

Math 180B HW1

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Problem 1.

(a)

$$\begin{aligned} E[e^{tX}] &= \int_{-\infty}^{\infty} e^{tx} \cdot \frac{\lambda}{\Gamma(\alpha)} (\lambda x)^{\alpha-1} e^{-\lambda x} dx \\ &= \int_0^{\infty} e^{tx} \cdot \frac{\lambda}{\Gamma(\alpha)} (\lambda x)^{\alpha-1} e^{-\lambda x} dx \quad (\because x > 0 \text{ for Gamma Distribution}) \\ &= \int_0^{\infty} \frac{\lambda}{\Gamma(\alpha)} (\lambda x)^{\alpha-1} e^{-(\lambda-t)x} dx. \end{aligned}$$

Let $u = (\lambda - t)x$ for u-substitution. Then $\frac{du}{\lambda-t} = dx$, and 1) $u \rightarrow \infty$ when $t < \lambda$ and $x \rightarrow \infty$; 2) $u \rightarrow -\infty$ when $t > \lambda$ and $x \rightarrow \infty$; 3) $u \rightarrow 0$ when $t = \lambda$.

Hence, when $t < \lambda$,

$$\begin{aligned} E[e^{tX}] &= \int_0^{\infty} \frac{\lambda}{\Gamma(\alpha)} \left(\lambda \frac{u}{\lambda-t}\right)^{\alpha-1} e^{-u} \frac{u}{\lambda-t} du \\ &= \left(\frac{\lambda}{\lambda-t}\right)^{\alpha} \cdot \frac{1}{\Gamma(\alpha)} \int_0^{\infty} u^{\alpha-1} e^{-u} du \quad (\text{notice } \int_0^{\infty} u^{\alpha-1} e^{-u} du = \Gamma(\alpha)) \\ &= \left(\frac{\lambda}{\lambda-t}\right)^{\alpha}. \end{aligned}$$

When $t = \lambda$, $\lambda - t = 0$, and

$$\begin{aligned} E[e^{tX}] &= \left(\frac{\lambda}{\lambda-t}\right)^{\alpha} \cdot \frac{1}{\Gamma(\alpha)} \int_0^0 u^{\alpha-1} e^{-u} du \\ &= \left(\frac{\lambda}{\lambda-t}\right)^{\alpha} \cdot \frac{1}{\Gamma(\alpha)} \\ &= \lim_{t \rightarrow \lambda} \left(\frac{\lambda}{\lambda-t}\right)^{\alpha} \cdot \frac{1}{\Gamma(\alpha)} \\ &= \infty. \end{aligned}$$

When $t > \lambda$,

$$\begin{aligned} E[e^{tX}] &= \left(\frac{\lambda}{\lambda-t}\right)^{\alpha} \cdot \frac{1}{\Gamma(\alpha)} \int_0^{-\infty} u^{\alpha-1} e^{-u} du \quad (\text{notice } \int_0^{-\infty} u^{\alpha-1} e^{-u} du \text{ diverges}) \\ &= \infty. \end{aligned}$$

Hence, we have reached the moment generating function of Gamma Distribution for all cases.

(b) Since we only care about $t = 0$ and $\lambda > 0$, we can consider $M_X(t) = \left(\frac{\lambda}{\lambda-t}\right)^\alpha$ when $t < \lambda$ only.

$$\begin{aligned}
M'_X(t) &= \alpha \left(\frac{\lambda}{\lambda-t}\right)^{\alpha-1} \left(\frac{-\lambda}{(\lambda-t)^2}\right) (-1) \\
&= \alpha \left(\frac{\lambda}{\lambda-t}\right)^\alpha \left(\frac{1}{\lambda-t}\right) \\
&= \alpha \cdot \lambda^\alpha \cdot (\lambda-t)^{-\alpha-1} \\
\mu &= M'_X(t=0) = \frac{\alpha}{\lambda}. \\
M''_X(t) &= \alpha \cdot \lambda^\alpha \cdot (-\alpha-1)(\lambda-t)^{-\alpha-2} \cdot (-1) \\
&= \alpha \cdot \lambda^\alpha \cdot (\alpha+1)(\lambda-t)^{-\alpha-2} \\
\sigma^2 &= M''_X(t=0) = \frac{\alpha(\alpha+1)}{\lambda^2}.
\end{aligned}$$

Problem 2.

$$\vec{\mu} = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}; \quad \Sigma = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}; \quad \vec{A}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}; \quad \vec{A}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Then

$$\begin{aligned}
Y_1 &\sim N(\vec{A}_1 \cdot \vec{\mu}, \vec{A}_1^\top \Sigma \vec{A}_1) \\
&= N(\mu_1 + 2\mu_2, \sigma_1^2 + 4\rho\sigma_1\sigma_2 + 4\sigma_2^2), \\
Y_2 &\sim N(\vec{A}_2 \cdot \vec{\mu}, \vec{A}_2^\top \Sigma \vec{A}_2) \\
&= N(\mu_1 - \mu_2, \sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2).
\end{aligned}$$

The *pdf* of a normal random variable $N(m, s)$ for all $x \in \mathbb{R}$ is simply

$$f(x) = \frac{1}{\sqrt{2\pi s}} e^{-\frac{(x-m)^2}{2s}}.$$

Since we have already found the mean and variance of Y_1 and Y_2 , and we have determined they are normal random variable, what's left is only plugging into the formula, which we will omit here for easier readability as a whole.

PK Problem 2.1.8 Let F be first ball, S be second ball.

$$\begin{aligned}
P(F = R|S = R) &= \frac{P(S = R|F = R)P(F = R)}{P(S = R|F = R)P(F = R) + P(S = R|F = G)P(F = G)} \\
&= \frac{\frac{2}{3} \cdot \frac{1}{2}}{\frac{2}{3} \cdot \frac{1}{2} + \frac{1}{3} \cdot \frac{1}{2}} \\
&= \frac{2}{3}.
\end{aligned}$$

PK Problem 2.1.9

$$\begin{aligned}
p_N(n) &= \begin{cases} \frac{e^{-1}}{n!}, & n \in \mathbb{Z}^{\geq} \\ 0, & \text{otherwise,} \end{cases} \\
p_{X|N=n}(x) &= \begin{cases} \frac{1}{n+2}, & x \in [0, n+1] \\ 0, & \text{otherwise.} \end{cases}
\end{aligned}$$

Then,

$$\begin{aligned}
p_X(x) &= \sum_{n=x-1}^{\infty} p_{X|N=n}(x) \cdot p_N(n) \\
&= \sum_{n=x-1}^{\infty} \frac{1}{n+2} \cdot \frac{e^{-1}}{n!} \\
&= \frac{1}{e} \sum_{n=x-1}^{\infty} \frac{n+1}{(n+2)!} \\
&= \frac{1}{e} \sum_{n=x}^{\infty} \frac{n}{(n+1)!} \\
&= \frac{1}{e} \sum_{n=1}^{\infty} \frac{n}{(n+1)!} - \sum_{n=1}^{x-1} \frac{n}{(n+1)!} \\
&= \frac{1}{e} \left(1 - \frac{x! - 1}{x!} \right) \\
&= \frac{1}{e} + \frac{1 - x!}{e \cdot x!}.
\end{aligned}$$

PK Exercise 2.3.1 Random Sum approach: Let $\xi_i \sim \text{Bernoulli}(\frac{1}{2})$, $Z = \xi_1 + \xi_2 + \dots + \xi_n$, and $N \sim \text{Uniform}(1, 6)$. Then

$$\begin{aligned}
E[Z] &= E[\xi_i]E[N] \\
&= \frac{1}{2} \cdot \frac{7}{2} \\
&= \frac{7}{4}, \\
\text{Var}(Z) &= \text{Var}(\xi_i)E[N] + \text{Var}(N)(E[\xi_i])^2 \\
&= \frac{1}{4} \cdot \frac{7}{2} + \frac{6^2 - 1}{12} \cdot \left(\frac{1}{2}\right)^2 \\
&= \frac{77}{48}.
\end{aligned}$$

pmf approach: Let $Z \sim \text{Binomial}(n, \frac{1}{2})$ and $N \sim \text{Uniform}(1, 6)$. Then

$$\begin{aligned}
p_Z(z) &= \sum_{n=z}^6 p_{Z|N=n}(z) p_N(n) \\
&= \sum_{n=z}^6 \binom{n}{z} \left(\frac{1}{2}\right)^n \cdot \frac{1}{6}, \\
E[Z] &= \sum_{z=0}^6 z \cdot p_Z(z) \\
&= \sum_{z=0}^6 z \cdot \left[\sum_{n=z}^6 \binom{n}{z} \left(\frac{1}{2}\right)^n \cdot \frac{1}{6} \right] \\
&= \frac{7}{4}, \\
\text{Var}(Z) &= E[Z^2] - (E[Z])^2 = \frac{14}{3} - \left(\frac{7}{4}\right)^2 \\
&= \frac{77}{48}.
\end{aligned}$$

```
In [19]: expectation = 0
for z in range(7):
    sum = 0
    for n in range(z,7):
        sum += math.comb(n, z)*(0.5**n)*1/6
    expectation += z*sum
print(expectation)
1.7500000000000002
```

$E[Z]$

```
In [18]: expectation = 0
for z in range(7):
    sum = 0
    for n in range(z,7):
        sum += math.comb(n, z)*(0.5**n)*1/6
    expectation += z**2*sum
print(expectation)
4.666666666666667
```

$E[Z^2]$

PK Exercise 2.3.5 Let N be a *Poisson* random variable representing number of accidents in a week, and ξ_i be a random variable representing the number of individuals injured in each accident (assume all ξ_i are independent and identically distributed). Then the number of individuals, Z , injured in a week can be written as $Z = \xi_1 + \xi_2 + \cdots + \xi_n$.

Hence,

$$\begin{aligned}
 E[Z] &= E[\xi_i]E[N] \\
 &= 3 \times 2 \\
 &= 6, \\
 Var(Z) &= Var(\xi_i)E[N] + Var(N)(E[\xi_i])^2 \\
 &= 4 \times 2 + 2 \times 9 \\
 &= 26.
 \end{aligned}$$