Math 154 HW6

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Problem 1. Use the greedy algorithm with vertex ordering A, B, C, D, E, F to color the graph below. Does a coloring with fewer colors exist? Why or why not?

Solution.

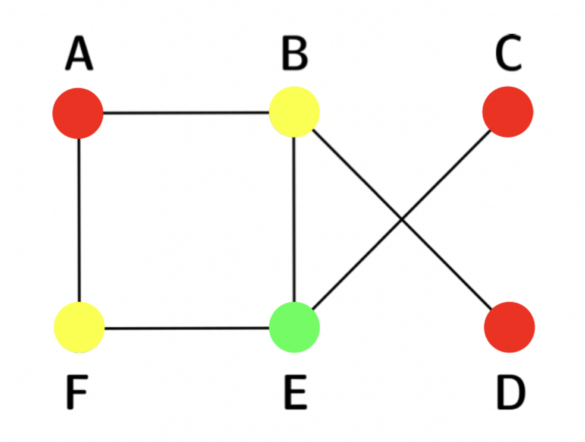

Figure 1: Coloring of the graph.

The greedy algorithm with vertex ordering A, B, C, D, E, F colors the graph with 3 colors. However, a coloring with fewer colors does exist. Notice that there is no odd cycle in the graph, so the graph is in fact bipartite. Hence, we can color the graph with 2 colors. $\square$

Problem 2. Let $\omega(G)$ be the maximum number of vertices in a complete subgraph of a graph G.

(a)

Proposition 1. *For every graph G, $\chi(G) \geq \omega(G)$.*

Proof. Let G' be the complete subgraph of G with $\omega(G)$ vertices. Then considering G' only, we know that $\chi(G') = \omega(G)$ by Brooks' theorem. Since G' is a subgraph of G, we know that $\chi(G') \leq \chi(G)$. Hence, $\chi(G) \geq \omega(G)$. $\square$

(b)

Proposition 2. *For every graph G, $\chi(G) \geq \frac{|V(G)|}{\alpha(G)}$.*

Proof. By definition, $\chi(G)$ is the minimum number of independent sets that partition $V(G)$. Let's denote $\{A_1, \ldots, A_{\chi(G)}\}$ to be the partition, in which $|A_k| \leq \alpha(G)$ $\forall k \in [1, \chi(G)]$. Hence, $\alpha(G)\chi(G) \geq \sum_{k=1}^{\chi(G)} A_k = |V(G)| \Leftrightarrow \chi(G) \geq \frac{|V(G)|}{\alpha(G)}$. $\square$

Problem 3. Suppose we have a computer program with 6 variables, as summarized in this table.

Variable	Steps used
a	$1-2$
b	$1-5$
c	$6-8$
d	$3-10$
e	$4-7$
f	$9-10$

(Here, a and b can't be stored in the same register, because both are used in Steps 1-2, but for example, a and c could be, since they're not used simultaneously.)

If we want to store each variable in a register, what is the minimum number of registers needed to run this program? (Note: you should convert this into a graph theory problem before solving it!)

Solution. The question can be framed as a graph theory problem concerned with the following graph G.

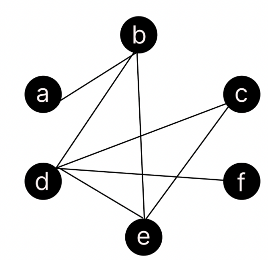

Figure 2: Graph G

The minimum number of registers needed to run this program is the chromatic number of G. Notice there is a triangle in G, so $\chi(G) \geq 3$. Now, notice that G is *2-degenerate*. Then, we can perform greedy coloring on G with the following ordering: a, f, b, c, d, e, which would take 3 colors. Therefore, $\chi(G) \leq 3$. Hence, the minimum number of registers needed to run this program is 3. □

Problem 4.

Proposition 3. *If m is the length of the longest path in a graph G, $\chi(G) \leq m+1$.*

Proof. Let v_1 be the vertex at the end of the longest path in G. Then, v_1 has at most m neighbors, otherwise contradiction and there exists a longer path by appending the extra neighbor to the path.

Now, we remove v_1 to get G'. Notice the longest path length in G' is at most m. Then, the end point, let v_2, of the longest path in G' has at most m neighbors. This is always true when we remove the end point of the longest path in G iteratively.

Therefore, we can form an arrangement of the vertices by iteratively removing the end point of the current longest path in the subgraph. Denote the arrangement as $(v_1, v_2, v_3, \ldots, v_n)$. Notice for each v_k, $k \in [1, n]$, v_k has at most m neighbors in the subgraph induced by $\{v_k, v_{k+1}, \ldots, v_n\}$. Then, we can perform greedy coloring starting from v_n, which would produce a coloring with at most $m+1$ colors.

Notice, there is a caveat that when we remove v_k from the subgraph, it may break into multiple components. However, this does not affect the proof, since we can just color the components separately, and as a matter of fact, it would make the coloring even easier. □