# Math 104 Practice

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## Chapter 14

**Proposition 1.** $\sum \frac{n^4}{2^n}$ *converges.*

*Proof.* We proceed with Ratio Test.

$$\begin{aligned}\lim \left| \frac{(n+1)^4}{2^{n+1}} \cdot \frac{2^n}{n^4} \right| &= \lim \frac{(n+1)^4}{2n^4} \\ &= \lim \frac{n^4 + O(n^3)}{2n^4} \\ &= \frac{1}{2} < 1.\end{aligned}$$

□

**Proposition 2.** $\sum \frac{2^n}{n!}$ *converges.*

*Proof.* We proceed with Ratio Test.

$$\begin{aligned}\lim \left| \frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n} \right| &= \lim \frac{2}{n+1} \\ &= 0 < 1.\end{aligned}$$

□

**Proposition 3.** $\sum \frac{n!}{n^4+3}$ *diverges.*

*Proof.* We proceed with Ratio Test.

$$\begin{aligned}\lim \left| \frac{(n+1)!}{(n+1)^4+3} \cdot \frac{n^4+3}{n!} \right| &= \lim \frac{n(n^4+3)}{(n+1)^4+3} \\ &= \lim \frac{n^5+3n}{n^4+O(n^3)} \\ &= \infty > 1.\end{aligned}$$

Hence,

$$\liminf \left| \frac{a_{n+1}}{a_n} \right| = \lim \left| \frac{a_{n+1}}{a_n} \right| = \infty > 1.$$

□

**Proposition 4.** $\sum \frac{cos^2 n}{n^2}$ *converges.*

*Proof.* We proceed with Comparison Test.

$$\left|\frac{cos^2 n}{n^2}\right| \leq \frac{1}{n^2}.$$

We know $\sum \frac{1}{n^2}$ converges. Hence, $\sum \frac{cos^2 n}{n^2}$ converges. $\square$

**Proposition 5.** $\sum_{n=2}^{\infty} \frac{1}{logn}$ *diverges.*

*Proof.* We proceed with Comparison Test.

$$\frac{1}{logn} \geq \frac{1}{n}.$$

We know $\sum_{n=2}^{\infty} \frac{1}{n}$ diverges to $+\infty$. Hence, $\sum_{n=2}^{\infty} \frac{1}{logn}$ diverges to $+\infty$. $\square$

**Proposition 6.** *Suppose* $\sum a_n = A, \sum b_n = B$ *where* $A, B \in \mathbb{R}$. *Then,* $\sum(a_n + b_n) = A + B$.

*Proof.* Define $(a'_n)$ as the partial sums of $(a_n)$, $(b'_n)$ as the partial sums of $(b_n)$, and $(c'_n)$ as the partial sums of $(a_n + b_n)$. Then

$$\begin{aligned} \sum(a_n + b_n) &= \lim c'_n \\ &= \lim(a'_n + b'_n) \\ &= \lim a'_n + \lim b'_n \\ &= A + B. \end{aligned}$$

$\square$

**Proposition 7.** *Suppose* $\sum a_n = A$ *for* $A \in \mathbb{R}$. *Then,* $\sum ka_n = kA$ *for* $k \in \mathbb{R}$.

*Proof.* Define $(a'_n)$ as the partial sums of $(a_n)$ and $(c'_n)$ as the partial sums of $(ka_n)$. Then

$$\begin{aligned} \sum(ka_n) &= \lim c'_n \\ &= \lim(ka'_n) \\ &= k \lim a'_n \\ &= kA. \end{aligned}$$

$\square$

**Proposition 8.** *Suppose* $\sum a_n = A, \sum b_n = B$ *where* $A, B \in \mathbb{R}$. *Then,* $\sum(a_n \cdot b_n) = AB$ *is not true in general.*

*Proof.* Define $(a_n) = (1, 0, 0, 0, \dots), (b_n) = (1/2)^n$. Then $A = 1, B = 2$ and $AB = 2$. But notice $a_n \cdot b_n = 0$ for all $n \neq 0$ and $\sum(a_n \cdot b_n) = a_0 \cdot b_0 = 1 \neq AB = 2$. $\square$

**Proposition 9.** *If $\sum |a_n|$ converges and $(b_n)$ is a bounded sequence, then $\sum a_n b_n$ converges. Note: Corollary 14.7 that absolutely convergent series are convergent is a special case when $(b_n)$ is taken to be 1 for all $n$.*

*Proof.* Since $(b_n)$ is bounded, we know there exists an supremum for $(|b_n|)$, denote $M = max\{\sup(|b_n|), 1\}$. Then, we know there exists $N \in \mathbb{N}$ such that for $n \geq m > N$, $\sum_{k=m}^{n} |a_k| < \frac{\epsilon}{M}$ for all $\epsilon > 0$. Now, take such $N$ and

$$\sum_{k=m}^{n} |a_k| < \frac{\epsilon}{M}$$

$$M \sum_{k=m}^{n} |a_k| < \epsilon$$

$$\left| \sum_{k=m}^{n} a_k b_k \right| \leq \sum_{k=m}^{n} |a_k||b_k| \leq \sum_{k=m}^{n} |a_k| M < \epsilon.$$

Hence, $\sum a_n b_n$ satifies the Cauchy criterion and thus converges. □

**Proposition 10.** *If $\sum a_n$ is a convergent series of nonnegative numbers and $p > 1$, then $\sum a_n^p$ converges.*

*Proof.* We know there exists $N$ such that for $n \geq m > N$, $|\sum_{k=m}^{n} a_k| < \sqrt[p]{\epsilon}$ for all $\epsilon > 0$. Take some $\epsilon > 0$ and such $N$, then

$$\left| \sum_{k=m}^{n} a_k \right| < \sqrt[p]{\epsilon}$$

$$\left| \sum_{k=m}^{n} a_k \right|^p < \epsilon$$

$$\left| \sum_{k=m}^{n} a_k^p \right| \leq \left| \left( \sum_{k=m}^{n} a_k \right)^p \right| < \epsilon. \tag{1}$$

Hence, $\sum a_n^p$ satifies Cauchy criterion and thus converges.

Note: the left inequality in (1) is true because $a_k \geq 0$ for all $k$ so there are simply extra nonnegative terms in $\left|(\sum_{k=m}^{n} a_k)^p\right|$. □

**Proposition 11.** *If $\sum a_n$ and $\sum b_n$ are convergent series of nonnegative numbers, then $\sum \sqrt{a_n b_n}$ converges. Hint: show $\sqrt{a_n b_n} \leq a_n + b_n$ for all $n$.*

*Proof.* Notice for all $n$

$$
\begin{aligned}
a_n^2 + b_n^2 + 2a_n b_n &\geq a_n b_n \\
(a_n + b_n)^2 &\geq a_n b_n \\
a_n + b_n &\geq \sqrt{a_n b_n}.
\end{aligned}
$$

Also, we know there exists $N_1$ such that for $n \geq m > N_1$, $|\sum_{k=m}^{n} a_k| < \epsilon/2$ and $N_2$ such that for $n \geq m > N_2$, $|\sum_{k=m}^{n} b_k| < \epsilon/2$. Now we take $N = \max\{N_1, N_2\}$ for some $\epsilon > 0$. Then, for all $n \geq m > N$

$$
\begin{aligned}
\left| \sum_{k=m}^{n} \sqrt{a_n b_n} \right| &\leq \left| \sum_{k=m}^{n} a_k + b_k \right| \\
&\leq \left| \sum_{k=m}^{n} a_k + \sum_{k=m}^{n} b_k \right| \\
&\leq \left| \sum_{k=m}^{n} a_k \right| + \left| \sum_{k=m}^{n} b_k \right| \\
&< \epsilon.
\end{aligned}
$$

Hence, $\sum \sqrt{a_n b_n}$ satisfied Cauchy criterion and thus converges. □

**Proposition 12.** *The convergence of a series does not depend on any finite number of terms, though the value of the limit does. More precisely, consider series $\sum a_n$ and $\sum b_n$ and suppose that the set $\{n \in \mathbb{N} : a_n \neq b_n\}$ is finite. Then the series both converge or else the both diverge.*

*Proof.* Without loss of generality, we will focus on $\sum a_n$ and conclude the convergence of $\sum b_n$ based on $\sum a_n$. Also, denote $M = \max\{n \in \mathbb{N} : a_n \neq b_n\}$.

Case 1: $\sum a_n$ converges. We know $\sum a_n$ satisfies Cauchy criterion, thus we know there exists $N_1$ such that for all $n \geq m > N_1$, $|\sum_{k=m}^{n} a_k| < \epsilon$ for some $\epsilon > 0$.

Then, let $N_2 = \max\{N_1, M\}$. Since we have set $N_2$ to be at least $M$, any terms after $N_2$ for $b_n$ is the same as $a_n$. Thus, any statement that holds true for $a_n$ is also true for $b_n$ after $N_2$ and we can conclude for all $n \geq m > N_2$ $|\sum_{k=m}^{n} b_k| < \epsilon$ for some $\epsilon > 0$.

Therefore, $\sum b_n$ satifies Cauchy criterion too and thus converges.

Case 2: $\sum a_n$ diverges. Assume for the sake of contradiction that $\sum b_n$ converges. Then there exists $N_2$ for all $\epsilon > 0$ such that for $n \geq m > N_2$, $|\sum b_n| < \epsilon$. Thus, we can take $N_1 = \max\{N_2, M\}$, which will make sure that for $n \geq m > N_1$, $|\sum_{k=m}^{n} a_n| < \epsilon$ for each $\epsilon$. But that contradicts that fact that $\sum a_n$ diverges. Hence, $\sum b_n$ must diverge. □

**Proposition 13.** *Let $(a_n)$ be a sequence of nonzero real numbers such that the sequence $\left(\frac{a_{n+1}}{a_n}\right)$ of ratios is a constance sequence, then $\sum a_n$ is a geometric series.*

*Proof.* Let $r = \frac{a_{n+1}}{a_n}$ for all $n$. Then we can define $(a_n)$ recursively such that $a_{n+1} = a_n \cdot r$. Hence, $a_n = a_0 \cdot r^n$. Indeed,

$$
\sum a_n = \sum_{k=0}^{n} a_0 \cdot r^k,
$$

which is a geometric series. □

**Proposition 14.** *Let* $(a_n)_{n\in\mathbb{N}}$ *be a sequence such that* $\liminf |a_n| = 0$*, then there is a subsequence such that* $\sum_{k=1}^{\infty} a_{n_k}$ *converges.*

*Proof.* Since $\liminf |a_n| = 0$, we know there exists a subsequence of $(|a_n|)$ that converges to 0. Hence, for each $\epsilon$, the set $\{n : \mathbb{N} : |a_n| < \epsilon\}$ is infinite. Then we can construct a subsequence such that $\sum_{k=1}^{\infty} a_{n_k}$ converges.

For each $k+1$, choose $n_{k+1} > n_k$ such that $|a_{n_{k+1}}| < \frac{1}{2^{k+1}} = b_{k+1}$. Then, for each $k$, $|a_{n_k}| \le b_k$. Apparently, $\sum b_k$ is a convergent geometric series, thus by comparison test, $\sum_{k=1}^{\infty} a_{n_k}$ converges. $\square$

**Proposition 15.** $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$. *Hint:* $\sum_{k=1}^{n} \frac{1}{k(k+1)} = \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1}\right)$.

*Proof.* Notice

$$\begin{aligned}
\sum_{k=1}^{n} \frac{1}{k(k+1)} &= \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1}\right) \\
&= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right) \\
&= 1 - \frac{1}{n+1}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{n(n+1)} &= \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k(k+1)} \\
&= \lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right) \\
&= 1.
\end{aligned}$$

$\square$

**Proposition 16.** $\sum_{n=1}^{\infty} \frac{n-1}{2^{n+1}} = \frac{1}{2}$. *Hint:* $\frac{k-1}{2^{k+1}} = \frac{k}{2^k} - \frac{k+1}{2^{k+1}}$.

*Proof.* Notice

$$\begin{aligned}
\sum_{k=1}^{n} \frac{k-1}{w^{k+1}} &= \sum_{k=1}^{n} \left(\frac{k}{2^k} - \frac{k+1}{2^{k+1}}\right) \\
&= \left(\frac{1}{2} - \frac{2}{2^2}\right) + \left(\frac{2}{2^2} - \frac{3}{2^3}\right) + \cdots + \left(\frac{n}{2^n} - \frac{n+1}{2^{n+1}}\right) \\
&= \frac{1}{2} - \frac{n+1}{2^{n+1}}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{n-1}{2^{n+1}} &= \lim_{n\to\infty} \sum_{k=1}^{n} \frac{k-1}{2^{k+1}} \\
&= \lim_{n\to\infty} \left( \frac{1}{2} - \frac{n+1}{2^{n+1}} \right) \\
&= \frac{1}{2} - \lim_{k\to\infty} \frac{k}{2^k} \\
&= \frac{1}{2} - \lim_{k\to\infty} \left( \frac{\sqrt[k]{k}}{2} \right)^k \\
&= \frac{1}{2}.
\end{aligned}$$

$\square$

**Proposition 17.** *Show that* $\sum_{n=1}^{\infty} \frac{1}{n}$ *diverges by comparing with the series* $\sum_{n=2}^{\infty} a_n$ *where* $(a_n)$ *is the sequence*

$$(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{16}, \frac{1}{16}, \frac{1}{16}, \frac{1}{16}, \frac{1}{16}, \frac{1}{16}, \frac{1}{16}, \frac{1}{16}, \frac{1}{32}, \frac{1}{32}, \cdots).$$

*Note: this is also known as the Cauchy Condensation Test.*

*Proof.* We will show that $\sum_{n=2}^{\infty} \frac{1}{n}$ diverges and thus $\sum_{n=1}^{\infty} \frac{1}{n}$, which differs only by the first term.

Notice for all $2^k < n \leq 2^{k+1}$, $a_n = \frac{1}{2^{k+1}} \leq \frac{1}{n}$. This is true for all $k \in \mathbb{N}$. Hence, $\frac{1}{n} \leq a_n$ for all $n$. Now observe within each interval $(2^k, 2^{k+1}]$, there are $2^k$ terms. Therefore, $\sum_{n=2^k}^{2^{k+1}} a_n = \frac{1}{2}$ and $\sum_{n=2}^{\infty} a_n = \lim_{k\to\infty} k\left(\frac{1}{2}\right) = \infty$.

Hence, by Comparison Test, $\sum_{n=1}^{\infty} \frac{1}{n}$ also diverges. $\square$

# Chapter 15

**Proposition 18.** $\sum \left[sin\left(\frac{n\pi}{6}\right)\right]^n$ *diverges.*

*Proof.* Notice that when $n = 12k + 3$, $\left[sin\left(\frac{n\pi}{6}\right)\right]^n = 1$. Hence, the summation never converges. $\square$

**Proposition 19.** $\sum \left[sin\left(\frac{n\pi}{7}\right)\right]^n$ *converges.*

*Proof.* We will show that the summation converges absolutely, hence converges.

Notice $\left|sin\left(\frac{n\pi}{7}\right)\right|$ is always between 0 and 1. In fact, it is bounded by above by some $r < 1$ such that $\left|sin\left(\frac{n\pi}{7}\right)\right| \leq r < 1$ and $\left|sin\left(\frac{n\pi}{7}\right)\right|^n \leq r^n < 1$. Then by Comparison Test, $\sum \left|sin\left(\frac{n\pi}{7}\right)\right|^n$ converges because $\sum r^n$ converges, which can be shown easily by Ratio Test or Root Test. $\square$

**Proposition 20.** $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$ *converges if and only if* $p > 1$.

*Proof.* We proceed with Integral Test with $f(x) = \frac{1}{x(\log x)^p}$. Notice $f(x)$ is continuous, positive, and decreasing for $x \geq 2$. Also, $f(n) = a_n$. Then for $p \neq 1$

$$\lim_{n\to\infty} \int_2^n \frac{1}{x(\log x)^p} dx = \lim_{n\to\infty} \left[\frac{(\log x)^{1-p}}{1-p}\right]_2^n. \tag{2}$$

For $p = 1$, we have

$$\lim_{n\to\infty} \int_2^n \frac{1}{x(\log x)} dx = \lim_{n\to\infty} [\log(\log x)]_2^n = \infty. \tag{3}$$

Then for ($\Rightarrow$) direction, we know that if $p = 1$, (2) goes to infinity, thus the summation diverges. If $p < 1$, (1) goes to infinity, thus the summation diverges again. Hence, forward direction is shown by contrapositive.

For ($\Leftarrow$) direction, we know that if $p > 1$, (1) converges, thus the summation converges. $\square$

**Proposition 21.** $\sum_{n=4}^{\infty} \frac{1}{n(\log n)(\log\log n)}$ *diverges.*

*Proof.* We proceed with Integral Test with $f(x) = \frac{1}{x(\log x)(\log\log x)}$. Notice $f(x)$ is decreasing, $f(n) = a_n$, and all $a_n$ are nonnegative. Then

$$\lim_{n\to\infty} \int_4^n \frac{1}{x(\log x)(\log\log x)} dx = \lim_{n\to\infty} [\log(\log(\log x))]_4^n = \infty.$$

$\square$

**Proposition 22.** $\sum_{n=2}^{\infty} \frac{\log n}{n^2}$ *converges.*

*Proof.* **Integral Test:**

We can integrate $f(x) = \frac{\log x}{x^2}$ to get

$$\begin{aligned}
\lim_{n\to\infty} \int_2^n \frac{\log x}{x^2} dx &= \lim_{n\to\infty} \int_2^n -(\log x) d\left(\frac{1}{x}\right) \\
&= \lim_{n\to\infty} \left[-\frac{\log x}{x}\right]_2^n + \lim_{n\to\infty} \int_2^n \frac{1}{x^2} dx \\
&= \lim_{n\to\infty} \left[-\frac{\log x}{x}\right]_2^n - \lim_{n\to\infty} \left[\frac{1}{x}\right]_2^n \\
&= \frac{1}{2},
\end{aligned}$$

and conclude that $\sum_{n=2}^{\infty} \frac{\log n}{n^2}$ converges.

**Comparison Test:**

We know that for $n > N$ where $N$ is some constant, $\sqrt{n} > \log n$. This can be proved by obversing that $\sqrt{n} > \log n$ when $n = 100$, and we see by first derivative that $\sqrt{n}$ has a higher increasing rate than $\log n$ for all $n$. Hence, we can conclude that $\sqrt{n} > \log n$ for all $n \geq 100$.

Then, we see $\frac{\log n}{n^2} < \frac{1}{n^{3/2}}$ for $n \geq 100$. We know that $\sum \frac{1}{n^{3/2}}$ converges for $p > 1 \Rightarrow \sum_{100}^{\infty} \frac{1}{n^{3/2}}$ converges $\Rightarrow \sum_{100}^{\infty} \frac{\log n}{n^2}$ converges $\Rightarrow \sum_{2}^{\infty} \frac{\log n}{n^2}$ converges. □

**Proposition 23.** *If $(a_n)$ is a decreasing sequence of real numbers and if $\sum a_n$ converges, then $\lim na_n = 0$.*

*Proof.* Since $\sum a_n$ converges, we know it satisfies Cauchy criterion. In other words, there exists $N$ such that for $n \geq m > N$,

$$\left|\sum_{k=m}^{n} a_k\right| = \sum_{k=m}^{n} a_k < \frac{\epsilon}{2}.$$

Notice $(a_n)$ is a decreasing sequence and $\lim a_n = 0$ is a necessary condition for $\sum a_n$ to converge, hence all $a_n$ are non-negative. Therefore, we can remove the absolute value.

In particular, for $m = N + 1$,

$$\begin{aligned}
(n - N)a_n &\leq \sum_{k=N+1}^{n} a_k \\
na_n &\leq \sum_{k=N+1}^{n} a_k + Na_n. \qquad (4)
\end{aligned}$$

Note that since $(a_n)$ converges to 0 and all $a_n$ are non-negative, we know there exists $N'$ such that

$$a_n = |a_n - 0| < \frac{\epsilon}{2N}.$$

Therefore, putting back to (4), as long as $n > \max\{N, N'\}$

$$|na_n - 0| = na_n \leq \sum_{k=N+1}^{n} a_k + Na_n < \frac{\epsilon}{2} + N\frac{\epsilon}{2N} = \epsilon.$$

□

# Chapter 17

**Proposition 24.** *$\sqrt{x}$ is continuous on its domain $[0, \infty)$.*

*Proof.* We consider non-negative sequences $(x_n)$ because any sequence with negative terms would not make sense on $f(x_n)$.

We have to show that if $(x_n)$ converges to some $x_0 \in [0, \infty)$, $f(x_n) = \sqrt{x_n}$ converges to $\sqrt{x_0}$. In other words, we seek to find $N$ such that for $\epsilon > 0$, as long as $n > N$,

$$|\sqrt{x_n} - \sqrt{x_0}| < \epsilon.$$

Consider $x_0 = 0$, since $(x_n) \to 0$, there exists some $N$ such that for $n > N$,

$$\begin{aligned} |x_n| &< \epsilon^2 \\ \sqrt{|x_n|} &< \epsilon \\ |\sqrt{x_n}| &< \epsilon. \end{aligned}$$

Therefore, $\sqrt{x}$ is continouous at $x = 0$.

For $x_0 > 0$, we need to show the existence of $N$ such that for $n > N$,

$$\begin{aligned} |\sqrt{x_n} - \sqrt{x_0}| &< \epsilon \\ \left| \frac{x_n - x_0}{\sqrt{x_n} + \sqrt{x_0}} \right| &< \epsilon \\ \left| \frac{x_n - x_0}{\sqrt{x_n} + \sqrt{x_0}} \right| \leq \left| \frac{x_n - x_0}{\sqrt{x_0}} \right| &< \epsilon \\ \left| \frac{x_n - x_0}{\sqrt{x_0}} \right| &< \epsilon \\ |x_n - x_0| &< \sqrt{x_0} \cdot \epsilon. \end{aligned}$$

Indeed, there exists such $N$ because $(x_n)$ converges to $x_0$. $\square$

**Proposition 25.** *If $m \in \mathbb{N}$, then the function $f(x) = x^m$ is continuous on $\mathbb{R}$.*

*Proof.* Suppose an arbitrary sequence $(x_n)$ that converges to $x_0 \in \mathbb{R}$, then

$$\begin{aligned} \lim f(x_n) &= \lim x_n^m \\ &= (\lim x_n)^m \\ &= x_0^m \\ &= f(x_0). \end{aligned}$$

$\square$

**Corollary 26.** *Every polynomial function $p(x) = a_0 + a_1 x + \cdots + a_n x^n$ is continuous on $\mathbb{R}$.*

*Proof.* Let $f_k = a_k x^k$ for $k \in [0, n]$. We have shown in previous proposition that $x^k$ is continuous for $k \in [1, n]$. Notice $x^0 = 1$ is continuous on $\mathbb{R}$ apparently because every sequence $(x_n)$ that converges to 1 will have $(x_n)^0 = 1$.

Therefore, $f_k$ is continuous on $\mathbb{R}$ because it's a scalar multiple of a continuous function. Consequently, $p(x)$ is continuous on $\mathbb{R}$ because it's a sum of continuous functions. $\square$

**Proposition 27.** *A rational function is a function $f$ of the form $p/q$ where $p, q$ are polynomial functions. The domain of $f$ is $\{x \in \mathbb{R} : q(x) \neq 0\}$. Then every rational function is continuous.*

*Proof.* From *Corollary 26*, we know every polynomial is continous on $\mathbb{R}$ and of course on $\mathbb{R}\backslash\{x : q(x) = 0\}$. Hence $p/q$ is continuous since it's a division of two continuous functions where $q \neq 0$. $\square$

**Proposition 28.** $\min(f, g) = \frac{1}{2}(f + g) - \frac{1}{2}|f - g|$.

*Proof.* Assume $f > g$, then

$$\begin{aligned}\frac{1}{2}(f+g) - \frac{1}{2}|f-g| &= \frac{1}{2}(f+g) - \frac{1}{2}(f-g) \\ &= g \\ &= \min(f, g).\end{aligned}$$

Assume $g \geq f$, then

$$\begin{aligned}\frac{1}{2}(f+g) - \frac{1}{2}|f-g| &= \frac{1}{2}(f+g) - \frac{1}{2}(g-f) \\ &= f \\ &= \min(f, g).\end{aligned}$$

$\square$

**Proposition 29.** $\min(f, g) = -\max(-f, -g)$.

*Proof.*

$$\begin{aligned}\max(f, g) &= \frac{1}{2}(f+g) + \frac{1}{2}|f-g| \\ \max(-f, -g) &= \frac{1}{2}(-f-g) + \frac{1}{2}|-f+g| \\ \max(-f, -g) &= -\frac{1}{2}(f+g) + \frac{1}{2}|g-f| \\ -\max(-f, -g) &= \frac{1}{2}(f+g) - \frac{1}{2}|f-g| \\ -\max(-f, -g) &= \min(f, g).\end{aligned}$$

$\square$

**Corollary 30.** *If $f$ and $g$ are continuous at $x_0$ in $\mathbb{R}$, then $\min(f, g)$ is continuous at $x_0$.*

*Proof.* $f, g \in C^0(x_0) \Rightarrow f + g, f - g \in C^0(x_0) \Rightarrow f + g, |f - g| \in C^0(x_0) \Rightarrow \frac{1}{2}(f + g), \frac{1}{2}|f - g| \in C^0(x_0) \Rightarrow \frac{1}{2}(f + g) + \frac{1}{2}|f - g| \in C^0(x_0) \Rightarrow \min(f, g) \in C^0(x_0)$. $\square$

**Proposition 31.** $g(x) = x^3$ *is continuous on* $\mathbb{R}$.

*Proof.* Consider arbitrary $c \in \mathbb{R}$. We seek to find $\delta$ such that for $|x - c| < \delta$, $|x^3 - c^3| < \epsilon$.

Let's evaluate $|x^3 - c^3|$:

$$\begin{aligned}|x^3 - c^3| &= |x - c||x^2 + xc + c^2| \\ &\leq |x - c| \left(|x^2| + |xc| + |c^2|\right).\end{aligned}$$

For $|x-c|<1$, then

$$|x^2|+|xc|+|c^2|<|c+1|^2+|c+1||c|+|c^2|.$$

Let $M=|c+1|^2+|c+1||c|+|c^2|>0$. Hence, let $|x-c|<\delta=\min\{\epsilon/M,1\}$, then

$$|x^3-c^3|<|x-c|M<\frac{\epsilon}{M}M=\epsilon.$$

$\square$

**Proposition 32.** *Let $f$ be a continuous real-valued function with domain $(a,b)$. If $f(r)=0$ for each rational number $r$ in $(a,b)$, then $f(x)=0$ for all $x\in(a,b)$.*

*Proof.* We will show that if there are some $x_0\in(a,b)$ such that $f(x_0)=c\neq 0$, then $f$ is not continuous at $x_0$, hence not continuous on $(a,b)$.

Assume for the sake of contradiction that there exists $x_0\in(a,b)\backslash\mathbb{Q}$ such that $f(x_0)=c\neq 0$. Let $\epsilon=\frac{|c|}{2}$. However for $|f(x_0)-f(x)|<\epsilon$, there does not exist $\delta$ such that $|x-x_0|<\delta\Rightarrow|f(x_0)-f(x)|<\epsilon$ because there is always a rational number $x'\in(x_0-\delta,x_0+\delta)$ according to the density of $\mathbb{Q}$, and $|f(x')-f(x_0)|=|c|>\epsilon$. Then, the contradiction that $f$ is not continuous is reached. $\square$

**Corollary 33.** *Let $f$ and $g$ be continuous real-valued functions on $(a,b)$ such that $f(r)=g(r)$ for each rational number $r$ in $(a,b)$, then $f(x)=g(x)$ for all $x\in(a,b)$.*

*Proof.* Let $f(r)=g(r)=c$ and define $f'(x)=f(x)-c$ and $g'(x)=g(x)-c$. Then from *Proposition 33*, we know $f'(x)=g'(x)=0$ because $f'$ and $g'$ are continuous and $f'(r)=g'(r)=0$. Therefore, $f'(x)=g'(x)=0\Rightarrow f'(x)+c=g'(x)+c=c\Rightarrow f(x)=g(x)=c$ for all $x\in(a,b)$. $\square$

**Proposition 34.** *Let $h(x)=x$ for rational numbers $x$ and $h(x)=0$ for irrational numbers, then $h$ in continuous only at $x=0$.*

*Proof.* $x_0=0$ : Let $\epsilon>0$. Then for $|x-x_0|<\epsilon$, $|h(x)-h(x_0)|=|h(x)-0|<\epsilon$ because for irrational numbers $x$, $|h(x)-0|=0<\epsilon$, and for rational numbers $x$, $|h(x)-0|=|x-0|<\epsilon$.

$x_0\neq 0$ **and irrational**: Let $\epsilon=\frac{|x_0|}{2}$. Assume for the sake of contradiction that there exists $\delta$ such that $|x-x_0|<\delta\Rightarrow|h(x)-0|<\epsilon$. However, by density of $\mathbb{Q}$ there exists a rational number $x'\in(x_0-\delta,x_0+\delta)$ such that $|x'|>|x_0|$ [$x'\in(x_0,x_0+\delta)$ if $x_0>0$, $x'\in(x_0-\delta,x_0)$ if $x_0\leq 0$]. Then $|h(x')-0|=|x'|>|x_0|>\epsilon$.

$x_0\neq 0$ **and rational**: Let $\epsilon=\frac{|x_0|}{2}$. Assume for the sake of contradiction that there exists $\delta$ such that $|x-x_0|<\delta\Rightarrow|h(x)-x_0|<\epsilon$. However, by density of irrationals there exists an irrational number $x'\in(x_0-\delta,x_0+\delta)$. Then $|h(x')-x_0|=|0-x_0|=|x_0|>\epsilon$. $\square$

**Proposition 35.** *For each nonzero rational number $x$, write $x=\frac{p}{q}$ where $p,q$ are integers with no common factors and $q>0$, and then define $f(x)=\frac{1}{q}$. Also define $f(0)=1$ and $f(x)=0$ for all $x\in\mathbb{R}\backslash\mathbb{Q}$. Thus $f(x)=1$ for each integer, $f(\frac{1}{2})=f(\frac{-1}{2})=f\frac{3}{2}=\cdots=\frac{1}{2}$, etc. Then $f$ is continuous at each point of $\mathbb{R}\backslash\mathbb{Q}$ and discontinuous at each point of $\mathbb{Q}$.*

*Proof.* **Discontinuous on** $\mathbb{Q}$: Let $x_0\in\mathbb{Q}$. Define a sequence of irrational numbers $(x_n)$ that converges to $x_0$, but $f(x_n)=0$ for all $n$ and obviously $\lim x_n=0\neq f(x_0)$ [notice $f(x_0)$ is never 0 for rational number $x_0$].

**Continuous on** $\mathbb{R}\backslash\mathbb{Q}$: Let $x_0\in\mathbb{R}\backslash\mathbb{Q}$ and $\epsilon>0$. Then let $N=\min\{n>\frac{1}{\epsilon}:n\in\mathbb{N}\}$. Now consider the interval $I=(x_0-1,x_0+1)$. We know there is only a finite number of $\frac{p}{q}\in I$ such that $q<N$ [consider $q=N-1$, there are only finite multiples of $\frac{1}{q}\in I$ because the multiple will eventually "step over" the interval; this is true all for $q<N$]. Since there is only a finite number of such fractions, we can take the

fraction that is closest to $x_0$, denote $r$, and let $\delta = |r - x_0|$. Then, the interval $(x_0 - \delta, x_0 + \delta)$ only contains either irrational numbers or $\frac{p}{q}$ with $q \geq N > \frac{1}{\epsilon}$.

Therefore, for rational $x$ such that $|x - x_0| < \delta$, $f(x) \leq \frac{1}{N} < \epsilon \Rightarrow |f(x) - f(x_0)| = |f(x) - 0| < \epsilon$. Obviously for irrational $x$, $|f(x) - f(x_0)|$ is always 0.

Intuition: We can zoom into $x_0$ to find an interval where only very very small mesh of fractions is contained in there because all larger fractions would "step over" the interval $\square$