Illinois Institute of Technology
Department of Computer Science

Solutions to Third Examination

CS 430 Introduction to Algorithms
Spring, 2016

Wednesday, April 27, 2016
10am–11:15am & 11:25am–12:40pm, 111 Life Sciences

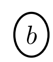
Exam Statistics

111 students took the exam. The range of scores was 0–95, with a mean of 45.85, a median of 46, and a standard deviation of 20.18.

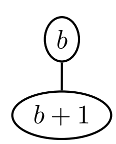
Problem Solutions

1. **Fibonacci Heaps**

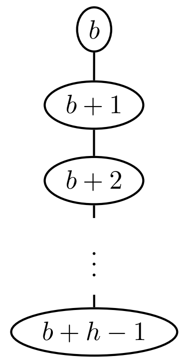
   (a) Tall-Heap$(1, b)$ clearly produces

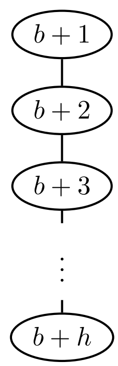
   because line 1 creates an empty Fibonacci heap, after which line 2 inserts b. Tall-Heap$(2, b)$ creates an empty Fibonacci heap T in line 5, after which lines 6–8 insert $b-1$, b, and $b+1$, respectively, all singleton nodes at the root level. The EXTRACT-MIN(T) deletes $b-1$, but then the consolidation phase then combines the two remaining roots each have degree 0 into

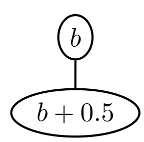
   That takes care of the base cases. Now assume that Tall-Heap(h, b) produces



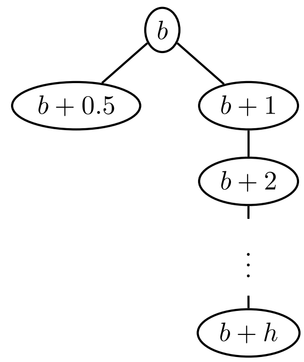
Tall-Heap$(h+1, b)$, for $h+1 > 2$ first calls Tall-Heap$(h, b+1)$ producing the heap



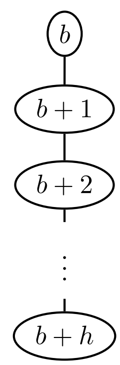
by induction. Inserting $b-0.5$, b, and $b+0.5$ adds these three values as roots of degree 0. Extracting the minimum deletes $b-0.5$, and the consolidation phase first merges degree 0 roots b, and $b+0.5$ into



and then combines the two roots of degree 1 to form



whereupon deleting $b+0.5$ produces



as desired.

(b) Although generally both etracting the minimum or deleting an element require amortized time $O(\log n)$ in a Fibonacci heap of n elements, here the consolidation phase takes only $O(1)$ worst-case time; making the heap initially and doing the insertions are all $O(1)$ worst-case time. Thus Tall-Heap$(h+1, b)$ takes time $O(h)$.

(c) By part (a), Tall-Heap$(n, 1)$ produces

2. **Depth First Search**

function DFS-visit(u)

$\mathit{color}[u] \leftarrow$ GRAY
$d[u] \leftarrow \mathit{time} \leftarrow \mathit{time} + 1$
for all $v \in \mathit{Adj}[u]$ **do**
  if $\mathit{color}[v] =$ WHITE **then**
    Tree edge
    $\pi[v] \leftarrow u$
    DFS-visit(v)
  else if $\mathit{color}[v] =$ GRAY **then**
    Back edge
  else if $d[u] < d[v]$ **then**
    Forward edge
  else
    Cross edge
  end if
end for
$\mathit{color}[u] \leftarrow$ BLACK
$f[u] \leftarrow \mathit{time} \leftarrow \mathit{time} + 1$

3. **Shortest Paths**

(a) Consider two paths: $P_1 = (u, u_1, u_2, \ldots, v)$ and $P_2 = (u, v_1, v_2, \ldots, v)$ of equal length, $L = |P_1| = |P_2|$. Under the transformation, the path length telescopes:

$$|P_1|' = \sum_{\text{edge } (a,b) \in P_1} \frac{w_V(a) + w_V(u_b)}{2} = |P_1| - \frac{w_V(u) + w_V(v)}{2} = L - \frac{w_V(u) + w_V(v)}{2}$$

and

$$|P_2|' = \sum_{\text{edge } (a,b) \in P_2} \frac{w_V(a) + w_V(u_b)}{2} = |P_2| - \frac{w_V(u) + w_V(v)}{2} = L - \frac{w_V(u) + w_V(v)}{2}$$

Thus, equal paths in the vertex-weighted graph also have equal path lengths in the edge-weighted graph.

(b) Reasoning as in part (a), the newly weighted paths differ by:

$$|P_2|' - |P_1|' = |P_2| - \frac{w_V(u) + w_V(v)}{2} - |P_1| + \frac{w_V(u) + w_V(v)}{2} = |P_2| - |P_1|$$

So that if P_1 is shorter than P_2 in the vertex-weighted graph, it will also be shorter in the edge-weighted graph.

4. **NP-Completeness**

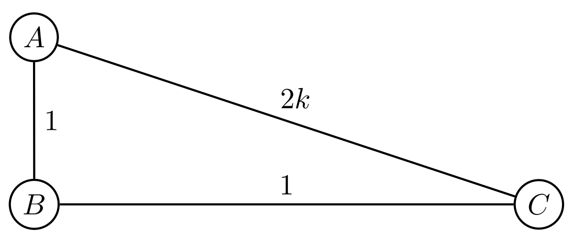
   The EXAM3-SCHEDULING decision problem is "Given an $n \times n$ graph job compatibility matrix C_{ij} and a constant d, can the n jobs be scheduled in d days?"

   EXAM3-SCHEDULING is clearly in the class NP because given C_{ij} and a proposed schedule using d days, we can easily check in time $O(n^2)$ whether the schedule has two incompatible jobs scheduled for the same day.

   To prove EXAM3-SCHEDULING is NP-hard, we reduce from GRAPH-COLORING (Problem 34-3 on page 1103 in CLRS; discussed at length in the lectures of April 11–13 and in HW 8). Given a graph $G = (V, E)$ and an integer $k > 0$, we can determine whether G can be k-colored by using the $|V| \times |V|$ Boolean matrix which is the adjacency matrix for G (see page 591 of CLRS or the lecture notes from March 28) as a compatibilty matrix for $|V|$ jobs and k as the number of days. If the jobs can be scheduled, assign a unique color to each of the k days and color each job (vertex) with a color of the day on which it is scheduled. Similarly, if the graph can be colored with k colors, the coloring gives a k-day schedule for the $|V|$ jobs.

5. **Spanning Tree Approximation**

   (a) The following graph achieves the desired result for any $k > 0$:

   The minimum spanning tree MST has edges AB and BC with cost 2. But the stupid algorithm could construct the spanning tree ST consisting of AC and BC with cost $2k + 1$. Now

   $$\frac{|\mathrm{ST}|}{|\mathrm{MST}|} = \frac{2k+1}{2} > k.$$

   (b) The new approximation bound is clearly 2 because *any* spanning tree has $|V| - 1$ edges, and hence the MST has cost at least $|V| - 1$. But no edge costs more than 2, so the stupid tree will have cost at most $2|V| - 2$. Thus

   $$\frac{|\mathrm{ST}|}{|\mathrm{MST}|} \leq \frac{2}{1} = 2.$$