Math 180B HW6

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PK Exercise 4.3.3 (b)

Solution. {0}, {1, 2}, {3, 4}, {5} are communicating classes. By observing whether the class is closed or not, we can immediately tell that {0}, {1, 2}, {5} are recurrent and {3, 4} are transient. □

PK Exercise 4.3.4

Solution. {2, 3, 4, 5} is a communicating class. By observing the period of {5} is 1, we can immediately tell that the period of {2, 3, 4, 5} is 1. The period of {0} is 1. The period of {1} is 0. □

PK Problem 4.3.3 *(for n = 1,2,3,4)*

Solution. Using MATLAB and computing the matrix multiplication P^n, we have

$$\begin{aligned}
P^{(0)} &= 1\\
P^{(1)} &= 0\\
P^{(2)} &= \frac{1}{4}\\
P^{(3)} &= \frac{1}{8}\\
P^{(4)} &= \frac{3}{8}\\
P^{(5)} &= \frac{7}{32}.
\end{aligned}$$

Then, using *equation (4.16)*,

$$\begin{aligned}
P^{(1)} &= f^{(1)}P^{(0)} \Rightarrow f^{(1)} = 0\\
P^{(2)} &= f^{(1)}P^{(1)} + f^{(2)}P^{(0)} \Rightarrow f^{(2)} = \frac{1}{4}\\
P^{(3)} &= f^{(1)}P^{(2)} + f^{(2)}P^{(1)} + f^{(3)}P^{(0)} \Rightarrow f^{(3)} = \frac{1}{8}\\
P^{(4)} &= f^{(1)}P^{(3)} + f^{(2)}P^{(2)} + f^{(3)}P^{(1)} + f^{(4)}P^{(0)} \Rightarrow f^{(4)} = \frac{5}{16}\\
P^{(5)} &= f^{(1)}P^{(4)} + f^{(2)}P^{(3)} + f^{(3)}P^{(2)} + f^{(4)}P^{(1)} + f^{(5)}P^{(0)} \Rightarrow f^{(5)} = \frac{5}{32}.
\end{aligned}$$

□

PK Exercise 4.4.2

(a)

$$\begin{aligned}0.1\pi_1 + 0.2\pi_2 + 0.3\pi_3 &= \pi_0\\ \pi_0 + 0.4\pi_1 + 0.2\pi_2 + 0.3\pi_3 &= \pi_1\\ 0.2\pi_1 + 0.5\pi_2 + 0.4\pi_3 &= \pi_2\\ \pi_0 + \pi_1 + \pi_2 + \pi_3 &= 1.\end{aligned}$$

By solving the system of equations, we have $\underline{\pi_0 \approx 0.1449}, \pi_1 \approx 0.4140, \pi_2 \approx 0.2880, \pi_3 \approx 0.1530$.

(b)

$$\begin{aligned}v_1 &= 1 + 0.4v_1 + 0.2v_2 + 0.3v_3\\ v_2 &= 1 + 0.2v_1 + 0.5v_2 + 0.1v_3\\ v_3 &= 1 + 0.3v_1 + 0.4v_2.\end{aligned}$$

By solving the system of equations, we have $\underline{v_1 \approx 5.90}, v_2 \approx 5.34, v_3 \approx 4.91$.

(c)

$$m_0 = 1 + 5.90 = 6.90.$$

Indeed,

$$\pi_0 = \frac{1}{m_0} = \frac{1}{6.90} \approx 0.1449.$$

PK Problem 4.4.6

Solution. $P_{00}^{(4)} > 0$ and $P_{00}^{(5)} > 0$. $gcd(4, 5) = 1$. Hence, $d(0) = 1$. □

PK Problem 4.4.8

Solution.

$$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & \cdots \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & \cdots \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Then, we can set up the system of linear equations:

$$\begin{aligned}\pi_0 + \pi_1 + \pi_2 + \pi_3 + \cdots &= 1\\ \frac{1}{2}\pi_0 + \frac{1}{3}\pi_1 + \frac{1}{4}\pi_2 + \cdots &= \pi_0\\ \frac{1}{2}\pi_0 + \frac{1}{3}\pi_1 + \frac{1}{4}\pi_2 + \cdots &= \pi_1\\ \frac{1}{3}\pi_1 + \frac{1}{4}\pi_2 + \cdots &= \pi_2\\ \frac{1}{4}\pi_2 + \cdots &= \pi_3\\ &\vdots\end{aligned}$$

Now, notice that by subtracting π_n from π_{n-1},

$$\pi_n = \pi_{n-1} - \frac{1}{n}\pi_{n-2}.$$

Then, we can set up the recurrence relation:

$$\begin{aligned}
\pi_1 &= \pi_0 \\
\pi_2 &= \pi_1 - \frac{1}{2}\pi_0 = \frac{1}{2}\pi_0 \\
\pi_3 &= \pi_2 - \frac{1}{3}\pi_1 = \frac{1}{2}\pi_0 - \frac{1}{3}\pi_0 = \frac{1}{6}\pi_0 \\
\pi_4 &= \pi_3 - \frac{1}{4}\pi_2 = \frac{1}{6}\pi_0 - \frac{1}{8}\pi_0 = \frac{1}{24}\pi_0 \\
\pi_5 &= \pi_4 - \frac{1}{5}\pi_3 = \frac{1}{24}\pi_0 - \frac{1}{30}\pi_0 = \frac{1}{120}\pi_0.
\end{aligned}$$

By observing the pattern, we claim that

$$\pi_n = \frac{1}{n!}\pi_0.$$

This can be proved rigorously by induction, but we will omit here.

Now, we can solve for π_0:

$$\begin{aligned}
\pi_0 + \pi_1 + \pi_2 + \pi_3 + \cdots &= 1 \\
\pi_0 + \frac{1}{1!}\pi_0 + \frac{1}{2!}\pi_0 + \frac{1}{3!}\pi_0 + \cdots &= 1 \\
\pi_0 \left(1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots\right) &= 1 \\
\pi_0 \left(e\right) &= 1 \\
\pi_0 &= \frac{1}{e}.
\end{aligned}$$

Then, we plug in $\pi_0 = \frac{1}{e}$ to get π_n

$$\begin{aligned}
\pi_n &= \frac{1}{n!}\pi_0 \\
\pi_n &= \frac{1}{n! \cdot e}.
\end{aligned}$$

$\square$