Math 180A HW5

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Problem 1. (b)

Problem 2. (a)

Problem 3. 4.

Problem 4.

$$\begin{align} E[X] &= \int_0^1 6x^2 - 6x^3 dx \tag{1}\\ &= \left[2x^3 - \frac{3}{2}x^4\right]_0^1 \tag{2}\\ &= \frac{1}{2} \tag{3}\end{align}$$

Problem 5.

(a)

$$\begin{align} E[X] &= \int_{-2}^0 \frac{x}{6} dx + \int_0^3 \frac{2x}{9} dx \tag{4}\\ &= \left[\frac{x^2}{12}\right]_{-2}^0 + \left[\frac{x^2}{9}\right]_0^3 \tag{5}\\ &= \frac{-1}{3} + 1 \tag{6}\\ &= \frac{2}{3} \tag{7}\end{align}$$

(b)

$$\begin{align} E[X^2] &= \int_{-2}^0 \frac{x^2}{6} dx + \int_0^3 \frac{2x^2}{9} dx \tag{8}\\ &= \left[\frac{x^3}{18}\right]_{-2}^0 + \left[\frac{2x^3}{27}\right]_0^3 \tag{9}\\ &= \frac{4}{9} + 2 \tag{10}\\ &= \frac{22}{9} \tag{11}\\ Var(X) &= E[X^2] - (E[X])^2 \tag{12}\\ &= \frac{22}{9} - \left(\frac{2}{3}\right)^2 \tag{13}\\ &= 2 \tag{14}\end{align}$$

(c)

$$
\begin{aligned}
E[(X-1)^2] &= E[X^2-2X+1] && (15)\\
&= E[X^2]-2E[X]+1 && (16)\\
&= \frac{22}{9}-\frac{4}{3}+1 && (17)\\
&= \frac{19}{9} && (18)
\end{aligned}
$$

Problem 6.

(a) Consider $x \in [0,\infty)$,

$$
\begin{aligned}
f_X(x) &= \frac{d}{dx}\frac{x}{1+x} && (19)\\
&= \frac{(1+x)-x}{(1+x)^2} && (20)\\
&= \frac{1}{(1+x)^2}. && (21)
\end{aligned}
$$

Hence,

$$
f_X(x) = \begin{cases} \frac{1}{(1+x)^2}, & x \geq 0 \\ 0, & x < 0. \end{cases} \qquad (22)
$$

(b)

$$
\begin{aligned}
P(2<X<3) &= P(X<3)-P(X<2) && (23)\\
&= F(3)-F(2) && (24)\\
&= \frac{3}{4}-\frac{2}{3} && (25)\\
&= \frac{1}{12} && (26)
\end{aligned}
$$

(c)

$$
\begin{aligned}
E\left[(1+X)^2e^{-2X}\right] &= E\left[X^2e^{-2X}+2Xe^{-2X}+e^{-2X}\right] && (27)\\
&= E\left[X^2e^{-2X}\right]+E\left[2Xe^{-2X}\right]+E\left[e^{-2X}\right] && (28)\\
&= \int_0^\infty \frac{x^2e^{-2x}}{(1+x)^2}+\int_0^\infty \frac{2xe^{-2x}}{(1+x)^2}+\int_0^\infty \frac{e^{-2x}}{(1+x)^2}dx && (29)\\
&= \int_0^\infty \frac{e^{-2x}(x^2+2x+1)}{(1+x)^2}dx && (30)\\
&= \int_0^\infty e^{-2x}dx && (31)\\
&= \frac{1}{-2}\left[e^{-2x}\right]_0^\infty && (32)\\
&= \frac{1}{2} && (33)
\end{aligned}
$$

Problem 7. 150 meteros per hour is equivalent to 2.5 meteros per minute. We can divide the first minute into n inifinitely small intervals, and assume each interval is independent. Thus the probability of a metero appearing in one interval is $\frac{2.5}{n}$. Then we can model the situation with $Bionomial(n,p)$, and approximate the wanted probability with $Poisson(2.5)$. Hence,

$$\begin{aligned}
P(X \geq 2) &= 1 - P(X=0) - P(X=1) - P(X=2) && (34)\\
&= 1 - e^{-2.5} - 2.5e^{-2.5} - \frac{2.5^2}{2}e^{-2.5} && (35)\\
&\approx 0.4562. && (36)
\end{aligned}$$

Problem 8.

(a) $f(x)$ is apparently non-negative because $\forall x \in \mathbb{R}, x^2 \geq 0 \Rightarrow \pi(1+x^2) \geq 0 \Rightarrow \frac{1}{\pi(1+x^2)} \geq 0$.

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{1}{\pi(1+x^2)}dx &= \frac{1}{\pi}\left[tan^{-1}(y)\right]_{-\infty}^{\infty} && (37)\\
&= \frac{1}{\pi}\left(\frac{\pi}{2} - \frac{-\pi}{2}\right) && (38)\\
&= 1 && (39)
\end{aligned}$$

(b)

$$\begin{aligned}
E[|X|] &= \int_{-\infty}^{0} \frac{-x}{\pi(1+x^2)}dx + \int_{0}^{\infty} \frac{x}{\pi(1+x^2)}dx && (40)\\
&= \frac{-1}{\pi}\int_{\infty}^{1} \frac{1}{2u}du + \frac{1}{\pi}\int_{1}^{\infty} \frac{1}{2u}du \qquad \left(\text{let } u = 1+x^2\right) && (41)\\
&= \frac{1}{2\pi}\int_{1}^{\infty} \frac{1}{u}du + \frac{1}{2\pi}\int_{1}^{\infty} \frac{1}{u}du && (42)\\
&= \lim_{t\to\infty} \frac{1}{\pi}\left[ln(u)\right]_1^t && (43)\\
&= \infty && (44)
\end{aligned}$$

(c) Looks like a 0.

(d) Maybe start by finding the CDF, which should be $\frac{1}{\pi}\left[tan^{-1}(y)\right]$: $\frac{1}{\pi}$ represents the weight of the angle, and $tan^{-1}(y)$ converts the slope to the angle. Then differentiate it to get $\frac{1}{\pi(1+x^2)}$.

Problem 9. I think the number of commits to a GitHub Repo would be well-modeled by Poisson distribution. data-engineering-zoomcap has an average of 4.87 commits per day. Let X be the number of commits per day, it can be modeled by $Poisson(4.87)$. $P(X=2) = \frac{4.87^2}{2}e^{-4.87}$, multiplied by the total of 210 days, we expect that there are 19 days that there are exactly 2 commits. In fact, there are a total of 17 days with total of 2 commits, which is pretty close in my opinion.
(https://github.com/DataTalksClub/data-engineering-zoomcamp/graphs/commit-activity)