

# Math 181A HW4

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## Problem 5.4.2

(a)

$$\begin{aligned}P(2.8 \leq \hat{\theta} \leq 3.2) &= P(\hat{\theta} \geq 2.8) = 1 - P(\{Y_1, \dots, Y_6\} \leq 2.8) \\&= 1 - P(Y_1 \leq 2.8)^6 \\&= 1 - \left(\frac{2.8}{3}\right)^6 \\&\approx 0.339.\end{aligned}$$

(b)

$$\begin{aligned}P(2.8 \leq \hat{\theta} \leq 3.2) &= P(\hat{\theta} \leq 2.8) = 1 - P(\{Y_1, \dots, Y_3\} \leq 2.8) \\&= 1 - P(Y_1 \leq 2.8)^3 \\&= 1 - \left(\frac{2.8}{3}\right)^3 \\&\approx 0.187.\end{aligned}$$

## Problem 5.4.15

$$\begin{aligned}E[\overline{W}^2] &= Var(\overline{W}) + (E[\overline{W}])^2 \\&= Var\left(\frac{1}{n} \sum_{i=1}^n W_i\right) + \left(E\left[\frac{1}{n} \sum_{i=1}^n W_i\right]\right)^2 \\&= \frac{1}{n^2} \sum_{i=1}^n Var(W_i) + \left(\frac{1}{n} \sum_{i=1}^n E[W_i]\right)^2 \\&= \frac{n\sigma^2}{n^2} + \left(\frac{n\mu}{n}\right)^2 \\&= \frac{\sigma^2}{n} + \mu^2, \\ \lim_{n \rightarrow \infty} E[\overline{W}^2] &= \lim_{n \rightarrow \infty} \frac{\sigma^2}{n} + \mu^2 \\&= \mu^2.\end{aligned}$$

**Problem 5.4.18** By symmetry,  $Var(Y_{min}) = Var(Y_{max})$ , let  $\sigma^2$ . Hence,

$$\begin{aligned}
 MSE(\hat{\theta}_1) &= Var(\hat{\theta}_1) + Bias(\hat{\theta}_1) \\
 &= Var(\hat{\theta}_1) \\
 &= \frac{36}{25}\sigma^2. \\
 MSE(\hat{\theta}_2) &= Var(\hat{\theta}_2) + Bias(\hat{\theta}_2) \\
 &= Var(\hat{\theta}_2) \\
 &= 36\sigma^2 \\
 &> MSE(\hat{\theta}_1).
 \end{aligned}$$

Therefore,  $\hat{\theta}_1$  is better than  $\hat{\theta}_2$ .

**Problem 5.4.21** With the same notation from last question, then

$$\begin{aligned}
 \frac{Var(\hat{\theta}_1)}{Var(\hat{\theta}_2)} &= \frac{(n+1)^2\sigma^2}{\left(\frac{n+1}{n}\right)^2\sigma^2} \\
 &= n^2.
 \end{aligned}$$

**Problem 5.5.2**

$$\begin{aligned} l(\lambda) &= \log\left(\frac{e^{-\lambda}\lambda^k}{k!}\right) \\ &= -\lambda + k \log(\lambda) - \log(k!), \end{aligned}$$

$$l'(\lambda) = -1 + \frac{k}{\lambda},$$

$$l''(\lambda) = -\frac{k}{\lambda^2}.$$

$$\begin{aligned} I(\lambda) &= -E[l''(\lambda)] \\ &= -E\left[\frac{-k}{\lambda^2}\right] \\ &= \sum_{k=0}^{\infty} \frac{k}{\lambda^2} \cdot e^{-\lambda} \frac{\lambda^k}{k!} \\ &= \frac{e^{-\lambda}}{\lambda} \sum_{k=0}^{\infty} \frac{\lambda^{k-1}}{(k-1)!} \\ &= \frac{e^{-\lambda}}{\lambda} \left( \frac{0 \cdot \lambda^{k-1}}{0!} + \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!} \right) \\ &= \frac{e^{-\lambda}}{\lambda} \left( \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \right) \\ &= \frac{e^{-\lambda}}{\lambda} \cdot e^{\lambda} \\ &= \frac{1}{\lambda}. \end{aligned}$$

$$\begin{aligned} \text{Var}(\hat{\lambda}) &= \text{Var}\left(\frac{1}{n} \sum_{i=1}^n X_i\right) \\ &= \frac{1}{n^2} \text{Var}\left(\sum_{i=1}^n X_i\right) \\ &= \frac{1}{n} \text{Var}(X_i) \\ &= \frac{1}{n} \lambda \\ &= \frac{1}{n(1/\lambda)} \\ &= \frac{1}{nI(\lambda)}. \end{aligned}$$