STAT 153 sketch

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Proof of null(X) contains at least one non-zero vector η:

Since $p > n$, the column vectors are linear dependent. Denote $(v_1, \cdots, v_p)$ as the column vectors of X. Then, there are non trivial coefficients $(c_1, \cdots, c_p)$ such that

$$\sum_{i=1}^{p} c_i v_i = 0.$$

Hence, $\eta = (c_1, \cdots, c_p)$ is a non-zero vector in null(X).

Proof of $\hat{\beta} = \tilde{\beta} + \eta$ is also a least squares solution for $\eta \in \text{null}(X)$:

Denote the prediction from $\tilde{\beta}$ as $\tilde{y} = X\tilde{\beta}$ with MSE $= y - \tilde{y}$. Then the prediction from $\hat{\beta}$

$$\begin{align}
\hat{y} &= X\hat{\beta} \tag{1}\\
&= X(\tilde{\beta} + \eta) \tag{2}\\
&= X\tilde{\beta} + X\eta \tag{3}\\
&= \tilde{y} + X\eta \tag{4}\\
&= \tilde{y}. \tag{5}
\end{align}$$

Therefore, they have the same MSE. Since $\hat{\beta}$ is a least squares solution, $\tilde{\beta} + \eta$ is also a least squares solution.

Since null(X) $\not\perp e_j$, there exists some $v \in \text{null}(X)$ that has non-zero j-th coorindate. Denote the j-th coordinate of v as a real number c. If $c > 0$, we can construct $\hat{\beta} = \tilde{\beta} - \left(\frac{\tilde{\beta}_j}{c}\right) v - v$, which has the j-th coordinate less than 0. If $c < 0$, we can construct $\hat{\beta} = \tilde{\beta} + \left(\frac{\tilde{\beta}_j}{c}\right) v - v$, which also has the j-th coordinate less than 0.

For either case, the prediction

$$\begin{aligned}
\hat{y} &= X\tilde{\beta}\\
&= X\left[\tilde{\beta} \pm \left(\frac{\tilde{\beta}_j}{c}\right) v - v\right]\\
&= X\tilde{\beta} \pm X\left(\frac{\tilde{\beta}_j}{c}\right) v - Xv\\
&= X\tilde{\beta} \pm \left(\frac{\tilde{\beta}_j}{c}\right) Xv - Xv\\
&= X\tilde{\beta} \qquad (\because v \in \text{null}(X))\\
&= \tilde{y}.
\end{aligned}$$

Hence, $\tilde{\beta}$ and $\hat{\beta}$ will have the same prediction under X, so as the MSE.

$$
\begin{aligned}
\mu_t &= \mathbb{E}(x_t) \\
&= \mathbb{E}\left[\sum_{j=1}^{p}\Big(U_{j1}\cos(2\pi\omega_j t) + U_{j2}\sin(2\pi\omega_j t)\Big)\right] \\
&= \mathbb{E}\left[\sum_{j=1}^{p}(U_{j1} + Uj2)\right] \\
&= 0.
\end{aligned}
$$

Notation: Use S to represent x_s and T to represent x_t, so $S_{k1} = U_{k1}\cos(2\pi\omega_i s)$, $S_{k2} = U_{k2}\sin(2\pi\omega_i s)$, while $T_{k1} = U_{k1}\cos(2\pi\omega_i t)$, $T_{k2} = U_{k2}\sin(2\pi\omega_i t)$. Notice the S and T are uncorrelated if they have different subscripts. Then the auto-covariance is

$$
\begin{aligned}
\gamma(s,t) &= \operatorname{Cov}(x_s, x_t) \\
&= \operatorname{Cov}\left(\sum_{j=1}^{p}\Big(U_{j1}\cos(2\pi\omega_j s) + U_{j2}\sin(2\pi\omega_j s)\Big), \sum_{j=1}^{p}\Big(U_{j1}\cos(2\pi\omega_j t) + U_{j2}\sin(2\pi\omega_j t)\Big)\right) \\
&= \sum_{j=1}^{p}\sum_{i=1}^{p}\operatorname{Cov}(S_{j1}, T_{i1}) + \sum_{j=1}^{p}\sum_{i=1}^{p}\operatorname{Cov}(S_{j1}, T_{i2}) + \sum_{j=1}^{p}\sum_{i=1}^{p}\operatorname{Cov}(S_{j2}, T_{i1}) + \sum_{j=1}^{p}\sum_{i=1}^{p}\operatorname{Cov}(S_{j2}, T_{i2}) \\
&= \sum_{i=1}^{p}\operatorname{Cov}(S_{i1}, T_{i1}) + \sum_{i=1}^{p}\operatorname{Cov}(S_{i2}, T_{i2}) \\
&= \sum_{i=1}^{p}\sigma^2\cos(2\pi\omega_i s)\cos(2\pi\omega_i t) + \sum_{i=1}^{p}\sigma^2\sin(2\pi\omega_i s)\sin(2\pi\omega_i t) \\
&= \sigma^2\sum_{i=1}^{p}\left[\cos(2\pi\omega_i s)\cos(2\pi\omega_i t) + \sin(2\pi\omega_i s)\sin(2\pi\omega_i t)\right] \\
&= \sigma^2\sum_{i=1}^{p}\cos(2\pi\omega_i s - 2\pi\omega_i t) \\
&= \sigma^2\sum_{i=1}^{p}\cos(2\pi\omega_i(s-t)),
\end{aligned}
$$

which after reparametrizing to the lag h is the same as

$$\gamma(h) = \sigma^2\sum_{i=1}^{p}\cos(2\pi\omega_i h).$$

Since the auto-covariance is dependent on the lag only and mean is a constant 0, the process is weakly stationary.