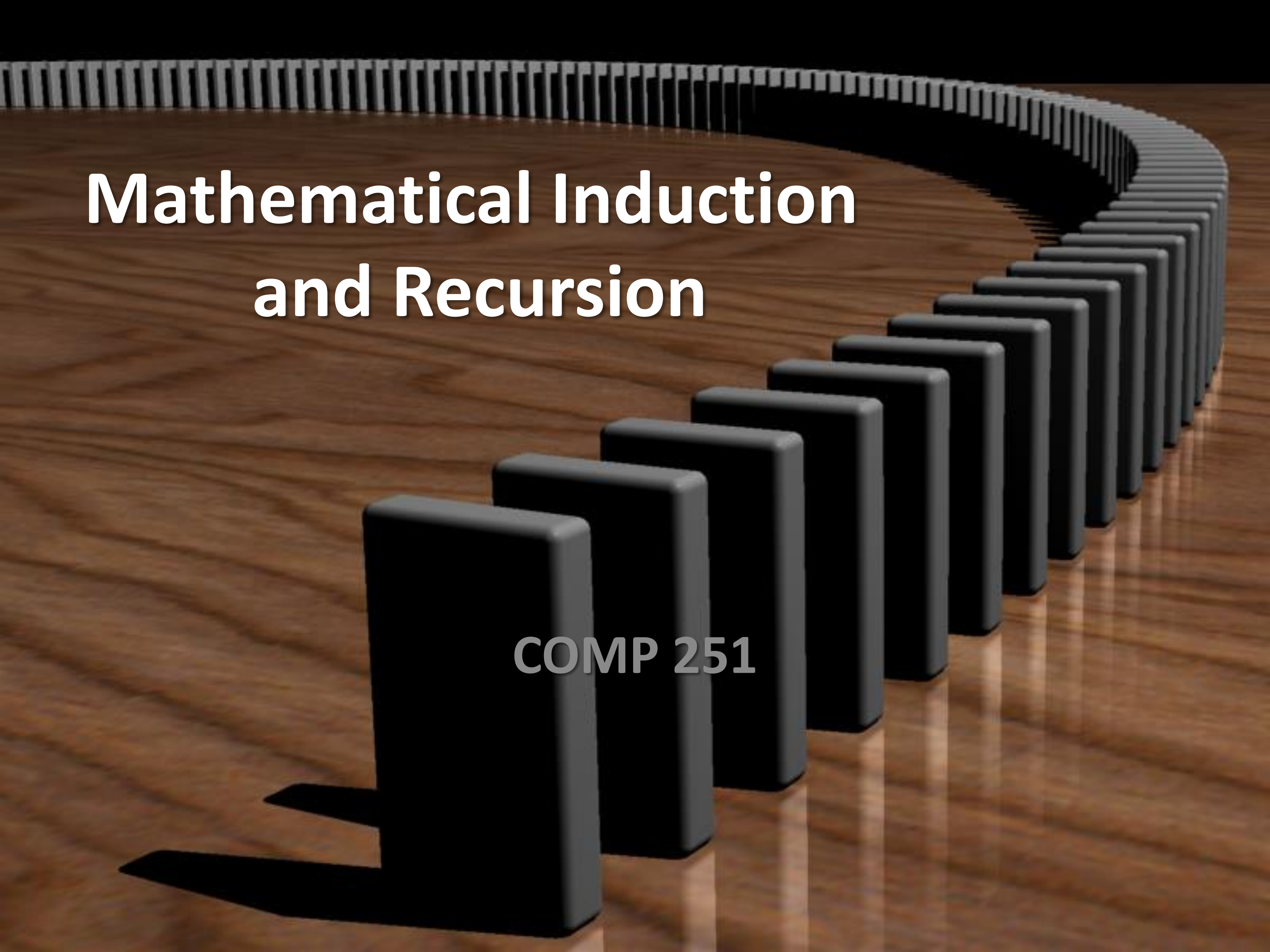
Mathematical Induction and Recursion



COMP 251

Goals

- To define the Principle of Mathematical Induction.
- To provide examples of it's application to $\boldsymbol{N}$ and $\boldsymbol{Z}^+$.
- To demonstrate the relationship between the Principle of Mathematical Induction and recursive sets.
- To introduce recursion from a mathematical perspective.

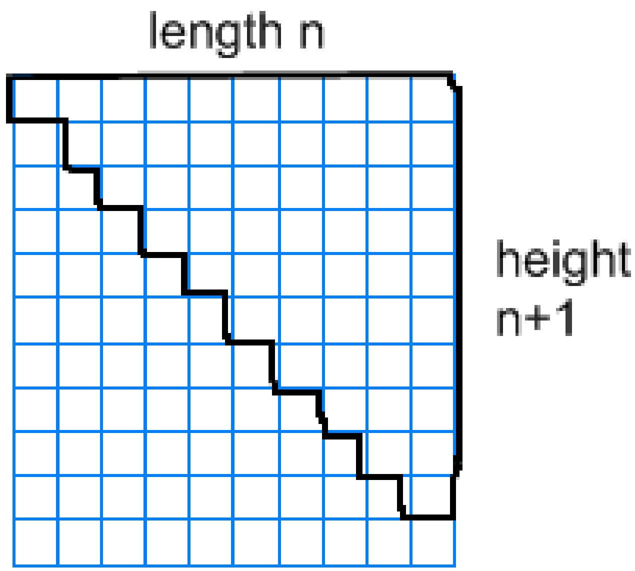
Consider the Sum of *n* Positive Integers

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

Consider the Sum of *n* Positive Integers

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

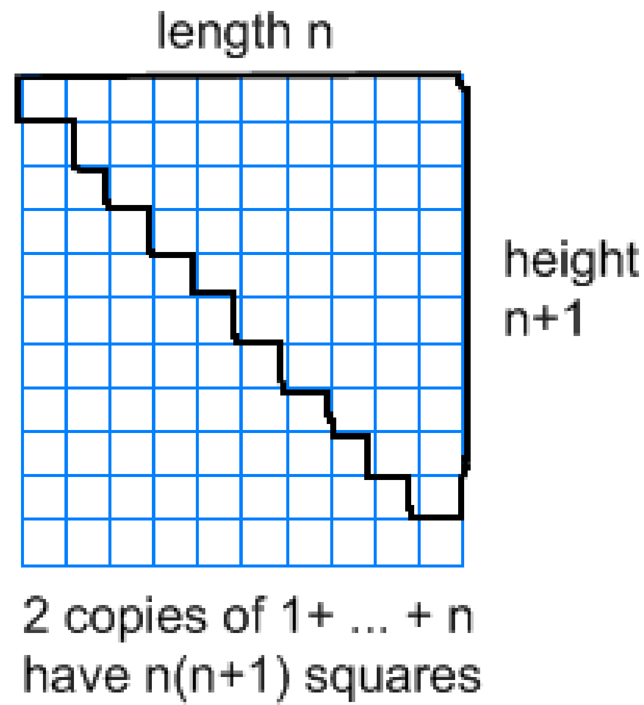
A Geometric Proof



2 copies of 1+ ... + n
have n(n+1) squares

Consider the Sum of n Positive Integers

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$



We require a formal proof technique for $\boldsymbol{Z}^{+}$ rather than a geometric proof.

We Begin With The Well-Ordering Principle

$$\exists \boldsymbol{m} \in \boldsymbol{Z}^{+}\ \forall \boldsymbol{n} \in \boldsymbol{Z}^{+}\ (\boldsymbol{m} \leq \boldsymbol{n})$$

Every *nonempty* subset of $\boldsymbol{Z}^{+}$ contains a smallest element.

We Begin With The Well-Ordering Principle

$$\exists m \in \mathbf{Z}^+ \ \forall n \in \mathbf{Z}^+ \ (m \leq n)$$

Every *nonempty* subset of $\mathbf{Z}^+$ contains a smallest element.

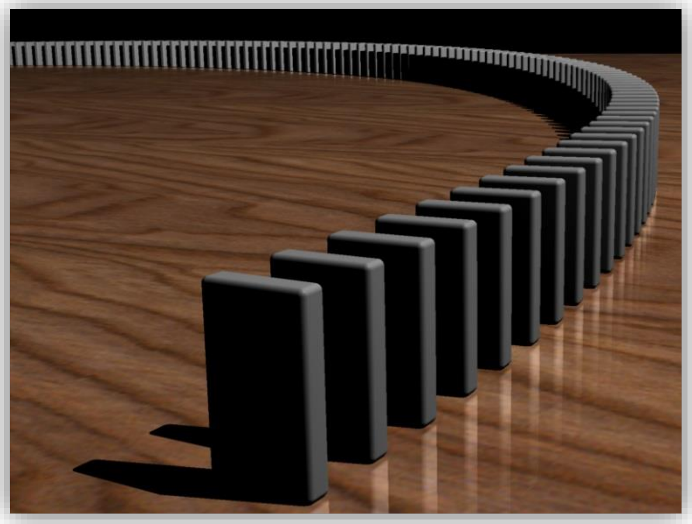
This is equivalent to PMI

(The Principle of Mathematical Induction)

The Principle of Mathematical Induction

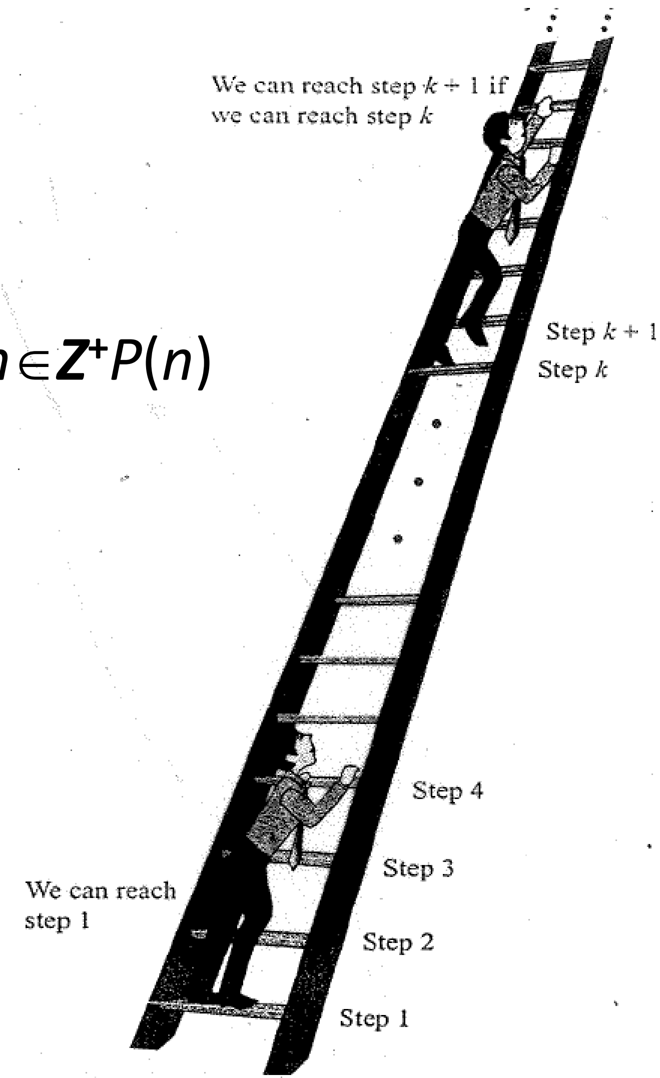
Basic idea:

- Two steps
  1. The first domino falls.
  2. When any domino falls, the next one will fall too.



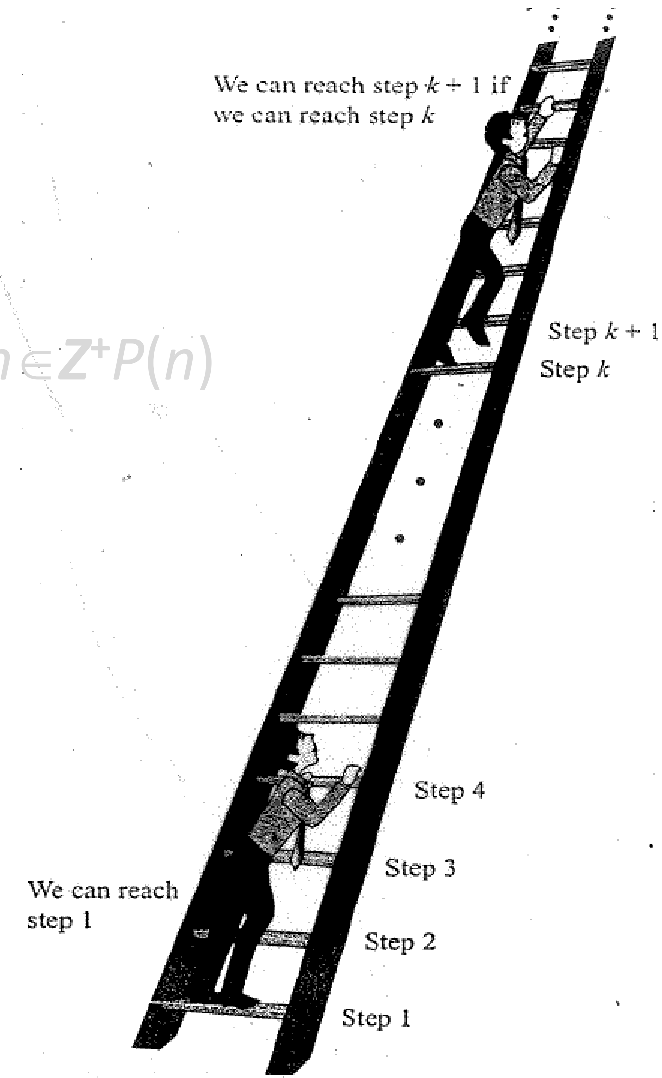
The Principle of Mathematical Induction

$$\forall P\left\{P(1) \wedge \forall k\in \boldsymbol{Z}^{+}[P(k) \Rightarrow P(k+1)]\right\} \Rightarrow \forall n\in \boldsymbol{Z}^{+}P(n)$$



The Principle of Mathematical Induction

$$\forall P\left\{P(1) \wedge \forall k \in \boldsymbol{Z}^{+}[P(k) \Rightarrow P(k+1)]\right\} \Rightarrow \forall n \in \boldsymbol{Z}^{+} P(n)$$



ALGORITHM (viewed as three steps):

1. Verify $P(1)$ is true – BASIS STEP (BS)
2. Assume $\forall n P(n)$ – INDUCTIVE HYPOTHESIS (IH)
3. Verify $P(k) \Rightarrow P(k+1)\ \forall k \in \boldsymbol{Z}^{+}$ **-** INDUCTIVE STEP (IS)

The Principle of Mathematical Induction

THIS IS REPEATED APPLICATIONS OF MODUS PONENS

$P(1)$	BS
$\underline{P(k) \Rightarrow P(k+1)\ \forall k \in \boldsymbol{Z}^{+}}$	IS
$P(n)\ \forall n \in \boldsymbol{Z}^{+} P(n)$	

$P(1)$

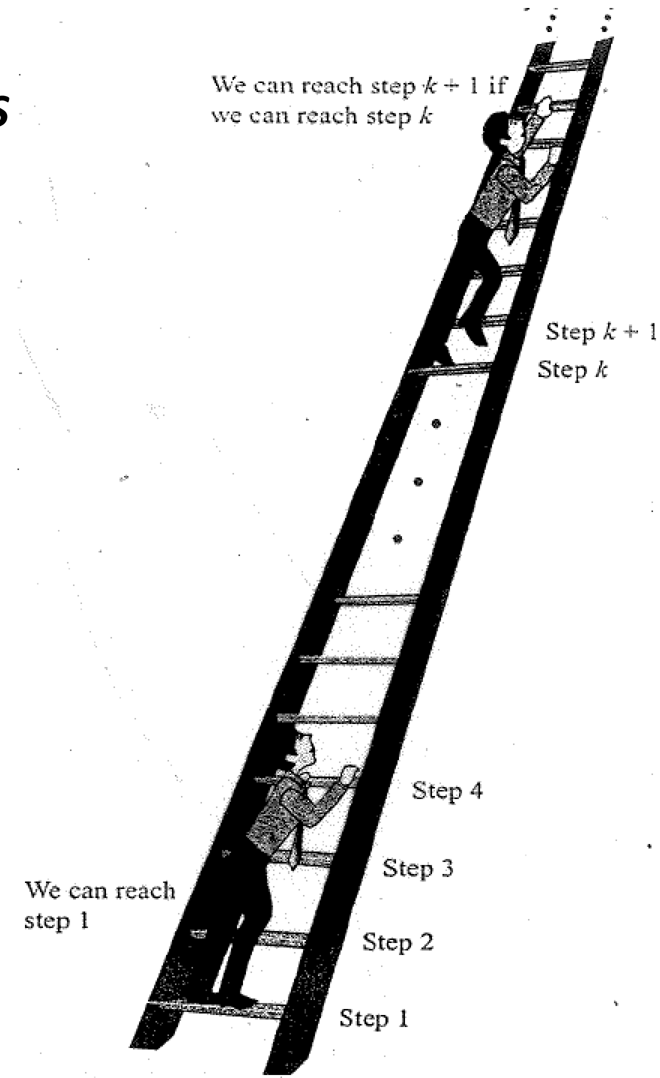
$\underline{P(1) \Rightarrow P(2)}$

$P(2)$

$\underline{P(2) \Rightarrow P(3)}$

$P(3)$

…



The Sum of the First *n* Integers

Mathematical induction can be used to prove that the following statement, which we will call $P(n)$, holds for all natural numbers n.

$$0 + 1 + 2 + \cdots + n = \frac{n(n+1)}{2}\,.$$

$P(n)$ gives a formula for the sum of the natural numbers less than or equal to number n. The proof that $P(n)$ is true for each natural number n proceeds as follows.

The Sum of the First *n* Integers

Basis: Show that the statement holds for $n = 0$.
$P(0)$ amounts to the statement:

$$0 = \frac{0 \cdot (0+1)}{2}\,.$$

In the left-hand side of the equation, the only term is 0, and so the left-hand side is simply equal to 0.
In the right-hand side of the equation, $0 \cdot (0 + 1)/2 = 0$.
The two sides are equal, so the statement is true for $n = 0$. Thus it has been shown that $P(0)$ holds.

The Sum of the First *n* Integers

Inductive step: Show that *if* $P(k)$ holds, then also $P(k + 1)$ holds. This can be done as follows.

Assume $P(k)$ holds (for some unspecified value of n). It must then be shown that $P(k + 1)$ holds, that is:

$$(0 + 1 + 2 + \cdots + k) + (k + 1) = \frac{(k+1)((k+1)+1)}{2}$$

Using the induction hypothesis that $P(k)$ holds, the left-hand side can be rewritten to:

$$\frac{k(k+1)}{2} + (k+1)\,.$$

Algebraically:

$$\begin{aligned}\frac{k(k+1)}{2} + (k+1) &= \frac{k(k+1) + 2(k+1)}{2} \\ &= \frac{(k+1)(k+2)}{2} \\ &= \frac{(k+1)((k+1)+1)}{2}.\end{aligned}$$

thereby showing that indeed $P(k + 1)$ holds.

The Sum of the First *n* Integers

Mathematical induction can be used to prove that the following statement, which we will call $P(n)$, holds for all natural numbers n.

$$0 + 1 + 2 + \cdots + n = \frac{n(n+1)}{2} .$$

$P(n)$ gives a formula for the sum of the natural numbers less than or equal to number n. The proof that $P(n)$ is true for each natural number n proceeds as follows.

Basis: Show that the statement holds for $n = 0$.
$P(0)$ amounts to the statement:

$$0 = \frac{0 \cdot (0+1)}{2} .$$

In the left-hand side of the equation, the only term is 0, and so the left-hand side is simply equal to 0.
In the right-hand side of the equation, $0 \cdot (0 + 1)/2 = 0$.
The two sides are equal, so the statement is true for $n = 0$. Thus it has been shown that $P(0)$ holds.

Inductive step: Show that *if* $P(k)$ holds, then also $P(k + 1)$ holds. This can be done as follows.

Assume $P(k)$ holds (for some unspecified value of n). It must then be shown that $P(k + 1)$ holds, that is:

$$(0 + 1 + 2 + \cdots + k) + (k+1) = \frac{(k+1)((k+1)+1)}{2}$$

Using the induction hypothesis that $P(k)$ holds, the left-hand side can be rewritten to:

$$\frac{k(k+1)}{2} + (k+1) .$$

Algebraically:

$$\begin{aligned} \frac{k(k+1)}{2} + (k+1) &= \frac{k(k+1) + 2(k+1)}{2} \\ &= \frac{(k+1)(k+2)}{2} \\ &= \frac{(k+1)((k+1)+1)}{2}. \end{aligned}$$

thereby showing that indeed $P(k + 1)$ holds.

Since both the basis and the inductive step have been proved, it has now been proved by mathematical induction that $P(n)$ holds for all natural n. Q.E.D.

Sum of Even Positive Integers

Problem: For any natural number $\boldsymbol{n}$, $\boldsymbol{2} + \boldsymbol{4} + \ldots + \boldsymbol{2n} = \boldsymbol{n(n+1)}$.

Sum of Even Positive Integers

Problem: For any natural number n, $2 + 4 + ... + 2n = n(n + 1)$.

Proof:
Basis Step: If $n = 0$, then $LHS = 0$, and $RHS = 0 * (0 + 1) = 0$.
Hence $LHS = RHS$.

Sum of Even Positive Integers

Induction: Assume that for an arbitrary natural number $\boldsymbol{n}$, $\boldsymbol{0 + 2 + \ldots + 2n = n(n + 1)}$. -------- *Induction Hypothesis*

To prove this for $\boldsymbol{n+1}$, first try to express ***LHS*** for $\boldsymbol{n+1}$ in terms of ***LHS*** for $\boldsymbol{n}$, and somehow use the induction hypothesis.

Here let us try

$$\textit{LHS} \text{ for } n + 1 = 0 + 2 + \ldots + 2n + 2(n + 1) = (0 + 2 + \ldots + 2n) + 2(n + 1) .$$

Using the induction hypothesis, the last expression can be rewritten as

$$n(n + 1) + 2(n + 1) .$$

Factoring $\boldsymbol{(n + 1)}$ out, we get

$$(n + 1)(n + 2) ,$$

which is equal to the ***RHS*** for $\boldsymbol{n+1}$.

Sum of Even Positive Integers

Problem: For any natural number n , $2 + 4 + ... + 2n = n(n + 1)$.

Proof:
Basis Step: If $n = 0$, then $LHS = 0$, and $RHS = 0 * (0 + 1) = 0$.
Hence $LHS = RHS$.
Induction: Assume that for an arbitrary natural number n, $0 + 2 + ... + 2n = n(n + 1)$. -------- *Induction Hypothesis*
To prove this for $n+1$, first try to express LHS for $n+1$ in terms of LHS for n, and somehow use the induction hypothesis.
Here let us try

LHS for $n + 1 = 0 + 2 + ... + 2n + 2(n + 1) = (0 + 2 + ... + 2n) + 2(n + 1)$.

Using the induction hypothesis, the last expression can be rewritten as

$n(n + 1) + 2(n + 1)$.

Factoring $(n + 1)$ out, we get

$(n + 1)(n + 2)$,

which is equal to the RHS for $n+1$.

QED

Sum of Squares

Problem: For any natural number n , $1^2 + 2^2 + ... + n^2 = n(n + 1)(2n + 1)/6$.

Sum of Squares

Problem: For any natural number $\boldsymbol{n}$, $1^2 + 2^2 + ... + n^2 = n(n + 1)(2n + 1)/6$.

Proof:

Basis Step: If $n = 0$, then $LHS = 0^2 = 0$, and $RHS = 0 * (0 + 1)(2*0 + 1)/6 = 0$.
Hence $LHS = RHS$.

Sum of Squares

Induction: Assume that for an arbitrary natural number n,

$1^2 + 2^2 + \ldots + n^2 = n(n+1)(2n+1)/6$. -------- *Induction Hypothesis*

To prove this for $n+1$, first try to express *LHS* for $n+1$ in terms of *LHS* for n, and use the induction hypothesis.

Here let us try

$$LHS \text{ for } n+1 = 1^2 + 2^2 + \ldots + n^2 + (n+1)^2 = (1^2 + 2^2 + \ldots + n^2) + (n+1)^2$$

Using the induction hypothesis, the last expression can be rewritten as

$$n(n+1)(2n+1)/6 + (n+1)^2$$

Factoring $(n+1)/6$ out, we get

$$(n+1)(n(2n+1) + 6(n+1))/6$$
$$= (n+1)(2n^2 + 7n + 6)/6$$
$$= (n+1)(n+2)(2n+3)/6,$$

which is equal to the *RHS* for $n+1$.

Thus *LHS* = *RHS* for $n+1$.

Sum of Squares

Problem: For any natural number n , $1^2 + 2^2 + ... + n^2 = n(n+1)(2n+1)/6$.

Proof:

Basis Step: If $n = 0$, then $LHS = 0^2 = 0$, and $RHS = 0 * (0+1)(2*0+1)/6 = 0$.
Hence $LHS = RHS$.
Induction: Assume that for an arbitrary natural number n,
$1^2 + 2^2 + ... + n^2 = n(n+1)(2n+1)/6$. -------- *Induction Hypothesis*
To prove this for $n+1$, first try to express LHS for $n+1$ in terms of LHS for n, and use the induction hypothesis.
Here let us try

$$LHS \text{ for } n+1 = 1^2 + 2^2 + ... + n^2 + (n+1)^2 = (1^2 + 2^2 + ... + n^2) + (n+1)^2$$

Using the induction hypothesis, the last expression can be rewritten as

$$n(n+1)(2n+1)/6 + (n+1)^2$$

Factoring $(n+1)/6$ out, we get

$$(n+1)(n(2n+1) + 6(n+1))/6$$
$$= (n+1)(2n^2 + 7n + 6)/6$$
$$= (n+1)(n+2)(2n+3)/6 ,$$

which is equal to the RHS for $n+1$.

Thus $LHS = RHS$ for $n+1$.

End of Proof.

Geometric Series

Problem: If r is a real number not equal to 1, then for every $n \geq 0$, $r^0 + r^1 + ... + r^n = (1 - r^{n+1})/(1 - r)$.

Geometric Series

Problem: If r is a real number not equal to 1, then for every $n \geq 0$, $r^0 + r^1 + ... + r^n = (1 - r^{n+1})/(1 - r)$.

Proof:

Basis Step: If $n = 0$, then $LHS = r^0 = 1$, and $RHS = (1 - r) / (1 - r) = 1$, since $r \neq 1$. Hence $LHS = RHS$.

Geometric Series

Induction: Assume that $r^0 + r^1 + \dots + r^n = (1 - r^{n+1})/(1 - r)$. -------- *Induction Hypothesis*

To prove this for $n+1$, first try to express *LHS* for $n+1$ in terms of *LHS* for n, and somehow use the induction hypothesis. Here let us try

$$LHS \text{ for } n + 1 = r^0 + r^1 + \dots + r^n + r^{n+1} = (r^0 + r^1 + \dots + r^n) + r^{n+1}.$$

Using the induction hypothesis, the last expression can be rewritten as $(1 - r^{n+1})/(1 - r) + r^{n+1}$.

Taking the common denominator, it is equal to $((1 - r^{n+1}) + (r^{n+1} - r^{n+2}))/(1 - r) = (1 - r^{n+2})/(1 - r)$,

which is equal to the *RHS* for $n+1$.

Thus *LHS* = *RHS* for $n+1$.

Geometric Series

Problem: If r is a real number not equal to 1, then for every $n \geq 0$, $r^0 + r^1 + ... + r^n = (1 - r^{n+1})/(1 - r)$.

Proof:

Basis Step: If $n = 0$, then $LHS = r^0 = 1$, and $RHS = (1 - r) / (1 - r) = 1$, since $r \neq 1$. Hence $LHS = RHS$.

Induction: Assume that $r^0 + r^1 + ... + r^n = (1 - r^{n+1})/(1 - r)$. -------- *Induction Hypothesis*

To prove this for $n+1$, first try to express LHS for $n+1$ in terms of LHS for n, and somehow use the induction hypothesis. Here let us try

$$LHS \text{ for } n + 1 = r^0 + r^1 + ... + r^n + r^{n+1} = (r^0 + r^1 + ... + r^n) + r^{n+1}.$$

Using the induction hypothesis, the last expression can be rewritten as $(1 - r^{n+1})/(1 - r) + r^{n+1}$.

Taking the common denominator, it is equal to $((1 - r^{n+1}) + (r^{n+1} - r^{n+2}))/(1 - r) = (1 - r^{n+2})/(1 - r)$,

which is equal to the RHS for $n+1$.

Thus $LHS = RHS$ for $n+1$.

End of Proof.

Factorial vs. Exponential Dominance

Problem: For every $n \geq 4$, $n! > 2^n$.

Factorial vs. Exponential Dominance

Problem: For every $n \geq 4$, $n! > 2^n$.

Proof:
In this problem $n_0 = 4$.

Basis Step: If $n = 4$, then $LHS = 4! = 24$, and $RHS = 2^4 = 16$.
Hence $LHS > RHS$.
Induction: Assume that $n! > 2^n$ for an arbitrary $n \geq 4$. -- Induction Hypothesis

Factorial vs. Exponential Dominance

To prove that this inequality holds for n+1, first try to express *LHS* for n+1 in terms of *LHS* for n and try to use the induction hypothesis.
Note here $(n + 1)! = (n + 1)\, n!$.
Thus using the induction hypothesis, we get $(n + 1)! = (n+1)n! > (n+1)2^n$.

Since $n \geq 4$, $(n+1) > 2$.

Hence $(n+1)2^n > 2^{(n+1)}$.

Hence $(n+1)! > 2^{(n+1)}$.

Factorial vs. Exponential Dominance

Problem: For every $n \geq 4$, $n! > 2^n$.

Proof:
In this problem $n_0 = 4$.

Basis Step: If $n = 4$, then $LHS = 4! = 24$, and $RHS = 2^4 = 16$.
Hence $LHS > RHS$.
Induction: Assume that $n! > 2^n$ for an arbitrary $n \geq 4$. -- Induction Hypothesis

To prove that this inequality holds for $n+1$, first try to express LHS for $n+1$ in terms of LHS for n and try to use the induction hypothesis.
Note here $(n + 1)! = (n + 1)\, n!$.
Thus using the induction hypothesis, we get $(n + 1)! = (n+1)n! > (n+1)2^n$.

Since $n \geq 4$, $(n+1) > 2$.

Hence $(n+1)2^n > 2^{(n+1)}$.

Hence $(n+1)! > 2^{(n+1)}$.

End of Proof.

Factorial vs. Exponential Dominance

Problem: For every $n \geq 4$, $n! > 2^n$.

Proof:
In this problem $n_0 = 4$.

Basis Step: If $n = 4$, then $LHS = 4! = 24$, and $RHS = 2^4 = 16$.
Hence $LHS > RHS$.
Induction: Assume that $n! > 2^n$ for an arbitrary $n \geq 4$. -- Induction Hypothesis

To prove that this inequality holds for $n+1$, first try to express *LHS* for $n+1$ in terms of *LHS* for n and try to use the induction hypothesis.
Note here $(n+1)! = (n+1)\, n!$.
Thus using the induction hypothesis, we get $(n+1)! = (n+1)n! > (n+1)2^n$.

Since $n \geq 4$, $(n+1) > 2$.

Hence $(n+1)2^n > 2^{(n+1)}$.

Hence $(n+1)! > 2^{(n+1)}$.

End of Proof.

Recursion

Definition

- **Recursion** occurs when a thing is defined in terms of itself or of its type.
- **Recursion** in *computing science* is a method where the solution to a problem depends on solutions to smaller instances of the same problem (as opposed to *iteration*) itself or of its type.

Recursively Defined Functions

Classic example – the Factorial Function:

$$n! = \begin{cases} 1 & \text{if } n = 0, \\ (n-1)! \times n & \text{if } n > 0. \end{cases}$$

Recursively Defined Sequences

Consider a physical process in which a given population, *x*, doubles over a particular time, *t*:

$$x_{t+1} = kx_t$$

The process requires an *initial condition*:

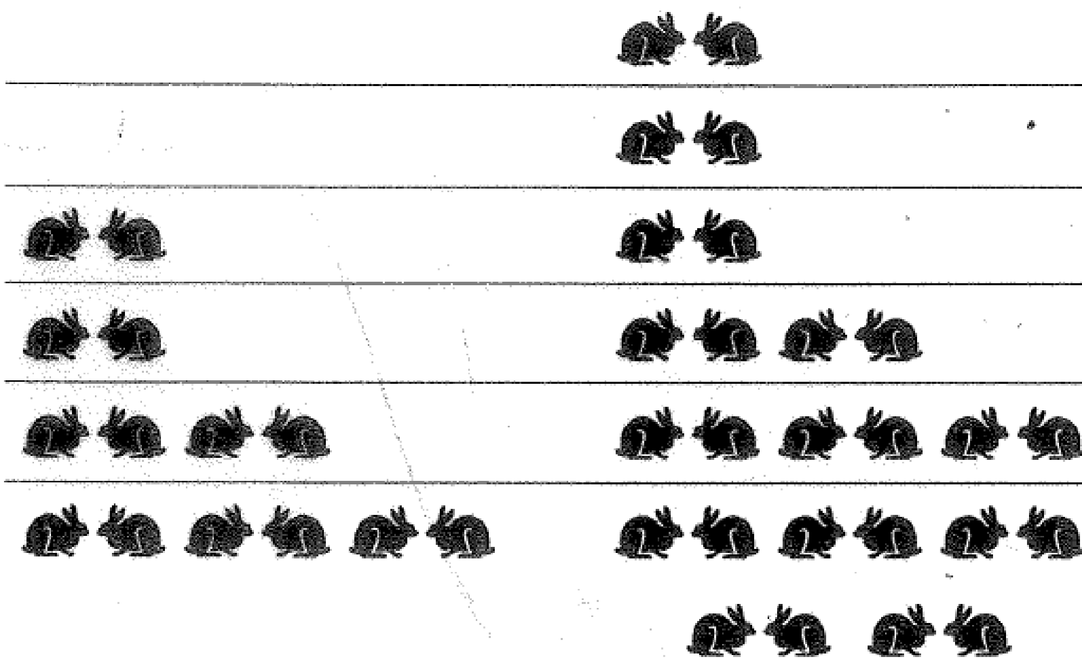
$$x_0 = a_0$$

This is known as a recursive set (a relation that is defined in terms of smaller units of itself and requiring a condition to get it started).

This is the discrete description of exponential growth/decay.

Recursively Defined Sequences

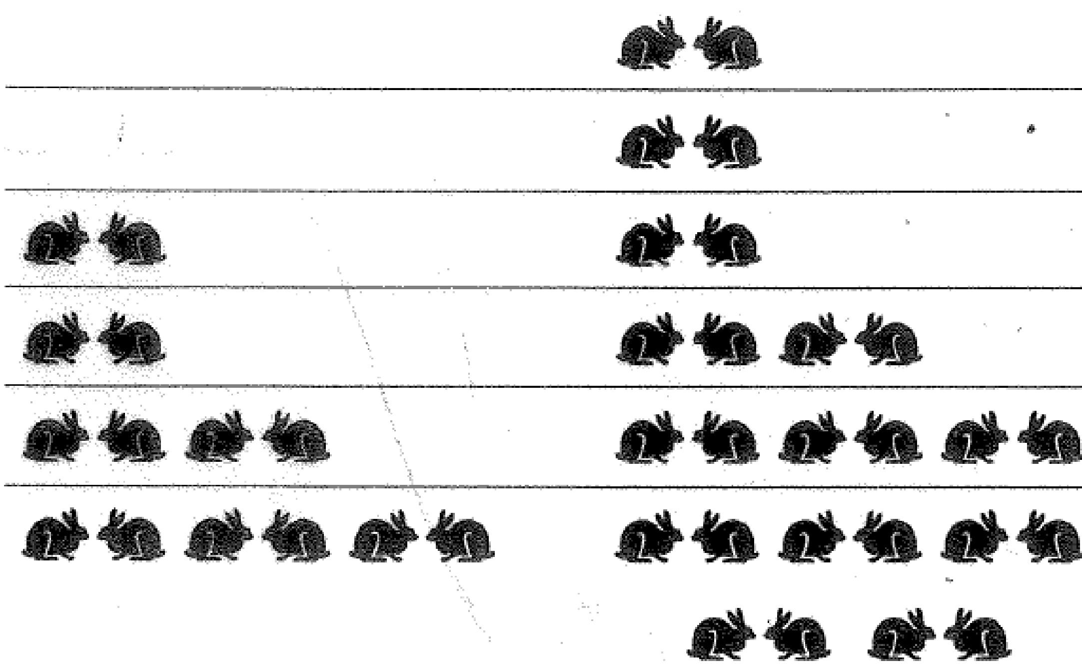
Reproducing rabbits – The Fibonacci Sequence



Reproducing pairs (at least two months old)	Young pairs (less than two months old)	Month	Reproducing pairs	Young pairs	Total pairs
		1	0	1	1
		2	0	1	1
		3	1	1	2
		4	1	2	3
		5	2	3	5
		6	3	5	8

Recursively Defined Sequences

Recall reproducing rabbits – The Fibonacci Sequence



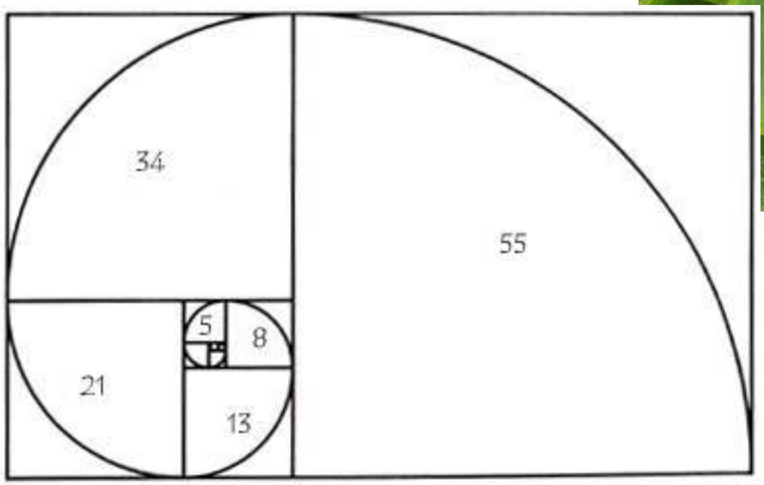
Reproducing pairs (at least two months old)	Young pairs (less than two months old)	Month	Reproducing pairs	Young pairs	Total pairs
		1	0	1	1
		2	0	1	1
		3	1	1	2
		4	1	2	3
		5	2	3	5
		6	3	5	8

Recursive Set

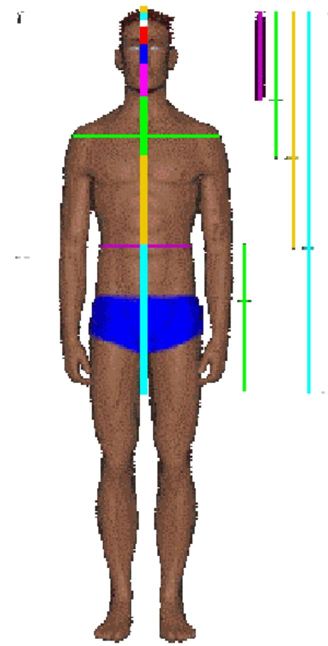
$$\begin{cases} f_1=1,\ f_2=1, \\ f_n = f_{n-1} + f_{n-2} \end{cases}$$

Recursively Defined Sequences

Fibonacci numbers appear throughout nature



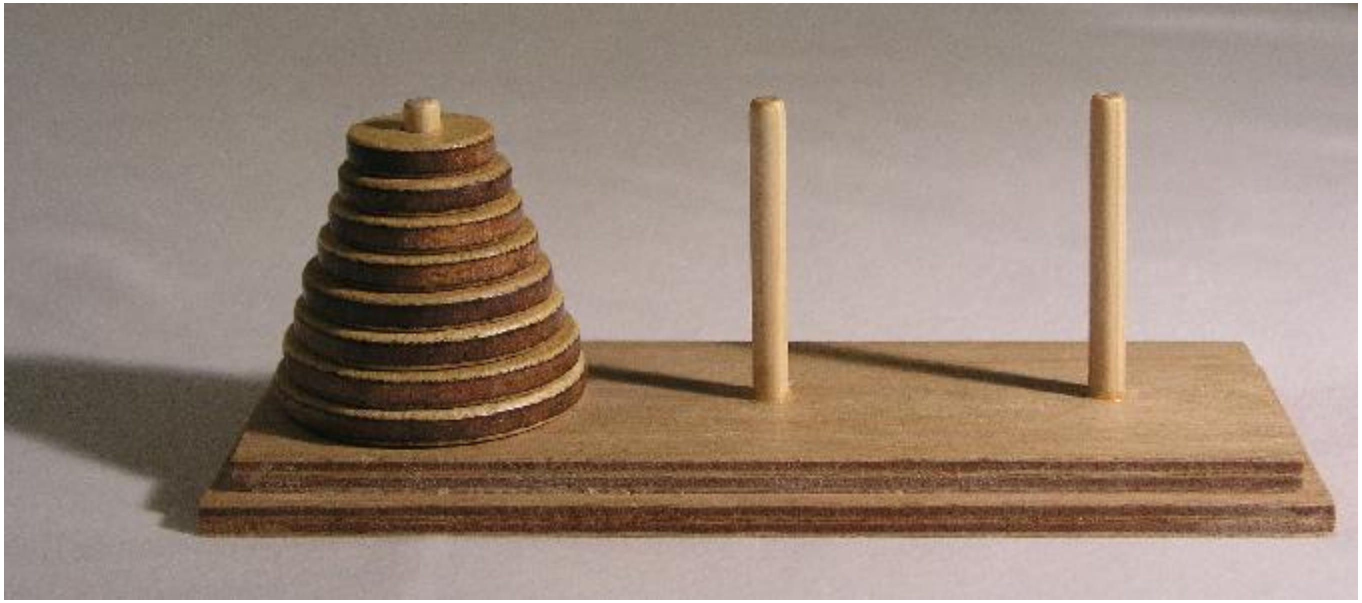
Is there an Biological Basis for Beauty?



The Golden Ratio (Fibonacci Numbers)

$$F(n) = \frac{\varphi^n - (1-\varphi)^n}{\sqrt{5}} = \frac{\varphi^n - (-1/\varphi)^n}{\sqrt{5}},$$ where φ is the **golden ratio.**

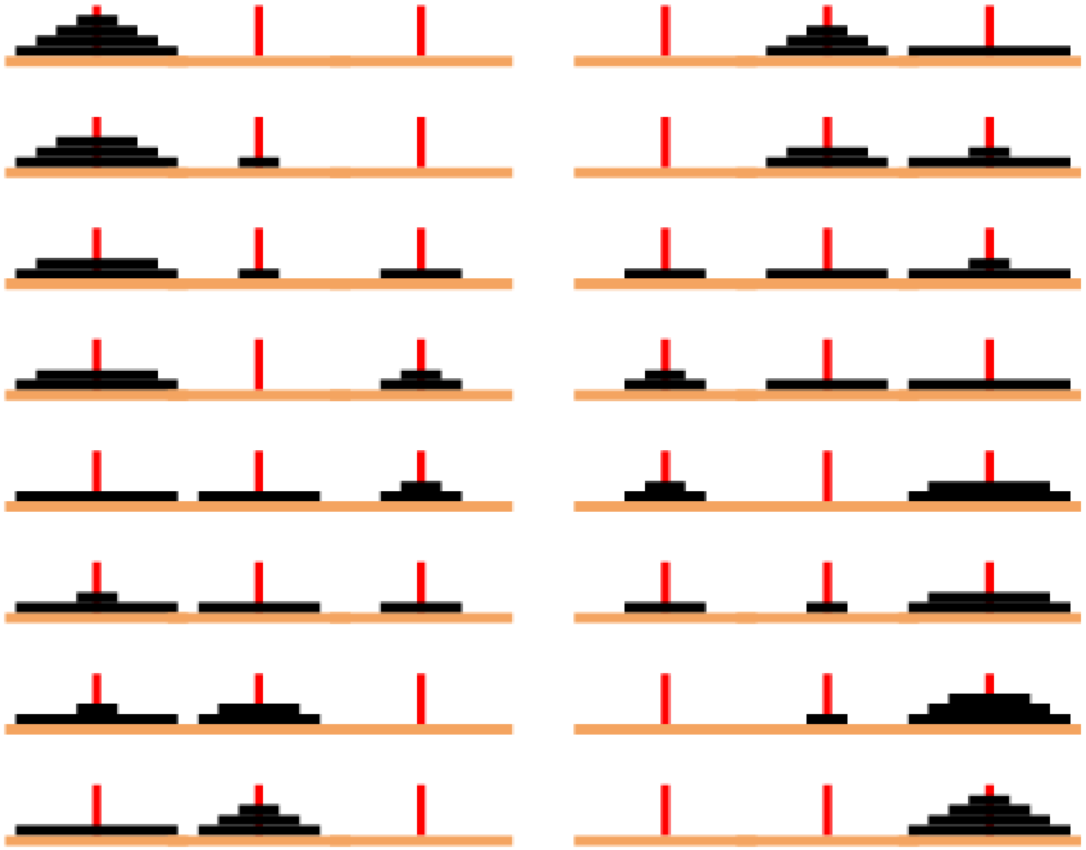
Recursive Sets – Tower of Hanoi



The objective of the puzzle is to **move the entire stack to another rod**, obeying the following **rules**:

- Only **one disk may be moved at a time**.
- Each move consists of taking the upper disk from one of the rods and sliding it onto another rod, on top of the other disks that may already be present on that rod.
- **No disk may be placed on top of a smaller disk**.

Recursive Sets – Tower of Hanoi



Recursive Sets – Tower of Hanoi

Let H_n denote the number of moves needed to solve the Tower of Hanoi problem with n disks. Set up a recurrence relation for the sequence $\{H_n\}$.

Recursive Sets – Tower of Hanoi

Let H_n denote the number of moves needed to solve the Tower of Hanoi problem with n disks. Set up a recurrence relation for the sequence $\{H_n\}$.

Solution: Begin with n disks on peg 1. We can transfer the top $n-1$ disks, following the rules of the puzzle, to peg 3 using H_{n-1} moves (see Figure 3 for an illustration of the pegs and disks at this point). We keep the largest disk fixed during these moves. Then, we use one move to transfer the largest disk to the second peg. We can transfer the $n-1$ disks on peg 3 to peg 2 using H_{n-1} additional moves, placing them on top of the largest disk, which always stays fixed on the bottom of peg 2. Moreover, it is easy to see that the puzzle cannot be solved using fewer steps. This shows that

$$H_n = 2H_{n-1} + 1.$$

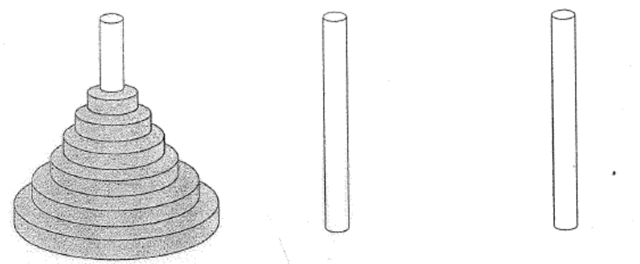


FIGURE 2 **The Initial Position in the Tower of Hanoi.**

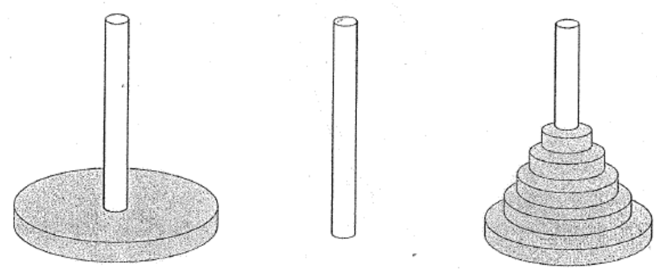
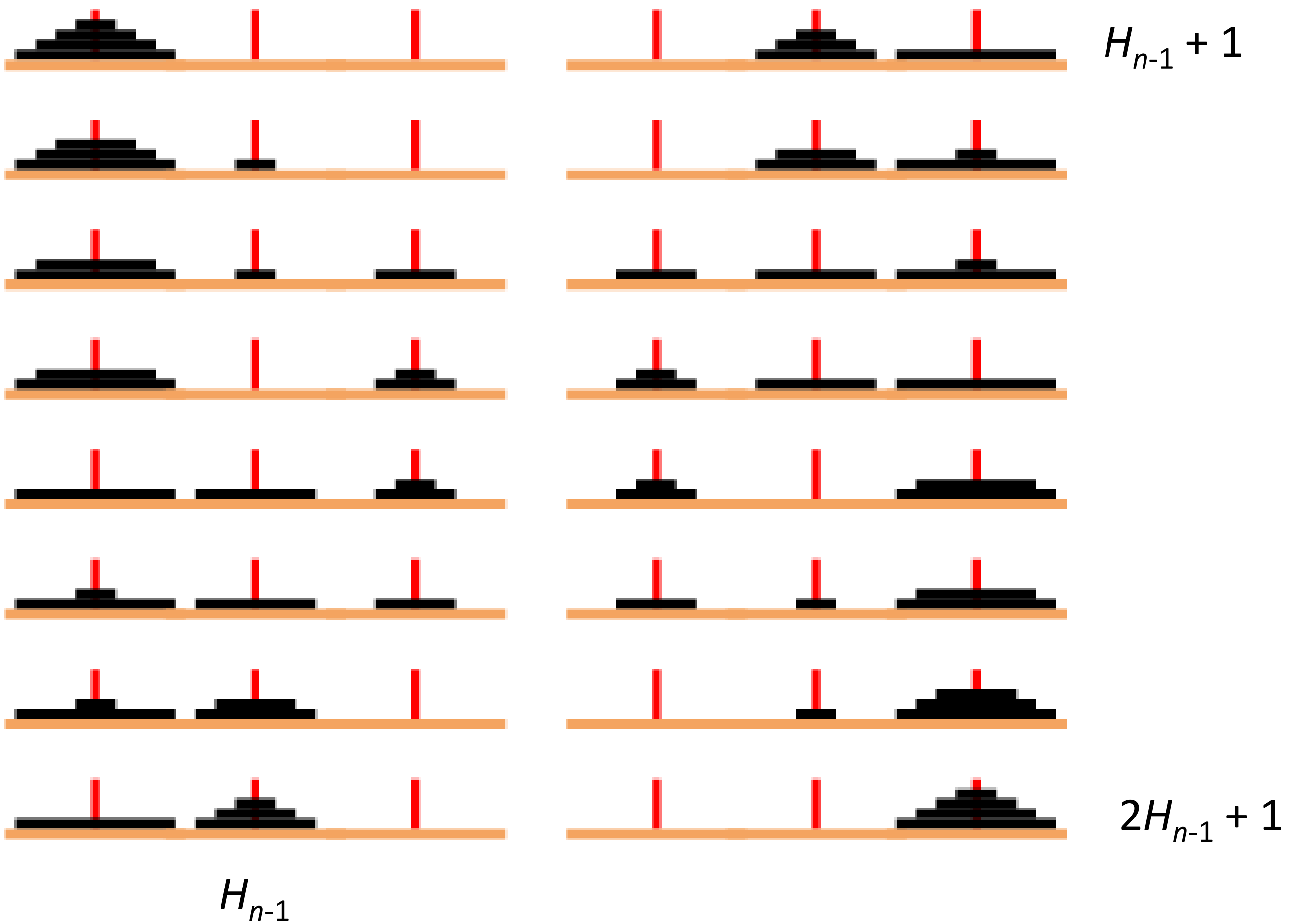


FIGURE 3 **An Intermediate Position in the Tower of Hanoi.**

Recursive Sets – Tower of Hanoi



Recursive Sets – Tower of Hanoi

Let H_n denote the number of moves needed to solve the Tower of Hanoi problem with n disks. Set up a recurrence relation for the sequence $\{H_n\}$.

Solution: Begin with n disks on peg 1. We can transfer the top $n-1$ disks, following the rules of the puzzle, to peg 3 using H_{n-1} moves (see Figure 3 for an illustration of the pegs and disks at this point). We keep the largest disk fixed during these moves. Then, we use one move to transfer the largest disk to the second peg. We can transfer the $n-1$ disks on peg 3 to peg 2 using H_{n-1} additional moves, placing them on top of the largest disk, which always stays fixed on the bottom of peg 2. Moreover, it is easy to see that the puzzle cannot be solved using fewer steps. This shows that

$$H_n = 2H_{n-1} + 1.$$

"Open form expression"

The initial condition is $H_1 = 1$, because one disk can be transferred from peg 1 to peg 2, according to the rules of the puzzle, in one move.

We can use an iterative approach to solve this recurrence relation. Note that

$$\begin{aligned} H_n &= 2H_{n-1} + 1 \\ &= 2(2H_{n-2} + 1) + 1 = 2^2 H_{n-2} + 2 + 1 \\ &= 2^2(2H_{n-3} + 1) + 2 + 1 = 2^3 H_{n-3} + 2^2 + 2 + 1 \\ &\vdots \\ &= 2^{n-1} H_1 + 2^{n-2} + 2^{n-3} + \cdots + 2 + 1 \\ &= 2^{n-1} + 2^{n-2} + \cdots + 2 + 1 \\ &= 2^n - 1. \end{aligned}$$

"Closed form expression"

Recursive Sets – Tower of Hanoi

Let H_n denote the number of moves needed to solve the Tower of Hanoi problem with n disks. Set up a recurrence relation for the sequence $\{H_n\}$.

Solution: Begin with n disks on peg 1. We can transfer the top $n-1$ disks, following the rules of the puzzle, to peg 3 using H_{n-1} moves (see Figure 3 for an illustration of the pegs and disks at this point). We keep the largest disk fixed during these moves. Then, we use one move to transfer the largest disk to the second peg. We can transfer the $n-1$ disks on peg 3 to peg 2 using H_{n-1} additional moves, placing them on top of the largest disk, which always stays fixed on the bottom of peg 2. Moreover, it is easy to see that the puzzle cannot be solved using fewer steps. This shows that

$$H_n = 2H_{n-1} + 1.$$

The initial condition is $H_1 = 1$, because one disk can be transferred from peg 1 to peg 2, according to the rules of the puzzle, in one move.

We can use an iterative approach to solve this recurrence relation. Note that

$$\begin{aligned}
H_n &= 2H_{n-1} + 1 \\
&= 2(2H_{n-2} + 1) + 1 = 2^2H_{n-2} + 2 + 1 \\
&= 2^2(2H_{n-3} + 1) + 2 + 1 = 2^3H_{n-3} + 2^2 + 2 + 1 \\
&\ \vdots \\
&= 2^{n-1}H_1 + 2^{n-2} + 2^{n-3} + \cdots + 2 + 1 \\
&= 2^{n-1} + 2^{n-2} + \cdots + 2 + 1 \\
&= 2^n - 1.
\end{aligned}$$

We have used the recurrence relation repeatedly to express H_n in terms of previous terms of the sequence. In the next to last equality, the initial condition $H_1 = 1$ has been used. The last equality is based on the formula for the sum of the terms of a geometric series, which can be found in Theorem 1 in Section 2.4.

The iterative approach has produced the solution to the recurrence relation $H_n = 2H_{n-1} + 1$ with the initial condition $H_1 = 1$. This formula can be proved using mathematical induction. This is left for the reader as an exercise at the end of the section.

A myth created to accompany the puzzle tells of a tower in Hanoi where monks are transferring 64 gold disks from one peg to another, according to the rules of the puzzle. The myth

Require Proof

Recursive Sets – Tower of Hanoi

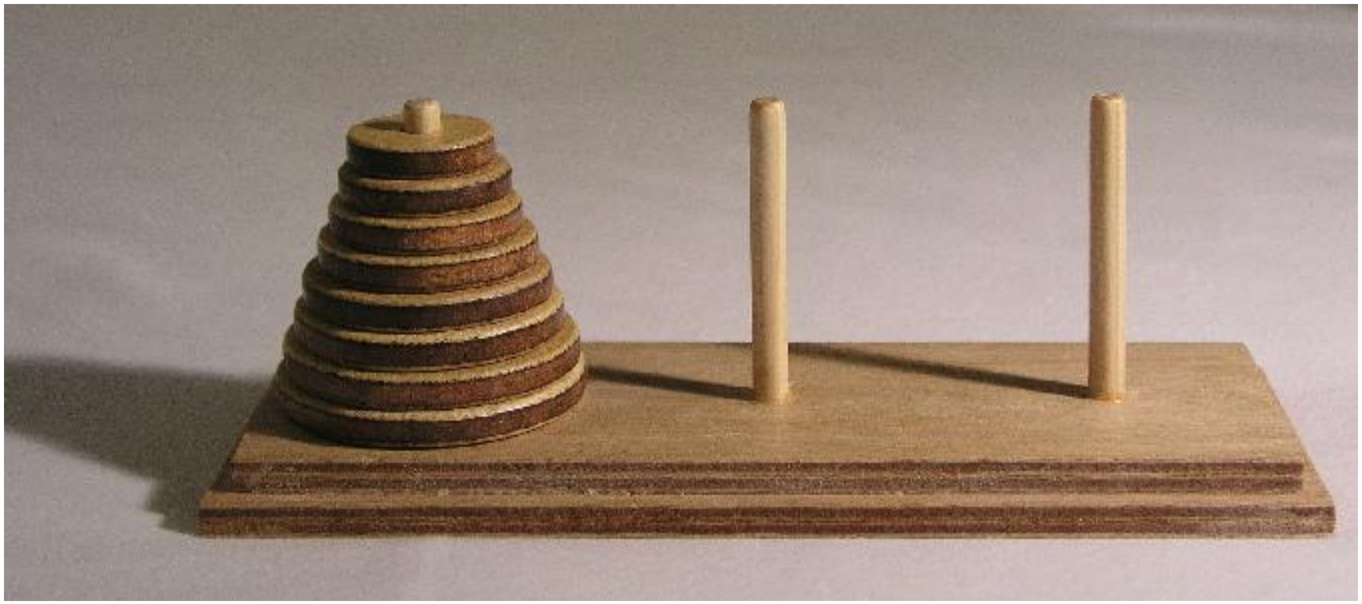
- The minimum number *Hn* of moves required to transfer a tower of *n* disks satisfies the open form expression, $H_n = 2H_{n\text{-}1} + 1\ \forall n>1$ with $H_1 = 1$ (one move for the case of one disk).
- We want to prove the closed form expression, $\underline{H_n = 2^n - 1}\ \forall n \in \boldsymbol{Z}^+$.

- **PROOF** (by PMI):
  - BS, n=1: $H_1 = 1 = 2^1 - 1 = 1$ (OK)
  - IH: Assume $H_n = 2^n - 1\ \forall k \in \boldsymbol{Z}^+$.
  - IS:

$H_{n+1} = 2H_n + 1$	Derive the $(n+1)^{st}$ step from the n^{th} step.
$= 2[2^n - 1] + 1$	Apply the IH.
$= 2^{n+1} - 2 + 1$	Basic algebra.
$= 2^{n+1} - 1$	

QED

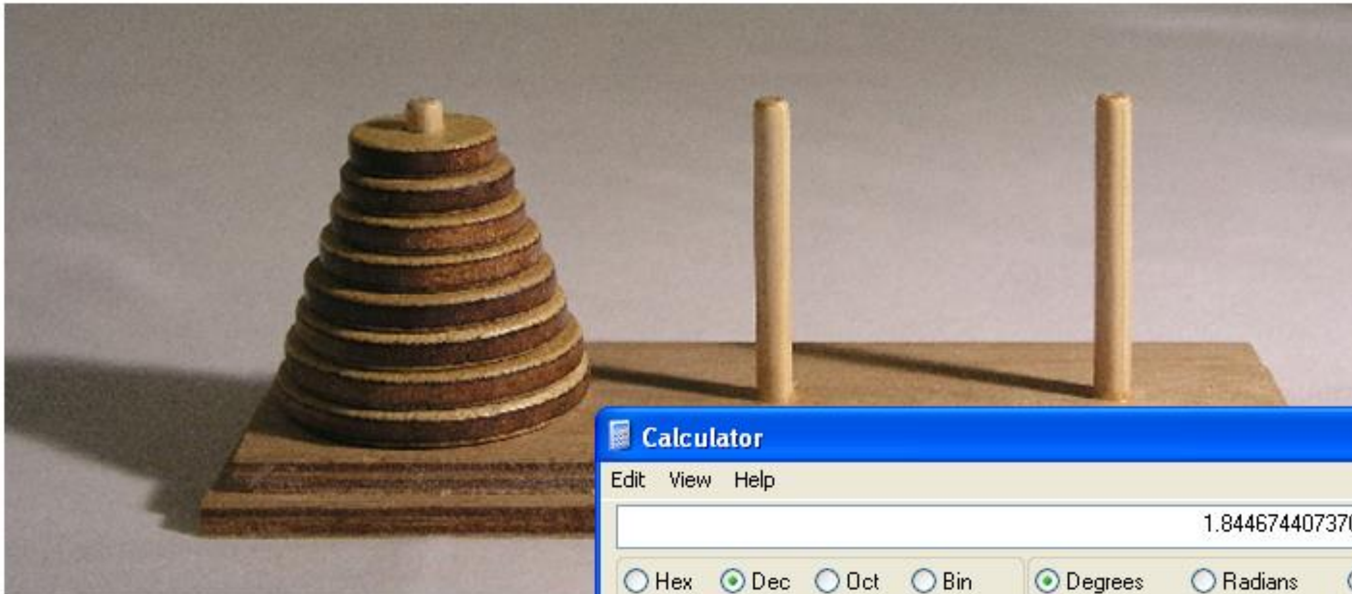
How Many Moves For 64 Disks?

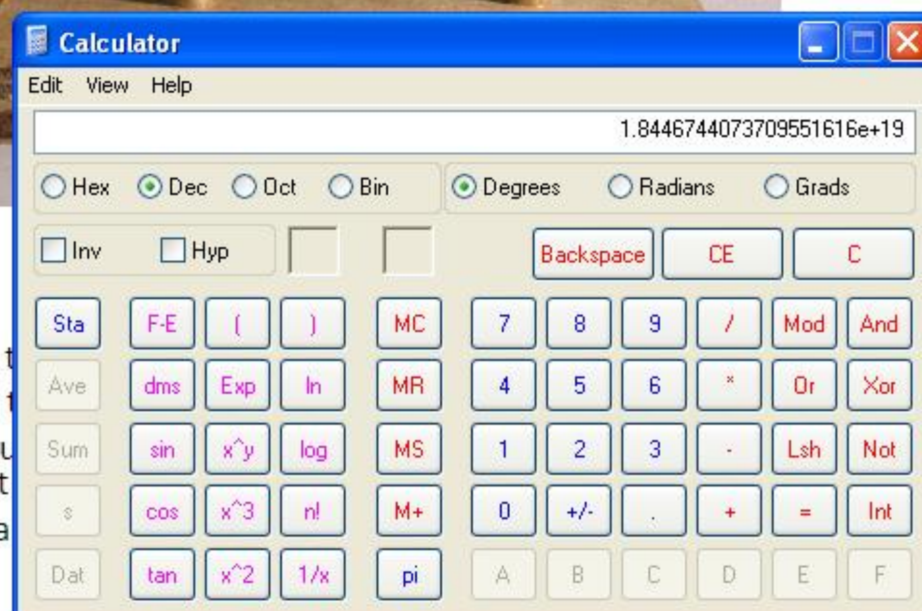


The objective of the puzzle is to move the entire stack to another rod, obeying the following rules:

- Only one disk may be moved at a time.
- Each move consists of taking the upper disk from one of the rods and sliding it onto another rod, on top of the other disks that may already be present on that rod.
- No disk may be placed on top of a smaller disk.

How Many Moves For 64 Disks?





The objective of the puzzle is to move t

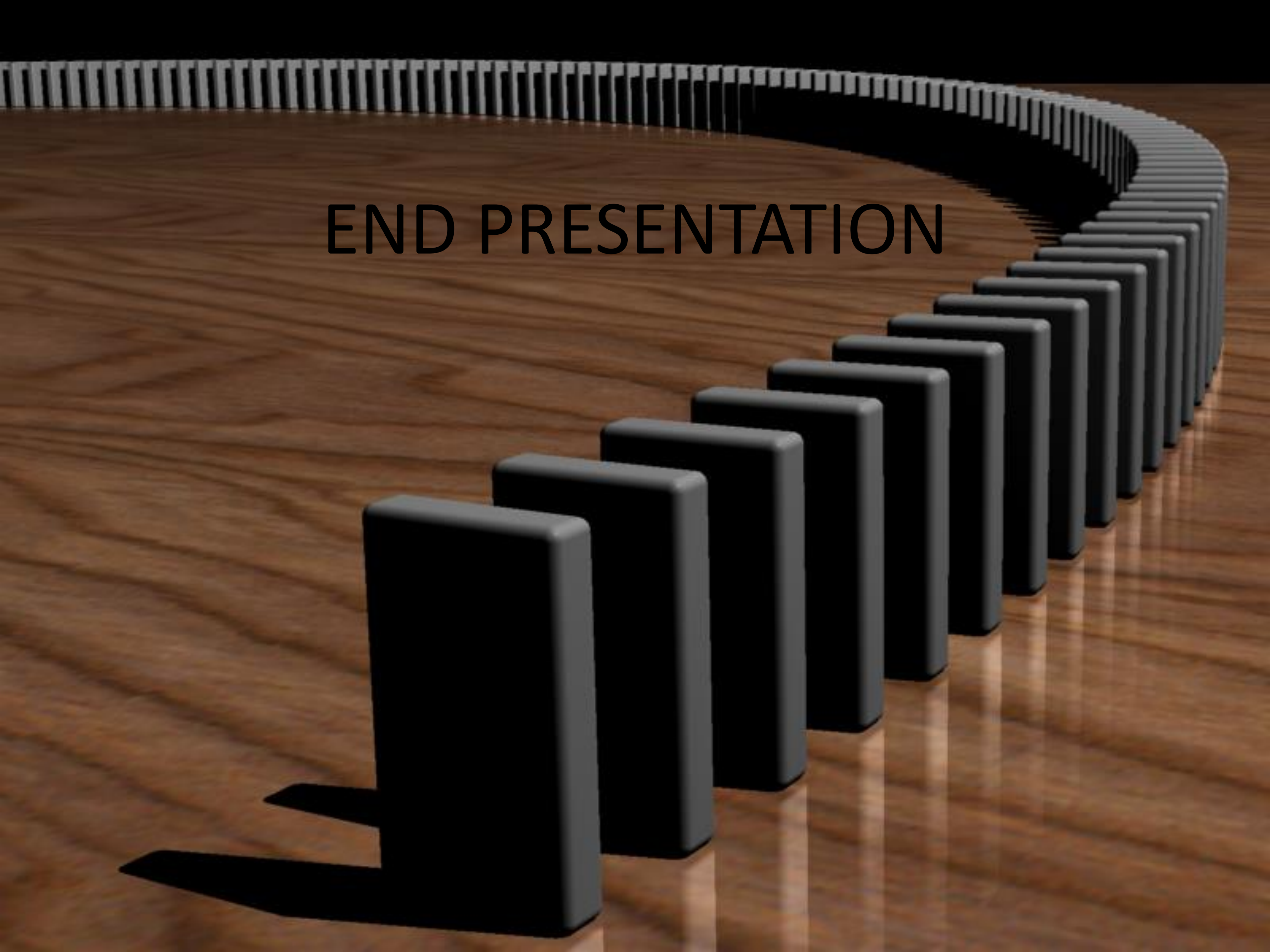
- Only one disk may be moved at a t
- Each move consists of taking the u rod, on top of the other disks that
- No disk may be placed on top of a

Recursion

- We see how the Principle of Mathematical Induction lends itself naturally to recursion (initial condition/base step and recursive step/inductive step).

Recursion

- We see how the Principle of Mathematical Induction lends itself naturally to recursion (initial condition/base step and recursive step/inductive step).
- Some computer programs are recursive in nature and, hence, can be proven correct using the PMI.


END PRESENTATION