Math 104 HW10

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Exercise 25.7

Proposition 1. $\sum_{n=1}^{\infty} \frac{1}{n^2} \cos nx$ *converges uniformly on* $\mathbb{R}$ *to a continuous function.*

Proof. Let $(M_n) = \frac{1}{n^2}$ and $g_n = \frac{1}{n^2} \cos nx$, then $|g_n| \leq \frac{1}{n^2} = M_m$ because $|\cos nx| \leq 1$. Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by *Theorem 15.1*, by *Weierstrass M-test*, $\sum_{n=1}^{\infty} \frac{1}{n^2} \cos nx$ converges uniformly on $\mathbb{R}$. Since cos is continuous, a constant times a continuous function $g_n = \frac{1}{n^2} \cos nx$ for $n \in \mathbb{N}$ is continuous. By *Theorem 25.5*, $\sum_{n=1}^{\infty} \frac{1}{n^2} \cos nx$ is continuous. $\square$

Exercise 25.10

Proposition 2.

(a) *$\sum \frac{x^n}{1+x^n}$ converges for $x \in [0,1)$.*

(b) *The series converges uniformly on $[0,a]$ for each $a \in (0,1)$.*

(c) *Does the series converge uniformly on $[0,1)$? Explain.*

Proof.

(a) For $x \in (0,1)$,

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{n+1}}{1+x^{n+1}} \cdot \frac{1+x^n}{x^n}\right| = \left|\frac{1+x^n}{1+x^{n+1}} \cdot x\right|$$
$$\lim x^{n+1} = \lim x^n = 0 \Longrightarrow \lim \left|\frac{1+x^n}{1+x^{n+1}} \cdot x\right| = \frac{1}{1} \cdot x = x < 1$$
$$\Longrightarrow \limsup |a_{n+1}/a_n| = \lim |a_{n+1}/a_n| < 1.$$

By *Ratio Test*, $\sum \frac{x^n}{1+x^n}$ converges for $x \in (0,1)$. For $x = 0$, the series obviously converges to 0. Hence, $\sum \frac{x^n}{1+x^n}$ converges for $x \in [0,1)$.

Alternatively, notice $\frac{x^n}{1+x^n} \leq x^n$ for $x \in [0,1)$. By *Comparison Test* with $\sum x^n$, which converges because $|x| < 1$, the series converges.

(b) We show that the series satisfies Cauchy Criterion uniformly on $[0,a]$. Notice for all $n \geq m$,

$$\left|\sum_{k=m}^{n} \frac{x^k}{1+x^k}\right| \leq \left|\sum_{k=m}^{n} x^k\right| \leq \left|\sum_{k=m}^{n} a^k\right|,$$

which means we only need to find N such that for all $n \geq m > N$,

$$\left|\sum_{k=m}^{n} a^k\right| < \epsilon.$$

We already know such N exists because $\sum a^k$ converges as $|a| < 1 \Longrightarrow \sum a^k$ satisfies Cauchy Criterion $\Longrightarrow$ such N exists. Hence, $\sum \frac{x^n}{1+x^n}$ converges uniformly on $[0,a]$.

(c) No, the series does not converge uniformly on $[0,1)$. Denote $f_n(x) = \sum_{k=0}^{n} \frac{x^k}{1+x^k}$. Assume for the sake of contradiction that the series converges uniformly on $[0,1)$, then there exists N such that for all $n > N$,

$$|f_n(x) - f(x)| < \epsilon \qquad \text{for all } x \in [0,1),$$

where

$$f(x) = \sum_{k=0}^{\infty} \frac{x^k}{1+x^k}.$$

Specifically for $n = N+1$

$$\left|\sum_{k=0}^{n} \frac{x^k}{1+x^k} - \sum_{k=0}^{\infty} \frac{x^k}{1+x^k}\right| = \left|\sum_{k=N+2}^{\infty} \frac{x^k}{1+x^k}\right| < \epsilon \qquad \text{for all } x \in [0,1).$$

Now denote $g(x) = \left|\sum_{k=N+2}^{\infty} \frac{x^k}{1+x^k}\right|$. However, notice as $x \to 1$, $g(x) \to \infty$. Therefore, there always exists $x \in [0, 1)$ such that $g(x) > \epsilon$, which is a contradiction. Hence, the series does not converge uniformly on $[0, 1)$.

$\square$

Exercise 28.4

Proposition 3. *Let $f(x) = x^2 \sin \frac{1}{x}$ for $x \neq 0$ and $f(0) = 0$.*

(a) *f is differentiable at each $a \neq 0$ and calculate $f'(a)$. Prove using* Theorem 28.3, 28.4.

(b) *f is differentiable at $x = 0$ and $f'(0) = 0$. Prove using the definition.*

(c) *f' is not continuous at 0.*

Proof. **(a)** We have $\frac{1}{x}$ is differentiable for $x \neq 0$ due to *Example 4*, and sin is differentiable, then by *Theorem 28.4*, $\sin \frac{1}{x}$ is differentiable at $a \neq 0$ and the derivative is $-\cos \frac{1}{x} \cdot \frac{1}{x^2}$. We also know x^2 is differentiable due to *Example 3*. By *Theorem 28.3 (iii)*, f is differentiable at $a \neq 0$ and

$$f'(a) = 2a \sin \frac{1}{a} - \cos \frac{1}{a}.$$

(b)

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{x^2 \sin \frac{1}{x}}{x} = \lim_{x \to 0} x \sin \frac{1}{x},$$

which we have shown in previous homework that the limit is 0 because we can take $\delta = \epsilon$, then since $|\sin \frac{1}{x}| \leq 1$ for $x \neq 0$,

$$|x| < \delta \Longrightarrow \left| x \sin \frac{1}{x} \right| < \delta = \epsilon.$$

(c) Consider the sequence $(x_n) = \frac{1}{n}$, which has limit equal to 0. Then,

$$f'(x_n) = \frac{2}{n} \sin n - \cos n.$$

Assume for the sake of contradiction that $\lim f'(x_n) = 0$, which means there exists N such that for all $n > N$,

$$\left| \frac{2}{n} \sin n - \cos n \right| < \epsilon.$$

More concretely, take $\epsilon = 0.1$. Now, notice $\lim \frac{2}{n} \sin n = 0$ because $\left| \frac{2}{n} \sin n \right| \leq \left| \frac{2}{n} \right|$, which has a limit of 0. This means there exists M such that for all $n > M$, $\left| \frac{2}{n} \sin n \right| < \epsilon$. However, notice there exists $n > \max\{N, M\}$ such that $\cos n > 2\epsilon$, which means

$$\left| \frac{2}{n} \sin n - \cos n \right| > \epsilon,$$

which is a contradiction. Hence, $\lim f'(x_n) \neq 0$, which means f' is not continuous at 0.

$\square$

Exercise 28.8

Proposition 4. *Let $f(x) = x^2$ for x rational and $f(x) = 0$ for x irrational.*

(a) *f is continuous at $x = 0$.*

(b) *f is discontinuous at each $x \neq 0$.*

(c) *f is differentaible at $x = 0$.*

Proof.

(a) Take $\delta = \min\{1, \epsilon\}$, then for all x irrational such that $|x - 0| < \delta \implies |f(x) - f(0)| = |0 - 0| < \epsilon$. Now, for all x rational such that $|x - 0| < \delta, \implies |f(x) - f(0)| = |x^2| < \epsilon^2 < \epsilon$ when $\epsilon < 1$, and $|f(x) - f(0)| = |x^2| < 1 \leq \epsilon$ when $\epsilon \geq 1$.

(b) For $x_0 \neq 0$, we can take $\epsilon = \frac{x_0^2}{2}$, then for all $\delta > 0$, there exists $x \in (x_0 - \delta, x_0 + \delta)$ and $|x_0| < |x|$ that is rational and $y \in (x_0 - \delta, x_0 + \delta)$ that is irrational. If x_0 is irrational, then $|x - x_0| < \delta$ but $|f(x) - f(x_0)| = |x^2| > |x_0^2| > \epsilon$. If x_0 is rational, then $|y - x_0| < \delta$ but $|f(y) - f(x_0)| = |x_0^2| > |x_0^2/2| = \epsilon$.

(c)

$$\frac{f(x) - f(0)}{x - 0} = \frac{x^2}{x} = x \qquad \text{if } x \text{ is rational}$$

$$\frac{f(x) - f(0)}{x - 0} = \frac{0}{x} = 0 \qquad \text{if } x \text{ is irrational.}$$

Then

$$\lim f'(x) = \lim \frac{f(x) - f(0)}{x - 0} = 0,$$

because we can take $\delta = \epsilon$ and $|x| < \delta \implies |x| < \epsilon$.

□

Exercise 28.14

Proposition 5. *Suppose f is differentiable at a,*

(a) $\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = f'(a)$,

(b) $\lim_{h\to 0} \frac{f(a+h)-f(a-h)}{2h} = f'(a)$.

Proof.

(a) Notice we can write $x = a + h$, then $x - a = h$ and $x \to a \equiv h \to 0$. Then,

$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{x\to a} \frac{f(x)-f(a)}{x-a} = f'(a).$$

(b)

$$\begin{aligned}\lim_{h\to 0} \frac{f(a+h)-f(a-h)}{2h} &= \lim_{h\to 0} \frac{f(a+h)-f(a)+f(a)-f(a-h)}{2h} \\ &= \lim_{h\to 0} \frac{f(a+h)-f(a)}{2h} + \frac{f(a)-f(a-h)}{2h} \\ &= \lim_{h\to 0} \frac{1}{2}\left(\frac{f(a+h)-f(a)}{h} + \frac{f(a)-f(a-h)}{h}\right) \\ &= \lim_{h\to 0} \frac{1}{2}\left(\frac{f(a+h)-f(a)}{h} + \frac{f(a-h)-f(a)}{-h}\right).\end{aligned}$$

Now notice we can write $x = a - h$, then $x - a = -h$ and $x \to a \equiv h \to 0$. Then,

$$\lim_{h\to 0} \frac{f(a-h)-f(a)}{-h} = \lim_{x\to a} \frac{f(x)-f(a)}{x-a} = f'(a).$$

We know f is differentiable at a, which means $f'(a) = L$ for finite L, then

$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{h\to 0} \frac{f(a-h)-f(a)}{-h} = L,$$

and

$$\begin{aligned}\lim_{h\to 0} \frac{1}{2}\left(\frac{f(a+h)-f(a)}{h} + \frac{f(a-h)-f(a)}{-h}\right) &= \frac{1}{2}\left(\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} + \lim_{h\to 0} \frac{f(a-h)-f(a)}{-h}\right) \\ &= \frac{1}{2}(L+L) = L = f'(a).\end{aligned}$$

$\square$