MATH 110 Notes

Book: Linear Algebra Done Right

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Contents

Chapter 1

Isometry

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Definition 1.1: Isometry and Unitary

$T \in \mathcal{L}(V, w)$ is called an isometry if $\|T\vec{v}\| = \|\vec{v}\|$ for all $\vec{v} \in V$.

Note:

In case $T \in \mathcal{L}(V)$ is an *invertible* isometry, it is called *unitary*.

Note:

If $T \in \mathcal{L}(v)$ is unitary, then any eigen-pair $(\lambda, \vec{v})$ of T satifies

$$\|T\vec{v}\| = \|\lambda\vec{v}\| = |\lambda|\|\vec{v}\| = \|\vec{v}\|.$$

So $|\lambda| = 1$.

Proposition 1.1 Isometry only has eigenvalues of modulus 1.

If T is an isometry, then

$$\|T\vec{e}_j\| = \|\vec{e}_j\| = 1,$$

but also

$$\|T\vec{e}_j\| = \|s_j \cdot \vec{f}_j\| = s_j = 1.$$

Proposition 1.2 Isometry is injective.

Proof: Recall $\operatorname{null} T^*T = \operatorname{null} T$. So if $\vec{v} \in \operatorname{null} T \implies T\vec{v} = \vec{0}$, then

$$\|\vec{v}\| = \|T\vec{v}\| = \|\vec{0}\| = 0 \implies \vec{v} = \vec{0}.$$

This also shows isometries do not have zero singular values. 😊

Definition 1.2: Jordan Normal Form

Note:

The goal of Jordan Normal Form is to get a matrix representation of T that is as simple (diagonal) as possible. This way, we get sparse matrices that are easy to compute with or understand.

Setting: Let V be a finite-dimensional vector space over $\mathbb{C}$ and $T \in \mathcal{L}(V)$.

First, represent $V = U \oplus W$ where U, W are invariant subspaces of T, and T is nilpotent on U and invertible on W. Here nilpotent on U means $T^k\vec{v} = \vec{0}$ for all $\vec{v} \in U$ and some $k \in \mathbb{N}$. We already get a block diagonal matrix representation of T when writing $\vec{v} = \vec{u} + \vec{w}$ for $\vec{v} \in V$, $\vec{u} \in U$, and $\vec{w} \in W$.

$\{0\} \subseteq \operatorname{null} T \subseteq \operatorname{null} T^2 \subseteq \cdots$ is a chain of subspaces of V. Since V is finite-dimensional, there exists $k \in \mathbb{N}$ such that $\operatorname{null} T^k = \operatorname{null} T^{k+1}$. In other words, $\operatorname{null} T^k$ will stabilize at some point.

Take $U = \operatorname{null} T^k$ and $W = \operatorname{range} T^k$ where $\operatorname{null} T^k$ stabilize. Notice, by the rank-nullity theorem, $\dim U + \dim W = \dim V$. If $\vec{x} \in U \cap W$, i.e., $\vec{x} = T^k\vec{v}$ for some $\vec{v} \in V$, then $T^k\vec{x} = T^{2k}\vec{v} = \vec{0}$, so $\vec{v} \in \operatorname{null} T^{2k} = \operatorname{null} T^k$. Thus, $\vec{x} = T^k\vec{v} = \vec{0}$.

Within the subspace U, $T^k = 0$. Witout loss of generality, U is our whole space. Then there exists $\vec{u_0} \in U$ such that $T^{k-1}\vec{u_0} \neq 0$ but $T^k\vec{u_0} = 0$. Then, there exists $v_0 \in V$ such that $\langle T^{k-1}\vec{v_0}, v_0\rangle \neq 0$. Consider the matrix

$$\langle T^{j-1}\vec{u_0}, T^{*^{k-i}}\vec{v_0}\rangle \text{ for } i, j = 1, 2, \ldots, k,$$

which is an invertible matrix.

This implies that $\operatorname{span}(\vec{u_0}, T\vec{u_0}, \ldots, T^{k-1}\vec{u_0}) \oplus \operatorname{span}(\vec{v_0}, T^*\vec{v_0}, \ldots, T^{*^{k-1}}\vec{v_0})^\perp = U$. Indeed, if $\vec{v}$ is in the intersection of the two subspaces, this creates a linear system precisely with the matrix above, which is invertible. Thus, $\vec{v} = \vec{0}$.

Then, represent U as a direct sum of generalized eigenspaces of T.