# Math 104 HW4

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## Exercise 9.12

Assume all $s_n \neq 0$ and that the limit $L = \lim \left|\frac{s_n+1}{s_n}\right|$ exists.

### (a)

**Proposition 1.** *If* $L < 1$, *then* $\lim s_n = 0$.

*Proof.* Since $L < 1$, $L$ can be written as $L + \delta = 1$ for some $\delta > 0$. We know there exists $N$ such that for all $n > N$

$$\left|\frac{s_{n+1}}{s_n} - L\right| < \frac{\delta}{2}$$
$$\left|\frac{s_{n+1}}{s_n}\right| < L + \frac{\delta}{2} = a < 1.$$

Then, for $n > N$, we have

$$|s_n| < a^{n-N}|S_N|.$$

So, now we only have to show that there exists $M$ such that for $n > M, \epsilon > 0$,

$$\begin{aligned} a^{n-N}|S_N| &< \epsilon \\ a^n &< \frac{\epsilon}{|S_N|}a^N \\ |a^n| &< \frac{\epsilon}{|S_N|}a^N \\ |a^n| &< C \qquad (\textit{for some } C > 0). \end{aligned}$$

Notice $\lim a^n = 0$ since $|a| < 1$ (Theorem 9.7). So, by definition of limit, indeed there exists such $M$. Therefore, for $n > \max\{M, N\}$,

$$|s_n - 0| = |s_n| < a^{n-N}|S_N| < \epsilon.$$

□

**(b)**

**Proposition 2.** *If* $L > 1$*, then* $\lim |s_n| = \infty$.

*Proof.* Define $t_n := \frac{1}{|s_n|}$. Then,

$$\begin{aligned} \lim \left| \frac{t_{n+1}}{t_n} \right| &= \lim \left| \frac{s_n}{s_{n+1}} \right| \\ &= \lim \frac{1}{\left| \frac{s_n+1}{s_n} \right|} \\ &= \frac{1}{L} < 1. \end{aligned}$$

Hence, from (a), $\lim t_n = 0$. Therefore, $\lim |s_n| = \infty$ by Theorem 9.10. □

## Exercise 9.15

**Proposition 3.** $\lim_{n\to\infty} \frac{a^n}{n!} = 0$ *for all* $a \in \mathbb{R}$.

*Proof.* If $a = 0$, then the limit is 0 trivially. If $a \neq 0$, denote $s_n = \frac{a^n}{n!}$, and

$$\begin{aligned} \left| \frac{s_{n+1}}{s_n} \right| &= \left| \frac{a^{n+1}}{(n+1)!} \frac{n!}{a^n} \right| \\ \left| \frac{s_{n+1}}{s_n} \right| &= \frac{|a|}{n+1} \\ \lim_{n\to\infty} \left| \frac{s_{n+1}}{s_n} \right| &= 0. \end{aligned}$$

Then, by (a) of Exercise 9.12, $\lim s_n = 0$. □

## Exercise 9.18

**(a)** Verify $1 + a + a^2 + \cdots + a^n = \frac{1-a^{n+1}}{1-a}$ for $a \neq 1$.

*Solution.* We prove by induction.

*Base case:* $n = 1$. LHS: $1 + a$. RHS: $\frac{1-a^2}{1-a} = \frac{(1-a)(1+a)}{1-a} = 1 + a$.

*Inductive step:* Assume the statement is true for $n = k$. Then,

$$\begin{aligned} 1 + a + a^2 + \cdots + a^k + a^{k+1} &= \frac{1 - a^{k+1}}{1-a} + a^{k+1} \\ &= \frac{1 - a^{k+1} + a^{k+1} - a^{k+2}}{1-a} \\ &= \frac{1 - a^{k+2}}{1-a}. \end{aligned}$$

Therefore, the statement is true for all $n \in \mathbb{N}$. □

**(b)** Find $\lim_{n\to\infty}(1 + a + a^2 + \cdots + a^n)$ for $|a| < 1$.

*Solution.*

$$\begin{aligned}\lim_{n\to\infty}(1 + a + a^2 + \cdots + a^n) &= \lim_{n\to\infty}\frac{1 - a^{n+1}}{1 - a}\\ &= \left(\lim 1 - a^{n+1}\right)\left(\lim \frac{1}{1-a}\right)\\ &= \frac{1}{1-a}. \qquad (\because \lim a^{n+1} = 0)\end{aligned}$$

Notice, this is just a gerometric series with $r = a$. $\square$

**(c)** Calculate $\lim_{n\to\infty}\frac{1}{n}\left(1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots + \frac{1}{3^n}\right)$.

*Solution.*

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{n}\left(1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots + \frac{1}{3^n}\right) &= \frac{1}{1 - 1/3}\\ &= \frac{3}{2}.\end{aligned}$$

$\square$

**(d)** What is $\lim_{n\to\infty}(1 + a + a^2 + \cdots + a^n)$ for $a \geq 1$.

*Solution.* For $a \geq 1$,

$$\begin{aligned}\lim_{n\to\infty}(1 + a + a^2 + \cdots + a^n) \geq \lim_{n\to\infty}(1 + 1 + \cdots + 1) &= \lim_{n\to\infty} n + 1\\ &= \infty.\end{aligned}$$

$\square$

## Exercise 10.6

**(a)**

**Proposition 4.** *Let $(s_n)$ be a sequence such that*

$$|s_{n+1} - s_n| < 2^{-n} \qquad \text{for all } n \in \mathbb{N},$$

*then $(s_n)$ is a Cauchy sequence and hence a convergent sequence.*

*Proof.* We need to show the existence of $N$ such that for all $n, m > N$, $|s_n - s_m| < \epsilon$. Without loss of generality, assume $n \geq m$. Also, $|s_n - s_m| < \epsilon$ is always true for $n = m$ so we consider $n > m$.

Notice

$$\begin{aligned} |s_n - s_m| &\leq |s_n - s_{n-1}| + |s_{n-1} - s_{n-2}| + \cdots + |s_{m+1} - s_m| \\ &\leq \sum_{k=m}^{n-1} |s_{k+1} - s_k| \\ &\leq \sum_{k=m}^{n-1} 2^{-k} \\ &< \lim_{n\to\infty} \sum_{k=N+1}^{n} 2^{-k}. \end{aligned}$$

So now we just need to find $N$ such that $\sum_{k=N+1}^{\infty} 2^{-k} < \epsilon$.

$$\begin{aligned} \lim_{n\to\infty} \sum_{k=N+1}^{n} 2^{-k} &< \epsilon \\ \frac{1}{2^{N+1}} + \frac{1}{2^{N+2}} + \cdots &< \epsilon \qquad \textit{(geometric series)} \\ \frac{1}{2^{N+1}} \left( \frac{1}{1 - \frac{1}{2}} \right) &< \epsilon \\ \frac{1}{2^N} &< \epsilon \\ 2^N &> \frac{1}{\epsilon} \\ N &> \log_2 \frac{1}{\epsilon}. \end{aligned}$$

Hence, we can take $N = \max\{\lceil \log_2 \frac{1}{\epsilon} \rceil + 1, 1\}$ and we will get

$$|s_n - s_m| < \lim_{n\to\infty} \sum_{k=N+1}^{n} 2^{-k} < \epsilon$$

for all $n, m > N, \epsilon > 0$. $\square$

**(b)** Is the result in (a) true if we only assume $|s_{n+1} - s_n| < \frac{1}{n}$ for all $n \in \mathbb{N}$.

*Solution.* Not necessarily. Let $s_n = \sum_{k=1}^{n} \frac{1}{2k}$. Then $|s_{n+1} - s_n| = \frac{1}{2n} < \frac{1}{n}$ but there does not exists $N$ such that for all $n, m > N$, $|s_n - s_m| < \epsilon$. Since $\lim_{n\to\infty} \sum_{N+1}^{n} \frac{1}{2k} = \lim_{n\to\infty} \frac{1}{2} \sum_{N+1}^{n} \frac{1}{k} > \frac{1}{2} \lim_{n\to\infty} \int_{N+1}^{n} \frac{1}{x} dx = \infty$, we can always set $m = N + 1$ and find $n$ such that $|s_n - s_m| > \epsilon$. $\square$

## Exercise 10.9

Let $s_1 = 1$ and $s_{n+1} = \left(\frac{n}{n+1}\right) s_n^2$ for $n \geq 1$.

**(a)** Find $s_2, s_3, s_4$.

*Solution.*

$$\begin{aligned} s_2 &= \left(\frac{1}{2}\right) s_1^2 = \frac{1}{2} \\ s_3 &= \left(\frac{2}{3}\right) s_2^2 = \frac{1}{6} \\ s_4 &= \left(\frac{3}{4}\right) s_3^2 = \frac{1}{48}. \end{aligned}$$

□

**(b)** Show $\lim s_n$ exists.

*Proof.* Observe that $(s_n)$ is monotonically decreasing, which can be proved by induction. But in fact look at the equation $s_{n+1} = \left(\frac{n}{n+1}\right) s_n^2$ careful, $s_n < 1$ starting from $n = 2$ so a square of it will only get smaller while $\frac{n}{n+1}$ is always less than 1. So, $s_{n+1} < s_n$ for all $n \geq 2$ [also true for $n = 1$].

Also, $(s_n)$ is bounded below by 0 because every $s_{n+1}$ is defined by multiplication of positive numbers, so $s_{n+1} > 0$ for all $n \in \mathbb{N}$.

Hence, $(s_n)$ is monotonically decreasing and bounded below, so it converges and $\lim s_n$ exists. □

**(c)** Prove $\lim s_n = 0$.

*Proof.* Notice since $\lim s_n$ exists, we can let $\lim s_n = s = \lim s_{n+1}$. Then according to the recusive definition of $(s_n)$,

$$\begin{aligned} s_{n+1} &= \left(\frac{n}{n+1}\right) s_n^2 \\ \lim s_{n+1} &= \lim \left(\frac{n}{n+1}\right) s_n^2 \\ \lim s_{n+1} &= \lim \left(\frac{n}{n+1}\right) \cdot \lim s_n^2 \\ s &= s^2 \\ s^2 - s &= 0. \end{aligned}$$

So, $s = 0$ or $s = 1$. But since $(s_n)$ is monotonically decreasing and $s_2$ is already less than 1, $s = 1$ is not possible.

*Alternatively,* we can prove by using squeeze theorem. We prove by induction that $0 \leq s_n \leq \frac{1}{n}$.

*Base case:* $n = 1$. $s_1 = 1$ and $\frac{1}{1} = 1$.

*Inductive step:* Assume $0 \leq s_k \leq \frac{1}{k}$. Then,

$$0 \leq s_{k+1} = \left(\frac{k}{k+1}\right) s_k^2 \leq \left(\frac{k}{k+1}\right) \left(\frac{1}{k}\right)^2 = \frac{1}{k(k+1)} \leq \frac{1}{k+1}.$$

Therefore, $0 \leq s_n \leq \frac{1}{n}$ for all $n \in \mathbb{N}$. Then, by squeeze theorem, $\lim s_n = 0$ because $\lim 0 = 0$ and $\lim \frac{1}{n} = 0$. $\square$