Math 104 HW9

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Exercise 23.1 (a, b, c)

Find the radius of convergence and determine the exact interval of convergence for

(a) $\sum n^2 x^n$

(b) $\sum \left(\frac{x}{n}\right)^n$

(c) $\sum \left(\frac{2^n}{n^2}\right) x^n$

Solution.

(a)

$$\lim \left|\frac{(n+1)^2}{n^2}\right| = \lim \left|\frac{n^2+2n+1}{n^2}\right| = \lim \left|1+\frac{2}{n}+\frac{1}{n^2}\right| = 1$$
$$\implies \beta = \limsup |x^2|^{\frac{1}{n}} = 1$$
$$\implies R = \frac{1}{\beta} = 1.$$

For both $x = 1$ and $x = -1$, the series diverges because for $x = 1$, $\lim n^2 x^n = \infty$ and for $x = -1$, the limit does not exist. Hence, the convergence interval is $(-1, 1)$.

(b)

$$\sum \left(\frac{x}{n}\right)^n = \sum \frac{1}{n^n} x^n$$
$$\implies \beta = \limsup \left|\frac{1}{n^n}\right|^{\frac{1}{n}} = \limsup \frac{1}{n} = 0$$
$$\implies R = \infty.$$

Hence, the series converges for all $x \in \mathbb{R}$.

(c)

$$\lim \left|\frac{2^{n+1}}{(n+1)^2} \cdot \frac{n^2}{2^n}\right| = \lim \left|\frac{2}{\left(1+\frac{1}{n}\right)^2}\right| = 2$$
$$\implies \beta = \limsup \left|\frac{2^n}{n^2}\right|^{\frac{1}{n}} = \lim \left|\frac{2^{n+1}}{(n+1)^2} \cdot \frac{n^2}{2^n}\right| = 2$$
$$\implies R = \frac{1}{\beta} = \frac{1}{2}.$$

For $x = \frac{1}{2}$,

$$\sum \left(\frac{2^n}{n^2}\right) \frac{1}{2^n} = \sum \frac{1}{n^2},$$

which we know converges. For $x = -\frac{1}{2}$,

$$\sum \left(\frac{2^n}{n^2}\right) \left(-\frac{1}{2}\right)^n = \sum \frac{(-1)^n}{n^2},$$

which we know converges by the alternating series test. Hence, the convergence interval is $\left[-\frac{1}{2}, \frac{1}{2}\right]$.

$\square$

Exercise 23.2 (b)

Find the radius of convergence and determine the exact interval of convergence for $\sum \frac{1}{n^{\sqrt{n}}} x^n$.

Solution.

$$\beta = \limsup \left|\frac{1}{n^{\sqrt{n}}}\right|^{\frac{1}{n}} = \limsup \frac{1}{n^{\frac{1}{\sqrt{n}}}} = \limsup \sqrt{\frac{1}{n^{\frac{1}{\sqrt{n}}}} \cdot \frac{1}{n^{\frac{1}{\sqrt{n}}}}} = \limsup \sqrt{\frac{1}{n^{1/n}}} = \sqrt{\lim \frac{1}{n^{1/n}}} = 1$$

$$\implies R = \frac{1}{\beta} = 1.$$

For $x = 1$,

$$\frac{1}{n^{\sqrt{n}}} \leq \frac{1}{n^2} \qquad \text{for } n \geq 4,$$

then by Comparision Test with $\sum \frac{1}{n^2}$, the series converges. For $x = -1$, the series converges by the Alternating Series Test. Hence, the convergence interval is $[-1, 1]$. $\square$

Exercise 23.8

Proposition 1. *For each $n \in \mathbb{N}$, let $f_n(x) = \frac{1}{n}\sin nx$, where each f_n is a differentiable function, then*

(a) $\lim f_n(x) = 0$ *for all* $x \in \mathbb{R}$*,*

(b) *but* $\lim f_n'(x)$ *need not exist [at* $x = \pi$ *for instance].*

Solution.

(a) For all $x \in \mathbb{R}$, $|f_n(x)| \leq |\frac{1}{n}|$ because $|\sin nx| \leq 1$ for all $n \in \mathbb{N}$. Then, taking the same N as for the ϵ-proof of the limit of $\frac{1}{n}$, we have

$$|f_n(x) - 0| < \epsilon,$$

thus $\lim f_n(x) = 0$ for all $x \in \mathbb{R}$.

(b)

$$f_n'(x) = \cos nx.$$

Specifically, $f_n'(\pi) = (-1)^n$, which does not have a limit.

□

Exercise 24.4

For $x \in [0,\infty)$, let $f_n(x) = \frac{x^n}{1+x^n}$,

(a) Find $f(x) = \lim f_n(x)$.

(b) Determine whether $f_n \to f$ uniformly on $[0,1]$.

(c) Determine whether $f_n \to f$ uniformly on $[0,\infty)$.

Solution.

(a) For $x < 1$, $\lim x^n = 0$, thus

$$\lim f_n(x) = 0.$$

For $x = 1$,

$$\lim f_n(x) = \frac{1}{2}.$$

For $x > 1$,

$$\lim f_n(x) = \lim \frac{1}{1/x^n + 1} = 1.$$

Hence,

$$f(x) = \begin{cases} 0 & \text{if } x < 1 \\ \frac{1}{2} & \text{if } x = 1 \\ 1 & \text{if } x > 1. \end{cases}$$

(b) For $x \in [0,1]$, x^n is continuous $\implies 1 + x^n$ is continuous and nonzero $\implies \frac{1}{1+x^n}$ is continuous. Hence, f_n is continuous on $[0,1]$ for all $n \in \mathbb{N}$. However, f is not continuous at $x = 1$ because

$$\lim_{x\to 1^-} f(x) = 0 \neq \frac{1}{2} = f(1).$$

Therefore, by *Theorem 24.3*, f_n does not converge uniformly to f on $[0,1]$.

(c) Similarly, f_n is continuous on $[0,\infty)$ for all $n \in \mathbb{N}$. However, f is not continuous at $x = 1$ because

$$\lim_{x\to 1^-} f(x) = 0 \neq \frac{1}{2} = f(1).$$

Therefore, by *Theorem 24.3*, f_n does not converge uniformly to f on $[0,\infty)$.

More simply, if f_n converges uniformly to f on $[0,\infty)$, then f_n will converge to f uniformly on $[0,1]$, which we have shown to be false. Hence, f_n does not converge uniformly to f on $[0,\infty)$.

$\square$

Exercise 24.11

Proposition 2. *Let $f_n(x) = x$ and $g_n(x) = \frac{1}{n}$ for all $x \in \mathbb{R}$. Let $f(x) = x$ and $g(x) = 0$ for $x \in \mathbb{R}$, then*

(a) *$f_n \to f$ uniformly on $\mathbb{R}$ and $g_n \to g$ uniformly on $\mathbb{R}$;*

(b) *The sequence $f_n g_n$ does not converge uniformly to fg on $\mathbb{R}$;*

Proof.

(a) For $f_n \to f$, take $N = 1$, then for all $n > N$ and $x \in \mathbb{R}$,

$$|f_n(x) - f(x)| = |x - x| = 0 < \epsilon.$$

For $g_n \to g$, take $N = \frac{1}{\epsilon}$, then for all $n > N$ and $x \in \mathbb{R}$,

$$|g_n(x) - g(x)| = \left|\frac{1}{n} - 0\right| = \frac{1}{n} < \epsilon.$$

(b)

$$(f_n g_n) = \frac{x}{n}, \qquad fg = 0.$$

We show that there does not exist N such that for $n > N$ and all $x \in \mathbb{R}$,

$$\left|\frac{x}{n} - 0\right| < 1.$$

Simply take $x = 2N$, then for $n = N + 1$,

$$\left|\frac{x}{n} - 0\right| = \left|\frac{2N}{N+1}\right| > 1.$$

□