

# Math 180A HW1

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## Problem 3.

- (a)  $\Omega = \{(1, 2, 3, 4, 5), (1, 2, 3, 4, 6), (1, 2, 3, 4, 7), \dots\}$  with no repetition of numbers in each element and orders do not matter, which means  $(1, 2, 3, 4, 5)$  is the same as  $(5, 4, 3, 2, 1)$ . There are a total of  $\binom{39}{5} = 575757$  elements in  $\Omega$ .

Since each number are picked randomly and uniformly, the distribution of each element is uniform with  $P = \frac{1}{|\Omega|} = \frac{1}{\binom{39}{5}}$ .

(b)  $P = \frac{\binom{5}{3} \times \binom{39-5}{2}}{\binom{39}{5}} = \frac{10 \times 561}{575757} = \frac{5610}{575757}$

We first choose 3 numbers out of the 5 numbers we have in our ticket. Then we choose 2 numbers out of the  $(39 - 5) = 34$  numbers that are not on our ticket.

## Problem 4.

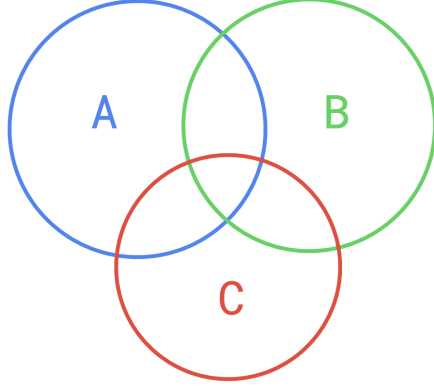
- (a)  $\Omega = \{1C, 2C, 3C, \dots, 1D, 2D, \dots, 1H, 2H, \dots, 1S, 2S, \dots, 13S\}$  with C, D, H, S representing club, diamond, heart, and spade respectively. Therefore, there are a total of 52 elements in  $\Omega$ .

Since one card is picked randomly and uniformly, the probability measure is uniform for each event with  $P = \frac{1}{|\Omega|} = \frac{1}{52}$ .

- (b) We can define event  $A$  as  $(1S \cup 2S \cup 3S)$ .

Since  $1S, 2S, 3S$  are disjoint,  $P(A) = P(1S \cup 2S \cup 3S) = P(1S) + P(2S) + P(3S) = \frac{1}{52} + \frac{1}{52} + \frac{1}{52} = \frac{3}{52}$ .

**Problem 5.**



(a)

$$(b) \quad P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$

Let  $p_1 = P(A \cap B^c \cap C^c)$ ,  $p_2 = P(A^c \cap B \cap C^c)$ ,  $p_3 = P(A^c \cap B^c \cap C)$ ,  $p_{12} = P(A \cap B \cap C^c)$ ,  $p_{13} = P(A \cap B^c \cap C)$ ,  $p_{23} = P(A^c \cap B \cap C)$ ,  $p_{123} = P(A \cap B \cap C)$ . Note that all sets are pairwise disjoint.

$$\begin{aligned} & P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C) \\ &= (p_1 + p_{12} + p_{13} + p_{123}) + (p_2 + p_{12} + p_{23} + p_{123}) + (p_3 + p_{13} + p_{23} + p_{123}) \\ &\quad - (p_{12} + p_{123}) - (p_{13} + p_{123}) - (p_{23} + p_{123}) + p_{123} \\ &= p_1 + p_2 + p_3 + p_{12} + p_{13} + p_{23} + p_{123} \\ &= P(A_1 \cup A_2 \cup A_3) \end{aligned}$$

**Problem 6.**

$$1 \geq P(A \cup B) \tag{1}$$

$$1 \geq P(A) + P(B) - P(AB) \tag{2}$$

$$1 \geq 0.4 + 0.7 - P(AB) \tag{3}$$

$$-0.1 \geq -P(AB) \tag{4}$$

$$0.1 \leq P(AB) \tag{5}$$

We have proved that  $0.1 \leq P(AB)$ .

Since  $(A \cap B) \subseteq A$  and  $(A \cap B) \subseteq B$ ,  $P(A \cap B) \leq P(A)$  and  $P(A \cap B) \leq P(B)$ . Therefore,  $P(A \cap B) \leq 0.4$ .

In conclusion,  $0.1 \leq P(AB) \leq 0.4$ .

**Problem 7.** Let  $G = \{\text{exactly two balls are green}\}$ ,  $R = \{\text{exactly two balls are red}\}$ ,  $Y = \{\text{exactly two balls are yellow}\}$ ,  $W = \{\text{exactly two balls are white}\}$ .

$$P(G \cup R \cup Y \cup W) = P(G) + P(R) + P(Y) + P(W) \quad (6)$$

$$- P(G \cup R) - P(G \cup Y) - \dots \quad (7)$$

$$+ P(G \cup R \cup Y) + P(G \cup R \cup W) + \dots \quad (8)$$

$$- P(G \cup R \cup Y \cup W) \quad (9)$$

Now let's calculate the individual terms:

$$P(G) = P(R) = \dots = \frac{\binom{4}{2} \times 3 \times 3}{4^4} = \frac{54}{256} = \frac{27}{128} \quad (10)$$

$$P(G \cup R) = P(G \cup Y) = \dots = \frac{\binom{4}{2}}{4^4} = \frac{3}{128} \quad (11)$$

Note that it's impossible for more than two of the events happen in the same time, therefore:

$$P(G \cup R \cup Y) = P(G \cup R \cup W) = \dots = P(G \cup R \cup Y \cup W) = 0 \quad (12)$$

Therefore,

$$P(G \cup R \cup Y \cup W) = 4 \times \frac{27}{128} - \binom{4}{2} \times \frac{3}{128} \quad (13)$$

$$= \frac{45}{64} \quad (14)$$

**Problem 8.**

By inclusion-exclusion,

$$P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2). \quad (15)$$

Since  $P(A_1 \cap A_2) \geq 0$ ,

$$P(A_1 \cup A_2) \leq P(A_1) + P(A_2). \quad (16)$$

Therefore,  $P(A_1 \cup \dots \cup A_k) \leq P(A_1) + \dots + P(A_k)$  is true for  $k = 2$ .

Suppose  $P(A_1 \cup \dots \cup A_k) \leq P(A_1) + \dots + P(A_k)$  is true for some natural numbers  $k \geq 2$ . We get:

$$P(A_1 \cup \dots \cup A_k \cup A_{k+1}) = P(A_1 \cup \dots \cup A_k) + P(A_{k+1}) - P((A_1 \cup \dots \cup A_k) \cap A_{k+1}) \quad (17)$$

Since  $P((A_1 \cup \dots \cup A_k) \cap A_{k+1}) \geq 0$ ,

$$P(A_1 \cup \dots \cup A_k \cup A_{k+1}) \leq P(A_1 \cup \dots \cup A_k) + P(A_{k+1}) \quad (18)$$

Since  $P(A_1) + \dots + P(A_k) \geq P(A_1 \cup \dots \cup A_k \cup A_{k+1})$ , we can replace  $P(A_1 \cup \dots \cup A_k \cup A_{k+1})$  with  $P(A_1) + \dots + P(A_k)$ , and we can get:

$$P(A_1 \cup \dots \cup A_k \cup A_{k+1}) \leq P(A_1) + \dots + P(A_k) + P(A_{k+1}). \quad (19)$$

Thus,  $P(A_1 \cup \dots \cup A_k \cup A_{k+1}) \leq P(A_1) + \dots + P(A_k) + P(A_{k+1})$  if the Induction Hypothesis is true.

Therefore, by Mathematical Induction,  $P(A_1 \cup \dots \cup A_n) \leq P(A_1) + \dots + P(A_n) = \sum_{k=1}^n P(A_k)$ .

**Problem 9.**