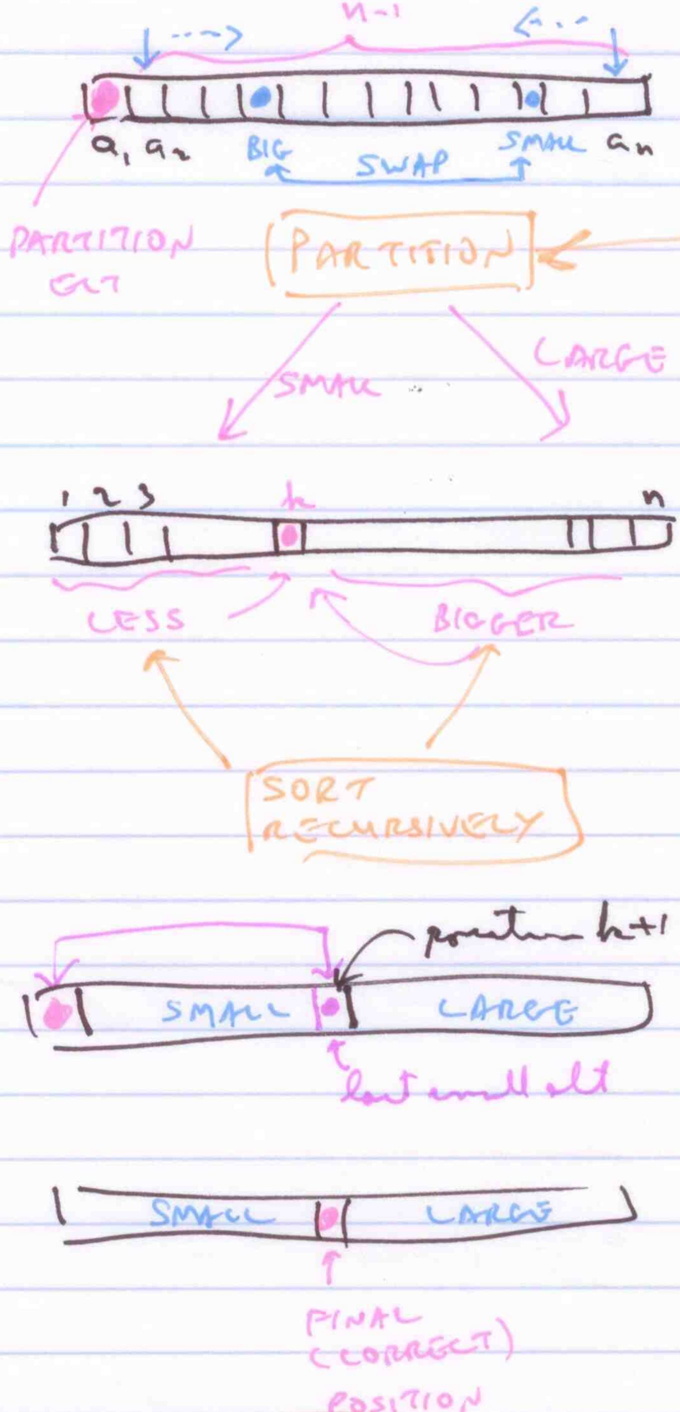
QUICKSORT



$a_i \neq a_j$ if $i \neq j$

n-1 comparisons

ANALSIS — # OF COMPARISONS

$c(n) =$ # of comparisons to sort $a_1 \dots a_n$ by quicksort

$$\boxed{\begin{aligned} c(n) &= \underbrace{(n-1)}_{\text{partition}} + \underbrace{c(k)}_{\text{small}} + \underbrace{c(n-1-k)}_{\text{large}} \\ c(1) &= 1 \end{aligned}}$$

Best / Worst / Avg cases

$\boxed{\text{even split}}$ (Best) — most uneven split (Worst)

$\boxed{\text{Best case}}$

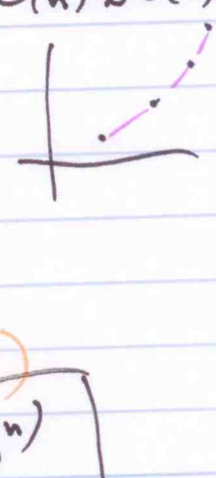
$$c(n) = (n-1) + \min_{0 \le k \le n-1} \left(c(k) + c(n-1-k) \right)$$

$$k = n-k-1 = \frac{n-1}{2} \quad \text{(calculus)}$$

(Knuth vol III)

$$c(n) = (n-1) + c\left(\left\lceil \frac{n-1}{2} \right\rceil\right) + c\left(\left\lfloor \frac{n-1}{2} \right\rfloor\right)$$

$$c'(n) = (n-1) + 2c'(n/2) \qquad c'(n) \approx c(n)$$



$\boxed{n = 2^k} \Rightarrow k = \lg n$

$$t_k = \underbrace{2^k - 1}_{(E-2)} + \underbrace{2t_{k-1}}_{(E-2)}$$

$(E-2)^2$

$\Rightarrow t_k = \Theta(k 2^k)$

$$\boxed{c'(n) = \Theta(n \lg n)}$$

Worst Case — Already sorted!

$$c(n) = (n-1) + \max_k \big(c(k) + c(n-1-k) \big)$$

$$(n-1) + (n-2) + \dots + 1 = \frac{n(n-1)}{2}$$

Guess $c(n) = \frac{n^2}{2}$ Too Small

$$?? \quad \underset{\frac{n^2}{2}}{c(n)} = \underbrace{(n-1) + \max_k \left(\frac{k^2}{2} + \frac{(n-1-k)^2}{2} \right)}_{\geq \frac{n^2}{2}}$$

$c(n) = n^2$ Too big

$$\boxed{c(n) = \frac{n^2}{2} + \frac{3}{2} n}$$

$$\frac{n^2}{2} + \frac{3}{2} n = (n-1) + \max_k \left(\frac{k^2}{2} + \frac{3}{2} k + \dots \right)$$

BINGO

Avg Case

$$c(n) = (n-1) + c(n/3) + c(2n/3)$$

$$\Downarrow$$

$$c(n) = \Theta(n \lg n)$$

$a_1 | a_2 | \ | \ | \ | \ | \ | \ | \ | a_n$ $\qquad (a_i : a_j)$

$z_i \ \cdots \ x \ \cdots \ z_j$

$z_1 | z_2 | \ | \ | \ | \ | \ | \ | \ | \ | z_n$ $\qquad \sum_{i,j}$ Prob $\frac{2}{j-i+1}$

$z_1 < z_2 < \cdots < z_n$ $\qquad \boxed{z_i : z_j}$

$i < j \qquad z_i < z_j$ $\qquad z_i < x < z_j$

$z_i : x \qquad x : z_j$

$$\boxed{\sum_{0<i<j\le n} \frac{1}{\underbrace{j-i+1}}} \approx 2n \ln n \qquad = \sum_k k H_k \qquad \int_1^n x \ln x \, dx$$

$$H_k = \sum_{m=1}^{k} \frac{1}{m}$$ (Harmonic NO's

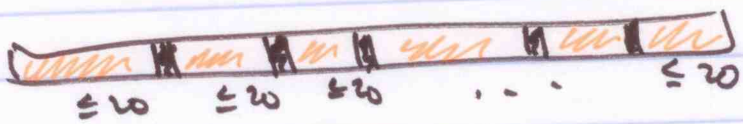
EULER'S SUMMATION

$H_k \approx \ln k + O(1)$

$$\sum_{m=1}^{k} \frac{1}{m} \approx \int_1^k \frac{1}{x} dx$$

$$c(n) = n-1 + \sum_{k=0}^{n-1} \left(c(k) + c(n-1-k) \right) \Pr\Big(\underbrace{\text{partition elt (med of 3) was } k^{th} \text{ largest}}_{1/n} \Big)$$

$$C(n) = n + \frac{2}{n} \sum_{k=0}^{n-1} c(k)$$



$$\boxed{\begin{array}{l} t_0 \quad t_1 \quad \dots \quad t_{n_0} \\ t_n = an + b + \frac{2}{n} \sum_{i=0}^{n-1} t_i \qquad n > n_0 \end{array}}$$

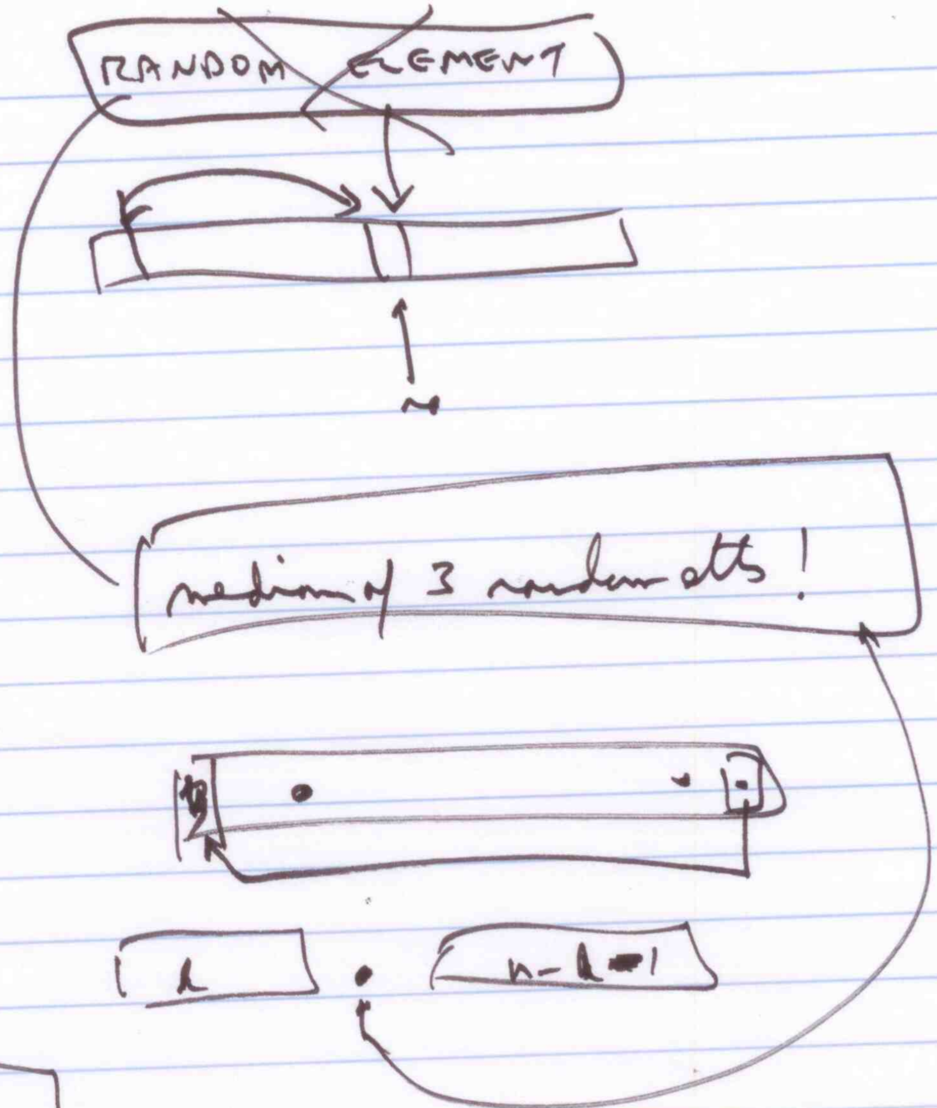
$$n t_n = an^2 + bn + 2 \sum_{i=0}^{n-1} t_i \qquad \forall n \geq n_0$$

$$- \quad (n-1) t_{n-1} = a(n-1)^2 + b(n-1) + 2 \sum_{i=0}^{n-2} t_i$$

$$n t_n - (n+1) t_{n-1} = a(n^2 - (n-1)^2) + b(n - (n-1)) + 2 t_{n-1}$$

$$\sum \left(\frac{t_n}{n+1} - \frac{t_{n-1}}{n} \right) = \boxed{}$$

$$\frac{t_n}{n+1} - \frac{t_0}{1} = \sum \approx 2a H_n \Rightarrow t_n \approx \boxed{2an\ln}$$



CS
330

$$\frac{k(n-1-k)}{\binom{n}{3}}$$

$$c(n) \approx \frac{12}{7} n H_n$$