Math 110 HW5

Neo Lee

09/30/2023

Problem 1.

Let $V := \mathbb{C}^3$. Give an example of a map $T \in \mathcal{L}(V, V)$ such that $V = \text{null}T \oplus \text{range}T$, with both $\text{null}T$ and $\text{range}T$ non-zero, or prove that none such exists.

Solution. Let $T : (x, y, z) \mapsto (x, 0, 0)$ for $x, y, z \in \mathbb{C}$. Then the range of T is $\{(x, 0, 0) : x \in \mathbb{C}\}$, and the null space of T is $\{(0, y, z) : y, z \in \mathbb{C}\}$. It is obvious that $\text{null}T + \text{range}T = V$ because every $v \in V$ can be written as a sum of a vector in the null space and a vector in the range. Also, $\text{null}T \cap \text{range}T = \{(0, 0, 0)\}$. Therefore, it is also a direct sum. $\square$

Problem 2.

Given an example of a map $T \in \mathcal{L}(\mathbb{R}^6, \mathbb{R}^2)$ such that

$$\text{null}T = \{(x_1, x_2, x_3, x_4, x_5, x_6) : x_1 = -x_2, \quad x_3 + x_5 = 0, \quad x_1 - x_4 - x_5 = 0\}$$

or prove that none such exists.

Solution. We claim that such a map does not exist. Assume for the sake of contradiction that such a map T exists.

Let's rewrite the null space of T as

$$\text{null}T = \{(x_1, -x_1, -x_5, x_1 - x_5, x_5, x_6) : x_1, x_5, x_6 \in \mathbb{R}\}.$$

In other words, null$T$ is the span of the vectors $(1, -1, 0, 1, 0, 0)$, $(0, 0, -1, -1, 1, 0)$, and $(0, 0, 0, 0, 0, 1)$. We can check whether these three vectors are linearly independent to find the exact dimension of null$T$ but it is unnecessary for this problem.

Since null$T$ is defined by a span of three vectors, the maximum dimension of null$T$ is 3. Then, by the rank-nullity theorem, the dimension of the range of T is at least $6 - 3 = 3$. However, the range of T is a subspace of $\mathbb{R}^2$, so the dimension of the range of T is at most 2. This is a contradiction. Therefore, such a map T does not exist. $\square$

Problem 3.

Suppose $T : \mathcal{P}_3(\mathbb{R}) \to \mathcal{P}_2(\mathbb{R})$ is defined by the formula $(Tf)(x) = 4xf''(x) - f'(x)$. Check that $T \in \mathcal{L}(\mathcal{P}_3(\mathbb{R}), \mathcal{P}_2(\mathbb{R}))$ and find a basis for the null space and a basis for the range of T.

Solution. To check whether T is a linear map, we need to check whether T follows additivity and homogeneity.

Additivity: Let $f, g \in \mathcal{P}_3(\mathbb{R})$. Then

$$\begin{aligned} T(f+g)(x) &= 4x(f+g)''(x) - (f+g)'(x) \\ &= 4x(f''+g'')(x) - (f'+g')(x) \\ &= 4xf''(x) + 4xg''(x) - f'(x) - g'(x) \\ &= T(f)(x) + T(g)(x). \end{aligned}$$

Homogeneity: Let $f \in \mathcal{P}_3(\mathbb{R})$ and $c \in \mathbb{R}$. Then

$$\begin{aligned} T(cf)(x) &= 4x(cf)''(x) - (cf)'(x) \\ &= 4x(c(f'')'(x)) - cf'(x) \\ &= 4cx(f'')'(x) - cf'(x) \\ &= c(4xf''(x) - f'(x)) \\ &= cT(f)(x). \end{aligned}$$

Range: Let $v \in \mathcal{P}_3(\mathbb{R})$, which can be written as $ax^3 + bx^2 + cx + d$ for $a, b, c, d \in \mathbb{R}$. Then, after applying T to v, we get

$$\begin{aligned} T(v)(x) &= 4xv''(x) - v'(x) \\ &= 4x(6ax + 2b) - (3ax^2 + 2bx + c) \\ &= 24ax^2 + 8bx - 3ax^2 - 2bx - c \\ &= 21ax^2 + 6bx - c, \end{aligned}$$

which is a linear combination of x^2, x, and 1. Therefore, the range of T is $\operatorname{span}\{x^2, x, 1\}$.

Null Space: Let $v \in \mathcal{P}_3(\mathbb{R})$, $ax^3 + bx^2 + cx + d$, such that $T(v)(x) = 0$. From what we have derived for range$T$, it means for all x

$$21ax^2 + 6bx - c = 0.$$

This is only possible when $21a = 6b = c = 0$ because $x^2, x, 1$ are linearly independent, in fact basis. Therefore, all vectors in the null space of T are of the form $ax^3 + bx^2 + cx + d$ where $a = b = c = 0$. In other words, the null space of T is $\operatorname{span}\{1\}$, or equivalently $\mathbb{R}$. Hence, the basis is $\{1\}$. $\square$

Problem 4.

Let $T : f(x) \mapsto (x-1)^2 f'''(x) - 3(x-1)f''(x) + f'(x)$. Write down its matrix representation:

(a) as a map in $\mathcal{L}(\mathcal{P}_3, \mathcal{P}_2)$ using the standard monomial bases both for the domain and codomain;

(b) as a map in $\mathcal{L}(\mathcal{P}_3, \mathcal{P}_3)$ using the standard monomial basis both for the domain and codomain;

(c) as a map in $\mathcal{L}(\mathcal{P}_3, \mathcal{P}_3)$ using the shifted monomial basis $1, x-1, (x-1)^2, (x-1)^3$ for the domain and for the codomain.

Solution.

(a)

$$\begin{aligned}
T(x^3) &= (x-1)^2(6) - 3(x-1)(6x) + 3x^2 \\
&= 6x^2 - 12x + 6 - 18x^2 + 18x + 3x^2 \\
&= -9x^2 + 6x + 6 \\
T(x^2) &= -3(x-1)(2) + 2x \\
&= -6x + 6 + 2x \\
&= -4x + 6 \\
T(x) &= 1 \\
T(1) &= 0.
\end{aligned}$$

Hence, the matrix representation is

$$\begin{bmatrix} -9 & 0 & 0 & 0 \\ 6 & -4 & 0 & 0 \\ 6 & 6 & 1 & 0 \end{bmatrix}.$$

(b) The matrix representation is simply prepending an empty row at the top of the matrix from (a):

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ -9 & 0 & 0 & 0 \\ 6 & -4 & 0 & 0 \\ 6 & 6 & 1 & 0 \end{bmatrix}.$$

(c)

$$\begin{aligned}
T\left((x-1)^3\right) &= (x-1)^2(6) - 3(x-1)(6(x-1)) + 3(x-1)^2 \\
&= 6(x-1)^2 - 18(x-1)^2 + 18(x-1)^2 \\
&= 6(x-1)^2 \\
T\left((x-1)^2\right) &= -3(x-1)(2) + 2(x-1) \\
&= -6(x-1) + 2(x-1) \\
&= -4(x-1) \\
T\left((x-1)\right) &= 1 \\
T\left(1\right) &= 0.
\end{aligned}$$

Hence, the matrix representation is

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 6 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

$\square$

Problem 5.

Suppose V and W are finite-dimensional vector spaces. Let v be a fixed vector in V, and consider

$$E_v := \{T \in \mathcal{L}(V, W) : Tv = 0\}.$$

(a) Show that E_v is a subspace of $\mathcal{L}(V, W)$.

(b) Suppose $v \neq 0$. What is $\dim E_v$?

Solution.

(a) Firstly, E_v is obviously a subset of $\mathcal{L}(V, W)$.

Additive identity: The zero map from V to W is obviously in E_v because it maps every vector, including v, to 0. Hence, the additive identity, which is the zero map, is in E_v.

Closed under addition: Let $T_1, T_2 \in E_v$. Then

$$\begin{aligned}(T_1 + T_2)v &= T_1 v + T_2 v \qquad (1)\\ &= 0 + 0\\ &= 0.\end{aligned}$$

(1) is possible because T_1, T_2 are in $\mathcal{L}(V, W)$. Therefore, $T_1 + T_2 \in E_v$.

Closed under scalar multiplication: Let $T \in E_v$ and $c \in \mathbb{R}$. Then

$$\begin{aligned}(cT)v &= c(Tv) \qquad (2)\\ &= c0\\ &= 0.\end{aligned}$$

(2) is possible because T is in $\mathcal{L}(V, W)$. Therefore, $cT \in E_v$.

(b) We can construct a basis of V that includes v, denoted as $(v, v_2, \ldots, v_n)$. Then consider arbitrary $T \in \mathcal{L}(V, W)$ and $z \in V$,

$$\begin{aligned}T(z) &= T(c_1 v + c_2 v_2 + \cdots + c_n v_n)\\ &= c_1 T(v) + c_2 T(v_2) + \cdots + c_n T(v_n)\\ &= c_2 T(v_2) + \cdots + c_n T(v_n)\\ &= T(c_2 v_2 + \cdots + c_n v_n).\end{aligned}$$

We can see that T is completely determined by its action on $v_2, \ldots, v_n$. Therefore, we can define a map $\Phi : E_v \to \mathcal{L}(V', W)$, $V' = \operatorname{span}\{v_2, \ldots, v_m\}$ by restricting the domain of T to V'. This map is well-defined as we saw from the above derivation. We will show that Φ is an isomorphism.

Linearity of Φ:

(i) Let $T_1, T_2 \in \mathcal{L}(V, W)$ and $z \in V'$. Then

$$\begin{aligned}\Phi(T_1 + T_2)(z) &= (T_1 + T_2)(z)\\ &= T_1(z) + T_2(z)\\ &= \Phi(T_1)(z) + \Phi(T_2)(z).\end{aligned}$$

(ii) Let $T \in \mathcal{L}(V, W)$, $c \in \mathbb{R}$, and $z \in V'$. Then

$$\begin{aligned} \Phi(cT)(z) &= (cT)(z) \\ &= cT(z) \\ &= c\Phi(T)(z). \end{aligned}$$

Injectivity of Φ: Let $\Phi(T) = T' \in \mathcal{L}(V', W)$ be the zero map. Then T' maps every vector of V', including the basis to 0. Then, T must map every v_i for $i \in [2, n]$, also the fixed v obviously, to 0 because T have the same actions as T' on V'. Hence, T maps every basis in V to zero, consequently every vectors in V to zero. Therefore, T is the zero map and Φ is injective.

Surjectivity of Φ: We know Φ has to be surjective because of how we defined Φ. Every $T' \in \mathcal{L}(V', W)$ has a corresponding $T \in \mathcal{L}(V, W)$ that perform the same action on V' as T' does. Therefore, Φ is surjective.

Φ is isomorphism: We have shown that Φ is linear, injective, and surjective. Therefore, Φ is an isomorphism between E_v and $\mathcal{L}(V', W)$. Therefore, $\dim E_v = \dim \mathcal{L}(V', W) = \dim W \times \dim V' = \dim W \times (\dim V - 1)$

$\square$