

Homework 1: Neo Lee

Introduction to Time Series, Fall 2023

Due Tuesday September 5 at 5pm

The total number of points possible for this homework is 41. The number of points for each question is written below, and questions marked as “bonus” are optional. Submit the **knitted html file** from this Rmd to Gradescope.

If you collaborated with anybody for this homework, put their names here:

Correlation and independence

1. (3 pts) Give an example to show that two random variables can be uncorrelated but not independent. You must explicitly prove that they are uncorrelated but not independent (for the latter, you may invoke any property that you know is equivalent to independence).

SOLUTION GOES HERE

Consider a uniform random variable X over the space of $\{-1, 0, 1\}$, and another random variable $Y = |X|$. Then, $Cov(X, Y) = E[XY] - E[X]E[Y] = (1 \times P(X = 1, Y = 1) - 1 \times P(X = -1, Y = 1) + 0 \times P(X = 0, Y = 0)) - 0 \times \frac{2}{3} = 0$. Hence, their correlation is also 0. However, we know X, Y are definitely not independent, because we have explicitly defined $Y = |X|$. We can also check by looking at $P(Y = 1|X = -1) = 1 \neq P(Y = 1) = \frac{2}{3}$.

2. (2 pts) If (X, Y) has a multivariate Gaussian distribution, and X, Y are uncorrelated: $Cov(X, Y) = 0$, then show that X, Y are independent.

SOLUTION GOES HERE

We just have to show that $f_{X,Y}(x, y) = f_X(x) \cdot f_Y(y)$. Now we evaluate the joint distribution

$$\begin{aligned} f_{X,Y}(x, y) &= \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \cdot \exp\left(-\frac{1}{2(1-\rho^2)}\left[\frac{(x-\mu_X)^2}{\sigma_X^2} - 2\rho\frac{(x-\mu_X)(y-\mu_Y)}{\sigma_X\sigma_Y} + \frac{(y-\mu_Y)^2}{\sigma_Y^2}\right]\right) \\ &= \frac{1}{2\pi\sigma_X\sigma_Y} \cdot \exp\left(-\frac{1}{2}\left[\frac{(x-\mu_X)^2}{\sigma_X^2} + \frac{(y-\mu_Y)^2}{\sigma_Y^2}\right]\right) \quad (\rho = 0 \because X, Y \text{ are uncorrelated}) \\ &= \frac{1}{\sqrt{2\pi}\sigma_X} \cdot \exp\left(-\frac{1}{2}\left[\frac{(x-\mu_X)^2}{\sigma_X^2}\right]\right) \cdot \frac{1}{\sqrt{2\pi}\sigma_Y} \cdot \exp\left(-\frac{1}{2}\left[\frac{(y-\mu_Y)^2}{\sigma_Y^2}\right]\right) \\ &= f_X(x) \cdot f_Y(y). \end{aligned}$$

3. (3 pts) Give an example to show that two random variables X, Y can be marginally Gaussian (meaning, X is Gaussian, and Y is Gaussian) and uncorrelated but *not* independent. Hint: (X, Y) cannot be multivariate Gaussian in this case.

SOLUTION GOES HERE

Define $X \sim N(0, 1)$. Then define

$$Y = \begin{cases} X & \text{with probability } \frac{1}{2} \\ -X & \text{with probability } \frac{1}{2}. \end{cases}$$

We can verify that Y is Gaussian by checking its *CDF*

$$\begin{aligned} P(Y \leq y) &= P(X \leq y | Y = X)P(Y = X) + P(-X \leq y | Y = -X)P(Y = -X) \\ &= P(X \leq y) \cdot \frac{1}{2} + P(X \geq -y) \cdot \frac{1}{2} \quad (\text{notice } P(X \leq y) = P(X \geq -y)) \\ &= P(X \leq y). \end{aligned}$$

Hence, $Y \sim N(0, 1)$. Now we check that X, Y are uncorrelated by evaluating their covariance

$$\begin{aligned} \text{Cov}(X, Y) &= E[XY] - E[X]E[Y] \\ &= E[XY] - 0 \\ &= E[XY | Y = X]P(Y = X) + E[XY | Y = -X]P(Y = -X) \\ &= E[X^2] \cdot \frac{1}{2} + E[-X^2] \cdot \frac{1}{2} \\ &= 0. \end{aligned}$$

However, we know X, Y are not independent, because $P(Y \leq -1 | -0.1 \leq X \leq 0.1) = 0 \neq P(Y \leq -1)$.

Random walks

4. (4 pts) Let $x_t, t = 1, 2, 3, \dots$ be a random walk with drift:

$$x_t = \delta + x_{t-1} + \epsilon_t,$$

where (say) $\epsilon_t \sim N(0, \sigma^2)$ for $t = 1, 2, 3, \dots$. Suppose that both δ and σ^2 are unknown. Devise a test statistic for the null hypothesis that $\delta = 0$. This should be based on a standard test that you know (have learned in a past course) for testing whether the mean of Gaussian is zero, with unknown variance, based on i.i.d. samples from this Gaussian.

State what the null distribution is for this test statistic, and how you would compute it in R (a function name is sufficient if the test statistic is implemented as a function in base R). Hint: consider taking differences along the sequence ... after that, what you want sounds like “c-test”, or “p-test”, or “ ϕ -test”, or ...

SOLUTION GOES HERE

We will proceed by testing the difference of the random walk sequence. Let $d_t = x_t - x_{t-1} = \delta + \epsilon_t$. Then $d_t \sim N(0 + \delta, \sigma^2)$ and d_t is *i.i.d.* Since σ is unknown, we can use the *t-test* to test whether the mean of d_t is zero, which means $\delta = 0$. The null distribution is t_{n-1} , where n is the length of the random walk sequence. The test statistic is

$$t = \frac{\bar{d}}{\hat{\sigma}/\sqrt{n}}.$$

We can compute it in R by using the function `t.test()`.

5. (2 pts) Simulate a random walk of length 100 *without* drift, i.e., $\delta = 0$, and compute the test statistic you devised in Q4 and report its value. Then repeat, but using a large nonzero value δ .

```
# CODE GOES HERE
# Generate some data
n <- 101
delta <- 0
sigma <- 1
epsilon <- rnorm(n, 0, sigma)
x <- rep(0, n)
for (t in 2:n) {
```

```

    x[t] <- delta + x[t - 1] + epsilon[t]
  }

  # First differences
  diff_x <- diff(x)

  # t-test
  test_result <- t.test(diff_x, mu = 0, alternative = "two.sided")
  t_value <- test_result$statistic
  p_value <- test_result$p.value

  print(test_result)

##
## One Sample t-test
##
## data: diff_x
## t = 0.64714, df = 99, p-value = 0.519
## alternative hypothesis: true mean is not equal to 0
## 95 percent confidence interval:
## -0.1394043 0.2743469
## sample estimates:
## mean of x
## 0.06747132
print(paste("test statistic:", t_value))

## [1] "test statistic: 0.64714132140939"
print(paste("likelihood of null hypothesis", p_value))

## [1] "likelihood of null hypothesis 0.519038207910816"
# CODE GOES HERE
# Generate some data
# let drift be 10
n <- 101
delta <- 10
sigma <- 1
epsilon <- rnorm(n, 0, sigma)
x <- rep(0, n)
for (t in 2:n) {
  x[t] <- delta + x[t - 1] + epsilon[t]
}

# First differences
diff_x <- diff(x)

# t-test
test_result <- t.test(diff_x, mu = 0, alternative = "two.sided")
t_value <- test_result$statistic
p_value <- test_result$p.value

print(test_result)

##

```

```
## One Sample t-test
##
## data: diff_x
## t = 109.38, df = 99, p-value < 2.2e-16
## alternative hypothesis: true mean is not equal to 0
## 95 percent confidence interval:
## 9.800815 10.162966
## sample estimates:
## mean of x
## 9.98189

print(paste("test statistic:", t_value))

## [1] "test statistic: 109.38111283583"

print(paste("likelihood of null hypothesis", p_value))

## [1] "likelihood of null hypothesis 4.53186535491104e-105"
```

6. (4 pts) Simulate 100 random walks each of length 500, with nonzero drift, and plot them on the same plot using transparent coloring, following the code used in the lecture notes from week 2 (“Measures of dependence and stationarity”). Calculate the sample mean $\hat{\mu}_t$ at each time t , across the repetitions, and plot as a dark line on the same plot. Then, calculate the sample standard deviation $\hat{\sigma}_t$ at each time t , and plot the mean plus or minus one standard deviation: $\hat{\mu}_t \pm \hat{\sigma}_t$, as dark dotted lines on the same plot. Describe what you see (you should see that both the mean and variance increase over time).

```
# CODE GOES HERE
```

Stationarity

7. (3 pts) Compute the mean, variance, auto-covariance, and auto-correlation functions for the process

$$x_t = w_t w_{t-1},$$

where each $w_t \sim N(0, \sigma^2)$, independently. Is x_t , $t = 1, 2, 3, \dots$ stationary?

SOLUTION GOES HERE

8. (3 pts) Repeat the same calculations in Q7, but where each $w_t \sim N(\mu, \sigma^2)$, independently, for $\mu \neq 0$. Is x_t , $t = 1, 2, 3, \dots$ stationary?

SOLUTION GOES HERE

9. (3 pts) Simulate the processes from Q7 (with $\mu = 0$) and Q8 (with $\mu \neq 0$) and plot the results. Compute the sample mean and sample variance for each simulated process and check that it is close to the population mean and variance from Q7 and Q8. Compute and plot the sample auto-correlation function using `acf()` and check again that it agrees with the population auto-correlation function from Q7 and Q8.

```
# CODE GOES HERE
```

10. (2 pts) Give an example of a weakly stationary process that is not strongly stationary.

SOLUTION GOES HERE

11. (Bonus) A function κ is said to be *positive semidefinite* (PSD) provided that

$$\sum_{i,j=1}^n a_i a_j \kappa(t_i - t_j) \geq 0, \quad \text{for all } n \geq 1, \text{ all } a_1, \dots, a_n, \text{ and all } t_1, \dots, t_n.$$

Prove that $x_t, t = 1, 2, 3, \dots$ is stationary, and $\gamma_x(h)$ is its auto-covariance function (as a function of lag h), then γ_x is PSD. You may use any equivalences between PSD and other linear-algebraic properties that you want, as long as you state clearly what you are using.

SOLUTION GOES HERE

Joint stationarity

Notions of joint stationarity, between two time series, can be defined in an analogous way to how we defined stationarity in lecture. We say that two time series $x_t, t = 1, 2, 3, \dots$ and $y_t, t = 1, 2, 3, \dots$ are *strongly jointly stationary* provided that:

$$(x_{s_1}, x_{s_2}, \dots, x_{s_k}, y_{t_1}, y_{t_2}, \dots, y_{t_\ell}) \stackrel{d}{=} (x_{s_1+h}, x_{s_2+h}, \dots, x_{s_k+h}, y_{t_1+h}, y_{t_2+h}, \dots, y_{t_\ell+h}),$$

for all $k, \ell \geq 1$, all s_1, \dots, s_k and t_1, \dots, t_ℓ , and all h .

Here $\stackrel{d}{=}$ means equality in distribution. In other words, any collection of variates from the two sequences has the same joint distribution after we shift the time indices forward or backwards in time. Meanwhile, we say that $x_t, t = 1, 2, 3, \dots$ and $y_t, t = 1, 2, 3, \dots$ are *weakly jointly stationary* or simply *jointly stationary* provided that each series is stationary, and:

$$\gamma_{xy}(s, t) = \gamma_{xy}(s + h, t + h), \quad \text{for all } s, t, h.$$

Here γ_{xy} is the cross-covariance function between x, y . In other words, the cross-covariance function must be invariant to shifts forward or backwards in time, and is only a function of the lag $h = s - t$. For jointly stationary series, we can hence abbreviate their cross-covariance function by $\gamma_{xy}(h)$.

SOLUTION GOES HERE

12. (2 pts) Give an example of two time series that are weakly jointly stationary but not strongly jointly stationary.

SOLUTION GOES HERE

13. (3 pts) If $x_t, t = 1, 2, 3, \dots$ and $y_t, t = 1, 2, 3, \dots$ form a *joint Gaussian process*, which means that any collection $(x_{s_1}, x_{s_2}, \dots, x_{s_k}, y_{t_1}, y_{t_2}, \dots, y_{t_\ell})$ of variates along the series has a multivariate Gaussian distribution, then prove that weak joint stationarity implies strong joint stationarity.

SOLUTION GOES HERE

14. (3 pts) Write down explicit formulas that shows how to estimate the cross-covariance and cross-correlation function of two finite time series $t = 1, \dots, n$ and $y_t, t = 1, \dots, n$, under the assumption of joint stationarity. Hint: these should be analogous to the *sample auto-covariance and sample auto-correlation functions* that we covered in lecture.

SOLUTION GOES HERE

15. (4 pts) Following the code used in the lecture notes from week 2 (“Measures of dependence and stationarity”), use the `ccf()` function to compute and plot the sample cross-correlation function between Covid-19 cases and deaths, separately, for each of Florida, Georgia, New York, Pennsylvania, and Texas. (The lecture code does this for California.) Comment on what you find: do the cross-correlation patterns look similar across different states?

Also, follow the lecture code to plot the case and death signals together, on the same plot, for each state (the lecture code provides a way to do this so that they are scaled dynamically to attain the same min and max, and hence look nice when plotted together). Comment on whether the estimated cross-correlation patterns agree with what you see visually between the case and death signals.

```
# CODE GOES HERE
```