Math 128A HW2

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Section 2.2

Problem 1c

Use algebraic manipulation to show that

$$g_3(x) = \left(\frac{x+3}{x^2+2}\right)^{1/2}$$

has a fixed point at p precisely when $f(p) = 0$, where $f(x) = x^4 + 2x^2 - x - 3$.

Proof.

$$\begin{aligned} f(p) = 0 &= p^4 + 2p^2 - p - 3 \\ p^2(p^2+2) &= p + 3 \\ p &= \left(\frac{p+3}{p^2+2}\right)^{1/2} = g(p). \end{aligned}$$

□

Problem 8

Use a fixed-point iteration method to determine a solution accurate to within 10^{-2} for $x^3 - x - 1 = 0$ on $[1, 2]$. Use $p_0 = 1$.

Solution. Define $g(x) := (x+1)^{1/3}$. Then $g(x) = x$ when $x^3 - x - 1 = 0$. Indeed, $g(x) \in [1, 2]$ for $x \in [1, 2]$ and $|g'(x)| < 1$ for $x \in [1, 2]$. By the fixed-point theorem, we are sure that the fixed-point iteration will converge to the solution.

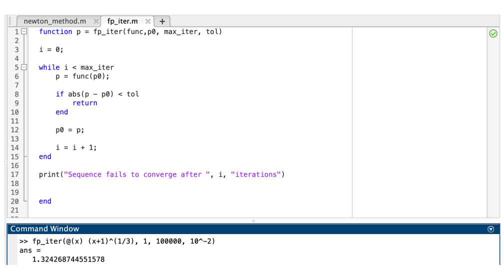


Figure 1: $x = 1.3243$

□

Problem 19

Let $g \in C^1[a,b]$ and p be in (a,b) with $g(p) = p$ and $|g'(p)| > 1$. Show that there exists a $\delta > 0$ such that if $0 < |p_0 - p| < \delta$, then $|p_0 - p| < |p_1 - p|$. Thus, no matter how close the initial approximation p_0 is to p, the next iteration p_1 is farther away, so the fixed-point iteration does not converge if $p_0 \neq p$.

Proof. $|g'(p) > 1| \Rightarrow |g'(p)| = 1 + \epsilon$ for some $\epsilon > 0$. Since g' is continuous, there exists $\delta > 0$ such that for all $c \in (p - \delta, p + \delta)$, $|g'(c) - g'(p)| < \epsilon \Rightarrow |g'(c)| > 1$. Now consider such $p_0 \in (p - \delta, p + \delta) - \{p\}$,

$$\begin{aligned}
|p_1 - p| &= |g(p_0) - p| \\
|p_1 - p| &= |g'(\xi)(p_0 - p)| \qquad (\textit{for } \xi \textit{ between } p_0, p) \\
\frac{|p_1 - p|}{|p_0 - p|} &= |g'(\xi)| \\
\frac{|p_1 - p|}{|p_0 - p|} &> 1. \qquad (\because \xi \in (p - \delta, p + \delta))
\end{aligned}$$

□

Problem 20

Let A be a given positive constant and $g(x) = 2x - Ax^2$.

a. Show that if fixed-point iteration converges to a nonzero limit, then the limit is $p = 1/A$, so the inverse of a number can be found by using only multiplications and subtractions.

   Solution. Let $g(p) = p = 2p - Ap^2$,

$$\begin{aligned}
p &= 2p - Ap^2 \\
Ap^2 - p &= 0 \\
p(Ap - 1) &= 0.
\end{aligned}$$

   Hence, $p = 0$ or $Ap = 1 \Rightarrow \underline{p = 1/A}$. □

b. Find an interval about $1/A$ for which fixed-point iteration converges, provided p_0 is in that interval.

   Solution. Define the interval $I = (\frac{1}{A} - |\frac{1}{2A}|, \frac{1}{A} + |\frac{1}{2A}|)$. Now consider $c \in I$, c can be written as $\frac{1}{A} + \epsilon$ such that $|\epsilon| < |\frac{1}{2A}|$.

$$\begin{aligned}
|g'(c)| &= |2 - 2Ac| \\
&= |2 - 2A\left(\frac{1}{A} + \epsilon\right)| \\
&= |2A\epsilon| \\
&< |2A \cdot \frac{1}{2A}| \\
&< 1.
\end{aligned}$$

Now consider $g(c) - g(\frac{1}{A})$,

$$\begin{aligned} |g(c) - g(\frac{1}{A})| &= |g'(\xi)(c - \frac{1}{A})| \qquad (\xi \textit{ between } c, \frac{1}{A}) \\ |g(c) - \frac{1}{A}| &< 1 \cdot |c - \frac{1}{A}| \\ |g(c) - \frac{1}{A}| &< |\frac{1}{2A}|. \end{aligned}$$

Hence, $g(c) \in I$.

We have verified that for all $c \in I$, $g'(c) < 1$ and $g(c) \in I$. Thus, by fixed point theorem, the fixed-point iteration with $p_0 \in I$ will converge. ☐

Section 2.3

Problem 6c

Use Newton's method to find solutions accurate to within 10^{-5} for

$$f(x) = 2xcos(2x) - (x-2)^2 = 0 \qquad \textit{for } 2 \le x \le 3 \textit{ and } 3 \le x \le 4.$$

Solution.

$$f'(x) = -4xsin(2x) + 2cos(2x) - 2(x-2).$$

Hence, for Newton's method,

$$g(x) = x - \frac{2xcos(2x) - (x-2)^2}{-4xsin(2x) + 2cos(2x) - 2(x-2)}.$$

With the fixed-point iteration function coded in Section 2.2 problem 8, we have ☐

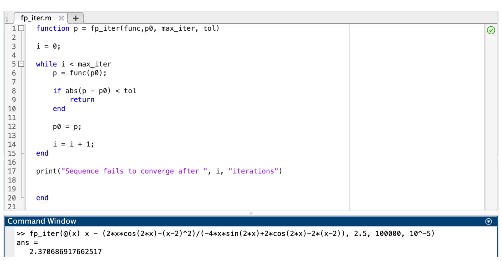


Figure 2: $p_0 = 2.5 \in [2, 3]$

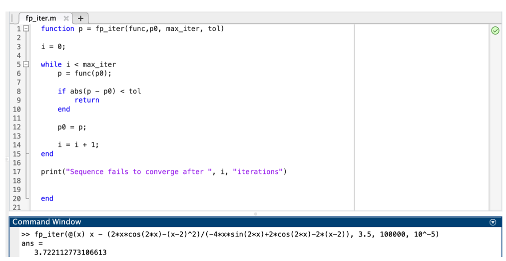


Figure 3: $p_0 = 3.5 \in [3, 4]$

Problem 8c

Repeat 6c using the Secant method.

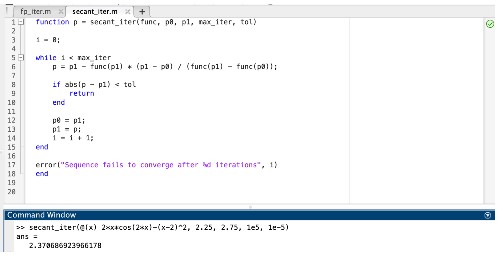


Figure 4: $p_0 = 2.25, p_1 = 2.75$

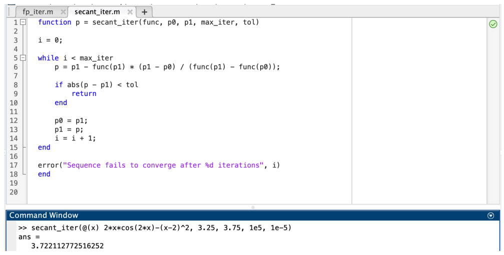


Figure 5: $p_0 = 3.25, p_1 = 3.75$

Solution. □

Problem 16

Use Newton's method to approximate the solution of $f(x) = x^2 - 10cos(x) = 0$ to within 10^{-5} with $\mathbf{(a)} p_0 = -100$ and $\mathbf{(d)} p_0 = 25$.

Solution.

$$f'(x) = 2x + 10sin(x)$$
$$g(x) = x - \frac{x^2 - 10cos(x)}{2x + 10sin(x)}.$$

□

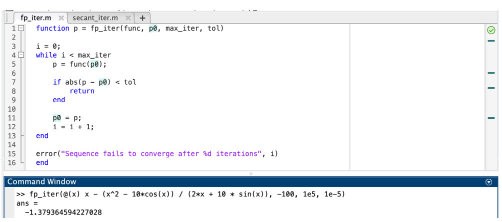


Figure 6: $\mathbf{(a)} p_0 = -100$

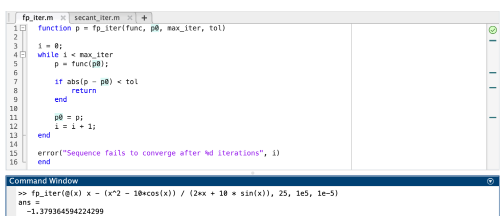


Figure 7: $\mathbf{(d)} p_0 = 25$

Section 2.4

Problem 2c

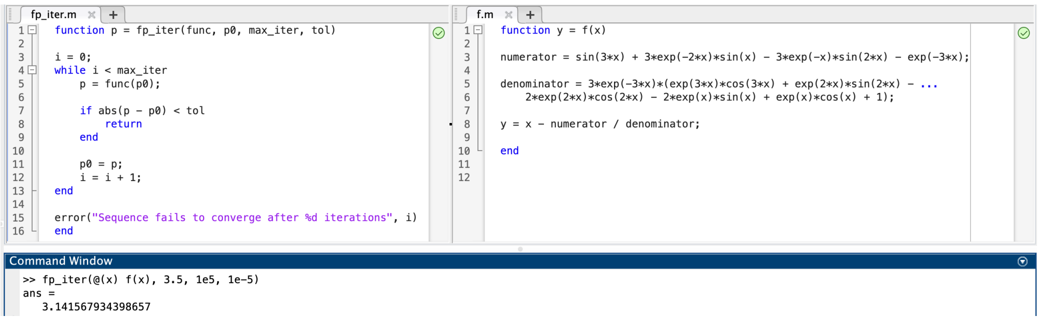
Use Newton's method to find solutions accurate to within 10^{-5} to

$$f(x) = sin(3x) + 3e^{-2x} sin(x) - 3e^{-x} sin(2x) - e^{-3x} = 0 \qquad for\ 3 \leq x \leq 4.$$

Solution.

$$f'(x) = 3e^{-3x} \left(e^{3x} cos(3x) + e^{2x} sin(2x) - 2e^{2x} cos(2x) - 2e^{x} sin(x) + e^{x} cos(x) + 1\right)$$
$$g(x) = x - \frac{sin(3x) + 3e^{-2x} sin(x) - 3e^{-x} sin(2x) - e^{-3x}}{3e^{-3x} \left(e^{3x} cos(3x) + e^{2x} sin(2x) - 2e^{2x} cos(2x) - 2e^{x} sin(x) + e^{x} cos(x) + 1\right)}$$

□



fp_iter.m

```
function p = fp_iter(func, p0, max_iter, tol)

i = 0;
while i < max_iter
    p = func(p0);

    if abs(p - p0) < tol
        return
    end

    p0 = p;
    i = i + 1;
end

error("Sequence fails to converge after %d iterations", i)
end
```

f.m

```
function y = f(x)

numerator = sin(3*x) + 3*exp(-2*x)*sin(x) - 3*exp(-x)*sin(2*x) - exp(-3*x);

denominator = 3*exp(-3*x)*(exp(3*x)*cos(3*x) + exp(2*x)*sin(2*x) - ...
    2*exp(2*x)*cos(2*x) - 2*exp(x)*sin(x) + exp(x)*cos(x) + 1);

y = x - numerator / denominator;

end
```

Command Window

```
>> fp_iter(@(x) f(x), 3.5, 1e5, 1e-5)
ans =
   3.141567934398657
```

Figure 8: $x = 3.1415679$

Problem 8

a. Show that the sequence $p_n = 10^{-2^n}$ converges quadratically to 0.

Proof.

$$\begin{aligned}
10^{-2^{n+1}} &= 10^{-2^n \cdot 2} \\
10^{-2^{n+1}} &= \left(10^{-2^n}\right)^2 \\
|10^{-2^{n+1}} - 0| &= |\left(10^{-2^n}\right) - 0|^2 \\
\frac{|10^{-2^{n+1}} - 0|}{|\left(10^{-2^n}\right) - 0|^2} &= 1 \\
\lim_{n\to\infty} \frac{|p_{n+1} - 0|}{|p_n - 0|^2} &= 1.
\end{aligned}$$

□

b. Show that the sequence $p_n = 10^{-n^k}$ does not converge to 0 quadratically, regardless of the size of the exponent $k > 1$.

Proof. If the sequence converge quadratically, $\lim_{n\to\infty} \frac{|p_{n+1}|}{|p_n|^2} = C$ for some positive constant C. Consider

$$\begin{aligned}\lim_{n\to\infty} \frac{|10^{-(n+1)^k}|}{|10^{-n^k}|^2} &= \lim_{n\to\infty} \frac{10^{-(n+1)^k}}{10^{-2n^k}} \\ &= \lim_{n\to\infty} 10^{-(n+1)^k+2n^k}.\end{aligned}$$

Now we evaluate

$$\begin{aligned}\lim_{n\to\infty} -(n+1)^k + 2n^k &= \lim_{n\to\infty} \frac{-(n+1)^k + 2n^k}{n^k} \cdot n^k \\ &= \lim_{n\to\infty} \left(-\left(\frac{n+1}{n}\right)^k + 2\right) \cdot n^k \\ &= \infty.\end{aligned}$$

Hence, the limit

$$\lim_{n\to\infty} \frac{|10^{-(n+1)^k}|}{|10^{-n^k}|^2} = \infty \neq C,$$

and the sequence does not converge quadratically. $\square$

Problem 9

a. Construct a sequence that converges to 0 of order 3.

Solution. Let $p_n = 10^{-3^n}$.

$$|10^{-3^{n+1}}| = |10^{-3^n}|^3$$

$$\lim_{n\to\infty} \frac{|10^{-3^{n+1}}|}{|10^{-3^n}|^3} = 1.$$

$\square$

b. Suppose $\alpha > 1$. COnstruct a sequence that converges to 0 of order α.

Solution. Let $p_n = 10^{-\alpha^n}$.

$$|10^{-\alpha^{n+1}}| = |10^{-\alpha^n}|^\alpha$$

$$\lim_{n\to\infty} \frac{|10^{-\alpha^{n+1}}|}{|10^{-\alpha^n}|^\alpha} = 1.$$

$\square$

Discussion question 4

What is the difference between the rate of convergence and the order of convergence? Have they any relationship to each other? Could two sequences have the same rates of convergence but different orders of convergence, or vice versa?

Extra

Question 1

Question 2