Math 110 HW3

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Problem 1

Let $U := \{p \in \mathcal{P}_2(\mathbb{R}) : \int_{-1}^{1}(xp''(x) + p'(x))dx = 0\}$.

(a) Find a basis for U.

(b) Extend your basis in part (a) to a basis of $\mathcal{P}_3(\mathbb{R})$.

(c) Find a subspace W of $\mathcal{P}_3(\mathbb{R})$ such that $\mathcal{P}_3(\mathbb{R}) = U \oplus W$.

Problem 2

Suppose $v_1, \ldots, v_m$ are linearly independent in V and $w \in V$. Prove that

$$\dim span(v_1 - w, v_2 - w, \ldots, v_m - w) \geq m - 1.$$

Problem 3

Does the 'inclusion-exclusion formula' hold for three subspaces, i.e., is it always true that

$$\begin{aligned}\dim(U_1 + U_2 + U_3) &= \dim(U_1) + \dim(U_2) + \dim(U_3) \\ &\quad - \dim(U_1 \cap U_2) - \dim(U_1 \cap U_3) - \dim(U_2 \cap U_3) \\ &\quad + \dim(U_1 \cap U_2 \cap U_3)?\end{aligned}$$

Prove this formula or provide a counterexample.

Problem 4

What is the dimension and the 'canonical' basis of:

(a) $\mathbb{C}$ as a vector space over $\mathbb{C}$?

(b) $\mathbb{C}$ as a vector space over $\mathbb{R}$?

(c) $\mathbb{C}^5$ as a vector space over $\mathbb{C}$?

(d) $\mathbb{C}^7$ as a vector space over $\mathbb{R}$?

Problem 5

Suppose U and W are subspaces of V such that $U + W = V$, suppose $u_1, \ldots, u_m$ is a basis of U and $w_1, \ldots, w_n$ is a basis of W. Disprove that $u_1, \ldots, u_m, w_1, \ldots, w_n$ is necessarily a basis of V. What additional condition on the sum $U + W$ makes this implication true? Explain.