 0$.

Proof. Since $\sum a_n$ converges, we know it satisfies Cauchy criterion. In other words, there exists N such that for $n \geq m > N$,

$$\left| \sum_{k=m}^n a_k \right| = \sum_{k=m}^n a_k < \frac{\epsilon}{2}.$$

Notice (a_n) is a decreasing sequence and $\lim a_n = 0$ is a necessary condition for $\sum a_n$ to converge, hence all a_n are non-negative. Therefore, we can remove the absolute value.

In particular, for $m = N + 1$,

$$\begin{aligned} (n - N)a_n &\leq \sum_{k=N+1}^n a_k \\ na_n &\leq \sum_{k=N+1}^n a_k + Na_n. \end{aligned} \tag{4}$$

Note that since (a_n) converges to 0 and all a_n are non-negative, we know there exists N' such that

$$a_n = |a_n - 0| < \frac{\epsilon}{2N}.$$

Therefore, putting back to (4), as long as $n > \max\{N, N'\}$

$$|na_n - 0| = na_n \leq \sum_{k=N+1}^n a_k + Na_n < \frac{\epsilon}{2} + N \frac{\epsilon}{2N} = \epsilon.$$

\square

Chapter 17

Proposition 24. \sqrt{x} is continuous on its domain $[0, \infty)$.

Proof. We consider non-negative sequences (x_n) because any sequence with negative terms would not make sense on $f(x_n)$.

We have to show that if (x_n) converges to some $x_0 \in [0, \infty)$, $f(x_n) = \sqrt{x_n}$ converges to $\sqrt{x_0}$. In other words, we seek to find N such that for $\epsilon > 0$, as long as $n > N$,

$$|\sqrt{x_n} - \sqrt{x_0}| < \epsilon.$$

Consider $x_0 = 0$, since $(x_n) \rightarrow 0$, there exists some N such that for $n > N$,

$$\begin{aligned} |x_n| &< \epsilon^2 \\ \sqrt{|x_n|} &< \epsilon \\ |\sqrt{x_n}| &< \epsilon. \end{aligned}$$

Therefore, \sqrt{x} is continuous at $x = 0$.

For $x_0 > 0$, we need to show the existence of N such that for $n > N$,

$$\begin{aligned} |\sqrt{x_n} - \sqrt{x_0}| &< \epsilon \\ \left| \frac{x_n - x_0}{\sqrt{x_n} + \sqrt{x_0}} \right| &< \epsilon \\ \left| \frac{x_n - x_0}{\sqrt{x_n} + \sqrt{x_0}} \right| &\leq \left| \frac{x_n - x_0}{\sqrt{x_0}} \right| < \epsilon \\ \left| \frac{x_n - x_0}{\sqrt{x_0}} \right| &< \epsilon \\ |x_n - x_0| &< \sqrt{x_0} \cdot \epsilon. \end{aligned}$$

Indeed, there exists such N because (x_n) converges to x_0 . □

Proposition 25. If $m \in \mathbb{N}$, then the function $f(x) = x^m$ is continuous on \mathbb{R} .

Proof. Suppose an arbitrary sequence (x_n) that converges to $x_0 \in \mathbb{R}$, then

$$\begin{aligned} \lim f(x_n) &= \lim x_n^m \\ &= (\lim x_n)^m \\ &= x_0^m \\ &= f(x_0). \end{aligned}$$

□

Corollary 26. Every polynomial function $p(x) = a_0 + a_1x + \cdots + a_nx^n$ is continuous on \mathbb{R} .

Proof. Let $f_k = a_kx^k$ for $k \in [0, n]$. We have shown in previous proposition that x^k is continuous for $k \in [1, n]$. Notice $x^0 = 1$ is continuous on \mathbb{R} apparently because every sequence (x_n) that converges to 1 will have $(x_n)^0 = 1$.

Therefore, f_k is continuous on \mathbb{R} because it's a scalar multiple of a continuous function. Consequently, $p(x)$ is continuous on \mathbb{R} because it's a sum of continuous functions. □

Proposition 27. A rational function is a function f of the form p/q where p, q are polynomial functions. The domain of f is $\{x \in \mathbb{R} : q(x) \neq 0\}$. Then every rational function is continuous.

Proof. From Corollary 26, we know every polynomial is continuous on \mathbb{R} and of course on $\mathbb{R} \setminus \{x : q(x) = 0\}$. Hence p/q is continuous since it's a division of two continuous functions where $q \neq 0$. \square

Proposition 28. $\min(f, g) = \frac{1}{2}(f + g) - \frac{1}{2}|f - g|$.

Proof. Assume $f > g$, then

$$\begin{aligned} \frac{1}{2}(f + g) - \frac{1}{2}|f - g| &= \frac{1}{2}(f + g) - \frac{1}{2}(f - g) \\ &= g \\ &= \min(f, g). \end{aligned}$$

Assume $g \geq f$, then

$$\begin{aligned} \frac{1}{2}(f + g) - \frac{1}{2}|f - g| &= \frac{1}{2}(f + g) - \frac{1}{2}(g - f) \\ &= f \\ &= \min(f, g). \end{aligned}$$

\square

Proposition 29. $\min(f, g) = -\max(-f, -g)$.

Proof.

$$\begin{aligned} \max(f, g) &= \frac{1}{2}(f + g) + \frac{1}{2}|f - g| \\ \max(-f, -g) &= \frac{1}{2}(-f - g) + \frac{1}{2}|-f + g| \\ \max(-f, -g) &= -\frac{1}{2}(f + g) + \frac{1}{2}|g - f| \\ -\max(-f, -g) &= \frac{1}{2}(f + g) - \frac{1}{2}|f - g| \\ -\max(-f, -g) &= \min(f, g). \end{aligned}$$

\square

Corollary 30. If f and g are continuous at x_0 in \mathbb{R} , then $\min(f, g)$ is continuous at x_0 .

Proof. $f, g \in C^0(x_0) \Rightarrow f + g, f - g \in C^0(x_0) \Rightarrow f + g, |f - g| \in C^0(x_0) \Rightarrow \frac{1}{2}(f + g), \frac{1}{2}|f - g| \in C^0(x_0) \Rightarrow \frac{1}{2}(f + g) + \frac{1}{2}|f - g| \in C^0(x_0) \Rightarrow \min(f, g) \in C^0(x_0)$. \square

Proposition 31. $g(x) = x^3$ is continuous on \mathbb{R} .

Proof. Consider arbitrary $c \in \mathbb{R}$. We seek to find δ such that for $|x - c| < \delta$, $|x^3 - c^3| < \epsilon$.

Let's evaluate $|x^3 - c^3|$:

$$\begin{aligned} |x^3 - c^3| &= |x - c||x^2 + xc + c^2| \\ &\leq |x - c|(|x^2| + |xc| + |c^2|). \end{aligned}$$

For $|x - c| < 1$, then

$$|x^2| + |xc| + |c^2| < |c + 1|^2 + |c + 1||c| + |c^2|.$$

Let $M = |c + 1|^2 + |c + 1||c| + |c^2| > 0$. Hence, let $|x - c| < \delta = \min\{\epsilon/M, 1\}$, then

$$|x^3 - c^3| < |x - c|M < \frac{\epsilon}{M}M = \epsilon.$$

□

Proposition 32. *Let f be a continuous real-valued function with domain (a, b) . If $f(r) = 0$ for each rational number r in (a, b) , then $f(x) = 0$ for all $x \in (a, b)$.*

Proof. We will show that if there are some $x_0 \in (a, b)$ such that $f(x_0) = c \neq 0$, then f is not continuous at x_0 , hence not continuous on (a, b) .

Assume for the sake of contradiction that there exists $x_0 \in (a, b) \setminus \mathbb{Q}$ such that $f(x_0) = c \neq 0$. Let $\epsilon = \frac{|c|}{2}$. However for $|f(x_0) - f(x)| < \epsilon$, there does not exist δ such that $|x - x_0| < \delta \Rightarrow |f(x_0) - f(x)| < \epsilon$ because there is always a rational number $x' \in (x_0 - \delta, x_0 + \delta)$ according to the density of \mathbb{Q} , and $|f(x') - f(x_0)| = |c| > \epsilon$. Then, the contradiction that f is not continuous is reached. □

Corollary 33. *Let f and g be continuous real-valued functions on (a, b) such that $f(r) = g(r)$ for each rational number r in (a, b) , then $f(x) = g(x)$ for all $x \in (a, b)$.*

Proof. Let $f(r) = g(r) = c$ and define $f'(x) = f(x) - c$ and $g'(x) = g(x) - c$. Then from Proposition 33, we know $f'(x) = g'(x) = 0$ because f' and g' are continuous and $f'(r) = g'(r) = 0$. Therefore, $f'(x) = g'(x) = 0 \Rightarrow f'(x) + c = g'(x) + c = c \Rightarrow f(x) = g(x) = c$ for all $x \in (a, b)$. □

Proposition 34. *Let $h(x) = x$ for rational numbers x and $h(x) = 0$ for irrational numbers, then h is continuous only at $x = 0$.*

Proof. $x_0 = 0$: Let $\epsilon > 0$. Then for $|x - x_0| < \epsilon$, $|h(x) - h(x_0)| = |h(x) - 0| < \epsilon$ because for irrational numbers x , $|h(x) - 0| = 0 < \epsilon$, and for rational numbers x , $|h(x) - 0| = |x - 0| < \epsilon$.

$x_0 \neq 0$ and **irrational**: Let $\epsilon = \frac{|x_0|}{2}$. Assume for the sake of contradiction that there exists δ such that $|x - x_0| < \delta \Rightarrow |h(x) - 0| < \epsilon$. However, by density of \mathbb{Q} there exists a rational number $x' \in (x_0 - \delta, x_0 + \delta)$ such that $|x'| > |x_0|$ [$x' \in (x_0, x_0 + \delta)$ if $x_0 > 0$, $x' \in (x_0 - \delta, x_0)$ if $x_0 \leq 0$]. Then $|h(x') - 0| = |x'| > |x_0| > \epsilon$.

$x_0 \neq 0$ and **rational**: Let $\epsilon = \frac{|x_0|}{2}$. Assume for the sake of contradiction that there exists δ such that $|x - x_0| < \delta \Rightarrow |h(x) - x_0| < \epsilon$. However, by density of irrationals there exists an irrational number $x' \in (x_0 - \delta, x_0 + \delta)$. Then $|h(x') - x_0| = |0 - x_0| = |x_0| > \epsilon$. □

Proposition 35. *For each nonzero rational number x , write $x = \frac{p}{q}$ where p, q are integers with no common factors and $q > 0$, and then define $f(x) = \frac{1}{q}$. Also define $f(0) = 1$ and $f(x) = 0$ for all $x \in \mathbb{R} \setminus \mathbb{Q}$. Thus $f(x) = 1$ for each integer, $f(\frac{1}{2}) = f(\frac{-1}{2}) = f\frac{3}{2} = \dots = \frac{1}{2}$, etc. Then f is continuous at each point of $\mathbb{R} \setminus \mathbb{Q}$ and discontinuous at each point of \mathbb{Q} .*

Proof. **Discontinuous on \mathbb{Q} :** Let $x_0 \in \mathbb{Q}$. Define a sequence of irrational numbers (x_n) that converges to x_0 , but $f(x_n) = 0$ for all n and obviously $\lim x_n = x_0 \neq f(x_0)$ [notice $f(x_0)$ is never 0 for rational number x_0].

Continuous on $\mathbb{R} \setminus \mathbb{Q}$: Let $x_0 \in \mathbb{R} \setminus \mathbb{Q}$ and $\epsilon > 0$. Then let $N = \min\{n > \frac{1}{\epsilon} : n \in \mathbb{N}\}$. Now consider the interval $I = (x_0 - 1, x_0 + 1)$. We know there is only a finite number of $\frac{p}{q} \in I$ such that $q < N$ [consider $q = N - 1$, there are only finite multiples of $\frac{1}{q} \in I$ because the multiple will eventually "step over" the interval; this is true all for $q < N$]. Since there is only a finite number of such fractions, we can take the

fraction that is closest to x_0 , denote r , and let $\delta = |r - x_0|$. Then, the interval $(x_0 - \delta, x_0 + \delta)$ only contains either irrational numbers or $\frac{p}{q}$ with $q \geq N > \frac{1}{\epsilon}$.

Therefore, for rational x such that $|x - x_0| < \delta$, $f(x) \leq \frac{1}{N} < \epsilon \Rightarrow |f(x) - f(x_0)| = |f(x) - 0| < \epsilon$. Obviously for irrational x , $|f(x) - f(x_0)|$ is always 0.

Intuition: We can zoom into x_0 to find an interval where only very very small mesh of fractions is contained in there because all larger fractions would "step over" the interval □