HW1 Solutions

#1 Solution
We have to make sure the worst case cost is the least possible # of attempts, and so we try and equalize the cost over all possibilities. Say we have N floors, and drop the first egg from floor n. If the first egg breaks, the worst case # of attempts will be 1+ (n-1) (1st egg on n^{th} floor, the remaining on 1, 2, 3... n-1).
Now, if the egg doesn't break on the n^{th} floor and we drop the first egg from the $(n + x)^{th}$ floor next, the worst case number of attempts if the first egg breaks will be $1 + 1 + ((n + x) - (n) - 1)$
Equating the two values gives $n = x - 1$. Therefore, each attempt for the first egg increases by a value of 1 less than the previous value. Thus, the floors we attempt dropping eggs from are n, n-1, n-2, ... and so on until

$$n + (n - 1) + (n - 2) + ... + 1 = N$$

which gives us the quadratic equation $n(n+1) = 2N \implies n^2 + n - 2N = 0$ giving us 2 roots, $\frac{-1+-\sqrt{1+8N}}{2}$. The positive root gives us the answer.

#2 Solution
The point at which the inequality changes can be obtained by solving the equation:

$$\begin{aligned} 200n^2 &= 1.5^n \\ \implies log(200n^2) &= log(1.5^n) \\ \implies log(200) + 2log(n) &= n * (log(1.5) \\ \implies n = log(200)/log(1.5) + 2/log(1.5)\ log(n) \end{aligned}$$

We know $log(200)/log(1.5) = 13.067$ and $2/log(200) = 3.419$

$$\therefore n = 13.067 + 3.419 * log(n) - -(eqn1)$$

Now, subsitute n on the R.H.S as 13.067 and calculate the value of n for the first iteration $n = 13.067 + 3.419 * log(13.067) = 25.745$
Again, substitute in R.H.S of eqn 1 to obtain a new value for n $n = 13.067 + 3.419 * log(25.745) = 29.08$
Repeatedly iterating makes the values of n converge to 29.8. Therefore, the integral smallest value of n at which 1.5^n is greater is 30.

#3 Solution

(a)

A counter example suffices to disprove this statement: $f(n) = 1, g(n) = n$. In this case, $f(n) = \hat{O}(g(n))$ because the following is true for $c = 1, n_0 = 2$

$$0 \leq 1 \leq cn \log n, \forall n \geq n_0$$

but no c', n'_0 can satisfy the following inequalities.

$$0 \leq n \leq c' \log n, \forall n \geq n'_0$$

(b)

A counter example suffices to disprove this statement: $f(n) = 2, g(n) = 1$. In this case, $\log f(n) = 1, \log g(n) = 0$, and $\log f(n) > \log g(n)$ at the first place, which contradicts the statement.

However, when $g(n)$ is an increasing function of n, then simple algebra shows the statement to be correct.

** Full credits are given to both types of answers.

(c)

Suppose $t(n) = o(f(n))$. By definition,

$$\forall c : \exists n_0 : \forall n \geq n_0 : 0 \leq t(n) < cf(n)$$

Then, the statement is true because for $a = 1, b = 2$, we have the following inequalities.

$$\forall n \geq n_0 : af(n) \leq f(n) + t(n) \leq bf(n)$$

First inequality is obvious since $t(n) \geq 0$. Second inequality holds because $t(n) < cf(n)$ for any $c > 0$, then obviously $t(n) < f(n)$ as well for a sufficiently large n.

(d)

A counter example would suffice to disprove this statement: $f(n) = 1, g(n) = n$.

$$\begin{aligned}
&\Theta(\min(f(n), g(n))) = \Theta(1), f(n) + g(n) = 1 + n \\
\Rightarrow &\nexists a, b, n_0 : \forall n \geq n_0 : a \cdot 1 \leq 1 + n \leq b \cdot 1
\end{aligned}$$