Math 104 HW11

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Exercise 29.2

Proposition 1. $|\cos x - \cos y| \leq |x - y|$ *for all* $x, y \in \mathbb{R}$.

Proof. Notice that the inequality looks similar to the slope of the function $\cos x$ evaluated on the interval $[x, y]$.

Since $\cos x$ is continuous and differentiable on $\mathbb{R}$, by mean value value theorem, there exists $c \in (x, y)$ such that

$$\begin{aligned} \frac{\cos x - \cos y}{x - y} &= \cos'(c) \\ \frac{\cos x - \cos y}{x - y} &= -\sin(c). \end{aligned}$$

Then notice that $|\sin(c)| \leq 1$ for all $c \in \mathbb{R}$, so we have for all $x, y \in \mathbb{R}$,

$$\begin{aligned} \left| \frac{\cos x - \cos y}{x - y} \right| &= |\sin(c)| \\ \left| \frac{\cos x - \cos y}{x - y} \right| &\leq 1 \\ |\cos x - \cos y| &\leq |x - y|. \end{aligned}$$

$\square$

Exercise 29.5

Proposition 2. *Let f be defined on $\mathbb{R}$, and suppose $|f(x) - f(y)| \leq (x - y)^2$ for all $x, y \in \mathbb{R}$, then f is a constant function.*

Proof. Fix arbitrary $y \in \mathbb{R}$, then we have for all $x \in \mathbb{R}$,

$$
\begin{aligned}
|f(x) - f(y)| &\leq (x - y)^2 \\
\left| \frac{f(x) - f(y)}{x - y} \right| &\leq |x - y|.
\end{aligned}
$$

Since $\lim_{x \to y} |x - y| = 0$, for all $\epsilon > 0$, there exists $\delta > 0$ such that $|x - y| < \delta$ implies $|x - y| < \epsilon$. Then we have for all $x \in \mathbb{R}$ such that $|x - y| < \delta$,

$$
\left| \frac{f(x) - f(y)}{x - y} \right| \leq |x - y| < \epsilon,
$$

and

$$
\lim_{x \to y} \left| \frac{f(x) - f(y)}{x - y} \right| = 0.
$$

Then by definition of derivative, we have $f'(y) = 0$. Since y is arbitrary, we have $f'(x) = 0$ for all $x \in \mathbb{R}$. Then by *Corollary 29.4*, we have f is a constant function. □

Exercise 29.11

Proposition 3. $\sin x \leq x$ *for all* $x \geq 0$.

Proof. Notice $x - \sin x$ is differentiable on $[0, \infty)$ because both x and $\sin x$ are differentiable on $[0, \infty)$. Then we have for all $x \geq 0$,

$$(x - \sin x)' = 1 - \cos x$$
$$(x - \sin x)' = 1 - \cos x \geq 0.$$

Then by *Corollary 29.7*, we have $x - \sin x$ is increasing on $[0, \infty)$, so for $x \geq 0$, we have $x - \sin x \geq 0$ because $x - \sin x = 0$ for $x = 0$. Then we have for all $x \geq 0$, $\sin x \leq x$. $\square$

Exercise 29.18

Proposition 4. *Let f be differentiable on $\mathbb{R}$ with $a = \sup\{|f'(x) : x \in \mathbb{R}|\} < 1$.*

(a) *Select $s_0 \in \mathbb{R}$ and define $s_n = f(s_{n-1})$ for $n \geq 1$. Thus $s_1 = f(s_0)$, $s_2 = f(s_1)$, etc. Then (s_n) is a convergent sequence.*

(b) *f has a fixed point, i.e., $f(s) = s$ for some $s \in \mathbb{R}$.*

Proof. **(a)** Since f is differentiable and hence continuous on $\mathbb{R}$, there exists c between s_n and s_{n-1} such that

$$\frac{f(s_n) - f(s_{n-1})}{s_n - s_{n-1}} = f'(c).$$

Then, since $|f'(c)| \leq a < 1$, we have for all $n \in \mathbb{N}$,

$$|s_{n+1} - s_n| = |f(s_n) - f(s_{n-1})| = |f'(c)||s_n - s_{n-1}| \leq a|s_n - s_{n-1}|.$$

In particular, $|s_{n+1} - s_n| \leq a^n|(s_1 - s_0)|$. Therefore, without loss of generality, for some $n > m \in \mathbb{N}$,

$$\begin{aligned}
|s_n - s_m| &= |(s_n - s_{n-1}) + (s_{n-1} - s_{n-1}) \cdots + (s_{m+1} - s_m)| \\
|s_n - s_m| &\leq |s_n - s_{n-1}| + |s_{n-1} - s_{n-1}| \cdots + |s_{m+1} - s_m| \\
|s_n - s_m| &\leq a^{n-1}|(s_1 - s_0)| + a^{n-2}|(s_1 - s_0)| \cdots + a^m|(s_1 - s_0)| \\
|s_n - s_m| &\leq |(s_1 - s_0)| \sum_{k=m}^{n-1} a^k \\
|s_n - s_m| &\leq \lim_{n\to\infty} |(s_1 - s_0)| \sum_{k=m}^{n-1} a^k = a^m \frac{|(s_1 - s_0)|}{1 - a}.
\end{aligned}$$

Now notice

$$\lim_{m\to\infty} a^m \frac{|(s_1 - s_0)|}{1 - a} = 0,$$

so there exists $N \in \mathbb{N}$ such that for all $m > N$,

$$a^m \frac{|(s_1 - s_0)|}{1 - a} < \epsilon,$$

therefore, for all $n \geq m > N$,

$$|s_n - s_m| < \epsilon.$$

Then (s_n) is a Cauchy sequence hence a convergent sequence.

(b) Since (s_n) is a convergent sequence, there exists $s \in \mathbb{R}$ such that

$$\lim_{n\to\infty} s_n = s.$$

Then since f is continuous, we have

$$\lim_{n\to\infty} f(s_n) = f(s).$$

Now recall that $s_{n+1} = f(s_n)$ for all $n \geq 1$, so we have

$$s = \lim_{n\to\infty} s_{n+1} = \lim_{n\to\infty} f(s_n) = f(s).$$

$\square$

Exercise 30.2

Find the following limits if they exist,

(a) $\lim_{x\to 0} \frac{x^3}{\sin x - x}$

(b) $\lim_{x\to 0} \frac{\tan x - x}{x^3}$

(c) $\lim_{x\to 0} \left(\frac{1}{\sin x} - \frac{1}{x}\right)$

(d) $\lim_{x\to 0} (\cos x)^{1/x^2}$

Solution.

(a)

$$\begin{aligned}\lim_{x\to 0} \frac{x^3}{\sin x - x} &= \lim_{x\to 0} \frac{3x^2}{\cos x - 1} \\ &= \lim_{x\to 0} \frac{6x}{-\sin x} \\ &= \lim_{x\to 0} \frac{6}{-\cos x} \\ &= -6.\end{aligned}$$

(b)

$$\begin{aligned}\lim_{x\to 0} \frac{\tan x - x}{x^3} &= \lim_{x\to 0} \frac{\sec^2 x - 1}{3x^2} \\ &= \lim_{x\to 0} \frac{2\sec^2 x \tan x}{6x} \\ &= \lim_{x\to 0} \frac{-4\sec^2 x + 6\sec^4 x}{6} \\ &= \frac{1}{3}.\end{aligned}$$

(c)

$$\begin{aligned}\lim_{x\to 0} \left(\frac{1}{\sin x} - \frac{1}{x}\right) &= \lim_{x\to 0} \frac{x - \sin x}{x \sin x} \\ &= \lim_{x\to 0} \frac{1 - \cos x}{\sin x + x\cos x} \\ &= \lim_{x\to 0} \frac{\sin x}{2\cos x - x \sin x} \\ &= 0.\end{aligned}$$

(d)

$$\lim_{x\to 0} \lim_{x\to 0} (\cos x)^{1/x^2} = \lim_{x\to 0} e^{\ln(\cos x)/x^2}.$$

Now,

$$\begin{aligned}
\lim_{x_to0} \frac{\ln(\cos x)}{x^2} &= \lim_{x\to 0} \frac{-\tan x}{2x} \\
&= \lim_{x\to 0} \frac{-\sec^2 x}{2} \\
&= -\frac{1}{2}.
\end{aligned}$$

Since e^x is continuous, we have

$$\lim_{x\to 0} \lim_{x\to 0} (\cos x)^{1/x^2} = \lim_{x\to 0} e^{\ln(\cos x)/x^2} = e^{-1/2}.$$

$\square$

Exercise 30.5

Find the limits

(a) $\lim_{x\to 0}(1+2x)^{1/x}$

(b) $\lim_{y\to\infty}(1+\frac{2}{y})^y$

(c) $\lim_{x\to\infty}(e^x+x)^{1/x}$

Solution.

(a)

$$\lim_{x\to 0}(1+2x)^{1/x} = \lim_{x\to 0} e^{\ln(1+2x)/x}.$$

Now

$$\begin{aligned}\lim_{x\to 0}\frac{\ln(1+2x)}{x} &= \lim_{x\to 0}\frac{2}{1+2x}\\ &= 2.\end{aligned}$$

Since e^x is continuous, we have

$$\lim_{x\to 0}(1+2x)^{1/x} = \lim_{x\to 0} e^{\ln(1+2x)/x} = e^2.$$

(b)

$$\lim_{y\to\infty}(1+\frac{2}{y})^y = \lim_{y\to\infty} e^{y\ln(1+\frac{2}{y})}.$$

Now,

$$\begin{aligned}\lim_{y\to\infty} y\ln(1+\frac{2}{y}) &= \lim_{y\to\infty}\frac{\ln(1+\frac{2}{y})}{\frac{1}{y}}\\ &= \lim_{y\to\infty}\frac{\frac{-2}{y^2(1+2/y)}}{\frac{-1}{y^2}}\\ &= \lim_{y\to\infty}\frac{2}{1+2/y}\\ &= 2.\end{aligned}$$

Since e^x is continuous, we have

$$\lim_{y\to\infty}(1+\frac{2}{y})^y = \lim_{y\to\infty} e^{y\ln(1+\frac{2}{y})} = e^2.$$

(c)

$$\lim_{x\to\infty}(e^x+x)^{1/x} = \lim_{x\to\infty} e^{\ln(e^x+x)/x}.$$

Now,

$$\begin{aligned}
\lim_{x\to\infty} \frac{\ln(e^x + x)}{x} &= \lim_{x\to\infty} \frac{e^x + 1}{e^x + x} \\
&= \lim_{x\to\infty} \frac{e^x}{e^x + x} + \lim_{x\to\infty} \frac{1}{e^x + x} \\
&= \lim_{x\to\infty} \frac{e^x}{e^x + 1} + 0 \\
&= \lim_{x\to\infty} \frac{e^x}{e^x} \\
&= 1.
\end{aligned}$$

Since e^x is continuous, we have

$$\lim_{x\to\infty} (e^x + x)^{1/x} = \lim_{x\to\infty} e^{\ln(e^x+x)/x} = e^1 = e.$$

$\square$