

Math 180B HW7

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PK Exercise 5.1.4 Customers arrive at a service facility according to a Poisson process of rate λ customer/hour. Let $X(t)$ be the number of customers that have arrived up to time t .

- (a) What is $P\{X(t) = k\}$ for $k = 0, 1, \dots$?

Solution. $P\{X(t) = k\}$ is simply *Poisson*(λt). Hence, $P\{X(t) = k\} = \frac{(\lambda t)^k}{k!} e^{-\lambda t}$. \square

- (b) Consider fixed times $0 < s < t$. Determine the conditional probability $P\{X(t) = n + k | X(s) = n\}$ and the expected value $E[X(t)X(s)]$.

Solution.

$$\begin{aligned} P\{X(t) = n + k | X(s) = n\} &= P\{X(t - s) = k\} \\ &= \frac{(\lambda(t - s))^k}{k!} e^{-\lambda(t - s)}. \\ E[X(t)X(s)] &= E[(X(t) - X(s) + X(s)) \cdot X(s)] \\ &= E[(X(t) - X(s)) \cdot X(s)] + E[X(s)^2] \\ &= E[X(t) - X(s)] \cdot E[X(s)] + E[X(s)^2] \\ &= E[X(t) - X(s)] \cdot E[X(s)] + [E[X(s)]^2 + \text{Var}(X(s))] \\ &= \lambda(t - s) \cdot \lambda s + \lambda^2 s^2 + \lambda s \\ &= \lambda^2 ts + \lambda s. \end{aligned}$$

\square

PK Exercise 5.1.5 Suppose that a random variable X is distributed according to a Poisson distribution with parameter λ . The parameter λ is itself a random variable, exponentially distributed with density $f(x) = \theta e^{-\theta x}$ for $x \geq 0$. Find the probability mass function for X .

Solution.

$$\begin{aligned}
 P(X = k) &= \int_0^\infty P(X = k | \lambda = x) f(x) dx \\
 &= \int_0^\infty \frac{x^k}{k!} e^{-x} \cdot \theta e^{-\theta x} dx \\
 &= \frac{\theta}{k!} \int_0^\infty x^k e^{-(\theta+1)x} dx \\
 &= \frac{\theta}{k!} \int_0^\infty \left(\frac{t}{\theta+1} \right)^k e^{-t} \cdot \frac{1}{\theta+1} dt \quad (\text{let } (\theta+1)x = t) \\
 &= \frac{\theta}{k!} \cdot \frac{1}{(\theta+1)^{k+1}} \int_0^\infty t^k e^{-t} dt \\
 &= \frac{\theta}{k!} \cdot \frac{1}{(\theta+1)^{k+1}} \cdot \Gamma(k+1) \\
 &= \frac{\theta}{k!} \cdot \frac{1}{(\theta+1)^{k+1}} \cdot k! \\
 &= \frac{\theta}{(\theta+1)^{k+1}}.
 \end{aligned}$$

□

PK Exercise 5.1.7 Suppose that customers arrive at a facility according to a Poisson process having rate $\lambda = 2$. Let $X(t)$ be the number of customers that have arrived up to time t . Determine the following probabilities and conditional probabilities:

(a) $P\{X(1) = 2\}$.

Solution. $P\{X(1) = 2\} = \frac{2^2}{2!} e^{-2} = 2e^{-2}$.

□

(b) $P\{X(1) = 2 \text{ and } X(3) = 6\}$.

Solution.

$$\begin{aligned}
 P\{X(1) = 2 \text{ and } X(3) = 6\} &= P\{X(1) = 2\} \cdot P\{X(2) = 4\} \\
 &= \frac{2^2}{2!} e^{-2} \cdot \frac{(2 \times 2)^4}{4!} e^{-(2 \times 2)} \\
 &= \frac{2^{10}}{2!4!} e^{-6} \\
 &= \frac{2^6}{3} e^{-6} \\
 &= \frac{64}{3} e^{-6}.
 \end{aligned}$$

□

(c) $P\{X(1) = 2|X(3) = 6\}$.

Solution.

$$\begin{aligned} P\{X(1) = 2|X(3) = 6\} &= \frac{P\{X(1) = 2 \text{ and } X(3) = 6\}}{P\{X(3) = 6\}} \\ &= \frac{\frac{2^6}{3}e^{-6}}{\frac{(2 \times 3)^6}{6!}e^{-6}} \\ &= \frac{6!}{3^7} \\ &= \frac{80}{243}. \end{aligned}$$

□

(d) $P\{X(3) = 6|X(1) = 2\}$.

$$\text{Solution. } P\{X(3) = 6|X(1) = 2\} = P\{X(2) = 4\} = \frac{(2 \times 2)^4}{4!}e^{-(2 \times 2)} = \frac{2^5}{3}e^{-4} = \frac{32}{3}e^{-4}.$$

□

PK Problem 5.1.5

Proposition 1. For each value of $h > 0$, let $X(h)$ have a Poisson distribution with parameter λh . Let $p_k(h) = P\{X(h) = k\}$ for $k = 0, 1, \dots$, then

$$p_0(h) = 1 - \lambda h + o(h)$$

$$p_1(h) = \lambda h + o(h)$$

$$p_2(h) = o(h).$$

Proof.

$$\begin{aligned} p_0(h) &= \frac{(\lambda h)^0 e^{-\lambda h}}{0!} \\ &= e^{-\lambda h} \\ &= 1 - \lambda h + \frac{(\lambda h)^2}{2!} - \frac{(\lambda h)^3}{3!} + \dots \\ &= 1 - \lambda h + o(h). \\ p_1(h) &= \frac{(\lambda h)^1 e^{-\lambda h}}{1!} \\ &= \lambda h e^{-\lambda h} \\ &= \lambda h(1 - \lambda h + o(h)) \\ &= \lambda h + o(h). \\ p_2(h) &= \frac{(\lambda h)^2 e^{-\lambda h}}{2!} \\ &= \frac{(\lambda h)^2}{2} e^{-\lambda h} \\ &= \frac{(\lambda h)^2}{2} (1 - \lambda h + o(h)) \\ &= \frac{(\lambda h)^2}{2} - \frac{(\lambda h)^3}{2} + o(h) \\ &= o(h). \end{aligned}$$

□

PK Problem 5.1.7 Shocks occur to a system according to a Poisson process of rate λ . Suppose that the system survives each shock with probability α , independently of other shocks, so that its probability of surviving k shocks is α^k . What is the probability that the system is surviving at time t ?

Solution.

$$\begin{aligned} P(\text{Surviving at time } t) &= \sum_{k=0}^{\infty} \alpha^k \cdot \frac{(\lambda t)^k e^{-\lambda t}}{k!} \\ &= e^{-\lambda t} \sum_{k=0}^{\infty} \frac{(\alpha \lambda t)^k}{k!} \\ &= e^{-\lambda t} e^{\alpha \lambda t} \\ &= e^{\lambda t(\alpha-1)}. \end{aligned}$$

□

PK Exercise 5.2.1 Determine numerical values to three decimal places for $P\{X = k\}, k = 0, 1, 2$, when
(a) X has a binomial distribution with parameters $n = 20$ and $p = 0.06$.

Solution.

$$\begin{aligned} p_0 &= \binom{20}{0} 0.94^{20} \approx 0.290 \\ p_1 &= \binom{20}{1} 0.06^1 0.94^{19} \approx 0.370 \\ p_2 &= \binom{20}{2} 0.06^2 0.94^{18} \approx 0.225. \end{aligned}$$

□

(b) X has a binomial distribution with parameters $n = 40$ and $p = 0.03$.

Solution.

$$\begin{aligned} p_0 &= \binom{40}{0} 0.97^{40} \approx 0.296 \\ p_1 &= \binom{40}{1} 0.03^1 0.97^{39} \approx 0.366 \\ p_2 &= \binom{40}{2} 0.03^2 0.97^{38} \approx 0.221. \end{aligned}$$

□

(c) X has a Poisson distribution with parameter $\lambda = 1.2$.

Solution.

$$\begin{aligned} p_0 &= \frac{1.2^0 e^{-1.2}}{0!} \approx 0.301 \\ p_1 &= \frac{1.2^1 e^{-1.2}}{1!} \approx 0.361 \\ p_2 &= \frac{1.2^2 e^{-1.2}}{2!} \approx 0.217. \end{aligned}$$

□