Math 104 HW2

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Exercise 4.1

For each set below that is bounded above, list three upper bounds for the set. Otherwise write "NOT BOUNDED ABOVE".

(a) [0,1]

Solution. $\{2, 3, 4\}$ $\square$

(c) $\{2,7\}$

Solution. $\{8, 9, 10\}$ $\square$

(e) $\{\frac{1}{n} : n \in \mathbb{N}\}$

Solution. $\{8, 9, 10\}$ $\square$

(g) $[0,1] \cup [2,3]$

Solution. $\{8, 9, 10\}$ $\square$

(i) $\cap_{n=1}^{\infty} \left[\frac{-1}{n}, 1 + \frac{1}{n}\right]$

Solution. $\{8,9,10\}$ $\square$

(k) $\{n + \frac{(-1)^n}{n} : n \in \mathbb{N}\}$

Solution. NOT BOUNDED ABOVE $\square$

(m) $\{r \in \mathbb{Q} : r^2 < 4\}$

Solution. $\{8,9,10\}$ $\square$

(o) $\{x \in \mathbb{R} : x < 0\}$

Solution. $\{8,9,10\}$ $\square$

(q) $\{0, 1, 2, 4, 8, 16\}$

Solution. $\{20, 30, 40\}$ $\square$

(s) $\{\frac{1}{n} : n \in \mathbb{N} \textit{ and } n \textit{ is prime}\}$

Solution. $\{20, 30, 40\}$ $\square$

(u) $\{x^2 : x \in \mathbb{R}\}$

Solution. NOT BOUNDED ABOVE ☐

(w) $\{sin\left(\frac{n\pi}{3}\right) : n \in \mathbb{N}\}$

Solution. {20, 30, 40} ☐

Exercise 4.2

Repeat Exercise 4.1 for lower bounds.
(a) [0,1]

Solution. {-2, -3, -4} ☐

(c) {2,7}

Solution. {-8, -9, -10} ☐

(e) $\{\frac{1}{n} : n \in \mathbb{N}\}$

Solution. {-8, -9, -10} ☐

(g) $[0,1] \cup [2,3]$

Solution. {-8, -9, -10} ☐

(i) $\cap_{n=1}^{\infty}\left[\frac{-1}{n}, 1+\frac{1}{n}\right]$

Solution. {-8,-9,-10} ☐

(k) $\{n + \frac{(-1)^n}{n} : n \in \mathbb{N}\}$

Solution. {-8,-9,-10} ☐

(m) $\{r \in \mathbb{Q} : r^2 < 4\}$

Solution. {-8,-9,-10} ☐

(o) $\{x \in \mathbb{R} : x < 0\}$

Solution. NOT BOUNDED BELOW ☐

(q) {0, 1, 2, 4, 8, 16}

Solution. {-20, -30, -40} ☐

(s) $\{\frac{1}{n} : n \in \mathbb{N} \textit{ and } n \textit{ is prime}\}$

Solution. {-20, -30, -40} ☐

(u) $\{x^2 : x \in \mathbb{R}\}$

Solution. {-20, -30, -40} ☐

(w) $\{sin\left(\frac{n\pi}{3}\right) : n \in \mathbb{N}\}$

Solution. {-20, -30, -40} ☐

Exercise 4.8

Let S and T be nonempty subsets of R with the following property: $s \leq t$ for all $s \in S$ and $t \in T$.

(a) Oberserve that S is bounded above and T is bounded below.

Proof. $T \subseteq U(S), S \subseteq L(T)$. □

(b)

Proposition 1. *Sup $S \leq$ inf T.*

Proof. Assume for the sake of contradiction that sup $S >$ inf T. Then inf T can be written as inf $T =$ sup $S - \epsilon$ for some $\epsilon > 0$. Notice that there exists $s \in S$ such that $s >$ sup $S - \epsilon$ [otherwise sup$S - \epsilon$ would be a smaller upper bound]. This implies that there exists $s \in S$ such that $s >$ inf T. That means inf T is not the largest lower bound of T [s is a larger lower bound], which is a contradiction. Hence, sup $S \leq$ inf T. □

(c) Give an exampe of such sets S and T where $S \cap T$ is nonempty.

Solution. $S = \{s \leq 0 : s \in \mathbb{R}\}$, $T = \{t \geq 0 : t \in \mathbb{R}\}$, $S \cap T = \{0\}$. □

(d) Give an example of sets S and T where sup $S =$ inf T and $S \cap T$ is an empty set.

Solution. $S = \{s < 0 : s \in \mathbb{R}\}$, $T = \{t > 0 : t \in \mathbb{R}\}$, $S \cap T = \emptyset$. □

Exercise 4.14

Let A and B be nonempty bounded subsets of $\mathbb{R}$, and let $A + B$ be the set of all sums $a + b$ where $a \in A$ and $b \in B$.

(a)

Proposition 2. *Sup $(A + B) =$ sup A + sup B. Hint: To show sup A + sup $B \leq$ sup $(A + B)$, show that for each $b \in B$, sup $(A + B) - b$ is an upper bound for A, hence sup $A \leq$ sup $(A + B) - b$. Then show sup $(A + B)$ - sup A is an upper bound for B.*

Proof. We proceed by first showing $sup(A+B) \leq supA+supB$, then showing $sup(A+B) \geq supA+supB$.

$\underline{sup(A + B) \leq supA + supB}$. For all $x \in A + B$, $x = a + b$ for $a \in A, b \in B$. Hence, $x = a + b \leq supA + supB \Rightarrow supA + supB \subseteq U(A + B) \Rightarrow sup(A + B) \leq supA + supB$.

$\underline{sup(A + B) \geq supA + supB}$. Assume for the sake of contradiction that $sup(A + B) < supA + supB$. Then $sup(A+B) = supA+supB-\epsilon$ for some $\epsilon > 0$. Notice $\exists b \in B$ and $\exists a \in A$ such that $supA-\epsilon/2 < a$ and $supB - \epsilon/2 < b$. Then $supA + supB - \epsilon = sup(A + B) < a + b$, which is a contradiction. Hence, $sup(A + B) \geq supA + supB$. □

(b)

Proposition 3. *Inf $(A + B) =$ inf A + inf B.*

Proof. We proceed by first showing $inf(A+B) \geq infA + infB$, then showing $inf(A+B) \leq infA + infB$.

$\underline{inf(A+B) \geq infA + infB}$. For all $x \in A+B$, $x = a+b$ for $a \in A, b \in B$. Hence, $x = a+b \geq infA + infB \Rightarrow infA + infB \subseteq L(A+B) \Rightarrow inf(A+B) \geq infA + infB$.

$\underline{inf(A+B) \leq infA + infB}$. Assume for the sake of contradiction that $inf(A+B) > infA + infB$. Then $inf(A+B) = infA + infB + \epsilon$ for some $\epsilon > 0$. Notice $\exists b \in B$ and $\exists a \in A$ such that $infA + \epsilon/2 > a$ and $infB + \epsilon/2 > b$. Then $infA + infB + \epsilon = inf(A+B) > a+b$, which is a contradiction. Hence, $inf(A+B) \leq infA + infB$. □

Exercise 4.16

Proposition 4. *Sup* $\{r \in \mathbb{Q} : r < a\} = a$ *for each* $a \in \mathbb{R}$.

Proof. Denote $A = \{r \in \mathbb{Q} : r < a\}$. We proceed by first showing a is an upper bound of A, then showing a is the least upper bound of A.

$\underline{a \text{ is an upper bound of } A}$. For all $r \in A$, $r < a \Rightarrow r \leq a$. Hence, a is an upper bound of A. Trivial.

$\underline{a \text{ is the least upper bound of } A}$. Assume for the sake of contradiction that $supA < a$, then $supA = a - \epsilon$ for some $\epsilon > 0$. Now by the Archimedean Property, there exists $n \in \mathbb{N}$ such that $1/n < \epsilon$. Then, we can take $r = supA + 1/n = a - \epsilon + 1/n < a$. This implies that $r \in A$ and $r > supA$, which is a contradiction. Hence, a is the least upper bound of A. □

Exercise 5.5

Proposition 5. *Inf* $S \leq$ *sup* S *for every nonempty subset of* $\mathbb{R}$. *Consider both bounded and unbounded sets.*

Proof.

Case 1: S is bounded above and below. Then $infS \leq s \in S$ and $supS \geq s \in S$ for $infS, supS \in \mathbb{R}$. Hence, $infS \leq s \leq supS$.

Case 2: S is bounded above and unbounded below. Then $infS = -\infty \leq s \in S$ and $supS \in \mathbb{R} \geq s \in S$. Obviously, $-\infty \leq supS$.

Case 3: S is unbounded above and bounded below. Then $infS \in \mathbb{R} \leq s \in S$ and $supS = \infty \geq s \in S$. Obviously, $infS \leq \infty$.

Case 4: S is unbounded above and below. Then $infS = -\infty$ and $supS = \infty$. Obviously, $-\infty \leq \infty$.

□