Math 128A HW1

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Section 1.1

Problem 2c

Proposition 1. *$f(x) = -3 \cdot tan(2x) + x = 0$ has at least one solution for $x \in [0, 1]$.*

Proof. Note that the interval is end point inclusive. We have $f(0) = 0$, which is immediately one solution to the equation. $\square$

Problem 2d

Proposition 2. *$f(x) = ln(x) - x^2 + \frac{5}{2}x - 1 = 0$ has at least one solution for $x \in [\frac{1}{2}, 1]$.*

Proof. $f(\frac{1}{2}) \approx -0.693, f(1) = 0.5$. Hence, by the intermediate value theorem, there exists a solution in the interval. $\square$

Problem 4d

Find interval containing solutions to $x^3 + 4.001x^2 + 4.002x + 1.101 = 0$.

Solution. Since there is no further intructions on the error bound of the interval, we will proceed with the simplest calculus method by finding critical points. Consider the first derivative and set it equal to zero.

$$\begin{aligned} 3x^2 + 8.002x + 4.002 = 0 \\ x &= \frac{-8.002 \pm \sqrt{8.002^2 - 4 \cdot 3 \cdot 4.002}}{2 \cdot 3} \\ &\approx \frac{-8 \pm 4}{6} = \frac{-4 \pm 2}{3}. \end{aligned}$$

Now, we can look at both points respectively. $f(-2) = 1.101, f(\frac{-2}{3}) \approx -0.0851$. By intermediate theorem, we know that there exists a solution in $[-2, \frac{-2}{3}]$. Now, by simple observation, we can easily conclude that the equation tends to $-\infty$ when $x \to -\infty$ and tends to ∞ when $x \to \infty$. Hence, by intermediate theorem, there exists one solution in $[-\infty, -2]$ and one in $[\frac{-2}{3}, \infty]$. $\square$

Problem 6a

Find $max_{a \le x \le b}|f(x)|$ for $f(x) = \frac{2x}{x^2+1}$ on $[0, 2]$.

Solution. We proceed by finding the critical points of $f(x)$ on $[0, 2]$.

$$
\begin{aligned}
f'(x) &= \frac{2(x^2+1) - 2x(2x)}{(x^2+1)^2} \\
&= \frac{2-2x^2}{(x^2+1)^2} \\
&= 0 \text{ when } x = \pm 1.
\end{aligned}
$$

Then, we have $f(0) = 0, f(1) = 1, f(2) = \frac{4}{5}$. Hence, the maximum value of $f(x)$ on $[0, 2]$ is 1 when $x = 1$. □

Problem 14

Let $f(x) = 2x \cdot cos(2x) - (x-2)^2$ and $x_0 = 0$.

(a) Find the third Taylor polynomial $P_3(x)$ and use it to approximate $f(0.4)$.

Solution.

$$
\begin{aligned}
f'(x) &= 2cos(2x) - 4xsin(2x) - 2(x-2), \\
f''(x) &= -8sin(2x) - 8xcos(2x) - 2 \\
f'''(x) &= 16xsin(2x) - 24cos(2x).
\end{aligned}
$$

Now, we have $f(0) = -4, f'(0) = 6, f''(0) = -2, f'''(0) = -24$. Hence, the third Taylor polynomial $P_3(x) = -4 + 6x - x^2 - 4x^3$, and $f(0.4) \approx -2.016$. □

(b) Use the error formula in Taylor's Theorem to find an upper bound for the error $|f(0.4) - P_3(0.4)|$

Solution.

$$
\begin{aligned}
f^4(x) &= 64sin(2x) + 32xcos(2x) \\
R_3(x) &= \frac{f^4(\xi(x))}{4!}(0.4)^4 \qquad \textit{for } 0 \le \xi(x) \le 0.4.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
|f(0.4) - P_3(0.4)| = |R_3(0.4)| &= \frac{f^4(\xi(x))}{4!}(0.4)^4 \\
&= \frac{64sin(2\xi(x)) - 32 \cdot \xi(x)cos(2\xi(x))}{24} \times 0.0256 \\
&\le \frac{64 - 32 \cdot \xi(x)}{24} \times 0.0256 \qquad (\textit{notice } 0 \le sin(2\xi(x)), cos(2\xi(x)) \le 1) \\
&\le \frac{64}{24} \times 0.0256 \\
&\le 0.06827.
\end{aligned}
$$

□

(c) Find the third Taylor polynomial $P_4(x)$ and use it to approximate $f(0.4)$.

Solution.

$$\begin{aligned} f^4(x) &= 64sin(2x) + 32xcos(2x) \\ f^4(0) &= 0 \\ P_4(x) &= -4 + 6x - x^2 \\ f(0.4) &\approx P_4(0.4) = -2.016. \end{aligned}$$

□

(d) Use the error formula in Taylor's Theorem to find an upper bound for the error $|f(0.4) - P_4(0.4)|$. Compute the actual error.

Solution.

$$\begin{aligned} f^5(x) &= 160cos(2x) - 64xsin(2x) \\ |f(0.4) - P_4(0.4)| = R_4(x) &= \frac{f^5(\xi(x))}{5!}(0.4)^5 \qquad \textit{for } 0 \le \xi(x) \le 0.4. \\ &\le \frac{160}{5!} \times 0.4^5 \\ &\le 0.01366. \end{aligned}$$

□

Problem 26

a. Use Rolle's Theorem to show that $f'(z_i) = 0$ for n numbers in $[a, b]$ with $a < z_1 < z_2 < \cdots < z_n < b$.

Solution. We are unable to show that without further assumption. Since our goal is proving the Generalized Rolle's Theorem, we will proceed with the assumption that $f(x)$ is n times differentiable on (a, b), and that $f(x) = f(a) = f(b)$ at $n - 1$ distinct numbers such that $a < x_2 < x_3 \cdots < x_n < b$. For clarity, we will also denote $x_1 = a$ and $x_{n+1} = b$ such that $a = x_1 < x_2 < \cdots < x_{n+1} = b$.

Now consider each interval $[x_i, x_{i+1}]$ for all $i \in [0, 1, \ldots, n]$. Notice that there are n such intervals and $f(x_i) = f(x_{i+1})$. Hence, by Rolle's Theorem, there exists $z_i \in (x_i, x_{i+1})$ such that $f'(z_i) = 0$. Therefore, we can conclude that $f'(z_i) = 0$ for n numbers in $[a, b]$ with $a < z_1 < z_2 < \cdots < z_n < b$. □

b. Use Rolle's Theorem to show that $f''(w_i) = 0$ for $n - 1$ numbers in $[a, b]$ with $z_1 < w_1 < z_2 < w_2 \cdots w_n < z_n < b$.

Solution. We have shown from (a) that $f'(z_i) = 0$ for $i \in [1, 2, \ldots, n]$. Now consider each interval (z_i, z_{i+1}) for $i \in [1, 2, \ldots, n - 1]$. There are $n - 1$ such intervals and $f'(z_i) = f'(z_{i+1})$. Hence, by Rolle's Theorem, there exists $w_i \in (z_i, z_{i+1})$ such that $f''(w_i) = 0$. Therefore, we can conclude that $f''(w_i) = 0$ for $n - 1$ numbers in $[a, b]$ with $a < z_1 < w_1 < z_2 < w_2 \cdots w_{n-1} < z_n < b$. □

c. Continue the arguments in parts (a) and (b) to show that for each $j = 1, 2, \ldots, n$, there are $n + 1 - j$ distinct numbers in $[a, b]$, where $f^{(j)}$ is 0.

Solution. We can easily show by induction on j. Base case is already established in (a). Now, assume that $f^{(k)}(y_i) = 0$ for $n + 1 - k$ distinct numbers in $[a, b]$. Consider each interval (y_i, y_{i+1}) for $i \in$

$[1, 2, \ldots, n-k]$. There are $n-k$ such intervals and $f^{(k)}(y_i) = f^{(k)}(y_{i+1})$. Hence, by Rolle's Theorem, there exists $n-k$ numbers of $u_i \in (y_i, y_{i+1})$ such that $f^{(k+1)}(u_i) = 0$. Therefore, we can conclude that $f^{(k+1)}(u_i) = 0$ for $n-k = n+1-(k+1)$ distinct numbers in $[a, b]$.

Hence, by induction, we can conclude that for each $j = 1, 2, \ldots, n$, there are $n+1-j$ distinct numbers in $[a, b]$, where $f^{(j)}$ is 0. □

d. Show that part (c) implies the conclusion of the Generalized Rolle's Theorem.

Solution. Notice our only assumptions are that $f(x)$ is n times differentiable on (a, b), and that $f(x) = f(a) = f(b)$ at $n-1$ distinct numbers such that $a < x_2 < x_3 \cdots < x_n < b$, which is equivalent to saying $f(x)$ is constant for $n+1$ distinct numbers in $[a, b]$.

With the mentioned assumptions, we concluded that there are $n+1-j$ distinct numbers in $[a, b]$, where $f^{(j)}$ is 0. Now take $j = n$, we have $n+1-n = 1$ distinct number in $[a, b]$, where $f^{(n)}$ is 0. Hence, we can conclude that there exists $c \in (a, b)$ such that $f^{(n)}(c) = 0$.

Notice this is exactly what the Generalized Rolle's Theorem says. □

Section 1.2

Problem 2c

Compute the absolute error and relative error in approximations of p by p^*: $p = 8!, p^* = 39900$.

Solution.

$$|p - p^*| = 8! - 39900 = 40320 - 39900 = 420.$$
$$\frac{|p - p^*|}{p} = \frac{420}{8!} = \frac{1}{96} \approx 0.0104.$$

□

Problem 4b

Find the largest interval in which p^* must lie to approximate p with relative error at most 10^{-4} for $p = e$.

Solution.

$$\frac{|p^* - e|}{e} \leq 10^{-4}$$
$$|p^* - e| \leq 10^{-4}e$$
$$|p^* - e| \leq 10^{-4}e$$
$$(1 - 10^{-4}) \cdot e \leq p^* \leq (1 + 10^{-4}) \cdot e$$

□

Problem 12

Problem 22

Section 1.3

Problem 8

Problem 15

Discussion Question 2 (p. 38)

Section 2.1

Problem 6d

Problem 8

Problem 20