

Math 180A HW7

Neo Lee

02/22/2023

Problem 1.

(a) 0.3.

(b) $\frac{7}{30}$.

Problem 2.

(a)

$$f_X(x) = \int_0^1 \frac{12}{7}(xy + y^2)dy \quad (1)$$

$$= \frac{12}{7} \left[\frac{1}{2}xy^2 + \frac{1}{3}y^3 \right]_0^1 \quad (2)$$

$$= \frac{12}{7} \left(\frac{x}{2} + \frac{1}{3} \right) \quad (3)$$

$$= \frac{6x + 4}{7} \quad (4)$$

for $x \in [0, 1]$ and 0 otherwise.

(b)

$$f_Y(y) = \int_0^1 \frac{12}{7}(xy + y^2)dx \quad (5)$$

$$= \frac{12}{7} \left[\frac{1}{2}x^2y + xy^2 \right]_0^1 \quad (6)$$

$$= \frac{12}{7} \left(\frac{y}{2} + y^2 \right) \quad (7)$$

$$= \frac{6y + 12y^2}{7} \quad (8)$$

for $y \in [0, 1]$ and 0 otherwise.

(c)

$$P(X < Y) = \int_0^1 \int_0^y \frac{12}{7} (xy + y^2) dx dy \quad (9)$$

$$= \frac{12}{7} \int_0^1 \left[\frac{1}{2} x^2 y + xy^2 \right]_{x=0}^{x=y} dy \quad (10)$$

$$= \frac{12}{7} \int_0^1 \frac{1}{2} y^3 + y^3 dy \quad (11)$$

$$= \frac{12}{7} \left[\frac{1}{8} y^4 + \frac{1}{4} y^4 \right]_0^1 \quad (12)$$

$$= \frac{12}{6} \left(\frac{3}{8} \right) \quad (13)$$

$$= \frac{3}{4}. \quad (14)$$

Problem 3.

$$P(X = x, Y = y) = P(X = x)P(Y = y) \quad (15)$$

$$= (p(1-p)^{x-1}) (r(1-r)^{y-1}). \quad (16)$$

Then,

$$P(X < Y) = P(X \in [1, \infty))P(Y > X) \quad (17)$$

$$= \sum_{n=1}^{\infty} p(1-p)^{n-1}(1-r)^n. \quad (18)$$

Problem 4.

(a)

$$\int_0^{\pi} \int_0^{\pi} c(1 - \cos(x)\cos(y)) dx dy = 1 \quad (19)$$

$$c \int_0^{\pi} \int_0^{\pi} dx dy - c \int_0^{\pi} \int_0^{\pi} \cos(x)\cos(y) dx dy = 1 \quad (20)$$

$$c\pi^2 - c \int_0^{\pi} [\sin(x)\cos(y)]_{x=0}^{x=\pi} dy = 1 \quad (21)$$

$$c\pi^2 = 1 \quad (22)$$

$$c = \frac{1}{\pi^2}. \quad (23)$$

(b)

$$f_X(x) = \frac{1}{\pi^2} \int_0^{\pi} 1 - \cos(x)\cos(y) dy \quad (24)$$

$$= \frac{1}{\pi^2} \left(\pi - [\cos(x)\sin(y)]_{y=0}^{y=\pi} \right) \quad (25)$$

$$= \frac{1}{\pi} \quad (26)$$

for $x \in [0, \pi]$ and 0 otherwise. Similarly,

$$f_Y(y) = \frac{1}{\pi} \quad (27)$$

for $y \in [0, \pi]$ and 0 otherwise. The probability distribution is uniform.

(c)

$$f(0,0) = \frac{1}{\pi^2} (1 - \cos(0)\cos(0)) \quad (28)$$

$$= 0. \quad (29)$$

On the other hand,

$$f_X(0)f_Y(0) = \left(\frac{1}{\pi}\right) \left(\frac{1}{\pi}\right) \quad (30)$$

$$= \frac{1}{\pi^2} \quad (31)$$

$$\neq f(0,0) \quad (32)$$

Hence, X and Y are not independent.

Problem 6. The convolution formula tells us

$$P_{X+Y}(z) = \sum_X P_X(x)P_Y(z-x) \quad (33)$$

$$= P(X=0)P(Y=z-0) + P(X=1)P(Y=z-1) \quad (34)$$

$$= (1-p)P(Y=z) + pP(Y=z-1). \quad (35)$$

Hence,

$$P_{X+Y}(z) = \begin{cases} (1-p)(1-r) & z=0 \\ p(1-r) + r(1-p) & z=1 \\ pr & z=2 \\ 0 & \text{otherwise.} \end{cases} \quad (36)$$

Problem 7. *Convolution Approach*

We know $f_Y(y) = 1$ for $y \in (1,2)$ and 0 otherwise. For $z \leq 1$, $f_{X+Y}(z) = 0$. For $z \in [1,2]$,

$$f_{X+Y}(z) = \int_1^z f_Y(y)f_X(z-y)dy \quad (37)$$

$$= \int_1^z 2(z-y)dy \quad (38)$$

$$= \int_1^z 2z - 2ydy \quad (39)$$

$$= [2yz]_{y=1}^{y=z} - [y^2]_1^z \quad (40)$$

$$= 2z^2 - 2z - z^2 + 1 \quad (41)$$

$$= z^2 - 2z + 1. \quad (42)$$

For $z \in [2,3]$,

$$f_{X+Y}(z) = \int_{z-2}^1 f_Y(z-x)f_X(x)dx \quad (43)$$

$$= \int_{z-2}^1 2xdx \quad (44)$$

$$= [x^2]_{z-2}^1 \quad (45)$$

$$= -z^2 + 4z - 3. \quad (46)$$

For $z \geq 3$, $f_{X+Y}(z) = 0$.

CDF Approach

Since X and Y are independent, $f(x, y) = f_X(x)f_Y(y) = 2x$. For $z \leq 1$, $f_{X+Y}(z) = 0$. For $z \in [1, 2]$,

$$P(X + Y \leq Z) = \int_1^z \int_0^{z-y} 2x dx dy \quad (47)$$

$$= \int_1^z [x^2]_0^{z-y} dy \quad (48)$$

$$= \int_1^z z^2 - 2zy + y^2 dy \quad (49)$$

$$= \left[z^2 y - zy^2 + \frac{1}{3} y^3 \right]_1^z \quad (50)$$

$$= z^3 - z^3 + \frac{1}{3} z^3 - z^2 + z - \frac{1}{3} \quad (51)$$

$$= \frac{1}{3} z^3 - z^2 + z - \frac{1}{3}. \quad (52)$$

Hence, for $z \in [1, 2]$,

$$f_{X+Y}(z) = \frac{d}{dz} \left(\frac{1}{3} z^3 - z^2 + z - \frac{1}{3} \right) \quad (53)$$

$$= z^2 - 2z + 1. \quad (54)$$

For $z \in [2, 3]$,

$$P(X + Y \leq Z) = \int_0^1 \int_1^2 2x dy dx - \int_{z-2}^1 \int_{z-x}^2 2x dy dx \quad (55)$$

$$= \int_0^1 [2xy]_{y=1}^{y=2} dx - \int_{z-2}^1 [2xy]_{y=z-x}^{y=2} dx \quad (56)$$

$$= \int_0^1 2x dx - \int_{z-2}^1 2x^2 + 4x - 2zx dx \quad (57)$$

$$= 1 - \left[\frac{2}{3} x^3 \right]_{z-2}^1 - [2x^2]_{z-2}^1 + [zx^2]_{z-2}^1 \quad (58)$$

$$= -\frac{1}{3} z^3 + 2z^2 - 3z + 1. \quad (59)$$

Hence, for $z \in [2, 3]$,

$$f_{X+Y}(z) = \frac{d}{dz} \left(-\frac{1}{3} z^3 + 2z^2 - 3z + 1 \right) \quad (60)$$

$$= -z^2 + 4z - 3. \quad (61)$$

For $z \geq 3$, $f_{X+Y}(z) = 0$.