# Math 110 Practice

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## Midterm

**Proposition 1.** *Let $V = \mathcal{P}_2(\mathbb{R})$. Let $I$ denote the identity map on $V$, $D$ the differentiation map, $D^2 = D \circ D$ the second differentiation map, and $T$ the map $f(x) \mapsto f(x-1)$. Then the list $(I, D, D^2, T)$ are linearly dependent in $\mathcal{L}(V)$.*

*Proof.* We can write $ax^2 + bx + c \in \mathbb{P}_2(\mathbb{R})$. We want to find non-trivial solution to the equation for any arbitrary $a, b, c \in \mathbb{R}$.

$$
\begin{aligned}
\alpha I(ax^2+bx+c) + \beta D(ax^2+bx+c) + \gamma D^2(ax^2+bx+c) + \delta T(ax^2+bx+c) &= 0 \\
\alpha(ax^2+bx+c) + \beta(2ax+b) + \gamma(2a) + \delta(a(x-1)^2 + b(x-1) + c) &= 0 \\
\alpha(ax^2+bx+c) + \beta(2ax+b) + \gamma(2a) + \delta(ax^2 - 2ax + bx + a - b + c) &= 0 \\
a(\alpha x^2 + 2\beta x + 2\gamma + \delta x^2 - 2\delta x + \delta) + b(\alpha x + \beta + \delta x - \delta) + c(\alpha + \delta) &= 0 \\
a\left((\alpha+\delta)x^2 + (2\beta - 2\delta)x + 2\gamma + \delta\right) + b\left((\alpha+\delta)x + \beta - \delta\right) + c(\alpha+\delta) &= 0.
\end{aligned}
$$

Since $a, b, c$ are free variables, we must have

$$
\begin{cases}
(\alpha+\delta)x^2 + (2\beta-2\delta)x + 2\gamma + \delta & = 0 \\
(\alpha+\delta)x + \beta - \delta & = 0 \\
\alpha + \delta & = 0
\end{cases}
$$

$$
\implies \begin{cases}
\alpha + \delta & = 0 \\
2\beta - 2\delta & = 0 \\
2\gamma + \delta & = 0 \\
\alpha + \delta & = 0 \\
\beta - \delta & = 0 \\
\alpha + \delta & = 0
\end{cases} \quad \because x^2, x, 1 \text{ are linearly independent}
$$

$$
\implies \begin{cases}
\alpha + \delta & = 0 \\
2\beta - 2\delta & = 0 \\
2\gamma + \delta & = 0.
\end{cases}
$$

$$
\implies \begin{cases}
\delta & = \delta \\
\alpha & = -\delta \\
\beta & = \delta \\
\gamma & = -\frac{1}{2}\delta.
\end{cases}
$$

Therefore, there exists infinitely non-trivial solutions to the equation, and the list is linearly dependent. $\square$

**Proposition 2.** *Let* $V = \mathbb{R}^4$*, let* $W_1 = \{(x_1, x_2, x_3, x_4) : x_2 + x_4 = 0, x_j \in \mathbb{R}$ *for all* $j\}$*, and let* $W_2 = \{(x_1, x_2, x_3, x_4) : x_1 + x_2 + x_3 = 0, x_j \in \mathbb{R}$ *for all* $j\}$*. Then*

*(a)* $W_1$ *and* $W_2$ *are subspaces of* $V$*;*

*(b)* $\dim W_1 \cap W_2 = 2$*;*

*(c)* $W_1 + W_2$ *is not a direct sum and* $\dim(W_1 + W_2)$

*Proof.*

(a) $W_1$ and $W_2$ obviously contain the zero vector by letting $x_1 = x_2 = x_3 = x_4 = 0$.

Both spaces are closed under addition. For $W_1$, let $u = (u_1, u_2, u_3, u_4), v = (v_1, v_2, v_3, v_4) \in W_1$. Then

$$u + v = (u_1 + v_1, u_2 + v_2, u_3 + v_3, u_4 + v_4),$$

and

$$(u + v)_2 + (u + v)_4 = (u_2 + v_2) + (u_4 + v_4) = (u_2 + u_4) + (v_2 + v_4) = 0.$$

Similarly, for $W_2$, let $u = (u_1, u_2, u_3, u_4), v = (v_1, v_2, v_3, v_4) \in W_2$. Then

$$u + v = (u_1 + v_1, u_2 + v_2, u_3 + v_3, u_4 + v_4),$$

and

$$(u+v)_1 + (u+v)_2 + (u+v)_3 = (u_1 + v_1) + (u_2 + v_2) + (u_3 + v_3) = (u_1 + u_2 + u_3) + (v_1 + v_2 + v_3) = 0.$$

Also, both spaces are closed under scalar multiplication. For $W_1$, let $u = (u_1, u_2, u_3, u_4) \in W_1$ and $\lambda \in \mathbb{R}$. Then

$$\lambda u = (\lambda u_1, \lambda u_2, \lambda u_3, \lambda u_4),$$

and

$$(\lambda u)_2 + (\lambda u)_4 = \lambda u_2 + \lambda u_4 = \lambda(u_2 + u_4) = 0.$$

Similarly, for $W_2$, let $u = (u_1, u_2, u_3, u_4) \in W_2$ and $\lambda \in \mathbb{R}$. Then

$$\lambda u = (\lambda u_1, \lambda u_2, \lambda u_3, \lambda u_4),$$

and

$$(\lambda u)_1 + (\lambda u)_2 + (\lambda u)_3 = \lambda u_1 + \lambda u_2 + \lambda u_3 = \lambda(u_1 + u_2 + u_3) = 0.$$

(b) We can write $W_1 = \{(x_1, -x_4, x_3, x_4 : x_j \in \mathbb{R})\} = \operatorname{span}\left\{\begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\-1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\end{bmatrix}\right\}$. Similarly, we can write $W_2 = \{(x_1, -x_1 - x_3, x_3, x_4 : x_j \in \mathbb{R})\} = \operatorname{span}\left\{\begin{bmatrix}1\\-1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\-1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right\}$. Putting all their basis vectors together and reducing the matrix, we have 4 linearly independent vectors, and therefore $\dim(W_1 + W_2) = 4$. Hence,

$$\dim W_1 \cap W_2 = \dim W_1 + \dim W_2 - \dim(W_1 + W_2) = 3 + 3 - 4 = 2.$$

*Alternatively,* we can denote the intersection $W_1 \cap W_2 = \{(x_1, x_2, x_3, x_4) : x_2 + x_4 = 0, x_1 + x_2 + x_3 = 0\}$, then

$$W_1 \cap W_2 = \begin{bmatrix} x_1 \\ x_2 \\ -x_1 - x_2 \\ -x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \\ -x_1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ x_2 \\ -x_2 \\ -x_2 \end{bmatrix} = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ -1 \end{bmatrix} \right\}.$$

Therefore, $\dim W_1 \cap W_2 = 2$.

(c) $\dim W_1 \cap W_2 = 2 \implies W_1 \cap W_2 \neq \{0\} \implies W_1 \not\oplus W_2$. $\dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim W_1 \cap W_2 = 4$.

$\square$

**Proposition 3.** *Let $V$ be the vector space of all trigonometric polynomials (in $x$) with real coefficients of degree at most 2, i.e. $V := \operatorname{span}\{1, \sin x, \cos x, \sin(2x), \cos(2x)\}$. The list $(1, \sin x, \cos x, \sin(2x), \cos(2x))$ is a basis of $V$. Consider the linear operator*

$$T \in \mathcal{L}(V) : (Tf)(x) = f''(x) + f(x).$$

*(a) Find the matrix representation of $T$ in this basis used for the domain and the codomain.*

*(b) Show* $\dim \operatorname{null} T = 2, \dim \operatorname{range} T = 3$.

*Proof.*

(a)

$$\begin{aligned}
(T1)(x) &= 0 + 1 = 1 \\
(T \sin x)(x) &= -\sin x + \sin x = 0 \\
(T \cos x)(x) &= -\cos x + \cos x = 0 \\
(T \sin(2x))(x) &= -4\sin(2x) + \sin(2x) = -3\sin(2x) \\
(T \cos(2x))(x) &= -4\cos(2x) + \cos(2x) = -3\cos(2x).
\end{aligned}$$

We have determined the image of the basis vectors. Now we can write the matrix representation of $T$ in this basis used for the domain and the codomain.

$$\mathcal{M}(T) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & -3 \end{bmatrix}.$$

(b) Notice the image of $T$ is determined by $\mathcal{M}(T)\vec{v}$ for $\vec{v} \in V$ represented as a column vector in terms of the basis. Therefore, range$T$ is the same as the column space of $\mathcal{M}(T)$. We can see obviously that the column space has dimension 3 since first column has non-zero entry at different coordinates. Therefore,

$$\dim \operatorname{range} T = 3$$

and

$$\dim \operatorname{null} T = \dim V - \dim \operatorname{range} T = 5 - 3 = 2.$$

$\square$

**Proposition 4.** *Consider the linear map* $T : \mathcal{P}_2(\mathbb{R}) \to \mathcal{P}_4(\mathbb{R}) : f(x) \mapsto f(x^2)$ *and the linear functional* $\varphi : \mathcal{P}_4(\mathbb{R}) \to \mathbb{R} : f(x) \mapsto f''(0)$. *Then*

*(a)* $T'(\varphi) : \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}$;

*(b)* $T'(\varphi) : ax^2 + bx + c \mapsto 2b$;

*(c)* $\dim \mathrm{null} T' = 2$ *and* $T'$ *is not an isomorphism.*

*Proof.*

(a) $T'(\varphi) = \varphi(T) : \mathcal{P}_2(\mathbb{R}) \to \mathcal{P}_4(\mathbb{R}) \to \mathbb{R} = \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}$.

(b) Consider arbitrary $ax^2 + bx + c \in \mathbb{P}_2(\mathbb{R})$,

$$\begin{aligned} T'(\varphi)(ax^2 + bx + c) = \varphi(T)(ax^2 + bx + c) &= \varphi(ax^4 + bx^2 + c) \\ &= (ax^4 + bx^2 + c)''(0) \\ &= (12ax^2 + 2b)_{x=0} \\ &= 2b. \end{aligned}$$

(c) Notice $\mathrm{null} T' = (\mathrm{range} T)^0$, the annihilator of $\mathrm{range} T \in \mathcal{P}_4(\mathbb{R})$, and $\mathrm{range} T = \mathrm{span}\{1, x^2, x^4\}$. Then

$$\begin{aligned} \dim \mathrm{null} T' &= \dim (\mathrm{range} T)^0 \\ &= \dim \mathcal{P}_4(\mathbb{R}) - \dim \mathrm{range} T \\ &= 5 - 3 \\ &= 2. \end{aligned}$$

Since $\dim \mathrm{null} T' = 2 \neq 0$, $T'$ is not injective and therefore not an isomorphism.

$\square$