Math 180B HW4

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PK Exercise 3.5.6

$$\begin{aligned}
E[T] &= E[T|\xi_1 \geq 100]P(\xi_1 \geq 100) + E[T|\xi_1 < 100]P(\xi_1 < 100) \\
E[T] &= 1 + E[T]P(\xi_1 < 100) \\
E[T](1 - P(\xi_1 < 100)) &= 1 \\
E[T] &= \frac{1}{P(\xi_1 \geq 100)} \\
E[T] &= \frac{1}{\sum_{k=100}^{\infty} 0.01(0.99)^k} \\
&= \frac{1}{0.01(0.99)^{100} \sum_{k=0}^{\infty}(0.99)^k} \\
&= \frac{1}{0.01(0.99)^{100} \frac{1}{1-0.99}} \\
&= \frac{1}{0.99^{100}}.
\end{aligned}$$

PK Problem 3.5.2

(a) Define p_i to be the probability of the component failing at $T = i + 1$, which means it has a lifetime of $T = i + 1$ given $T > i$, so $p_i = \frac{a_{i+1}}{\sum\limits_{k=i+1}^{\infty} a_k}$. Define q_i to be the probability that the component is not failing at $T = i+1$, which is just the complement of p_i, so $q_i = 1-p_i$. Define the probability transition matrix to be:

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \vdots \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & \cdots \end{array} \\ \left\Vert \begin{array}{cccc} p_0 & q_0 & 0 & \cdots \\ p_1 & 0 & q_1 & \cdots \\ p_2 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{array} \right\Vert \end{array},$$

(b) For $i \in [0, N-2], p_i = \frac{a_{i+1}}{\sum\limits_{k=i+1}^{\infty} a_k}, q_i = 1 - p_i$. For $i = N-1, p_{N-1} = 1, q_{N-1} = 0$.

PK Exercise 3.6.1

(a) Define $P(R)$ to be the probability that the rat finds the food before getting shocked given it is in box r at the current state. Then, by first step analysis,

$$\begin{aligned}
P(R = r) &= 0.5P(R = r - 1) + 0.5P(R = r + 1) \\
P(R) - P(R = r - 1) &= P(R = r + 1) - P(R = r).
\end{aligned}$$

By observing the equation, we can see that $P(R)$ is a linear function of r. Then, let $P(R=5)=1, P(R=0)=0$. Hence, $P(R=3)=3\times\frac{1}{5}=\frac{3}{5}$.

(b) Simply plugging p, q into the formula, we get

$$P(R=3)=\frac{(p/q)^{5-3}-(p/q)^5}{1-(p/q)^5}$$

PK Exercise 3.8.1

$$\begin{aligned}
E[\xi] &= \frac{1}{2}\cdot 0+\frac{1}{2}\cdot 2 = 1\\
Var(\xi) &= \frac{1}{2}(-1)^2+\frac{1}{2}(2-1)^2 = 1\\
E[X_n] &= E[\xi]^n\\
&= 1^n = 1.\\
Var(X_n) &= n\cdot Var(\xi)E[\xi]^{n-1}\\
&= n\cdot 1\cdot 1^{n-1} = n.
\end{aligned}$$

PK Exercise 3.8.3

$$\begin{aligned}
u_0 &= 0\\
u_1 &= \frac{1}{2}+\frac{1}{2}(u_0)^2 = \frac{1}{2}\\
u_2 &= \frac{1}{2}+\frac{1}{2}(u_1)^2 = \frac{5}{8}\\
u_3 &= \frac{1}{2}+\frac{1}{2}(u_2)^2 = \frac{89}{128}\approx 0.695\\
u_4 &= \frac{1}{2}+\frac{1}{2}(u_3)^2 = \frac{24305}{32768}\approx 0.742\\
u_5 &= \frac{1}{2}+\frac{1}{2}(u_4)^2 \approx 0.775.
\end{aligned}$$

PK Problem 3.8.2 *Correction to problem statement in the textbook:* $Z=\sum_{n=0}^{\infty}X_n$

$$\begin{aligned}
E[Z] &= E\left[\sum_{n=0}^{x}X_n\right]\\
&= E[X_0]+E[X_1]+E[X_2]+\dots\\
&= 1+\mu+\mu^2+\dots\\
&= \frac{1}{1-\mu}.
\end{aligned}$$