Math 110 HW9

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11/04/2023

Problem 1.

Suppose V is a complex finite-dimensional vector space, $T \in \mathcal{L}(V)$, and $\lambda \in \mathbb{C}$. Prove or disprove: if λ is an eigenvalue of T, then it is also an eigenvalue of T'. Prove or disprove the converse.

Proof. **Forward direction:** Let λ be the eigenvalue, we want to show that there exists $\varphi \neq 0 \in V'$ such that

$$\begin{aligned}
(T' - \lambda I)(\varphi) &= 0 \\
T'(\varphi) - \lambda\varphi &= 0 \\
\varphi \circ T(v) - \varphi(\lambda v) &= 0 \qquad \forall v \in V \\
\varphi\,(T - \lambda I)\,(v) &= 0.
\end{aligned}$$

Notice $(T - \lambda I) \in \mathcal{L}(V)$ is not injective because the $\operatorname{null}(T - \lambda I) \neq \{0\}$, hence not surjective. Therefore, $\dim \operatorname{range}(T - \lambda I) \leq \dim V - 1$. Hence, we can construct φ such that φ vanishes on $\operatorname{range}(T - \lambda I)$, and φ is not zero. For example, extend the basis of $\operatorname{range}(T - \lambda I)$ to the basis of V, denote the extended basis as $\{v_1, \dots\}$, then the dual basis of v_1 is a suitable φ.

Backward direction: We will prove by contrapositive. Suppose λ is an eigenvalue of T' but not T, then $\operatorname{null}(T - \lambda I) = \{0\}$. Thus, $T - \lambda I$ is injective hence surjective, which means $\operatorname{range}(T - \lambda I) = V$. Therefore, there does not exists $\varphi \neq 0 \in V'$ such that

$$\varphi\,(T - \lambda I)\,(v) = 0$$

for all $v \in V$. Following the above derivation (in the forward direction proof) backwards, there does not exists $\varphi \neq 0$ such that

$$(T' - \lambda I)(\varphi) = 0,$$

which means λ is not an eigenvalue of T'. $\square$

Problem 2.

Let V be the complex vector space of bivariate polynomials of total degree at most 2, and let T be the linear operator $T : p \mapsto \frac{\partial p}{\partial x} - \frac{\partial p}{\partial y}$. Determine, with proof, (a) the minimal polynomial, (b) all eigenvalues, and (c) the corresponding eigenvectors of T.

Solution.

(a) Let $p = ax^2 + by^2 + cxy + dx + ey + f$, then

$$\begin{aligned}
T(p) &= 2ax + cy + d - 2by - cx - e = x(2a - c) + y(c - 2b) + (d - e) \\
T^2(p) &= 2a - c - c + 2b = 2a + 2b - 2c \\
T^3(p) &= 0.
\end{aligned}$$

Notice the minimal polynomial must be of degree at least 3 because $I(p), T(p), T^2(p)$ are linearly independent due to different degrees. In fact, the minimal polynomial is of degree 3 because $T^3(p) = 0$. Therefore, the coefficients for the minimal polynomial satisfy

$$\begin{aligned}
T^3(p) + \beta T^2(p) + \gamma T(p) + \delta I(p) &= 0 \\
\beta T^2(p) + \gamma T(p) + \delta I(p) &= 0 \\
\implies \beta = \gamma = \delta &= 0.
\end{aligned}$$

Hence, the minimal polynomial is $q(z) = z^3$.

(b) By *Theorem 5.27*, the eigenvalues of T are the zeros of the minimal polynomial, hence the only eigenvalues are 0.

(c) We want to find $E(0, T)$.

$$\begin{aligned}
T(p) = 0 &= x(2a - c) + y(c - 2b) + (d - e) \\
&\implies \begin{cases} 2a - c = 0 \\ c - 2b = 0 \\ d - e = 0 \end{cases} \\
&\implies \begin{cases} a = b = \frac{c}{2} \\ d = e. \end{cases}
\end{aligned}$$

Hence,

$$\begin{aligned}
p &= cx^2 + cy^2 + 2cxy + dx + dy + f \\
&= c(x + y)^2 + d(x + y) + f \\
&\implies E(0, T) = \operatorname{span}\{1, x + y, (x + y)^2\}.
\end{aligned}$$

The corresponding eigenvectors are $\{1, x + y, (x + y)^2\}$.

□

Problem 3.

Suppose V is a finite-dimensional vector space. Prove or disprove: if two linear operators T and S from $\mathcal{L}(V)$ commute, then T is diagonalizable if and only if S is.

Solution. We will provide a counter example. Consider $V = \mathcal{P}_2(\mathbb{R})$, $S = I$ and $T = D$, the differentiation operator. Obviously, $ID = DI$, and S is diagonalizable, but T is not because all non-zero vectors in V will be reduced by one degree after mapping with T, hence no non-zero vectors is a scalar multiple itself after mapping with T. $\square$

Problem 4.

Let $V := \mathcal{P}_3(\mathbb{R})$ and let $T \in \mathcal{L}(V)$ be the operator $f(x) \mapsto f(x-1) + f(x+1)$. Is T triangularizable? If yes, is T also diagonalizable? Justify your answers.

Solution. Let $f = ax^3 + bx^2 + cx + d$, then

$$\begin{aligned} T(f) &= a(x-1)^3 + b(x-1)^2 + c(x-1) + d + a(x+1)^3 + b(x+1)^2 + c(x+1) + d \\ &= 2ax^3 + 2bx^2 + (6a+2c)x + (2b+2d). \end{aligned}$$

Hence, the matrix representation of T with respect to $(x^3, x^2, x, 1)$ is

$$\mathcal{M}(T) = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 6 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \end{bmatrix},$$

which is already triangular. Therefore, T is triangularizable.

However, T is not diagonalizable. By *Theorem 5.41*, the eigenvalues of $T = \{2\}$. Then we want to find the eigenspace $E(2, T)$,

$$\begin{aligned} T(f) = 2ax^3 + 2bx^2 + (6a+2c)x + (2b+2d) &= 2ax^3 + 2bx^2 + 2cx + 2d \\ (6a+2c)x + (2b+2d) &= 2cx + 2d \end{aligned}$$

$$\implies \begin{cases} 6a + 2c = 2c \\ 2b + 2d = 2d \end{cases}$$

$$\implies \begin{cases} a = 0 \\ b = 0. \end{cases}$$

Hence, $E(2, T) = \operatorname{span}\{x, 1\}$, which does not span $\mathcal{P}_3(\mathbb{R})$. Therefore, T is not diagonalizable. $\square$

Problem 5.

Determine whether or not the function taking the pair $((x_1, x_2, x_3), (y_1, y_2, y_3)) \in \mathbb{R}^3 \times \mathbb{R}^3$ to $x_1 y_1 + x_2 y_2 - 3x_2 y_3 + 3x_3 y_2 + x_3 y_3$ is an inner product.

Proof. **Positivity:** Let $v = (a, b, c)$,

$$\langle v, v\rangle = a^2 + b^2 - 3bc + 3bc + c^2 = a^2 + b^2 + c^2 \geq 0.$$

Definiteness: Let $v = (a, b, c)$,

$$\begin{aligned} \langle v, v\rangle &= 0 \\ a^2 + b^2 + c^2 &= 0 \\ &\implies a^2 = b^2 = c^2 = 0 \\ &\implies a = b = c = 0. \end{aligned}$$

Additivity in first slot: Let $u = (a_1, b_1, c_1), v = (a_2, b_2, c_2), w = (a_3, b_3, c_3)$,

$$\begin{aligned} \langle u + v, w\rangle &= (a_1 + a_2)(a_3) + (b_1 + b_2)(b_3) + (c_1 + c_2)(c_3) - 3(b_1 + b_2)(c_3) + 3(c_1 + c_2)(b_3) \\ &= (a_1 a_3 + b_1 b_3 + c_1 c_3 - 3b_1 c_3 + 3c_1 b_3) + (a_2 a_3 + b_2 b_3 + c_2 c_3 - 3b_2 c_3 + 3c_2 b_3) \\ &= \langle u, w\rangle + \langle v, w\rangle. \end{aligned}$$

Homogeneity in first slow: Let $u = (a_1, b_1, c_1), v = (a_2, b_2, c_2)$,

$$\begin{aligned} \langle \lambda u, w\rangle &= \lambda a_1 a_2 + \lambda b_1 b_2 + \lambda c_1 c_2 - 3\lambda b_1 c_2 + 3\lambda c_1 b_2 \\ &= \lambda(a_1 a_2 + b_1 b_2 + c_1 c_2 - 3b_1 c_2 + 3c_1 b_2) \\ &= \lambda\langle u, w\rangle. \end{aligned}$$

Conjugate symmetry: Let $u = (1, 2, 3), v = (4, 5, 6)$,

$$\begin{aligned} \langle u, v\rangle &= 1 \times 4 + 2 \times 5 + 3 \times 6 - 3 \times 2 \times 6 + 3 \times 3 \times 5 \\ &\neq 1 \times 4 + 2 \times 5 + 3 \times 6 - 3 \times 3 \times 5 + 3 \times 2 \times 6 = \langle v, u\rangle. \end{aligned}$$

Hence, the function is not an inner product. $\square$