Math 181A HW5

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Problem 10-1

$$\begin{aligned}
E[Z] &= E[\sum_{i=1}^{n} \frac{\partial}{\partial\theta} \log f_X(X_i;\theta)] \\
&= nE[\frac{\partial}{\partial\theta} \log f_X(X_i;\theta)] \\
&= nE[\frac{\frac{\partial}{\partial\theta} f_X(X_i;\theta)}{f_X(X_i;\theta)}] \\
&= n\int_{\mathbb{R}} \frac{\frac{\partial}{\partial\theta} f_X(X_i;\theta)}{f_X(X_i;\theta)} \cdot f_X(X_i;\theta)dx \\
&= n\int_{\mathbb{R}} \frac{\partial}{\partial\theta} f_X(X_i;\theta)dx \\
&= n\frac{\partial}{\partial\theta}\int_{\mathbb{R}} f_X(X_i;\theta)dx \\
&= n\frac{\partial}{\partial\theta}1 \\
&= 0. \\
Var(Z) &= E[Z^2] - E[Z]^2 \\
&= E[Z^2] \\
&= E\left[\left(\sum_{i=1}^{n} \frac{\partial}{\partial\theta} \log f_X(X_i;\theta)\right)^2\right] \\
&= E\left[\sum_{i=1}^{n} \left(\frac{\partial}{\partial\theta} \log f_X(X_i;\theta)\right)^2 - 2\left(\sum_{i<j} \log f_X(X_i;\theta) \cdot \log f_X(X_j;\theta)\right)\right] \\
&= E\left[\sum_{i=1}^{n} \left(\frac{\partial}{\partial\theta} \log f_X(X_i;\theta)\right)^2\right] \\
&= nE\left[\left(\frac{\partial}{\partial\theta} \log f_X(X_i;\theta)\right)^2\right] \\
&= nI(\theta).
\end{aligned}$$

Problem 10-2

$$
\begin{aligned}
I(\mu) &= -E\left[\frac{\partial^2}{\partial\mu^2}\log\frac{1}{2\sqrt{2\pi}}e^{\frac{-1}{2\cdot 4}(x-\mu)^2}\right] \\
&= -E\left[\frac{\partial^2}{\partial\mu^2}\left(-\log 2\sqrt{2\pi}-\frac{1}{8}(x-\mu)^2\right)\right] \\
&= -E\left[\frac{\partial}{\partial\mu}\left(\frac{1}{4}(x-\mu)\right)\right] \\
&= E\left[\frac{1}{4}\right] \\
&= \frac{1}{4}. \\
Var(\hat{\mu}) &= \frac{1}{nI(\mu)} \\
&= \frac{4}{n}
\end{aligned}
$$

Hence, the 95% confidence interval is $\hat{\mu}\pm 1.96\sqrt{\frac{4}{n}} = 5\pm 1.96\frac{2}{\sqrt{n}}$.

Problem 11-1

$$
\begin{aligned}
s^2 &= \frac{1}{n-1}\sum_{i=1}^{n}(Y_i-\overline{Y})^2 \\
&= \frac{1}{n-1}\sum_{i=1}^{n}(Y_i^2-2Y_i\overline{Y}+\overline{Y}^2) \\
&= \frac{1}{n-1}\left(\sum_{i=1}^{n}Y_i^2-2\sum_{i=1}^{n}Y_i\overline{Y}+\sum_{i=1}^{n}\overline{Y}^2\right) \\
&= \frac{1}{n-1}\left(\sum_{i=1}^{n}Y_i^2-2\overline{Y}\frac{n}{n}\sum_{i=1}^{n}Y_i+n\overline{Y}^2\right) \\
&= \frac{1}{n-1}\left(\sum_{i=1}^{n}Y_i^2-2n\overline{Y}^2+n\overline{Y}^2\right) \\
&= \frac{1}{n-1}\left(\sum_{i=1}^{n}Y_i^2-n\overline{Y}^2\right) \\
&= \frac{n}{n-1}\left(\hat{\mu}_2-\hat{\mu}_1\right).
\end{aligned}
$$

Then, we want to find

$$
\begin{aligned}
\lim_{n\to\infty}P(|s^2-\sigma^2|>\epsilon) &= \lim_{n\to\infty}P(|s^2-(\mu_2-\mu_1)\,|>\epsilon) \\
&= \lim_{n\to\infty}P(|\frac{n}{n-1}\left(\hat{\mu}_2-\hat{\mu}_1\right)-(\mu_2-\mu_1)\,|>\epsilon) \\
&= \lim_{n\to\infty}P(|\left(\hat{\mu}_2-\hat{\mu}_1\right)-(\mu_2-\mu_1)\,|>\epsilon) \qquad (\because \lim_{n\to\infty}\frac{n}{n-1}=1) \\
&= \lim_{n\to\infty}P(|\left(\hat{\mu}_2-\mu_2\right)-(\hat{\mu}_1-\mu_1)\,|>\epsilon) \\
&= 0. \qquad (\textit{weak law of large number: } \hat{\mu}_2\to\mu_2, \hat{\mu}_1\to\mu_1)
\end{aligned}
$$

Problem 5.7.6

$$\begin{aligned}
\lim_{n\to\infty} P(|Y'_{n+1}-\mu|>\epsilon) &= \lim_{n\to\infty} P(|Y'_{n+1}-E[Y'_{n+1}]|>\epsilon) \\
&\le \frac{Var(Y'_{n+1})}{\epsilon^2} \qquad (\textit{Chebyshev's inequality}) \\
&= \frac{1}{8\left[f_Y(\mu;\mu)\right]^2 n}\cdot\frac{1}{\epsilon^2} \\
&= 0. \qquad (\lim_{n\to\infty}\frac{1}{8\left[f_Y(\mu;\mu)\right]^2 n}=0)
\end{aligned}$$

Problem 5.6.4

$$\begin{aligned}
L(\sigma^2) &= \prod_{i=1}^{n}\frac{1}{\sqrt{2\pi}\sigma}e^{\frac{-y^2}{2\sigma^2}} \\
&= \frac{1}{(2\pi)^{n/2}(\sigma^2)^{n/2}}e^{\frac{-1}{2\sigma^2}\sum_{i=1}^{n}y_i^2} \\
&= \frac{1}{(2\pi)^{n/2}(\sigma^2)^{n/2}}e^{\frac{-1}{2\sigma^2}\hat{\sigma}^2}.
\end{aligned}$$

Then,

$$\begin{aligned}
g(\hat{\sigma}^2;\sigma^2) &= \frac{1}{(2\pi)^{n/2}(\sigma^2)^{n/2}}e^{\frac{-1}{2\sigma^2}\hat{\sigma}^2}, \\
b(Y_1,\ldots,Y_n) &= 1.
\end{aligned}$$

Problem 5.6.8 $\hat{\theta}=max(Y_1,\ldots,Y_n)$ is sufficient for θ.

$$\begin{aligned}
L(\theta) &= \prod_{i=1}^{n}\frac{1}{\theta} \\
&= \frac{1}{\theta}\mathbb{I}_{\{0\le y_1\le\theta\}}\cdots\frac{1}{\theta}\mathbb{I}_{\{0\le y_n\le\theta\}} \qquad (\textit{let } \mathbb{I} \textit{ be an indicator function}) \\
&= \frac{1}{\theta^n}\mathbb{I}_{\{0\le max(y_i)\le\theta\}}\cdot\mathbb{I}_{\{0\le min(y_i)\le\theta\}}.
\end{aligned}$$

Then,

$$\begin{aligned}
g(\hat{\theta};\theta) &= \frac{1}{\theta^n}\mathbb{I}_{\{0\le max(y_i)\le\theta\}}, \\
b(Y_1,\ldots,Y_n) &= \mathbb{I}_{\{0\le min(y_i)\le\theta\}}.
\end{aligned}$$