Math 154 HW7

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Problem 1. The complement $\overline{G}$ of a graph G is the graph on the same vertex set where $\{u, v\} \in E(\overline{G})$ if and only if $\{u, v\} \notin E(G)$. (In other words, to obtain the complement $\overline{G}$ of G, we fill in all the edges missing from G to form a complete graph, then delete the edges originally present in G.)

(a)

Proposition 1. *For every graph G on 11 or more vertices, at most one of G and $\overline{G}$ can be planar.*

Proof. Assume for the sake of contradiction that both G and $\overline{G}$ can be planar.

Let n be the number of vertices in G. Since $E(G)$ and $E(\overline{G})$ are disjoint, $E(G) \cup E(\overline{G}) = E(K_n)$. Assume without loss of generality that $|E(G)| \geq |E(\overline{G})|$. Then $|E(G)| \geq \frac{1}{2}|E(K_n)| = \frac{1}{4}n(n-1)$. Performing simple algebric operations, we get

$$\frac{1}{4}n(n-1) \leq |E(G)| \leq 3|V(G)| - 6$$
$$\frac{1}{2}n^2 - \frac{13}{2}n + 12 \leq 0.$$

The above inequality is false when $n \geq 11$. Hence, contradiction. □

(b) Give an example of a graph G on 11 or more vertices where both G and $\overline{G}$ are nonplanar.

Solution. Let G be a graph with 12 vertices and 33 arbitrary edgees. □

Problem 2. Determine whether the graph G below is planar or not planar. If it is planar, prove it by explicitly drawing a planar embedding of G. If it is not planar, use Kuratowski's Theorem: identify a subgraph that is a subdivision of K_5 or $K_{3,3}$, and draw G with that subgraph clearly highlighted.

Solution. The induced subgraph with vertices of all the boundary vertices (colored in blue, green, red) is a subdivision of $K_{3,3}$. Contract the yellow vertices and we will get a $K_{3,3}$ (blue and red are the two partitions). Hence, G is not planar. □

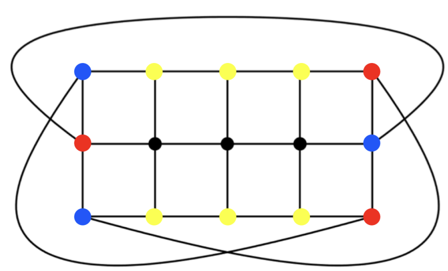

Figure 1: Graph G

Problem 3.

Proposition 2. *Every triangle-free planar graph is 4-colorable.*

Proof. Let n be the number of vertices in the graph. Let an arbitrary triangle-free planar graph G,

$$\begin{aligned}
&\frac{1}{2}\sum_{v\in V(G)} d(v) = |E(G)| \leq \frac{4}{4-2}(n-2) = 2n-4 \\
&\Rightarrow \frac{1}{2}\sum_{v\in V(G)} d(v) \leq 2n-4 \\
&\Rightarrow \sum_{v\in V(G)} d(v) \leq 4n-8 \\
&\Rightarrow \frac{\sum_{v\in V(G)} d(v)}{n} \leq 4-\frac{8}{n} \\
&\Rightarrow \textit{average degree} < 4 \\
&\Rightarrow \exists w \in V(G), d(w) \leq 3.
\end{aligned}$$

Since every subgraph of G is also triangle-free planar, every subgraph of G has a vertex of degree at most 3. Hence, G is 4-degenerate and thus 4-colorable. □