STAT 153 sketch

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Proof of null(X) contains at least one non-zero vector η:

Since $p > n$, the column vectors are linear dependent. Denote $(v_1, \cdots, v_p)$ as the column vectors of X. Then, there are non trivial coefficients $(c_1, \cdots, c_p)$ such that

$$\sum_{i=1}^{p} c_i v_i = 0.$$

Hence, $\eta = (c_1, \cdots, c_p)$ is a non-zero vector in null(X).

Proof of $\hat{\beta} = \tilde{\beta} + \eta$ is also a least squares solution for $\eta \in \text{null}(X)$:

Denote the prediction from $\tilde{\beta}$ as $\tilde{y} = X\tilde{\beta}$ with MSE $= y - \tilde{y}$. Then the prediction from $\hat{\beta}$

$$\begin{aligned}
\hat{y} &= X\hat{y} && (1)\\
&= X(\tilde{y} + \eta) && (2)\\
&= X\tilde{y} + X\eta && (3)\\
&= \tilde{y} + X\eta && (4)\\
&= \tilde{y}. && (5)
\end{aligned}$$

Therefore, they have the same MSE. Since $\hat{\beta}$ is a least squares solution, $\tilde{\beta} + \eta$ is also a least squares solution.