Math 180A HW8

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Problem 1. True

Problem 2.

(a) 4

(b) $\sigma_x = 3, \sigma_y = 2$. $\text{Cov}(X,Y) = 4$. $\text{Corr}(X,Y) = \frac{4}{3\times 2} = \frac{2}{3}$.

(c) 8

Problem 3.

(a) Area of the square $= \sqrt{1+1}^2 = 2$. For $x \in [-1,0], y \in [-x-1, x+1]$ and $x \in [0,1], y \in [x-1, -x+1]$,

$$f_{X,Y}(x,y) = \frac{1}{2} \tag{1}$$

and 0 otherwise.

(b) $\text{Corr}(X,Y) = \frac{E[XY]-E[X]E[Y]}{\sigma_X \sigma_Y}$.

$$E[XY] = \int_{-1}^{0}\int_{-x-1}^{x+1} xy\frac{1}{2}dydx + \int_{0}^{1}\int_{x-1}^{-1+1} xy\frac{1}{2}dydx \tag{2}$$

By symmetry,

$$\int_{-1}^{0}\int_{-x-1}^{x+1} xy\frac{1}{2}dydx = 0 \tag{3}$$

and

$$\int_{0}^{1}\int_{x-1}^{-1+1} xy\frac{1}{2}dydx = 0. \tag{4}$$

Hence, $E[XY] = 0$.
For $x \in [-1,0]$,

$$\begin{aligned} f_X(x) &= \int_{-x-1}^{x+1} \frac{1}{2}dy \qquad (5)\\ &= x+1. \qquad (6) \end{aligned}$$

For $x \in [0,-1]$,

$$\begin{aligned} f_X(x) &= \int_{x-1}^{-x+1} \frac{1}{2}dy \qquad (7)\\ &= -x+1. \qquad (8) \end{aligned}$$

Hence,

$$E[X] = \int_{-1}^{0} x(x+1)dx + \int_{0}^{1} x(-x+1)dx \tag{9}$$
$$= \frac{-1}{6} + \frac{1}{6} \tag{10}$$
$$= 0. \tag{11}$$

By symmetry, $E[Y] = 0$. Thus, $\mathrm{Corr}(X,Y) = \frac{E[XY]-E[X]E[Y]}{\sigma_X \sigma_Y} = 0$.

(c) X and Y are not independent. Let $x = -1, f_X(-1) = 0 \Rightarrow f_X(-1)f_Y(y) = 0$ for all $y \in \mathbb{R}$. Yet, $f_{X,Y}(-1,0) = \frac{1}{2}$.

Problem 4.

(a)

$$M_Y'(t) = \frac{-5}{9}e^{-5t} + \frac{1}{18}e^{t} + e^{2t}. \tag{12}$$

Then,

$$E[Y] = M_Y'(0) \tag{13}$$
$$= \frac{-5}{9} + \frac{1}{18} + 1 \tag{14}$$
$$= 0.5. \tag{15}$$

(b) $M_Y(t) = E[e^{tY}]$. Let

$$f_Y(y) = \begin{cases} \frac{1}{3} & \text{if } y = 0, \\ \frac{1}{9} & \text{if } y = -5, \\ \frac{1}{18} & \text{if } y = 1, \\ \frac{1}{2} & \text{if } y = 2, \\ 0 & \text{otherwise.} \end{cases} \tag{16}$$

Then,

$$E[Y] = \frac{1}{9} \times -5 + \frac{1}{18} + \frac{1}{2} \times 2 \tag{17}$$
$$= 0.5. \tag{18}$$

Problem 5.

(a)

$$M_Y(t) = E[e^{tY}] = E[e^{t(aX+b)}] \tag{19}$$
$$= E[e^{t(aX)} \cdot e^{tb}] \tag{20}$$
$$= e^{tb}E[e^{t(aX)}] \tag{21}$$
$$= e^{tb}M_X(at). \tag{22}$$

(b)

$$\begin{aligned}
M_Y(t) &= e^t M_X(2t) && (23)\\
&= e^t E[e^{2tX}] && (24)\\
&= e^t \cdot \frac{1}{5}\int_0^\infty e^{2tx}e^{-\frac{1}{5}x}dx && (25)\\
&= \frac{e^t}{5}\int_0^\infty e^{(2t-\frac{1}{5})x}dx && (26)\\
&= \frac{e^t}{5(2t-\frac{1}{5})}\lim_{z\to\infty}\left[e^{(2t-\frac{1}{5})x}\right]_{x=0}^{x=z} && (27)\\
&= \frac{e^t}{10t-1}\lim_{z\to\infty}\left[e^{(2t-\frac{1}{5})x}\right]_{x=0}^{x=z}. && (28)
\end{aligned}$$

Hence,

$$M_Y(t) = \begin{cases} \infty & \text{if } t > 10, \\ \frac{e^t}{1-10t} & \text{if } t < 10, \\ 0 & \text{if } t = 10. \end{cases} \tag{29}$$

Porblem 6.

(a)

$$\begin{aligned}
M_X(t) = E[e^{tX}] &= \sum_{n=1}^{\infty} e^{tn}p(1-p)^{n-1} && (30)\\
&= p\left[e^t + e^{2t}(1-p) + e^{3t}(1-p)^2 + e^{4t}(1-p)^3 + ...\right] && (31)\\
&= p\left[\lim_{n\to\infty} e^t \cdot \frac{1-[e^t(1-p)]^n}{1-e^t(1-p)}\right]. && (32)
\end{aligned}$$

Hence,

$$M_X(t) = \begin{cases} \infty & \text{if } e^t(1-p) \geq 1 \Rightarrow t \geq -ln(1-p), \\ \frac{pe^t}{1-e^t+pe^t} & \text{otherwise.} \end{cases} \tag{33}$$

(b)

$$\begin{aligned}
M_X'(t) &= \frac{pe^t(1-e^t+pe^t) - pe^t(-e^t+pe^t)}{(1-e^t+pe^t)^2} = \frac{pe^t}{(1-e^t+pe^t)^2}, && (34)\\
E[X] = M_X'(0) &= \frac{p}{(1-1+p)^2} = \frac{1}{p}, && (35)\\
M_X''(t) &= \frac{pe^t(1-e^t+pe^t)^2 - pe^t \cdot 2(1-e^t+pe^t)(-e^t+pe^t)}{(1-e^t+pe^t)^4}, && (36)\\
E[Y^2] = M_X''(0)&\frac{p(1-1+p)^2 - 2p(1-1+p)(-1+p)}{(1-1+p)^4} = \frac{2-p}{p^2}, && (37)\\
Var(X) = E[Y^2] - (E[Y])^2 &= \frac{2-p}{p^2} - \frac{1}{p^2} = \frac{1-p}{p^2}. && (38)
\end{aligned}$$

Problem 7.

(a) $X = \text{exponential}(\frac{1}{10})$. By Markov's inequality, $P(X \geq 30) \leq \frac{E[X]}{30} = \frac{10}{30} = \frac{1}{3}$.

(b) $E[X] = \mu = 10$. $Var(X) = \sigma^2 = 100$. By Chebyshev's inequality, $P(X \geq 30) \leq P(|X - 10| \geq 20) \leq \frac{100}{20^2} \leq \frac{1}{4}$.

(c) $P(X \geq 30) = e^{-\frac{1}{10}\times 30} = e^{-3} \approx 0.04979$.

Problem 8. $P(X \geq a) = P(e^X \geq e^a) \leq \frac{E[e^x]}{e^a} = e^{-a}M_X(1)$.