Math 181A HW4

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Problem 5.4.2

(a)

$$\begin{aligned}
P(2.8 \leq \hat{\theta} \leq 3.2) = P(\hat{\theta} \geq 2.8) &= 1 - P(\{Y_1, \ldots, Y_6\} \leq 2.8) \\
&= 1 - P(Y_1 \leq 2.8)^6 \\
&= 1 - \left(\frac{2.8}{3}\right)^6 \\
&\approx 0.339.
\end{aligned}$$

(b)

$$\begin{aligned}
P(2.8 \leq \hat{\theta} \leq 3.2) = P(\hat{\theta} \leq 2.8) &= 1 - P(\{Y_1, \ldots, Y_3\} \leq 2.8) \\
&= 1 - P(Y_1 \leq 2.8)^3 \\
&= 1 - \left(\frac{2.8}{3}\right)^3 \\
&\approx 0.187.
\end{aligned}$$

Problem 5.4.15

$$\begin{aligned}
E[\overline{W}^2] &= Var(\overline{W}) + (E[\overline{W}])^2 \\
&= Var(\frac{1}{n}\sum_{i=1}^{n} W_i) + \left(E[\frac{1}{n}\sum_{i=1}^{n} W_i]\right)^2 \\
&= \frac{1}{n^2}\sum_{i=1}^{n} Var(W_i) + \left(\frac{1}{n}\sum_{i=1}^{n} E[W_i]\right)^2 \\
&= \frac{n\sigma^2}{n^2} + \left(\frac{n\mu}{n}\right)^2 \\
&= \frac{\sigma^2}{n} + \mu^2, \\
\lim_{n\to\infty} E[\overline{W}^2] &= \lim_{n\to\infty} \frac{\sigma^2}{n} + \mu^2 \\
&= \mu^2.
\end{aligned}$$

Problem 5.4.18 By symmetry, $Var(Y_{min}) = Var(Y_{max})$, let σ^2. Hence,

$$\begin{aligned}
MSE(\hat{\theta}_1) &= Var(\hat{\theta}_1) + Bias(\hat{\theta}_1) \\
&= Var(\hat{\theta}_1) \\
&= \frac{36}{25}\sigma^2. \\
MSE(\hat{\theta}_2) &= Var(\hat{\theta}_2) + Bias(\hat{\theta}_2) \\
&= Var(\hat{\theta}_2) \\
&= 36\sigma^2 \\
&> MSE(\hat{\theta}_1).
\end{aligned}$$

Therefore, $\hat{\theta}_1$ is better than $\hat{\theta}_2$.

Problem 5.4.21 With the same notation from last question, then

$$\begin{aligned}
\frac{Var(\hat{\theta}_1)}{Var(\hat{\theta}_2)} &= \frac{(n+1)^2\sigma^2}{\left(\frac{n+1}{n}\right)^2\sigma^2} \\
&= n^2.
\end{aligned}$$

Problem 5.5.2

$$
\begin{aligned}
l(\lambda) &= \log(\frac{e^{-\lambda}\lambda^k}{k!}) \\
&= -\lambda + k\log(\lambda) - \log(k!), \\
l'(\lambda) &= -1 + \frac{k}{\lambda}, \\
l''(\lambda) &= -\frac{k}{\lambda^2}. \\
I(\lambda) &= -E[l''(\lambda)] \\
&= -E[\frac{-k}{\lambda^2}] \\
&= \sum_{k=0}^{\infty} \frac{k}{\lambda^2} \cdot e^{-\lambda}\frac{\lambda^k}{k!} \\
&= \frac{e^{-\lambda}}{\lambda} \sum_{k=0}^{\infty} \frac{\lambda^{k-1}}{(k-1)!} \\
&= \frac{e^{-\lambda}}{\lambda} \left(\frac{0 \cdot \lambda^{k-1}}{0!} + \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!} \right) \\
&= \frac{e^{-\lambda}}{\lambda} \left(\sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \right) \\
&= \frac{e^{-\lambda}}{\lambda} \cdot e^{\lambda} \\
&= \frac{1}{\lambda}. \\
Var(\hat{\lambda}) &= Var\left(\frac{1}{n} \sum_{i=1}^{n} X_i \right) \\
&= \frac{1}{n^2} Var\left(\sum_{i=1}^{n} X_i \right) \\
&= \frac{1}{n} Var(X_i) \\
&= \frac{1}{n}\lambda \\
&= \frac{1}{n(1/\lambda)} \\
&= \frac{1}{nI(\lambda)}.
\end{aligned}
$$