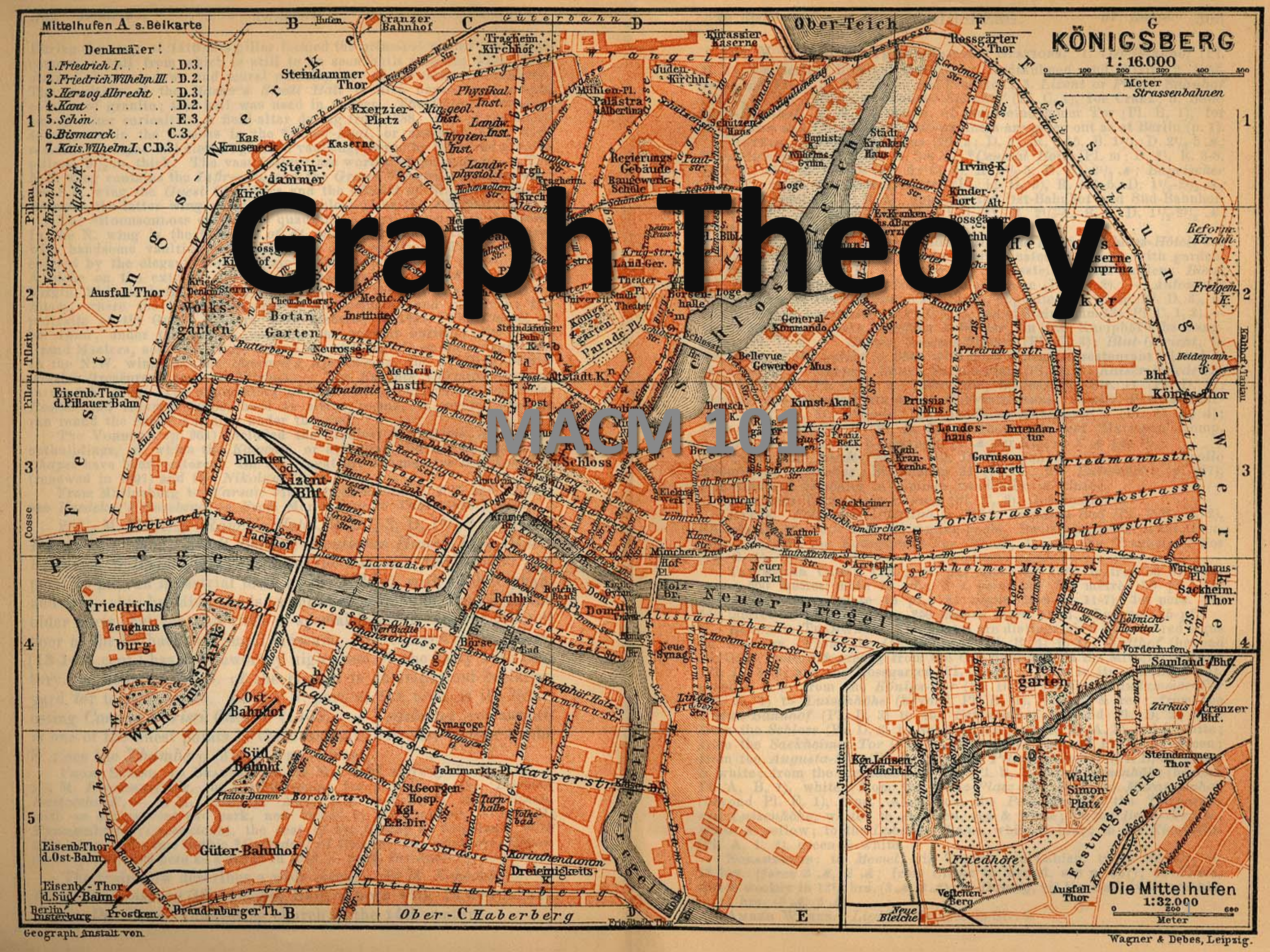
Graph Theory

MACM 101



Goals For This Unit

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- Discuss Weighted Trees and Prefix Codes (Section 12.3).

What is Graph Theory?

- The study of graphs.
  - Informally, graphs are mathematical structures used to model binary relations between elements of a given set.

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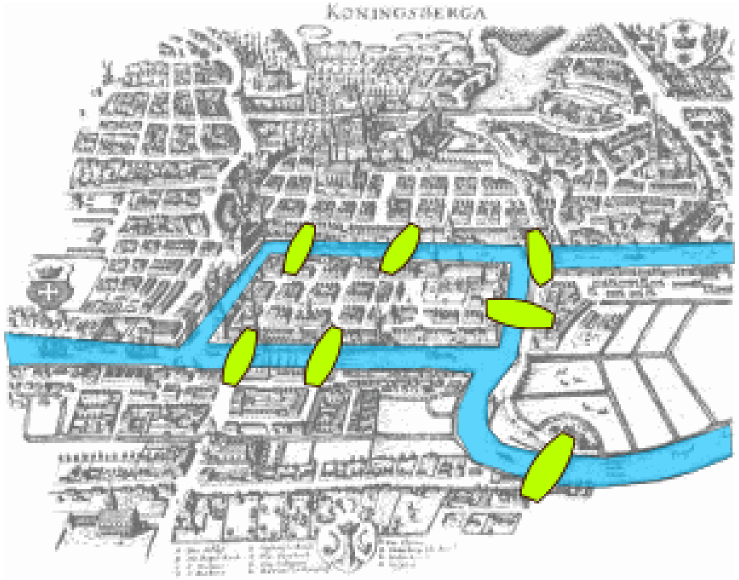
- The origin of the theory is credited to Euler, 1736 – The Königsberg Bridge Problem.

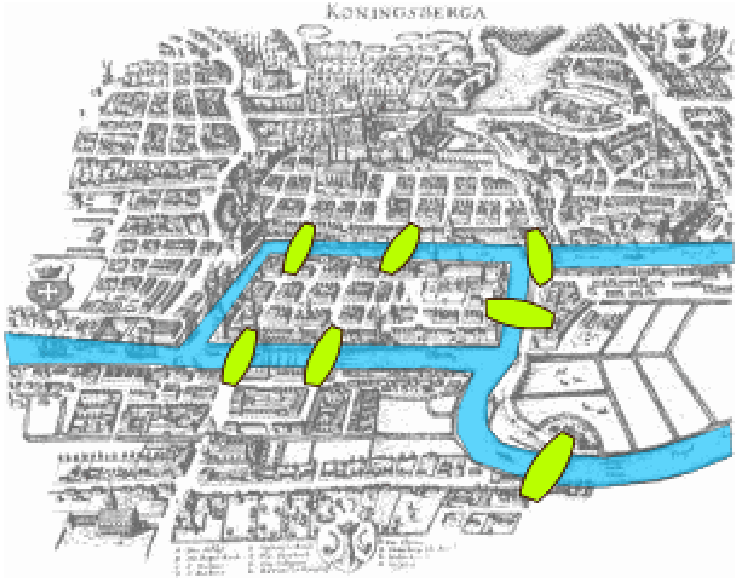
The Königsberg Bridge Problem

- Is there a route which crosses each of the seven bridges exactly once?



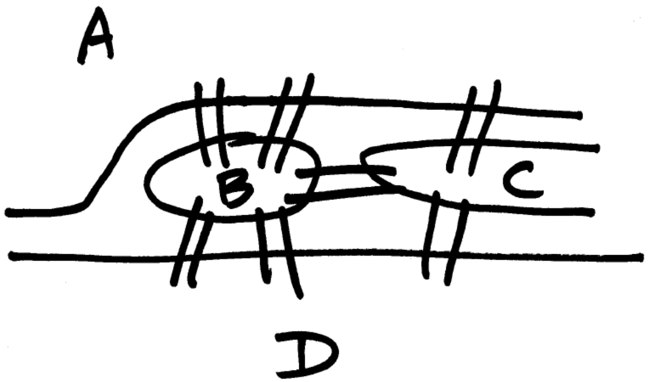
The Königsberg Bridge Problem

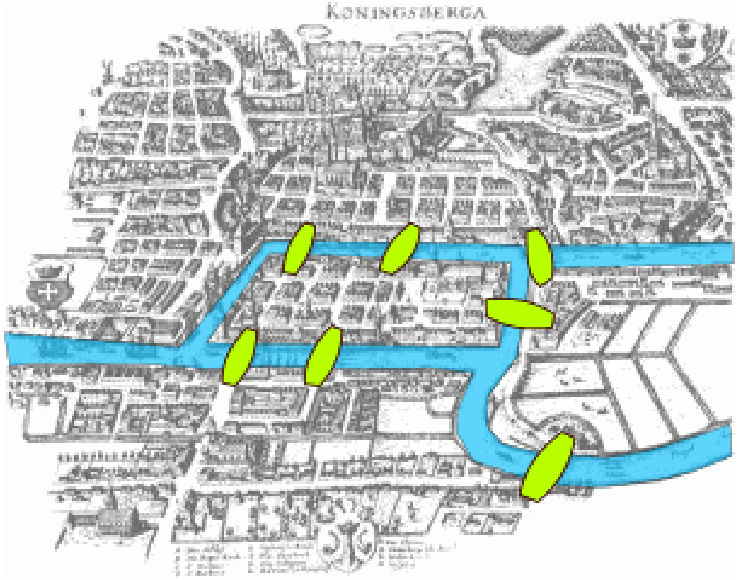
- Is there a route which crosses each of the seven bridges exactly once?
- NO.



The Königsberg Bridge Problem

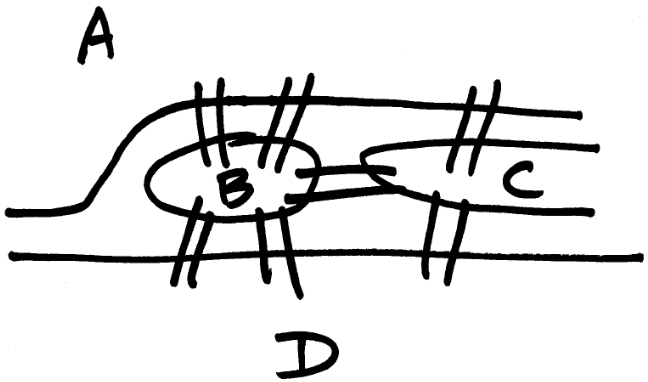
- Schematic:



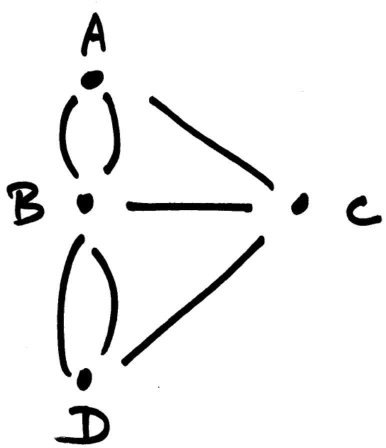


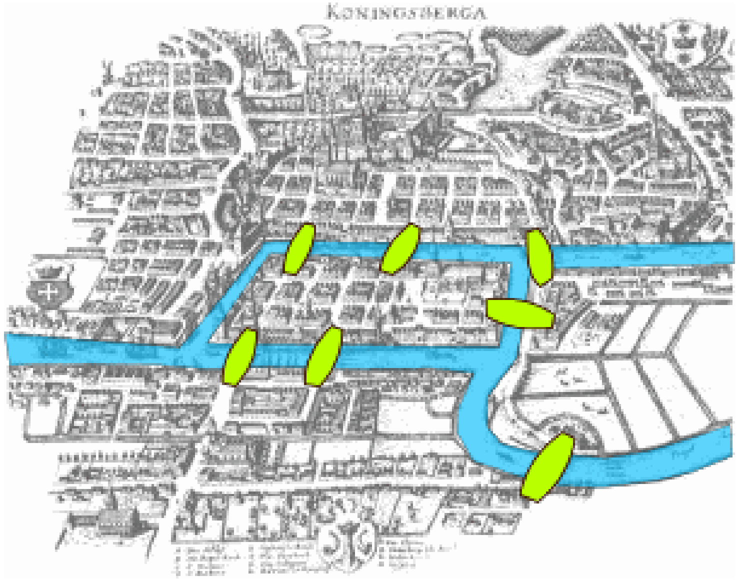
The Königsberg Bridge Problem

- **Schematic**:



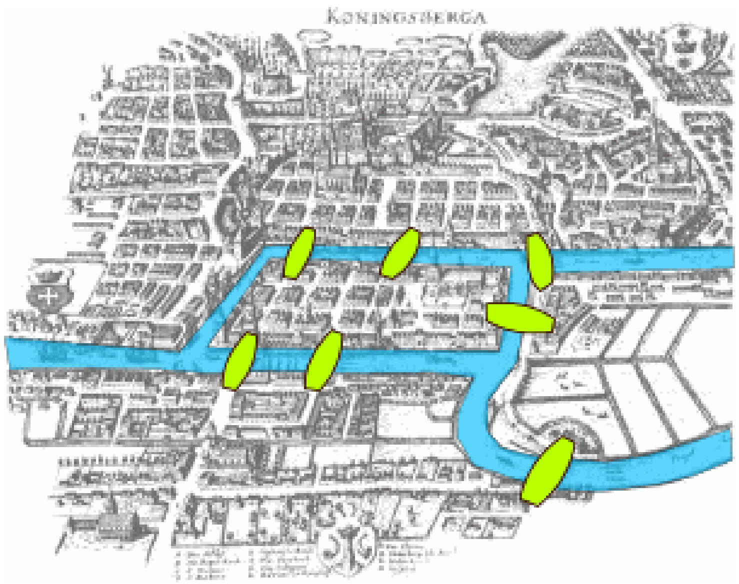
- **Graph**:

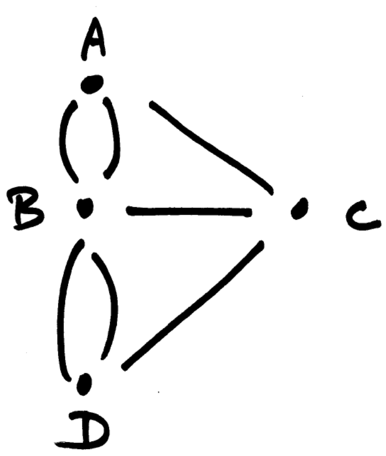




The Königsberg Bridge Problem

- **Question**: Can one trace this graph starting and ending at the same point without lifting the pencil from the paper, traversing each edge exactly once?





Aside: Terminology

- You must be familiar with the basic terminology – there's lots of it in Graph Theory.
- You are responsible for the nomenclature handout emailed to you today.
- Again, see Wikipedia.

<u>Basic Def^{ns}</u>

Consider a graph G, and let v and w be vertices in G.

- A <u>walk</u> from v to w is a finite alternating sequence of adjacent vertices and edges of G and has the form

$$v_0\, e_1\, v_1\, e_2 \ldots v_{n-1}\, e_n\, v_n$$

where the v's represent vertices, the e's edges, $v_0 = v$, $v_n = w$, and $\forall i = 1, 2, \ldots, n$, v_{i-1} & v_i are the endpoints of e_i.

- The <u>trivial walk</u> from v to v consists of the single vertex v.
- A <u>path</u> from v to w is a walk from v to w that does <u>not</u> contain a repeated edge. (a walk s.t. $e_i \neq e_k$ $\forall i \neq k$).
- A <u>simple path</u> from v to w is a path that does <u>not</u> contain a repeated vertex. (a walk s.t. $e_i \neq e_k$, $i \neq k$ & $v_j \neq v_m$, $j \neq m$)
- A <u>closed walk</u> is a walk that starts and ends @ same vertex
- A <u>circuit</u> is a closed walk that does <u>not</u> contain a repeated edge (a walk where $v_0 = v_n$, e_i distinct).
- A <u>simple circuit</u> is a circuit w/ <u>no</u> repeated vertex except the 1st and last.

The Königsberg Bridge Problem

Informally :

- Begin at vertex A.

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- Thus, if it is possible to find such a route that uses all the edges of the graph and starts and ends at A, then the *total* number of arrival and departures from each vertex must be a multiple of two (*i.e.,* $\deg(v_i) = 2n$)

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- But, deg(B) = 5, deg(C) = deg(D) = 3 (5+3+3 not even).
- Hence, there is no such route. Thus, it is impossible to travel around Königsberg crossing each bridge exactly once.

Fundamental Concepts

Definition: Graph

A graph G is an ordered pair of disjoint sets, (V, E) such that E is a subset of the set of unordered pairs of V.

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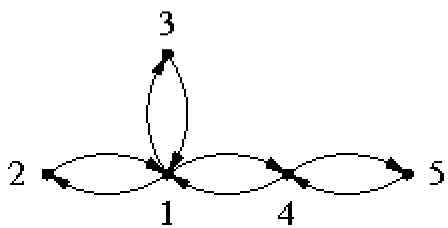
- Edges connect pairs of vertices.

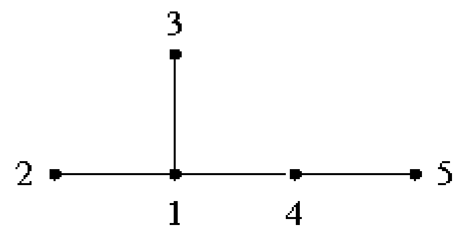
Fundamental Concepts

Definition: Digraph (directed graph)

A graph *D* with ordered pairs of nodes called arcs.



Digraph



Graph

Fundamental Concepts

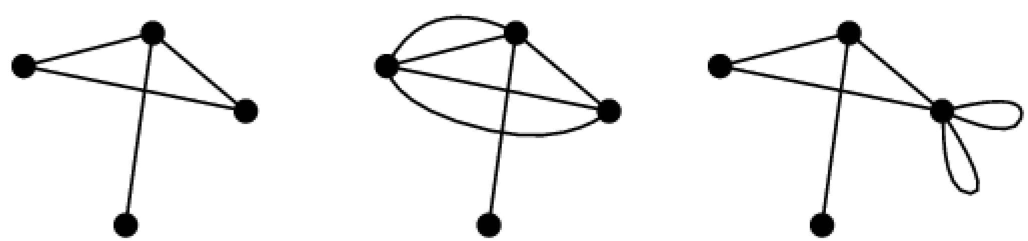
Definition: Simple Graph

A graph with no multiple edges or loops.

Fundamental Concepts

Definition: Multigraph

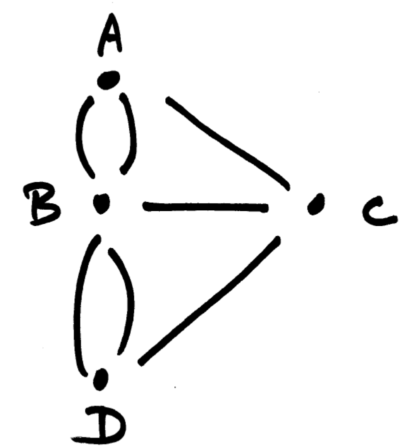
A graph with multiple edges (two nodes connected by more than one edge).



simple graph | *nonsimple graph with multiple edges* | *nonsimple graph with loops*

The Königsberg Bridge Problem

- This is a multigraph.



The Königsberg Bridge Problem

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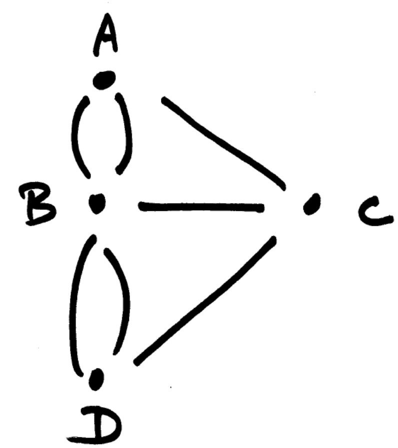
- **FACT**: A path that uses all edges in a multigraph exactly once is called an **Eulerian path**.



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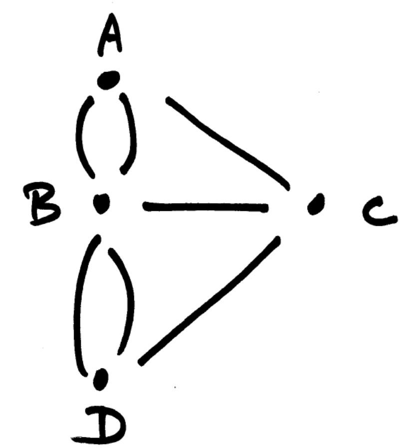

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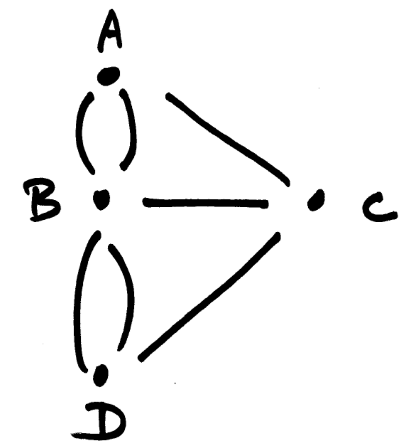
- The multigraph to the right has 4 vertices with an odd number of edges leaving them, so it has no Eulerian path.



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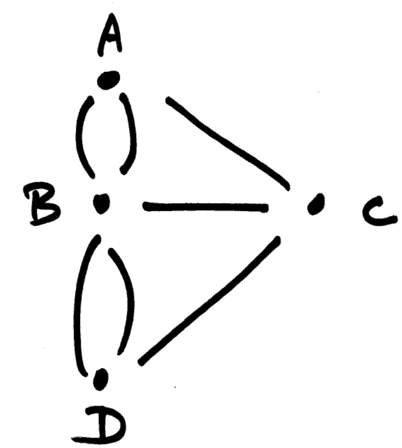

A Related Problem

Theorem: The Handshake Theorem

If *G* is any undirected graph, then the sum of the degrees of all the *n* vertices, *v*, of *G* equals twice the number of edges, *m*, of *G*. That is,

$$\sum_{v \in V(G)} \deg(v) = 2m$$

A Related Problem

Proof: The Handshake Theorem

- Crux: Each edge has two end vertices.

A Related Problem

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- Let G be an arbitrarily chosen undirected graph.

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- If G has n vertices, v_1, v_2,…, v_n, and m edges, where $n \in \mathbf{Z}^+$, $m \in \mathbf{N}$, we claim that each edge of G contributes 2 to the total degree of G.

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- This edge contributes 1 to the degree of v_i *and* 1 *to* the degree of v_j.

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- Therefore, e contributes 2 to the total degree of G.

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- For suppose *e* is an arbitrarily chosen edge with endpoints v_i *and* v_j.
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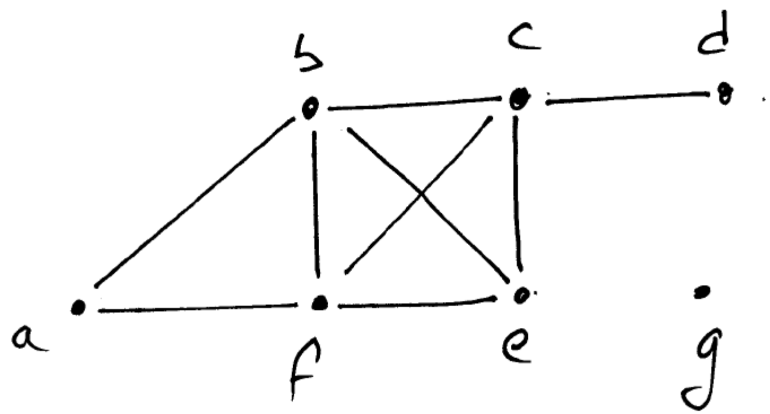
- This is true even if $v_i = v_j$ (an edge that is a loop is counted twice in computing the degree on which it is incident).
- Therefore, *e* contributes 2 to the total degree of *G*.
- Since *e* was arbitrarily chosen, this shows that each edge of *G* contributes 2 to the total degree of *G.*

QED

Example

- Apply the Handshake Theorem to the following:



edges = 9 ✓ OK

$\sum \deg = 18$

Corollary

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  - Known as The Handshaking “Lemma”.
  - Actually, “The number of vertices of odd degree is even”.
  - Example: Is the graph with 4 vertices of degrees 1, 1, 2, and 3 possible?

    NO!

Presentation Terminated