Math 110 HW11

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Problem 1.

Find a polynomial $p \in \mathcal{P}_3(\mathbb{R})$ such that

$$q'(1) = \int_0^1 p(t)q(t)dt \qquad \text{for all } \; q \in \mathcal{P}_3(\mathbb{R}).$$

Solution. We only need to determine the action on the basis $q \in \{1, x, x^2, x^3\}$ because both differentiation and integration are linear. For example, if $q = \alpha x^3 + \beta x^2 + \gamma x + \delta$, then

$$\begin{aligned}
q'(1) &= \left(\alpha(x^3)' + \beta(x^2)' + \gamma(x)' + \delta(1)'\right)(1) \\
&= \int_0^1 \left(\alpha x^3\right) p(x)dx + \int_0^1 \left(\beta x^2\right) p(x)dx + \int_0^1 (\gamma x)\, p(x)dx + \int_0^1 (\delta)\, p(x)dx \\
&= \int_0^1 q(x)p(x)dx.
\end{aligned}$$

Let

$$p = ax^3 + bx^2 + cx + d,$$

then we solve the system of linear equations

$$\begin{aligned}
&\begin{cases}
0 = \int_0^1 \left(ax^3 + bx^2 + x + d\right) dx \\
1 = \int_0^1 \left(ax^4 + bx^3 + cx^2 + dx\right) dx \\
2 = \int_0^1 \left(ax^5 + bx^4 + cx^3 + dx^2\right) dx \\
3 = \int_0^1 \left(ax^6 + bx^5 + cx^4 + dx^3\right) dx
\end{cases} \\
\implies &\begin{cases}
0 = \frac{1}{4}a + \frac{1}{3}b + \frac{1}{2}c + d \\
1 = \frac{1}{5}a + \frac{1}{4}b + \frac{1}{3}c + \frac{1}{2}d \\
2 = \frac{1}{6}a + \frac{1}{5}b + \frac{1}{4}c + \frac{1}{3}d \\
3 = \frac{1}{7}a + \frac{1}{6}b + \frac{1}{5}c + \frac{1}{4}d
\end{cases} \\
\implies &\begin{cases}
a = 1680 \\
b = -2340 \\
c = 840 \\
d = -60
\end{cases} \\
\implies &p = 1680x^3 - 2340x^2 + 840x - 60.
\end{aligned}$$

$\square$

Problem 2.

Let $V = C[-\pi, \pi]$ with the inner product

$$\langle f, g \rangle := \int_{-\pi}^{\pi} f(t)\overline{g(t)}dt.$$

Determine the orthogonal projection of the function $h(x) = \exp(2ix)$ on the subspace

(a) $\operatorname{span}(1, \cos x, \sin x)$;

(b) $\operatorname{span}(1, \cos x, \sin x, \cos 2x, \sin 2x)$;

(c) $\operatorname{span}(1, \cos x, \sin x, \dots, \cos nx, \sin nx)$ for $n > 2$.

Solution. Notice

$$h(x) = e^{2ix} = \cos(2x) + i\sin(2x),$$

and all

$$(1, \cos x, \sin x, \cos 2x, \sin 2x, \dots)$$

are orthogonal to each other (a property of Fourier Series). In fact, all the trig functions have length $\sqrt{\pi}$, so the orthonormal version would be

$$\left(\frac{1}{\sqrt{2\pi}}, \frac{\cos x}{\sqrt{\pi}}, \frac{\sin x}{\sqrt{\pi}}, \frac{\cos 2x}{\sqrt{\pi}}, \frac{\sin 2x}{\sqrt{\pi}}, \dots\right),$$

but we don't need to make use of the orthonormal version in this question.

(a) $h(x)$ is orthogonal to all the basis because $\cos(2x), i(\sin 2x)$ are orthogonal to the basis. So the projection is 0.

(b) Notice $h(x)$ is in the span of the basis as noted in the very first line, so the projection is $h(x)$.

(c) Again, $h(x)$ is in the span of the basis, so the projection is $h(x)$.

$\square$

Problem 3.

Find $p \in \mathcal{P}_3(\mathbb{R})$ such that $p(-1) = 0$, $p'(-1) = 0$, and the following is minimized:

$$\int_0^1 (1 - 5x - p(x))^2 dx.$$

Proof. Denote

$$V = \{p \in \mathcal{P}_3(\mathbb{R}) : p(-1) = 0, p'(-1) = 0\}.$$

Let $p = ax^3 + bx^2 + cx + d$, then solving the equations

$$\begin{cases} 0 = p(-1) \\ 0 = p'(-1) \end{cases}$$

yields

$$\begin{cases} a = a \\ b = b \\ c = 2b - 3a \\ d = b - 2a \end{cases},$$

which implies $p = a(x^3 - 3x - 2) + b(x^2 + 2x + 1) \implies V = \operatorname{span}\left\{(x+1)^3, (x+1)^2\right\}$. We want to find the orthogonal projection of $(1 - 5x)$ onto V, which would minimize the integral. The orthonoaml basis are

$$\begin{aligned} e_1 &= \frac{(x+1)^2}{\sqrt{\int_0^1 (x+1)^4 dx}} = \sqrt{\frac{5}{31}}(x+1)^2 \\ e_2 &= \frac{(x+1)^3 - \frac{5}{31}(x+1)^2 \int_0^1 (x+1)^5 dx}{\|(x+1)^3 - \langle (x+1)^3, e_1 \rangle e_1\|} \\ &= \sqrt{\frac{868}{313}} \left((x+1)^3 - \frac{105}{62}(x+1)^2 \right). \end{aligned}$$

Then the projection is

$$\begin{aligned} P_u &= \langle (1 - 5x), e_1 \rangle e_1 + \langle (1 - 5x), e_2 \rangle e_2 \\ &= \frac{-95}{124}(x+1)^2 - \frac{791}{626}\left((x+1)^3 - \frac{105}{62}(x+1)^2 \right) \\ &= \frac{430}{313}(x+1)^2 - \frac{791}{626}(x+1)^3. \end{aligned}$$

$\square$

Problem 4.

Find $p \in \mathcal{P}_3(\mathbb{R})$ such that $p(-1) = 0$, $p'(-1) = 0$, and the following is minimized:

$$p(0)^2 + \int_0^1 (1 - 5x - p'(x))^2 dx.$$

Solution. We can view this as minimizing the orthogonal residual of projecting $x - \frac{5}{2}x^2$ onto V where

$$V = \operatorname{span}\left\{(x+1)^2, (x+1)^2\right\},$$

and the inner product is

$$\langle f, g \rangle = f(0)g(0) + \int_0^1 f'(x)g'(x)dx.$$

The orthonormal basis are

$$\begin{aligned}
e_1 &= \frac{(x+1)^2}{\sqrt{1 + \int_0^1 4(x+1)^2 dx}} \\
&= \sqrt{\frac{3}{31}}(x+1)^2 \\
e_2 &= \frac{(x+1)^3 - \sqrt{\frac{3}{31}}(x+1)^2\left(\sqrt{\frac{3}{31}} + \sqrt{\frac{3}{31}}\int_0^1 6(x+1)^3 dx\right)}{\|(x+1)^3 - \langle (x+1)^3, e_1 \rangle e_1\|} \\
&= \sqrt{\frac{620}{2081}}\left((x+1)^3 - \frac{141}{62}(x+1)^2\right).
\end{aligned}$$

So the projection is

$$\begin{aligned}
P_u &= \left\langle \left(x - \frac{5}{2}x^2\right), e_1 \right\rangle e_1 + \left\langle \left(x - \frac{5}{2}x^2\right), e_2 \right\rangle e_2 \\
&= \frac{-16}{31}(x+1)^2 - \frac{1315}{2081}\left((x+1)^3 - \frac{141}{62}(x+1)^2\right) \\
&= \frac{3833}{4162}(x+1)^2 - \frac{1315}{2081}(x+1)^3.
\end{aligned}$$

□

Problem 5.

Let V be the vector space $\mathbb{R}^3$ equipped with the standard inner product. Prove or disprove: any linear operator $P \in \mathcal{L}(V)$ such that $P^2 = P$ is an orthogonal projector.

Solution. Define

$$T : (x, y, z) \mapsto (y, x, z).$$

Consider $v = (1, 2, 3)$, then

$$\langle T(v), v - T(v)\rangle = \langle(2, 1, 3), (-1, 1, 0)\rangle = -1 \neq 0,$$

which means the residual is not orthogonal to the image of T, so T is not an orthogonal projector. $\square$