HW4 Solutions

#1 Solution

Prove that an AVL tree of height h has at least n nodes, where F_h is the h^{th} Fibonacci number.

We use induction on the number of nodes (N_h) in the tree for height h. For height 0, $N_0 = 1 \geq F_0 = 0$. For height 1, $N_1 \geq 2 \geq F_1 = 1$. An AVL tree of height h has, for every node x, the left and right sub trees of x with heights differing by at most 1. So, the least number of nodes in such an AVL tree will have the root having two child sub trees of heights h and $h-1$. Now, for each of the roots of the two sub trees, the least number of nodes will be obtained when the two child sub trees have heights $h-1$ and $h-2$ AND $h-2$ and $h-3$ respectively, and so on. Note that both the sub trees of any x are AVL trees as well.

$\therefore$ we conclude that the sub tree with height $h-1$ has at least F_{h-1} nodes and the sub tree with height $h-2$ has at least F_{h-2} nodes. $\therefore$ the total number of nodes $\geq F_{h-1} + F_{h-2} + 1 \implies n \geq F_h$

We know that $F_h = \frac{1}{\sqrt{5}}\Big[\big(\frac{1+\sqrt{5}}{2}\big)^h - \big(\frac{1-\sqrt{5}}{2}\big)^h\Big] \implies \frac{1}{\sqrt{5}}\Big[\big(\frac{1+\sqrt{5}}{2}\big)^h - \big(\frac{1-\sqrt{5}}{2}\big)^h\Big] \approx \Theta(2^h) \leq n \implies h \leq log(n)$

#2 Solution

The linked list is replaced by a red-black tree. We denote $\alpha = n/m$.

In the case of an unsuccessful search, the Expected time $E(unsuccessful) = 1 + (max\ height\ of\ the\ red\ black\ tree)$

$= 1 + E(height\ of\ tree)$

$= 1 + 2log(\alpha + 1)$ $\because$ the max height of a red black tree with α nodes is $2log(\alpha + 1)$

For successful searches:

$E(successful\ search\ for\ 1\ item) = 1 + E(depth\ of\ the\ item)$

$\therefore E(successful\ search\ over\ all\ items) = \frac{1}{\alpha}\Big[\sum_{i=1}^{\alpha} 1 + E(depth\ of\ the\ i^{th}\ item)\Big]$

$= 1 + \frac{1}{\alpha}\Big[\sum_{i=1}^{\alpha} E(depth\ of\ the\ i^{th}\ item)\Big]$

$= 1 + \frac{1}{\alpha}\big[1 + 2.2 + 2^2.3 + 2^3.4 + \cdots + 2^{log(\alpha)-1}.log(\alpha)\big]$

Let $t = \big[1 + 2.2 + 2^2.3 + 2^3.4 + \cdots + 2^{log(\alpha)-1}.log(\alpha)\big]$
$\therefore 2t = \big[2 + 2^2.2 + 2^3.3 + 2^4.4 + \cdots + 2^{log(\alpha)}.log(\alpha)\big]$
$Giving\ t = 1 - 2 - 2^2 - 2^3 - \cdots - 2^{log(\alpha)-1} + 2^{log(\alpha)}log(\alpha)$
$= -\frac{2^{log(\alpha)}-1}{2-1} + log(\alpha).2^{log(\alpha)}$
$= 1 + \alpha(log(\alpha) - 1) \implies E(successful) = 1 + \frac{1}{\alpha}\big(1 + \alpha(log(\alpha) - 1)\big)$
$\approx O\big(log(\alpha) + \frac{1}{\alpha}\big)$