MATH 185 Notes

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Contents

Chapter 1

First chapter

1.1 Lecture 1 (skipped)

1.2 Lecture 2

Definition 1.2.1: Integral powers of complex numbers from geomtery

If $z = r \cdot e^{i\theta}$, so $|z| = r$ and define $\arg z = \theta$, then $z^n = r^n \cdot e^{in\theta}$.

Note:

$|z| = r$ can be shown by writing $e^{i\theta} = \cos\theta + i \sin\theta$ and taking absolute value.

Note:

Because of our definition, integral powers of complex numbers can be easily visualized in polar coordinates. In particular, for $z = r \cdot e^{i\theta}$,

$$|z| = r^n \quad \text{and} \quad \arg z^n = n\theta.$$

Example 1.2.1 (Map from $z \to z^2$)

Let $z = x + yi$, then $\operatorname{Re} z$ is mapped from x to x^2, while $\operatorname{Im} z$ is mapped from yi to $-y^2$.

$$z = x + yi \mapsto x^2 - y^2 + 2xyi.$$

Hence, $\operatorname{Re} z^2 = x^2 - y^2$ and $\operatorname{Im} z = 2xy$. If we plot z^2 on the real and imaginary axis, then it's a parabola.

Theorem 1.2.1 De Moivre formula

$$(\cos\theta + i \sin\theta)^n = \cos(n\theta) + i \sin(n\theta).$$

Proof: This falls immediately by taking nth power of $z = e^i\theta$ and following *definition 1.1.* 😊

Note:

By applying binomial expansion to the left hand side of De Moivre formula, we get the following consequences immediately:

$$\cos^n\theta - \binom{n}{2}\cos^{n-2}\theta\cdot\sin^2\theta + \cdots = \cos(n\theta)$$
$$i\left(n\cdot\cos^{n-1}\theta\cdot\sin\theta - \binom{n}{3}\cos^{n-3}\theta\cdot\sin^3\theta + \cdots\right) = i\sin(n\theta).$$

From the first equation, we can see $\cos^2\theta - \sin^2\theta = \cos(2\theta)$ is a special case of De Moivre formula. From the second equation, we can see $2\cos\theta\cdot\sin\theta = \sin(2\theta)$ is a special case as well.

Definition 1.2.2: n^{th} roots of complex number

The n^{th} roots of complex number are the solutions z to the equation $z^n = x$.

Note:

We call the solutions z to the equation $z^n = 1$ the n^{th} roots of unity.

Proposition 1.2.1 There are n^{th} distinct roots of unity

There are n^{th} distinct roots z to the equation $z^n = 1$, and they are

$$1, \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}, \cos\frac{4\pi}{n} + i\sin\frac{4\pi}{n}, \cdots, \cos\frac{(2n-2)\pi}{n} + i\sin\frac{(2n-2)\pi}{n}.$$

Let $\xi_n = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$, then the list is

$$1, \xi_n, \xi_n^2, \ldots, \xi_n^{n-1}.$$

Note:

The roots are essentially cutting the unit circle into n parts, and the cuts lying on the unit circle are the roots.

Corollary 1.2.1 Linear factorization of $x^n - 1$

From the previous proposition, we have the linear factorization

$$x^n - 1 = (x-1)(x-\xi_n)(x-\xi_n^2)\cdots(x-\xi_n^{n-1}).$$

Proof: From Bezout's theorem, if $x = a$ is a solution to $P(x) = 0$, then $(x-a)$ divides $P(x)$. Now from *proposition 1.1*, we found all the roots to $x^n - 1 = 0$, so we can linearly factor $x^n - 1$. ☺

Definition 1.2.3: Primitive roots of unity

An n^{th} root of unity ξ is called primitive if $\xi^n = 1$ but $\xi^k \neq 1$ for all $0 < k < n$. In other words ξ is the n^{th} primitive root of unity if ξ raised to the power n is the first time equating to 1.

Example 1.2.2 (6^{th} roots of unity)

Consider the equation $z^6 = 1$ and let $\xi_n = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$. From *proposition 1.1*, we know all the roots are raising ξ to the power until 5. Then by drawing it geometrically as points on the unit circle, and raising power is equavalent to rotating along the circle, the 6^{th} primitive roots are ξ and ξ^5.

Theorem 1.2.2 n^{th} roots of unity can be partition into divisors of n

Let $\{d_1, \ldots, d_k\}$ be the divisors of n, then the set of n^{th} roots of unity can be partitioned into groups of primitive d_k^{th} roots of unity.

Theorem 1.2.3 Factorization of $x^n - 1$ in integer coefficient terms

Define the d^{th} cyclotomic polynomial as $\Phi_d(x)$, which is an irreducible polynomial over rationals and with integer coefficients. Then, grouping the factors of

$$x^n - 1 = (x-1)(x-\xi_n)\cdots(x-\xi_n^{n-1})$$

according to divisors of n with primitive roots gives a factorization

$$x^n - 1 = \prod_{d|n} \Phi_d(x).$$

Example 1.2.3 (Factorization of $x^3 - 1$ and $x^6 - 1$)

$$\begin{aligned}
x^3 - 1 &= \underbrace{(x-1)}_{\Phi_1(x)}\underbrace{(x^2+x+1)}_{\Phi_3(x)} \\
x^6 - 1 &= \underbrace{(x-1)}_{\Phi_1(x)}\underbrace{(x+1)}_{\Phi_2(x)}\underbrace{(x^2+x+1)}_{\Phi_3(x)}\underbrace{(x^2-x+1)}_{\Phi_6(x)}.
\end{aligned}$$

Definition 1.2.4: "Cyclotomic integers"

Cyclotomic integers are $\mathbb{Z}[\xi_2, \xi_3, \ldots] = \mathbb{Z}[\xi_n]$ for a fixed n. Equivalently, it's formed by adjoining all roots of units to $\mathbb{Z}$.

1.3 Lecture 3 (skipped)

1.4 Lecture 4

1.4.1 Multi-variable real differentiation

Note:

Naturally constructed functions will often have singularities, for example, the inversion

$$z \mapsto \frac{1}{z}.$$

As $z \to 0$ in $\mathbb{C}$, $|z| \to 0$ in $\mathbb{R}$, so $|z^{-1}| = |z|^{-1} \to \infty$. This can be fixed by the extended complex plane $\mathbb{C} \cup \{\infty\}$.

Definition 1.4.1: Real differentiation in $\mathbb{R}$

Let $I \in \mathbb{R}$ be an open interval. A function $f : I \to \mathbb{R}$ is differentiable at a point $x_0 \in I$ if

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

exists.

Example 1.4.1 (Motivating example of new defintion of $\mathbb{R}^2$ differentiation)

We say $f(x_0) + f'(x_0)(x - x_0)$ is the best linear apprixmation to f at point x_0, and

$$\frac{|f(x) - f(x_0) - f'(x_0)(x - x_0)|}{|x - x_0|} \to 0 \quad \text{as} \quad x \to x_0.$$

Note:

With some knowledge of multivariable analysis, we can get this definition by rearranging the definition of multi-variable differentiation involving the Taylor remainder $R(x - x_0)$.

Definition 1.4.2: Real differentiation from $\mathbb{R}^2 \to \mathbb{R}$

Let $U \in \mathbb{R}^2$ be an open subset. A function $f : U \to \mathbb{R}$ is differentiable at $\langle x_0, y_0 \rangle \in U$ if there exists a linear map $Df : \mathbb{R}^2 \to \mathbb{R}$ such that

$$\frac{|f(\langle x, y \rangle) - f(\langle x_0, y_0 \rangle) - Df(\langle x - x_0, y - y_0 \rangle)|}{\|\langle x - x_0, y - y_0 \rangle\|} \to 0 \quad \text{as} \quad \langle x, y \rangle \to \langle x_0, y_0 \rangle.$$

Note:

Df is the derivative at $\langle x_0, y_0 \rangle$, which gives the best linear approximation to f at $\langle x_0, y_0 \rangle$.

Definition 1.4.3: Partial differentiation from $\mathbb{R}^2 \to \mathbb{R}$

Let $f : U \to \mathbb{R}$ where U is an open subset of $\mathbb{R}^2$, then

$$\frac{\partial}{\partial x} f = \lim_{x \to x_0} \frac{f(\langle x, y_0 \rangle) - f(\langle x_0, y_0 \rangle)}{x - x_0} \quad \text{and} \quad \frac{\partial}{\partial y} f = \lim_{x \to x_0} \frac{f(\langle x_0, y \rangle) - f(\langle x_0, y_0 \rangle)}{y - y_0}.$$

Note:

- This is reducible to the definition of real differentiation in $\mathbb{R}$ by fixing $x = x_0$ or $y = y_0$.
- Df as the derivative at $\langle x_0, y_0 \rangle$ in this case is given by the (matrix) multiplication by the vector $\langle \frac{\partial}{\partial x} f, \frac{\partial}{\partial y} f \rangle$.

Definition 1.4.4: Differentiation from $\mathbb{R}^2$ to $\mathbb{R}^2$

Obvious adoptation of the previous definition, where f is a vector valued function, say $\mathbb{R}^2 \to \mathbb{R}^2$. If $f = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$, Df is the Jacobian matric $\begin{bmatrix} \frac{\partial}{\partial x} f_1 & \frac{\partial}{\partial y} f_1 \\ \frac{\partial}{\partial x} f_2 & \frac{\partial}{\partial y} f_2 \end{bmatrix}$.

Theorem 1.4.1 Sufficient condition for total differentiability

If all partial derivatives of f exist and are continuous in a neighborhood of $\langle x_0, y_0 \rangle$, then f is totally differentiable at $\langle x_0, y_0 \rangle$.

Note:

For limits and convergence questions in $\mathbb{C}$, we can consider it as limits and convergence in $\mathbb{R}^2$. However, note that we cannot treat differentiability in $\mathbb{C}$ directly as differentiability in $\mathbb{R}^2$, which we will later prove that there are extra conditions.

Note:

Addition and multiplication of complexnumbesr are continuous functions $\mathbb{C} \times \mathbb{C} \to \mathbb{C}$, and inversion map $z \to z^{-1} = \frac{z}{|z|^2}$ is also continuous away from $z = 0$.

1.4.2 Complex differentiation

Definition 1.4.5: Complex differentiable

f is complex differentiable at z_0 if

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists.

Note:

The limit is the (complex) derivative of f at z_0, $f'(z_0)$.

Theorem 1.4.2 Complex differentiability implies continuity

If f is complex differentiable at x_0, then f is continuous at z_0.

Proof:

$$\begin{aligned}\lim_{z \to z_0} f(x) - f(x_0) &= \lim_{z \to z_0} (f(x) - f(x_0)) \frac{z - z_0}{z - z_0} \\ &= \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \cdot \lim_{z \to z_0} z - z_0 \\ &= f'(z_0) \cdot 0 = 0.\end{aligned}$$

😊

Example 1.4.2 (Derivative of monomial)

Let $f(z) = z^n$, then notice

$$\frac{f(z) - f(z_0)}{z - z_0} = \frac{z^n - z_0^n}{z - z_0} = z^{n-1} + z^{n-2} z_0 + \cdots + z_0^{n-1}.$$

Since addition and multiplication are continuous on $\mathbb{C}$ and $\lim_{z \to z_0} z = z_0$, the limit of the RHS is $n z_0^{n-1}$. Hence, $f'(z_0) = n z_0^{n-1}$.

Theorem 1.4.3 Properties of complex differentiation

If f, g are complex differentiable at z_0, then

1. $f + g$ is complex differentiable at z_0 and $(f + g)'(z_0) = f'(z_0) + g'(z_0)$.
2. fg is complex differentiable at z_0 and $(fg)'(z_0) = f'(z_0)g(z_0) + f(z_0)g'(z_0)$.
3. If $g(z_0) \neq 0$, then $\frac{f}{g}$ is complex differentiable at z_0 and

$$\left(\frac{f}{g}\right)'(z_0) = \frac{f'(z_0)g(z_0) - f(z_0)g'(z_0)}{g(z_0)^2}.$$

Proof: The same proof as in $\mathbb{R}$ using absolute value of complex numbers and triangle inequality. ☺

Corollary 1.4.1 Complex differentiable of rational functions

Let

$$z = \frac{P(z)}{Q(z)} \quad \text{for some polynomials } P, Q \text{ and } Q \neq 0.$$

We can omplex differentiate z using the same algebraic manipulation as in $\mathbb{R}$.

Proof: From *example 1.4.2*, we know how to differentiate monomials just like in the real case, and from *theorem 1.4.3*, we can differentiate polynomials just like in the real case by linear combination of monomials. Hence, we can differentiate $P(z)$ and $Q(z)$, and then apply *theorem 1.4.3 (3)*. ☺

Example 1.4.3 ($f = \frac{1}{z}$)

$f'(z) = \frac{-1}{z^2}$ when $z \neq 0$.

Definition 1.4.6: Holomorphic

$f : U \to \mathbb{C}$ is holomorphic if it is complex differentiable at every point in the open set $U \subseteq \mathbb{C}$.

1.4.3 Complex vs real differentiability

Proposition 1.4.1 Complex differentiabilit implies $\mathbb{R}^2$ differentiability

Consider complex number $z = x + yi$ and $f(z) = u(x, y) + iv(x, y)$. If f is complex differentiable at $z_0 = x_0 + y_0 i$, then the map

$$\begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix} : \mathbb{R}^2 \to \mathbb{R}^2$$

is real differentiable at $\langle x_0, y_0 \rangle$.

Proof: If f is differentiable at z_0, then

$$\lim_{z \to z_0} \frac{f(z) - f(z_0) - f'(z_0)(z - z_0)}{z - z_0} = 0,$$

which is equivalent to

$$\lim_{z \to z_0} \frac{|f(z) - f(z_0) - f'(z_0)(z - z_0)|}{|z - z_0|} = 0.$$

Now notice that in face $f'(z_0)$ is a linear homogeneous function from $\mathbb{C} \to \mathbb{C}$. Again if we consider

$z_0 = x_0 + y_0 i$, then $f'(z_0)$ is a linear function of $\langle x_0, y_0 \rangle$ from $\mathbb{R}^2 \to \mathbb{R}^2$. Hence, we have found the derivative $D\begin{bmatrix} u(x_0,y_0) \\ v(x_0,y_0) \end{bmatrix}$, which is $f'(z_0)$ when writing in vector $\mathbb{R}^2$ form.

Note:

Since $f'(z_0)$ is a linear function from $\mathbb{C} \to \mathbb{C}$, it is simply scaling z_0 by a complex number. if $f'(z_0) = (p+qi)z_0$, then $\operatorname{Re} f'(z_0) = p \cdot \operatorname{Re} z_0 - q \cdot \operatorname{Im} z_0$ and $\operatorname{Im} f'(z_0) = p \cdot \operatorname{Im} z_0 + q \cdot \operatorname{Re} z_0$. Writing it in matrix form and representing complex numbers as vectors in $\mathbb{R}^2$, we get the matrix $\begin{bmatrix} p & -q \\ q & p \end{bmatrix}$.

☺

Theorem 1.4.4 Equivalence of complex and $\mathbb{R}^2$ differentiability

$$(z = x + yi) \mapsto (f(z) = w = u(x,y) + v(x,y)i)$$

is complex differentiable at $z_0 = x_0 + y_0 i$ if and only if the map

$$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix}$$

is real differentiable at (x_0, y_0) AND satisfies the Cauchy-Reimann equations

$$u_x = v_y, \quad v_x = -u_y$$

at (x_0, y_0).

Example 1.4.4 (Exponential)

The complex exponential $f : z \mapsto e^z$ is holomorphic in $\mathbb{C}$.

Proof:

$$f : z \mapsto e^z \iff f : (x+yi) \mapsto e^x(\cos y + i \sin y).$$

If we write it in $\mathbb{R}^2$, then

$$g : \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} e^x \cos y \\ e^x \sin y \end{bmatrix}.$$

We know the function is differentiable in $\mathbb{R}^2$ because the partial derivatives exist and are continuous. In particular, the derivative is

$$Dg : \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{bmatrix}.$$

Indeed, the Cauchy-Reimann equations are satisfied, and hence f is real differentiable in $\mathbb{R}^2$, hence complex differentiable in $\mathbb{C}$, which is the definition of holomorphic. ☺

Example 1.4.5 (Other holomorphic / non-holomorphic examples)

- $(x,y) \mapsto \left(x^2 + y^2, 2xy\right)$ non-holomorphic
- $(x,y) \mapsto \left(x^2 - y^2, 2xy\right)$
- $(x,y) \mapsto \left(\frac{1}{2}\log\left(x^2+y^2\right), \arctan\frac{y}{x}\right)$

Proof: Conside as $U \longrightarrow \mathbb{R}^2 : U \subseteq \mathbb{R}^2$ map and check that the Jacobian matrix satisfies the Cauchy-Reimann equations. 😄

1.4.4 Some general properties of holomorphic functions

Proposition 1.4.2 Set of holomorphic maps is an algebra

The set of holomorphic maps $U \longrightarrow \mathbb{C}$ is an algebra:

- If f, g are holomorphic, then $f + g$ is holomorphic.
- If $k \in \mathbb{C}$, then kf is holomorphic.
- If f, g are holomorphic, then fg is holomorphic.
- If f, g are holomorphic and $g(z) \neq 0$ for all $z \in U$, then $\frac{f}{g}$ is holomorphic.

Theorem 1.4.5 Holomorphic function implies holomorphic derivative

If $f : U \longrightarrow \mathbb{C}$ is holomorphic and twice real differentiable, then f' is also holomorphic.

Note:

In fact, the assumption of twice real differentiable can be omitted because holomorphic maps are infinitely differentiable.

Proof: Let $f(x + yi) = u(x, y) + v(x, y)i$ where $u, v : U \longrightarrow \mathbb{R}$. Then the Jacobian matrix of f is

$$Df = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} = \begin{bmatrix} u_x & -v_x \\ v_x & u_x \end{bmatrix}$$

Or we can write $Df = (u_x - v_x) + (v_x + u_x)i$, then the Hessian matrix (derivative of f') of f is

$$D^2 f = \begin{bmatrix} u_{xx} - v_{xx} & u_{xy} - v_{xy} \\ v_{xx} + u_{xx} & v_{xy} + u_{xy} \end{bmatrix} = \begin{bmatrix} u_{xx} - v_{xx} & \overbrace{u_{yx} - v_{yx}}^{\text{Clairaut's}} \\ u_{xx} + v_{xx} & \underbrace{v_{yx} + u_{yx}}_{\text{Clairaut's}} \end{bmatrix} = \begin{bmatrix} u_{xx} - v_{xx} & \overbrace{-u_{xx} - v_{xx}}^{u_x = v_y, v_x = -u_y} \\ u_{xx} + v_{xx} & \underbrace{u_{xx} - v_{xx}}_{u_x = v_y, v_x = -u_y} \end{bmatrix}.$$

Hence, we have shown that for $D^2 f = f''$ is also real differentiable satisfying the Cauchy-Reimann equations. Hence, f' is holomorphic. 😄

Chapter 2

Starting a new chapter

2.1 Demo of commands

Definition 2.1.1: Some defintion

yap

Question 1: Some question

yap

Solution

Some proof: yap ☺

Note:

Some note

Theorem 2.1.1 Some theorem

yap

Wrong Concept 2.1.1: Some wrong concept

yap

Lemma 2.1.1 Some lemma

yap

Proposition 2.1.1 Some proposition

yap

Example 2.1.1 (Some example)

yap

Claim 2.1.1 Some claim

yap

Corollary 2.1.1 Some corollary

yap

Some unlabeled theorem

This is a new paragraph

Algorithm 1: Some algorithm

```
Input: input
Output: output
/* This is a comment */
This is first line ;                                // This is also a comment
if x > 5 then
|   do nothing
else if x < 5 then
|   do nothing
else
|   do nothing
end
while x == 5 do
|   still do nothing
end
foreach x = 1 : 5 do
|   do nothing
end
return return nothing
```