

# Probability Distribution Properties

## 1 Discrete Distributions

### 1.1 Uniform Random Variable

$$E[X] = \frac{x_1 + \cdots + x_n}{n}; \quad Var(X) = \frac{x_1^2 + \cdots + x_n^2}{n} - \left( \frac{x_1 + \cdots + x_n}{n} \right)^2.$$
$$E[X] = \frac{n+1}{2}; \quad Var(X) = \frac{n^2-1}{12}; \quad \text{only when } x_i \in [1, 2, \dots, n].$$

### 1.2 Bernoulli Random Variable

$$I = \begin{cases} 1 & \text{if } X = 1, \\ 0 & \text{if } X = 0. \end{cases}$$
$$E[I] = p; \quad Var(I) = p(1-p).$$

### 1.3 Binomial Random Variable

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k} \quad \text{for } k \in [0, n].$$
$$E[X] = np; \quad Var(X) = np(1-p).$$

### 1.4 Poisson Random Variable

$$P(X = k) = \frac{\lambda^k e^{-\lambda}}{k!} \quad \text{for } k \in \mathbb{Z}^{\geq}.$$
$$E[X] = \lambda; \quad Var(X) = \lambda.$$

### 1.5 Geometric Random Variable

$$P(X = n) = (1-p)^{n-1} p \quad \text{for } n \in \mathbb{N}.$$
$$E[X] = \frac{1}{p}; \quad Var(X) = \frac{1-p}{p^2}; \quad P(X > n) = (1-p)^n; \quad P(X > n+k | X > k) = P(X > n).$$

## 2 Continuous Distributions

### 2.1 Uniform Random Variable

$$f(x) = \begin{cases} \frac{1}{\beta - \alpha} & \text{if } x \in [\alpha, \beta], \\ 0 & \text{otherwise.} \end{cases}$$
$$E[X] = \frac{\alpha + \beta}{2}; \quad \text{Var}(X) = \frac{(\beta - \alpha)^2}{12}.$$

### 2.2 Exponential Random Variable

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$
$$E[X] = \frac{1}{\lambda}; \quad \text{Var}(X) = \frac{1}{\lambda^2}; \quad P(X \geq x) = e^{-\lambda x}; \quad P(X \geq x + y | X \geq y) = e^{-\lambda x}.$$

### 2.3 Normal Random Variable

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \quad \text{for } x \in \mathbb{R}.$$
$$E[X] = \mu; \quad \text{Var}(X) = \sigma^2; \quad Z \sim N(0, 1) = \frac{X - \mu}{\sigma}.$$

### 2.4 Gamma Random Variable

$$f(x) = \frac{\lambda}{\Gamma(\alpha)} (\lambda x)^{\alpha-1} e^{-\lambda x} \quad \text{for } \alpha, \lambda \in \mathbb{R}^+, x \geq 0.$$
$$\Gamma(\alpha) = \int_0^\infty u^{\alpha-1} e^{-u} du; \quad \Gamma(\alpha) = (\alpha - 1)! \quad \text{if } \alpha \in \mathbb{Z}^+.$$
$$E[X] = \frac{\alpha}{\lambda}; \quad \text{Var}(X) = \frac{\alpha}{\lambda^2}.$$

### 3 Multivariate Distributions

#### 3.1 Multivariate Normal Distribution

$$f(\vec{x}) = \frac{1}{(2\pi)^{n/2} |\Sigma|^{1/2}} e^{-\frac{1}{2}(\vec{x}-\vec{\mu})^\top \Sigma^{-1}(\vec{x}-\vec{\mu})} \quad \text{for } x \in \mathbb{R}^n.$$

$$E[X] = \vec{\mu}; \quad \text{Var}(X) = \Sigma.$$

Let random vector  $\begin{bmatrix} X_1 \\ \vdots \\ X_n \end{bmatrix}$  have a multivariate normal distribution and  $Z$  be the linear combination of the

random vector with coefficients  $\vec{A} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$ . Then,  $Z \sim N(\vec{A} \cdot \vec{\mu}, \vec{A}^\top \Sigma \vec{A})$ . For bivariate random vector  $\begin{bmatrix} X \\ Y \end{bmatrix}$

with  $\vec{A} = \begin{bmatrix} a \\ b \end{bmatrix}$ ,

$$\text{Var}(Z) = a^2 \sigma_X^2 + 2ab\rho\sigma_X\sigma_Y + b^2 \sigma_Y^2.$$

#### 3.2 Multinomial Distribution

$$P(\vec{x}) = \frac{n!}{x_1! \cdots x_k!} p_1^{x_1} \cdots p_k^{x_k} \quad \text{for } \vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_k \end{bmatrix}, \quad x_i \in \mathbb{Z}^{\geq}, \quad \sum_{i=1}^k x_i = n, \quad p_i \in [0, 1].$$

$$E[\vec{x}] = n\vec{p}; \quad \text{Var}(\vec{x}) = \overrightarrow{np(1-p)}.$$

#### 3.3 Random Sums

$$X = \xi_1 + \cdots + \xi_N; \quad \text{where } N \text{ is a R.V. and } \xi_i \text{ are i.i.d. R.Vs.}$$

$$E[X] = E[\xi]E[N]; \quad \text{Var}(X) = E[N]\text{Var}(\xi) + E[\xi]^2\text{Var}(N).$$

## 4 Theorems

### 4.1 Chebyshev's Inequality

$$P(|X - E[X]| \geq k) \leq \frac{Var(X)}{k^2}.$$

### 4.2 Markov Inequality

For a non-negative R.V.  $X$ ,

$$P(X \geq k) \leq \frac{E[X]}{k}.$$

### 4.3 Central Limit Theorem

Let  $S_n = X_1 + \cdots + X_n$ . When  $n$  is large,  $S_n$  is approximately normally distributed with mean  $n\mu$  and variance  $n\sigma^2$ .

$$\frac{S_n - n\mu}{\sqrt{n}\sigma} = \frac{\bar{S}_n - \mu}{\sigma/\sqrt{n}} \sim N(0,1) \quad \text{as } n \rightarrow \infty.$$

### 4.4 Weak Law of Large Numbers

Let  $S_n = X_1 + \cdots + X_n$ .

$$\lim_{n \rightarrow \infty} P(|\bar{S}_n - \mu| < \epsilon) = 1.$$

### 4.5 Strong Law of Large Numbers

Let  $S_n = X_1 + \cdots + X_n$ .

$$P(\lim_{n \rightarrow \infty} \bar{S}_n = \mu) = 1.$$

### 4.6 Moment Generating Function

$$M_X(t) = E[e^{tX}] = \int_{-\infty}^{\infty} e^{tx} f(x) dx.$$

## 5 Special Markov Chains

### 5.1 Two State Markov Chain

$$P = \begin{matrix} & \begin{matrix} 0 & 1 \end{matrix} \\ \begin{matrix} 0 \\ 1 \end{matrix} & \left\| \begin{array}{cc} 1-a & a \\ b & 1-b \end{array} \right\| \end{matrix}; \text{ state } 0, 1 \text{ are independent R.V. iff } a = 1 - b.$$

$$P^n = \frac{1}{a+b} \begin{bmatrix} b & a \\ b & a \end{bmatrix} + \frac{(1-a-b)^n}{a+b} \begin{bmatrix} a & -a \\ -b & b \end{bmatrix}; \quad \lim_{n \rightarrow \infty} P^n = \begin{matrix} & \begin{matrix} 0 & 1 \end{matrix} \\ \begin{matrix} 0 \\ 1 \end{matrix} & \left\| \begin{array}{cc} \frac{b}{a+b} & \frac{a}{a+b} \\ \frac{b}{a+b} & \frac{a}{a+b} \end{array} \right\| \end{matrix}.$$

### 5.2 i.i.d. Markov Chain

$$\xi_1, \dots, \xi_n \text{ are i.i.d. R.V.s.; } P(\xi = k) = a_k \text{ for } k \in \mathbb{Z}^+.$$

#### 5.2.1 Independent Random Variables

$$X_n = \xi; P = \left\| \begin{array}{cccc} a_0 & a_1 & a_2 & \cdots \\ a_0 & a_1 & a_2 & \cdots \\ a_0 & a_1 & a_2 & \cdots \\ a_0 & a_1 & a_2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{array} \right\|.$$

#### 5.2.2 Successive Maxima

$$X_n = \max(\xi_1, \dots, \xi_n) \text{ for } n \in \mathbb{N}.$$

$$P = \left\| \begin{array}{cccccc} A_0 & a_1 & a_2 & a_3 & \cdots \\ 0 & A_1 & a_2 & a_3 & \cdots \\ 0 & 0 & A_2 & a_3 & \cdots \\ 0 & 0 & 0 & A_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array} \right\| \text{ where } A_k = P(X_{n+1} \leq k | X_n = k) = P(\xi_{n+1} \leq k).$$

Let  $T$  be the number of trials needed for  $X_n = \max(\xi_1, \dots, \xi_n) \geq M$  for an arbitrary value  $M$ ,

$$E[T] = \frac{1}{P(\xi \geq M)}.$$

*Successive Maxima* can be interpreted as a *Geometric*( $p$ ) random variable with  $p = P(\xi \geq M)$ .

### 5.2.3 Partial Sums

$$X_n = \xi_1 + \cdots + \xi_n \text{ for } n \in \mathbb{N}.$$

$$P = \left\| \begin{array}{ccccc} a_0 & a_1 & a_2 & a_3 & \cdots \\ 0 & a_0 & a_1 & a_2 & \cdots \\ 0 & 0 & a_0 & a_1 & \cdots \\ 0 & 0 & 0 & a_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array} \right\| \text{ if } P(\xi = k) = a_k \text{ for } k \in \mathbb{Z}^{\geq}.$$

$$P = \left\| \begin{array}{ccccc} \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdots & a_{-1} & a_0 & a_1 & a_2 & \cdots \\ \cdots & a_{-2} & a_{-1} & a_0 & a_1 & \cdots \\ \cdots & a_{-3} & a_{-2} & a_{-1} & a_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array} \right\| \text{ if } P(\xi = k) = a_k \text{ for } k \in \mathbb{Z}.$$

### 5.2.4 Branching Processes

$$X_{n+1} = \xi_1^{(n)} + \cdots + \xi_{X_n}^{(n)} \text{ for } n \in \mathbb{N}; \quad X_0 = 1.$$

*u<sub>n</sub> = probability of extinction by n-th generation*

$$= \sum_{k=0}^{\infty} P(\xi = k)(u_{n-1})^k \text{ for } n \in \mathbb{N};$$

$$u_0 = 0.$$

From random sums, we know

$$E[X_{n+1}] = E[\xi]E[X_n]; \quad \text{Var}(X_{n+1}) = \text{Var}(\xi)E[X_n] + E[\xi]^2 \text{Var}(X_n).$$

After derivation,

$$E[X_n] = E[\xi]^n; \quad \text{Var}(X_n) = \text{Var}(\xi)E[\xi]^{n-1} \times \begin{cases} n & \text{if } E[\xi] = 1, \\ \frac{1-E[\xi]^n}{1-E[\xi]} & \text{otherwise.} \end{cases}$$

## 5.3 One-dimensional Random Walk

$$P = \begin{array}{c} \begin{array}{ccccc} & 0 & 1 & 2 & 3 & \cdots \\ 0 & r_0 & p_0 & 0 & 0 & \cdots \\ 1 & q_1 & r_1 & p_1 & 0 & \cdots \\ 2 & 0 & q_2 & r_2 & p_2 & \cdots \\ 3 & 0 & 0 & q_3 & r_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array} \end{array} \left\| \begin{array}{c} \\ \\ \\ \\ \\ \end{array} \right\|.$$

### 5.3.1 Simple Random Walk

$$r_i = r = 0; \quad q_i = q = \frac{1}{2}; \quad p_i = p = \frac{1}{2}.$$

### 5.3.2 Gambler's Ruin

$$P = \begin{array}{c} 0 \\ 1 \\ 2 \\ \vdots \\ N-1 \\ N \end{array} \left\| \begin{array}{cccccc} 0 & 1 & 2 & 3 & \cdots & N-1 & N \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ q & 0 & p & 0 & \cdots & 0 & 0 \\ 0 & q & 0 & p & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & p \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \end{array} \right\|.$$

Let  $u_i$  be the probability of  $X_n$  reaching state 0 before state  $N$  given current state  $i$ .

$$u_i = \begin{cases} \frac{N-i}{N} & \text{if } p = q = \frac{1}{2}, \\ \frac{(q/p)^i - (q/p)^N}{1 - (q/p)^N} & \text{if } p \neq q. \end{cases}$$

$$\lim_{N \rightarrow \infty} u_i = \begin{cases} 1 & \text{if } p \leq q, \\ \left(\frac{q}{p}\right)^i & \text{if } p > q. \end{cases}$$

Let  $T$  be the time needed to reach state 0.

$$E[T] = i(N-i) \text{ if } p = q = \frac{1}{2}.$$

### 5.4 Success Runs

$$P = \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ \vdots \end{array} \left\| \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & \cdots \\ p_0 & q_0 & 0 & 0 & 0 & \cdots \\ p_1 & r_1 & q_1 & 0 & 0 & \cdots \\ p_2 & 0 & r_2 & q_2 & 0 & \cdots \\ p_3 & 0 & 0 & r_3 & q_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array} \right\|.$$

#### 5.4.1 Two Outcomes Repeated Trials

$$r_i = r = 0; \quad q_i = q = \frac{1}{2}; \quad p_i = p = \frac{1}{2}.$$

#### 5.4.2 Renewal Process

Model a light bulb's lifetime as a random variable  $\xi$ , where

$$P(\xi = k) = a_k \text{ for } k \in \mathbb{N},$$

and  $X_n$  as the age of the bulb in service at time  $n$ .

$$r_k = 0; \quad p_k = \frac{a_{k+1}}{a_{k+1} + a_{k+2} + \cdots}; \quad q_k = 1 - p_k \text{ for } k \in \mathbb{Z}^{\geq}.$$

## 6 Estimators

### 6.1 Unbiasedness

$$Bias(\hat{\theta}) = E[\hat{\theta}] - \theta.$$

### 6.2 Efficiency

Let  $\hat{\theta}_1$  and  $\hat{\theta}_2$  be two unbiased estimators of  $\theta$ .  $\hat{\theta}_1$  is more efficient than  $\hat{\theta}_2$  if

$$Var(\hat{\theta}_1) < Var(\hat{\theta}_2).$$

The relative efficiency of  $\hat{\theta}_1$  to  $\hat{\theta}_2$  is

$$\frac{Var(\hat{\theta}_2)}{Var(\hat{\theta}_1)}.$$

### 6.3 Consistency

$\hat{\theta}$  is a consistent estimator of  $\theta$  if

$$\lim_{n \rightarrow \infty} P(|\hat{\theta} - \theta| \geq \epsilon) = 0.$$

### 6.4 Sufficiency

An estimator  $\hat{\theta}$  is sufficient for  $\theta$  if the conditional distribution of  $X_1, \dots, X_n$  given  $\hat{\theta}$  does not depend on  $\theta$ ; equivalently  $\hat{\theta}$  is sufficient for  $\theta$  if the likelihood function  $L(\theta) = f(X; \theta)$  can be factorized as

$$L(\theta) = g(\hat{\theta}; \theta)b(X_1, \dots, X_n).$$

### 6.5 Mean Squared Error (MSE)

$$MSE(\hat{\theta}) = E[(\hat{\theta} - \theta)^2] = Var(\hat{\theta}) + Bias(\hat{\theta})^2.$$

### 6.6 Minimum Variance Unbiased Estimator

#### 6.6.1 Fishers Information

$$I(\theta) = E \left[ \left( \frac{\partial}{\partial \theta} \log f(X; \theta) \right)^2 \right] = -E \left[ \frac{\partial^2}{\partial \theta^2} \log f(X; \theta) \right].$$

#### 6.6.2 Cramer-Rao Lower Bound

Let  $\hat{\theta}$  be an unbiased estimator of  $\theta$ , then

$$Var(\hat{\theta}) \geq \frac{1}{I(\theta)}.$$

If  $Var(\hat{\theta}) = \frac{1}{I(\theta)}$ , then  $\hat{\theta}$  is a minimum variance unbiased estimator, aka asymptotically efficient.

### 6.7 Asymptotic Properties of Maximum Likelihood Estimators

#### 6.7.1 Asymptotic Normality

$$\lim_{n \rightarrow \infty} \hat{\theta} \sim N(\theta, \frac{1}{nI(\theta)}).$$



### 6.7.2 Asymptotic Unbiasedness

$$\lim_{n \rightarrow \infty} E[\hat{\theta}] = \theta.$$

### 6.7.3 Asymptotic Efficiency

$$\lim_{n \rightarrow \infty} Var(\hat{\theta}) = \frac{1}{nI(\theta)}.$$