Math 180B HW7

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PK Exercise 5.1.4 Customers arrive at a service facility according to a Poisson process of rate λ customer/hour. Let $X(t)$ be the number of customers that have arrived up to time t.

(a) What is $P\{X(t) = k\}$ for $k = 0, 1, \ldots$?

Solution. $P\{X(t) = k\}$ is simply $Poisson(\lambda t)$. Hence, $P\{X(t) = k\} = \frac{(\lambda t)^k}{k!} e^{-\lambda t}$. □

(b) Consider fixed times $0 < s < t$. Determine the conditional probability $P\{X(t) = n + k | X(s) = n\}$ and the expected value $E[X(t)X(s)]$.

Solution.

$$\begin{aligned}
P\{X(t) = n + k | X(s) = n\} &= P\{X(t-s) = k\} \\
&= \frac{(\lambda(t-s))^k}{k!} e^{-\lambda(t-s)}. \\
E[X(t)X(s)] &= E[(X(t) - X(s) + X(s)) \cdot X(s)] \\
&= E[(X(t) - X(s)) \cdot X(s)] + E[X(s)^2] \\
&= E[X(t) - X(s)] \cdot E[X(s)] + E[X(s)^2] \\
&= E[X(t) - X(s)] \cdot E[X(s)] + \left[E[X(s)]^2 + Var(X(s))\right] \\
&= \lambda(t-s) \cdot \lambda s + \lambda^2 s^2 + \lambda s \\
&= \lambda^2 ts + \lambda s.
\end{aligned}$$

□

PK Exercise 5.1.5 Suppose that a random variable X is distributed according to a Poisson distribution with parameter λ. The parameter λ is itself a random variable, exponentially distributed with density $f(x) = \theta e^{-\theta x}$ for $x \geq \theta$. Find the probability mass function for X.

Solution.

$$\begin{aligned}
P(X = k) &= \int_0^\infty P(X = k|\lambda = x) f(x) dx \\
&= \int_0^\infty \frac{x^k}{k!} e^{-x} \cdot \theta e^{-\theta x} dx \\
&= \frac{\theta}{k!} \int_0^\infty x^k e^{-(\theta+1)x} dx \\
&= \frac{\theta}{k!} \int_0^\infty \left(\frac{t}{\theta+1}\right)^k e^{-t} \cdot \frac{1}{\theta+1} dt \qquad (\textit{let } (\theta+1)x = t) \\
&= \frac{\theta}{k!} \cdot \frac{1}{(\theta+1)^{k+1}} \int_0^\infty t^k e^{-t} dt \\
&= \frac{\theta}{k!} \cdot \frac{1}{(\theta+1)^{k+1}} \cdot \Gamma(k+1) \\
&= \frac{\theta}{k!} \cdot \frac{1}{(\theta+1)^{k+1}} \cdot k! \\
&= \frac{\theta}{(\theta+1)^{k+1}}.
\end{aligned}$$

□

PK Exercise 5.1.7 Suppose that customers arrive at a facility according to a Poisson process having rate $\lambda = 2$. Let $X(t)$ be the number of customers that have arrived up to time t. Determine the following probabilities and conditional probabilities:

(a) $P\{X(1) = 2\}$.

Solution. $P\{X(1) = 2\} = \frac{2^2}{2!} e^{-2} = 2e^{-2}$. □

(b) $P\{X(1) = 2 \textit{ and } X(3) = 6\}$.

Solution.

$$\begin{aligned}
P\{X(1) = 2 \textit{ and } X(3) = 6\} &= P\{X(1) = 2\} \cdot P\{X(2) = 4\} \\
&= \frac{2^2}{2!} e^{-2} \cdot \frac{(2 \times 2)^4}{4!} e^{-(2\times 2)} \\
&= \frac{2^{10}}{2!4!} e^{-6} \\
&= \frac{2^6}{3} e^{-6} \\
&= \frac{64}{3} e^{-6}.
\end{aligned}$$

□

(c) $P\{X(1)=2|X(3)=6\}$.

Solution.

$$\begin{aligned}
P\{X(1)=2|X(3)=6\} &= \frac{P\{X(1)=2 \text{ and } X(3)=6\}}{P\{X(3)=6\}} \\
&= \frac{\frac{2^6}{3}e^{-6}}{\frac{(2\times 3)^6}{6!}e^{-6}} \\
&= \frac{6!}{3^7} \\
&= \frac{80}{243}.
\end{aligned}$$

□

(d) $P\{X(3)=6|X(1)=2\}$.

Solution. $P\{X(3)=6|X(1)=2\} = P\{X(2)=4\} = \frac{(2\times 2)^4}{4!}e^{-(2\times 2)} = \frac{2^5}{3}e^{-4} = \frac{32}{3}e^{-4}$. □

PK Problem 5.1.5

Proposition 1. *For each value of $h > 0$, let $X(h)$ have a Poisson distribution with parameter λh. Let $p_k(h) = P\{X(h) = k\}$ for $k = 0, 1, \dots,$ then*

$$\begin{aligned}
p_0(h) &= 1 - \lambda h + o(h) \\
p_1(h) &= \lambda h + o(h) \\
p_2(h) &= o(h).
\end{aligned}$$

Proof.

$$\begin{aligned}
p_0(h) &= \frac{(\lambda h)^0 e^{-\lambda h}}{0!} \\
&= e^{-\lambda h} \\
&= 1 - \lambda h + \frac{(\lambda h)^2}{2!} - \frac{(\lambda h)^3}{3!} + \dots \\
&= 1 - \lambda h + o(h). \\
p_1(h) &= \frac{(\lambda h)^1 e^{-\lambda h}}{1!} \\
&= \lambda h e^{-\lambda h} \\
&= \lambda h(1 - \lambda h + o(h)) \\
&= \lambda h + o(h). \\
p_2(h) &= \frac{(\lambda h)^2 e^{-\lambda h}}{2!} \\
&= \frac{(\lambda h)^2}{2} e^{-\lambda h} \\
&= \frac{(\lambda h)^2}{2}(1 - \lambda h + o(h)) \\
&= \frac{(\lambda h)^2}{2} - \frac{(\lambda h)^3}{2} + o(h) \\
&= o(h).
\end{aligned}$$

□

PK Problem 5.1.7 Shocks occur to a system according to a Poisson process of rate λ. Suppose that the system survives each shock with probability α, independently of other shocks, so that its probability of surviving k shocks is α^k. What is the probability that the system is surviving at time t?

Solution.

$$\begin{aligned}
P(\textit{Surviving at time } t) &= \sum_{k=0}^{\infty} \alpha^k \cdot \frac{(\lambda t)^k e^{-\lambda t}}{k!} \\
&= e^{-\lambda t} \sum_{k=0}^{\infty} \frac{(\alpha \lambda t)^k}{k!} \\
&= e^{-\lambda t} e^{\alpha \lambda t} \\
&= e^{\lambda t(\alpha - 1)}.
\end{aligned}$$

□

PK Exercise 5.2.1 Determine numerical values to three decimal places for $P\{X = k\}, k = 0, 1, 2$, when

(a) X has a binomial distribution with parameters $n = 20$ and $p = 0.06$.

Solution.

$$\begin{aligned}
p_0 &= \binom{20}{0} 0.94^{20} \approx 0.290 \\
p_1 &= \binom{20}{1} 0.06^1 0.94^{19} \approx 0.370 \\
p_2 &= \binom{20}{2} 0.06^2 0.94^{18} \approx 0.225.
\end{aligned}$$

□

(b) X has a binomial distribution with parameters $n = 40$ and $p = 0.03$.

Solution.

$$\begin{aligned}
p_0 &= \binom{40}{0} 0.97^{40} \approx 0.296 \\
p_1 &= \binom{40}{1} 0.03^1 0.97^{39} \approx 0.366 \\
p_2 &= \binom{40}{2} 0.03^2 0.97^{38} \approx 0.221.
\end{aligned}$$

□

(c) X has a Poisson distribution with parameter $\lambda = 1.2$.

$$\begin{aligned}
p_0 &= \frac{1.2^0 e^{-1.2}}{0!} \approx 0.301 \\
p_1 &= \frac{1.2^1 e^{-1.2}}{1!} \approx 0.361 \\
p_2 &= \frac{1.2^2 e^{-1.2}}{2!} \approx 0.217.
\end{aligned}$$