Math 110 HW13

Neo Lee

12/02/2023

Problem 1.

Let T be a self-adjoint operator on a finite-dimensional inner product space (real or complex) such that $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ are the only eigenvalues of T. Prove that $p(T) = 0$ where $p(\lambda) := (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3)$. Give a counterexample to this statement for an operator which is not self-adjoint.

Proof. Since T is self-adjoint, hence normal, by either the Real or Complex spectral theorem, V has an orthonormal basis consisting of eigenvectors of T, denote them as $e_1, \dots, f_1, \dots, g_1, \dots,$ where e, f, g are the eigenvectors corresponding to $\lambda_1, \lambda_2, \lambda_3$ respectively. Then, for any $v \in V$, we can write v as a linear combination of these eigenvectors, i.e.

$$v = \sum_{i=1}^{n} \alpha_i e_i + \sum_{i=1}^{n} \beta_i f_i + \sum_{i=1}^{n} \gamma_i g_i.$$

We use the property that polynomials applied on operators are commutative under composition, then,

$$\begin{aligned}
p(T)v &= (T - \lambda_1 I)(T - \lambda_2 I)(T - \lambda_3 I) \left(\sum_{i=1}^{n} \alpha_i e_i + \sum_{i=1}^{n} \beta_i f_i + \sum_{i=1}^{n} \gamma_i g_i \right) \\
&= (T - \lambda_1 I)(T - \lambda_2 I)(T - \lambda_3 I) \left(\sum_{i=1}^{n} \alpha_i e_i \right) \\
&\quad + (T - \lambda_1 I)(T - \lambda_2 I)(T - \lambda_3 I) \left(\sum_{i=1}^{n} \beta_i f_i \right) \\
&\quad + (T - \lambda_1 I)(T - \lambda_2 I)(T - \lambda_3 I) \left(\sum_{i=1}^{n} \gamma_i g_i \right) \\
&= (T - \lambda_2 I)(T - \lambda_3 I)(T - \lambda_1) \left(\sum_{i=1}^{n} \alpha_i e_i \right) \\
&\quad + (T - \lambda_1 I)(T - \lambda_3)(T - \lambda_2 I) \left(\sum_{i=1}^{n} \beta_i f_i \right) \\
&\quad + (T - \lambda_1 I)(T - \lambda_2 I)(T - \lambda_3) \left(\sum_{i=1}^{n} \gamma_i g_i \right) \\
&= 0,
\end{aligned}$$

because $e \in \operatorname{null}(T - \lambda_1 I), f \in \operatorname{null}(T - \lambda_2 I), g \in \operatorname{null}(T - \lambda_3 I)$. Therefore, $p(T)v = 0$ for all $v \in V$, i.e. $p(T) = 0$.

Counterexample: Let $V = \mathbb{R}^4$ and T be an opeartor with the action

$$T(e_1) = e_1, \quad T(e_2) = 2e_2, \quad T(e_3) = 3e_3, \quad T(e_4) = e_1 + e_4.$$

Clearly this is not self-adjoint as can be seen from the matrix form that it does not equal its conjugate transpose. Also, the eigenvalues are $1, 2, 3$ as can be seen from the matrix form as an upper triangular matrix with $1, 2, 3$ as diagonal entries.

Now, we apply $p(T)$ on e_4,

$$\begin{aligned}
p(T)e_4 &= (T - I)(T - 2I)(T - 3I)e_4 \\
&= (T - I)(T - 2I)(e_1 + e_4 - 3e_4) \\
&= (T - I)(T - 2I)(e_1 - 2e_4) \\
&= (T - I)[(e_1 - 2(e_1 + e_4)) - 2(e_1 - 2e_4)] \\
&= (T - I)(-3e_1 + 2e_4) \\
&= 2e_1 \neq 0.
\end{aligned}$$

□

Problem 2.

Let $T \in \mathcal{L}(V)$. Show that

$$\langle v, u \rangle_T := \langle Tv, u \rangle$$

is an inner product on V if and only if T is positive (per our definition of positivity).

Proof. ($\Longrightarrow$) Assume $\langle v, u \rangle_T$ is an inner product on V. Then,

$$\langle Tv, v \rangle = \langle v, v \rangle_T \geq 0.$$

Also,

$$\begin{aligned}
\langle v, Tu \rangle &= \overline{\langle Tu, v \rangle} \\
&= \overline{\langle u, v \rangle_T} \\
&= \langle v, u \rangle_T \\
&= \langle Tv, u \rangle \\
&= \langle v, T^* u \rangle.
\end{aligned}$$

Hence, by the uniqueness of Riesz representation theorem, $T = T^*$, i.e. T is self-adjoint. Therefore, T is positive.

($\Longleftarrow$) Assume T is positive.

Positivity: For any $v \in V$,

$$\langle v, v \rangle_T = \langle Tv, v \rangle \geq 0.$$

Definiteness:

$$\langle v, v \rangle_T = 0 \iff \langle Tv, v \rangle = 0.$$

Per our definition of strict positivity, $\langle Tv, v \rangle > 0$ for all $v \neq 0 \in V$. Hence,

$$\langle Tv, v \rangle = 0 \iff v = 0.$$

Additivity on the first slot: For any $u, v, w \in V$,

$$\langle u + v, w \rangle_T = \langle T(u + v), w \rangle = \langle Tu + Tv, w \rangle = \langle Tu, w \rangle + \langle Tv, w \rangle = \langle u, w \rangle_T + \langle v, w \rangle_T.$$

Homogeneity on the first slot: For any $u, v \in V$ and $\alpha \in \mathbb{F}$,

$$\langle \alpha u, v \rangle_T = \langle T(\alpha u), v \rangle = \langle \alpha Tu, v \rangle = \alpha \langle Tu, v \rangle = \alpha \langle u, v \rangle_T.$$

Conjugate symmetry: For any $u, v \in V$,

$$\langle u, v \rangle_T = \langle Tv, u \rangle = \overline{\langle u, Tv \rangle} = \overline{\langle Tu, v \rangle} = \overline{\langle u, v \rangle_T}.$$

$\square$

Problem 3.

Show that the operator $T = -D^2$ is nonnegative on the space $V := \operatorname{span}(1, \cos x, \sin x)$ over $\mathbb{R}$, with the inner product

$$\langle f, g\rangle := \int_{-\pi}^{\pi} f(x)g(x)dx.$$

Find

(a) its square root operator $\sqrt{T}$;

(b) an example of a self-adjoint operator $R \neq \sqrt{T}$ such that $R^2 = T$;

(c) an example of a non-self-adjoint operator S such that $S^*S = T$.

Proof. Let $f = a + b\cos x + c\sin x$. Then,

$$T(f) = -D^2(a + b\cos x + c\sin x) = -D^2(a) - D^2(b\cos x) - D^2(c\sin x) = b\cos x + c\sin x,$$

and

$$\begin{aligned}
\langle Tf, f\rangle &= \langle b\cos x + c\sin x, a + b\cos x + c\sin x\rangle \\
&= \int_{-\pi}^{\pi} (b\cos x + c\sin x)(a + b\cos x + c\sin x)dx \\
&= \int_{-\pi}^{\pi} b^2\cos^2 x + \int_{-\pi}^{\pi} c^2\sin^2 x dx \qquad (\textit{everything else are orthogonal}) \\
&= b^2\pi + c^2\pi \geq 0.
\end{aligned}$$

(a) $T(1) = 0, T(\cos x) = \cos x, T(\sin x) = \sin x$. Hence, T has eigenvalues $0, 1, 1$ with corresponding eigenvectors $1, \cos x, \sin x$ respectively. Then, by the spectral theorem, T has orthonormal eigenbasis $\{\frac{1}{\sqrt{2\pi}}, \frac{\cos x}{\sqrt{\pi}}, \frac{\sin x}{\sqrt{\pi}}\}$. $\sqrt{T}$ is uniquely determined by its action on the eigenbasis by scaling with the square root of the corresponding eigenvalues, i.e.

$$\sqrt{T}\left(\frac{1}{\sqrt{2\pi}}\right) = 0, \quad \sqrt{T}\left(\frac{\cos x}{\sqrt{\pi}}\right) = \frac{\cos x}{\sqrt{\pi}}, \quad \sqrt{T}\left(\frac{\sin x}{\sqrt{\pi}}\right) = \frac{\sin x}{\sqrt{\pi}}.$$

We can see that $\sqrt{T} = T$.

(b) Define R with the action on the eigenbasis as

$$R\left(\frac{1}{\sqrt{2\pi}}\right) = 0, \quad R\left(\frac{\cos x}{\sqrt{\pi}}\right) = \frac{\cos x}{\sqrt{\pi}}, \quad R\left(\frac{\sin x}{\sqrt{\pi}}\right) = -\frac{\sin x}{\sqrt{\pi}}.$$

(c) Define S with the action on the eigenbasis as

$$S\left(\frac{1}{\sqrt{2\pi}}\right) = 0, \quad S\left(\frac{\cos x}{\sqrt{\pi}}\right) = \frac{1}{\sqrt{2\pi}}, \quad S\left(\frac{\sin x}{\sqrt{\pi}}\right) = \frac{\cos x}{\sqrt{\pi}}.$$

The adjoint of S is

$$S^*\left(\frac{1}{\sqrt{2\pi}}\right) = \frac{\cos x}{\sqrt{\pi}}, \quad S^*\left(\frac{\cos x}{\sqrt{\pi}}\right) = \frac{\sin x}{\sqrt{\pi}}, \quad S^*\left(\frac{\sin x}{\sqrt{\pi}}\right) = 0.$$

$\square$

Problem 4.

Let T_1 and T_2 be normal operators on an n-dimensional inner product space V. Suppose both have n distinct eigenvalues $\lambda_1, \dots, \lambda_n$. Show that there is an isometry $S \in \mathcal{L}(V)$ such that $T_1 = S^* T_2 S$.

Proof. Since both operators have n distinct eiganvalues, their eigenvectors are linearly independent and span the whole space. Besides, by *Theorem 7.22*, the eigenvectors are orthogonal since T_1 and T_2 are normal. We normalize the eigenvectors and denote them as $e_1, \dots, e_n$ and $f_1, \dots, f_n$ for T_1 and T_2 respectively.

Then, we can construct an isometry S that maps from the eigen-basis of T_1 to the eigen-basis of T_2 by

$$S : e_i \mapsto f_i.$$

In fact, this isometry is invertible and hence a unitary operator. Then, for any $v \in V$, we can write v as a linear combination of these eigenvectors, i.e.

$$v = \sum_{i=1}^{n} \alpha_i e_i.$$

Then,

$$T_1(v) = \sum_{i=1}^{n} \alpha_i \lambda_i e_i.$$

On the other hand,

$$\begin{aligned}
S^* T_2 S(v) &= S^* T_2 S \left(\sum_{i=1}^{n} \alpha_i e_i \right) \\
&= S^* T_2 \left(\sum_{i=1}^{n} \alpha_i f_i \right) \\
&= S^* \left(\sum_{i=1}^{n} \alpha_i \lambda_i f_i \right) \\
&= S^{-1} \left(\sum_{i=1}^{n} \alpha_i \lambda_i f_i \right) \qquad (S^* = S^{-1} \because S \text{ is unitary}) \\
&= \sum_{i=1}^{n} \alpha_i \lambda_i e_i = T_1(v).
\end{aligned}$$

□

Problem 5.

Find the singular values of the operator $T \in \mathcal{P}_3(\mathbb{C}) : p(x) \mapsto 2xp'(x) - x^2p''(x)$ if the inner product on $\mathcal{P}_3(\mathbb{C})$ is defined as

$$\langle p, q\rangle := \int_{-1}^{1} p(x)\,\overline{q(x)}\,dx.$$

Solution. We first orthonormalize the basis $\{1, x, x^2, x^3\}$.

$$\begin{aligned}
v_1 &= 1\\
\|v_1\| &= \sqrt{\langle v_1, v_1\rangle} = \sqrt{\int_{-1}^{1} 1 \cdot 1dx} = \sqrt{2}\\
e_1 &= \frac{1}{\sqrt{2}}\\
v_2 &= x - \langle x, e_1\rangle e_1 = x\\
&= x - \frac{1}{2}\int_{-1}^{1} xdx\\
&= x\\
\|v_2\| &= \sqrt{\langle v_2, v_2\rangle} = \sqrt{\int_{-1}^{1} x^2dx} = \sqrt{\frac{2}{3}}\\
e_2 &= \sqrt{\frac{3}{2}}x\\
v_3 &= x^2 - \langle x^2, e_1\rangle e_1 - \langle x^2, e_2\rangle e_2\\
&= x^2 - \frac{1}{2}\int_{-1}^{1} x^2dx - \frac{3}{2}x\int_{-1}^{1} x^3dx\\
&= x^2 - \frac{1}{3}\\
\|v_3\| &= \sqrt{\langle v_3, v_3\rangle} = \sqrt{\int_{-1}^{1} (x^2 - \frac{1}{3})^2dx} = \sqrt{\frac{8}{45}}\\
e_3 &= \sqrt{\frac{45}{8}}(x^2 - \frac{1}{3})\\
v_4 &= x^3 - \langle x^3, e_1\rangle e_1 - \langle x^3, e_2\rangle e_2 - \langle x^3, e_3\rangle e_3\\
&= x^3 - \frac{1}{2}\int_{-1}^{1} x^3dx - \frac{3}{2}x\int_{-1}^{1} x^4dx - \frac{45}{8}(x^2 - \frac{1}{3})\int_{-1}^{1} x^5dx\\
&= x^3 - \frac{3}{5}x\\
\|v_4\| &= \sqrt{\langle v_4, v_4\rangle} = \sqrt{\int_{-1}^{1} (x^3 - \frac{3}{5}x)^2dx} = \sqrt{\frac{8}{175}}\\
e_4 &= \sqrt{\frac{175}{8}}(x^3 - \frac{3}{5}x).
\end{aligned}$$

Now we construct $\mathcal{M}(T,(e_1,e_2,e_3,e_4))$ by inspecting T's action on the orthonormal basis.

$$\begin{aligned}
T(e_1) &= 0 \\
T(e_2) &= 2x \cdot \sqrt{\frac{3}{2}} = 2e_2 \\
T(e_3) &= 2x\left(2 \cdot \sqrt{\frac{45}{8}}x\right) - x^2\left(2 \cdot \sqrt{\frac{45}{8}}\right) \\
&= 4\sqrt{\frac{45}{8}}x^2 - 2\sqrt{\frac{45}{8}}x^2 \\
&= 2e_3 + \sqrt{5}e_1 \\
T(e_4) &= 2x\left(3\sqrt{\frac{175}{8}}x^2 - \frac{3\sqrt{7}}{8}\right) - x^2\left(6\sqrt{\frac{175}{8}}x\right) \\
&= 6\sqrt{\frac{175}{8}}x^3 - 2\frac{3\sqrt{7}}{\sqrt{8}}x - 6\sqrt{\frac{175}{8}}x^3 \\
&= -\sqrt{21}e_2.
\end{aligned}$$

Hence, the matrix representation of T with respect to the orthonormal basis is

$$\mathcal{M}(T,(e_1,e_2,e_3,e_4)) = \begin{bmatrix} 0 & 0 & \sqrt{5} & 0 \\ 0 & 2 & 0 & -\sqrt{21} \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then the matrix representation of T^* is the conjugate transpose, which is

$$\mathcal{M}(T^*,(e_1,e_2,e_3,e_4)) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ \sqrt{5} & 0 & 2 & 0 \\ 0 & -\sqrt{21} & 0 & 0 \end{bmatrix}.$$

Then,

$$\mathcal{M}(T^*)\mathcal{M}(T) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & -2\sqrt{21} \\ 0 & 0 & 9 & 0 \\ 0 & -2\sqrt{21} & 0 & 21 \end{bmatrix}.$$

By solving the characteristic equation

$$-\lambda(9-\lambda)\left[(4-\lambda)(21-\lambda) - 84\right] = 0,$$

we get $\lambda = 0$ with multiplicity 2, $\lambda = 9$ with multiplicity 1, and $\lambda = 25$ with multiplicity 1. Hence, the singular values of T are $\sqrt{25} = 5, \sqrt{9} = 3, 0$. $\square$