Dynamic Table Slides

Mitsunori Ogihara

A dynamic table is a table of variable size, where an expansion (or a contraction) is caused when the load factor has become larger (or smaller) than a fixed threshold.

Let the expansion threshold be 1 and the expansion rate be 2; that is, *the table size is doubled when an item is to be inserted when the table is full*.

Let the contraction threshold be $1/4$ and the contraction rate be $1/2$; that is, *the table size is halved when an item is to be eliminated when the table is exactly* $1/4$ *full*.

When these operations take place we *create a new table* and move all the elements from the old one to the new one.

Suppose that there are n calls of insertion and deletion are made, what is the average cost of each operation?

If the size is kept the same the cost is $O(1)$.

If the size is doubled from M to $2M$, the actual cost is $M + 1$. The time that it takes for the next table size change to occur is *at least M steps for doubling* and *at least $M/2$ steps for halving*. So the actual cost can be spread over the next $M/2$ "normal" steps. This gives an amortized cost of $O(1)$.

If the size is halved from M to $M/2$, the actual cost is $M/4$. The time that it takes for the next table size change to occur is *at least $M/4$ steps for doubling* and *at least $M/8$ steps for halving*. So the actual cost can be spread over the next $M/8$ steps to yield an amortized cost of $O(1)$.

For each i, $1 \le i \le n$, define c_i to be the number of insertions and deletions that are executed at the i-th operation, and define

$$\Phi_i = \begin{cases} 2\mathsf{num}_i - \mathsf{size}_i & \text{if } \alpha_i \ge \frac{1}{2}, \\ \frac{\mathsf{size}_i}{2} - \mathsf{num}_i & \text{if } \alpha_i < \frac{1}{2}, \end{cases}$$

Here size_i is the table size, num_i is the number of elements in the table, and α_i is the ratio $\mathsf{num}_i/\mathsf{size}_i$ after the i-th operation. Note that

- at time 0, the table is empty, so $\Phi_0 = 0$,
- for all i, $\Phi_i \ge 0$, and thus, $\Phi_n \ge \Phi_0$, and
- $\Phi_n \le 2n - n = n$, so the contribution of the potential function to the amortized cost is at most 1.

Here $m = \text{num}_{i-1}$ and $s = \text{size}_{i-1}$

(a) $\alpha_{i-1} = 1$: Here $m = s$.

c_i	Φ_i	Φ_{i-1}	$\widehat{c_i}$
$m+1$	$2(m+1) - 2s$	$2m - s$	3

(b) $\frac{1}{2} \leq \alpha_{i-1} < 1$:

c_i	Φ_i	Φ_{i-1}	$\widehat{c_i}$
1	$2(m+1) - s$	$2m - s$	3

(c) $\alpha_i = \frac{1}{2}$: Here $m + 1 = \frac{s}{2}$.

c_i	Φ_i	Φ_{i-1}	$\widehat{c_i}$
1	$2(m+1) - s$	$s/2 - m$	0

(d) $\alpha_i < \frac{1}{2}$:

c_i	Φ_i	Φ_{i-1}	$\widehat{c_i}$
1	$s/2 - m - 1$	$s/2 - m$	0

So the amortized cost of insertion is $O(1)$.

(a) $\alpha_i \geq \frac{1}{2}$:

c_i	Φ_i	Φ_{i-1}	$\widehat{c_i}$
1	$2(m-1)-s$	$2m-s$	-1

(b) $\alpha_{i-1} = \frac{1}{2}$: Here $2m = s$.

c_i	Φ_i	Φ_{i-1}	$\widehat{c_i}$
1	$\frac{s}{2}-(m-1)$	$2m-s$	2

(c) $\frac{1}{4} < \alpha_{i-1} \leq \frac{1}{2}$:

c_i	Φ_i	Φ_{i-1}	$\widehat{c_i}$
1	$s/2-(m-1)$	$s/2-m$	2

(d) $\alpha_{i-1} = \frac{1}{4}$: $m = \frac{s}{4}$ and $\alpha_i < \frac{1}{2}$.

c_i	Φ_i	Φ_{i-1}	$\widehat{c_i}$
m	$s/4-(m-1)$	$s/2-m$	1

So the amortized cost of deletion is $O(1)$.