MATH 110 Notes

Book: Linear Algebra Done Right (4th Edition)

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Chapter 4

Polynomials

4.1 A quick introduction to polynomials

Definition 4.1: Complex numbers

Suppose $z = a + bi$, where a and b are real numbers.

- The *real part* of z, is defined by $\operatorname{Re} z = a$.
- The *imaginary part* of z, is defined by $\operatorname{Im} z = b$.

Definition 4.2: Complex conjugate and absolute value

Suppose $z \in \mathbb{C}$.

- The *complex conjugate* of z, denoted by $\overline{z}$, is defined by

$$\overline{z} = \operatorname{Re} z - (\operatorname{Im} z)i.$$

- The *absolute value* of a complex number z, denoted by $|z|$, is defined by

$$|z| = \sqrt{(\operatorname{Re} z)^2 + (\operatorname{Im} z)^2} = \sqrt{z\overline{z}}.$$

Proposition 4.1 Properties of complex numbers

Suppose $w, z \in C$. Then the following equalites and inequalities hold.

- **sum of z and $\overline{z}$**

$$z + \overline{z} = 2\operatorname{Re} z.$$

- **difference of z and $\overline{z}$**

$$z - \overline{z} = 2(\operatorname{Im} z)i.$$

- **product of z and $\overline{z}$**

$$z\overline{z} = |z|^2.$$

- **additivity and multiplicativity of complex conjugate**

$$\overline{z + w} = \overline{z} + \overline{w}, \qquad \overline{zw} = \overline{z}\,\overline{w}.$$

- **double conjugate**
$$\overline{\overline{z}} = z.$$
- **real and imaginary parts are bounded by $|z|$**
$$|\mathrm{Re}\, z| \leq |z|, \qquad |\mathrm{Im}\, z| \leq |z|.$$
- **absolute value of the complex conjugate**
$$|\overline{z}| = |z|.$$
- **multiplicativity of absolute value**
$$|wz| = |w||z|.$$
- **triangle inequality**
$$|w + z| \leq |w| + |z|.$$

4.1.1 Division Algorithm for Polynomials

Proposition 4.2 If a polynomial is the zero function, then all coefficients are 0

Suppose $a_0, \ldots, a_m \in \mathbb{F}$. If
$$a_0 + a_1 z + \cdots + a_m z^m = 0$$
for every $z \in F$, then $a_0 = \cdots = a_m = 0$.

Note:

Note:

Think of polynomial as a function that maps from a domain to a codomain, and the actions of this function can be encapsulated by a list of coefficients.

This result implies that the coefficients of a polynomial are uniquely determined because if a polynomial had two different sets of coefficients, then subtracting the two representations of the polynomial would give a contradiction to the result.

Lemma 4.1 Division algorithm for polynomials

Suppose that $p, s \in \mathcal{P}(\mathbb{F})$, with $s \neq 0$. Then there exist unique polynomials $q, r \in \mathcal{P}(\mathbb{F})$ such that
$$p = sq + r$$
and $\deg r < \deg s$.

4.1.2 Zeros of Polynomials

Definition 4.3: Zero of a polynomial

A number $\lambda \in F$ is called a *zero* (or root) of a polynomial $p \in p(\mathbb{F})$ if
$$p(\lambda) = 0.$$

Definition 4.4: Factor

A polynomial $s \in \mathcal{P}(\mathbb{F})$ is called a *factor* of $p \in \mathcal{P}(\mathbb{F})$ if there exists a polynomial $q \in \mathcal{P}(\mathbb{F})$ such that $p = sq$.

Proposition 4.3 Each zero of a polynomial corresponds to a degree-one factor

Suppose $p \in \mathcal{P}(\mathbb{F})$ and $\lambda \in \mathbb{F}$. Then $p(\lambda) = 0$ if and only iff there is a polynomial $q \in \mathcal{P}(\mathbb{F})$ such that

$$p(z) = (z - \lambda)q(z)$$

for every $z \in \mathbb{F}$.

Note:

$(z - \lambda)$ is called a *linear factor* of p. It is also a degree one polynomial.

Proposition 4.4 A polynomial has at most as many zeros as its degree

Suppose $p \in \mathcal{P}(F)$ is a polynomial with degree $m \geq 0$. Then p has at most m distinct zeros in $\mathbb{F}$.

4.1.3 Factorization of Polynomials over $\mathbb{C}$

Theorem 4.1 Fundamental theorem of algebra, first version

Every nonconstant polynomial with complex coefficients has a zero in $\mathbb{C}$.

Theorem 4.2 Fundamental theorem of algebra, second version

If $p \in \mathcal{P}(\mathbb{C})$ is a nonconstant polynomial, then p has a unique factorization (up to rearrangement of the factors) of the form

$$p(z) = c(z - \lambda_1)\cdots(z - \lambda_m)$$

where $c, \lambda_1, \ldots, \lambda_m \in \mathbb{C}$ and $c \neq 0$.

4.1.4 Factorization of Polynomials over $\mathbb{R}$

Note:

We like to use x to denote the real variable of a polynomial over $\mathbb{R}$, and z to denote the complex variable of a polynomial over $\mathbb{C}$.

Proposition 4.5 Polynomials with real coefficients have nonreal zeros in pairs

Suppose $p \in \mathcal{P}(\mathbb{C})$ is a polynomial with real coefficients. If $\lambda \in \mathbb{C}$ is a zero of p, then so is $\overline{\lambda}$.

Proposition 4.6 Factorization of a quadratic polynomial

Suppose $b, c \in \mathbb{R}$. Then there is a polynomial factorization of the form

$$x^2 + bx + c = (x - \lambda_1)(x - \lambda_2)$$

with $\lambda_1, \lambda_2 \in \mathbb{R}$ if and only if $b^2 - 4c \geq 0$.

Theorem 4.3 Factorization of a polynomial over $\mathbb{R}$

Suppose $p \in \mathcal{P}(\mathbb{R})$ is a nonconstant polynomial. Then p has a unique factorization (up to rearrangement of the factors) of the form

$$p(x) = c(x - \lambda_1) \cdots (x - \lambda_m)(x^2 + b_1 x + c_1) \cdots (x^2 + b_n x + c_n)$$

where $c, \lambda_1, \ldots, \lambda_m, b_1, \ldots, b_n, c_1, \ldots, c_n \in \mathbb{R}$, $c \neq 0$, and $b_k^2 - 4c < 0$ for each k.

Chapter 5

Eigenvalues and Eigenvectors

5.1 Invairant Subspaces

5.1.1 Eigenvalues

Definition 5.1: Operator

A linear map from a vector space to itself is called an *operator*.

Note:

We denote $\mathcal{L}(V)$ as the set of all operators on V, which is the same as $\mathcal{L}(V, V)$.

Definition 5.2: Invariant subspace

Suppose $T \in \mathcal{L}(V)$. A subspace U of V is called *invariant* under T if

$$Tu \in U \qquad \text{for every } u \in U.$$

Example 5.1 (Invariant subspace of differentiation operator)

Suppose $V = \mathcal{P}(\mathbb{F})$ and $D \in \mathcal{L}(V)$ is the differentiation operator. Then $U = \mathcal{P}_n(\mathbb{F})$ is an invariant subspace of D for every $n \in \mathbb{N}$ because for every $v \in \mathcal{P}(\mathbb{F})$, Dv is a polynomial of degree one less than v, and hence $Dv \in \mathcal{P}_n(\mathbb{F})$.

Example 5.2 (Four invariant subspaces, not necessarily all different)

If $T \in \mathcal{L}(V)$, then the following subspaces are all invariant under T:

- $\{0\}$.
- V.
- $\operatorname{null} T$.
- $\operatorname{range} T$.

Definition 5.3: Eigenvalue

Suppose $T \in \mathcal{L}(V)$. A number $\lambda \in \mathbb{F}$ is called an *eigenvalue* of T if there exists a nonzero vector $v \in V$ such that $Tv = \lambda v$.

Example 5.3 (Eiganvalue in $\mathbb{R}^3$)

Define an opeartor $T \in \mathcal{L}(\mathbb{R}^3)$ by

$$T(x, y, z) = (7x + 3z, 3x + 6y + 9z, -6y).$$

Then

$$T(3, 1, -1) = (18, 6, -6) = 6(3, 1, -1).$$

Thus 6 is an eigenvalue of T.

Proposition 5.1 Equivalent conditions to be an eigenvalue

Suppose V is finite-dimensional, $T \in \mathcal{L}(V)$, and $\lambda \in \mathbb{F}$. Then the following are equivalent:

(a) λ is an eigenvalue of T.

(b) $T - \lambda I$ is not injective.

(c) $T - \lambda I$ is not surjective.

(d) $T - \lambda I$ is not invertible.

Proof: $Tv = \lambda v$ is equivalent to $(T - \lambda I)v = 0$, which means $T - \lambda I$ (which is an operator on V) has non-trivial null space, and all the equivalent conditions follow from the rank-nullity theorem. ☺

Definition 5.4: Eigenvector

Suppose $T \in \mathcal{L}(V)$ and $\lambda \in \mathbb{F}$ is an eigenvalue of T. A vector $v \in V$ is called an *eigenvector* of T corresponding to λ if $v \neq 0$ and $Tv = \lambda v$.

Note:

Because $Tv = \lambda v$ if an only if $(T - \lambda I)v = 0$, a vector $v \in V$ with $v \neq 0$ is an eigenvector of T corresponding to λ if and only if $v \in \operatorname{null}(T - \lambda I)$.

Lemma 5.1 Linearly independent eigenvectors

Suppose $T \in \mathcal{L}(V)$. Then every list of eigenvectors of T corresponding to *distinct* eigenvalues is linearly independent.

Lemma 5.2 Operator cannot have more eigenvalues than dimension of vector space

Suppose V is finite-dimensional. Then each operator on V has at most $\dim V$ distinct eigenvalues.

5.1.2 Polynomials Applied to Operators

Definition 5.5: Notation: T^m

Suppose $T \in \mathcal{L}(V)$ and m is a positive integer.

- $T^m \in \mathcal{L}(V)$ is defined by $T^m = \underbrace{T \cdots T}_{m \text{ times}}$.
- T^0 is defined to be the identity operator I on V.
- If T is invertible with inverse T^{-1}, then $T^{-m} \in \mathcal{L}(V)$ is defined by $T^{-m} = (T^{-1})^m$.

Definition 5.6: Notation: $\mathcal{P}(T)$

Suppose $T \in \mathcal{L}(V)$ and $p \in \mathcal{P}(\mathbb{F})$ is a polynomial given by

$$p(z) = a_0 + a_1 z + \cdots + a_m z^m \qquad \text{for all } z \in \mathbb{F}.$$

Then $\mathcal{P}(T) \in \mathcal{L}(V)$ is defined by

$$\mathcal{P}(T) = a_0 I + a_1 T + \cdots + a_m T^m.$$

Note:

If we fix an opeartor $T \in \mathcal{L}(V)$, then the fucntion $\Phi : \mathcal{P}(\mathbb{F}) \to \mathcal{L}(V)$ given by $p \mapsto p(T)$ is linear because

$$\Phi(p+q) = (p+q)(T) = p(T) + q(T) = \Phi(p) + \Phi(q), \qquad \Phi(cp) = cp(T) = c\Phi(p)$$

for every $p, q \in \mathcal{P}(\mathbb{F})$ and $c \in \mathbb{F}$, which can be checked by writing out the coefficients of $p(T)$ and $q(T)$.

Example 5.4 (A polynomial applied to the differentiation opeartor)

Suppose $D \in \mathcal{L}(\mathcal{P}(\mathbb{R}))$ is the differentiation operator and $p \in \mathcal{P}(\mathbb{R})$ is given by

$$p(x) = 7 - 3x + 5x^2.$$

Then

$$(p(D))\, q = 7q - 3Dq + 5D^2 q = 7 - 3q' + 5q'' \qquad \text{for all } q \in \mathcal{P}(\mathbb{R}).$$

Definition 5.7: Product of polynomials

If $p, q \in \mathcal{P}(\mathbb{F})$, then $pq \in \mathcal{P}(\mathbb{F})$ is the polynomial defined by

$$(pq)(z) = p(z)q(z) \qquad \text{for all } z \in \mathbb{F}.$$

Proposition 5.2 Multiplicative properties

Suppose $p, q \in \mathcal{P}(\mathbb{F})$ and $T \in \mathcal{L}(V)$. Then

(a) $(pq)(T) = p(T)q(T)$.

(b) $p(T)q(T) = q(T)p(T)$.

Proposition 5.3 Null space and range of $p(T)$ are invariant under T

Suppose $T \in \mathcal{L}(V)$ and $p \in \mathcal{P}(\mathbb{F})$. Then null $p(T)$ and range $p(T)$ are invariant under T.

5.2 Eigenvalues and the Minimal Polynomial

5.2.1 Existence of Eigenvalues on Complex Vector Space

Theorem 5.1 Operators on complex vector spaces have an eigenvalue

Every operator on a finite-dimensional, nonzero, complex vector space has an eigenvalue.

5.2.2 Minimal Polynomial

Definition 5.8: Monic polynomial

A *monic polynomial* is a polynomial whose highest-degree coefficient equals 1.

Proposition 5.4 Existence, uniqueness, and degree of minimal polynomial

Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Then there is a unique monic polynomial $p \in \mathcal{P}(\mathbb{F})$ of smallest degree such that $p(T) = 0$. Futhermore, $\deg p \le \dim V$.

Note:

Because the existence and uniquess of such monic polynomial is guaranteed, we can define a *minimal polynomial* of T.

Definition 5.9: Minimal polynomial

Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Then the *minimal polynomial* of T is the unique monic polynomial $p \in \mathcal{P}(\mathbb{F})$ of smallest degree such that $p(T) = 0$.

Note:

To compute the minimal polynomial of an operator $T \in \mathcal{L}(V)$, we need to find the smallest positive integer m such that the equation

$$c_0 I + c_1 T + \cdots + c_{m-1}T^{m-1} = -T^m$$

has a solution $(c_0, \ldots, c_{m-1}) \in \mathbb{F}$. We can pick a basis of V and represent T as a matrix, then the equation above can be thought of as a system of $(\dim V)^2$ linear equations, and we can use Gaussian elimination to see if a unique solution exists for successive values of $m = 1, \ldots, \dim V - 1$ until a unique solution is found.

Even faster (usually), pick $v \in V$ (can be a basis vector) and consider the equation

$$c_0 v + c_1 Tv + \cdots + c_{\dim V-1}T^{\dim V-1}v = -T^{\dim V}v.$$

Use a basis of V to represent the vectors $T^k v$ in the equation above, then it is a system of $\dim V$ linear equations. If this system has a unique solution $c_0, c_1, \ldots, c_{\dim V-1}$ (as happens most of the time), then this unique solution is the coefficients of the minimal polynomial of T because the uniqueness of the solution guarantees that the degree of the minimal polynomial is $\dim V$ and the minimal polynomial can have degree at most $\dim V$.

Theorem 5.2 Eigenvalues are the zeros of the minimal polynomial

Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$.

(a) The zeros of the minimal polynomial of T are the eigenvalues of T.

(b) If V is a complex vector space, then the minimal polynomial of T has the form

$$p(z) = (z - \lambda_1)^{m_1} \cdots (z - \lambda_k)^{m_k}$$

where $\lambda_1, \ldots, \lambda_k$ are the distinct eigenvalues of T and $m_1, \ldots, m_k$ are positive integers.

Note:

- The result in (b) is only true for complex vector spaces because only $\mathbb{C}$ can guarantee a linear factorization of a polynomial.
- The powers in (b) mean that the minimal polynomial of T may or may not have repeated factors.
- (a) and (b) combined to gether means that all the zeros of the minimal polynomial of T are eigenvalues of T, and all the eigenvalues of T are zeros of the minimal polynomial of T.

Proposition 5.5 $q(T) = 0 \iff q$ is a polynomial multiple of the minimal polynomial

Suppose V is finite-dimensional, $T \in \mathcal{L}(V)$, and $q \in \mathcal{P}(F)$. Then $q(T) = 0$ if and only if q is a polynomial multiple of the minimal polynomial of T.

Note:

This means that the characteristic polynomial of T is a polynomial multiple of the minimal polynomial of T.

Proposition 5.6 Minimal polynomial of a restriction operator

Suppose V is finite-dimensional, $T \in \mathcal{L}(V)$, and U is a subspace of V that is invariant under T. Then the minimal polynoial of T is a polynomimal multiple of the minimal polynomial of $T|_U$.

Proof: The minimal polynomial of T acting on $T|_U$ is also a zero function, and the previous proposition implies that the minimal polynomial of T is a polynomial multiple of the minimal polynomial of $T|_U$. ☺

Proposition 5.7 T not invertible $\iff$ constant term of minimal polynomial of T is 0

Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Then T is not invertible if and only if the constant term of the minimal polynomial of T is 0.

Proof: Denote the minimal polynomial of T by p.

$$\begin{aligned}
T \text{ is not invertible} &\iff T \text{ is not injective} \\
&\iff \operatorname{null} T \neq \{0\} \\
&\iff 0 \text{ is an eigenvalue of } T \text{ with eigenspace} = \operatorname{null} T \\
&\iff 0 \text{ is a zero of } p \\
&\iff p(0) = 0 \\
&\iff \text{the constant term of } p \text{ is } 0.
\end{aligned}$$

☺

5.2.3 Real Vector Spaces: Invariant Subspaces and Eigenvalues

Theorem 5.3 Eigenvalue or invariant subspace of dimension 2

Every operator on a finite-dimensional non zero vector space has an invariant subspace with dimension 1 or dimension 2.

Theorem 5.4 Operators on odd-dimensional vector spaces have eigenvalues

Every operator on an odd-dimensional vector space has an eigenvalue.

5.3 Upper-Triangular Matrices

5.3.1 Definitions

Definition 5.10: Matrix of an operator, $\mathcal{M}(T)$

Suppose $T \in \mathcal{L}(V)$ and $v_1, \ldots v_n$ is a basis of V. The matrix of T with respect to this basis is the n-by-n matrix

$$\mathcal{M}(T) = \begin{bmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & \ddots & \vdots \\ A_{n,1} & \cdots & A_{n,n} \end{bmatrix}$$

whose entries $A_{j,k}$ are defined by

$$Tv_k = A_{1,k}v_1 + \cdots + A_{n,k}v_n.$$

If the basis is not clear from the context, then the notation $\mathcal{M}(T, (v_1, \ldots, v_n))$ is used.

Definition 5.11: Diagonal of a matrix

The *diagonal* of a square matrix consists of the entries on the line from the upper left corner to the bottom right corner.

Note:
Only square matrices have diagonals.

Definition 5.12: Upper-triangular matrix

A square matrix is called *upper triangular* if all entries below the diagonal are zero.

5.3.2 Conditions for Upper-Triangular Matrices

Proposition 5.8 Conditions for upper-triangular matrix

Suppose $T \in \mathcal{L}(V)$ and $v_1, \ldots, v_n$is a basis of V. Then the following are equivalent.

(a) $\mathcal{M}(T, (v_1, \ldots, v_n))$ is upper triangular.

(b) $\operatorname{span}(v_1, \ldots, v_k)$ is invariant under T for each $k = 1, \ldots, n$.

(c) $Tv_k \in \operatorname{span}(v_1, \ldots, v_k)$ for each $k = 1, \ldots, n$.

Theorem 5.5 Determination of eigenvalues from upper-triangular matrix

Suppose $T \in \mathcal{L}(V)$ has an upper-triangular matrix with respect to some basis of V. Then the eigenvalues of T are the entries on the diagonal of this matrix.

Note:
If $\mathcal{M}(T, v_1, \ldots, v_n)$ is an upper-triangular matrix, v_1 is angenvector of T. However, $v_2, \ldots, v_n$ need not be eigenvectors of T. A basis vector v_k is an eigenvector of T if and only if all the entries in the k^{th} column of the matrix are 0, except possibly the k^{th} entry.

Theorem 5.6 Necessary and sufficient condition to have an upper-triangular matrix

Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Then T has an upper-triangular matrix with respect to some basis of V if and only if the minimal polynomial of T equals $(z-\lambda_1)\cdots(z-\lambda_n)$ for some $\lambda_1, \dots, \lambda_n \in \mathbb{F}$.

Note:

In other words, T has an upper-triangular matrix if and only if the minimal polynomial of T can be linearly factorized. (The linear factors can be repeated.)

Theorem 5.7 If $\mathbb{F} = \mathbb{C}$, then every operator on V has an upper-triangular matrix

Suppoes V is finite-dimensional and $T \in \mathcal{L}(V)$.

(a) If $\mathbb{F} = \mathbb{C}$, then T has an upper-triangular matrix with respect to some basis of V.

(b) If $\mathbb{F} = \mathbb{R}$, then T has an upper-triangular matrix with respect to some basis of V if and only if every zero of the minimal polynomial of T, thought of as a polynomial with complex coefficients, is real.

Proof: Follows directly from the previous proposition and the fundamental theorem of algebra and its consequence on factorization of polynomial over $\mathbb{C}$ and $\mathbb{R}$. 😄

Note:

If starting with a square matrix, the matrix in row echelon form will be an upper-triangular matrix. However, do not confuse this upper-triangular matrix with the upper-triangular matrix of an operator with respect to some basis of V. There is no connection between the two.

5.4 Diagonalizable Operators

5.4.1 Definitions

Definition 5.13: Diagonal matrix

A *diagonal matrix* is a square matrix that is zero everywhere except possibly on the diagonal.

Definition 5.14: Diagonalizable

An operator $T \in \mathcal{L}(V)$ is called *diagonalizable* if the operator has a diagonal matrix with respect to some basis of V.

Note:
This is true if and only if V has a basis of eigenvectors of T.

Definition 5.15: Eigenspace, $E(\lambda, T)$

Supose $T \in \mathcal{L}(V)$ and $\lambda \in \mathbb{F}$. The *eigenspace* of T corresponding to λ is the subspace $E(\lambda, T)$ of V defined by

$$E(\lambda, T) = \operatorname{null}(T - \lambda I) = \{v \in V : Tv = \lambda v\}.$$

In other words, $E(\lambda, T)$ is the set of all eigenvectors of T corresponding to λ, along with the zero vector.

Note:
If λ is an eigenvalue of T, then T restricted to $E(\lambda, T)$ is just multiplication by λ.

5.4.2 Conditions for Diagonalizability

Proposition 5.9 Sum of eigenspaces is a direct sum

Suppose $T \in \mathcal{L}(V)$ and $\lambda_1, \ldots, \lambda_m$ are distinct eigenvalues of T. Then

$$E(\lambda_1, T) + \cdots + E(\lambda_m, T)$$

is a direct sum. further more, if V is finite-dimensional, then

$$\dim E(\lambda_1, T) + \cdots + \dim E(\lambda_m, T) \leq \dim V.$$

Proposition 5.10 Conditions equivalent to diagonalizability

Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Let $\lambda_1, \ldots, \lambda_m$ denote the distinct eigenvalues of T. Then the following are equivalent.

(a) T is diagonalizable.

(b) V has a basis consisting of eigenvectors of T.

(c) $V = E(\lambda_1, T) \oplus \cdots \oplus E(\lambda_m, T)$.

(d) $\dim V = \dim E(\lambda_1, T) + \cdots + \dim E(\lambda_m, T)$.

Proposition 5.11 Enough eigenvalues implies diagonalizability

Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$ has $\dim V$ distinct eigenvalues. Then T is diagonalizable.

Theorem 5.8 Necessary and sufficient condition for diagonalizability

Suppose V is finite-dimensional and $t \in \mathcal{L}(V)$. Then T is diagonalizable if and only if the minimal polynomial of T equals $(z - \lambda_1) \cdots (z - \lambda_m)$ for some list of distinct numbers $\lambda_1, \ldots, \lambda_m \in \mathbb{F}$.

Note:

In other words, T is diagonalizable if and only if the minimal polynomial of T can be linearly factorized into distinct linear factors.

Proposition 5.12 Restriction of diagonalizable to invariant subspace is diagonalizable

Suppose $T \in \mathcal{L}(V)$ is diagonalizable and U is a subspace of V that is invariant under T. Then $T|_U$ is a diagonalizable operator on U.

Proof: Denote the minimal polynomial of $T|_U$ as q and the minimal polynomial of T as p. Then p is a polynomial multiple of q because $p(T|_U) = 0$. Since p can be linearly factorized, q can also be linearly factorized, otherwise p as a polynomial multiple of q would have repeated factors. Thus $T|_U$ is diagonalizable. 😄

5.5 Commuting Operators

Definition 5.16: Commute

- Two operators $S, T \in \mathcal{L}(V)$ are said to *commute* if $ST = TS$.
- Two square matrices $A, B \in \mathbb{F}^{n\times n}$ are said to *commute* if $AB = BA$.

Note:

Commuting matrices are unusual. For example, there are 214,358,881 pairs of 2×2 matrices all of whose entries are integers in the interval $[-5, 5]$. About 0.3% of these pairs of matrices commute.

Proposition 5.13 Commuting operators correspond to commuting matrices

Suppose $S, T \in \mathcal{L}(V)$ and $v_1, \ldots, v_n$is a basis of V. Then S and T commute if and only if $\mathcal{M}(S,(v_1, \ldots, v_n))$ and $\mathcal{M}(T,(v_1, \ldots, v_n))$ commute.

Lemma 5.3 Eigenspace is invariant under commuting operator

Suppose $S, T \in \mathcal{L}(V)$ commute and $\lambda \in \mathbb{F}$. Then $E(\lambda, S)$ is invariant under T.

Theorem 5.9 Simultaneous diagonalizability $\iff$ commutativity

Two diagonalizable operators on the same vector space have diagonal matrices with respect to the same basis if and only if they commute.

Proposition 5.14 Common eigenvector for commuting operators

Every pair of commuting operatos on a finite-dimensional, nonzero, complex vector space has a common eigenvector.

Theorem 5.10 Commuting operators are simultaneously upper triangularizable

Suppose V is a finite-dimensional complex vector space and S, T are commuting operators on V. Then there is a basis of V with respect to which both S and T have upper-triangular matrices.

Theorem 5.11 Eigenvalues of sum and product of commuting operators

Suppose V is a finite-dimensional complex vector space and S, T are commuting operators on V. Then

- every eigenvalue of $S + T$ is an eigenvalue of S plus an eigenvalue of T.
- every eigenvalue of ST is a an eigenvalue of S times an eigenvalue of T.

Chapter 6

Inner Product Spaces

6.1 Inner Products and Norms

6.1.1 Inner Products

Definition 6.1: Inner product

An *inner product* on V is a function that takes each ordered pair (u, v) of elements of V to a number $\langle u, v\rangle \in \mathbb{F}$ and has the following properties.

- **positivity**
$$\langle v, v\rangle \geq 0 \qquad \text{for all } v \in V.$$
- **definiteness**
$$\langle v, v\rangle = 0 \iff v = 0.$$
- **additivity in the first slot**
$$\langle u + v, w\rangle = \langle u, w\rangle + \langle v, w\rangle \qquad \text{for all } u, v, w \in V.$$
- **homogeneity in the first slot**
$$\langle cu, v\rangle = c\langle u, v\rangle \qquad \text{for all } u, v \in V \text{ and } c \in \mathbb{F}.$$
- **conjugate symmetry**
$$\langle u, v\rangle = \overline{\langle v, u\rangle} \qquad \text{for all } u, v \in V.$$

Definition 6.2: Inner product space

An *inner product space* is a vector space equipped with an inner product on V.

Proposition 6.1 Basic properties of an inner product

(a) For each fixed $u \in V$, the function that maps $v \in V$ to $\langle u, v\rangle$ is a linear map from V to F, specifically a linear functional on V.

(b) $\langle 0, v\rangle = \langle v, 0\rangle$ for all $v \in V$.

(c) $\langle u, v + w\rangle = \langle u, v\rangle + \langle u, w\rangle$ for all $u, v, w \in V$.

(d) $\langle u, \lambda v\rangle = \overline{\lambda}\langle u, v\rangle$ for all $u, v \in V$ and $\lambda \in \mathbb{F}$.

6.1.2 Norms

Definition 6.3: Norm, $\|v\|$

For $v \in V$, the *norm* of v, denoted $\|v\|$, is defined by

$$\|v\| = \sqrt{\langle v, v\rangle}.$$

Proposition 6.2 Basic properties of the norm

Suppose $v \in V$.

(a) $\|v\| = 0 \iff v = 0$.

(b) $\|\lambda v\| = |\lambda|\|v\|$ for all $\lambda \in \mathbb{F}$.

Definition 6.4: Orthogonal

Two vectors $u, v \in V$ are called *orthogonal* if $\langle u, v\rangle = 0$.

Note:
Hence, orthogonality is dependent on the inner product.

Proposition 6.3 Orthogonality and 0

(a) 0 is orthogonal to every vector in V.

(b) 0 is the only vector in V that is orthogonal to itself.

6.1.3 Consequent Properties of Inner Products and Norms

Theorem 6.1 Pythagorean theorem

Suppose $u, v \in V$. If u and v are orthogonal, then

$$\|u + v\|^2 = \|u\|^2 + \|v\|^2.$$

Note:
This is equivalent to saying their self inner products (norm squared) can be split into the sum of the self inner products of each vector.

$$\langle u + v, u + v\rangle = \langle u, u\rangle + \langle v, v\rangle.$$

But this is not necessarily true for the norm of the sum of two vectors.

Proposition 6.4 An orthogonal decomposition

Suppose $u, v \in V$, with $v \neq 0$. Set $c = \frac{\langle u,v\rangle}{\|v\|^2}$ and $w = u - cv$. Then

$$\langle w, v\rangle = 0 \text{ and } u = w + cv.$$

Theorem 6.2 Cauchy-Schwarz inequality

Suppose $u, v \in V$. Then

$$|\langle u, v\rangle| \leq \|u\|\|v\|.$$

This inequality is an equality if and only if u and v are linearly dependent.

Theorem 6.3 Triangle inequality

Suppose $u, v \in V$. Then

$$\|u + v\| \leq \|u\| + \|v\|.$$

This inequality is an equality if and only if on of u and v is a nonnegative multiple of the other.

Theorem 6.4 Reverse triangle inequality

Suppose $u, v \in V$. Then

$$\big|\|u\| - \|v\|\big| \leq \|u - v\|.$$

This inequality is an equality if and only if one of u and v is a nonnegative multiple of the other.

Theorem 6.5 Parallelogram equality

Suppose $u, v \in V$. Then

$$\|u + v\|^2 + \|u - v\|^2 = 2\|u\|^2 + 2\|v\|^2.$$

Proof: Just expand by the definition of norm. 😄

6.2 Orthonormal Bases

6.2.1 Orthonormal Lists and the Gram-Schmidt Procedure

Definition 6.5: Orthonormal

- A list of vectors is called *orthonormal* if each vector in the list has norm 1 and is orthogonal to all the other vectors in the list.
- In other words, a list $e_1, \ldots, e_m$ is orthonormal if and only if

$$\langle e_i, e_j \rangle = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

Proposition 6.5 Norm of an orthonormal linear combination

Suppose $e_1, \ldots, e_m$is an orthonormal list of vectors in V. Then

$$\|a_1 e_1 + \cdots + a_m e_m\|^2 = |a_1|^2 + \cdots + |a_m|^2 \qquad \text{for all } a_1, \ldots, a_m \in \mathbb{F}.$$

Proof: It follows directly from a repeated application of the Pythagorean theorem and the basic properties of the norm. ☺

Corollary 6.1 Orthonormal lists are linearly independent

Every orthonormal list of vectors is linearly independent.

Note:

This implies every orthogonal list of vectors (excluding the 0 vector) is linearly independent.

Theorem 6.6 Bessel's inequality

Suppose $e_1, \ldots, e_m$ is an orthonormal list of vectors in V. If $v \in V$ then

$$|\langle v, e_1 \rangle|^2 + \cdots + |\langle v, e_m \rangle|^2 \leq \|v\|^2.$$

Definition 6.6: Orthonormal basis

An *orthonormal basis* of V is a basis of V that is orthonormal.

Proposition 6.6 Orthonormal lists of the right length are orthonormal bases

Suppose V is finite-dimensional. Then every orthonormal list of vectors in V with $\dim V$ is an orthonoral basis of V.

Proposition 6.7 Writing a vector as linear combination of orthonromal basis

Suppose $e_1, \ldots, e_n$ is an orthonormal basis of V and $u, v \in V$. Then

(a) $v = \langle v, e_1 \rangle e_1 + \cdots + \langle v, e_n \rangle e_n$.

(b) $\|v\|^2 = |\langle v, e_1 \rangle|^2 + \cdots + |\langle v, e_n \rangle|^2$.

(c) $\langle u, v \rangle = \langle u, e_1 \rangle \overline{\langle v, e_1 \rangle} + \cdots + \langle u, e_n \rangle \overline{\langle v, e_n \rangle}$.

Proof: The first equality is from the orthogonal decomposition. The second equality is from (a) and Pythagorean theorem. The third equality is by taking the inner product of u with each side of (a) and then using the conjugate symmetry. 😊

Theorem 6.7 Gram-Schmidt procedure

Suppose $v_1, \ldots, v_m$ is a linearly independent list of vectors in V. Let $e_1 = \frac{v_1}{\|v_1\|}$. For $k = 2, \ldots, m$, define e_k inductively by

$$e_k = \frac{v_k - \langle v_k, e_1 \rangle e_1 - \cdots - \langle v_k, e_{k-1} \rangle e_{k-1}}{\|v_k - \langle v_k, e_1 \rangle e_1 - \cdots - \langle v_k, e_{k-1} \rangle e_{k-1}\|}.$$

Then $e_1, \ldots, e_m$ is an orthonormal list of vectors in V such that

$$\operatorname{span}(v_1, \ldots, v_k) = \operatorname{span}(e_1, \ldots, e_k) \qquad \text{for } k = 1, \ldots, m.$$

Note:

We can split this procedure into two steps. First, we find residual of v_k after projecting v_k onto the subspace spanned by $e_1, \ldots, e_{k-1}$

$$u_k = v_k - \langle v_k, e_1 \rangle e_1 - \cdots \langle v_k, e_{k-1} \rangle e_{k-1},$$

then we normalize u_k to get e_k by

$$e_k = \frac{u_k}{\|u_k\|}.$$

Proposition 6.8 Existence of orthonormal basis

Every finite-dimensional inner product space has an orthonormal basis.

Proof: Choose a basis of V and apply Gram-Schmidt procedure, which will obtain an orthonormal list with the same span as the original basis. 😊

Proposition 6.9 Every orthonormal list extends to orthonoral basis

Suppose V is finite-dimensional. Then every orthonormal list of vectors in V can be extended to an orthonormal basis of V.

Proof: Suppose $e_1, \ldots, e_m$ is an orthonormal list of vectors in V, then we extend this orthonormal hence linearly independent list of vectors to a basis $e_1, \ldots, e_m, v_1, \ldots, v_n$ of V. Now we apply Gram-Schmidt procedure to $e_1, \ldots, e_m, v_1, \ldots, v_n$, which the first $e_1, \ldots, e_m$ will remain unchanged since they are already orthonormal. Then we will obtain

$$e_1, \ldots, e_m, u_1, \ldots, u_n,$$

where the list is orthonormal and spans V. 😊

Proposition 6.10 Upper-triangular matrix with respect to orthonormal basis

Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Then T has an upper-triangular matrix with respect to some orthonormal basis of V if and only if the minimal polynomial of T equals $(z - \lambda_1) \cdots (z - \lambda_n)$ for some $\lambda_1, \ldots, \lambda_n \in \mathbb{F}$.

Proof: We show T is upper-triangularizable with respect to some basis if and only if T is upper-triangularizable with respect to some orthonormal basis. Then the result follows from the necessary and sufficient conditions of upper-triangularizable matrix.

Suppose T is upper-triangularizable with respect to some basis of V. Then $\operatorname{span}(v_1, \ldots, v_k)$ is invariant under T for each $k = 1, \ldots, n$ by the equivalent conditions of upper-triangular matrix. Then we apply Gram-Schmidt procedure to $v_1, \ldots, v_k$ to obtain an orthonormal basis $e_1, \ldots, e_k$ of V. These two list of vectors have the same span

$$\operatorname{span}(v_1, \ldots, v_k) = \operatorname{span}(e_1, \ldots, e_k) \qquad \text{for } k = 1, \ldots, n.$$

Hence, $\operatorname{span}(e_1, \ldots, e_k)$ is invariant under T for each $k = 1, \ldots, n$. By the equivalent conditions of upper-triangular matrix again, T is upper-triangularizable with respect to the orthonormal basis $e_1, \ldots, e_n$.

The other direction is always true. 😊

Theorem 6.8 Schur's theorem

Every operator on a finite-dimensional complex inner product space has an upper-triangular matrix with respect to some orthonormal basis.

Proof: This is a direct consequence of the previous proposition and fundamental theorem of algebra that all polynomials can be linearly factorized over $\mathbb{C}$. 😊

6.2.2 Linear Functionals on Inner Product Spaces

Definition 6.7: Linear functional, dual space, V'

- A *linear functional* on V is a linear map from V to $\mathbb{F}$.
- The *dual space* of V, denoted V', is the vector space of all linear Functionals on V, in other words, $V' = \mathcal{L}(V, \mathbb{F})$.

Theorem 6.9 Riesz representation theorem

Suppose V is finite-dimensional and φ is a linear functional on V. Then there is a unique vector $v \in V$ such that

$$\varphi(u) = \langle u, v\rangle \qquad \text{for all } u \in V.$$

Proof: Let $e_1, \ldots, e_n$ be an orthonormal basis of V, then

$$\begin{aligned}
\varphi(u) &= \varphi(\langle u, e_1\rangle e_1 + \cdots + \langle u, e_n\rangle e_n) \\
&= \langle u, e_1\rangle \varphi(e_1) + \cdots + \langle u, e_n\rangle \varphi(e_n) \\
&= \langle u, \overline{\varphi(e_1)} e_1 + \cdots + \overline{\varphi(e_n)} e_n\rangle.
\end{aligned}$$

Hence, the vector v is

$$v = \overline{\varphi(e_1)} e_1 + \cdots + \overline{\varphi(e_n)} e_n.$$

Now we prove the uniqueness of v. Suppose there is another vector $w \in V$ such that

$$\varphi(u) = \langle u, w\rangle \qquad \text{for all } u \in V.$$

Then

$$\langle u, v\rangle - \langle u, w\rangle = \langle u, v - w\rangle = 0 \qquad \text{for all } u \in V.$$

Take $u = v - w$, then the above equation is equivalent to

$$\|v - w\|^2 = 0,$$

which by the definiteness of the inner product implies $v - w = 0 \iff v = w$. 😊

Note:

The uniqueness of the Riesz representation theorem is very useful in proving the adjoint of an operator in later chapters.

6.3 Orthogonal Complements and Minimization Problems

6.3.1 Orthogonal Complements

Definition 6.8: Orthogonal complement, $U^\perp$

If U is a subset of V, then the *orthogonal complement* of U, denoted $U^\perp$, is the set of all vectors in V that are orthogonal to every vector in U:

$$U^\perp = \{v \in V : \langle u, v\rangle = 0 \text{ for all } u \in U\}.$$

Proposition 6.11 Properties of orthogonal complement

(a) If U is a subset of V, then $U^\perp$ is a subspace of V.

(b) $\{0\}^\perp = V$.

(c) $V^\perp = \{0\}$.

(d) If U is a subset of V, then $U \cap U^\perp \subseteq \{0\}$.

Note: The intersection can be empty if U does not contain the zero vector.

(e) If G and H are subsets of V and $G \subseteq H$, then $H^\perp \subseteq G^\perp$.

Proposition 6.12 Direct sum of a subspace and its orthogonal complement

Suppose U is a finite-dimensional subspace of V. Then

$$V = U \oplus U^\perp.$$

Proposition 6.13 Dimension of orthogonal complement

Suppose U is a finite-dimensional subspace of V. Then

$$\dim U + \dim U^\perp = \dim V.$$

Proposition 6.14 Orthogonal complement of the orthogonal complement

Suppose U is a finite-dimensional subspace of V. Then

$$(U^\perp)^\perp = U.$$

Proposition 6.15 $U^\perp = \{0\} \iff U = W$ (for U a finite-dimensional subspace of W)

Suppose U is a finite-dimensional subspace of V. Then

$$U^\perp = \{0\} \iff U = W.$$

Definition 6.9: Orthogonal projection, P_U

Suppose U is a finite-dimensional subspace of V. The *orthogonal projection* of V onto U is the operator $P_U \in \mathcal{L}(V)$ defined as follows: For each $v \in V$, write $v = u + w$, where $u \in U$ and $w \in U^\perp$. Then $P_U v = u$.

Note:

The direct sum decomposition $V = U \oplus U^\perp$ guarantees thet every vector $v \in V$ can be written uniquely as $v = u + w$, where $u \in U$ and $w \in U^\perp$.

Proposition 6.16 Properties of orthogonal projection P_U

Suppose U is a finite-dimensional subspace of V. Then

(a) $P_U \in \mathcal{L}(V)$.

(b) $P_U u = u$ for all $u \in U$.

(c) $P_U w = 0$ for all $w \in U^\perp$.

(d) $\operatorname{range} P_U = U$.

(e) $\operatorname{null} P_U = U^\perp$.

(f) $v - P_U v \in U^\perp$ for all $v \in V$.

(g) $P_U^2 = P_U$.

(h) $\|P_U v\| \leq \|v\|$ for all $v \in V$.

(i) If $e_1, \ldots, e_m$ is an orthonormal basis of U and $v \in V$, then

$$P_U v = \langle v, e_1 \rangle e_1 + \cdots + \langle v, e_m \rangle e_m.$$

Theorem 6.10 Riesz representation theorem, revisited

Suppose V is finite-dimensional. For each $v \in V$, define $\varphi_v \in V'$ by

$$\varphi_v(u) = \langle u, v \rangle \qquad \text{for all } u \in V.$$

Then the map $V \to V'$ defined by $v \mapsto \varphi_v$ is injective.

Note:

This map is not isophormism because this is not a linear map if $\mathbb{F} = \mathbb{C}$ due to

$$\lambda v \mapsto \varphi_{\lambda v} = \langle u, \lambda v \rangle = \overline{\lambda} \langle u, v \rangle = \overline{\lambda} \varphi_v \neq \lambda \varphi_v.$$

6.3.2 Minimization Problems

Proposition 6.17 Minimizing distance to a subspace

Suppose U is a finite-dimensional subspace of V, $v \in V$, and $u \in U$. Then

$$\|v - P_U v\| \leq \|v - u\|.$$

Furthermore, the inequality above is an equality if and only if $u = P_U v$.

Note:

This is equivalent to saying the distance from v to U is minimized when $u = P_U v$. Or the residual of v after orthogonal projection is the smallest among all residuals from other kinds of projection.

Proof:

$$\begin{aligned} \|v - P_U v\|^2 &\le \|v - P_U v\|^2 + \|P_U v - u\|^2 \\ &= \|(v - P_U v) + (P_U v - u)\|^2 \\ &= \|v - u\|^2. \end{aligned}$$

The second equality holds because $v - P_U v \in U^\perp$ and $P_U v - u \in U$, hence they are orthogonal to each other and we can apply the Pythagorean theorem. 😄

6.3.3 Pseudoinverses (skipped)

Proposition 6.18 Restrictoin of a linear map to obtain a bijective function

Suppose V is finite-dimensional and $T \in \mathcal{L}(V, W)$. Then $T|_{(\operatorname{null} T)^\perp}$ is a bijective map of $(\operatorname{null} T)^\perp$ onto $\operatorname{range} T$.

Definition 6.10: Pseudoinverse, $T^\dagger$

Suppose that V is finite dimensional and $T \in \mathcal{L}(V, W)$. The *pseudoinverse* $T^\dagger \in \mathcal{L}(W, V)$ of T is the linear map from W to V defined by

$$T^\dagger w = (T|_{(\operatorname{null} T)^\perp})^{-1} P_{\operatorname{range} T} w \qquad \text{for all } w \in W.$$

Note:

- The pseudoinverse is also called the *Moore-Penrose inverse*.
- The pseudoinverse is equivalent to first projecting $w \in W$ to the $\operatorname{range} T$, denote the projection as u, then applying the inverse of $T|_{(\operatorname{null} T)^\perp}$ on u to get the pre-image of u in the subspace $(\operatorname{null} T)^\perp$. The inverse of $T|_{(\operatorname{null} T)^\perp}$ is guaranteed by the previous proposition.

Proposition 6.19 Algebraic properties of the pseudoinverse

Suppose V is finite-dimensional and $T \in \mathcal{L}(V, W)$.

(a) If T is invertible, then $T^\dagger = T^{-1}$.

(b) $T^\dagger T = P_{(nulT)^\perp}$ = the orthogonal projection of V onto $(\operatorname{null} T)^\perp$.

(c) $TT^\dagger = P_{\operatorname{range}} T$ = the orthogonal projection of W onto $\operatorname{range} T$.

Note:

The second and third property can be understood intuitively by drawing a diagram involving the subspaces.

Theorem 6.11 Pseudoinverse provides best approximate solution or best solution

Suppose V is finite-dimensional, $T \in \mathcal{L}(V, W)$, and $b \in W$.

(a) If $x \in V$, then

$$\|T(T^\dagger b) - b\| \leq \|Tx - b\|,$$

with equality if and only if $x \in T^\dagger b + \operatorname{null} T$.

(b) If $x \in T^\dagger b + \operatorname{null} T$, then

$$\|T^\dagger b\| \leq \|x\|,$$

with equality if and only if $x = T^\dagger b$.

Note:

- The first property means that $T^\dagger$ applied to b gives a $x \in V$ such that $Tx \in \operatorname{range} T$ is closest to b among all $w \in \operatorname{range} T$.
- The second property means that out of all $x \in V$ such that Tx is closest to b, $T^\dagger b$ has the smallest norm.

Chapter 7

Operators on Inner Product Spaces

7.1 Self-Adjoint and Normal Operators

7.1.1 Adjoints

Definition 7.1: Adjoint, T^*

Suppose $T \in \mathcal{L}(V, W)$. The *adjoint* of T is the function $T^* : W \to V$ such that

$$\langle Tv, w \rangle = \langle v, T^* w \rangle \qquad \text{for all } v \in V \text{ and } w \in W.$$

Proposition 7.1 Adjoint of a linear map is a linear map

If $T \in \mathcal{L}(V, W)$, then $T^* \in \mathcal{L}(W, V)$.

Proposition 7.2 Properties of the adjoint

Suppose $T \in \mathcal{L}(V, W)$. Then

(a) $(S + T)^* = S^* + T^*$ for all $S \in \mathcal{L}(V, W)$.

(b) $(\lambda T)^* = \overline{\lambda} T^*$ for all $\lambda \in \mathbb{F}$.

(c) $(T^*)^* = T$.

(d) $(ST)^* = T^* S^*$ for all $S \in \mathcal{L}(W, U)$ (here U is an arbitrary inner product space).

(e) $I^* = I$.

(f) if T is invertible, then T^* is invertible and $(T^*)^{-1} = (T^{-1})^*$.

Proposition 7.3 Null space and range of T^*

Suppose $T \in \mathcal{L}(V, W)$. Then

(a) $\operatorname{null} T^* = (\operatorname{range} T)^{\perp}$;

(b) $\operatorname{range} T^* = (\operatorname{null} T)^{\perp}$;

(c) $\operatorname{null} T = (\operatorname{range} T^*)^{\perp}$;

(d) $\operatorname{range} T = (\operatorname{null} T^*)^{\perp}$.

Definition 7.2: Conjugate transpose, A^*

The *conjugate transpose* of an m-by-n matrix A is the n-by-m matrix A^* obtained by interchanging the rows and columns and then taking the complex conjugate of each entry, In other words, $(A^*)_{j,k} = \overline{A_{k,j}}$.

Proposition 7.4 Matrix of T^* equals conjugate transpose of matrix of T

Let $T \in \mathcal{L}(V, W)$. Suppose $e_1, \ldots, e_n$ is an orthonormal basis of V and $f_1, \ldots, f_m$ is an orthonormal basis of W. Then $\mathcal{M}(T^*, (f_1, \ldots, f_m), (e_1, \ldots, e_n))$ is the conjugate tranpose of $\mathcal{M}(T, (e_1, \ldots, e_n), (f_1, \ldots, f_m))$. In other words,

$$\mathcal{M}(T^*) = (\mathcal{M}(T))^*.$$

7.2 Spectral Theorem

7.3 Positiver Operators

7.4 Isometries, Unitary Operators, QR Factorization

7.5 Singular Value Decomposition

7.6 Consequences of Singular Value Decomposition

Chapter 8

Operators on Complex Vector Spaces

8.1 Generalized Eigenvectors and Nilpotent Operators

8.2 Generalized Eigenspace Decomposition

8.3 Consequences of Generalized Eigenspace Decomposition

8.4 Trace: A Connection Between Operators and Matrices

Chapter 9

Multilinear Algebra and Determinants (Skipped)