

Math 109 HW5

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Problem 1 Let A, B, C be the set of students that like Reasoning, Algebra, and Calculus respectively. The question is asking for $|(A \cup B \cup C)^c|$.

$$\begin{aligned} |(A \cup B \cup C)^c| &= 182 - |A \cup B \cup C| \\ &= 182 - [|A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|] \\ &= 182 - [129 + 129 + 129 - 85 - 89 - 86 + 54] \\ &= 1. \end{aligned}$$

Problem 2. Let $X = (A \cap B); Y = (A \cap C); Z = (B \cap C); W = (A \cap B \cap C)$, and $x = |X|; y = |Y|; z = |Z|; w = |W|$. The question is asking for w . We know

$$A = (A \cap B^c \cap C^c) \cup (X \cup Y) \quad (1)$$

$$B = (A^c \cap B \cap C^c) \cup (X \cup Z) \quad (2)$$

$$C = (A^c \cap B^c \cap C) \cup (Y \cup Z). \quad (3)$$

and notice that the sets enclosed with parentheses are disjoint. Now, let us apply the inclusion-exclusion principle,

$$\begin{aligned} 170 - |A \cup B \cup C| = 2 &= 170 - [|A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|] \\ 2 &= 170 - [124 + 124 + 124 - x - y - z + w] \\ 204 &= x + y + z + -w, \end{aligned} \quad (4)$$

which is our first equation. Then we can apply the addition principle and inclusion-exclusion principle on (1), (2), and (3), and we get

$$\begin{aligned} |A| = 124 &= 10 + (x + y - w) \\ 114 &= x + y - w \end{aligned} \quad (5)$$

$$|B| = 124 = x + z - w \quad (6)$$

$$\begin{aligned} |C| = 124 &= 4 + y + z - w \\ 120 &= y + z - w. \end{aligned} \quad (7)$$

Solving (4), (5), (6), and (7), we get $w = 50, x = 84, y = 80, z = 90$.

Problem 3.

(i) Let A, B, C be the set of people that speak English, Spanish, and Swahili respectively. Let

$$\begin{aligned} a &= |(A \cap B^c \cap C^c)| \\ b &= |(A^c \cap B \cap C^c)| \\ c &= |(A^c \cap B^c \cap C)| \\ x &= |(A \cap B)| \\ y &= |(A \cap C)| \\ z &= |(B \cap C)| \\ w &= |(A \cap B \cap C)|. \end{aligned}$$

Our goal is to maximize $a + b + c$. We know

$$\begin{aligned} |A \cup B \cup C| &= 100 = [|A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|] \\ 100 &= 75 + 60 + 45 - x - y - z + w \\ 80 &= x + y + z - w. \end{aligned} \tag{8}$$

Also,

$$|A| = 75 = a + x + y - w \tag{9}$$

$$|B| = 60 = b + x + z - w \tag{10}$$

$$|C| = 45 = c + y + z - w. \tag{11}$$

Then, from (9) to (11),

$$\begin{aligned} |A| + |B| + |C| &= 75 + 60 + 45 = a + b + c + 2x + 2y + 2z - 3w \\ 180 &= a + b + c + 2x + 2y + 2z - 3w. \end{aligned} \tag{12}$$

(12) $- 2 \times$ (8):

$$a + b + c = 20 + w. \tag{13}$$

Hence, we need to maximize w . Note that $x, y, z \geq w$. Thus, by (8), $w \leq 40$, and $a + b + c \leq 60$. Taking $w, x, y, z = 40$ and solving (9) to (11), $a = 35, b = 20, c = 5$.

(ii) (9) $-$ (8):

$$\begin{aligned} -5 &= a - z \\ a &= z - 5. \end{aligned}$$

Hence, maximizing a is equivalent to maximizing z . $z = |B \cap C| \leq |C| = 45 \Rightarrow a \leq 40$. Thus, we take $a = 40, z = 45$. Notice that $c = |(A^c \cap B^c \cap C)|$ and $z = |B \cap C|$, so if $z = 45 = |C|$, then $c = 0$. Then, solving (9) to (11), we can conclude with the following equations:

$$\begin{aligned} a &= 40 \\ z &= 45 \\ c &= 0 \\ x &= 35 \\ b &= w - 20 = y - 20 \\ 0 &\leq b \leq 15. \quad (\because w = y \leq |A| - a = 35) \end{aligned}$$

Maximum number of people who speak only English = 40. In this case, the number of people who speak only Spanish = 0, and the number of people who speak only Swahili can range from 0 to 15.

(iii) Already proved in (i).