Math 181A HW2

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04/11/2023

Problem 1-1

(3.4.13) For $y \in [0, 2]$,

$$
\begin{aligned}
f_Y(y) &= \frac{d}{dy} F_Y(y) \\
f_Y(y) &= \frac{d}{dy} \frac{1}{12}(y^2 + y^3) \\
f_Y(y) &= \frac{1}{12}\left[\frac{y^3}{3} + \frac{y^4}{4}\right] \\
f_Y(y) &= \frac{y^3}{36} + \frac{y^4}{48}.
\end{aligned}
$$

(3.4.17)

$$
\begin{aligned}
&P(-a \le Y \le a) = 1 - P(Y \le -a) - P(Y \ge a) \\
&P(-a \le Y \le a) = 1 - 2P(Y \ge a) \qquad (\because P(Y \ge a) = P(Y \le -a)) \\
&P(-a \le Y \le a) = 1 - 2(1 - F_Y(a)) \\
&P(-a \le Y \le a) = 2F_Y(a) - 1.
\end{aligned}
$$

(3.6.16)

$$
\begin{aligned}
E\left(\frac{W-\mu}{\sigma}\right) &= E\left(\frac{W}{\sigma}\right) - E\left(\frac{\mu}{\sigma}\right) \\
&= \frac{1}{\sigma}E(W) - \frac{\mu}{\sigma} \\
&= \frac{\mu}{\sigma} - \frac{\mu}{\sigma} \\
&= 0.
\end{aligned}
$$

$$
\begin{aligned}
Var\left(\frac{W-\mu}{\sigma}\right) &= \frac{1}{\sigma^2}Var(W-\mu) \\
&= \frac{Var(W)}{\sigma^2} \\
&= \frac{\sigma^2}{\sigma^2} \\
&= 1.
\end{aligned}
$$

Problem 2-1 For discrete case,

$$
\begin{aligned}
E[X] &= \sum_{x \geq a} xP(X=x) + \sum_{x<a} xP(X=x) \\
E[X] &\geq \sum_{x \geq a} aP(X=x) + 0 \\
E[X] &\geq a \sum_{x \geq a} P(X=x) \\
E[X] &\geq aP(X \geq a) \\
P(X \geq a) &\leq \frac{E[X]}{a}.
\end{aligned}
$$

It is analogous for continuous case, but just replacing the summation with integral.

Problem 2-2 Let $\bar{S_n} = \frac{1}{n}\sum\limits_{i=1}^{n} X_i$, and all X_i are i.i.d. Then,

$$
\begin{aligned}
E[\bar{S_n}] &= E[\frac{1}{n}\sum_{i=1}^{n} X_i] \\
E[\bar{S_n}] &= \frac{1}{n}\sum_{i=1}^{n} E[X_i] \\
E[\bar{S_n}] &= \frac{1}{n} nE[X_i] \\
E[\bar{S_n}] &= \mu,
\end{aligned}
$$

and

$$
\begin{aligned}
Var(\bar{S_n}) &= Var(\frac{1}{n}\sum_{i=1}^{n} X_i) \\
Var(\bar{S_n}) &= \frac{1}{n^2} Var(\sum_{i=1}^{n} X_i) \\
Var(\bar{S_n}) &= \frac{1}{n^2}\sum_{i=1}^{n} Var(X_i) \\
Var(\bar{S_n}) &= \frac{1}{n^2} nVar(X_i) \\
Var(\bar{S_n}) &= \frac{\sigma^2}{n}.
\end{aligned}
$$

Then,

$$
\begin{aligned}
P(|\bar{S_n} - E[\bar{S_n}]| \geq a) &\leq \frac{Var(\bar{S_n})}{a^2} \\
P(|\bar{S_n} - \mu| \geq a) &\leq \frac{\sigma^2}{na^2}.
\end{aligned}
$$

Since σ and a are constants,

$$
\begin{aligned}
&\lim_{n \to \infty} P(|\bar{S_n} - \mu| \geq a) \leq 0 \qquad (= 0 \because P \geq 0) \\
&\lim_{n \to \infty} P(|\bar{S_n} - \mu| \leq a) = 1.
\end{aligned}
$$

Problem 3-1

(5.2.23)

$$
\begin{aligned}
E[X] = \mu = \sum_{k \in K} k P_X(k;\theta) \\
\mu = 0 \times \left[\theta^0(1-\theta)^{1-0}\right] + 1 \times \left[\theta^1(1-\theta)^{1-1}\right] \\
\mu = \theta.
\end{aligned}
$$

Then, from the sample, we can estimate

$$
\begin{aligned}
\hat{\mu} &= \frac{1+1}{5} \\
\hat{\mu} &= \frac{2}{5} \\
\theta = \mu \approx \hat{\mu} &= \frac{2}{5}.
\end{aligned}
\tag{1}
$$

Problem R

(1)

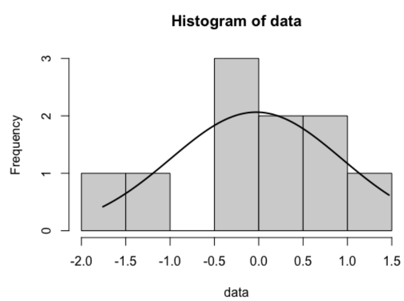
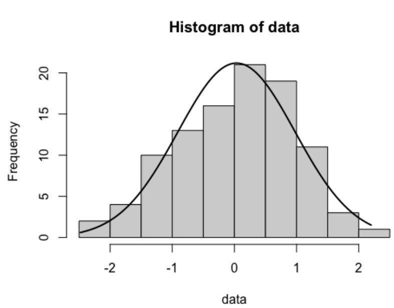
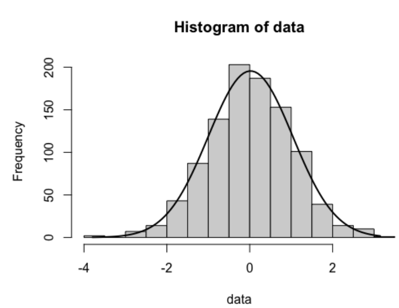


Figure 1: $n = 10$

(2) I expect the distribution of the samples would trace closer and closer to a normal distribution, a result directly from Central Limit Theorem. We can see from the plot, the result is what we expected.



$n = 100$



$n = 1000$