Math 180B HW8

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PK Problem 5.2.1 Let $X(n,p)$ have a binomial distribution with parameters n and p. Let $n \to \infty$ and $p \to 0$ in such a way that $np = \lambda$. Show that

$$\lim_{n\to\infty} P\{X(n,p) = 0\} = e^{-\lambda}$$

and

$$\lim_{n\to\infty} \frac{P\{X(n,p) = k+1\}}{P\{X(n,p) = k\}} = \frac{\lambda}{k+1} \text{ for } k = 0, 1, \ldots.$$

Solution. The first equation is immediate from the Poisson approximation to the binomial distribution with the error bounded by np^2:

$$\begin{aligned}
|P(X=0) - e^{-\lambda}| \leq np^2 \leq n\left(\frac{1}{n}\right)^2 \qquad \left(p \leq \frac{1}{n} \text{ as } p \to 0\right) \\
\lim_{n\to\infty} |P(X=0) - e^{-\lambda}| \leq \lim_{n\to\infty} n\left(\frac{1}{n}\right)^2 = 0 \\
\lim_{n\to\infty} P(X=0) = e^{-\lambda}.
\end{aligned}$$

For the second equation, we can use the Poisson approximation to the binomial distribution again, then similarly, we get:

$$\lim_{n\to\infty} P(X = k+1) = \frac{\lambda^{k+1}e^{-\lambda}}{(k+1)!}$$
$$\lim_{n\to\infty} P(X = k) = \frac{\lambda^{k}e^{-\lambda}}{k!}$$

and we can get the second equation by dividing the two equations above:

$$\begin{aligned}
\lim_{n\to\infty} \frac{P(X=k+1)}{P(X=k)} &= \lim_{n\to\infty} \frac{\lambda^{k+1}e^{-\lambda}}{(k+1)!} \cdot \frac{k!}{\lambda^k e^{-\lambda}} \\
&= \frac{\lambda}{k+1}.
\end{aligned}$$

□

PK Problem 5.2.7 N bacteria are spread independently with uniform distribution on a microscope slide of area A. An arbitrary region having area a is selected for observation. Determine the probability of k bacteria within the region of area a. Show that as $N \to \infty$ and $a \to 0$ such that$(a/A)N \to c(0 < c < \infty)$,then $p(k) \to e^{-c}c^k/k!$.

Solution. The probability interested is essentially a binomial distribution with N trials and $p = a/A$. Hence, we can use the Poisson approximation to the binomial distribution to get:

$$p(k) = \frac{c^k e^{-c}}{k!}.$$

□

PK Exercise 5.3.2 A radioactive source emits particles according to a Poisson process of rate $\lambda = 2$ particles per minute.

(a) What is the probability that the first particle appears some time after 3 min but before 5 min?

Solution. We are interested in $P(3 \leq W_1 \leq 5)$, where W_1 is the waiting time for the first particle, following an exponential distribution with parameter $\lambda = 2$.

$$\begin{aligned} P(3 \leq W_1 \leq 5) &= \int_3^5 2e^{-2t} dt \\ &= \left[-e^{-2t}\right]_3^5 \\ &= e^{-6} - e^{-10}. \end{aligned}$$

□

(b) What is the probability that exactly one particle is emitted in the interval from 3 to 5 min?

$$\begin{aligned} P(N(3,5) = 1) &= \frac{e^{-2\times 2}(2 \times 2)^1}{1!} \\ &= 4e^{-4}. \end{aligned}$$

PK Exercise 5.3.6 For $i = 1, ..., n$, let $\{X_i(t); t \geq 0\}$ be independent Poisson processes, each with the same parameter λ. Find the distribution of the first time that at least one event has occurred in every process.

Solution. Let $Y = min(t; X_i(t) \geq 1)$. We will find $f_Y(y)$ by finding $P(Y \leq y)$.

$$\begin{aligned} P(Y \leq y) &= P(W_1^{(1)} \leq y, W_1^{(2)} \leq y, \ldots, W_1^{(n)} \leq y) \\ &= (1 - e^{-\lambda y})^n \\ f_Y(y) &= \frac{d}{dy} P(Y \leq y) \\ &= n(1 - e^{-\lambda y})^{n-1} \cdot \lambda e^{-\lambda y}. \end{aligned}$$

□

PK Problem 5.3.3 The joint probability density function for the waiting times W_1 and W_2 is given by

$$f(w_1, w_2) = \lambda^2 e^{-\lambda w_2} \ for \ 0 \leq w_1 \leq w_2.$$

Change variables according to

$$S_0 = W_1 \ and \ S_1 = W_2 - W_1$$

and determine the joint distribution of the first two sojourn times. Compare with Theorem 5.5.

Solution.

$$\begin{aligned} f(w_1, w_2) &= \lambda^2 e^{-\lambda w_2} \\ &= \lambda^2 e^{-\lambda(w_1 + w_2 - w_1)} \\ &= \lambda^2 e^{-\lambda(s_0 + s_1)} \\ &= \lambda e^{-\lambda s_0} \cdot \lambda e^{-\lambda s_1} \\ &= f(s_0, s_1). \end{aligned}$$

Hence, the joint distribution of the first two sojourn times is two independent exponential distribution with parameter λ, which agrees with Theorem 5.5. □

PK Problem 5.3.7 A critical component on a submarine has an operating lifetime that is exponentially distributed with mean 0.50 years. As soon as a component fails, it is replaced by a new one having statistically identical properties. What is the smallest number of spare components that the submarine should stock if it is leaving for a one-year tour and wishes the probability of having an inoperable unit caused by failures exceeding the spare inventory to be less than 0.02?

Solution. Let $S_n = X_1 + \cdots + X_n$, where X_k represents the life time of the k-th component. Notice that S_n is a sum of n independent exponential random variables with parameter $\lambda = 2$, so S_n follows a gamma distribution with parameter $\alpha = n$ and $\lambda = 2$, which interestingly is the waiting time of the n-th event in a Poisson process with parameter $\lambda = 2$.

We are interested in finding n such that $P(S_n < 1) < 0.02$.

$$\begin{aligned} P(S_n < 1) &= \int_0^1 \frac{2^n}{(n-1)!} t^{n-1} e^{-2t} dt \\ &= \int_0^2 \frac{2^n}{(n-1)!} \left(\frac{u}{2}\right)^{n-1} e^{-u} \cdot \frac{1}{2} du \qquad (u = 2t) \\ &= \frac{1}{(n-1)!} \int_0^2 u^{n-1} e^{-u} du \\ &= \frac{1}{(n-1)!} \Gamma(n). \end{aligned}$$

With the help of a calculator, $P(S_5 < 1) \approx 0.0527, P(S_6 < 1) \approx 0.0166$. Hence, $n = 6$ is the smallest number of spare components that the submarine should stock. $\square$