Math 104 HW8

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Exercise 19.1

Which of the following continuous functions are uniformly continuous on the specified set?
(b) $f(x) = x^3$ on [0, 1] **(d)** $f(x) = x^3$ on $\mathbb{R}$ **(f)** $f(x) = \sin \frac{1}{x^2}$ on (0,1] **(g)** $f(x) = x^2 \sin \frac{1}{x}$ on (0,1].

Solution.

(b) $f(x) = x^3$ is continous on $\mathbb{R}$, and hence continuous on [0, 1]. By *Theorem 19.2*, f is uniformly continuous on [0,1].

(d) $f(x) = x^3$ is not uniformly continuous on $\mathbb{R}$. Assume for the sake of contradiction that for any $\epsilon > 0$ there exists $\delta > 0$ such that for all $x, y \in \mathbb{R}$, $|x - y| < \delta \implies |f(x) - f(y)| = |x^3 - y^3| < \epsilon$. However, consider $y > 0$ and $x = y + \frac{\delta}{2}$, then $|x^3 - y^3| = x^3 - y^3 = (x - y)(x^2 + xy + y^2) = \frac{\delta}{2}\left((y + \frac{\delta}{2})^2 + (y + \frac{\delta}{2})y + y^2\right)$, which is unbounded as $y \to \infty$. Hence, there exists $y > 0$ and $x = y + \frac{\delta}{2}$ such that $|x - y| < \delta \not\Longrightarrow |f(x) - f(y)| = |x^3 - y^3| < \epsilon$.

(f) Let a sequence $(s_n) = \frac{1}{n}$, which is obviously convergent and hence Cauchy on (0, 1]. However, notice $(f(s_n)) = \sin n^2$ is not convergent [there are always $n > N$ such that $\sin n^2$ is not within the ϵ neighborhood of any potential limit]. Hence, $(f(s_n))$ is not Cauchy, and by *Theorem 19.4*, f is not uniformly continuous.

(g) Extend f to $\tilde{f}$ by defining $\tilde{f}(0) = 0$, then $\tilde{f}$ is continuous at 0 because for $x \neq 0$ and $|x - 0| < \sqrt{\epsilon}$, $|x^2 \sin \frac{1}{x} - 0| \leq |x^2| < \epsilon$. Hence, $\tilde{f}$ is continuous on [0, 1] and by *Theorem 19.5*, f is uniformly continuous on $(0, 1)$. Then also notice f is continuous on the interval $[0.5, 1]$, hence is uniformly continuous on $[0.5, 1]$ by *Theorem 19.2*. Then take $\delta = \min\{\delta_1, \delta_2, 0.5\}$, where δ_1 and δ_2 represent the guaranteed δ for f on $(0, 1)$ and $[0.5, 1]$ respectively. Since $\delta \leq 0.5$, x, y must both in $[0.5, 1]$ or $(0, 1)$, and $|x - y| < \delta \implies |f(x) - f(y)| < \epsilon$ for $x, y \in (0, 1]$.

□

Exercise 19.4

(a)

Proposition 1. *If f is uniformly continuous on a bounded set S, then f is a bounded function on S.*

Proof. Assume for the sake of contradiction that f is unbounded on S. We will consider the case for unbounded above here, and the case for unbounded below is analogous, which will be omitted. Since f is unbounded above on S, for any $n \in \mathbb{N}$, there exists $x_n \in S$ such that $f(x_n) > n$. Since S is bounded,

(x_n) is bounded, and hence by Bolzano-Weierstrass theorem, there exists a convergent subsequence (x_{n_k}) of (x_n).

Now consider this subsequence (x_{n_k}), which is Cauchy since it is convergent. We already know f is uniformly continuous on S, hence by *Theorem 19.4*, $(f(x_{n_k}))$ is a Cauchy sequence, hence convergent. However, since $f(x_{n_k}) > n_k$, $(f(x_{n_k}))$ is not convergent, which is a contradiction. Hence, f must be bounded on S. $\square$

(b) Use **(a)** to give a proof that $\frac{1}{x^2}$ is not uniformly continuous on (0, 1).

Proof. Consider $f = \frac{1}{x^2}$ on (0, 1), which is a bounded set. However, f is unbounded above on $(0,1)$ because consider the sequence $(x_n) = \frac{1}{n}$, $\lim f(x_n) = \lim n^2 = \infty$, which means there always $\exists x_n \in (0,1)$ such that $f(x_n) > M \in \mathbb{R}^+$. Hence, by **(a)**, f is not uniformly continuous on $(0,1)$. □

Exercise 20.6

Proposition 2. *Let* $f = \frac{x^3}{|x|}$, *then* $\lim_{x\to\infty} f(x) = \infty, \lim_{x\to-\infty} f(x) = -\infty, \lim_{x\to 0} f(x) = \lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x) = 0$.

Proof. Notice

$$f = \begin{cases} x^2 & x > 0 \\ -x^2 & x < 0. \end{cases}$$

To prove $\lim_{x\to\infty} f(x) = \infty$, consider the interval $(1,\infty)$. Then for any sequence $(x_n) \in (1,\infty)$ with $\lim x_n = \infty$, we have $\lim f(x_n) = \infty$ because for each $M > 0$, there exists $N \in \mathbb{N}$ such that $x_n > M$ for all $n > N \Longrightarrow f(x_n) = x_n^2 \geq x_n > M$ for all $n > \mathbb{N}$.

Similarly, to prove $\lim_{x\to-\infty} f(x) = -\infty$, consider the interval $(-\infty,-1)$. Then for any sequence $(x_n) \in (-\infty,-1)$ with $\lim x_n = -\infty$, we have $\lim f(x_n) = -\infty$ because for each $M < 0$, there exists $N \in \mathbb{N}$ such that $x_n < M$ for all $n > N \Longrightarrow f(x_n) = -x_n^2 \leq x_n < M$ for all $n > \mathbb{N}$.

To prove $\lim_{x\to 0^+} f(x) = 0$, consider the interval $(0,1)$, then for any sequence $(x_n) \in (0,1)$ with $\lim x_n = 0$, we have $\lim f(x_n) = \lim x_n \cdot x_n = 0 \cdot 0 = 0$.

Similarly, to prove $\lim_{x\to 0^-} f(x) = 0$, consider the interval $(-1,0)$, then for any sequence $(x_n) \in (-1,0)$ with $\lim x_n = 0$, we have $\lim f(x_n) = \lim x_n \cdot x_n = 0 \cdot 0 = 0$.

Finally, by *Theorem 20.10*, since $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x) = 0$, $\lim_{x\to 0} f(x) = 0$. □

Exercise 20.16

Proposition 3. *Suppose that limits* $L_1 = \lim_{x\to a^+} f_1(x)$ *and* $L_2 = \lim_{x\to a^+} f_2(x)$ *exists.*

(a) *If* $f_1(x) \leq f_2(x)$ *for all* x *in some intervale* (a,b), *then* $L_1 \leq L_2$.

(b) *If* $f_1(x) < f_2(x)$ *for all* x *in some interval* (a,b), L_1 *is not necessarily less than* L_2.

Proof. **(a)** Assume for the sake of contradiction that $L_1 > L_2$. Then, there exists some sequence $(x_n) \in (a,b)$ with $\lim x_n = a$ such that $\lim f_1(x_n) = L_1$ while $\lim f_2(x) = L_2 < L_1$.

Now denote $\epsilon = L_1 - L_2$, then there exists $N_1 \in \mathbb{N}$ such that for all $n > N_1$, $|f_1(x_n) - L_1| < \frac{\epsilon}{2} \Longrightarrow f_1(x_n) \in (L_1 - \frac{\epsilon}{2}, L_1 + \frac{\epsilon}{2})$. Also, there exists $N_2 \in \mathbb{N}$ such that for all $n > N_2$, $|f_2(x_n) - L_2| < \frac{\epsilon}{2} \Longrightarrow f_2(x_n) \in (L_2 - \frac{\epsilon}{2}, L_2 + \frac{\epsilon}{2})$. Then, for all $n > \max\{N_1, N_2\}$, $f_1(x_n) \in (L_1 - \frac{\epsilon}{2}, L_1 + \frac{\epsilon}{2})$ and $f_2(x_n) \in (L_2 - \frac{\epsilon}{2}, L_2 + \frac{\epsilon}{2})$, which means $f_1(x_n) > f_2(x_n)$, which is a contradiction. Hence, $L_1 \leq L_2$.

(b) We will provide a counterexample. Consider $f_1(x) = x$ and $f_2(x) = x^2$ on $(1,2)$, then obviously $f_1(x) < f_2(x)$ for all $x \in (1,2)$. However, we know $\lim_{x\to 1^+} f_1(x) = f_1(1) = 1$ since f_1 is continuous at $x = 1$, and $\lim_{x\to 1^+} f_2(x) = f_2(1) = 1$ since f_2 is continuous at $x = 1$. Hence, $L_1 = L_2 = 1, \Longrightarrow L_1 \not< L_2$.

□

Exercise 20.18

Proposition 4. *Let* $f(x) = \frac{\sqrt{1+3x^2}-1}{x^2}$ *for* $x \neq 0$. *Then* $\lim_{x\to 0} f(x)$ *exists and determine its value.*

Proof. Rewrite

$$f(x) = \frac{\sqrt{1+3x^2}-1}{x^2} = \frac{\left(1+3x^2\right)-1}{x^2\left(\sqrt{1+3x^2}+1\right)} = \frac{3x^2}{x^2\left(\sqrt{1+3x^2}+1\right)} = \frac{3}{\sqrt{1+3x^2}+1} \quad \text{for } x \neq 0.$$

Now consider $\lim_{x\to 0^+} f(x)$. For any sequence on $(0,\infty)$ with $\lim x_n = 0$, $\lim 3x_n^2 = \lim 3 \cdot \lim x_n \cdot \lim x_n = 0 \implies \lim 1 + 3x_n^2 = 1 \implies \lim \sqrt{1+3x_n^2} = 1$[see Example 5 in Chapter 8] $\implies \lim \sqrt{1+3x^2} + 1 = 2 \implies \lim_{x\to 0^+} f(x) = \lim \frac{3}{\sqrt{1+3x^2}+1} = \frac{3}{2}$.

Similarly, consider any sequence on $(-\infty, 0)$ with $\lim x_n = 0$, $\lim_{x\to 0^-} f(x) = \frac{3}{2}$. Then by *Theorem 20.10*, $\lim_{x\to 0} f(x) = \frac{3}{2}$. □