MATH 105 Notes

Professor: Khalilah Beal-Uribe

Book: Real Mathematical Analysis[1] *by Pugh*

Neo Lee

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[1] An introductory but holistic, intuitive, and easy to read book.

Contents

Chapter 1

First chapter

1.1 Lecture 1

Definition 1.1.1: Norm

Given a vector space V over a subfield $\mathbb{F}$ of $\mathbb{C}$, a norm of V is a real-valued function $p : V \to \mathbb{R}$ satisfying the following properties:

1. **Triangle inequality:** $p(v + w) \leq p(v) + p(w)$,
2. **Absolute homogeneity:** $p(\alpha v) = |\alpha| p(v)$,
3. **Positive definiteness:** $p(v) \geq 0$ and $p(v) = 0$ iff $v = 0$.

Note:

Usually, we denote the norm of v by $\|v\|$, and for clarity of the underlying vector space, we may write $\|v\|_V$.

Proposition 1.1.1 Normed space is a metric space

Let V be a normed space. Then the function $d : V \times V \to \mathbb{R}$ defined by $d(v, w) = p(v - w) = \|v - w\|$ is a metric on V.

Definition 1.1.2: Isomorphism in vector spaces

A function $f : V \to W$ between two vector spaces V and W over the same field $\mathbb{F}$ is called an isomorphism if it is bijective and linear. If such an isomorphism exists, we say that the two vector spaces are isomorphic.

Definition 1.1.3: Homeomorphism

A function $f : X \longrightarrow Y$ between two topological spaces X and Y is called a homeomorphism if it satisfies the following properties:

1. f is bijective,
2. f is continuous,
3. f^{-1} is continuous.

If such a homeomorphism exists, we say that the two topological spaces are homeomorphic.

Note:

In general, isomorphism does not imply homeomorphism. However, in certain cases, they are equivalent, which will be discussed in details later.

Definition 1.1.4: Operator norm

Let $T : V \longrightarrow W$ be a linear oepration between normed spaces. Denote $\|\cdot\|_V$ and $\|\cdot\|_W$ be the norms in V and W respectively. The operator norm of A is defined by

$$\begin{aligned}\|T\| &= \sup\left\{\frac{\|Tv\|_W}{\|v\|_V} : v \neq 0, v \in V\right\}\\ &= \inf\left\{c \geq 0 : \|Tv\|_W \leq c\|v\|_V, \forall v \in V\right\}\end{aligned}$$

Note:

We say that T is bounded if $\|T\| < \infty$.

1.2 Lecture 2

Theorem 1.2.1 Multiplication of matrices are composition of linear maps

$T_A \circ T_b = T_{AB}$.

Theorem 1.2.2 Bounded operator is equivalent to continuous

Let $T : V \longrightarrow W$ be a linear transformation from one normed space to another. The following are equivalent:

1. $\|T\| < \infty$,
2. T is uniformly continuous,
3. T is continuous,
4. T is continuous at 0.

Proof: We show that (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) $\Longrightarrow$ (4) $\Longrightarrow$ (1).

- (1) $\implies$ (2): Let $M = \|T\| < \infty$, and let $\delta = \frac{\epsilon}{M}$. Then for any $x, y \in V$ such that $\|x - y\| < \delta$, we have

$$\begin{aligned}\|Tx - Ty\| &= \|T(x-y)\| \\ &\leq M\|x-y\| \\ &< M\delta \\ &= \epsilon.\end{aligned}$$

  Hence, T is uniformly continuous.

- (2) $\implies$ (3): Trivial. Uniformly continuous automatically implies continuous.

- (3) $\implies$ (4): Trivial. T is continuous over the whole domain implies that it is continuous at any point in the domain, including 0.

- (4) $\implies$ (1): Let $\epsilon = 1$, then there exists $\delta > 0$ such that $\|x\| < \delta$ implies $\|Tx\| < 1$. Then for any $v \neq 0$, define $v' = \frac{\delta}{2\|v\|}$, then $\|v'\| < \delta$ and hence $\|Tv'\| < 1$. Then we have

$$\begin{aligned}\|Tv'\| &< 1 \\ \left\|T\left(\frac{\delta}{2\|v\|}v\right)\right\| &< 1 \\ \frac{\delta}{2\|v\|}\|Tv\| &< 1 \\ \|Tv\| &< \frac{2}{\delta}\|v\|.\end{aligned}$$

  Then, from our *definition 1.1.4* of operator norm, we have $\|T\| < \frac{2}{\delta}$ and hence $\|T\| < \infty$.

😄

Theorem 1.2.3 Linear map from finite-dimensional Euclidean space to normed space is continuous

Let $T : \mathbb{R}^n \to W$, where T is linear and W is a normed space. Then

1. T is continuous,
2. if T is an isomorphism, then T is a homeomorphism.

Corollary 1.2.1 Linear maps from finite-dimensional normed space to normed space are continuous

All linear maps from finite-dimensional normed space to another normed space are continuous, and all isomorphisms from finite-dimensional space to normed space are homeomorphisms.

In particular, if a finite-dimensional vector spaces is equipped with two norms, then the identity map between them is a homeomorphism. For example, $T : \mathcal{M} \to \mathcal{L}$ is a homeomorphism.

Proof: Let V be a n-dimensional normed space and W be another normed space, and $T : V \to W$. Then, there exists an isoemorphism $S : V \to \mathbb{R}^n$. *Theorem 1.2.2* gaurentees that S and S^{-1} are homeomorphisms. Then, $T \circ S : \mathbb{R}^n \to W$ is also a continuous linear map guaranteed by *Theorem 1.2.2*. Then,

$$T = (T \circ S) \circ S^{-1}$$

is also a continuous linear because it is a composition of continuous linear maps. Hence, T is continuous.

Now, if $T : V \to W$ is an isomorphism where V is a finite-dimensional normed space. Then, W

is also a finite-dimensional normed space. Then, T is continuous by the above argument. Then, $T^{-1} : W \to V$ is a linear map from a finite-dimensional normed space, hence also continuous. Therefore, T is a homeomorphism.

Finally, let V be a finite-dimenisonal vector space equipped with two norms $\|\cdot\|_1$ and $\|\cdot\|_2$. Then, the identity map $I : V \to V$ is an isomorphism between the two finite-dimensional normed spaces. Then, I is a homeomorphism by the above argument. 😊

1.3 Lecture 3

Our goal is to generalize on-variable differentiation to multi-variable differentiation. In more precise terms, we want to:

Note:

Obtain a natural derivative of $F : U \to \mathbb{R}^m$ at a point $p \in$ open set $U \subseteq \mathbb{R}^n$ by generalizing the derivative of $f : U \to \mathbb{R}$ at a point $p \in U \subseteq \mathbb{R}$.

The key is to understand that f is differentiable at p if and only if f is "approximately linear" at p.

Consider an example in 2-dimensional Euclidean space to motivate our new definition of derivatives in multi-dimensional spaces.

Example 1.3.1 (Derivative in $\mathbb{R}^2$)

Is $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$f(x_1, x_2) := (x_1^2, x_2^2)$$

differentiable at the point $(1, 2) \in \mathbb{R}^2$?

Solution

Let's first try to use the definition of derivative in $\mathbb{R}$ to see if it works. Let $p = (1, 2)$, then we have

$$f'(p) = \lim_{h \to 0} \frac{f(p + h) - f(p)}{h} = \lim_{h \to 0} \frac{f(\langle 1, 2\rangle + h) - f(1, 2)}{h}$$

where h is a vector. But this does not make sense because we have not defined what it meant by division of a vector by a scalar. Hence, we need a new definition of derivative in $\mathbb{R}^2$, or more generally in multi-dimensional spaces.

Definition 1.3.1: Multi-variable derivative (aka total derivative or Frechet derivative)

Let $f : U \to \mathbb{R}^m$ be given where U is an open subset of $\mathbb{R}^n$. The function f is differentiable at $p \in U$ with derivative $(Df)_p = T$ if $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation and

$$f(p+v) = f(p) + T(v) + R(v) \implies \lim_{\|v\| \to 0} \frac{R(v)}{\|v\|} = 0.$$

Note:

The form is coming from the definition of derivative in $\mathbb{R}$ by rearranging the terms in

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

to

$$f(x+h) = f(x) + f'(x)h + R(h) \implies \lim_{h \to 0} \frac{R(h)}{\|h\|} = 0.$$

We say that they Taylor remainder R is *sublinear* because it tends to 0 faster than $\|v\|$.

Note:

Our definition of differentiability is coordinate free, which means we can study differentiation on spaces other than $\mathbb{R}^n$, e.g. differential manifolds, which is the natural next topic to study after $\mathbb{R}^n$.

Example 1.3.2 (Back to *example 1.3.1*)

Under our new definition, we can try to determine the differentiability of f at $p = (1, 2)$. Write

$$\begin{aligned} f(p+v) &= f(1+v_1, 2+v_2) \\ &= \langle (1+v_1)^2, (2+v_2)^2 \rangle \\ &= \langle 1 + 2v_1 + v_1^2, 4 + 4v_2 + v_2^2 \rangle \\ &= \langle 1, 4 \rangle + \langle 2v_1, 4v_2 \rangle + \langle v_1^2, v_2^2 \rangle \\ &= f(p) + \langle 2v_1, 4v_2 \rangle + \langle v_1^2, v_2^2 \rangle. \end{aligned}$$

Then, we can define $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T(v_1, v_2) = \langle 2v_1, 4v_2 \rangle$ and $R : \mathbb{R}^2 \to \mathbb{R}^2$ by $R(v_1, v_2) = \langle v_1^2, v_2^2 \rangle$. Now we just have to show that $\lim_{\|v\| \to 0} \frac{R(v)}{\|v\|} = 0$. We can check whether the norm of $\frac{R(v)}{\|v\|}$ go to 0 when $\|v\| \to 0$. It doesn't matter which norm we choose (the Euclidean norm, sum norm, or max norm, etc.), because they are equivalent in finite-dimensional spaces. Let's choose the Euclidean norm for simplicity. Then, we have

$$\begin{aligned} \left\| \frac{R(v)}{\|v\|} \right\| = \frac{1}{\|v\|} \|\langle v_1^2, v_2^2 \rangle\| &= \sqrt{\frac{v_1^4 + v_2^4}{v_1^2 + v_2^2}} \\ &= \sqrt{\frac{v_1^4}{v_1^2 + v_2^2} + \frac{v_2^4}{v_1^2 + v_2^2}} \\ &\le \sqrt{\frac{v_1^4}{v_1^2} + \frac{v_2^4}{v_2^2}} = \sqrt{v_1^2 + v_2^2} = \|v\|, \end{aligned}$$

which obviously goes to 0 when $\|v\| \to 0$. Hence, f is differentiable at $p = (1, 2)$ with derivative $T(v_1, v_2) =$

$\langle 2v_1, 4v_2 \rangle$.

Note:

Note: this T is only true for $p = (1, 2)$. For other points, we may have different derivatives.

Theorem 1.3.1 Derivative is unique

If f is differentiable at p, then it uniquely determins $(Df)_p$ according to the limit formula, valid for all $u \in \mathbb{R}^n$,

$$(Df)_p(u) = \lim_{t \to 0} \frac{f(p+tu) - f(p)}{t}.$$

Proof: Let T be a linear transformation that satisfies *definition 1.3.1*. Now fix any $u \in \mathbb{R}^n$ and take $v = tu$. Then

$$\frac{f(p+tu) - f(p)}{t} = \frac{T(tu) + R(tu)}{t} = T(u) + \frac{R(tu)}{t\|u\|}\|u\| = T(u) + \frac{R(tu)}{\|tu\|}\|u\|.$$

The last term converges to 0 as $t \to 0$ since $\|tu\| \to 0$. Limits, when they exist, are unique, so $T(u)$ is uniquely determined. 😊

Theorem 1.3.2 Differentiability implies continuity

Proof: Differentiability at p implies that

$$|f(p+v) - f(p)| = |(Df)_p(v) + R(v)| \leq |(Df)_p(v)| + |R(v)| = \|(Df)_p\||v| + |R(v)|,$$

which tends to 0 as $v \to 0$ since $\|(Df)_p\|$ is finite in a finite-dimensional space and R is sublinear. Hence, f is continuous at p. 😊

Corollary 1.3.1 Total derivative existsnce implies partial derivative existence

Example 1.3.3 (Some important concepts)

- All partial derivatives exist at a point does not imply total derivative exists at that point.
- All directional derivatives exist at a point does not imply total derivative exists at that point.
- Partial derivatives exist and are continuous at a point implies total derivative exists at that point.

Chapter 2

Starting a new chapter

2.1 Demo of commands

Definition 2.1.1: Some defintion

yap

Question 1: Some question

yap

Solution

Some proof: yap ☺

Note:

Some note

Theorem 2.1.1 Some theorem

yap

Wrong Concept 2.1.1: Some wrong concept

yap

Lemma 2.1.1 Some lemma

yap

Proposition 2.1.1 Some proposition

yap

Example 2.1.1 (Some example)

yap

Claim 2.1.1 Some claim

yap

Corollary 2.1.1 Some corollary

yap

Some unlabeled theorem

This is a new paragraph

Algorithm 1: Some algorithm

Input: input
Output: output

```
/* This is a comment */
This is first line ;                    // This is also a comment
if x > 5 then
    do nothing
else if x < 5 then
    do nothing
else
    do nothing
end
while x == 5 do
    still do nothing
end
foreach x = 1 : 5 do
    do nothing
end
return return nothing
```