All Shortest Paths from ~~Single~~ All Sources

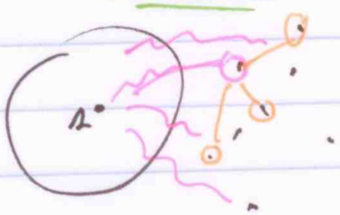
— No Neg. Cycles

Prim's Alg — MST

Warshall's Alg

Dijkstra's Alg — Fib Heap

Repeat for every starting vertex.



$\Theta(\lg|V|)$

Extract Min — 1 per vertex

Decrease Key — 1 per edge

$O(|V|\lg|V| + |E|)$

$\boxed{O(|V|^2\lg|V| + |V||E|)}$

$O(\cancel{|V|^2|E|})$

Consequences of Relaxation

shortest path

relaxation steps

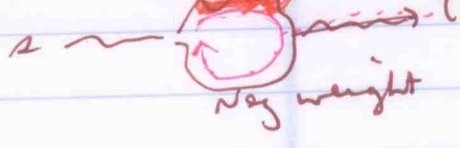
$\boxed{\Theta(|V|\cdot|E|)}$

Bellman-Ford

All shortest paths starting at s

Fine: All edges have non negative weight

Cycle of negative weight



Neg weight

All Shortest Path Problem

(all starting/ending vertices)

$|V|^2$ output

Adj space $O(|V|+|E|)$

$|V|^2$ input

Distance Matrix (generalization of Adj matrix)

input Graph

$$W = \begin{array}{c} \begin{array}{ccc} 1 \cdots & j & n \end{array} \\ \left(\begin{array}{ccc} \vdots & & \\ & w_{ij} & \\ n & & \end{array} \right) \end{array} \quad i$$

∞ no edge i to j

$= 0$ $i = j$

w_{ij} = length (weight) edge from vertex i to vertex j.

output distances

$$D = \left(\begin{array}{c} j \\ d_{ij} \end{array} \right) \quad i$$

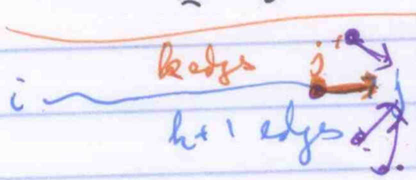
$= 0$ $i = j$

$= \infty$ no path $i \to j$

d_{ij} = length of shortest path from i to j.

Best Way

$i \xrightarrow[\underline{k} \text{ edges}]{\text{at most}} j \quad \forall i,j$



<u>n vertices</u>

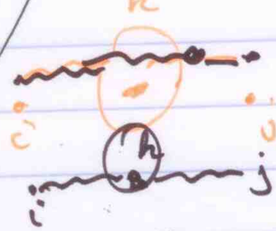
D^m $\quad d_{ij}^m$ = shortest distance from vertex i to vertex j using <u>at most</u> <u>m steps</u>

$$\min \begin{cases} \text{using } m-1 \text{ at most steps only} \\ \text{using } m \text{ steps} \end{cases}$$

all possible penultimate steps

$$d_{ij}^{\cancel{m}\ 2m-2} = \min\left(d_{ij}^{m-1},\ \min_{\substack{k \\ 1 \le k \le n}}\left(d_{ik}^{m-1} + d_{kj}^{m-1}\right)\right)$$

$\Theta(n)$



<u>Want</u> d_{ij}^n

$$D^0 \quad D' \to D^2 \to D^3 \to \cdots \to D^n$$

W

0 steps $\quad$ 1 step $\quad$ 2 steps $\quad$ n steps

$D^m \quad D^{2m}$

$\lg n$

$$\cancel{D^{m-1}} \longrightarrow D^m$$

$\Theta(n^3)$

$$\underbrace{\qquad\qquad}_{\Theta(n^4)}$$

$O(|V|^3 \lg |V|)$

$O(\cancel{|V|^4})$

$$\Pi \quad \left(\ \pi_{ij}\ \right)$$

penultimate vertex on shortest path from i to j

$$\underbrace{1 \to 2 \to 4 \to 8 \to \cdots\ 2^\ell}_{\lg n} \ge n$$

$\le m$ steps $\le m$ steps

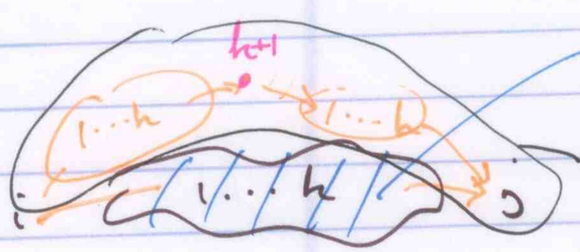
$$\underbrace{i \;\cdots\cdots\; \bullet \;\cdots\cdots\; j}_{\le 2m \text{ steps}}$$

vertices $1, 2, 3, \ldots, n$.

$$\boxed{d_{ij}^{k} = \begin{cases} \text{length of shortest} \\ \text{path from } i \text{ to } j \\ \text{using only vertices} \\ 1 \ldots k \end{cases}}$$

$k = n$

Π – exercise



$$\min \left\{ \begin{array}{l} d_{ij}^{k} \\ d_{i\,k+1}^{k} + d_{k+1\,j}^{k} \end{array} \right\}$$

$\underline{\underline{O(1)}}$

$$d_{ij}^{k+1} = \begin{cases} \text{length of shortest path} \\ \text{from } i \text{ to } j \text{ using only} \\ \text{vertices } 1 \ldots \underline{\underline{k+1}} \end{cases}$$

$\boxed{\text{FSM}}$

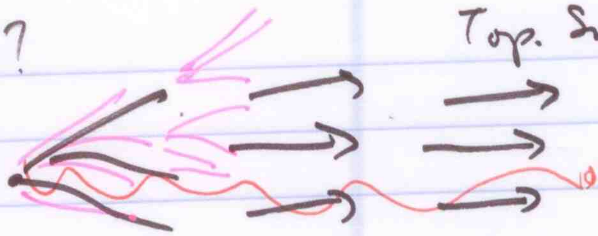
$D^k \longrightarrow D^{k+1}$ $O(1)$ per elt

$O(|V|^2)$ $\Longrightarrow \underline{\underline{O(|V|^2)}}$

$$\underbrace{D^0 \quad D^1 \quad \cdots\cdots \quad D^n}_{|V| \text{ steps}} \Longrightarrow \underline{\underline{O(|V|^3)}}$$

What about D.A.G?

Top. Sort

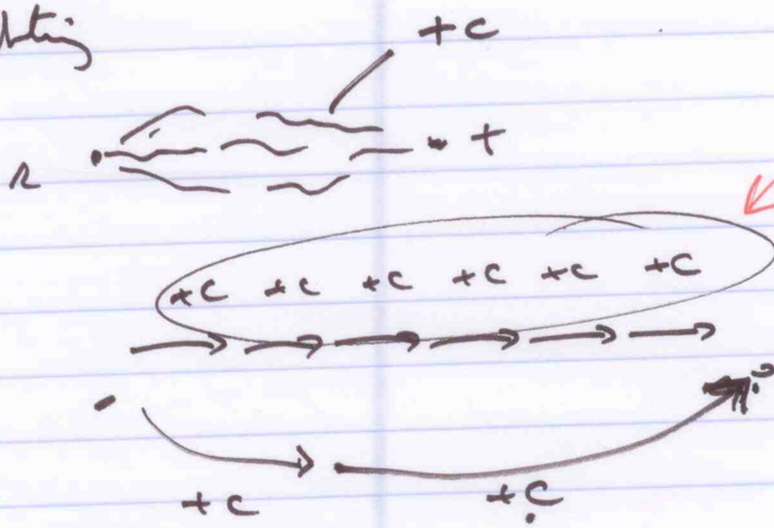


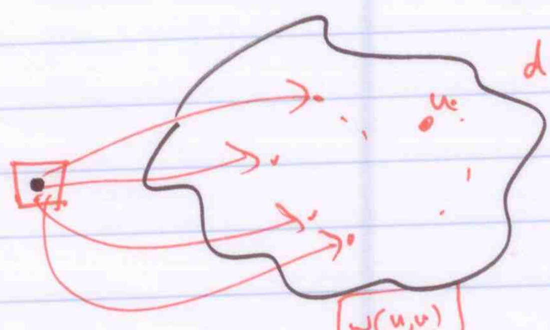
$O(|E|)$

$\boxed{O(|V|+|E|)}$

Negative Edges (All path case)

Reweighting

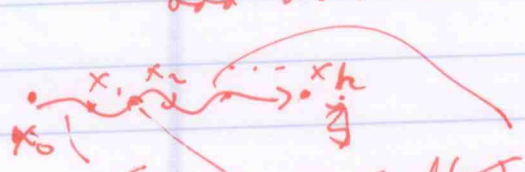




$\boxed{w(u,v)}$

$u \longrightarrow v$

add $d(v) - d(u) \geq 0$



$\cancel{d(x_1)} - \underline{d(x_0)}$ $\quad \cancel{d(x_2)} - \cancel{d(x_1)}$ $\quad d(x_3) - \cancel{d(x_2)}$.