Math 109 HW5

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Problem 1 Let A, B, C be the set of students that like Reasoning, Algebra, and Calculus respectively. The question is asking for $|(A \cup B \cup C)^c|$.

$$\begin{aligned}
|(A \cup B \cup C)^c| &= 182 - |A \cup B \cup C| \\
&= 182 - [|A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|] \\
&= 182 - [129 + 129 + 129 - 85 - 89 - 86 + 53] \\
&= 2.
\end{aligned}$$

Problem 2. Let $X = (A \cap B); Y = (A \cap C); Z = (B \cap C); W = (A \cap B \cap C)$, and $x = |X|; y = |Y|; z = |Z|; w = |W|$. The question is asking for w. We know

$$\begin{aligned}
A &= (A \cap B^c \cap C^c) \cup (X \cup Y) && (1) \\
B &= (A^c \cap B \cap C^c) \cup (X \cup Z) && (2) \\
C &= (A^c \cap B^c \cap C) \cup (Y \cup Z). && (3)
\end{aligned}$$

and notice that the sets enclosed with parentheses are disjoint. Now, let us apply the inclusion-exclusion principle,

$$\begin{aligned}
170 - |A \cup B \cup C| = 2 &= 170 - [|A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|] \\
2 &= 170 - [124 + 124 + 124 - x - y - z + w] \\
204 &= x + y + z + -w, && (4)
\end{aligned}$$

which is our first equation. Then we can apply the addition principle and inclusion-exclusion principle on (1), (2), and (3), and we get

$$\begin{aligned}
|A| = 124 &= 10 + (x + y - w) \\
114 &= x + y - w && (5) \\
|B| = 124 &= x + z - w && (6) \\
|C| = 124 &= 4 + y + z - w \\
120 &= y + z - w. && (7)
\end{aligned}$$

Solving (4), (5), (6), and (7), we get $w = 50, x = 84, y = 80, z = 90$.

Problem 3.

(i) Let A, B, C be the set of people that speak English, Spanish, and Swahili respectively. Let

$$\begin{aligned}
a &= |(A \cap B^c \cap C^c)| \\
b &= |(A^c \cap B \cap C^c)| \\
c &= |(A^c \cap B^c \cap C)| \\
x &= |(A \cap B)| \\
y &= |(A \cap C)| \\
z &= |(B \cap C)| \\
w &= |(A \cap B \cap C)|.
\end{aligned}$$

Our goal is to maximize $a+b+c$. We know

$$\begin{aligned}|A\cup B\cup C| = 100 &= [|A|+|B|+|C|-|A\cap B|-|A\cap C|-|B\cap C|+|A\cap B\cap C|]\\ 100 &= 75+60+45-x-y-z+w\\ 80 &= x+y+z-w. \qquad (8)\end{aligned}$$

Also,

$$\begin{aligned}|A| &= 75 = a+x+y-w\\ |B| &= 60 = b+x+z-w\\ |C| &= 45 = c+y+z-w.\end{aligned}$$

Then,

$$\begin{aligned}|A|+|B|+|C| = 75+60+45 &= a+b+c+2x+2y+2z-3w\\ 180 &= a+b+c+2x+2y+2z-3w. \qquad (9)\end{aligned}$$

(9) $-\, 2\times$ (8):

$$a+b+c = 20+w. \qquad (10)$$

Hence, we need to maximize w. Note that $x,y,z \geq w$. Thus, by (8), $w \leq 40$, and $a+b+c \leq 60$. Taking $w,x,y,z=40$, $a=35, b=20, c=5$.

(ii) From (10), $a = 20+w-b-c$. So we need to maximize $w-b-c$. Take $w=35, b=10, c=0$, $a=40$.

(iii) Already proved in (i).