

Math 180A HW3

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Problem 3. Obviously, $P(A) = \frac{1}{2}$ and $P(B) = \frac{1}{5}$. Then, $P(A \cap B) = P(A|B)P(B) = \frac{1}{2} \times \frac{1}{5} = P(A)P(B)$. Hence, A and B are indeed independent.

Problem 4.

- (a) $E(X)$ will be larger. For Y , each outcome is weighted evenly with $p(x_i) = \frac{1}{4}$. Yet, for X , the outcome of more students are weighted more while the outcome of fewer students are weighted less with $p(x_i) = \frac{x_i}{90}$. In other words, there's a higher probability of choosing a student from a larger class than from a smaller class in X .

(b)

$$E(X) = \sum x_i p(x_i) \quad (1)$$

$$= 21 \times \frac{21}{90} + 24 \times \frac{24}{90} + 17 \times \frac{17}{90} + 28 \times \frac{28}{90} \quad (2)$$

$$= \frac{209}{9} \quad (3)$$

$$E(Y) = 21 \times \frac{1}{4} + 24 \times \frac{1}{4} + 17 \times \frac{1}{4} + 28 \times \frac{1}{4} \quad (4)$$

$$= \frac{45}{2} \quad (5)$$

Problem 5.

- (a) Possible values of $Y = 0, 1$. $P(Y = 1) = \frac{1}{2} + \frac{1}{6} = \frac{2}{3}$ and $P(Y = 0) = \frac{1}{3}$.

$$E(|X|) = E(Y) \quad (6)$$

$$= 1 \times \frac{2}{3} + 0 \times \frac{1}{3} \quad (7)$$

$$= \frac{2}{3} \quad (8)$$

(b)

$$E(|X|) = |-1| \times \frac{1}{2} + |1| \times \frac{1}{6} + 0 \times \frac{1}{6} \quad (9)$$

$$= \frac{2}{3} \quad (10)$$

Problem 6. We can express X in terms of a collection of indicators $I_i = I_{\{i\text{th card is a pair with } (i+1)\text{th card}\}}$, for which $X = I_1 + I_2 + \dots + I_{51}$. Note that

$$E(I_1) = E(I_2) = \dots = E(I_{51}) = 1 \cdot P(\{\text{first card is a pair with second card}\}) \quad (11)$$

$$= \frac{3}{51}. \quad (12)$$

Therefore, by additivity, $E(X) = E(I_1) + E(I_2) + \dots + E(I_{51}) = 51 \times \frac{3}{51} = 3$.

Side note: this counting method counts 3-of-a-kind as two of two consecutive cards and 4-of-a-kind as three of two consecutive cards.

Problem 7.

- (a) Let A be the event of a die roll and B be the event of the other die roll. Note that A and B are independent because $P(A \cap B) = P(A)P(B)$.

Now let us consider $P(X \leq s) : 1 \leq s \leq 20, s \in \mathbb{Z}^+$, which is the probability that both the dice rolls are less than or equal to s . Therefore, it can be written as $P(X \leq s) = P((A \leq s) \cap (B \leq s)) = \frac{s}{20} \times \frac{s}{20} = \frac{s^2}{400}$.

Hence, CDF of X is

$$F_x(s) = \begin{cases} 0, & s \leq 0 \\ \frac{s^2}{400}, & 1 \leq s \leq 20 \\ 1, & 20 < s. \end{cases} \quad (13)$$

Now let us determine the pmf of X . Note that $P(X = k) = P(X \leq k) - P(X \leq k - 1)$. Therefore,

$$\forall k \in \mathbb{Z}, p(k) = P(X = k) = \begin{cases} 0, & k < 1 \text{ or } 20 < k \\ \frac{k^2}{400} - \frac{(k-1)^2}{400} = \frac{2k-1}{400}, & 1 \leq k \leq 20. \end{cases} \quad (14)$$

- (b) Let us first approach the question with basic combinatorics knowledge then we'll approach the question with CDF , which will converge to the same conclusion.

For $Y = k$, one die roll has to be equal to k , and another die roll can be between k and 20 (end point inclusive), which has a total of $(21 - k)$ possibilities. Then, let's take ordering into account. Since there are two die roll, for every combination there are two orders so we times 2 for $(21 - k)$. Note we have to minus 1 because there is only one ordering if the two roll die have the same outcome. Finally, we divide the event outcomes by all possible outcomes $20 \times 20 = 400$.

Hence,

$$\forall k \in \mathbb{Z}, P(Y = k) = \begin{cases} 0, & 1 < k \text{ or } 20 < k \\ \frac{2(21-k)-1}{20 \times 20} = \frac{41-2k}{400}, & 1 \leq k \leq 20. \end{cases} \quad (15)$$

Now let us find the pmf of Y from CDF just like how we did in (a). Let A be the event of a die roll and B be the event of the other die roll. Note that A and B are independent because $P(A \cap B) = P(A)P(B)$. For $1 \leq s \leq 20 : s \in \mathbb{Z}$,

$$P(Y \leq s) = 1 - P(Y > s) \quad (16)$$

$$= 1 - P(A > s)P(B > s) \quad (17)$$

$$= 1 - \left(\frac{20-s}{20}\right)\left(\frac{20-s}{20}\right) \quad (18)$$

$$= 1 - \frac{s^2 - 40s + 400}{400} \quad (19)$$

$$= \frac{40s - s^2}{400}. \quad (20)$$

Hence, the CDF of Y is

$$F_y(s) = \begin{cases} 0, & s \leq 0 \\ \frac{40s - s^2}{400}, & 1 \leq s \leq 20 \\ 1, & 20 < s. \end{cases} \quad (21)$$

By the same reasoning from (a), $P(Y = k) = P(Y \leq k) - P(Y \leq k - 1)$. Therefore,

$$\forall k \in \mathbb{Z}, P(Y = k) = \begin{cases} 0, & 1 < k \text{ or } 20 < k \\ \frac{40k - k^2 - (40(k-1) - (k-1)^2)}{400} = \frac{41-2k}{400}, & 1 \leq k \leq 20, \end{cases} \quad (22)$$

which is the same as the combinatorics approach.