Math 104 HW7

Neo Lee

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Exercise 15.1

Determine whether the following series converges: **(a)** $\sum \frac{(-1)^n}{n}$ **(b)** $\sum \frac{(-1)^n n!}{2^n}$.

Solution.

(a) Consider $(a_n) = \frac{1}{n}$, then a_n is an increasing sequence, and $\lim a_n = 0$. By Theorem 15.3, $\sum \frac{(-1)^n}{n}$ converges (Alternating Series).

(b)

$$\begin{aligned}\lim \left|\frac{a_{n+1}}{a_n}\right| &= \lim \left|\frac{(-1)^{n+1}(n+1)!}{2^{n+1}} \cdot \frac{2^n}{(-1)^n n!}\right| \\ &= \lim \frac{n+1}{2} \\ &= \infty.\end{aligned}$$

Hence, by Theorem 10.7, $\liminf \left|\frac{a_{n+1}}{a_n}\right| = \lim \left|\frac{a_{n+1}}{a_n}\right| = \infty$. By Theorem 14.8, $\sum \frac{(-1)^n n!}{2^n}$ containing non-zero terms diverges. diverges.

□

Exercise 17.2

Let $f(x) = 4$ for $x \geq 0, f(x) = 0$ for $x < 0$, and $g(x) = x^2$ for all x. Thus $\operatorname{dom}(f) = \operatorname{dom}(g) = \mathbb{R}$.

(a) Determine the following functions: $f + g, fg, f \circ g, g \circ f$. Be sure to specify their domain.

Solution.

$(f + g)$:

$$f + g : \mathbb{R} \to \mathbb{R} = \begin{cases} 4 + x^2 & x \geq 0 \\ x^2 & x < 0. \end{cases}$$

(fg):

$$fg : \mathbb{R} \to \mathbb{R} = \begin{cases} 4x^2 & x \geq 0 \\ 0 & x < 0. \end{cases}$$

$(f \circ g)$:

$$f \circ g : \mathbb{R} \to \mathbb{R} = \begin{cases} 4 & x \geq 0 \\ 0 & x < 0. \end{cases}$$

$(g \circ f)$:

$$g \circ f : \mathbb{R} \to \mathbb{R} = \begin{cases} 16 & x \geq 0 \\ 0 & x < 0. \end{cases}$$

□

(b) Which of the functions $f, g, f+g, fg, f \circ g, g \circ f$ is continuous?

Solution.

(f): Not continuous. Consider $x_0 = 0$ and two sequences $(s_n) = \frac{1}{n}, (t_n) = \frac{-1}{n}$. Then Then $\lim s_n = \lim t_n = 0$, but $\lim f(s_n) = \lim 4 = 4 \neq 0 = \lim f(t_n) = \lim 0$.

(g): Continuous. Consider $x_0 \in \mathbb{R}$. Then for any sequence $(s_n) \to x_0$, $\lim g(s_n) = \lim s_n^2 = \lim s_n \cdot \lim s_n = x_0^2 = g(x_0)$.

$(f+g)$: Not continuous. Consider $x_0 = 0$ and two sequences $(s_n) = \frac{1}{n}$, $(t_n) = \frac{-1}{n}$. Then $\lim s_n = \lim t_n = 0$, but $\lim(f+g)(s_n) = \lim 4 + \lim s_n \cdot \lim s_n = 4 \neq 0 = \lim(f+g)(t_n) = \lim t_n \cdot \lim t_n$.

(fg): Continuous. Consider $(-\infty, 0)$, $(0, \infty)$, and 0.

For $x_0 \in (-\infty, 0)$, let $\delta = |x_0 - 0|$, then $|x - x_0| < \delta \implies f(x) = 0 = f(x_0) \implies |f(x) - f(x_0)| < \epsilon$ for $\epsilon > 0$.

For $x_0 \in (0, \infty)$, consider any sequence $(s_n) \to x_0$. Let $\epsilon = |x_0 - 0|$, then $\exists N$ such that $|s_n - x_0| < \epsilon$ for $n > N$, which implies $s_n > 0$ for $n > N$. Then notice the limit of a sequence is independent of finite number of terms, hence $\lim fg(s_n) = \lim fg(s_{n|n>N}) = \lim 4 \cdot s_{n|n>N} \cdot s_{n|n>N} = 4x_0^2 = fg(x_0)$.

For $x_0 = 0$, take $\delta = \sqrt{\frac{\epsilon}{4}}$, then $|x - 0| < \delta \implies |4x^2 - 0| = |f(x) - f(x_0)| < \epsilon$ for $x \geq 0$. If $x < 0$, $f(x) = 0$ and obviously $|f(x) - f(x_0)| = 0 < \epsilon$.

$(f \circ g)$: Not continuous. Consider $x_0 = 0$ and two sequences $(s_n) = \frac{1}{n}$, $(t_n) = \frac{-1}{n}$. Then $\lim s_n = \lim t_n = 0$, but $\lim(f \circ g)(s_n) = \lim 4 \neq 0 = \lim(f \circ g)(t_n) = \lim 0$.

$(g \circ f)$: Not continuous. Consider $x_0 = 0$ and two sequences $(s_n) = \frac{1}{n}$, $(t_n) = \frac{-1}{n}$. Then $\lim s_n = \lim t_n = 0$, but $\lim(g \circ f)(s_n) = \lim 16 \neq 0 = \lim(g \circ f)(t_n) = \lim 0$.

□

Exercise 17.13

(a)

Proposition 1. *Let $f(x) = 1$ for $x \in \mathbb{Q}$ and $f(x) = 0$ for $x \in \mathbb{R} \setminus \mathbb{Q}$. Then f is discontinuous at every $x \in \mathbb{R}$.*

Proof.

Case 1: $x_0 \notin \mathbb{Q}$. Assume for the sake of contradiction that $\exists \delta > 0$ such that $|x - x_0| < \delta \implies |f(x) - f(x_0)| < 1$. However, by the density of $\mathbb{Q}$, $\exists x \in (x_0 - \delta, x_0)$ such that x is rational, then $|f(x) - f(x_0)| = |1 - 0| = 1 \not< 1$, which is a contradiction.

Case 2: $x_0 \in \mathbb{Q}$. Assume for the sake of contradiction that $\exists \delta > 0$ such that $|x - x_0| < \delta \implies |f(x) - f(x_0)| < 1$. However, by the density of irrationals, $\exists x \in (x_0 - \delta, x_0)$ such that x is irrational, then $|f(x) - f(x_0)| = |0 - 1| = 1 \not< 1$, which is a contradiction.

Note: density of irrationals was not covered in the book. But the proof idea is by considering $\frac{a}{\sqrt{2}} < \frac{b}{\sqrt{2}}$, then there exists rational x in between. Multiplying the inequality by $\sqrt{2}$, we get $a < \sqrt{2}x < b$, where $\sqrt{2}x$ is irrational.

□

(b)

Proposition 2. *Let $h(x) = x$ for rational numbers x and $h(x) = 0$ for irrational numbers, then h is continuous at $x = 0$ and at no other point.*

Proof.

$x_0 = 0$: Let $\delta = \epsilon$, then if $x \in \mathbb{Q}$, $|x - x_0| = |x - 0| < \delta \implies |h(x) - h(x_0)| < \epsilon$. If $x \notin \mathbb{Q}$, $h(x)$ is always 0, hence $|h(x) - h(x_0)| = 0 < \epsilon$.

$x_0 \neq 0, x_0 \in \mathbb{Q}$: Let $\epsilon = |x_0|$. Assume for the sake of contradiction that $\exists \delta > 0$ such that $|x - x_0| < \delta \implies |h(x) - h(x_0)| < \epsilon$. However, by the density of irrationals, $\exists x \in (x_0 - \delta, x_0)$ such that x is irrational, then $|h(x) - h(x_0)| = |0 - x_0| = |x_0| = \epsilon \not< \epsilon$, which is a contradiction.

$x_0 \neq 0, x_0 \notin \mathbb{Q}$: Let $\epsilon = \frac{1}{2}|x_0|$. Assume for the sake of contradiction that $\exists \delta > 0$ such that $|x - x_0| < \delta \implies |h(x) - h(x_0)| < \epsilon$. However, by the density of $\mathbb{Q}$, $\exists x \in (x_0 - \min\{\delta, \frac{1}{2}|x_0|\}, x_0)$ such that x is rational, then $|h(x) - h(x_0)| = |x - 0| > \frac{1}{2}|x_0| \not< \epsilon$, which is a contradiction.

□

Exercise 18.8

Proposition 3. *Suppose f is a real-valued continuous function on $\mathbb{R}$and $f(a)f(b) < 0$ for some $a, b \in \mathbb{R}$, then there exists $x \in (a, b)$ such that $f(x) = 0$.*

Proof. Note that $f(a) \neq 0$ and $f(b) \neq 0$ and $f(a), f(b)$ cannot have the same sign. Without loss of generality, assume $f(a) > 0$ and $0 > f(b)$. Then, by the Intermediate Value Theorem, $\exists x \in (a, b)$ such that $f(x) = 0$. □