Math 110 HW3

Neo Lee

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Problem 1

Let $U := \{p \in \mathcal{P}_2(\mathbb{R}) : \int_{-1}^{1}(xp''(x) + p'(x))dx = 0\}$.

(a) Find a basis for U.

Solution. Let's rewrite U in a simpler form. We know every $p \in \mathcal{P}_2(\mathbb{R})$ can be written as $p(x) = ax^2 + bx + c$. Then we have

$$\begin{aligned} p''(x) &= 2a \\ p'(x) &= 2ax + b. \end{aligned}$$

Now,

$$\begin{aligned} \int_{-1}^{1}(xp''(x) + p'(x))dx &= 0 \\ \int_{-1}^{1}(x(2a) + 2ax + b)dx &= 0 \\ \int_{-1}^{1}(2ax + 2ax + b)dx &= 0 \\ \int_{-1}^{1}(4ax)dx + \int_{-1}^{1} bdx &= 0 \\ 4a\int_{-1}^{1} xdx + b\int_{-1}^{1} dx &= 0 \\ 4a\left[\frac{x^2}{2}\right]_{-1}^{1} + b\left[x\right]_{-1}^{1} &= 0 \\ 4a\left(\frac{1}{2} - \frac{1}{2}\right) + b(1 - (-1)) &= 0 \\ 2b &= 0 \\ b &= 0. \end{aligned}$$

Therefore, $U = \{p \in \mathcal{P}_2(\mathbb{R}) : p(x) = ax^2 + c\} = \{ax^2 + c : a, c \in \mathbb{R}\}$. Now, we can see clearly that the basis of U is $\{x^2, 1\}$ as every $p \in U$ can be written as $p(x) = ax^2 + c$ for $a, c \in \mathbb{R}$ and $\{x^2, 1\}$ are linearly independent. $\square$

(b) Extend your basis in part (a) to a basis of $\mathcal{P}_3(\mathbb{R})$.

Solution. We can do this by appending the standard basis of $\mathcal{P}_3(\mathbb{R})$ to the basis of U in part (a), then remove all the dependent vectors.

After appending the basis, we get $\{x^2, 1, x^3, x^2, x, 1\}$. Now, we can remove the dependent vectors, which are very obvious because they are the same. Therefore, we are left with $\{x^3, x^2, x, 1\}$, which are independet and span the whole $\mathcal{P}_3(\mathbb{R})$. $\square$

(c) Find a subspace W of $\mathcal{P}_3(\mathbb{R})$ such that $\mathcal{P}_3(\mathbb{R}) = U \oplus W$.

Solution. Define $W := span\{x, x^3\}$. We can see that W is a subspace of $\mathcal{P}_3(\mathbb{R})$ obviously.

Now we check that it is indeed a direct sum by checking their intersection. We know that for $u \in U$, $u(x) = ax^2 + c$ for $a, c \in \mathbb{R}$. For $w \in W$, $w(x) = bx^3 + dx$ for $b, d \in \mathbb{R}$. Now if $u = w$,

$$\begin{aligned} ax^2 + c &= bx^3 + dx \\ -bx^3 + ax^2 + c - dx &= 0. \end{aligned}$$

And we know that $x^3, x^2, x, 1$ are linearly independent. Therefore, $a = b = c = d = 0$, and $u = w = 0$. Therefore, $U \cap W = \{0\}$, and $U \oplus W = \mathcal{P}_3(\mathbb{R})$. $\square$

Problem 2

Suppose $v_1, \dots, v_m$ are linearly independent in V and $w_0 \in V$. Prove that

$$\dim span(v_1 - w_0, v_2 - w_0, \dots, v_m - w_0) \geq m - 1.$$

Proof 1. Define $U := span(v_1 - w_0, v_2 - w_0, \dots, v_m - w_0)$ and $W := span(w_0)$. Now we know $U + W = span(v_1, \dots, v_m, w_0)$. To show this, we look at the linear combination of any $u \in U, w \in W$,

$$\begin{aligned} a_1(v_1 - w_0) + \cdots + a_m(v_m - w_0) + b_1 w_0 &= a_1 v_1 + \cdots + a_m v_m + (b_1 - a_1 - \cdots - a_m) \cdot w_0 \\ &= a_1 v_1 + \cdots + a_m v_m + c_0 w_0. \end{aligned}$$

Also, notice $\dim span(v_1, \dots, v_m, w_0) \geq m$ since $v_1, \dots, v_m$ are linearly independent. Therefore, $\dim(U + W) \geq m$.

Now, apply the inclusion-exclusion formula, we have

$$\begin{aligned} \dim(U + W) &= \dim(U) + \dim(W) - \dim(U \cap W) \\ \dim(U) &= \dim(U + W) + \dim(U \cap W) - \dim(W) \\ \dim(U) &\geq m + \dim(U \cap W) - 1 \\ \dim(U) &\geq m - 1. \end{aligned}$$

□

Proof 2. To simplify the notations, denote $U := span(v_1 - w_0, v_2 - w_0, \dots, v_m - w_0) = span(u_1, u_2, \dots, u_m)$ and $U' = span(u_1 - u_1, u_2 - u_1, u_3 - u_1, \dots, u_m - u_1) = span(0, v_2 - v_1, v_3 - v_1, \dots, v_m - v_1)$ [the 0 is trivial and can be removed].

Now we want to show that U' is a subspace of U, and $\dim(U') = m - 1$. From that we can say $\dim(U) \geq \dim(U') = m - 1$ because a vector space cannot have a dimension smaller than its subspace.

$\underline{U' \subseteq U}$: For arbitrary $u' \in U'$,

$$\begin{aligned} u' &= a_2(v_2 - v_1) + \cdots + a_m(v_m - v_1) \\ &= a_2(u_2 - u_1) + \cdots + a_m(u_m - u_1) \\ &= a_2 u_2 + \cdots + a_m u_m - (a_2 + \cdots + a_m) u_1, \end{aligned}$$

which is a linear combination of $(u_1, \dots, u_m)$. Therefore, $u' \in U$. Obviously, $0 \in U'$, and U' is closed under addition and scalar multiplication because U' is defined with *span*. Therefore, U' is a subspace of U.

$\underline{\dim(U') = m - 1}$: We will proceed by showing that $(v_2 - v_1, v_3 - v_1, \dots, v_m - v_1)$ are linearly independent and hence form a basis of U'.

$$\begin{aligned} a_2(v_2 - v_1) + a_3(v_3 - v_1) + \cdots + a_m(v_m - v_1) &= 0 \\ a_2 v_2 + a_3 v_3 + \cdots + a_m v_m - (a_2 + a_3 + \cdots + a_m) v_1 &= 0. \end{aligned}$$

Since $v_1, \dots, v_m$ are linearly independent, we have $a_2 = a_3 = \cdots = a_m = 0$. Therefore, $(v_2 - v_1, v_3 - v_1, \dots, v_m - v_1)$ are linearly independent. $\dim(U') = length(v_2 - v_1, v_3 - v_1, \dots, v_m - v_1) = m - 1$.

Hence, $\dim(U) \geq \dim(U') = m - 1$.

□

Problem 3

Does the 'inclusion-exclusion formula' hold for three subspaces, i.e., is it always true that

$$\begin{aligned}\dim(U_1 + U_2 + U_3) \quad &= \quad \dim(U_1) + \dim(U_2) + \dim(U_3) \\ &\quad - \dim(U_1 \cap U_2) - \dim(U_1 \cap U_3) - \dim(U_2 \cap U_3) \\ &\quad + \dim(U_1 \cap U_2 \cap U_3)?\end{aligned}$$

Prove this formula or provide a counterexample.

Solution. Define $U_1 := \{(x,0) : x \in \mathbb{R}\}, U_2 := \{(0,y) : y \in \mathbb{R}\}, U_3 := \{(z,z) : z \in \mathbb{R}\}$. Then, apparently, $U_1 + U_2$ span U_3, and $U_1 + U_2 + U_3 = \mathbb{R}^2$. Therefore,

$$\dim(U_1 + U_2 + U_3) = \dim(\mathbb{R}^2) = 2.$$

Now notice

$$U_i \cap_{i \neq k} U_k = \{0\},$$

$$U_1 \cap U_2 \cap U_3 = \{0\}.$$

Both claims can be observed easily and can be proved by considering arbitrary vectors from the spaces. We will omit the trivial proof here. Hence, the right hand side of the equation $= 3$. Therefore, the equation does not hold. $\square$

Problem 4

What is the dimension and the 'canonical' basis of:

(a) $\mathbb{C}$ as a vector space over $\mathbb{C}$?

Solution.

$$\mathbb{C} = \{c \cdot 1 : c \in \mathbb{C}\}.$$

Therefore, $\dim(\mathbb{C}) = 1$, and the canonical basis is $\{1\}$. $\square$

(b) $\mathbb{C}$ as a vector space over $\mathbb{R}$?

Solution.

$$\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}.$$

Therefore, $\dim(\mathbb{C}) = 2$, and the canonical basis is $\{1, i\}$. $\square$

(c) $\mathbb{C}^5$ as a vector space over $\mathbb{C}$?

Solution.

$$\mathbb{C}^5 = \{(a, b, c, d, e) : a, b, c, d, e \in \mathbb{C}\}.$$

Therefore, $\dim(\mathbb{C}) = 5$, and the canonical basis is e_1, e_2, e_3, e_4, e_5, where e_i is the list of length 5 with 1 at the ith position and 0 elsewhere. $\square$

(d) $\mathbb{C}^7$ as a vector space over $\mathbb{R}$?

$$\mathbb{C}^7 = \{(a + bi, c + di, e + fi, g + hi, j + ki, l + mi, n + oi) : a, b, c, d, e, f, g, h, j, k, l, m, n, o \in \mathbb{R}\}.$$

Therefore, $\dim(\mathbb{C}) = 14$, and the canonical basis is $e_{1,1}$, $e_{1,2}$, $e_{2,1}$, $e_{2,2}$, $e_{3,1}$, $e_{3,2}$, $e_{4,1}$, $e_{4,2}$, $e_{5,1}$, $e_{5,2}$, $e_{6,1}$, $e_{6,2}$, $e_{7,1}$, $e_{7,2}$, where $e_{k,1}$ is a list of length 7 with 1 at the kth position and $e_{k,2}$ is a list of length 7 with i at the k position.

Problem 5

Suppose U and W are subspaces of V such that $U+W=V$, suppose u_1, ..., u_m is a basis of U and w_1, ..., w_n is a basis of W. Disprove that u_1,..., u_m, w_1, ..., w_n is necessarily a basis of V. What additional condition on the sum $U+W$ makes this implication true? Explain.

Solution. Let $U := \{(a,b,0) : a,b \in \mathbb{R}\}, W := \{(0,c+d,d) : c,d \in \mathbb{R}\}$. Then $U+W=\mathbb{R}^3$. Now, U has a basis $\{(1,0,0),(0,1,0)\}$, and W has a basis $\{(0,1,0),(0,1,1)\}$. However, $\{(1,0,0),(0,1,0),(0,1,0),(0,1,1)\}$ is not a basis of $\mathbb{R}^3$ because it is not linearly independent [also obviously we cannot have a basis of length 4 in 3 dimension].

The implication is true if it is a direct sum. First, we show that $u_1,\ldots,v_m,w_1,\ldots,w_n$ always span V. For any vector $v \in V$, it can be written as a sum of some $u_0 \in U$ and some $w_0 \in W$. We write, for some $x_i, y_i \in \mathbb{F}$,

$$\begin{aligned} v &= u_0 + w_0 \\ &= (x_1u_1 + \cdots + x_mu_m) + (y_1w_1 + \cdots + y_nw_n), \end{aligned}$$

which is a linear combination of $u_1,\ldots,v_m,w_1,\ldots,w_n$. Therefore, $u_1,\ldots,v_m,w_1,\ldots,w_n$ span V.

Now we show that $u_1,\ldots,v_m,w_1,\ldots,w_n$ are linearly independent. For some $x_i, y_i \in \mathbb{F}$

$$\begin{aligned} x_1u_1 + \cdots + x_mu_m + y_1w_1 + \cdots + y_nw_n &= 0 \qquad (1) \\ (x_1u_1 + \cdots + x_mu_m) + (y_1w_1 + \cdots + y_nw_n) &= 0 \\ u + w &= 0. \qquad \textit{(for some } u \in U, w \in W\textit{)} \end{aligned}$$

Now, since it is a direct sum, $u = w = 0$. Therefore,

$$x_1u_1 + \cdots + x_mu_m = 0$$

and

$$y_1w_1 + \cdots + y_nw_n = 0.$$

Since $(u_1,\ldots,v_m)$ and $(w_1,\ldots,w_n)$ are both linearly independent [basis of U, W], we have $x_1 = \cdots = x_m = y_1 = \cdots = y_n = 0$. Therefore, $x_i = y_i = 0$ for all i in (1). Hence, $u_1,\ldots,v_m,w_1,\ldots,w_n$ are linearly independent.

Since $u_1,\ldots,v_m,w_1,\ldots,w_n$ are linearly independent and span V, they form a basis of V.

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