Math 180A HW6

Neo Lee

02/22/2023

Problem 1.

1. (b)
2. (a)
3.

$$\begin{aligned} E[X^2] &= Var(X) + (E[X])^2 && (1) \\ &= \sigma^2 + \mu^2 && (2) \\ &= 4 + 9 && (3) \\ &= 7 && (4) \end{aligned}$$

Problem 2. (c)

Problem 3.

(a)

$$\begin{aligned} P(X \geq 10 + 15 | X \geq 10) = P(X \geq 15) &= e^{-(\frac{1}{20} \times 15)} && (5) \\ &\approx 0.4724 && (6) \end{aligned}$$

(b)

$$\begin{aligned} P(X \geq 25) &= \int_{25}^{40} \frac{1}{40} dx && (7) \\ &= (40 - 25) \times \frac{1}{40} && (8) \\ &= \frac{3}{8} && (9) \end{aligned}$$

(c) Since exponential random variable is memoryless, the probability of waiting for at least 15 additional minutes is always the same given the time before has already passed.

On the other hand, uniform random vairable is not memoryless. In fact, the probability is eventually ditributed throughout the interval $[0, 40]$. Therefore, the probabily of waiting for at least 15 additional minutes decreases as time progresses.

Problem 4.

(a)

$$\begin{aligned}
E[Z^n] &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z^n e^{\frac{-z^2}{2}} dz && (10)\\
&= \frac{1}{\sqrt{2\pi}} \left[\left[-z^{n-1} e^{\frac{-z^2}{2}} \right]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} (n-1) z^{n-2} e^{\frac{-z^2}{2}} dz \right] && (11)\\
&= \frac{n-1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z^{n-2} e^{\frac{-z^2}{2}} dz && (12)\\
&= (n-1)E[Z^{n-2}] && (13)
\end{aligned}$$

Hence, $E[Z^3] = (3-1)E[Z] = 0$.

(b)

$$\begin{aligned}
E[X^3] &= E[(\sigma Z + \mu)^3] && (14)\\
&= E[\sigma^3 Z^3 + \mu^3] && (15)\\
&= \sigma^3 E[Z^3] + E[\mu^3] && (16)\\
&= E[\mu^3] && (17)\\
&= \mu^3 && (18)
\end{aligned}$$

Problem 5. For $a \in [0, 1]$,

$$\begin{aligned}
P(X \in [0,1] \cap X \in [a,2]) &= P(X \in [a,1]) && (19)\\
&= e^{-2a} - e^{-2}. && (20)
\end{aligned}$$

For the events to be independent,

$$\begin{aligned}
P(X \in [0,1] \cap X \in [a,2]) &= P(X \in [0,1])P(X \in [a,2]) && (21)\\
e^{-2a} - e^{-2} &= \left(1 - e^{-2}\right) \times \left(e^{-2a} - e^{-4}\right) && (22)\\
e^{-2a} - e^{-2} &= e^{-2a} - e^{-4} - e^{-2a-2} + e^{-6} && (23)\\
e^{-2a-2} &= e^{-6} - e^{-4} + e^{-2} && (24)\\
e^{-2a-2} &= e^{-2}\left(e^{-4} - e^{-2} + 1\right) && (25)\\
-2a - 2 &= -2 + ln(e^{-4} - e^{-2} + 1) && (26)\\
a &= \frac{ln(e^{-4} - e^{-2} + 1)}{-2} && (27)\\
&\approx 0.0622 && (28)
\end{aligned}$$

Problem 6.

(a) There can be two scenarios: 1) if the stove breaks within r years, profit = \$$C - 600$; 2) if the stove lasts longer than r years, profit = \$$200 + C$.

Let $g(x)$ be a function that calculates the profit, then

$$\begin{aligned}
E[g(x)] &= (C - 600) \times (1 - P(X \geq r)) + (200 + C) \times P(X \geq r) && (29)\\
&= (C - 600) \times (1 - e^{\frac{-r}{10}}) + (200 + C) \times e^{\frac{-r}{10}} && (30)\\
&= C - 600 + 600e^{\frac{-r}{10}} + 200e^{\frac{-r}{10}} && (31)\\
&= 800e^{\frac{-r}{10}} - 600 + C && (32)
\end{aligned}$$

(b)

$$800e^{\frac{-5}{10}} - 600 + C = 0 \tag{33}$$
$$C = 600 - 800e^{-0.5} \tag{34}$$
$$\approx 114.8 \tag{35}$$

Problem 7.

$$P((X, Y) \in [0, 1]) = 1 \tag{36}$$
$$c\int_0^1 \int_0^1 xy + y^2 dxdy = 1 \tag{37}$$
$$c\int_0^1 \left[\frac{1}{2}x^2y + y^2\right]_{x=0}^{x=1} = 1 \tag{38}$$
$$c\int_0^1 \frac{1}{2}y = 1 \tag{39}$$
$$c\left[\frac{1}{4}y^2\right]_0^1 = 1 \tag{40}$$
$$\frac{1}{4}c = 1 \tag{41}$$
$$c = 4 \tag{42}$$