<u>Lower Bounds</u>

INSERTION $\Theta(n^2)$ W.C./AVG
MERGE $\Theta(n \lg n)$ W.C.
Quicksort $\Theta(n^2)$ W.C. / $\Theta(n \lg n)$ Avg

what about $\Theta(n \lg n)$? — $\Theta(n)$

Don't exist!

$\Omega(n \lg n)$ is a lower bound on sorting W.C./Avg

$\boxed{\text{INFORMATION THEORETIC}}$ BOUND

Find one of $n!$ permutations

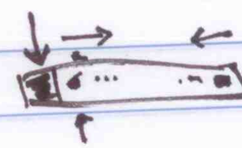
N possibilities — $\lg N$ bits
$\lg n! = \Theta(n \lg n)$

What does a comparison get us?

$a_1 \; a_2 \; \ldots \; a_n$

$\boxed{a_i : a_j}$

Q( )

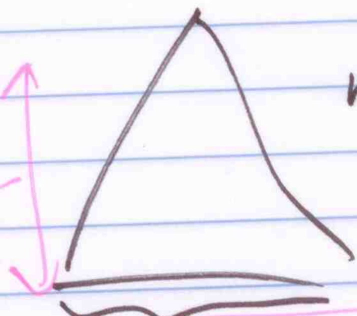


PARTITION( )
QUICK( )
QUICK( )

LATEX / PSTRICKS

SORTING ALG

⇒ DECISION TREE

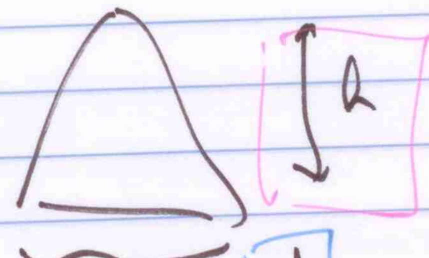


n elts

avg distance to a leaf $= \Omega(n \lg n)$

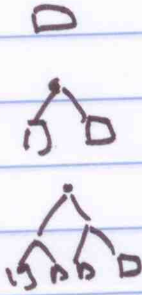
n! leaves

Pf by induction



h

? $\leq 2^h$ leaves



$$\geq \lg n! = \lg(1 \cdot 2 \cdot 3 \cdots n)$$

$$= \sum_{i=1}^{n} \lg i$$

$$\approx \int_1^n \lg x \, dx$$

EULER SUMMATION

$$= \Theta(n \lg n)$$

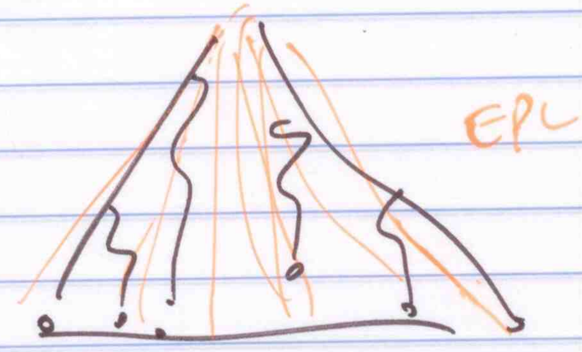
⇒ height is $\Omega(n \lg n)$

⇒ ∃ inputs that take $\Theta(n \lg n)$

Quicksort — $\Theta(n^2)$

(AVG behavior

Avg Distance to a leaf in a binary tree



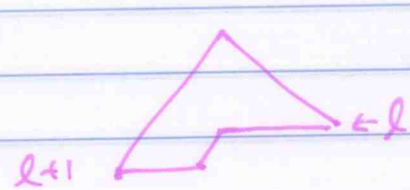
$$\min \boxed{\frac{1}{\#\text{ of leaves}} \sum_{\text{leaves}} \text{distance to the leaf}}$$

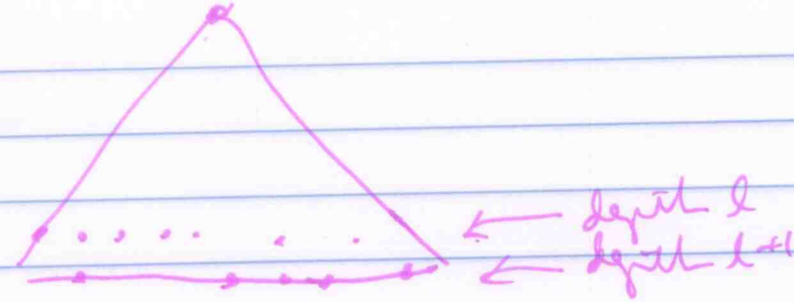
Fix # leaves — find binary tree w/ least avg dist to a leaf.

$$\min_{\substack{\text{Trees} \\ \text{w/ 100} \\ \text{leaves}}} \boxed{\sum_{\substack{\text{leaves} \\ x \in T}} \text{distance to leaf}(x)}$$

(EPL) EXTERNAL PATH LENGTH (T)

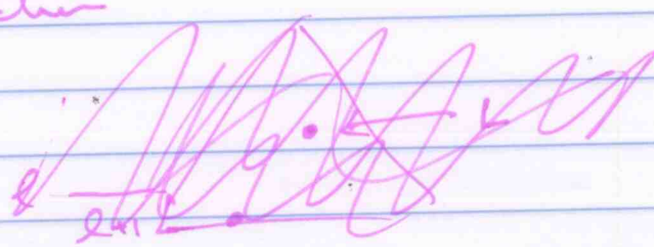
Lemma The binary T w/ min external path length EPL(T) has all of its leaves at some levels l and l+1 for some value of l.





giving min EPL.

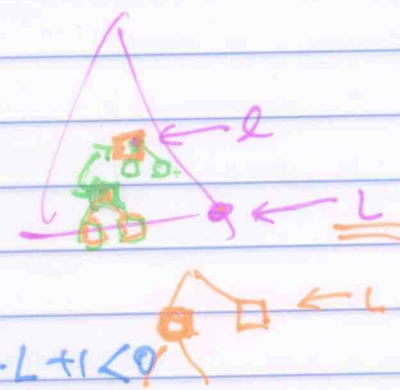
By contradiction



~~Assume~~ $\boxed{\text{MIN EPL}}$

→ deepest leaf at level L

→ highest " " " l



$\boxed{l < L-1} \Rightarrow l - L + 1 < 0$

SWAP ⇒

- lose 2 leaves at level L ✓
- gain 1 leaf at level L−1 ✓
- gain 2 leaves at level l+1 ✓
- lose 1 leaf at level l ✓

$\Rightarrow \Delta EPL \quad -2L + (L-1) + 2(l+1) - l$

$-L + l + 1 < 0$

NO!

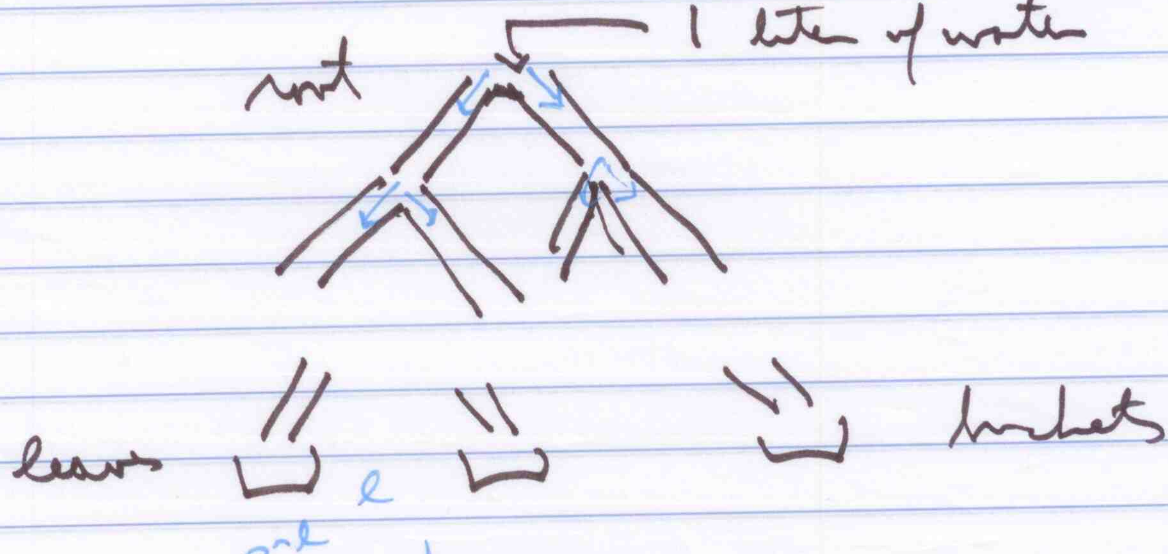
KRAFT'S INEQUALITY

Lemma If $l_1\ l_2\ l_3 \cdots l_N$ are the depths of the N leaves in a binary tree then

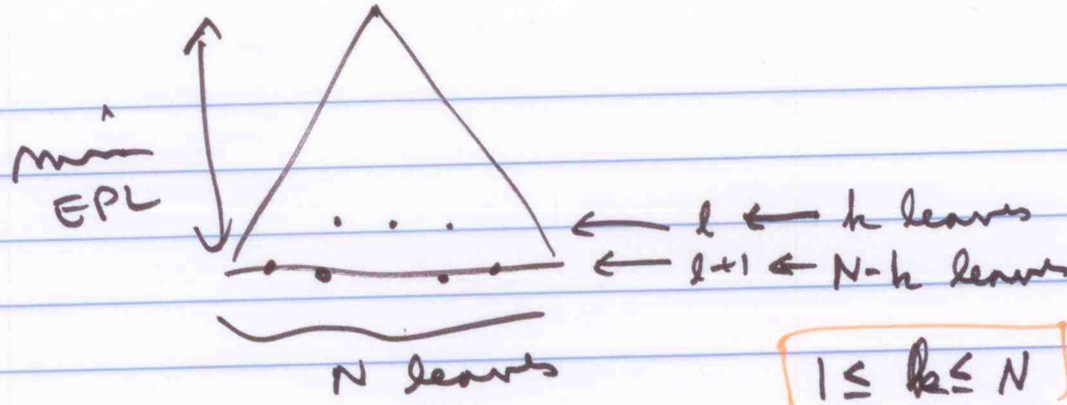
$$\sum_{i=1}^{N} 2^{-l_i} \le 1$$

Pf: Induction — HW

WATER POURING



$$\sum_{i=1}^{N} 2^{-l_i} \le 1$$



$1 \le k \le N$

(all leaves at level l)

Kraft's Inequality

$$\Rightarrow \quad k 2^{-l} + (N-k) 2^{-l-1} = 1$$

$$\Rightarrow \quad k = 2^{l+1} - N \qquad k \ge 1$$

$$2^{l+1} - N \ge 1$$

$$2^{l+1} \ge N+1$$

$$l+1 \ge \boxed{\lceil \lg(N+1) \rceil}$$

$\Rightarrow$ $n!$ leaves has EPL $\ge \lg n! = \Theta(n \lg n)$

INFO TH. LOWER BOUND

CANNOT SORT IN TIME $o(n \lg n)$
EITHER WORST-CASE OR AVERAGE

$\Theta(n)$