Math 154 HW2

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04/19/2023

Problem 1.

(a)
- $n = 2k + 1$ for $k \in \mathbb{N}$.
- $n = 2$.
- All $n \geq 3$.

(b)
- $r = 2n, s = 2k$ for $n, k \in \mathbb{N}$.
- $r = 2$, $s = 2k + 1$ or $s = 2, r = 2k + 1$ for $k \in \mathbb{Z}^{\geq}$, and $r = s = 1$.
- $r, s \geq 2; r = s$.

Problem 2.

Proposition 1. *Let P be a longest path in a connected graph G, and suppose there exists a cycle C such that $P \subseteq C \subseteq G$. Then G is Hamiltonian.*

Proof. Proof goals: P is a Hamiltonian path in $G \Rightarrow C$ is a Hamiltonian cycle $\Rightarrow G$ is Hamiltonian.

First, let $P = (v_1, v_2, \ldots, v_n)$. Then, we can construct $C = (v_1, v_2, \ldots, v_n, \ldots, v_1)$. Notice C can only be in the form of $(v_1, v_2, \ldots, v_n) + (v_n, \ldots, v_1)$ by appending a path to the end of P that connects v_n to v_1 because connecting any non-end points vertices would yield a cycle that P is not a subset of.

Then, we will prove that P contains all vertices in G. Assume to the contrary that there exists $w \in V(G)$ that is not contained in P. Since G is connected, there exists a path between w and an arbitrary point $v_i \in P$. Then, we can construct a path $P' = (w, \ldots, v_i, v_{i-1}, \ldots, v_1, v_n, v_{n-1}, \ldots, v_{i+1})$ that is longer than P. Hence, we reached contradiction and proved that P contains all vertices in G, which means P is a Hamiltonian path in G.

Next, we will prove that C is a Hamiltonian cycle. Since P is a Hamiltonian path and $P \subseteq C$, C is a cycle that contains all vertices in G too, which is the definition of a Hamiltonian cycle. Thus, by definition, G is Hamiltonian since it has a Hamiltonian cycle C. □

Problem 3.

Proposition 2. *A graph G of minimum degree at leas $k \geq 2$ contains no triangles contains a cycle of length at least $2k$.*

Proof. Let $P = (v_1, v_2 \ldots, v_n)$ be the longest path in G. We know P must contain all the neighbors of v_1, otherwise we can construct a longer path $P' = (w, v_1, \ldots, v_n)$ such that $w \in N(v_1)$ and $w \notin P$. Therefore, we know *length(P)* $\geq d_G(v_1) + 1 \geq k + 1$. By connecting v_1 with the further vertex $u \in N(v_1)$ in P, we can then form a cycle with length $\geq k + 1$. [Proved in class]

Then, notice that for $2 \leq i \leq n - 2$, $v_i, v_{i+1} \in P$ cannot be neighbor of v_1 at the same time, otherwise a triangle can be constructed with (v_1, v_i, v_{i+1}, v_1). Therefore, $P = (v_1, \ldots, v_i, w, \ldots, v_{i+1}, \ldots, v_n)$. In other words, any vertices in P that are neighbor of v_1 must be separated by at least one vertex that is non-neighbor of v_1. Finally, by connecting v_1 to the further vertex $u \in N(v_1)$ in P, we can actually form a even longer cycle with length $\geq 2k$. [Consider the minimum case that all the neighbors of v_1 are separated by exactly one vertex that is non-neighbor of v_1, namely $(v_2, w_1, v_3, w_2, \ldots, v_k)$. Then, this path has length $d_G(v_1) + (d_G(v_1) - 1)$. Finally, adding v_1 to the front and connecting the ends, we can guarantee to form the cycle with length $2d_G(v_1) \geq 2k$.] □

Problem 4.

Drawing pictures was definitely helpful. It helped me when I was brainstorming the proof for Problem 3. I first recalled how Dr. Gwen proved the the lemma that for graph with minimum degree at least k, there exists a cycle of length at least $k+1$. I first drew the picture of that path and cycle. Then I stared at it and looked for more clues from the question. I was just randomly doodling and drawing triangles and suddenly I just found the proof. It was like magic haha. I would say drawing pictures definitely helped me spot out some of the patterns and clues.

Also, considering related theorem is definitely helpful too. Just like the lemma I mentioned in the previous praragraph. I started by first noticing the similarity between the lemma and the question. Then, I tried to base my proof on that lemma.