Grafica pe calculator

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Reprezentarea curbelor si suprafetelor

Reprezentarea curbelor si suprafetelor

- Curbe si suprafete netede
  - Nu sunt fractali !

$$f : D \to \Re$$

$$f(x) \in C^{(n)}(D), n \in \aleph$$

- f este derivabila de n ori pe intreg domeniul de definitie D
- $f^{(n)}(x)$ este continua

- De ce ?
- Necesitate

Reprezentarea curbelor si suprafetelor

- De ce ?
  - Obiecte inerent netede
    - Obiecte CAD
    - Fonturi
    - Desene artisti
    - Traiectoria unei camere in animatie
    - Drumuri in spatii de culori

Reprezentarea curbelor si suprafetelor

- Necesitate
  - Modelarea unui obiect pentru care nu exista o descriere matematica iar modelarea prin memorarea coordonatelor tuturor (unui numar suficient) punctelor sale este imposibila
    - Aproximare pe portiuni
      - Plane, sfere, etc.
  - Modelarea iterativa a unui obiect

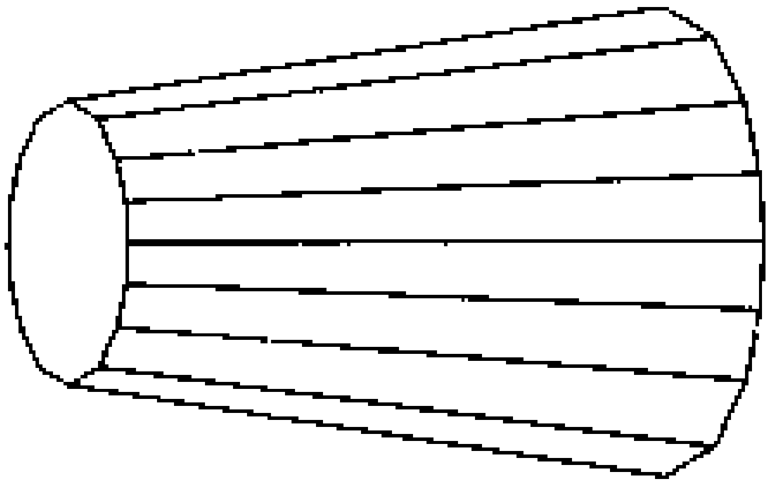
Reprezentarea curbelor si suprafetelor

- Retele de petice poligonale (polygon mesh)
- Curbe parametrice polinomiale (parametric polynomial curves)
  - Curbe parametrice cubice
- Petice de suprafete polinomiale parametrice bivariate (parametric bivariate polynomial surface patches)
  - Suprafete bicubice
- Suprafete cvadrice (quadric surfaces)

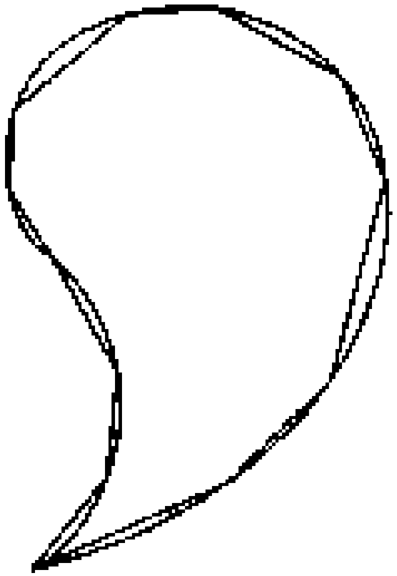
Retele de petice poligonale

- Definitie
  - O multime de suprafete poligonale (planare) marginite, conectate intre ele
- Modelare
  - Exacta
    - Volumele marginite de fete planare
  - Aproximare
    - Obiectele cu suprafete curbe

Retele de petice poligonale



Reprezentarea unui obiect 3D



Sectiune printr-un obiect avand o forma curba si reprezentarea sa poligonala

Curbe parametrice polinomiale

- Definitie
  - Puncte pe o curba 3D utilizand 3 polinoame in t

Curbe parametrice polinomiale

$$Q(t) = \begin{pmatrix} x(t) & y(t) & z(t) \end{pmatrix}$$

$$\begin{cases} x(t) = \sum_{k=0}^{n_x} x_k t^k \\ y(t) = \sum_{k=0}^{n_y} y_k t^k, \quad 0 \le t \le 1 \\ z(t) = \sum_{k=0}^{n_z} z_k t^k \end{cases}$$

Curbe parametrice polinomiale cubice

$$Q(t) = \begin{pmatrix} x(t) & y(t) & z(t) \end{pmatrix}$$

$$\begin{cases} x(t) = \sum\limits_{k=0}^{n_x} x_k t^k \\ y(t) = \sum\limits_{k=0}^{n_y} y_k t^k, \quad 0 \le t \le 1 \\ z(t) = \sum\limits_{k=0}^{n_z} z_k t^k \end{cases} \wedge \begin{cases} n_x = 3 \\ n_y = 3 \\ n_z = 3 \end{cases}$$

Suprafete polinomiale parametrice

- Parametric bivariate polynomial surface patches
- Definitie
  - Puncte pe o suprafata curba utilizand 3 polinoame bivariate in s si t
  - Limitele suprafetei sunt curbe parametrice polinomiale
- Comparatie SPP vs RPP
  - Reprezentare mai eficienta prin SPP decat prin RPP
    - Numar mai mic de petice
  - Algoritmi mai complecsi in cazul SPP in raport cu RPP
    - Descrierea mai complexa d.p.d.v. matematic

Suprafete polinomiale parametrice

$$Q(s,t) = \big(x(s,t) \quad y(s,t) \quad z(s,t)\big)$$

x, y, z polinoame in s si in t

$0 \le s, t \le 1$

Suprafete bicubice

$$Q(s,t) = \begin{pmatrix} x(s,t) & y(s,t) & z(s,t) \end{pmatrix}$$

x, y, z polinoame de grad cel mult 3 in s si in t

$0 \le s,t \le 1$

Suprafete cvadrice

- Suprafete definite implicit printr-o ecuatie de forma f(x,y,z) = 0, unde f este un polinom de grad cel mult 2 in x, y si z

$$f(x_1, x_2, x_3) = \sum_{i,j=1}^{3} Q_{i,j} x_i x_j + \sum_{i=1}^{3} P_i x_i + R$$

$$Q = \begin{pmatrix} a & d & f \\ d & b & e \\ f & e & c \end{pmatrix} \in \mathfrak{M}_{3\times 3}(\Re)$$

$$P = \begin{pmatrix} g & h & i \end{pmatrix} \in \mathfrak{M}_{1\times 3}(\Re)$$

$$R \in \Re$$

Suprafete cvadrice

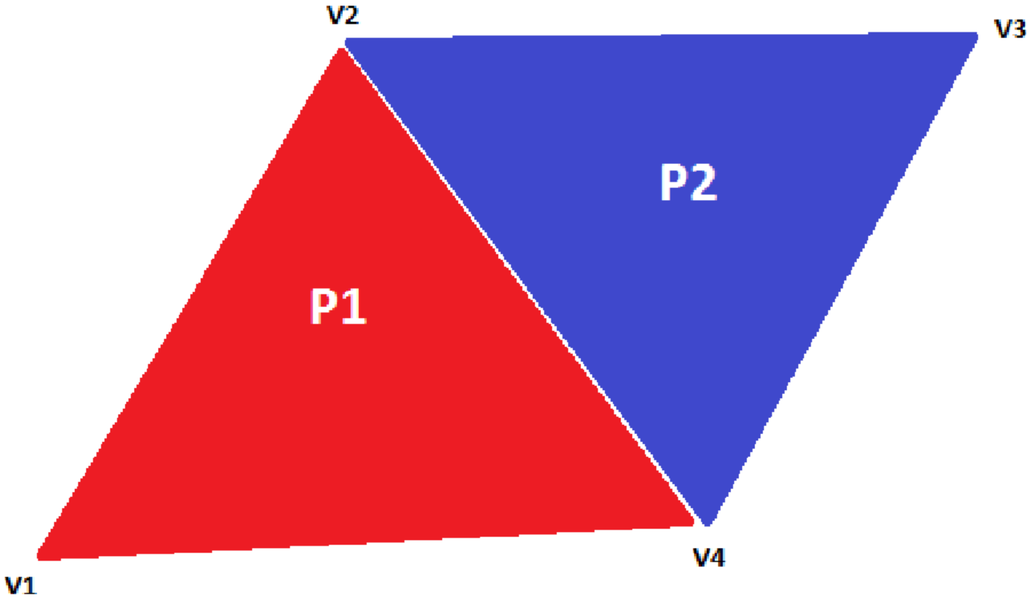
- Obiecte modelate exact
  - Sfere $$f(x,y,z)=\frac{x^2}{a^2}+\frac{y^2}{a^2}+\frac{z^2}{a^2}-1$$
  - Elipsoizi $$f(x,y,z)=\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1$$
  - Cilindri
    - Eliptici $$f(x,y,z)=\frac{x^2}{a^2}+\frac{y^2}{b^2}-1$$
    - Circulari $$f(x,y,z)=\frac{x^2}{a^2}+\frac{y^2}{a^2}-1$$

Retele de petice poligonale

- Reprezentare
  - Explicita
  - Pointeri la o lista de varfuri
  - Pointeri la o lista de muchii

Retele de petice poligonale

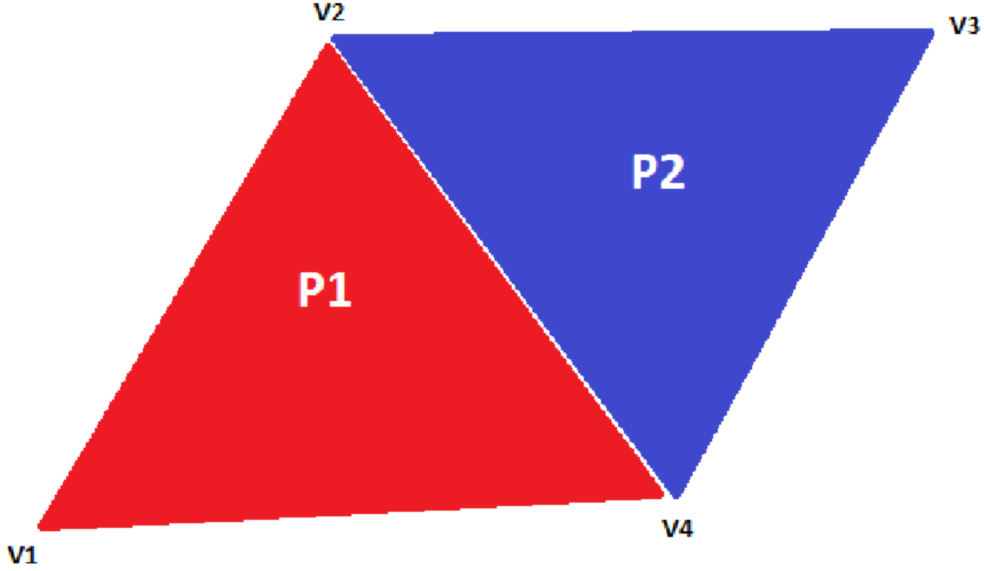
- Reprezentare explicita



P1 = (V1(x1,y1,z1), V2(x2,y2,z2), V4(x4,y4,z4))
P2 = (V2(x2,y2,z2), V3(x3,y3,z3), V4(x4,y4,z4))

Retele de petice poligonale

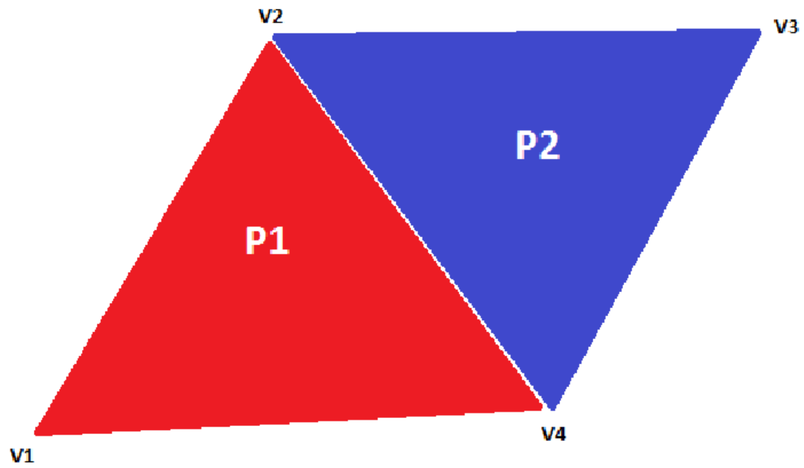
- Reprezentare prin pointeri la o lista de varfuri



V = (V1(x1,y1,z1), V2(x2,y2,z2), V3(x3,y3,z3), V4(x4,y4,z4))
P1 = (1, 2, 4)
P2 = (2, 3, 4)

Retele de petice poligonale

- Reprezentare prin pointeri la o lista de muchii



V = (V1(x1,y1,z1), V2(x2,y2,z2), V3(x3,y3,z3), V4(x4,y4,z4))
E = (E1, E2, E3, E4, E5)
E1 = (V1, V2, P1, NULL)
E2 = (V2, V3, P2, NULL)
E3 = (V3, V4, P2, NULL)
E4 = (V2, V4, P1, P2)
P1 = (1, 4, 5)
P2 = (2, 3, 4)

Retele de petice poligonale

- Consistenta
  - Toate poligoanele sunt inchise
  - Fiecare muchie este utilizata c.putin 1 data si c.mult un nr. fixat de ori
  - La fiecare varf fac referire c.putin 2 muchii
  - Suplimentar:
    - RPP sa fie un graf conex
    - RPP sa fie un graf planar
    - RPP sa nu aiba goluri (sa existe o unica frontiera)

Retele de petice poligonale

- Planaritate
  - Problema: daca un poligon este neplanar (de ex., are 4 varfuri iar unul dintre ele nu se afla in planul determinat de celelalte 3) cum se calculeaza normala poligonului ?

Retele de petice poligonale

- Planaritate

$$\pi : Ax + By + Cz + D = 0$$

$$D = 0 \Leftrightarrow O(0,0,0) \in \pi$$

$$Pp. \quad D \neq 0 \Rightarrow \pi : A'x + B'y + C'z + 1 = 0$$

Retele de petice poligonale

- Planaritate

Normala planului

$$\pi : Ax + By + Cz + 1 = 0$$

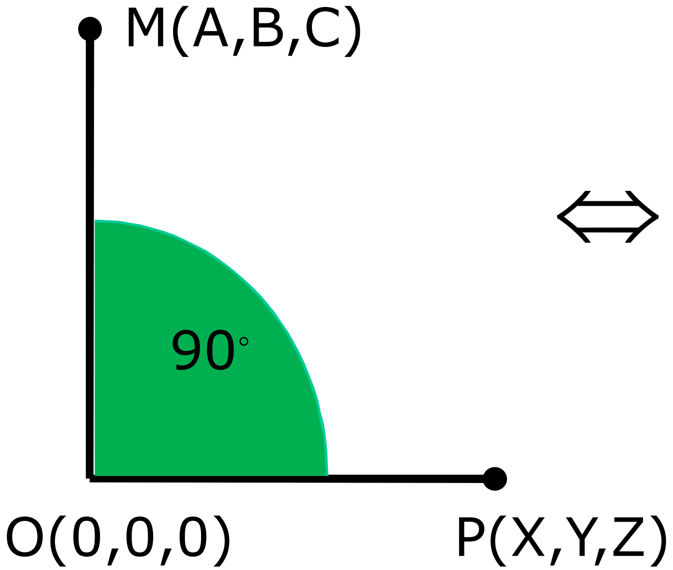
Este

$$N_\pi : \frac{A\vec{i} + B\vec{j} + C\vec{k}}{\sqrt{A^2 + B^2 + C^2}}$$

Si rezulta din: $P1 \wedge P2 \wedge P3$

Retele de petice poligonale

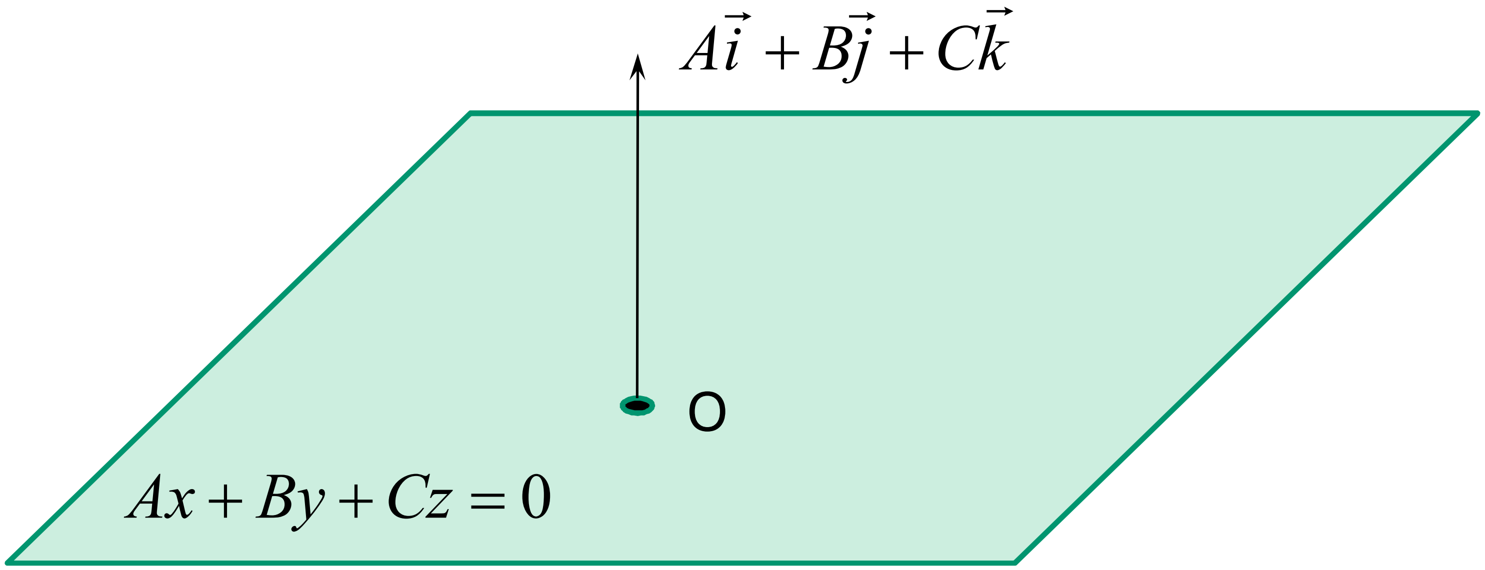
- Si rezulta din
  - P1



$$\Leftrightarrow \quad AX + BY + CZ = 0$$

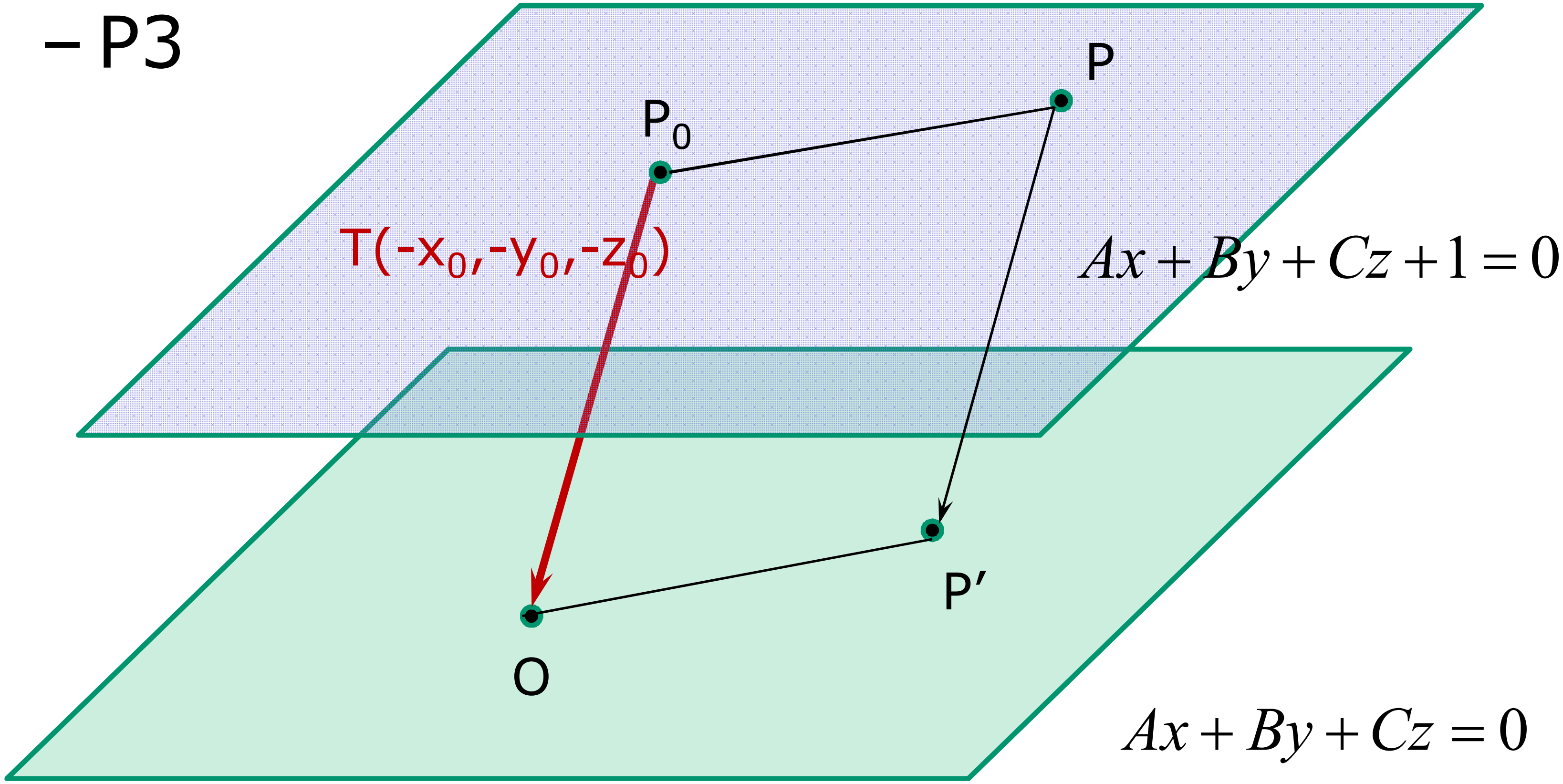
Retele de petice poligonale

- Si rezulta din
  - P2



Retele de petice poligonale

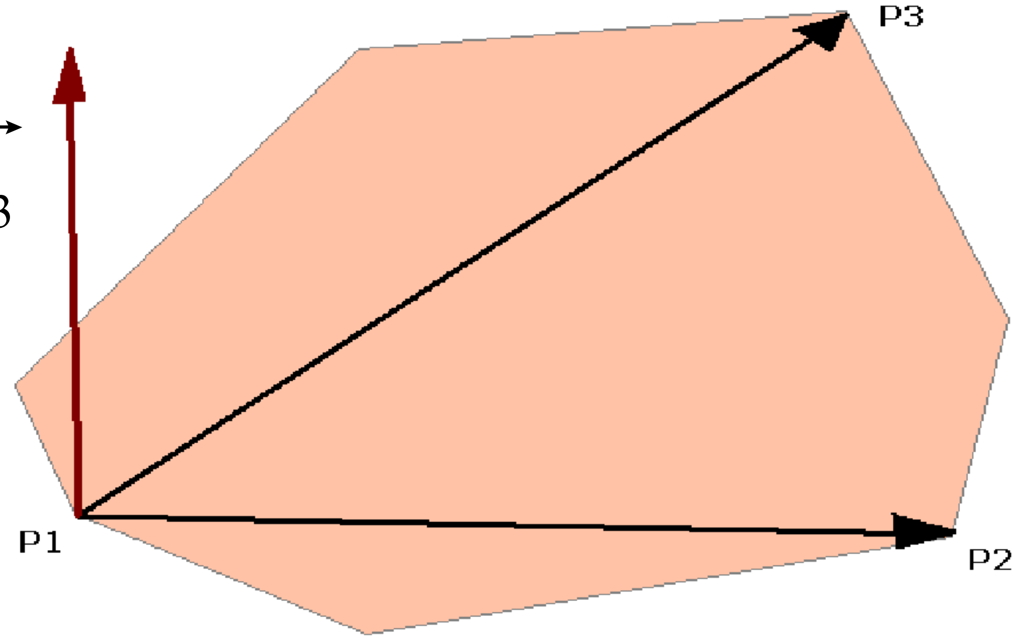
- Si rezulta din
  - P3



Retele de petice poligonale

- Calcul normala poligon – metoda 1

$$\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3}$$



Retele de petice poligonale

- Calcul normala poligon – metoda 2

$$A = \frac{1}{2} \cdot \sum_{i=1}^{n} \left(z_i + z_{i \oplus 1}\right) \cdot \left(y_{i \oplus 1} - y_i\right)$$

$$B = \frac{1}{2} \cdot \sum_{i=1}^{n} \left(x_i + x_{i \oplus 1}\right) \cdot \left(z_{i \oplus 1} - z_i\right)$$

$$C = \frac{1}{2} \cdot \sum_{i=1}^{n} \left(y_i + y_{i \oplus 1}\right) \cdot \left(x_{i \oplus 1} - x_i\right)$$

$$a \oplus 1 = \begin{cases} a + 1 & , \quad 1 \leq a < n \\ 1 & , \quad a = n \end{cases}$$

Retele de petice poligonale

- Masura neplanaritatii poligonului

  Distanta dintre $M_0(x_0,y_0,z_0)$ la planul

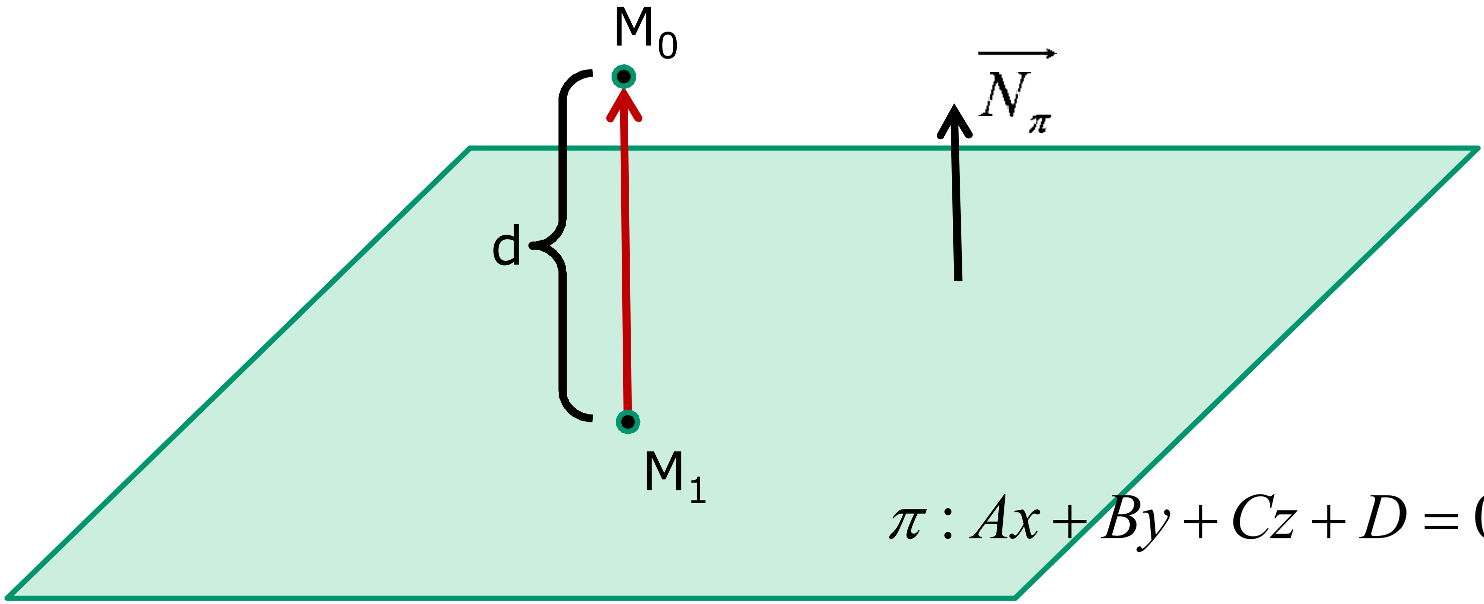
$$\pi : Ax + By + Cz + D = 0$$

este

$$d = \frac{Ax_0 + By_0 + Cz_0 + D}{\sqrt{A^2 + B^2 + C^2}}$$

Retele de petice poligonale

- Masura neplanaritatii poligonului



$\overrightarrow{N_\pi}$

$$\pi : Ax + By + Cz + D = 0$$

Curbe parametrice polinomiale cubice
Definitie

$$Q(t) = \begin{pmatrix} x(t) & y(t) & z(t) \end{pmatrix}$$

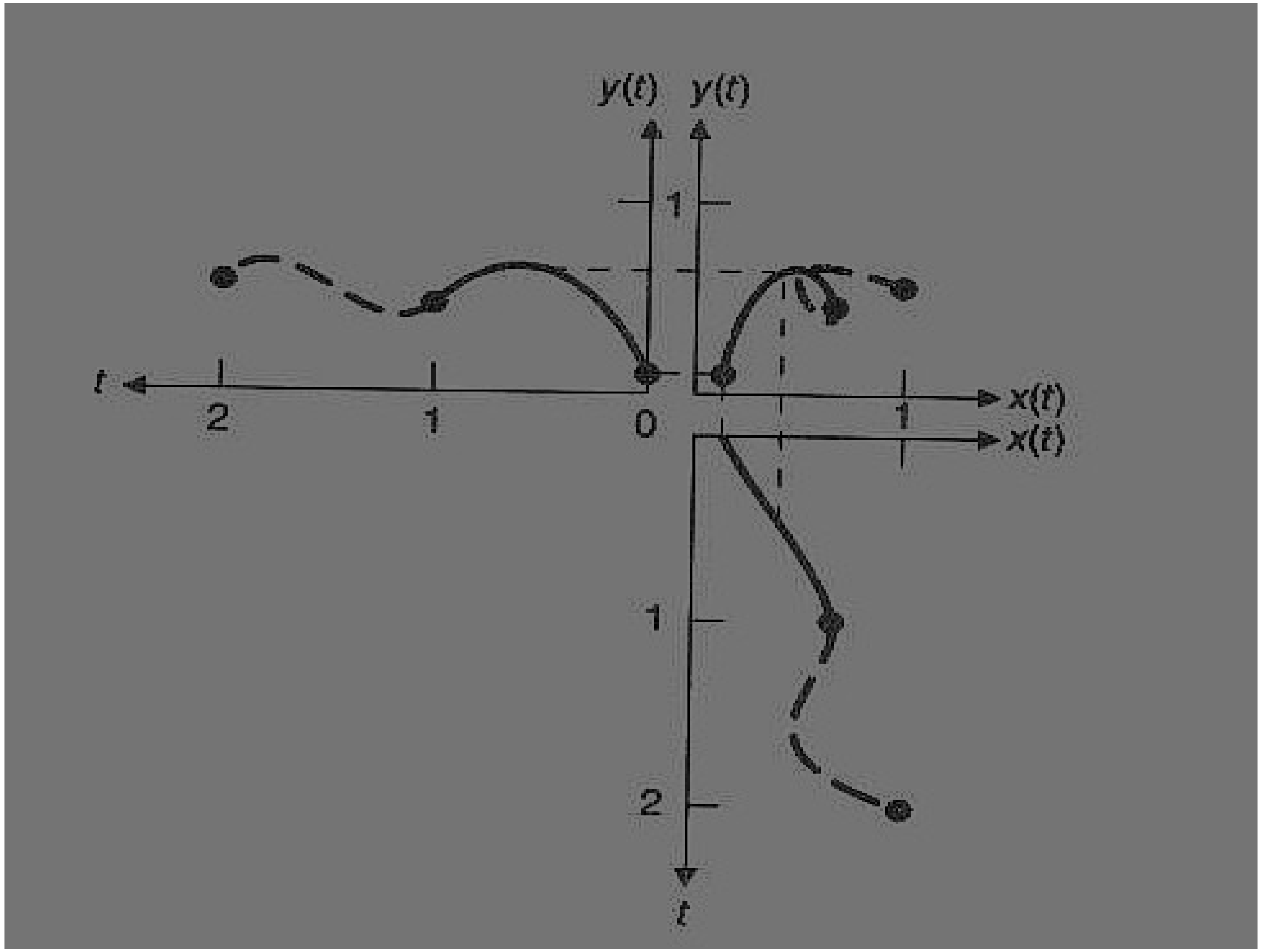
$$\begin{cases} x(t) = a_x t^3 + b_x t^2 + c_x t + d_x \\ y(t) = a_y t^3 + b_y t^2 + c_y t + d_y, \quad 0 \le t \le 1 \\ z(t) = a_z t^3 + b_z t^2 + c_z t + d_z \end{cases}$$

$$Q(t) = T \cdot C$$

$$T = \begin{pmatrix} t^3 & t^2 & t & 1 \end{pmatrix}$$

$$C = \begin{pmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \\ d_x & d_y & d_z \end{pmatrix}$$

Doua segmente de curbe parametrice 2D si polinoamele care le definesc. Linia punctata intre graficele (x,y) si x(t), y(t) arata corespondenta dintre punctele de pe curba (x,y) si polinoamele cubice care le definesc : x = x(t), y = y(t).



Curbe parametrice polinomiale cubice
Vectorul parametric tangent

$$Q'(t) = \frac{d}{dt} Q(t) = \begin{pmatrix} \frac{d}{dt} x(t) & \frac{d}{dt} y(t) & \frac{d}{dt} z(t) \end{pmatrix}$$

$$= \frac{d}{dt} T \cdot C = \begin{pmatrix} 3t^2 & 2t & 1 & 0 \end{pmatrix} \cdot C$$

$$= \begin{pmatrix} 3a_x t^2 + 2b_x t + c_x \\ 3a_y t^2 + 2b_y t + c_y \\ 3a_z t^2 + 2b_z t + c_z \end{pmatrix}^t$$

Continuitatea in punctul de contact

- Deoarece un segment de curba polinomiala cubica este continuu (cele 3 componente fiind polinoame) ne intereseaza continuitatea in punctul de contact a doua segmente de curba.

Continuitatea in punctul de contact

- Deoarece un segment de curba polinomiala cubica este continuu (cele 3 componente fiind polinoame) ne intereseaza continuitatea in punctul de contact a doua segmente de curba.
- Tipuri de continuitate
  - Continuitatea geometrica (G-continuitatea)
  - C-continuitatea

Continuitate geometrica G-continuitate

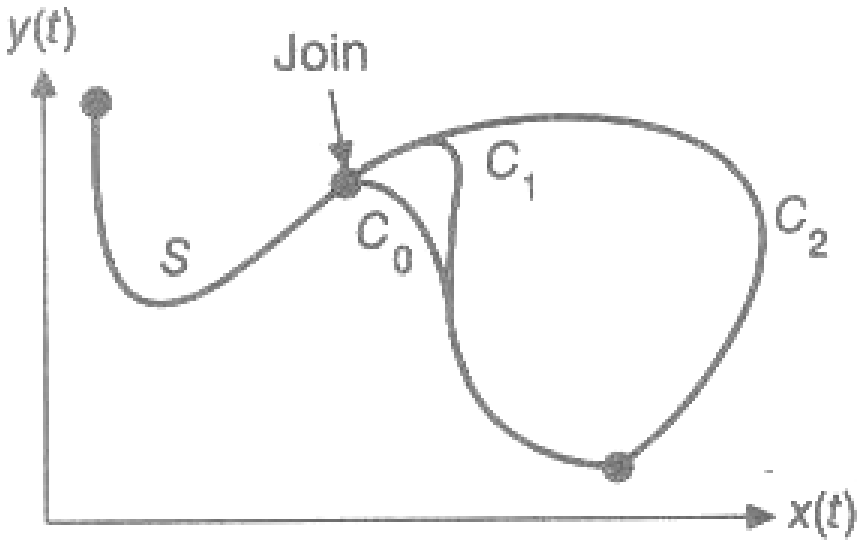
- 2 segmente de curba parametrica Q_1 si Q_2 sunt G^0-continue daca se unesc intr-o extremitate ($Q_1(1)= Q_2(0)$ sau $Q_1(0)= Q_2(1)$).
- 2 segmente de curba parametrica Q_1 si Q_2 sunt G^1-continue daca sunt G^0-continue si in punctul de contact cele 2 segmente de curba au aceeasi tangenta geometrica.

C-continuitatea

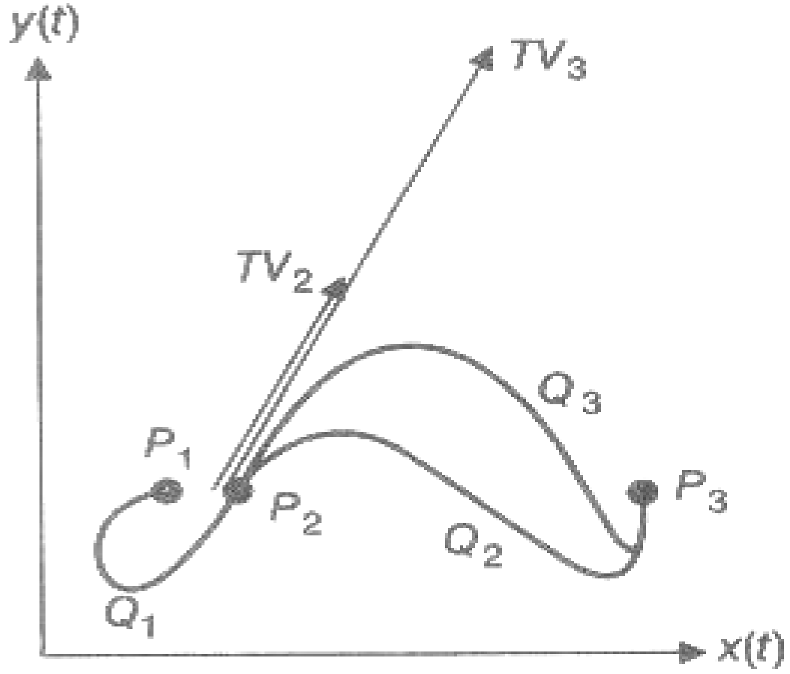
- 2 segmente de curba parametrica Q_1 si Q_2 sunt C^1-continue daca in punctul de contact vectorii parametrici tangenti la cele 2 segmente de curba sunt egali (aceeasi directie, sens, marime): $Q_1'(1) = Q_2'(0)$.
- 2 segmente de curba parametrica Q_1 si Q_2 sunt C^n-continue daca

$$Q_1^{(n)}(1) = Q_2^{(n)}(0), \quad \text{unde } \mathrm{Q}^{(\mathrm{n})}(t) = \frac{d^n}{dt^n}\left[Q(t)\right]$$

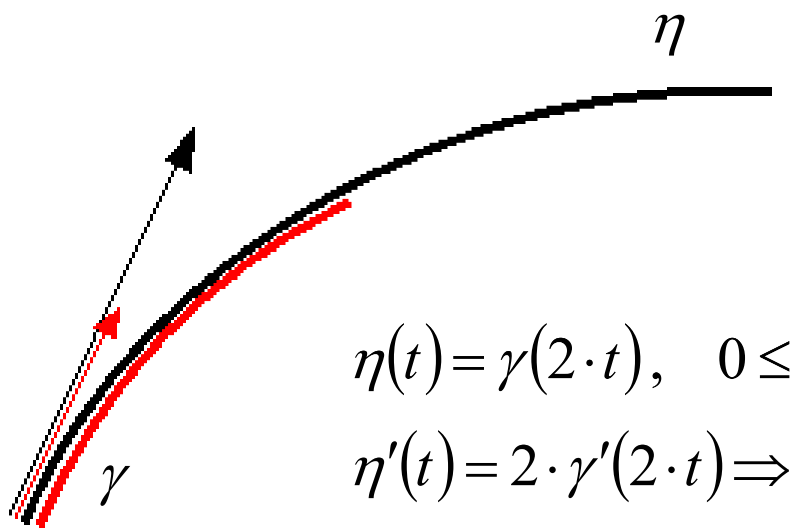
Punct de contact (join) cu continuitate C^0, C^1, C^2



Continuitatea G^1 nu implica continuitatea C^1. Segmentele de curba Q1, Q2 si Q3 se unesc in P2. Q1 si Q2 au vectori tangenti egali (deci sunt continue G^1 si C^1 in punctul de contact). Q1 si Q3 au vectori tangenti in aceeasi directie dar TV3 = 2 TV2 (si deci sunt doar continue G^1 in punctul de contact).



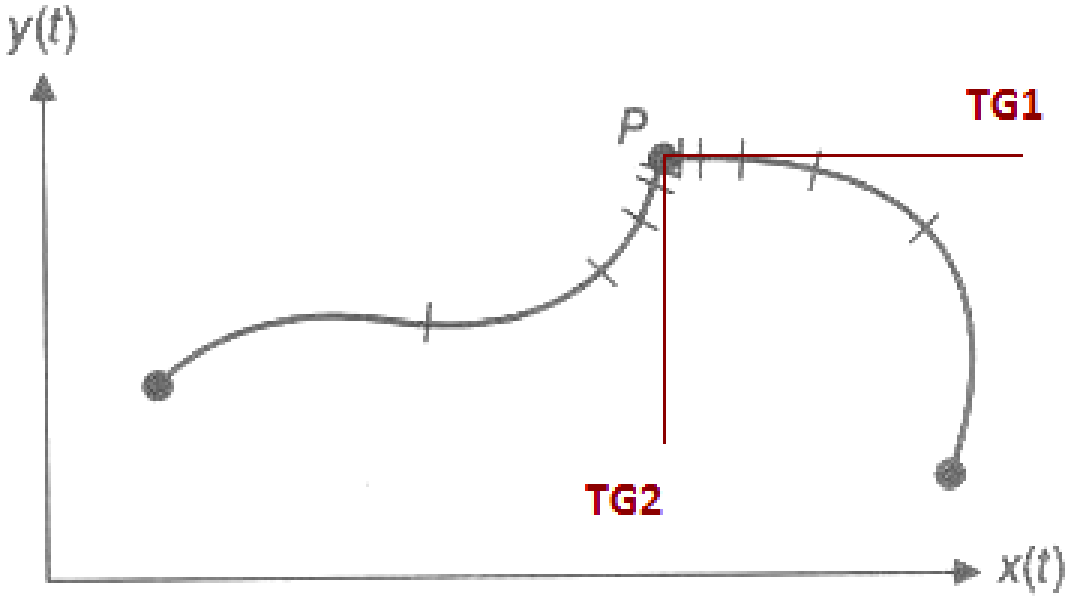
Continuitatea G^1 nu implica continuitatea C^1.



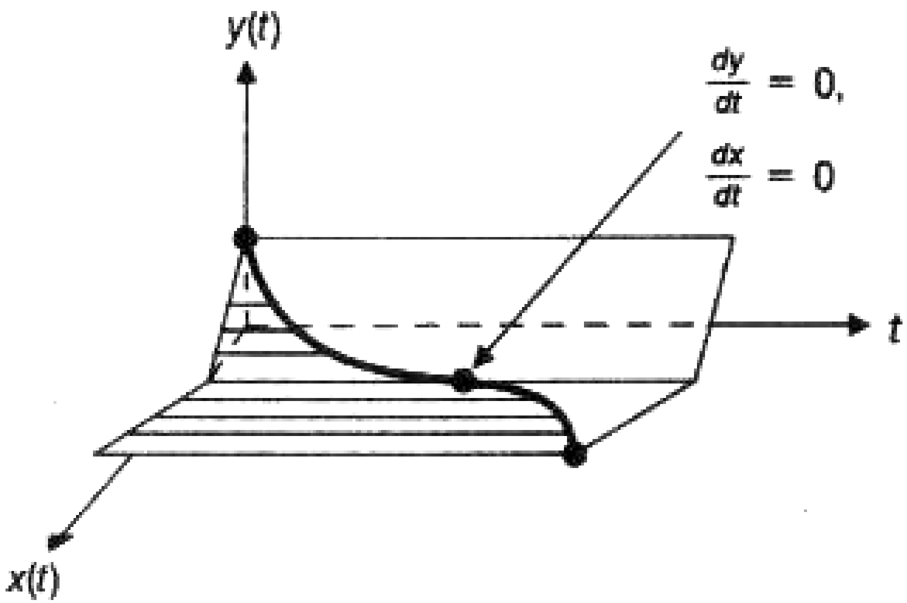
$$\eta(t) = \gamma(2 \cdot t), \quad 0 \le t \le 1/2$$

$$\eta'(t) = 2 \cdot \gamma'(2 \cdot t) \Rightarrow \eta'(0) = 2 \cdot \gamma'(0)$$

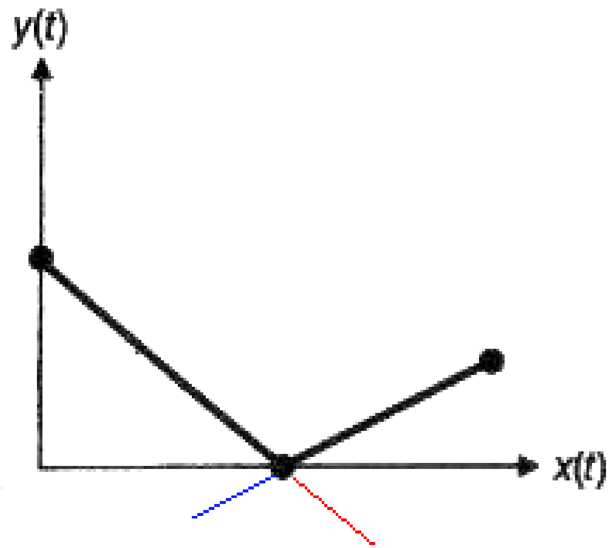
Continuitatea C^1 implica, in general, continuitatea G^1, cu exceptia cazului cand $Q'(t) = 0$. Avem C^1 dar nu G^1. Vectorul tangent (viteza parametrica de-a lungul curbei) este 0 in punctul de contact. Marcajele indica intervale egale de timp, deci pe masura ce ne apropiem, pe curba, de P viteza scade catre 0 si apoi creste din nou plecand de la 0.



Continuitatea C^1 implica, in general, pe cea G^1. Un caz in care nu implica: in punctul de contact vitezele parametrice sunt 0 (= dy / dt = dx / dt). Se poate observa ca in punctul de contact curba este || cu axa t si deci nu exista nici o schimbare in x sau in y. In punctul de contact curbele parametrice sunt continue C^1 dar nu sunt continue G^1.



O curba parametrica cubica 2D in spatiul (x,y,t)



Aceeasi curba parametrica cubica 2D in spatiul 2D (x(t), y(t))

Curbe parametrice polinomiale cubice Constrangeri

- Pentru determinarea coeficientilor C, in cazul curbelor parametrice polinomiale cubice, avem nevoie de 12 constrangeri (pentru putea formula 3 sisteme a cate 4 ecuatii cu 4 necunoscute)
- Constrangerile pot fi
  - Q(0), Q(1), Q'(0), Q'(1) pentru curbele Hermite
  - Q(0), Q(1) si alte 2 puncte de control pentru curbele Bézier

Curbe parametrice polinomiale cubice

$$Q(t) = T \cdot C = T \cdot M \cdot G = B \cdot G$$

$$M = \left(m_{ij}\right)_{\substack{1\le i\le 4\\ 1\le j\le 4}} \in \mathrm{M}_{4\times 4}(\mathbf{R}) \quad \text{matricea de baza}$$

$$G \in \mathrm{M}_{4\times 3}(\mathbf{R}) \quad \text{vectorul geometric (geometry vector)}$$

$$\mathrm{G} = \begin{pmatrix} g_{1x} & g_{1y} & g_{1z} \\ g_{2x} & g_{2y} & g_{2z} \\ g_{3x} & g_{3y} & g_{3z} \\ g_{4x} & g_{4y} & g_{4z} \end{pmatrix} = \begin{pmatrix} G_1 \\ G_2 \\ G_3 \\ G_4 \end{pmatrix} = \begin{pmatrix} G_x & G_y & G_z \end{pmatrix}$$

$$B \in \mathrm{M}_{1\times 4}(\mathbf{R}) \quad \text{functii de amestecare (blending functions)}$$

Curbe parametrice polinomiale cubice

$$Q(t) =$$

$$T \cdot M \cdot G =$$

$$\begin{pmatrix} t^3 & t^2 & t & 1 \end{pmatrix} \cdot \begin{pmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{pmatrix} \cdot \begin{pmatrix} G_1 \\ G_2 \\ G_3 \\ G_4 \end{pmatrix} =$$

$$\sum_{j=1}^{4} \left(\sum_{i=1}^{4} m_{ij} \cdot t^{4-i} \right) \cdot G_j$$

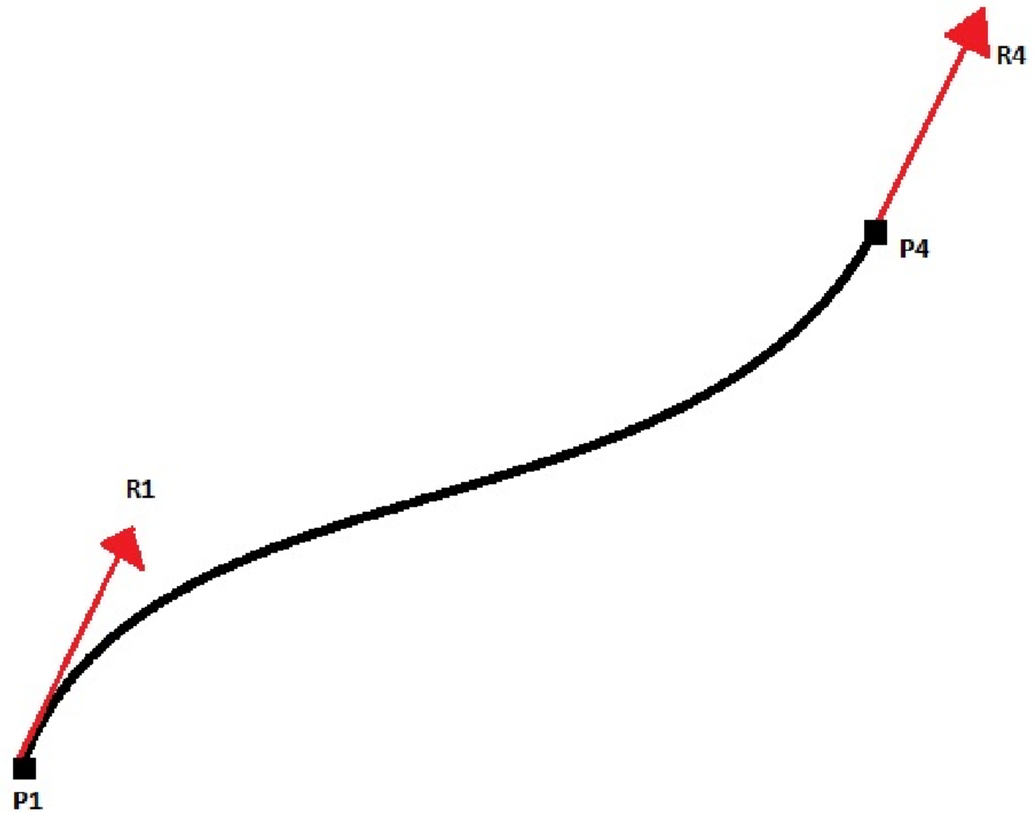
Curbe Hermite

$$Q(t) = T \cdot M_H \cdot G_H$$

$$G_H = \begin{pmatrix} P_1 \\ P_4 \\ R_1 \\ R_4 \end{pmatrix} = \begin{pmatrix} Q(0) \\ Q(1) \\ Q'(0) \\ Q'(1) \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{pmatrix} \cdot M_H \cdot G_H \Rightarrow$$

$$M_H = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

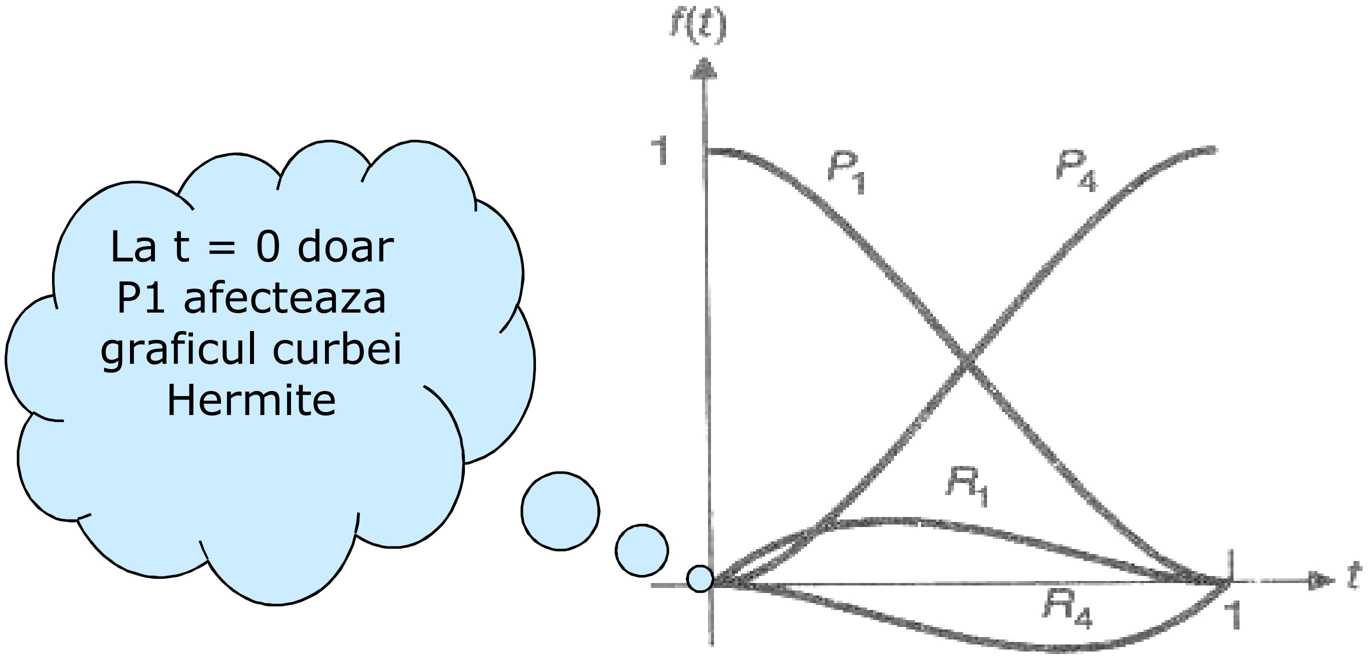
Curbe Hermite



Curbe Hermite
functiile de amestecare

$$B_H = T \cdot M_H = \begin{pmatrix} 2t^3 - 3t^2 + 1 \\ -2t^3 + 3t^2 \\ t^3 - 2t^2 + t \\ t^3 - t^2 \end{pmatrix}$$

Curbe Hermite
functiile de amestecare

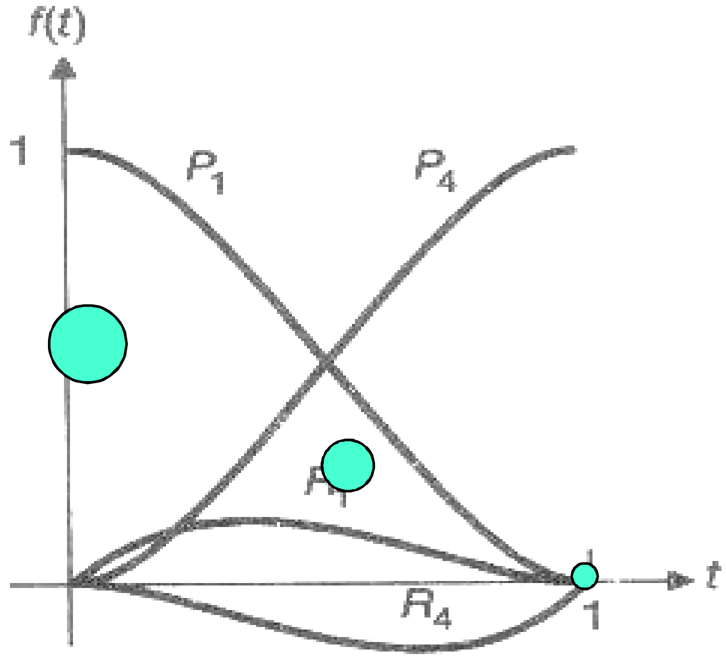


$$Q(t) = B_H \cdot G_H \Rightarrow$$

$$Q(t) = (2t^3 - 3t^2 + 1) \cdot P_1 + (-2t^3 + 3t^2) \cdot P_4 + (t^3 - 2t^2 + t) \cdot R_1 + (t^3 - t^2) \cdot R_4$$

Curbe Hermite
functiile de amestecare

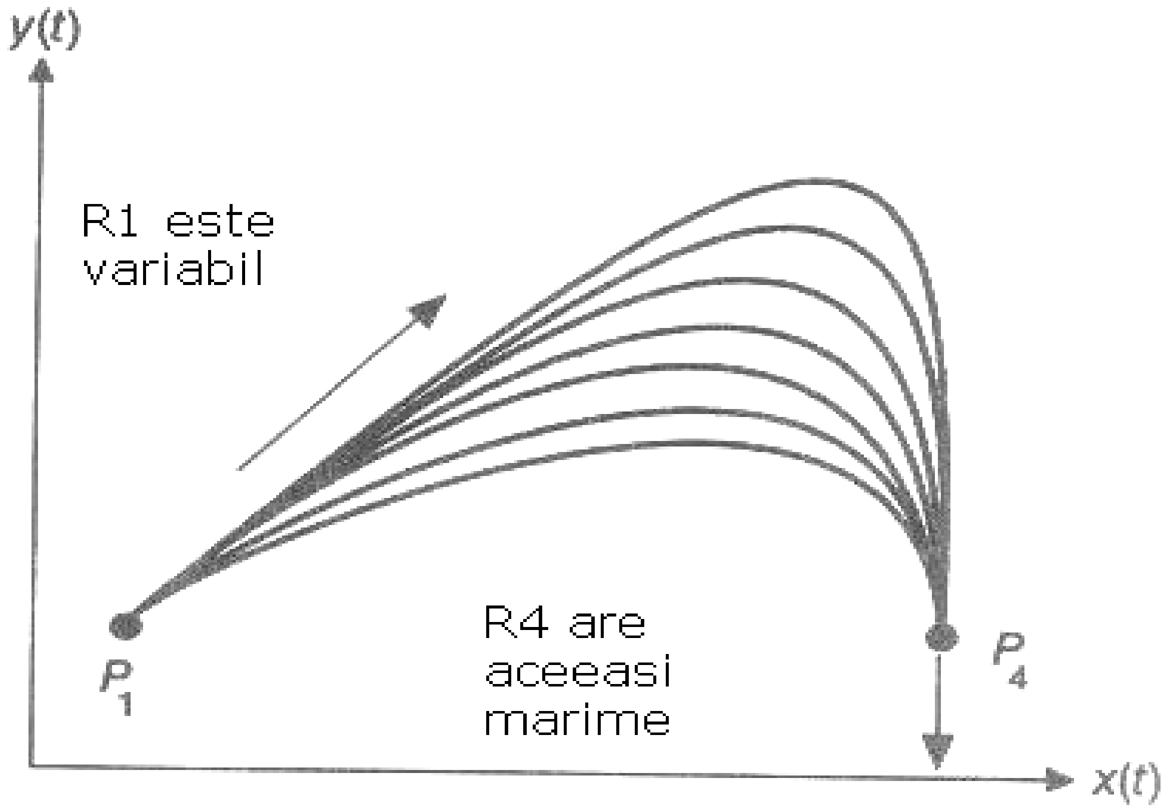
La t = 1 doar
P4 afecteaza
graficul curbei
Hermite



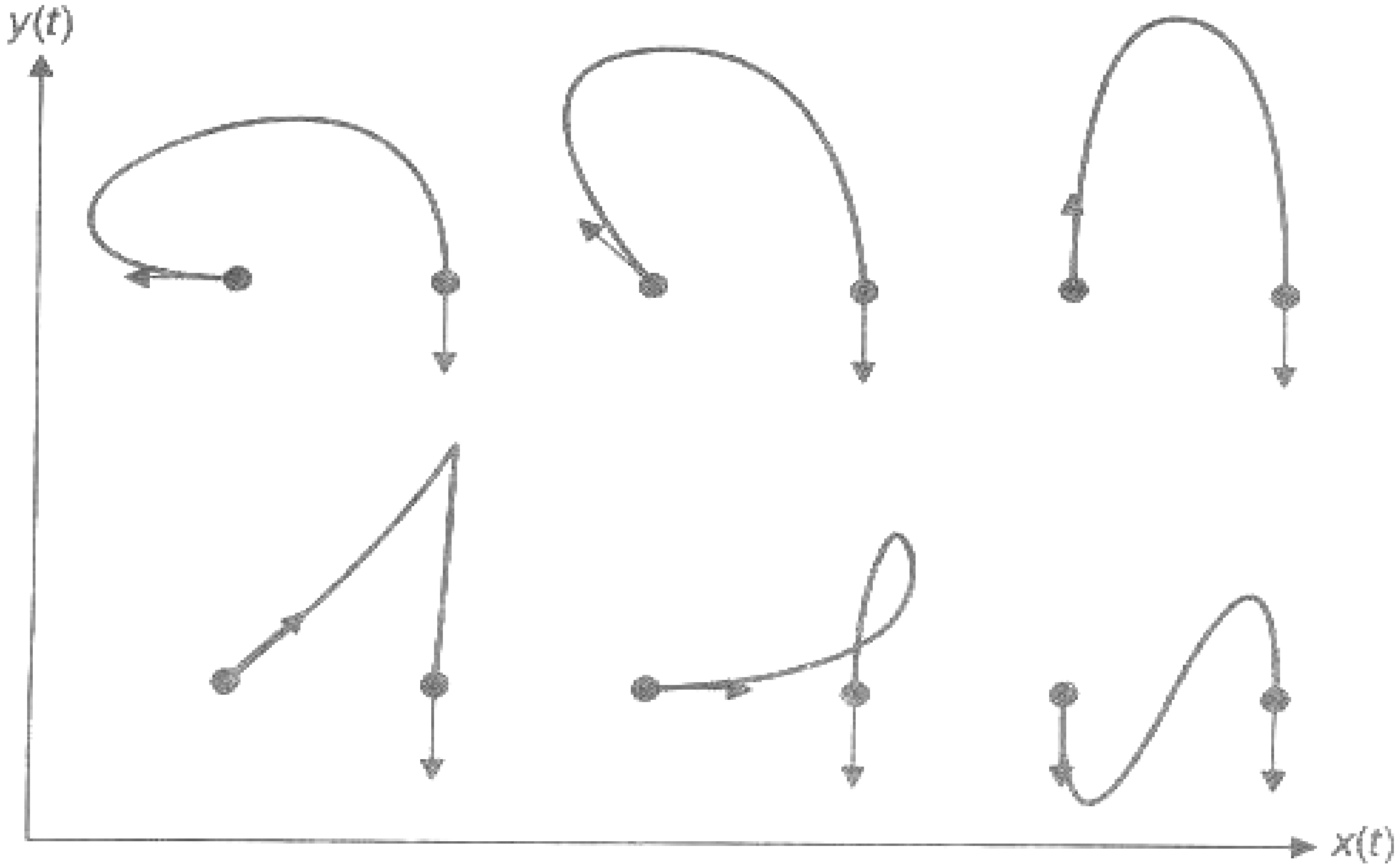
$$Q(t) = B_H \cdot G_H \Rightarrow$$

$$Q(t) = (2t^3 - 3t^2 + 1) \cdot P_1 + (-2t^3 + 3t^2) \cdot P_4 + (t^3 - 2t^2 + t) \cdot R_1 + (t^3 - t^2) \cdot R_4$$

O familie de curbe Hermite (care variaza prin lungimea lui R1)



O familie de curbe Hermite (care variaza prin directia lui R1)



Curbe Hermite

desenareHermite.cpp

Curbe Bézier

- O curba Bézier Q(t) este specificata prin intermediul a 4 puncte de control : P1, P2, P3, P4.
  - P1, P4 sunt extremitatile segmentului de curba Q(0) si Q(1)
  - P2, P3 sunt doua puncte de control care nu apartin curbei Bézier. Prin intermediul acestor puncte sunt specificati vectorii tangenti Q'(0) si Q'(1): o curba Bézier cu vectorul geometric ^{t}(P1 P2 P3 P4) este o curba Hermite avand vectorul geometric ^{t}(P1 P4 R1 R4), unde
    R1 = Q'(0) = 3 * P1P2 si
    R4 = Q'(1) = 3 * P3P4
- $Q(t) = T * M_B * G_B$
- Vectorul geometric G_H se obtine pe baza G_B

Curbe Bézier

- $Q(t) = T * M_B * G_B$
- Vectorul geometric G_H se obtine pe baza G_B:
  $G_H = M_{HB} * G_B$
- $Q(t) = T * M_H * G_H = T * M_H * M_{HB} * G_B =$
  $T * M_B * G_B = B_B * G_B =$
  $B(0,3) * P1 + B(1,3) * P2 + B(2,3) * P3 + B(3,3) * P4$
- $B(i,n)$ sunt polinoame Bernstein

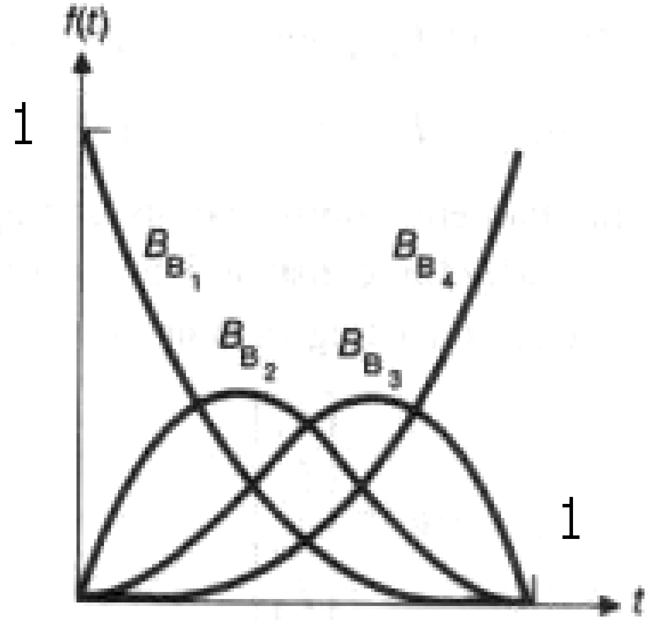
Curbe Bézier
polinoame Bernstein

$$B(i,n) = C_n^i \cdot t^i \cdot (1-t)^{n-i}$$

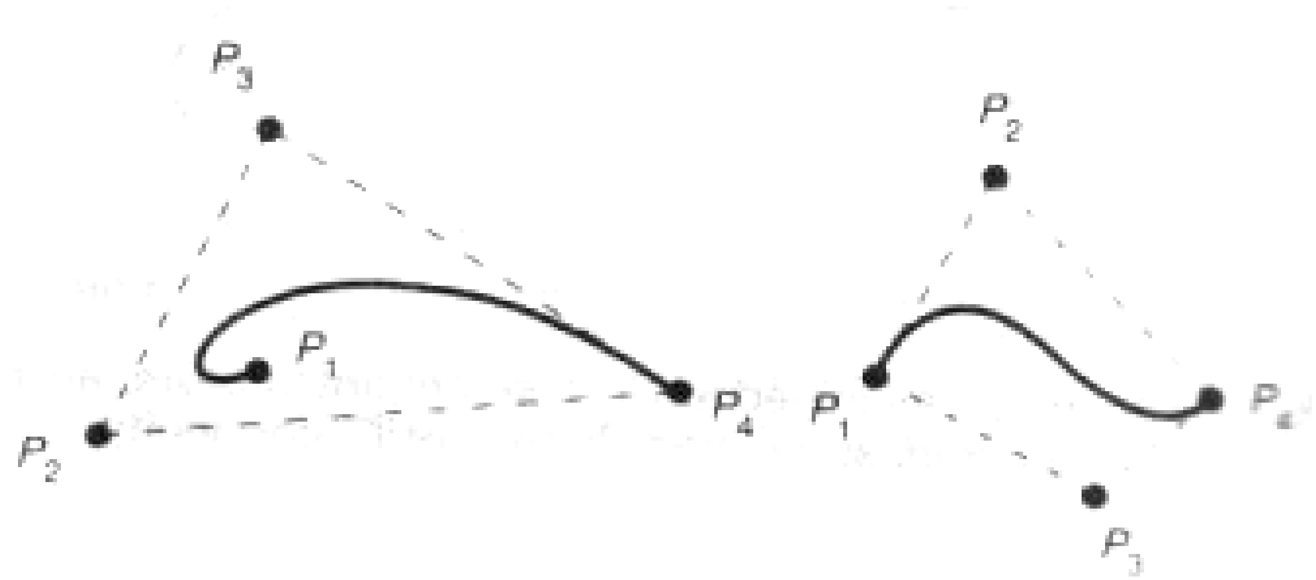
$$\sum_{i=0}^{n} B(i,n) = 1$$

$$(\forall 0 \leq i \leq n)(B(i,n) \geq 0)$$

Polinoame Bernstein – functii de ponderare pentru curbele Bezier

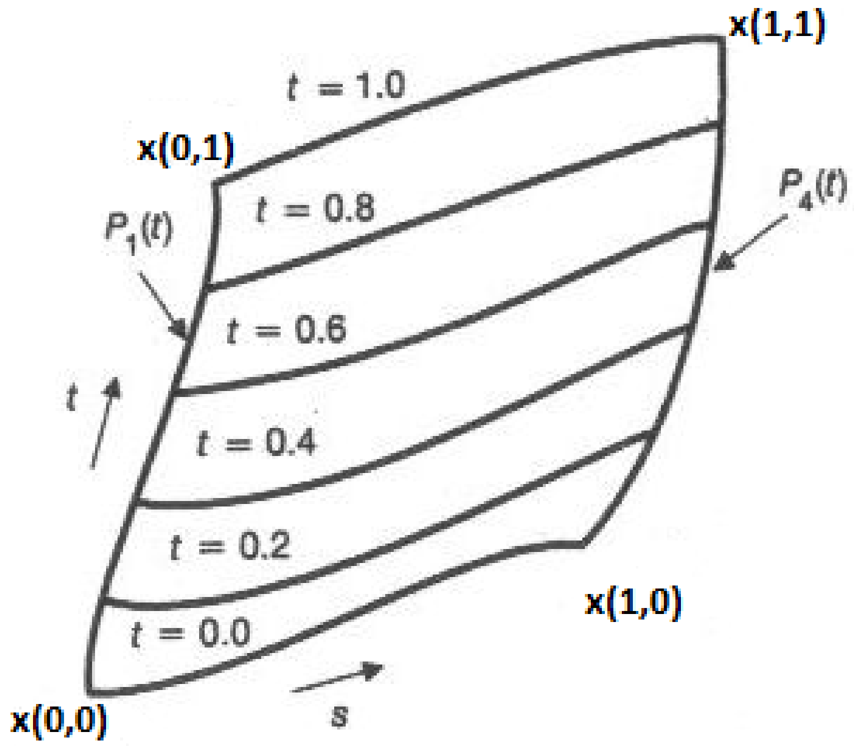


Doua curbe Bezier si punctele lor de control. Remarcati ca sunt complet incluse in infasuratoarea convexa a celor 4 puncte de control.

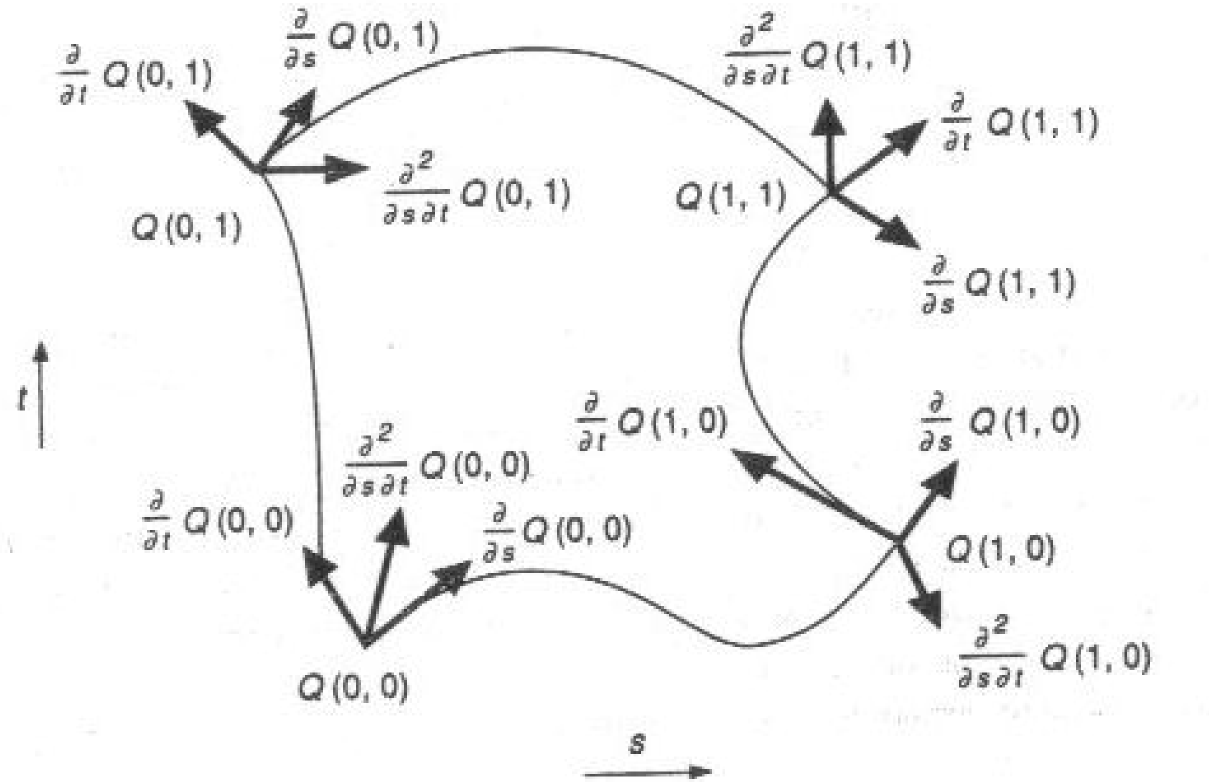


Suprafete parametrice polinomiale bivariate

$$P1(t) = Q(0, t)$$
$$P4(t) = Q(1, t)$$



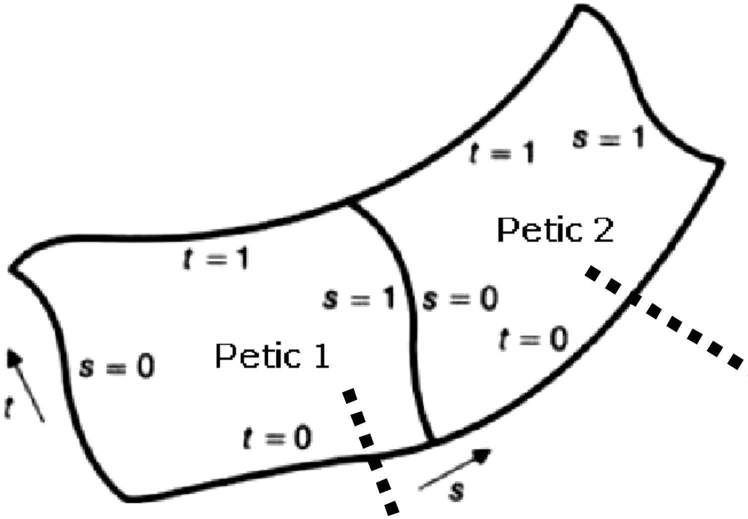
Elementele matricii geometrice pentru o suprafata Hermite



Elementele matricii geometrice pentru o suprafata Hermite

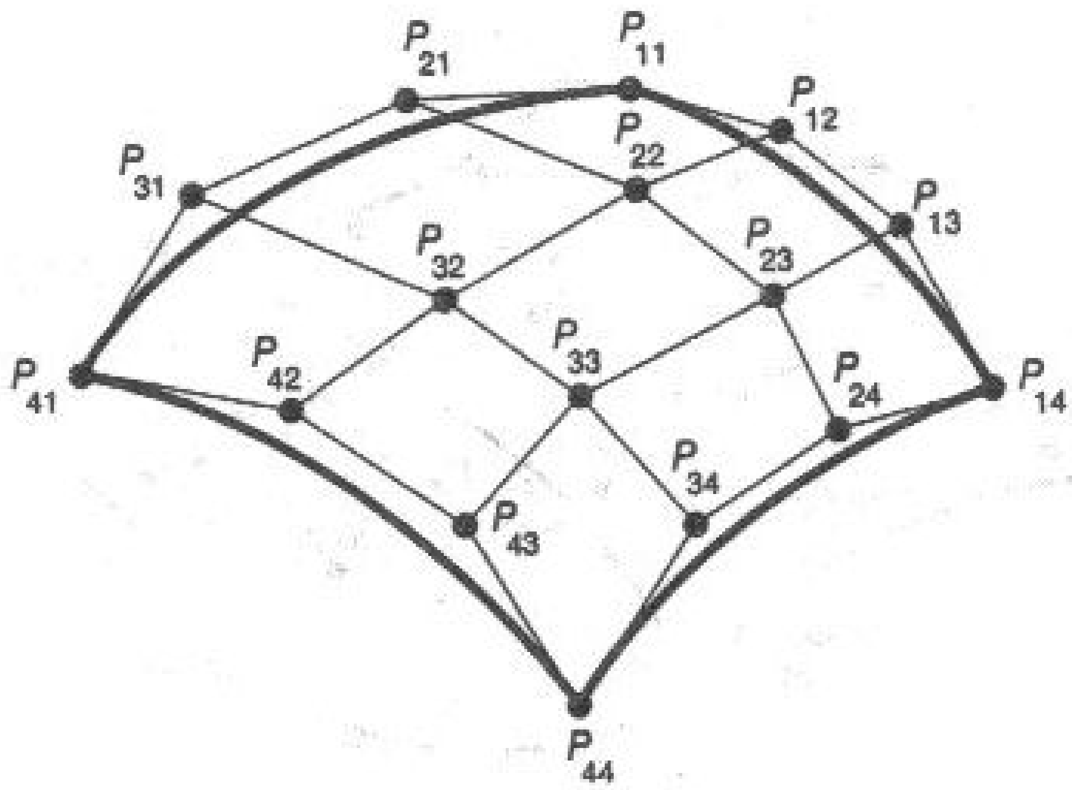
$$\overline{G_H} = \begin{pmatrix} Q(0,0) & Q(0,1) & \frac{\partial}{\partial t}Q(0,0) & \frac{\partial}{\partial t}Q(0,1) \\ Q(1,0) & Q(1,1) & \frac{\partial}{\partial t}Q(1,0) & \frac{\partial}{\partial t}Q(1,1) \\ \frac{\partial}{\partial s}Q(0,0) & \frac{\partial}{\partial s}Q(0,1) & \frac{\partial^2}{\partial s \partial t}Q(0,0) & \frac{\partial^2}{\partial s \partial t}Q(0,1) \\ \frac{\partial}{\partial s}Q(1,0) & \frac{\partial}{\partial s}Q(1,1) & \frac{\partial^2}{\partial s \partial t}Q(1,0) & \frac{\partial^2}{\partial s \partial t}Q(1,1) \end{pmatrix}$$

Suprafate bicubice Hermite. Continuitate.



$$\begin{bmatrix} - & - & - & - \\ g_{21} & g_{22} & g_{23} & g_{24} \\ - & - & - & - \\ g_{41} & g_{42} & g_{43} & g_{44} \end{bmatrix} \quad \begin{bmatrix} g_{21} & g_{22} & g_{23} & g_{24} \\ - & - & - & - \\ \mathrm{k}g_{41} & \mathrm{k}g_{42} & \mathrm{k}g_{43} & \mathrm{k}g_{44} \\ - & - & - & - \end{bmatrix} k > 0.$$

Suprafata bicubica Bezier. Remarcati cele 16 puncte de control.



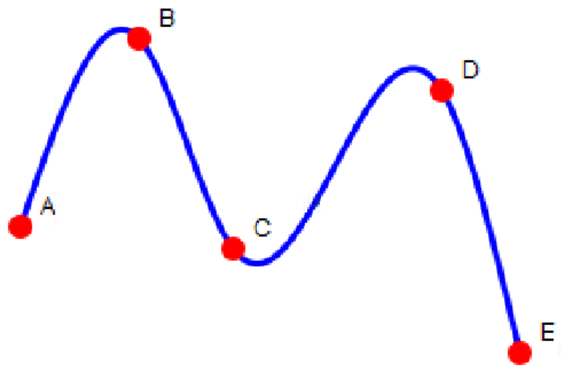
Curbe si suprafete Bézier in OpenGL

cap6p1.cpp

cap6p2.cpp

Curbe B-spline uniform nerationale

Curbe spline naturale cubice



- Polinoame de grad 3, continue C^0, C^1, C^2 in punctele de contact
  - mai netede decat Bézier, Hermite (C^0)

Curbe B-spline uniform nerationale

- Curbe spline naturale cubice
  - N puncte de control
  - Interpoleaza prin toate punctele de control
  - Coeficientii polinoamelor depind de toate punctele de control
    - Calculul lor pp. inversarea unei matrici din $\mathfrak{M}_{n+1 \times n+1}(\Re)$

Curbe B-spline uniform nerationale

- Curbe B-spline
  - Nu interpoleaza prin punctele de control
  - Coeficientii polinoamelor depind de cateva punctele de control: proprietatea controlului local
  - Aceeasi continuitate ca si curbele spline naturale

Curbe B-spline uniform nerationale

- Curbe B-spline
  - Puncte de control (aproximare, nu interpolare): m+1

    $$\left(P_i\right)_{0 \le i \le m} \quad , \quad m \ge 3$$

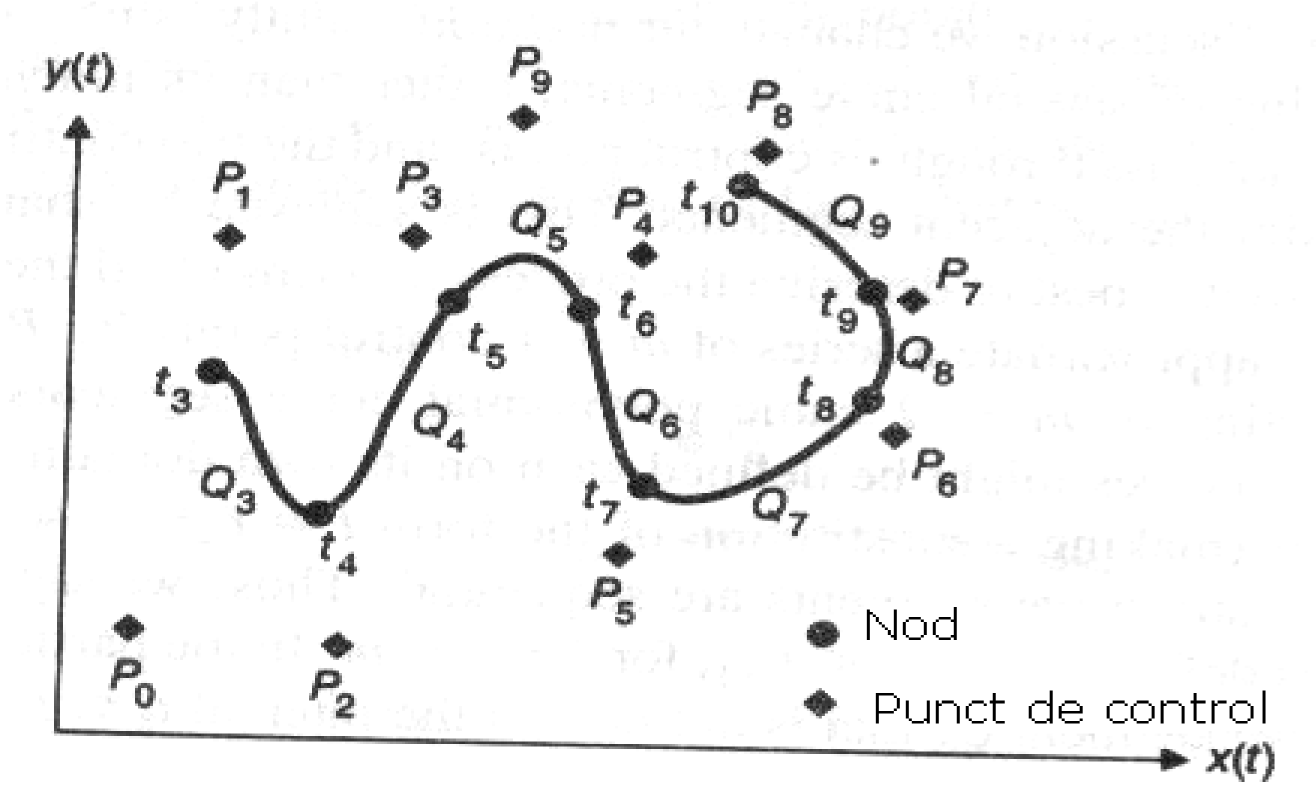
  - Segmente de curba: m-2

    $$\left(Q_j\right)_{3 \le j \le m}$$

Curbe B-spline uniform nerationale

- Curbe B-spline
  - Nod
  - Valoarea nodului

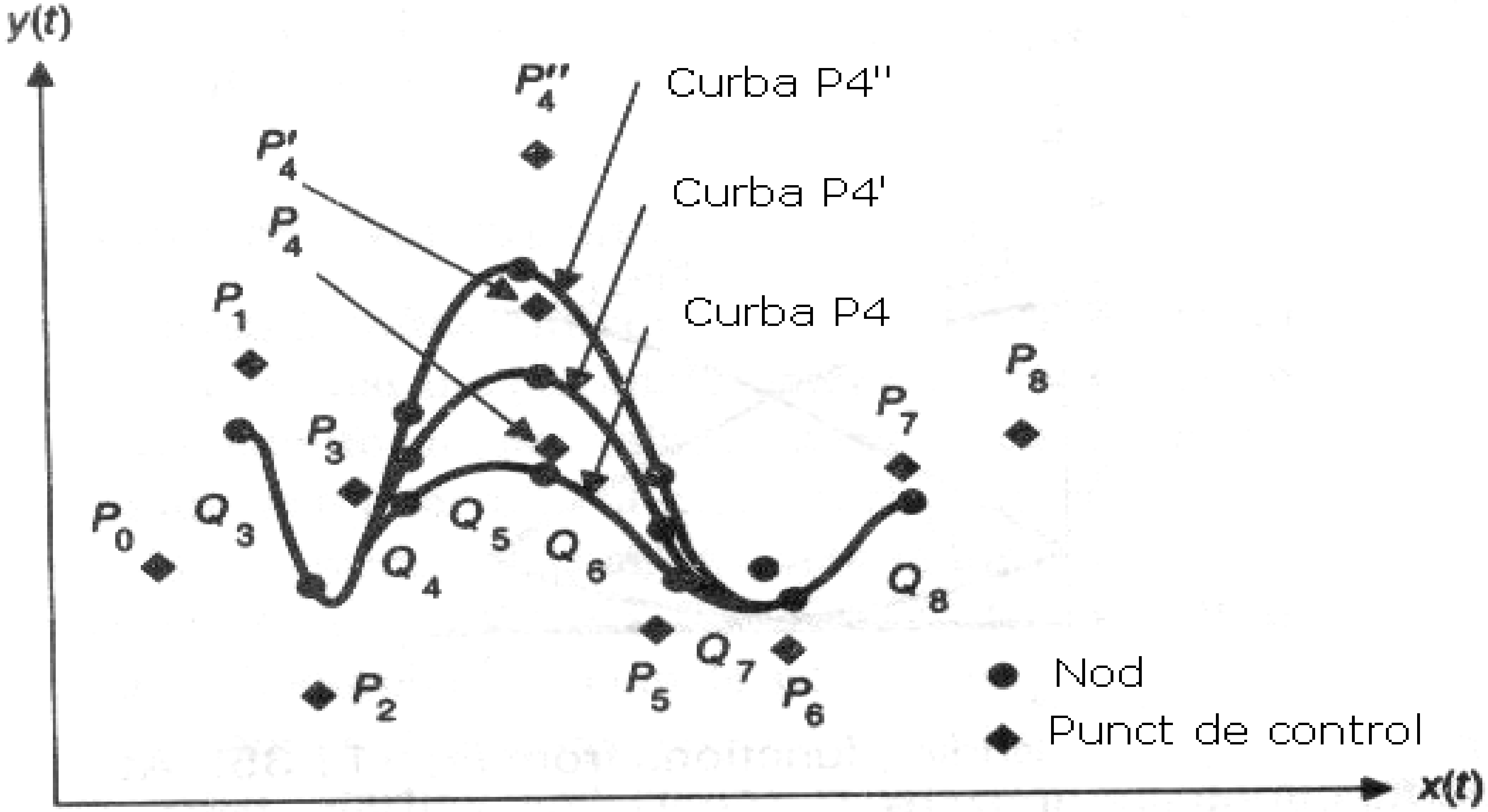
O curba B-spline uniform nerationala avand segmentele de curba Q3÷Q9.



Curbe B-spline uniform nerationale

- Curbe B-spline
  - Semnificatii
    - Uniform
    - Nerational
    - B

O curba B-spline uniform nerationala cu punctul de control P4 in diverse pozitii : P4, P4’, P4’’.



Curbe B-spline uniform nerationale

$$Q_i(t) = T_i \cdot M_{Bs} \cdot G_{Bs_i}$$

$$T_i = \left((t - t_i)^3 \quad (t - t_i)^2 \quad (t - t_i) \quad 1 \right)$$

$$t_i \leq t \leq t_i + 1$$

$$3 \leq i \leq m$$

Curbe B-spline uniform nerationale

$$Q_i(t) = T_i \cdot M_{\mathrm{Bs}} \cdot G_{\mathrm{Bs}_i}$$

$$M_{\mathrm{Bs}} = \frac{1}{6}\begin{pmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 1 & 4 & 1 & 0 \end{pmatrix}$$

Curbe B-spline uniform nerationale

$$Q_i(t) = T_i \cdot M_{Bs} \cdot G_{Bs_i}$$

$$G_{Bs_i} = \begin{pmatrix} P_{i-3} \\ P_{i-2} \\ P_{i-1} \\ P_i \end{pmatrix}$$

Curbe B-spline uniform nerationale

$$T = \begin{pmatrix} t^3 & t^2 & t & 1 \end{pmatrix} \quad , \quad 0 \le t < 1$$

$$B_{Bs} = T \cdot M_{Bs} = \begin{pmatrix} B_{Bs_{-3}} & B_{Bs_{-2}} & B_{Bs_{-1}} & B_{Bs_{-0}} \end{pmatrix} =$$

$$\frac{1}{6} \cdot \begin{pmatrix} (1-t)^3 & 3t^3 - 6t^2 + 4 & -3t^3 + 3t^2 + 3t + 1 & t^3 \end{pmatrix}$$

$$\Downarrow$$

$$\vdots$$

Curbe B-spline uniform nerationale

$$\vdots$$

$$\Downarrow$$

$$Q_i(t+t_i) = B_{Bs} \cdot G_{Bs_i} =$$

$$\frac{(1-t)^3}{6} \cdot P_{i-3} + \frac{3t^3 - 6t^2 + 4}{6} \cdot P_{i-2} + \frac{-3t^3 + 3t^2 + 3t + 1}{6} \cdot P_{i-1} + \frac{t^3}{6} \cdot P_i$$

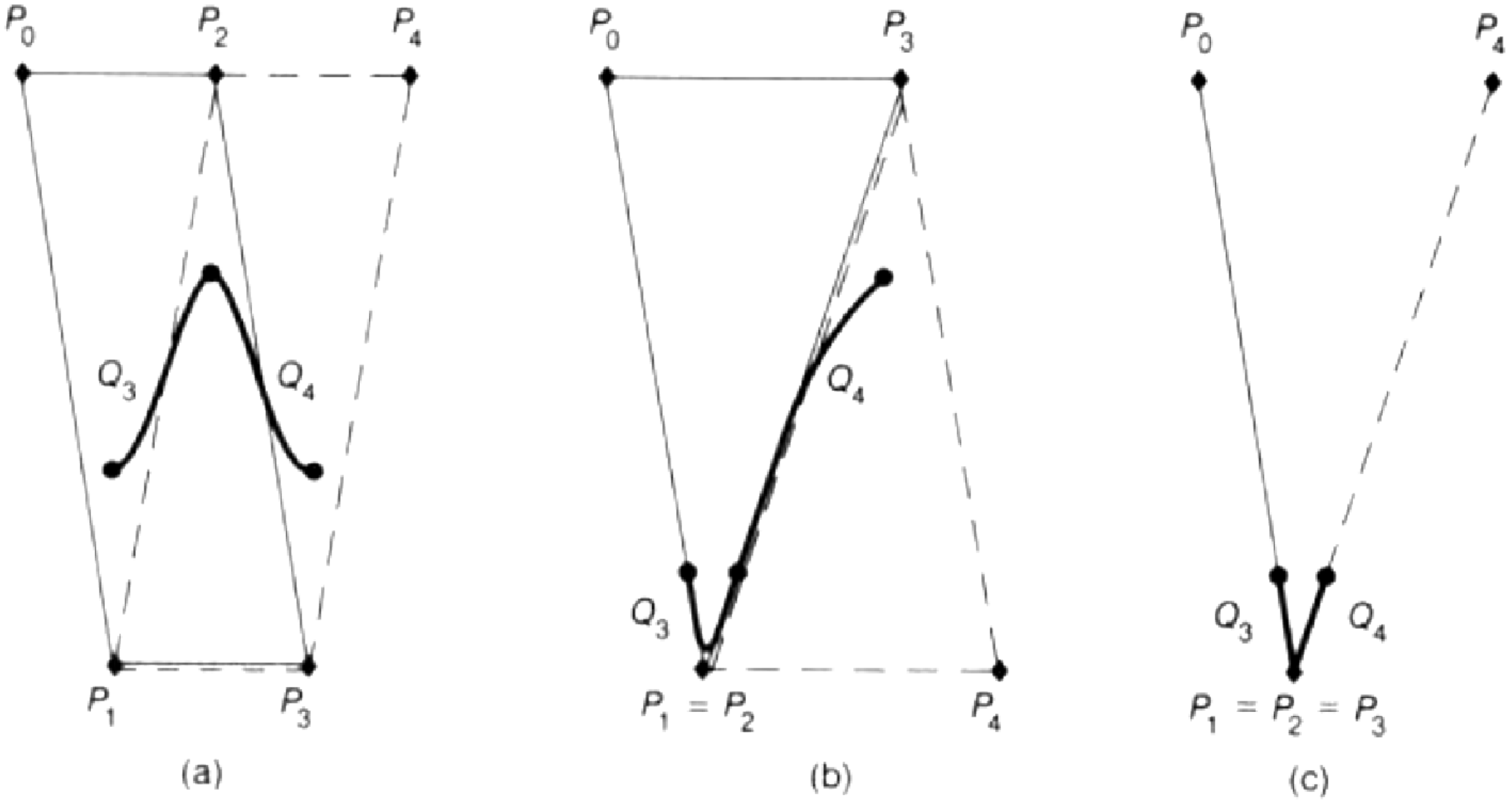
$$0 \le t < 1$$

Curbe B-spline uniform nerationale

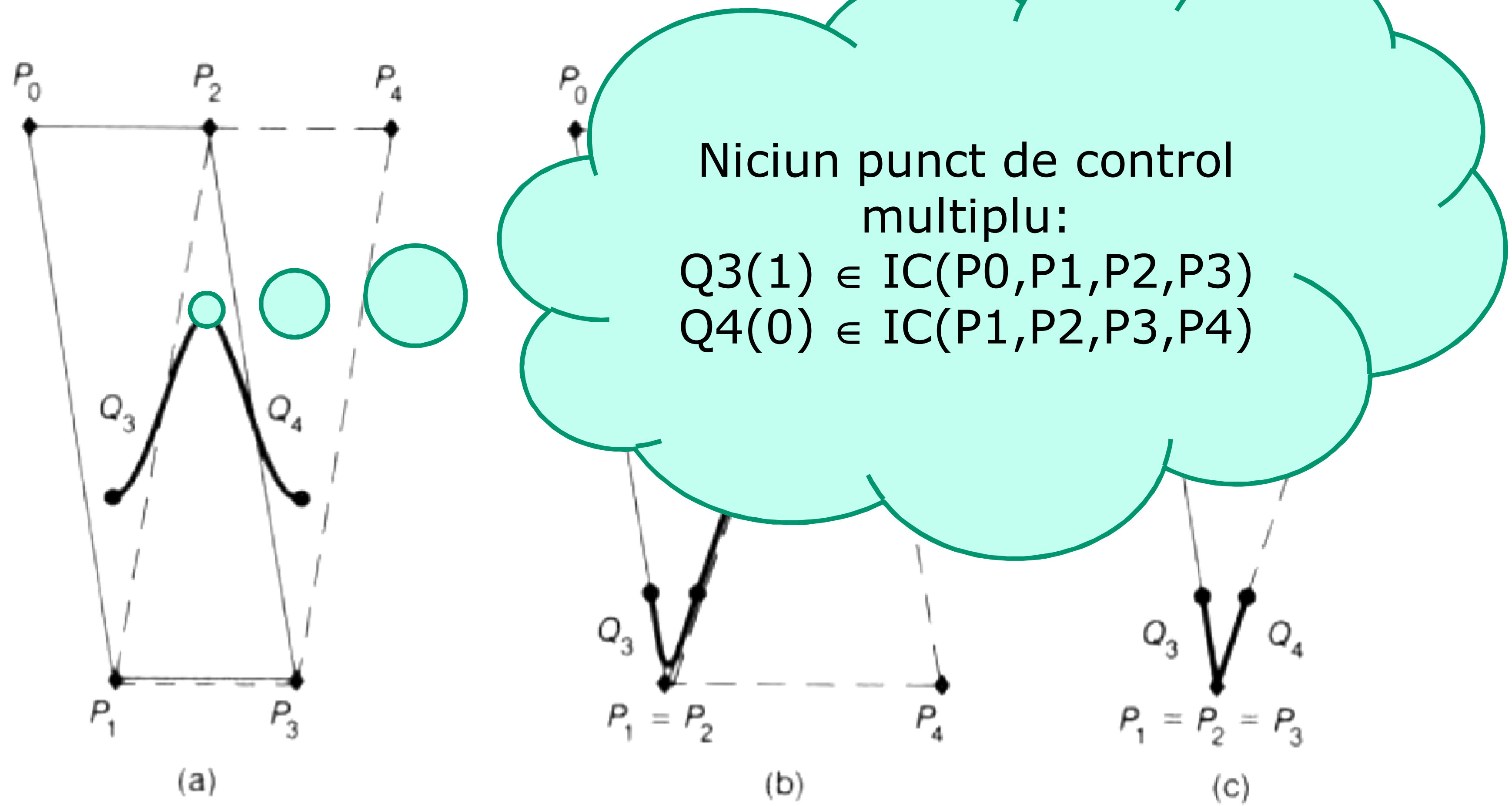
- Curbele B-spline uniform nerationale au continuitate C^0, C^1 si C^2 in punctele de contact

$$Q_i(1+t_i) = Q_{i+1}(0+t_{i+1})$$
$$Q_i'(1+t_i) = Q_{i+1}'(0+t_{i+1})$$
$$Q_i''(1+t_i) = Q_{i+1}''(0+t_{i+1})$$
$$f_i(t) \stackrel{\bullet}{\underset{\bullet}{=}} Q_i(t+t_i)$$

Efectul punctelor multiple de control asupra unei curbe B-spline uniform nerationale: fortarea interpolarii anumitor puncte de control



Efectul punctelor multiple de control asupra unei curbe B-spline uniform nerationale: fortarea interpolarii anumitor puncte de control



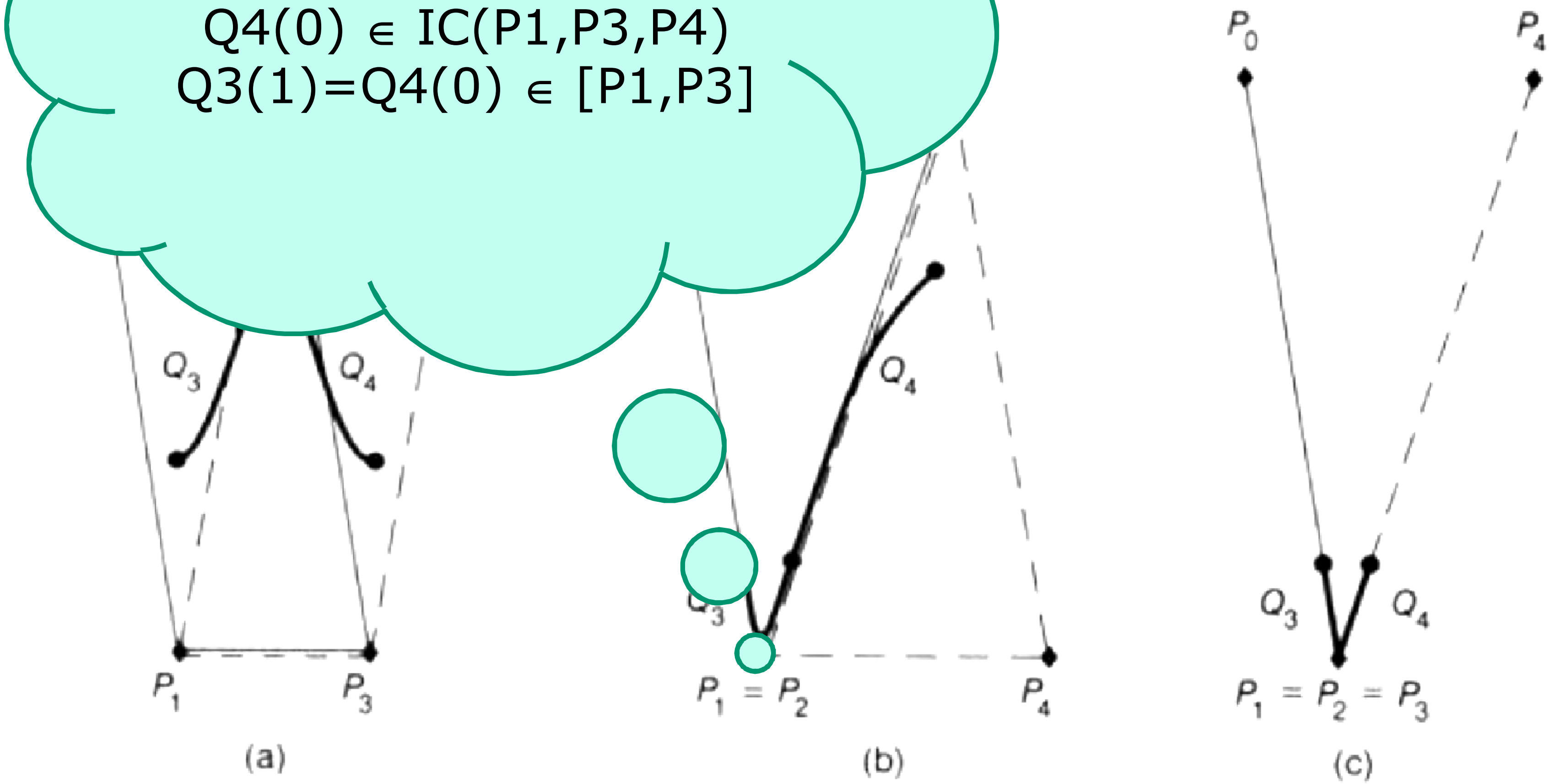
Efectul pu... de control asupra unei cu... ...erationale: fortarea i... ...e control

P1 = P2

Q3(1) ∈ IC(P0,P1,P3)

Q4(0) ∈ IC(P1,P3,P4)

Q3(1)=Q4(0) ∈ [P1,P3]



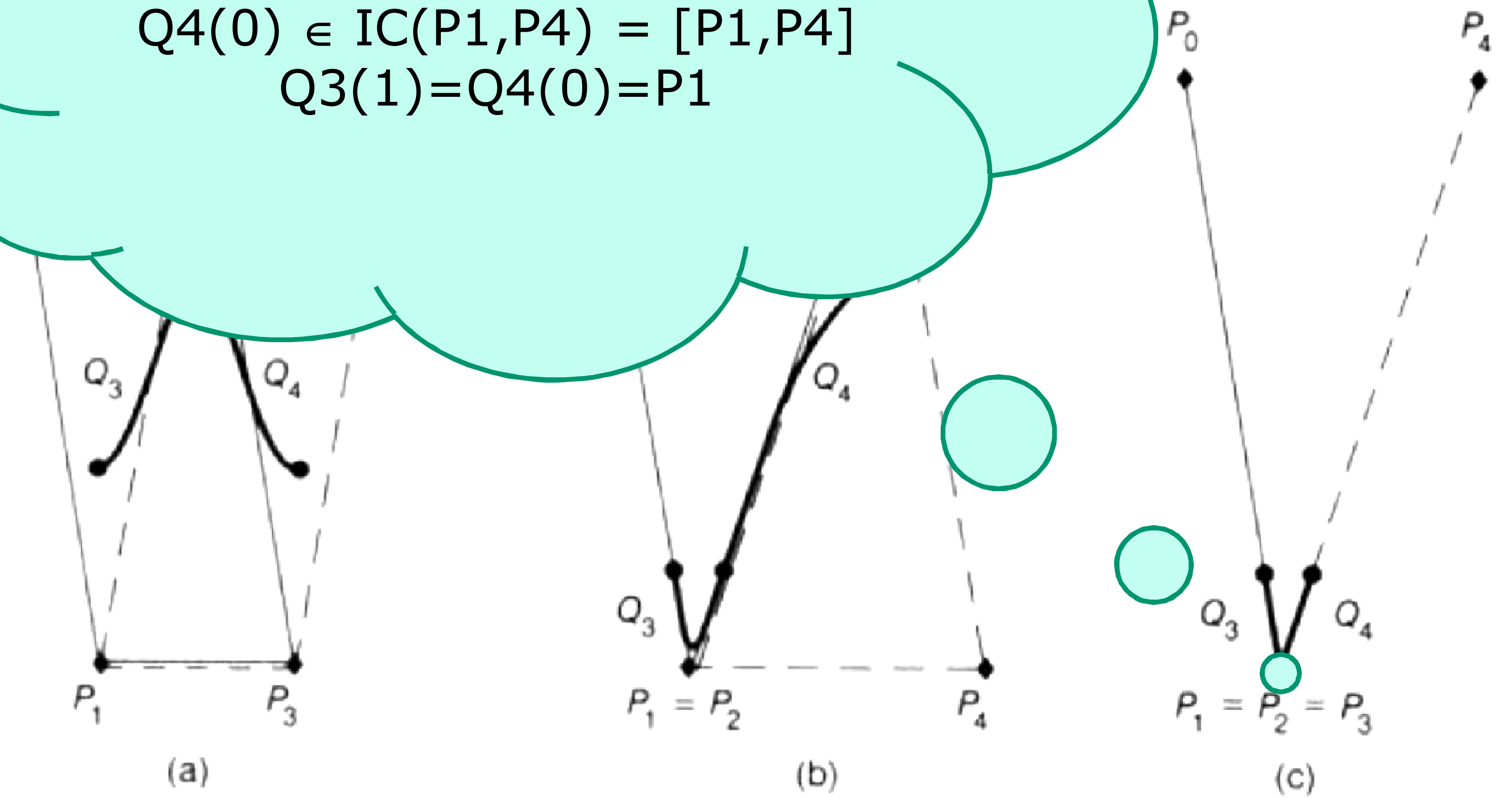
Efectul punct... m... ntrol asupra unei cu... ale: fortarea i... ol

P1 = P2 = P3

Q3(1) ∈ IC(P0,P3) = [P0,P1]

Q4(0) ∈ IC(P1,P4) = [P1,P4]

Q3(1)=Q4(0)=P1



Curbe B-spline neuniform nerationale

- Intervalele $[t_i, t_{i+1})$ nu mai sunt uniforme ca in cazul curbelor B-spline uniform nerationale
- Continuitatea in punctele de contact poate varia: C^2, C^1, C^0, necontinuitate

Curbe B-spline uniform nerationale

- Puncte de control (aproximare, nu interpolare): m+1

$$\left(P_i\right)_{0 \le i \le m} \quad , \quad m \ge 3$$

- Secventa de valori de nod: m+5

$$\left(t_i\right)_{0 \le i \le m+4}$$
$$\left(\forall\, 0 \le i < m+4\right)\left(t_i \le t_{i+1}\right)$$

Curbe B-spline uniform nerationale

- Segmente de curba: m-2

$$\left(Q_i(t)\right)_{3\le i\le m}$$

$$t_i \le t < t_{i+1}$$

- Noduri multiple de multiplicitate k

$$t_i = t_{i+1} = \ldots = t_{i+k-1}$$

Curbe B-spline neuniform nerationale

$$Q_i(t) = \sum_{j=0}^{3} P_{i-j} \cdot B_{i-j,4}(t),\ 3 \le i \le m$$

$$\left(B_{i-j,4}(t)\right)_{0 \le j \le 3} \quad \text{sunt functiile de amestecare}$$

$$B_{i,1}(t) = \begin{cases} 1 & ,t_i \le t < t_{i+1} \\ 0 & ,\text{altfel} \end{cases}$$

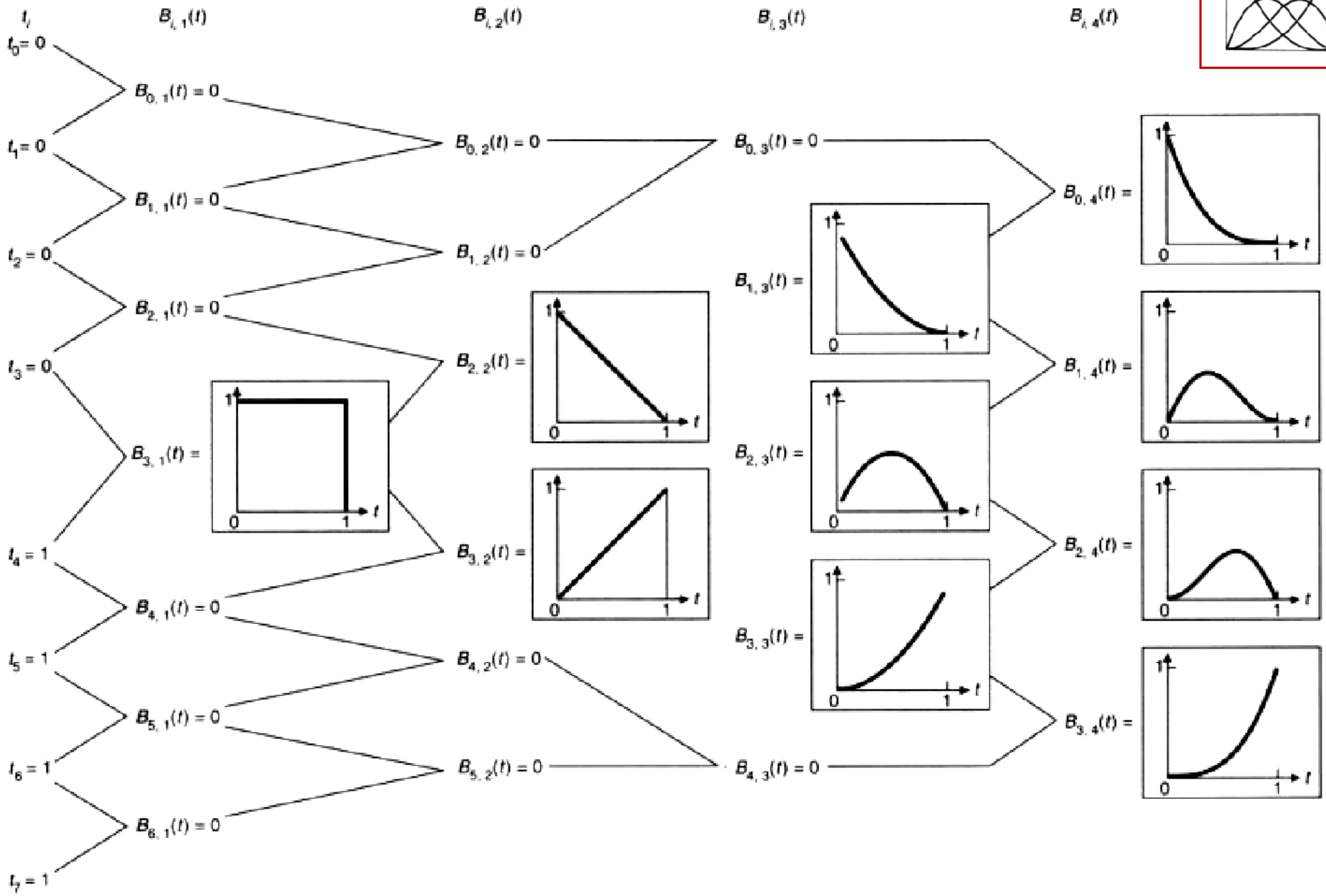
$$B_{i,2}(t) = \frac{t - t_i}{t_{i+1} - t_i} \cdot B_{i,1}(t) + \frac{t_{i+2} - t}{t_{i+2} - t_{i+1}} \cdot B_{i+1,1}(t)$$

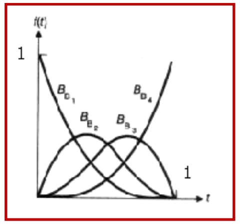
$$B_{i,3}(t) = \frac{t - t_i}{t_{i+2} - t_i} \cdot B_{i,2}(t) + \frac{t_{i+3} - t}{t_{i+3} - t_{i+1}} \cdot B_{i+1,2}(t)$$

$$B_{i,4}(t) = \frac{t - t_i}{t_{i+3} - t_i} \cdot B_{i,3}(t) + \frac{t_{i+4} - t}{t_{i+4} - t_{i+1}} \cdot B_{i+1,3}(t)$$

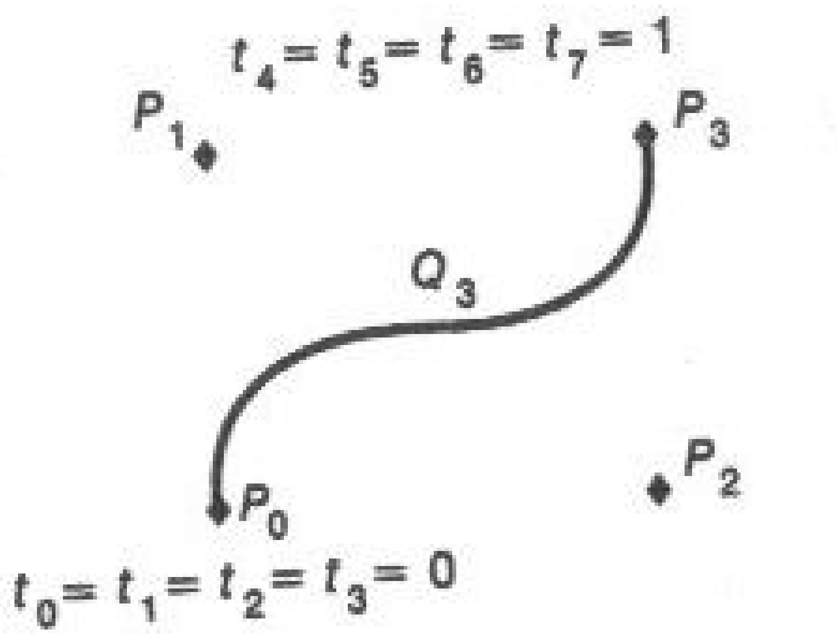
Curbe B-spline uniform nerationale

- Exemplu
  - m=3
  - secventa de valori de nod (0,0,0,0,1,1,1,1)


t_i
$B_{i,1}(t)$
$B_{i,2}(t)$
$B_{i,3}(t)$
$B_{i,4}(t)$
$t_0 = 0$
$t_1 = 0$
$t_2 = 0$
$t_3 = 0$
$t_4 = 1$
$t_5 = 1$
$t_6 = 1$
$t_7 = 1$
$B_{0,1}(t) = 0$
$B_{1,1}(t) = 0$
$B_{2,1}(t) = 0$
$B_{3,1}(t) =$
$B_{4,1}(t) = 0$
$B_{5,1}(t) = 0$
$B_{6,1}(t) = 0$
$B_{0,2}(t) = 0$
$B_{1,2}(t) = 0$
$B_{2,2}(t) =$
$B_{3,2}(t) =$
$B_{4,2}(t) = 0$
$B_{5,2}(t) = 0$
$B_{0,3}(t) = 0$
$B_{1,3}(t) =$
$B_{2,3}(t) =$
$B_{3,3}(t) =$
$B_{4,3}(t) = 0$
$B_{0,4}(t) =$
$B_{1,4}(t) =$
$B_{2,4}(t) =$
$B_{3,4}(t) =$



Curba neuniform nerationala B-spline cu noduri multiple. Secventa de noduri este (0,0,0,0,1,1,1,1). Practic avem un segment de curba Bezier.



Curbe B-spline uniform nerationale

- $\left(\forall 3 \leq i \leq m\right)\left(B_{i,4}\left(t_i\right) = 0\right)$
  - Corolar
    $Q_i(t_i) \in IC\ (P_{i-3}, P_{i-2}, P_{i-1})$
    - Daca $t_i = t_{i+1}$
      atunci $Q_i(t) = P \in IC\ (P_{i-3}, P_{i-2}, P_{i-1}) \cap IC\ (P_{i-2}, P_{i-1}, P_i) = [P_{i-2}\ P_{i-1}]$

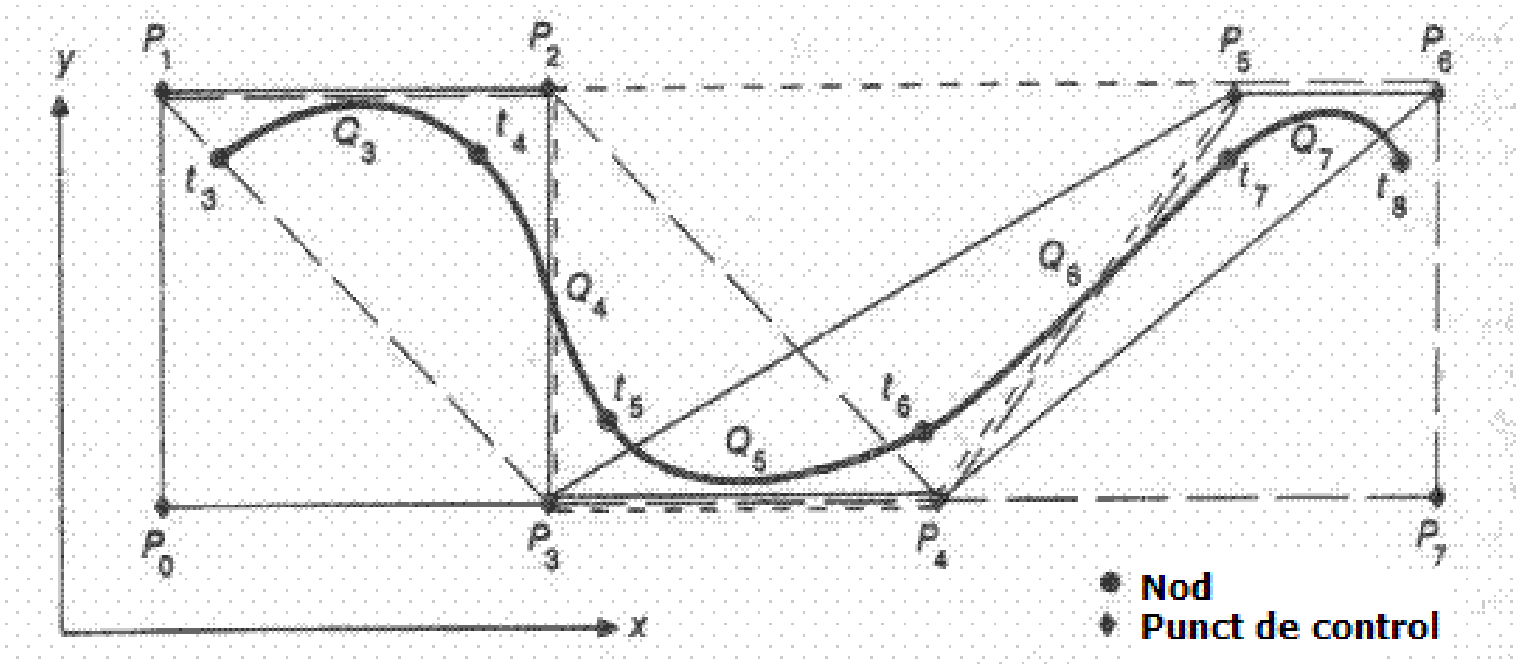
Curbe B-spline uniform nerationale

- $\left(\forall 3 \leq i \leq m\right)\left(B_{i,4}\left(t_i\right) = 0\right)$
  - Corolar
    $Q_i(t_i) \in IC\ (P_{i-3}, P_{i-2}, P_{i-1})$
    - Daca $t_i = t_{i+1} = t_{i+2}$
      atunci $Q_i(t) = Q_{i+1}(t) = P \in$
      $IC\ (P_{i-3}, P_{i-2}, P_{i-1}) \cap$
      $IC\ (P_{i-2}, P_{i-1}, P_i) \cap$
      $IC\ (P_{i-1}, P_i, P_{i+1}) = \{P_{i-1}\}$

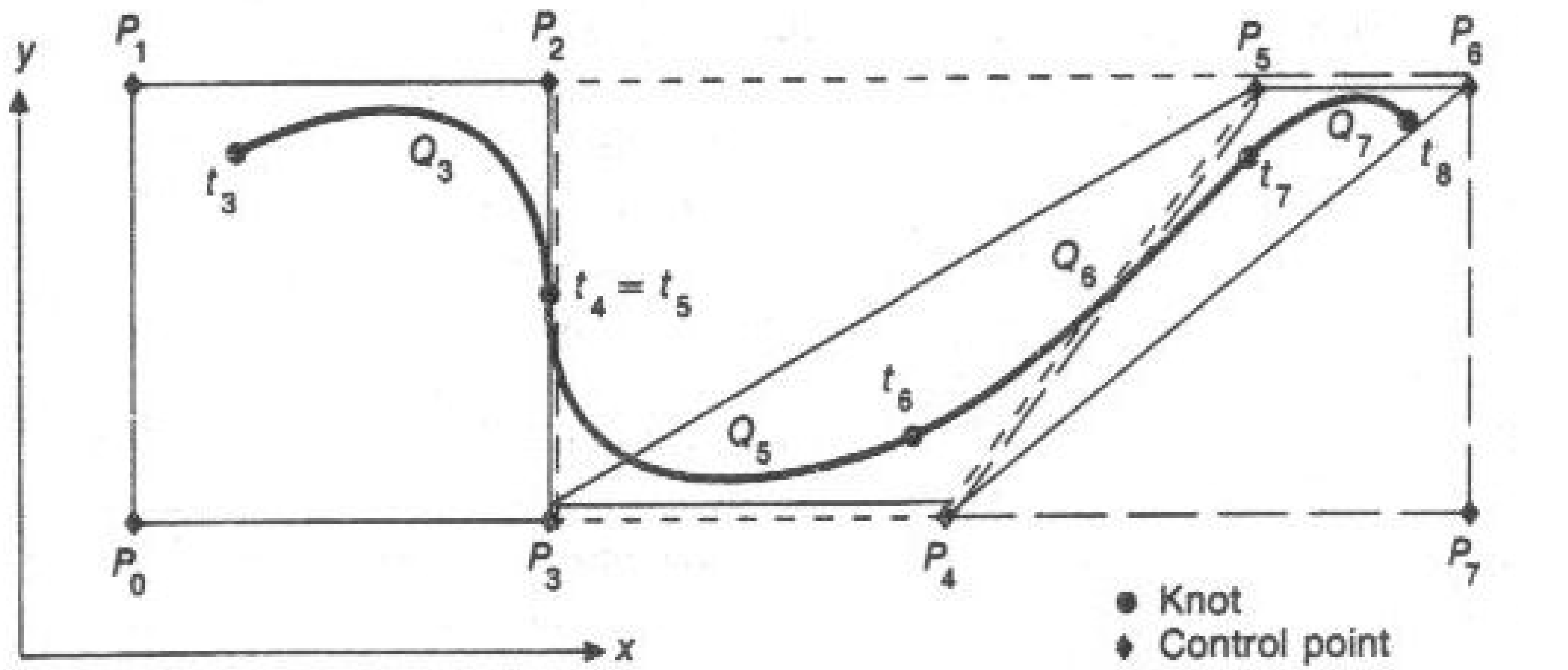
Curbe B-spline uniform nerationale

- $$\left(\forall 3 \le i \le m\right)\left(B_{i,4}\left(t_i\right) = 0\right)$$
  - Corolar
    $Q_i(t_i) \in IC\ (P_{i-3}, P_{i-2}, P_{i-1})$
    - Daca $t_i = t_{i+1} = t_{i+2} = t_{i+3}$ atunci necontinuitate

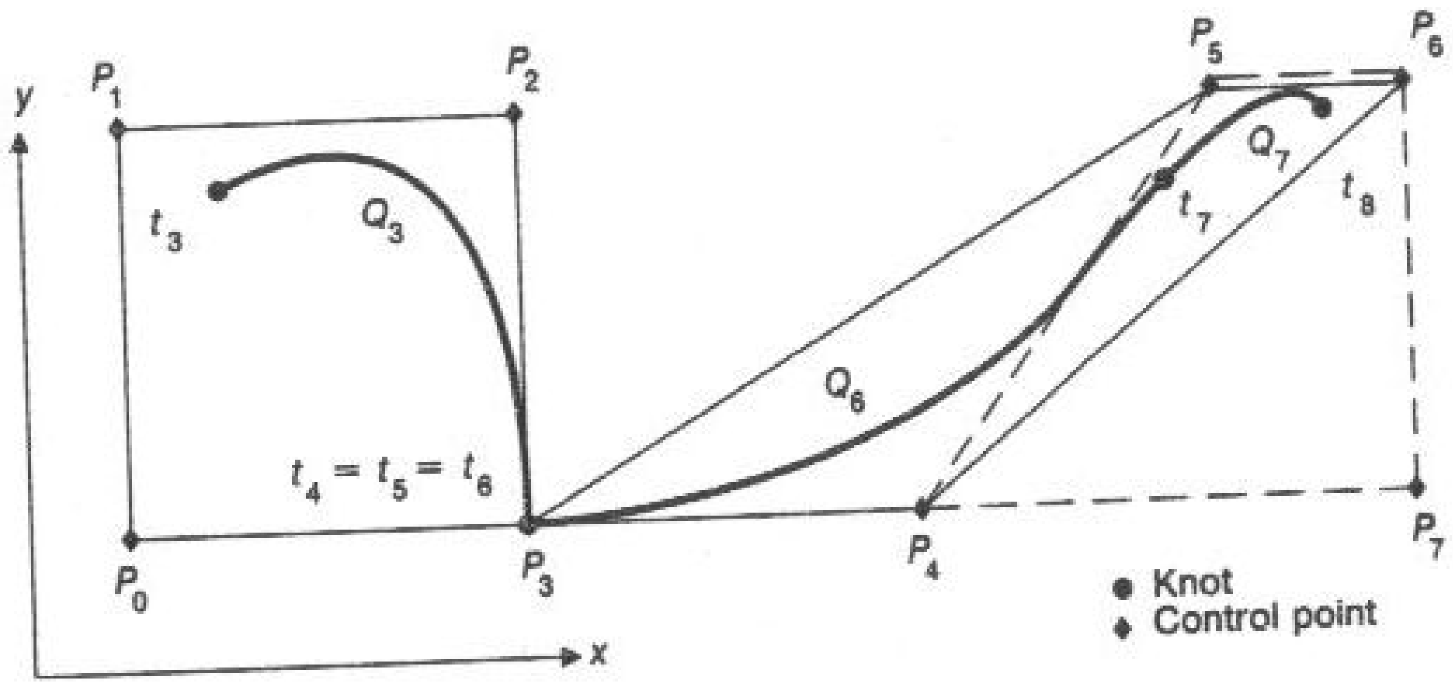
Efectul nodurilor multiple. In acest caz avem secventa de noduri (0,1,2,3,4,5) si deci nu avem noduri multiple. Continuitatea este C^2.



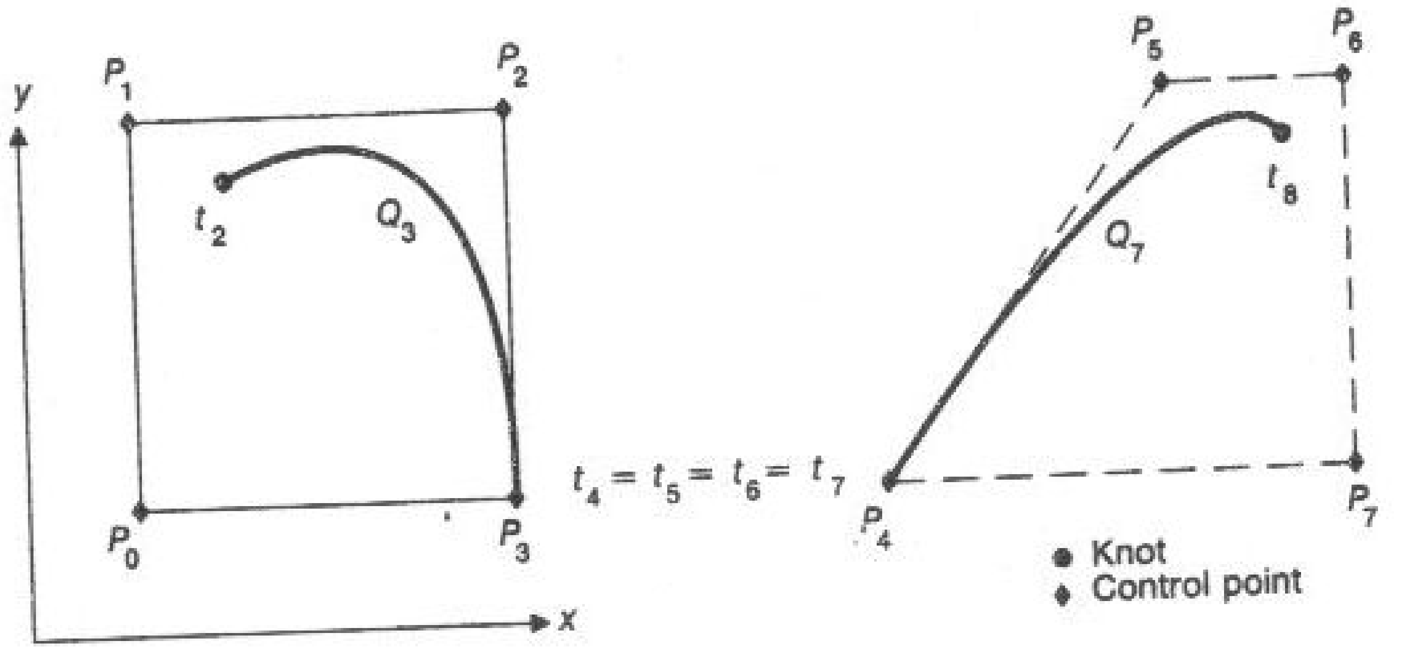
Efectul nodurilor multiple. In acest caz avem secventa de noduri (0,1,1,2,3,4) si deci avem un nod dublu si deci segmentul de curba Q4 este un punct. Punctul de contact dintre Q3 si Q4 se gaseste pe segmentul P2P3 care este intersectia dintre I.C. ce contin Q3 si Q4. Continuitatea in pct. de contact este C^1.



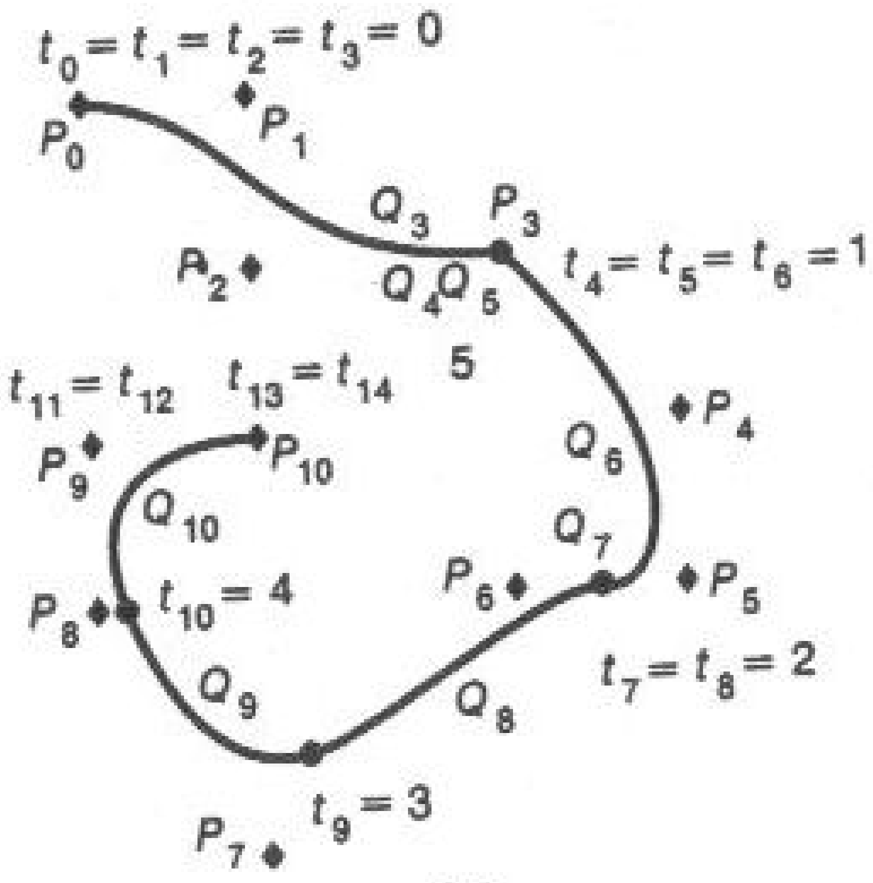
Efectul nodurilor multiple. In acest caz avem secventa de noduri (0,1,1,1,2,3) si deci avem un nod triplu si deci Q4 si Q5 sunt puncte. I.C.(Q3) se intersecteaza cu I.C.(Q6) doar in P3 si deci punctul de contact dintre Q3 si Q6 este P3. Continuitatea in pct. de contact este C^0.



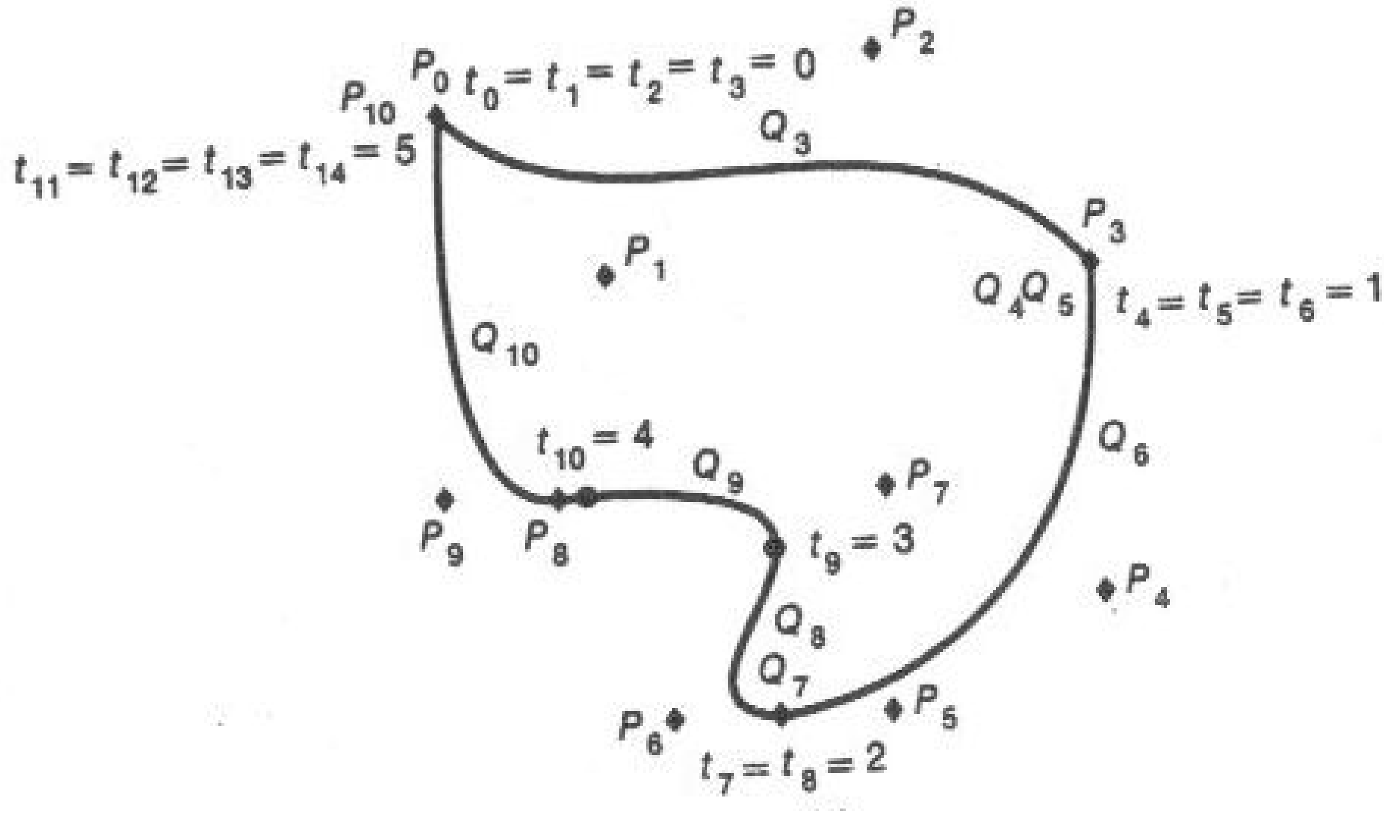
Efectul nodurilor multiple. In acest caz avem secventa de noduri (0,1,1,1,1,2) si deci avem un nod cvadruplu. Curba este discontinua.



Curba neuniform nerationala B-spline cu noduri multiple. Secventa de noduri este (0,0,0,0,1,1,1,2,2,3,4,5,5,5,5).



Curba neuniform nerationala B-spline cu noduri multiple. Secventa de noduri este (0,0,0,0,1,1,1,2,2,3,4,5,5,5,5). Difera de curba precedenta prin punctele de control.



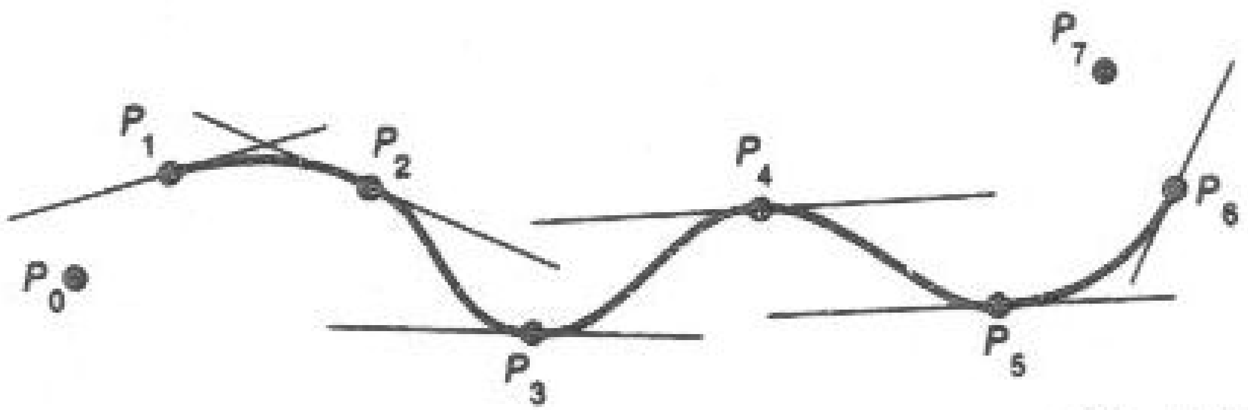
Curbe spline Catmull-Rom

- Puncte de control (aproximare, nu interpolare): m+1

$$\left(P_i\right)_{0 \le i \le m} \quad , \quad m \ge 3$$

  - Interpolare $\left(P_i\right)_{1 \le i \le m-1}$
  - Vectorul tangent in P_i este paralel cu $P_{i-1}P_{i+1}$

Curba spline Catmull-Rom



Curbe spline Catmull-Rom

$$Q_i(t) = T \cdot M_{CR} \cdot G_{Bs_i} =$$

$$\frac{1}{2} \cdot T \cdot \begin{pmatrix} -1 & 3 & -3 & 1 \\ 2 & -5 & 4 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} P_{i-3} \\ P_{i-2} \\ P_{i-1} \\ P_i \end{pmatrix}$$

Suprafete cvadrice

- Suprafete definite implicit printr-o ecuatie de forma f(x,y,z) = 0, unde f este un polinom de grad cel mult 2 in x, y si z

$$f(x_1, x_2, x_3) = \sum_{i,j=1}^{3} Q_{i,j} x_i x_j + \sum_{i=1}^{3} P_i x_i + R$$

$$Q = \begin{pmatrix} a & d & f \\ d & b & e \\ f & e & c \end{pmatrix} \in \mathfrak{M}_{3\times 3}(\Re)$$

$$P = \begin{pmatrix} g & h & i \end{pmatrix} \in \mathfrak{M}_{1\times 3}(\Re)$$

$$R \in \Re$$

Suprafete cvadrice

$$\boldsymbol{f(x, y, z) = ax^2 + by^2 + cz^2 + 2dxy + 2eyz + 2fxz + 2gx + 2hy + 2jz + k}$$

$${}^tP \cdot Q \cdot P = 0$$

$$Q = \begin{pmatrix} a & d & f & g \\ d & b & e & h \\ f & e & c & j \\ g & h & j & k \end{pmatrix} \in \mathfrak{M}_{4\times 4}(\Re)$$

$$P = \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

Suprafete cvadrice

- Transformarea M

$$ {}^{t}P \cdot Q' \cdot P = 0 $$

$$ Q' = {}^{t}\left(M^{-1}\right) \cdot Q \cdot \left(M^{-1}\right) $$