Math 104 HW12

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Exercise 31.2

Find the Taylor series for the functions $\sinh = \frac{1}{2}\left(e^x - e^{-x}\right)$ and $\cosh = \frac{1}{2}\left(e^x + e^{-x}\right)$ and indicate why they converge to sinh and cosh respectively for all $x \in \mathbb{R}$.

Solution. The even degree derivatives of sinh centered at 0 are all 0, and the odd degree derivatives are all 1. Thus, the Taylor series for sinh is

$$\sinh(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}.$$

On the other hand, the even degree derivatives of cosh centered at 0 are all 1, and the odd degree derivatives are all 0. Thus, the Taylor series for cosh is

$$\cosh(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}.$$

For all $x \in \mathbb{R}$, $x \in (-|x| - 1, |x| + 1)$, and both $\sinh^{(n)}$ and $\cosh^{(n)}$ are bounded on this interval (by $e^{|x|+1}$) for all $n \in \mathbb{N}$. Thus, by *Corollary 31.4*, the remainder term of Taylor series for both sinh and cosh tend to 0 as $n \to \infty$ for all $x \in \mathbb{R}$. Thus, the Taylor series for both sinh and cosh converges to sinh and cosh respectively for all $x \in \mathbb{R}$. $\square$

Exercise 31.5

Let $g(x) = e^{-1/x^2}$ for $x \neq 0$ and $g(0) = 0$.

(a) Show $g^{(n)}(0) = 0$ for all $n = 0, 1, 2, \ldots$.

(b) Show the Taylor series for g about 0 agrees with g only at $x = 0$.

Solution.

(a) Let $f(x) = e^{-1/x}$, then $g(x) = f(x^2)$ for all $x \in \mathbb{R}$. Since f is differentiable at 0, and x^2 is differentiable at 0. By the Chain rule, g is differentiable at 0, and

$$g'(x) = f'(x^2) \cdot 2x$$

for all $x = 0$. Since f^k is differentiable at 0 for all $k \in \mathbb{Z}^{\geq}$, and x^j is differentiable at 0 for all $j \in \mathbb{N}$, we can apply Chain rule and Product rule repeatedly to obtain higher degree derivatives of g at 0.

The formula for derivatives of g at 0 is

$$g^{(n)}(x) = \sum_{k=0}^{n} a_k f^{(k)}(x^2) \cdot x^{b_k}$$

for some non-negative integers a_k and b_k. From *example 3*, we know $f^{(n)}(0) = 0$ for $n \in \mathbb{Z}^{\geq}$. Hence, $g^{(n)}(x)$ evaluated at 0 is 0 for all $n \in \mathbb{Z}^{\geq}$.

(b) The Taylor series of g about 0 is

$$\sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!} x^n = 0.$$

Clearly, $g(x) = 0$ for all $x \neq 0$, but $g(0) = 0$. Thus, the Taylor series for g about 0 agrees with g only at $x = 0$.

□

Exercise 32.2

Let $f(x) = x$ for rational x and $f(x) = 0$ for irrational x.

(a) Calculate the upper and lower Darboux integrals for f on $[0, b]$.

(b) Is f integrable on $[0, b]$?

Solution.

(a) For any partition P of $[0, b]$, the supremum of f on any subinterval of P is the right endpoint of the subinterval, and the infimum of f on any subinterval of P is 0 because rationals and irrationals are dense in $\mathbb{R}$.

Therefore, with any partition P of $[0, b]$,

$$U(f, P) = \sum_{k=1}^{n} x_k \cdot (x_k - x_{k-1}).$$

In fact, f has the same upper Darboux integrals as $g(x) = x$ on $[0, b]$ because they have the same $U(f, P)$ for any P. From calculus, we know that g is integrable on $[0, b]$ and

$$\int_0^b g(x)dx = \frac{b^2}{2}.$$

Hence, the upper Darboux integral of g is $\frac{b^2}{2}$, and the upper Darboux integral of f is also $\frac{b^2}{2}$.

On the other hand, for any partition P of $[0, b]$,

$$L(f, P) = \sum_{k=1}^{n} 0 \cdot (x_k - x_{k-1}) = 0.$$

Hence, the lower Darboux integral of f is 0.

Alternatively for $g(x) = x$, we can show that it is integrable on $[0, b]$ by evaluating its upper and lower Darboux integrals by partitioning $[0, b]$ into $\left[\frac{(k-1)b}{n}, \frac{kb}{n}\right]$ and letting $n \to \infty$.

(b) Since the upper and lower Darboux integrals of f are not equal, f is not integrable.

$\square$

Exercise 32.7

Let f be integrable on $[a, b]$, and suppose g is a function on $[a, b]$ such that $g(x) = f(x)$ except for finitely many $x \in [a, b]$. Show that g is integrable on $[a, b]$ and $\int_a^b g(x)dx = \int_a^b f(x)dx$.

Solution. Let S be the set of points $x \in [a, b]$ such that $g(x) \neq f(x)$. Then for any partition P, define

$$P' = P \cup \{\cdots, t_k, x - \frac{1}{n}, x, t_{k+1}, \cdots\} \qquad \text{for } x \in S.$$

where the t's belong to the original P and n is chosen such that $x - \frac{1}{n} > t_k$. Since $P \subseteq P'$, by *Lemma 32.2*,

$$L(f, P) \leq L(f, P') \leq U(f, P') \leq U(f, P),$$

we can just focus on the partition P'.

Now for any partition P', $U(f, P')$ and $U(g, P')$ only differ by the finitely many subintervals that contain points in S. In fact, we can write

$$U(g, P') = U(f, P') - \sum_{x \in S} M(f, [x - \frac{1}{n}, x]) \cdot \frac{1}{n} + \sum_{x \in S} M(g, [x - \frac{1}{n}, x]) \cdot \frac{1}{n}$$

Since S is finite, the two terms on the right hand side equal 0 when $n \to \infty$, and hence

$$U(g, P') = U(f, P').$$

Similarly, we can show that $L(g, P') = L(f, P')$. Thus, g and f have the same upper and lower Darboux integrals, and g is integrable on $[a, b]$ with $\int_a^b g(x)dx = \int_a^b f(x)dx$.

Note: We have not considered the case where $M(f, [x - \frac{1}{n}, x])$ or $M(g, [x - \frac{1}{n}, x])$ is infinite. However, since f and g only differ by finite terms on that interval, if any of the two is infinite, the other must also be infinite. In that case, we don't do any manipulation within that interval, and the proof still holds. The same is true for the case where $m(f, [x - \frac{1}{n}, x])$ or $m(g, [x - \frac{1}{n}, x])$ is infinite. $\square$

Exercise 33.4

Give an example of a function f on $[0,1]$ that is not integrable for which $|f|$ is integrable.

Solution. Let $f(x) = -1$ for rationals and $f(x) = 1$ for irrationals. Then $|f(x)| = 1$ for all $x \in [0,1]$. From calculus (or we can go through the tedious upper and lower Darboux argument), we know that $|f|$ is integrable on $[0,1]$ and

$$\int_0^1 |f(x)|dx = 1.$$

However, for any partition P,

$$m(f,p) = -1 \qquad \text{and} \qquad M(f,p) = 1,$$

where p is any subinterval of P. Hence,

$$L(f,P) = \sum_{k=1}^{n} -1 \cdot (x_k - x_{k-1}) = -1 \qquad \text{and} \qquad U(f,P) = \sum_{k=1}^{n} 1 \cdot (x_k - x_{k-1}) = 1.$$

Thus, f is not integrable on $[0,1]$. $\square$