

# Math 109 HW2

Neo Lee

02/01/2023

(1)

**Proposition 1.**  $n^2 + n$  is even for all  $n \in \mathbb{N}$ .

*Proof.* Note that all natural numbers  $n$ ,  $n$  is either even or odd. If  $n$  is even, it can be written as  $n = 2k$  for some positive integer  $k$ . If  $n$  is odd, it can be written as  $n = 2k + 1$  for some whole number  $k$ .

Let  $n$  be even,  $n^2 + n = (2k)^2 + 2k = 4k^2 + 2k = 2(2k^2 + k)$ , which is divisible by 2. Therefore,  $n^2 + n$  is even if  $n$  is even.

Let  $n$  be odd,  $n^2 + n = (2k + 1)^2 + (2k + 1) = (4k^2 + 4k + 1) + (2k + 1) = 2(2k^2 + 3k + 1)$ , which is also divisible by 2. Therefore,  $n^2 + n$  is even if  $n$  is odd.

Hence,  $n^2 + n$  is always even for  $n \in \mathbb{N}$ . □

(2)

**Proposition 2.**  $\sqrt{xy} \leq \frac{x+y}{2}$  if  $x, y \geq 0$  are real numbers.

*Proof.*

$$\sqrt{xy} \leq \frac{x+y}{2} \Leftrightarrow 2\sqrt{xy} \leq x+y \tag{1}$$

$$\Leftrightarrow 4xy \leq (x+y)^2 \tag{2}$$

$$\Leftrightarrow 4xy \leq x^2 + 2xy + y^2 \tag{3}$$

$$\Leftrightarrow 0 \leq x^2 - 2xy + y^2 \tag{4}$$

$$\Leftrightarrow 0 \leq (x-y)^2 \tag{5}$$

$$\Leftrightarrow 0 \leq x, y \ (x, y \in \mathbb{R}) \tag{6}$$

Hence, for all real numbers  $x, y \geq 0$ ,  $(x-y)^2 \geq 0 \Rightarrow \sqrt{xy} \leq \frac{x+y}{2}$ . □

(3)

**Proposition 3.** For all real numbers  $x > 2$ ,  $\frac{x+1}{x-1} < \frac{x+2}{x-2}$ .

*Proof.*

$$\frac{x+1}{x-1} < \frac{x+2}{x-2} \Leftrightarrow (x-2)(x+1) < (x-1)(x+2) \tag{7}$$

$$\Leftrightarrow x^2 - x - 2 < x^2 + x - 2 \tag{8}$$

$$\Leftrightarrow 0 < 2x \tag{9}$$

$$\Leftrightarrow 4 < 2x \tag{10}$$

$$\Leftrightarrow 2 < x \ (x \in \mathbb{R}) \tag{11}$$

□

(4)

**Proposition 4.** Let  $n \geq 2$  be a natural number. Let  $k$  be the maximum integer such that  $2^k \leq n$ . Among the numbers  $1, \dots, n$ , the number  $2^k$  is the only one which is divisible by  $2^k$ .

*Proof.* Assume to the contrary that other than  $2^k$ , there exists  $i$  such that  $2^k | i$ , for which  $1 \leq i \leq n$  and  $i \in \mathbb{N}$ . Since  $2^k | i$ ,  $i$  can be written as  $i = 2^k \cdot b = 2^k \cdot (2 + b - 2) = 2^{k+1} + (b - 2)2^k$  for some positive integer  $b \geq 2$ . Note that  $b \geq 2 \Rightarrow b - 2 \geq 0 \Rightarrow i = 2^{k+1} + (b - 2)2^k \geq 2^{k+1}$ , which contradicts that  $k$  is the maximum integer such that  $2^k \leq n$ . Hence,  $2^k$  is the only number that is divisible by  $2^k$  within  $[1, n]$ .

The claim would not be true if  $2^k$  is replaced by  $3^k$ . For example, for  $n = 26$ , the greatest  $k$  such that  $3^k \leq n$  is 2. In this example, 18 is divisible by  $3^2 = 9$ .  $\square$

(5)

**Proposition 5.**  $\sum_{k=1}^{2^n} \frac{1}{k} \geq 1 + \frac{n}{2}$  for  $n \in \mathbb{N}$ .

*Proof.* Proving  $P(n) : \sum_{k=1}^{2^n} \frac{1}{k} \geq 1 + \frac{n}{2}$  for  $n \in \mathbb{N}$  by induction.

Base case:

$$P(1) : \sum_{k=1}^{2^1} \frac{1}{k} = \frac{1}{1} + \frac{1}{2} \quad (12)$$

$$= \frac{3}{2} \quad (13)$$

$$\geq 1 + \frac{1}{2}. \quad (14)$$

Thus,  $P(n)$  is true for  $n = 1$ .

Induction step: assuming  $P(m)$  is true,

$$P(m+1) : \sum_{k=1}^{2^{m+1}} \frac{1}{k} = \sum_{k=1}^{2^m} \frac{1}{k} + \sum_{k=2^m+1}^{2^{m+1}} \frac{1}{k} \quad (15)$$

$$\geq 1 + \frac{m}{2} + (2^{m+1} - (2^m + 1) + 1) \frac{1}{2^{m+1}} \quad (16)$$

$$\geq 1 + \frac{m}{2} + (2^{m+1} - \frac{2^{m+1}}{2}) \frac{1}{2^{m+1}} \quad (17)$$

$$\geq 1 + \frac{m}{2} + (\frac{1}{2} \cdot 2^{m+1}) \frac{1}{2^{m+1}} \quad (18)$$

$$\geq 1 + \frac{m}{2} + \frac{1}{2} \quad (19)$$

$$\geq 1 + \frac{m+1}{2}. \quad (20)$$

Therefore,  $P(m+1)$  is true.

By Mathematical Induction,  $P(n)$  is true for  $n \in \mathbb{N}$ .  $\square$

(6)

**Proposition 6.**  $3 | 4^n + 5$  for  $n \in \mathbb{Z}^+$ .

*Proof.* Proving  $P(n) : 3|4^n + 5$  for  $n \in \mathbb{Z}^+$  by induction.

Base case:

$$P(1) : 4^1 + 5 = 9 \quad (21)$$

$$= 3 \cdot 3. \quad (22)$$

Hence,  $4^1 + 5$  is divisible by 3 and  $P(n)$  is true for  $n = 1$ .

Induction step: assuming  $P(m)$  is true, which means  $4^m + 5 = 3 \cdot b$  for  $b \in \mathbb{Z}^+$ ,

$$P(m+1) : 4^{m+1} + 5 = 4 \cdot 4^m + 5 \quad (23)$$

$$= 3 \cdot 4^m + 4^m + 5 \quad (24)$$

$$= 3 \cdot 4^m + 3 \cdot b \quad (25)$$

$$= 3(4^m + b). \quad (26)$$

Since  $4^m + b$  is an integer,  $4^{m+1} + 5$  is divisible by 3 and  $P(m+1)$  is true.

By Mathematical Induction,  $P(n)$  is true for  $n \in \mathbb{Z}^+$ . □

(7)

**Proposition 7.**  $\sum_{i=1}^n \frac{1}{i(i+1)} = \frac{n}{n+1}$  for  $n \in \mathbb{Z}^+$ .

*Proof.* Proving  $P(n) : \sum_{i=1}^n \frac{1}{i(i+1)} = \frac{n}{n+1}$  for  $n \in \mathbb{Z}^+$  by induction.

Base case:

$$P(1) : \sum_{i=1}^1 \frac{1}{i(i+1)} = \frac{1}{1(1+1)} \quad (27)$$

$$= \frac{1}{2} \quad (28)$$

$$= \frac{1}{1+1}. \quad (29)$$

Hence,  $P(n)$  is true for  $n = 1$ .

Induction step: assuming  $P(m)$  is true,

$$P(m+1) : \sum_{i=1}^{m+1} \frac{1}{i(i+1)} = \sum_{i=1}^m \frac{1}{i(i+1)} + \frac{1}{(m+1)(m+2)} \quad (30)$$

$$= \frac{m}{m+1} + \frac{1}{(m+1)(m+2)} \quad (31)$$

$$= \frac{m(m+2) + 1}{(m+1)(m+2)} \quad (32)$$

$$= \frac{m^2 + 2m + 1}{(m+1)(m+2)} \quad (33)$$

$$= \frac{(m+1)^2}{(m+1)(m+2)} \quad (34)$$

$$= \frac{m+1}{m+2} \quad (35)$$

$$= \frac{m+1}{(m+1)+1}. \quad (36)$$

Thus,  $P(m+1)$  is true.

By Mathematical Induction,  $P(n)$  is true for  $n \in \mathbb{Z}^+$ . □

(8)

**Proposition 8.**  $\prod_{i=2}^n \left(1 - \frac{1}{i^2}\right) = \frac{n+1}{2n}$  for integers  $n \geq 2$ .

*Proof.* Proving  $P(n) : \prod_{i=2}^n \left(1 - \frac{1}{i^2}\right)$  for integers  $n \geq 2$  by induction.

Base case:

$$P(2) : \prod_{i=2}^2 \left(1 - \frac{1}{i^2}\right) = 1 - \frac{1}{2^2} \quad (37)$$

$$= 1 - \frac{1}{4} \quad (38)$$

$$= \frac{3}{4} \quad (39)$$

$$= \frac{2+1}{2 \cdot 2} \quad (40)$$

Hence,  $P(n)$  is true for  $n = 2$ .

Induction step: assuming  $P(m)$  is true,

$$P(m+1) : \prod_{i=2}^{m+1} \left(1 - \frac{1}{i^2}\right) = \prod_{i=2}^m \left(1 - \frac{1}{i^2}\right) \cdot \left(1 - \frac{1}{(m+1)^2}\right) \quad (41)$$

$$= \frac{m+1}{2m} \cdot \left(1 - \frac{1}{(m+1)^2}\right) \quad (42)$$

$$= \frac{m+1}{2m} - \frac{1}{(2m)(m+1)} \quad (43)$$

$$= \frac{(m+1)^2 - 1}{(2m)(m+1)} \quad (44)$$

$$= \frac{m^2 + 2m}{(2m)(m+1)} \quad (45)$$

$$= \frac{m(m+2)}{(2m)(m+1)} \quad (46)$$

$$= \frac{m+2}{2m+2} \quad (47)$$

$$= \frac{(m+1)+1}{2(m+1)} \quad (48)$$

Thus,  $P(m+1)$  is true.

By Mathematical Induction,  $P(n)$  is true for interger  $n \geq 2$ . □