Math 104 Practice

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Chapter 14

Proposition 1. $\sum \frac{n^4}{2^n}$ *converges.*

Proof. We proceed with Ratio Test.

$$\begin{aligned}\lim \left| \frac{(n+1)^4}{2^{n+1}} \cdot \frac{2^n}{n^4} \right| &= \lim \frac{(n+1)^4}{2n^4} \\ &= \lim \frac{n^4 + O(n^3)}{2n^4} \\ &= \frac{1}{2} < 1.\end{aligned}$$

$\square$

Proposition 2. $\sum \frac{2^n}{n!}$ *converges.*

Proof. We proceed with Ratio Test.

$$\begin{aligned}\lim \left| \frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n} \right| &= \lim \frac{2}{n+1} \\ &= 0 < 1.\end{aligned}$$

$\square$

Proposition 3. $\sum \frac{n!}{n^4+3}$ *diverges.*

Proof. We proceed with Ratio Test.

$$\begin{aligned}\lim \left| \frac{(n+1)!}{(n+1)^4+3} \cdot \frac{n^4+3}{n!} \right| &= \lim \frac{n(n^4+3)}{(n+1)^4+3} \\ &= \lim \frac{n^5+3n}{n^4+O(n^3)} \\ &= \infty > 1.\end{aligned}$$

Hence,

$$\liminf \left| \frac{a_{n+1}}{a_n} \right| = \lim \left| \frac{a_{n+1}}{a_n} \right| = \infty > 1.$$

$\square$

Proposition 4. $\sum \frac{cos^2 n}{n^2}$ *converges.*

Proof. We proceed with Comparison Test.

$$\left|\frac{cos^2 n}{n^2}\right| \leq \frac{1}{n^2}.$$

We know $\sum \frac{1}{n^2}$ converges. Hence, $\sum \frac{cos^2 n}{n^2}$ converges. $\square$

Proposition 5. $\sum_{n=2}^{\infty} \frac{1}{logn}$ *diverges.*

Proof. We proceed with Comparison Test.

$$\frac{1}{logn} \geq \frac{1}{n}.$$

We know $\sum_{n=2}^{\infty} \frac{1}{n}$ diverges to $+\infty$. Hence, $\sum_{n=2}^{\infty} \frac{1}{logn}$ diverges to $+\infty$. $\square$

Proposition 6. *Suppose* $\sum a_n = A, \sum b_n = B$ *where* $A, B \in \mathbb{R}$. *Then,* $\sum (a_n + b_n) = A + B$.

Proof. Define (a'_n) as the partial sums of (a_n), (b'_n) as the partial sums of (b_n), and (c'_n) as the partial sums of $(a_n + b_n)$. Then

$$\begin{aligned}\sum (a_n + b_n) &= \lim c'_n \\ &= \lim(a'_n + b'_n) \\ &= \lim a'_n + \lim b'_n \\ &= A + B.\end{aligned}$$

$\square$

Proposition 7. *Suppose* $\sum a_n = A$ *for* $A \in \mathbb{R}$. *Then,* $\sum ka_n = kA$ *for* $k \in \mathbb{R}$.

Proof. Define (a'_n) as the partial sums of (a_n) and (c'_n) as the partial sums of (ka_n). Then

$$\begin{aligned}\sum (ka_n) &= \lim c'_n \\ &= \lim(ka'_n) \\ &= k \lim a'_n \\ &= kA.\end{aligned}$$

$\square$

Proposition 8. *Suppose* $\sum a_n = A, \sum b_n = B$ *where* $A, B \in \mathbb{R}$. *Then,* $\sum (a_n \cdot b_n) = AB$ *is not true in general.*

Proof. Define $(a_n) = (1, 0, 0, 0, \dots), (b_n) = (1/2)^n$. Then $A = 1, B = 2$ and $AB = 2$. But notice $a_n \cdot b_n = 0$ for all $n \neq 0$ and $\sum (a_n \cdot b_n) = a_0 \cdot b_0 = 1 \neq AB = 2$. $\square$

Proposition 9. *If $\sum |a_n|$ converges and (b_n) is a bounded sequence, then $\sum a_n b_n$ converges. Note: Corollary 14.7 that absolutely convergent series are convergent is a special case when (b_n) is taken to be 1 for all n.*

Proof. Since (b_n) is bounded, we know there exists an supremum for $(|b_n|)$, denote $M = max\{\sup(|b_n|), 1\}$. Then, we know there exists $N \in \mathbb{N}$ such that for $n \geq m > N$, $\sum_{k=m}^{n} |a_k| < \frac{\epsilon}{M}$ for all $\epsilon > 0$. Now, take such N and

$$\sum_{k=m}^{n} |a_k| < \frac{\epsilon}{M}$$

$$M \sum_{k=m}^{n} |a_k| < \epsilon$$

$$\left| \sum_{k=m}^{n} a_k b_k \right| \leq \sum_{k=m}^{n} |a_k||b_k| \leq \sum_{k=m}^{n} |a_k| M < \epsilon.$$

Hence, $\sum a_n b_n$ satifies the Cauchy criterion and thus converges. □

Proposition 10. *If $\sum a_n$ is a convergent series of nonnegative numbers and $p > 1$, then $\sum a_n^p$ converges.*

Proof. We know there exists N such that for $n \geq m > N$, $|\sum_{k=m}^{n} a_k| < \sqrt[p]{\epsilon}$ for all $\epsilon > 0$. Take some $\epsilon > 0$ and such N, then

$$\left| \sum_{k=m}^{n} a_k \right| < \sqrt[p]{\epsilon}$$

$$\left| \sum_{k=m}^{n} a_k \right|^p < \epsilon$$

$$\left| \sum_{k=m}^{n} a_k^p \right| \leq \left| \left(\sum_{k=m}^{n} a_k \right)^p \right| < \epsilon. \tag{1}$$

Hence, $\sum a_n^p$ satifies Cauchy criterion and thus converges.

Note: the left inequality in (1) is true because $a_k \geq 0$ for all k so there are simply extra nonnegative terms in $\left|(\sum_{k=m}^{n} a_k)^p\right|$. □

Proposition 11. *If $\sum a_n$ and $\sum b_n$ are convergent series of nonnegative numbers, then $\sum \sqrt{a_n b_n}$ converges. Hint: show $\sqrt{a_n b_n} \leq a_n + b_n$ for all n.*

Proof. Notice for all n

$$\begin{aligned} a_n^2 + b_n^2 + 2a_n b_n &\geq a_n b_n \\ (a_n + b_n)^2 &\geq a_n b_n \\ a_n + b_n &\geq \sqrt{a_n b_n}. \end{aligned}$$

Also, we know there exists N_1 such that for $n \geq m > N_1$, $|\sum_{k=m}^n a_k| < \epsilon/2$ and N_2 such that for $n \geq m > N_2$, $|\sum_{k=m}^n b_k| < \epsilon/2$. Now we take $N = \max\{N_1, N_2\}$ for some $\epsilon > 0$. Then, for all $n \geq m > N$

$$\begin{aligned} \left|\sum_{k=m}^{n} \sqrt{a_n b_n}\right| &\leq \left|\sum_{k=m}^{n} a_k + b_k\right| \\ &\leq \left|\sum_{k=m}^{n} a_k + \sum_{k=m}^{n} b_k\right| \\ &\leq \left|\sum_{k=m}^{n} a_k\right| + \left|\sum_{k=m}^{n} b_k\right| \\ &< \epsilon. \end{aligned}$$

Hence, $\sum \sqrt{a_n b_n}$ satisfied Cauchy criterion and thus converges. □

Proposition 12. *The convergence of a series does not depend on any finite number of terms, though the value of the limit does. More precisely, consider series $\sum a_n$ and $\sum b_n$ and suppose that the set $\{n \in \mathbb{N} : a_n \neq b_n\}$ is finite. Then the series both converge or else the both diverge.*

Proof. Without loss of generality, we will focus on $\sum a_n$ and conclude the convergence of $\sum b_n$ based on $\sum a_n$. Also, denote $M = \max\{n \in \mathbb{N} : a_n \neq b_n\}$.

Case 1: $\sum a_n$ converges. We know $\sum a_n$ satisfies Cauchy criterion, thus we know there exists N_1 such that for all $n \geq m > N_1$, $|\sum_{k=m}^n a_k| < \epsilon$ for some $\epsilon > 0$.

Then, let $N_2 = \max\{N_1, M\}$. Since we have set N_2 to be at least M, any terms after N_2 for b_n is the same as a_n. Thus, any statement that holds true for a_n is also true for b_n after N_2 and we can conclude for all $n \geq m > N_2$ $|\sum_{k=m}^n b_k| < \epsilon$ for some $\epsilon > 0$.

Therefore, $\sum b_n$ satifies Cauchy criterion too and thus converges.

Case 2: $\sum a_n$ diverges. Assume for the sake of contradiction that $\sum b_n$ converges. Then there exists N_2 for all $\epsilon > 0$ such that for $n \geq m > N_2$, $|\sum b_n| < \epsilon$. Thus, we can take $N_1 = \max\{N_2, M\}$, which will make sure that for $n \geq m > N_1$, $|\sum_{k=m}^n a_n| < \epsilon$ for each ϵ. But that contradicts that fact that $\sum a_n$ diverges. Hence, $\sum b_n$ must diverge. □

Proposition 13. *Let (a_n) be a sequence of nonzero real numbers such that the sequence $\left(\frac{a_{n+1}}{a_n}\right)$ of ratios is a constance sequence, then $\sum a_n$ is a geometric series.*

Proof. Let $r = \frac{a_{n+1}}{a_n}$ for all n. Then we can define (a_n) recursively such that $a_{n+1} = a_n \cdot r$. Hence, $a_n = a_0 \cdot r^n$. Indeed,

$$\sum a_n = \sum_{k=0}^{n} a_0 \cdot r^k,$$

which is a geometric series. □

Proposition 14. *Let* $(a_n)_{n\in\mathbb{N}}$ *be a sequence such that* $\liminf |a_n| = 0$*, then there is a subsequence such that* $\sum_{k=1}^{\infty} a_{n_k}$ *converges.*

Proof. Since $\liminf |a_n| = 0$, we know there exists a subsequence of $(|a_n|)$ that converges to 0. Hence, for each ϵ, the set $\{n : \mathbb{N} : |a_n| < \epsilon\}$ is infinite. Then we can construct a subsequence such that $\sum_{k=1}^{\infty} a_{n_k}$ converges.

For each $k+1$, choose $n_{k+1} > n_k$ such that $|a_{n_{k+1}}| < \frac{1}{2^{k+1}} = b_{k+1}$. Then, for each k, $|a_{n_k}| \leq b_k$. Apparently, $\sum b_k$ is a convergent geometric series, thus by comparison test, $\sum_{k=1}^{\infty} a_{n_k}$ converges. $\square$

Proposition 15. $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$. *Hint:* $\sum_{k=1}^{n} \frac{1}{k(k+1)} = \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1}\right)$.

Proof. Notice

$$\begin{aligned}\sum_{k=1}^{n} \frac{1}{k(k+1)} &= \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1}\right) \\ &= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right) \\ &= 1 - \frac{1}{n+1}.\end{aligned}$$

Hence,

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{n(n+1)} &= \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k(k+1)} \\ &= \lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right) \\ &= 1.\end{aligned}$$

$\square$

Proposition 16. $\sum_{n=1}^{\infty} \frac{n-1}{2^{n+1}} = \frac{1}{2}$. *Hint:* $\frac{k-1}{2^{k+1}} = \frac{k}{2^k} - \frac{k+1}{2^{k+1}}$.

Proof. Notice

$$\begin{aligned}\sum_{k=1}^{n} \frac{k-1}{w^{k+1}} &= \sum_{k=1}^{n} \left(\frac{k}{2^k} - \frac{k+1}{2^{k+1}}\right) \\ &= \left(\frac{1}{2} - \frac{2}{2^2}\right) + \left(\frac{2}{2^2} - \frac{3}{2^3}\right) + \cdots + \left(\frac{n}{2^n} - \frac{n+1}{2^{n+1}}\right) \\ &= \frac{1}{2} - \frac{n+1}{2^{n+1}}.\end{aligned}$$

Hence,

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{n-1}{2^{n+1}} &= \lim_{n\to\infty} \sum_{k=1}^{n} \frac{k-1}{2^{k+1}} \\
&= \lim_{n\to\infty} \left(\frac{1}{2} - \frac{n+1}{2^{n+1}} \right) \\
&= \frac{1}{2} - \lim_{k\to\infty} \frac{k}{2^k} \\
&= \frac{1}{2} - \lim_{k\to\infty} \left(\frac{\sqrt[k]{k}}{2} \right)^k \\
&= \frac{1}{2}.
\end{aligned}$$

□

Proposition 17. *Show that* $\sum_{n=1}^{\infty} \frac{1}{n}$ *diverges by comparing with the series* $\sum_{n=2}^{\infty} a_n$ *where* (a_n) *is the sequence*

$$(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{16}, \frac{1}{16}, \frac{1}{16}, \frac{1}{16}, \frac{1}{16}, \frac{1}{16}, \frac{1}{16}, \frac{1}{16}, \frac{1}{32}, \frac{1}{32}, \dots).$$

Note: this is also known as the Cauchy Condensation Test.

Proof. We will show that $\sum_{n=2}^{\infty} \frac{1}{n}$ diverges and thus $\sum_{n=1}^{\infty} \frac{1}{n}$, which differs only by the first term.

Notice for all $2^k < n \leq 2^{k+1}$, $a_n = \frac{1}{2^{k+1}} \leq \frac{1}{n}$. This is true for all $k \in \mathbb{N}$. Hence, $\frac{1}{n} \leq a_n$ for all n. Now observe within each interval $(2^k, 2^{k+1}]$, there are 2^k terms. Therefore, $\sum_{n=2^k}^{2^{k+1}} a_n = \frac{1}{2}$ and $\sum_{n=2}^{\infty} a_n = \lim_{k\to\infty} k \left(\frac{1}{2}\right) = \infty$.

Hence, by Comparison Test, $\sum_{n=1}^{\infty} \frac{1}{n}$ also diverges. □

Chapter 15

Proposition 18. *$\sum \left[sin\left(\frac{n\pi}{6}\right)\right]^n$ diverges.*

Proof. Notice that when $n = 12k + 3$, $\left[sin\left(\frac{n\pi}{6}\right)\right]^n = 1$. Hence, the summation never converges. □

Proposition 19. *$\sum \left[sin\left(\frac{n\pi}{7}\right)\right]^n$ converges.*

Proof. We will show that the summation converges absolutely, hence converges.

Notice $\left|sin\left(\frac{n\pi}{7}\right)\right|$ is always between 0 and 1. In fact, it is bounded by above by some $r < 1$ such that $\left|sin\left(\frac{n\pi}{7}\right)\right| \leq r < 1$ and $\left|sin\left(\frac{n\pi}{7}\right)\right|^n \leq r^n < 1$. Then by Comparison Test, $\sum \left|sin\left(\frac{n\pi}{7}\right)\right|^n$ converges because $\sum r^n$ converges, which can be shown easily by Ratio Test or Root Test. □

Proposition 20. *$\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$ converges if and only if $p > 1$.*

Proof. We proceed with Integral Test with $f(x) = \frac{1}{x(\log x)^p}$. Notice $f(x)$ is continuous, positive, and decreasing for $x \geq 2$. Also, $f(n) = a_n$. Then for $p \neq 1$

$$\lim_{n\to\infty} \int_2^n \frac{1}{x(\log x)^p} dx = \lim_{n\to\infty} \left[\frac{(\log x)^{1-p}}{1-p}\right]_2^n. \tag{2}$$

For $p = 1$, we have

$$\lim_{n\to\infty} \int_2^n \frac{1}{x(\log x)} dx = \lim_{n\to\infty} [\log(\log x)]_2^n = \infty. \tag{3}$$

Then for ($\Rightarrow$) direction, we know that if $p = 1$, (2) goes to infinity, thus the summation diverges. If $p < 1$, (1) goes to infinity, thus the summation diverges again. Hence, forward direction is shown by contrapositive.

For ($\Leftarrow$) direction, we know that if $p > 1$, (1) converges, thus the summation converges. □

Proposition 21. *$\sum_{n=4}^{\infty} \frac{1}{n(\log n)(\log\log n)}$ diverges.*

Proof. We proceed with Integral Test with $f(x) = \frac{1}{x(\log x)(\log\log x)}$. Notice $f(x)$ is decreasing, $f(n) = a_n$, and all a_n are nonnegative. Then

$$\lim_{n\to\infty} \int_4^n \frac{1}{x(\log x)(\log\log x)} dx = \lim_{n\to\infty} [\log(\log(\log x))]_4^n = \infty.$$

□

Proposition 22. $\sum_{n=2}^{\infty} \frac{\log n}{n^2}$ *converges.*

Proof. **Integral Test:**

We can integrate $f(x) = \frac{\log x}{x^2}$ to get

$$\begin{aligned}
\lim_{n\to\infty} \int_2^n \frac{\log x}{x^2} dx &= \lim_{n\to\infty} \int_2^n -(\log x) d\left(\frac{1}{x}\right) \\
&= \lim_{n\to\infty} \left[-\frac{\log x}{x}\right]_2^n + \lim_{n\to\infty} \int_2^n \frac{1}{x^2} dx \\
&= \lim_{n\to\infty} \left[-\frac{\log x}{x}\right]_2^n - \lim_{n\to\infty} \left[\frac{1}{x}\right]_2^n \\
&= \frac{1}{2},
\end{aligned}$$

and conclude that $\sum_{n=2}^{\infty} \frac{\log n}{n^2}$ converges.

Comparison Test:

We know that for $n > N$ where N is some constant, $\sqrt{n} > \log n$. This can be proved by obversing that $\sqrt{n} > \log n$ when $n = 100$, and we see by first derivative that $\sqrt{n}$ has a higher increasing rate than $\log n$ for all n. Hence, we can conclude that $\sqrt{n} > \log n$ for all $n \geq 100$.

Then, we see $\frac{\log n}{n^2} < \frac{1}{n^{3/2}}$ for $n \geq 100$. We know that $\sum \frac{1}{n^{3/2}}$ converges for $p > 1 \Rightarrow \sum_{100}^{\infty} \frac{1}{n^{3/2}}$ converges $\Rightarrow \sum_{100}^{\infty} \frac{\log n}{n^2}$ converges $\Rightarrow \sum_{2}^{\infty} \frac{\log n}{n^2}$ converges. □

Proposition 23. *If* (a_n) *is a decreasing sequence of real numbers and if* $\sum a_n$ *converges, then* $\lim n a_n = 0$.

Proof. Since $\sum a_n$ converges, we know it satisfies Cauchy criterion. In other words, there exists N such that for $n \geq m > N$,

$$\left| \sum_{k=m}^{n} a_k \right| < \epsilon.$$

Then, □