Math 128A HW1

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Section 1.1

Problem 2c

Proposition 1. *$f(x) = -3 \cdot tan(2x) + x = 0$ has at least one solution for $x \in [0, 1]$.*

Proof. Note that the interval is end point inclusive. We have $f(0) = 0$, which is immediately one solution to the equation. $\square$

Problem 2d

Proposition 2. *$f(x) = ln(x) - x^2 + \frac{5}{2}x - 1 = 0$ has at least one solution for $x \in [\frac{1}{2}, 1]$.*

Proof. $f(\frac{1}{2}) \approx -0.693, f(1) = 0.5$. Hence, by the intermediate value theorem, there exists a solution in the interval. $\square$

Problem 4d

Find interval containing solutions to $x^3 + 4.001x^2 + 4.002x + 1.101 = 0$.

Solution. Since there is no further intructions on the error bound of the interval, we will proceed with the simplest calculus method by finding critical points. Consider the first derivative and set it equal to zero.

$$\begin{aligned} 3x^2 + 8.002x + 4.002 = 0 \\ x &= \frac{-8.002 \pm \sqrt{8.002^2 - 4 \cdot 3 \cdot 4.002}}{2 \cdot 3} \\ &\approx \frac{-8 \pm 4}{6} = \frac{-4 \pm 2}{3}. \end{aligned}$$

Now, we can look at both points respectively. $f(-2) = 1.101, f(\frac{-2}{3}) \approx -0.0851$. By intermediate theorem, we know that there exists a solution in $[-2, \frac{-2}{3}]$. Now, by simple observation, we can easily conclude that the equation tends to $-\infty$ when $x \to -\infty$ and tends to ∞ when $x \to \infty$. Hence, by intermediate theorem, there exists one solution in $[-\infty, -2]$ and one in $[\frac{-2}{3}, \infty]$. $\square$

Problem 6a

Find $max_{a \leq x \leq b}|f(x)|$ for $f(x) = \frac{2x}{x^2+1}$ on $[0, 2]$.

Solution. We proceed by finding the critical points of $f(x)$ on $[0, 2]$.

$$
\begin{aligned}
f'(x) &= \frac{2(x^2+1) - 2x(2x)}{(x^2+1)^2} \\
&= \frac{2-2x^2}{(x^2+1)^2} \\
&= 0 \text{ when } x = \pm 1.
\end{aligned}
$$

Then, we have $f(0) = 0, f(1) = 1, f(2) = \frac{4}{5}$. Hence, the maximum value of $f(x)$ on $[0, 2]$ is 1 when $x = 1$. $\square$

Problem 14

Let $f(x) = 2x \cdot cos(2x) - (x-2)^2$ and $x_0 = 0$.

(a) Find the third Taylor polynomial $P_3(x)$ and use it to approximate $f(0.4)$.

Solution.

$$
\begin{aligned}
f'(x) &= 2cos(2x) - 4xsin(2x) - 2(x-2), \\
f''(x) &= -8sin(2x) - 8xcos(2x) - 2 \\
f'''(x) &= 16xsin(2x) - 24cos(2x).
\end{aligned}
$$

Now, we have $f(0) = -4, f'(0) = 6, f''(0) = -2, f'''(0) = -24$. Hence, the third Taylor polynomial $P_3(x) = -4 + 6x - x^2 - 4x^3$, and $f(0.4) \approx -2.016$. $\square$

(b) Use the error formula in Taylor's Theorem to find an upper bound for the error $|f(0.4) - P_3(0.4)|$

Solution.

$$
\begin{aligned}
f^4(x) &= 64sin(2x) + 32xcos(2x) \\
R_3(x) &= \frac{f^4(\xi(x))}{4!}(0.4)^4 \qquad \textit{for } 0 \leq \xi(x) \leq 0.4.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
|f(0.4) - P_3(0.4)| = |R_3(0.4)| &= \frac{f^4(\xi(x))}{4!}(0.4)^4 \\
&= \frac{64sin(2\xi(x)) + 32 \cdot \xi(x)cos(2\xi(x))}{24} \times 0.0256 \\
&\leq \frac{64 + 32 \cdot \xi(x)}{24} \times 0.0256 \qquad (\textit{notice } 0 \leq sin(2\xi(x)), cos(2\xi(x)) \leq 1) \\
&\leq \frac{64 + 32 \times 0.4}{24} \times 0.0256 \\
&\leq 0.08192.
\end{aligned}
$$

$\square$

(c) Find the third Taylor polynomial $P_4(x)$ and use it to approximate $f(0.4)$.

Solution.

$$\begin{aligned}
f^4(x) &= 64sin(2x) + 32xcos(2x) \\
f^4(0) &= 0 \\
P_4(x) &= -4 + 6x - x^2 \\
f(0.4) &\approx P_4(0.4) = -2.016.
\end{aligned}$$

□

(d) Use the error formula in Taylor's Theorem to find an upper bound for the error $|f(0.4) - P_4(0.4)|$. Compute the actual error.

Solution.

$$\begin{aligned}
f^5(x) &= 160cos(2x) - 64xsin(2x) \\
|f(0.4) - P_4(0.4)| = |R_4(x)| &= \frac{f^5(\xi(x))}{5!}(0.4)^5 \qquad \textit{for } 0 \leq \xi(x) \leq 0.4. \\
&\leq \frac{160}{5!} \times 0.4^5 \\
&\leq 0.01366.
\end{aligned}$$

□

Problem 26

The *Generalized Rolle's Theorem* states that suppose $f \in C[a, b]$ is n times differentiable on (a, b). If $f(x) = 0$ at the $n + 1$ distinct number $a \leq x_0 < x_1 < \cdots < x_n \leq b$, then a number c in (x_0, x_n) and hence in (a, b) exists such that $f^{(n)}(c) = 0$.

a. Use Rolle's Theorem to show that $f'(z_i) = 0$ for n numbers in $[a, b]$ with $a < z_1 < z_2 < \cdots < z_n < b$.

   Solution. We are unable to prove solely with the information contained in (a). Therefore, we will proceed with the assumptions made in the *Generalized Rolle's Theorem.*

   Now consider each interval $[x_{i-1}, x_i]$ for all $i \in [1, 2, \ldots, n]$. Notice that there are n such intervals and $f(x_{i-1}) = f(x_i) = 0$. Hence, by Rolle's Theorem, there exists $z_i \in (x_{i-1}, x_i)$ such that $f'(z_i) = 0$. Therefore, we can conclude that $f'(z_i) = 0$ for n numbers in $[a, b]$ with $a < z_1 < z_2 < \cdots < z_n < b$. □

b. Use Rolle's Theorem to show that $f''(w_i) = 0$ for $n - 1$ numbers in $[a, b]$ with $z_1 < w_1 < z_2 < w_2 \cdots w_n < z_n < b$.

   Solution. We have shown from (a) that $f'(z_i) = 0$ for $i \in [1, 2, \ldots, n]$. Now consider each interval (z_i, z_{i+1}) for $i \in [1, 2, \ldots, n-1]$. There are $n - 1$ such intervals and $f'(z_i) = f'(z_{i+1})$. Hence, by Rolle's Theorem, there exists $w_i \in (z_i, z_{i+1})$ such that $f''(w_i) = 0$. Therefore, we can conclude that $f''(w_i) = 0$ for $n - 1$ numbers in $[a, b]$ with $a < z_1 < w_1 < z_2 < w_2 \cdots w_{n-1} < z_n < b$. □

c. Continue the arguments in parts (a) and (b) to show that for each $j = 1, 2, \ldots, n$, there are $n + 1 - j$ distinct numbers in $[a, b]$, where $f^{(j)}$ is 0.

Solution. We can easily show by induction on j up until n. Base case is already established in (a). Now, assume that $f^{(k)}(y_i) = 0$ for $n+1-k$ distinct numbers in $[a, b]$. Consider each interval (y_i, y_{i+1}) for $i \in [1, 2, \dots, n-k]$. There are $n-k$ such intervals and $f^{(k)}(y_i) = f^{(k)}(y_{i+1})$. Hence, by Rolle's Theorem, there exists $n-k$ numbers of $u_i \in (y_i, y_{i+1})$ such that $f^{(k+1)}(u_i) = 0$. Therefore, we can conclude that $f^{(k+1)}(u_i) = 0$ for $n-k = n+1-(k+1)$ distinct numbers in $[a, b]$.

Hence, by induction, we can conclude that for each $j = 1, 2, \dots, n$, there are $n+1-j$ distinct numbers in $[a, b]$, where $f^{(j)}$ is 0. □

d. Show that part (c) implies the conclusion of the Generalized Rolle's Theorem.

Solution. Notice our only assumptions are that $f(x)$ is n times differentiable on (a, b), and that $f(x) = 0$ at $n+1$ distinct numbers such that $a \leq x_0 < x_1 \cdots < x_n \leq b$.

With the mentioned assumptions, we concluded that there are $n+1-j$ distinct numbers in $[a, b]$, where $f^{(j)}$ is 0. Now take $j = n$, we have $n+1-n = 1$ distinct number in $[a, b]$, where $f^{(n)}$ is 0. Hence, we can conclude that there exists $c \in (a, b)$ such that $f^{(n)}(c) = 0$.

Notice this is exactly what the Generalized Rolle's Theorem says. □

Section 1.2

Problem 2c

Compute the absolute error and relative error in approximations of p by p^*: $p = 8!, p^* = 39900$.

Solution.

$$|p - p^*| = 8! - 39900 = 40320 - 39900 = 420.$$
$$\frac{|p - p^*|}{p} = \frac{420}{8!} = \frac{1}{96} \approx 0.0104.$$

□

Problem 4b

Find the largest interval in which p^* must lie to approximate p with relative error at most 10^{-4} for $p = e$.

Solution.

$$\begin{aligned}
\frac{|p^* - e|}{e} &\leq 10^{-4} \\
|p^* - e| &\leq 10^{-4}e \\
|p^* - e| &\leq 10^{-4}e \\
(1 - 10^{-4}) \cdot e \leq p^* &\leq (1 + 10^{-4}) \cdot e \\
0.9999 \cdot e \leq p^* &\leq 1.0001 \cdot e.
\end{aligned}$$

If we want to find a numerical bound, we can arbitrarily choose $e = 2.71828$, the most commonly used approximation of e. Then, we have

$$\begin{aligned}
2.71828 \cdot 0.9999 \leq p^* &\leq 2.71828 \cdot 1.0001 \\
2.71800 \leq p^* &\leq 2.71855.
\end{aligned}$$

$\square$

Problem 12

The number e can be defined by $e = \sum_{n=0}^{\infty}(1/n!)$ where $n! = n(n-1)\cdots 2\cdot 1$ for $n \neq 0$ and $0! = 1$. Compute the absolute error and relative error in the following approximations of e:

a. $\sum\limits_{n=0}^{5} \frac{1}{n!}$

Solution. We will just plug in matlab and use the default e value from matlab to compute the error.

$$|e - \sum_{n=0}^{5} \frac{1}{n!}| \approx 0.0016152$$
$$\frac{|e - \sum\limits_{n=0}^{5} \frac{1}{n!}|}{|e|} \approx 0.00059418.$$

$\square$

b. $\sum\limits_{n=0}^{10} \frac{1}{n!}$

Solution. We will just plug in matlab and use the default e value from matlab to compute the error.

$$|e - \sum_{n=0}^{10} \frac{1}{n!}| \approx 2.7313 \times 10^{-8}$$
$$\frac{|e - \sum\limits_{n=0}^{10} \frac{1}{n!}|}{|e|} \approx 1.0048 \times 10^{-8}.$$

$\square$

Problem 22

The Taylor polynomial of degree n for $f(x) = e^x$ is $\sum_{i=0}^{\infty}(x^i/i!)$. Use the Taylor polynomial of degree nine and three-digit chopping arithmetic to find an approximation to e^{-5} by each of the following methods.

a. $e^{-5} \approx \sum\limits_{i=0}^{9} \frac{(-5)^i}{i!} = \sum\limits_{i=0}^{9} \frac{(-1)^i 5^i}{i!}$.

Solution. With three-digit chopping arithmetic, we have

$$\begin{aligned}\sum_{i=0}^{9} \frac{(-1)^i 5^i}{i!} &= 1 - 5 + \frac{25}{2} - \frac{125}{6} + \frac{625}{24} - \frac{3120}{120} + \frac{15600}{720} - \frac{78000}{5040} + \frac{390000}{40300} - \frac{1950000}{362000} \\ &= 1 - 5 + 12.5 - 20.8 + 26.0 - 26.0 + 21.6 - 15.4 + 9.67 - 5.38 \\ &= -1.81.\end{aligned}$$

$\square$

b. $e^{-5} = \frac{1}{e^5} \approx \frac{1}{\sum_{i=0}^{9} \frac{5^i}{i!}}$.

Solution. With three-digit chopping arithmetic, we have

$$\sum_{i=0}^{9}\frac{(-1)^i5^i}{i!} = 1/\left(1+5+\frac{25}{2}+\frac{125}{6}+\frac{625}{24}+\frac{3120}{120}+\frac{15600}{720}+\frac{78000}{5040}+\frac{390000}{40300}+\frac{1950000}{362000}\right)$$
$$= 1/(1+5+12.5+20.8+26.0+26.0+21.6+15.4+9.67+5.38)$$
$$= 1/141$$
$$= 0.00709.$$

□

c. An approximate value of e^{-5} correct to three digits is 6.74×10^{-3}. Which formula,(a) or (b), gives the most accuracy, and why?

Solution. (b) obviously gives a more accurate answer.

Notice that e^{-5} is a very small number. However, with the (a) approach, it is adding and subtracting large numbers. This will cause a lot of cancellation to reach a small number. On the other hand, with the (b) approach, we are only adding the numbers and taking the reciprocal. Dividing by an increasing large number will immediately result in a small number, which would be closer to the actual value, thus converge faster. □

Section 1.3

Problem 8

Suppose that $0<q<p$ and that $\alpha_n = \alpha + O(n^{-p})$.

a. Show that $a_n = \alpha + O(n^{-q})$.

Proof. For $n \geq 1$,

$$\begin{aligned}
\frac{n^q}{n^p} &\leq 1\\
\frac{1}{n^p} &\leq \frac{1}{n^q}\\
n^{-p} &\leq n^{-q}\\
|a_n-\alpha| \leq k\cdot n^{-p} &\leq k\cdot n^{-q} \qquad \textit{(we know the existence of } k)\\
|a_n-\alpha| &\leq k\cdot n^{-q}\\
a_n &= \alpha + O(n^{-q}).
\end{aligned}$$

□

b. Make a table listing $1/n$, $1/n^2$, $1/n^3$, and $1/n^4$ for n = 5, 10, 100, and 1000 and discuss the varying rates of convergence of these sequences as n becomes large.

$n=$	5	10	100	1000
$1/n$	0.2	0.1	0.01	0.001
$1/n^2$	0.04	0.01	0.001	0.000001
$1/n^3$	0.008	0.001	10^{-6}	10^{-9}
$1/n^4$	0.0016	0.0001	10^{-8}	10^{-12}

They all converge to 0, but at different rates. The value converges faster with a larger exponent. For example, $1/n^4$ converges faster than $1/n^3$.

Problem 15

a. How many multiplications and additions are required to determine a sum of the form

$$\sum_{i=1}^{n}\sum_{j=1}^{i} a_i b_j$$

Solution. Let's expand the summation to count more easily.

$$\begin{aligned}\sum_{i=1}^{n}\sum_{j=1}^{i} a_i b_j =& a_1 \cdot b_1 + (a_2 \cdot b_1 + a_2 \cdot b_2) + (a_3 \cdot b_1 + a_3 \cdot b_2 + a_3 \cdot b_3) \\ &+ \cdots + (a_n \cdot b_1 + a_n \cdot b_2 + \cdots + a_n \cdot b_n).\end{aligned}$$

Then, we can just count the number of $\cdot$ and $+$. The number of multiplication is $1+2+\cdots+n = \frac{n(n+1)}{2}$. The number of addition is $(n-1)+(n-1)+\cdots+1 = \frac{n(n-1)}{2} + (n-1)$. Hence, the total number of operations is n^2+n-1. $\square$

b. Modify the sum in part (a) to an equivalent form that reduces the number of computations.

Solution.

$$\sum_{i=1}^{n}\sum_{j=1}^{i} a_i b_j = \sum_{i=1}^{n} a_i \cdot \left(\sum_{j=1}^{i} b_j\right).$$

There are still $(n-1)+(n-1)+\cdots+1 = \frac{n(n-1)}{2}+(n-1)$ additions. But the number of multiplication is reduced to n. Hence, the total number of operations is $\frac{n(n-1)}{2}+(n-1)+n = \frac{n^2+3n-2}{2}$. $\square$

Discussion Question 2 (p. 38)

Construct an algorithm that has as input an integer $n \geq 1$,numbers $x_0, x_1, \ldots, x_n$, and a number x and that produces as output the product $(x-x_0)(x-x_1)\cdots(x-x_n)$.

Solution.

INPUT $x_0, x_1, \ldots, x_n, x$

OUTPUT TOTAL

Step 1 Set TOTAL $= 1$

Step 2 For $i = 0, 1, \ldots, n$ do

Set TOTAL = TOTAL $*(x - x_i)$

Step 3 OUTPUT TOTAL;

STOP

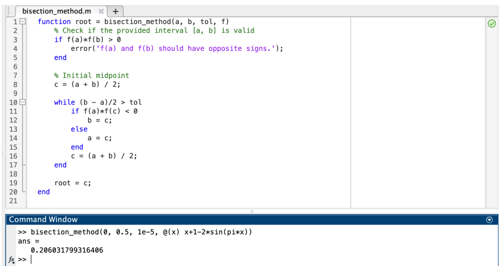
$\square$

Section 2.1

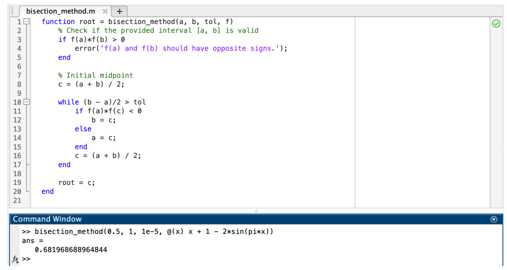
Problem 6d

Use the Bisection method to find solutions, accurate to within 10^{-5} for

$$x + 1 - 2sin(\pi x) = 0 \qquad (for\ 0 \leq x \leq 0.5\ \ and\ 0.5 \leq x \leq 1).$$



$0 \leq x \leq 0.5$



$0.5 \leq x \leq 1$

Problem 8

a. Sketch the graphs of $y = x$ and $y = tan(x)$.

Solution. $\square$

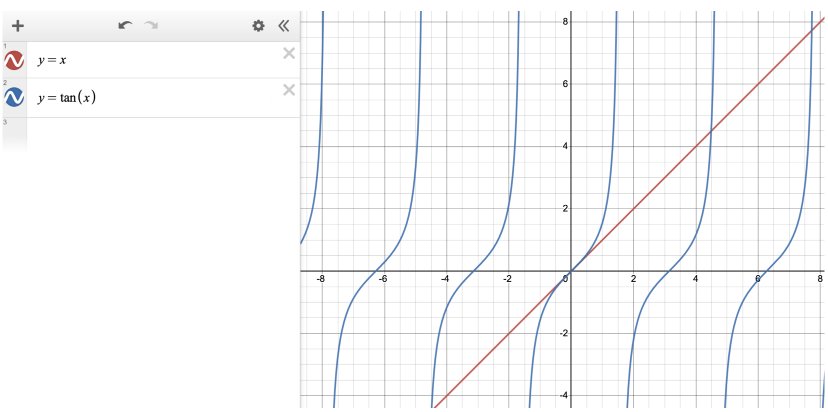


Figure 1: Too trivial to draw by hand.

b. Use the Bisection method to find an approximation to within 10^{-5} to the first positive value of x with $x = tan(x)$.

Solution. We use the same function coded from last problem to solve the equation $tan(x) - x = 0$ on the interval $[4, 4.6]$. $\square$

bisection_method.m

```
function root = bisection_method(a, b, tol, f)
    % Check if the provided interval [a, b] is valid
    if f(a)*f(b) > 0
        error('f(a) and f(b) should have opposite signs.');
    end

    % Initial midpoint
    c = (a + b) / 2;

    while (b - a)/2 > tol
        if f(a)*f(c) < 0
            b = c;
        else
            a = c;
        end
        c = (a + b) / 2;
    end

    root = c;
end
```

Command Window

```
>> bisection_method(4, 4.6, 1e-5, @(x) tan(x) - x)
ans =
   4.493405151367186
>>
```

Problem 20

Let $f(x) = (x-1)^{10}, p = 1$, and $p_n = 1 + \frac{1}{n}$. Show that $|f(p_n)| < 10^{-3}$ whenever $n > 1$ but that $|p - p_n| < 10^{-3}$ requires that $n > 1000$.

Proof. $f(p_n) = (1 + \frac{1}{n} - 1)^{10} = \left(\frac{1}{n}\right)^{10}$. Notice that $\left(\frac{1}{n}\right)^{10}$ is a positive function. Then $|f(p_n)| = \frac{1}{n^{10}} < 10^{-3} \Rightarrow n > \sqrt[10]{1000} \Rightarrow n > 1.996 \Rightarrow n > 1$ for $n \in \mathbb{N}$.

Notice $p_n > p$ for all n. Hence, $|p - p_n| = p_n - p = \frac{1}{n}$. Then, we have $\frac{1}{n} < 10^{-3} \Rightarrow n > 1000$. $\square$