Derivation of a discrete variational integrator for rigid body translational motion

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***Abstract*— This document provides a derivation of the variational integrator for linear motion**

This closely follows the derivation for rotational rigid body dynamics given in [1]. For more clarity, [2] is also referenced.

I. Free-body equations

First, the derivation for a free-body with no external forces is given. The action integral for linear motion is

$$S = \int_{t_0}^{t_f} \frac{m}{2} \mathbf{v}^T \mathbf{v} dt \tag{1}$$

where m is the mass of the rigid body and $\mathbf{v}$ is the vector of its linear velocities.Its discrete form, the action sum is given as

$$S = \sum_{k=0}^{N-1} \int_{t_k}^{t_{k+1}} \frac{m}{2} \mathbf{v}^T \mathbf{v} dt \tag{2}$$

The exact discrete Lagrangian is given by

$$L_d^E = \int_{t_k}^{t_{k+1}} \frac{m}{2} \mathbf{v}^T \mathbf{v} dt \tag{3}$$

The exact discrete Lagrangian is approximated by using the rectangle rule. First, the finite difference approximation of the velocity is used,

$$\mathbf{v}_k = (\mathbf{x}_{k+1} - \mathbf{x}_k)/h \tag{4}$$

where $h = t_{k+1} - t_k$, and $\mathbf{x}$ is the vector of positions. Applying the rectangle rule and using (4), we get the discrete Lagrangian

$$L_d = h \left(\frac{m}{h^2} (\mathbf{x}_{k+1} - \mathbf{x}_k)^T (\mathbf{x}_{k+1} - \mathbf{x}_k) \right) \tag{5}$$

The discrete equivalent of (1), the discrete action sum is

$$S_d = \sum_{k=0}^{N-1} L_d \tag{6}$$

A variation on $\mathbf{x_k}$ is used, where $\delta \mathbf{x}_k = \mathbf{x}_k + \epsilon \boldsymbol{\eta}_k$

Taking variational derivative of the action sum

$$\delta S_d = \frac{d}{d\epsilon}\bigg|_{\epsilon=0} \sum_{k=0}^{N-1} L_d = 0 \tag{7}$$

$$\delta S_d = \sum_{k=0}^{N-1} \frac{m}{h} (\mathbf{x}_{k+1} - \mathbf{x}_k) \left(\boldsymbol{\eta}_{k+1} - \boldsymbol{\eta}_k \right) = 0$$

To eliminate $\boldsymbol{\eta}_{k+1}$, discrete integration by parts (which amounts to index manipulation) is performed.

$$\delta S_d = \sum_{k=0}^{N-1} \frac{m}{h} (\mathbf{x}_{k+1} - \mathbf{x}_k) \boldsymbol{\eta}_{k+1} - (\mathbf{x}_{k+1} - \mathbf{x}_k) \boldsymbol{\eta}_k = 0 \tag{8}$$

$$\delta S_d = \frac{m}{h} \sum_{k=1}^{N-1} \left((\mathbf{x}_k - \mathbf{x}_{k-1}) \boldsymbol{\eta}_k - (\mathbf{x}_{k+1} - \mathbf{x}_k) \boldsymbol{\eta}_k \right) + (\mathbf{x}_N - \mathbf{x}_{N-1}) \boldsymbol{\eta}_N - (\mathbf{x}_1 - \mathbf{x}_0) \boldsymbol{\eta}_0 = 0$$

Variation at the end-points is zero,and the above equation should be true for all values of the variation $\boldsymbol{\eta_k}$ so the free-body discrete equations are given by

$$(\mathbf{x}_k - \mathbf{x}_{k-1}) - (\mathbf{x}_{k+1} - \mathbf{x}_k) = 0 \tag{9}$$

II. External forces

To incorporate external torques in the variational equations, Lagrange - D'Alembert's equations are used.

$$\delta \int_{t_0}^{t_f} L dt + \int_{t0}^{tf} \mathbf{F} \delta \mathbf{x} \, dt = 0 \tag{10}$$

The term on the left is the variation on the action, and the term on the right is the integral on the virtual work, done by (generalized) force $\mathbf{F}$.

The virtual work is approximated by

$$\int_{t0}^{tf} \mathbf{F} \delta \mathbf{x} \, dt = \sum_{k=0}^{N-1} \int_{tk}^{t_{k+1}} f \delta \mathbf{x} \, dt \tag{11}$$

$$\int_{tk}^{t_{k+1}} f \delta \mathbf{x} \, dt = f_k^- \cdot \delta \mathbf{x}_k + f_k^+ \cdot \delta \mathbf{x}_{k+1}$$

The discrete forces are assumed to be constant over each time interval, $[t_k, t_{k+1}]$, so they are transformed to the time node, t_k as $f_k^- = \frac{h}{2} f_k$ and $f_{k+1}^- = \frac{h}{2} f_{k+1}$.

The discrete form of the Lagrange-D'Alembert's equation is

$$\delta \sum_{k=0}^{N-1} L_d + \sum_{k=0}^{N-1} \left(\frac{h}{2} f_k \delta \mathbf{x}_k + \frac{h}{2} f_{k+1} \cdot \delta \mathbf{x}_{k+1} \right) = 0 \tag{12}$$

Carrying out variational derivatives for the first term of (12) is the same as in section I. Variational derivatives for the forced part is found as ($\delta \mathbf{x}_k = \mathbf{x}_k + \epsilon \boldsymbol{\eta}_k$):

$$\frac{h}{2} \sum_{k=0}^{N-1} \left(f_k \boldsymbol{\eta}_k + f_{k+1} \boldsymbol{\eta}_{k+1} \right) \tag{13}$$

Using discrete integration by parts/index manipulation gives

$$h \sum_{k=1}^{N-1} f_k \boldsymbol{\eta}_k + h f_N \boldsymbol{\eta}_N + h f_0 \boldsymbol{\eta}_0 \tag{14}$$

Noting that variations at end-points are zero, and using (8), 12 can be written as:

$$\frac{m}{h} \sum_{k=1}^{N-1} \left(\left(\mathbf{x}_k - \mathbf{x}_{k-1} \right) \boldsymbol{\eta}_k - \left(\mathbf{x}_{k+1} - \mathbf{x}_k \right) \boldsymbol{\eta}_k \right) + h \sum_{k=1}^{N-1} f_k \boldsymbol{\eta}_k = 0 \tag{15}$$

Equation (15) must hold true for all variations $\boldsymbol{\eta}_k$, which gives us the discrete time equations of motion for a forced rigid body in discrete time

$$\left(\mathbf{x}_k - \mathbf{x}_{k-1} \right) - \left(\mathbf{x}_{k+1} - \mathbf{x}_k \right) + \frac{h^2}{m} f_k = 0 \tag{16}$$

REFERENCES

[1] Z. R. Manchester, M. A. Peck, Quaternion variational integrators for spacecraft dynamics, Journal of Guidance, Control, and Dynamics 39 (2016) 69-76

[2] S. Leyendecker, S. Ober-Blöbaum and J.E. Marsden, Discrete mechanics and optimal control for constrained systems.Optim.Contr. Appl. Meth.(2009) DOI: 10.1002/oca.912