Math 104 HW1

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Exercise 1.3

Proposition 1. $1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2$ *for all positive integers n.*

Proof. We proceed by induction.

<u>Base case</u>: $n = 1$. We have $1^3 = 1^2$.

<u>Inductive step</u>: Assume that $1^3 + 2^3 + \cdots + k^3 = (1 + 2 + \cdots + k)^2$ for some $k \in \mathbb{N}$. Now consider $k + 1$,

$$\begin{aligned}
1^3 + 2^3 + \cdots + k^3 + (k+1)^3 &= (1 + 2 + \cdots + k)^2 + (k+1)^3 \\
&= \left(\frac{(k+1)\cdot k}{2}\right)^2 + (k+1)^3 \\
&= \frac{(k+1)^2 \cdot k^2 + (k+1)^2 \cdot 4(k+1)}{4} \\
&= \frac{(k+1)^2(k^2 + 4k + 4)}{4} \\
&= \frac{(k+1)^2(k+2)^2}{4} \\
&= \left(\frac{(k+1)\cdot(k+2)}{2}\right)^2 \\
&= (1 + 2 + \cdots + (k+1))^2.
\end{aligned}$$

Hence, by the principle of mathematical induction, $1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2$ for all positive integers n. $\square$

Exercise 1.5

Proposition 2. $1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n} = 2 - \frac{1}{2^n}$ *for all positive integers n.*

Proof. We again proceed by induction.

<u>Base case</u>: $n = 1$. We have $1 + \frac{1}{2} = 2 - \frac{1}{2}$.

Inductive step: Assume that $1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^k} = 2 - \frac{1}{2^k}$ for some $k \in \mathbb{N}$. Now consider $k+1$,

$$
\begin{aligned}
1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^k} + \frac{1}{2^{k+1}} &= 2 - \frac{1}{2^k} + \frac{1}{2^{k+1}} \\
&= 2 - \frac{1}{2^k} + \frac{1}{2^k} \cdot \frac{1}{2} \\
&= 2 - \frac{1}{2^k}\left(1 - \frac{1}{2}\right) \\
&= 2 - \frac{1}{2^k} \cdot \frac{1}{2} \\
&= 2 - \frac{1}{2^{k+1}}.
\end{aligned}
$$

Hence, by the principle of mathematical induction, $1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n} = 2 - \frac{1}{2^n}$ for all positive integers n. $\square$

Exercise 1.11

(a)

Proposition 3. *If $n^2 + 5n + 1$ is an even integer, then $(n+1)^2 + 5(n+1) + 1$ is also an even integer for $n \in \mathbb{N}$.*

Consider

$$
\begin{aligned}
(n+1)^2 + 5(n+1) + 1 &= n^2 + 2n + 1 + 5n + 5 + 1 \\
&= n^2 + 5n + 1 + 2n + 6 \\
&= (n^2 + 5n + 1) + 2(n+3) \\
&= 2k + 2(n+3) \qquad (\textit{for some } k \in \mathbb{Z} \because n^2 + 5n + 1 \textit{ is an even integer}) \\
&= 2(k + n + 3).
\end{aligned}
$$

Hence, $(n+1)^2 + 5(n+1) + 1$ is an even integer.

(b) For which $n \in \mathbb{N}$ is $n^2 + 5n + 1$ an even integer?

Solution. If n is even, then $n^2 + 5n + 1 = (2k)^2 + 5(2k) + 1 = 2(2k^2 + 5k) + 1$ for some $k \in \mathbb{Z}$, thus is an odd integer. If n is odd, then $n^2 + 5n + 1 = (2j+1)^2 + 5(2j+1) + 1 = 2(2j^2 + 7j + 3) + 1$ for some $j \in \mathbb{Z}$, thus is also an odd integer. Hence, $n^2 + 5n + 1$ is never an even integer.

The moral of the exercise is that even the inductive step is true, the proposition is not necessarily true without a proper and true base case. $\square$

Exercise 2.7

(a)

Proposition 4. *$\sqrt{4 + 2\sqrt{3}} - \sqrt{3}$ is rational.*

Proof. Let $x = \sqrt{4 + 2\sqrt{3}} - \sqrt{3}$. Now, evaluate

$$\begin{aligned}
x &= \sqrt{4 + 2\sqrt{3}} - \sqrt{3} \\
(x + \sqrt{3})^2 &= 4 + 2\sqrt{3} \\
x^2 + 2x\sqrt{3} + 3 &= 4 + 2\sqrt{3} \\
x^2 - 1 &= \sqrt{3}(2 - 2x) \\
(x^2 - 1)^2 &= 3(2 - 2x)^2 \\
x^4 - 2x^2 + 1 &= 12 - 24x + 12x^2 \\
x^4 - 14x^2 + 24x - 11 &= 0.
\end{aligned}$$

By the rational zeros theorem, the only possible rational roots are $\pm 1, \pm 11$. Indeed, $x = 1$ is a root of the equation, and 1 is obviously rational. □

(b)

Proposition 5. $\sqrt{6 + 4\sqrt{2}} - \sqrt{2}$ *is rational.*

Proof. Again, let $x = \sqrt{6 + 4\sqrt{2}} - \sqrt{2}$. Now, evaluate

$$\begin{aligned}
x &= \sqrt{6 + 4\sqrt{2}} - \sqrt{2} \\
(x + \sqrt{2})^2 &= 6 + 4\sqrt{2} \\
x^2 + 2x\sqrt{2} + 2 &= 6 + 4\sqrt{2} \\
x^2 - 4 &= \sqrt{2}(4 - 2x) \\
(x^2 - 4)^2 &= 2(4 - 2x)^2 \\
x^4 - 8x^2 + 16 &= 32 - 32x + 8x^2 \\
x^4 - 16x^2 + 32x - 16 &= 0.
\end{aligned}$$

By the rational zeros theorem, the only possible rational roots are $\pm 1, \pm 2, \pm 4, \pm 8, \pm 16$. Indeed, $x = 2$ is a root of the equation, and 2 is obviously rational. □

Exercise 2.8

Find all rational solutions of the equation $x^8 - 4x^5 + 13x^3 - 7x + 1 = 0$.

Solution. By rational zeros theorem, the only possible rational candidates to the equation is only ± 1. Only -1 satisfies the equation, thus -1 is the only rational solution. □

Exercise 3.1

(a) Which of the ordered field properties A1-A4, M1-M4, DL, O1-05 fail for $\mathbb{N}$.

Solution. A3: $\mathbb{N}$ does not have additive identity, e.g. $\nexists n \in \mathbb{N}$ such that $n + 2 = 2$.
A4: $\mathbb{N}$ does not have additive inverse, e.g. $\nexists n \in \mathbb{N}$ such that $n + 2 = 0$.
M4: $\mathbb{N}$ does not have multiplicative inverse, e.g. $\nexists n \in \mathbb{N}$ such that $n \times 2 = 1$.

□

(b) Which of the ordered field properties A1-A4, M1-M4, DL, O1-05 fail for $\mathbb{Z}$.

Solution. M4: $\mathbb{Z}$ does not have multiplicative inverse, e.g. $\nexists z \in \mathbb{Z}$ such that $z \times 2 = 1$.

□

Exercise 3.6a

Proposition 6. $|a+b+c| \leq |a|+|b|+|c|$ *for all* $a, c, b \in \mathbb{R}$.

Proof.

$$\begin{aligned} |a+b+c| &= |(a+b)+c| \\ &\leq |a+b|+|c| \qquad (\textit{triangle inequality on } (a+b), c) \\ &\leq |a|+|b|+|c|. \qquad (\textit{triangle inequality on } a, b) \end{aligned}$$

□