Numerical Solution of an Initial Value Problem using Euler's Method

Assignment: Step-by-Step Computation and Analysis

Problem Statement

Given IVP

$$\frac{dy}{dt} = \cos(t) - y, \qquad y(0) = 1, \qquad 0 \leq t \leq 5$$

Our goal is to approximate the solution to this initial value problem (IVP) using the classical Euler method.

Step Size and Discretization

We partition the interval $[0, 5]$ into $n = 20$ equal subintervals. The step size is:

$$h = \frac{5 - 0}{20} = 0.25$$

Let $t_k = kh$ for $k = 0, 1, \ldots, 20$.

Euler's Method Formula

Euler's method updates the solution iteratively as follows:

$$y_{k+1} = y_k + h \cdot f(t_k, y_k)$$

where $f(t, y) = \cos(t) - y$.

Initialization

$$t_0 = 0$$
$$y_0 = 1$$

Iterative Computation

At each step, we compute:

$$y_{k+1} = y_k + 0.25\left[\cos(t_k) - y_k\right]$$

Below, we show the calculations for the first few steps for clarity:

$$y_1 = 1 + 0.25\left[\cos(0) - 1\right] = 1 + 0.25(1 - 1) = 1.0000$$

$$\begin{aligned} y_2 &= y_1 + 0.25\left[\cos(0.25) - y_1\right] \\ &= 1.0000 + 0.25(0.9689 - 1.0000) = 0.9922 \end{aligned}$$

$$\begin{aligned} y_3 &= y_2 + 0.25\left[\cos(0.5) - y_2\right] \\ &= 0.9922 + 0.25(0.8776 - 0.9922) = 0.9636 \end{aligned}$$

The process continues in this manner until $t_{20} = 5$.

Summary Table of Results

Step (k)	t_k	y_k
0	0.00	1.0000
1	0.25	1.0000
2	0.50	0.9922
3	0.75	0.9636
4	1.00	0.9056
5	1.25	0.8167
6	1.50	0.6918
7	1.75	0.5195
8	2.00	0.3221
9	2.25	0.1151
10	2.50	-0.0495
11	2.75	-0.2227
12	3.00	-0.3875
13	3.25	-0.5298
14	3.50	-0.6367
15	3.75	-0.6967
16	4.00	-0.7142
17	4.25	-0.7005
18	4.50	-0.6275
19	4.75	-0.5234
20	5.00	**-0.3913**

Conclusion and Remarks

Final Approximation

$$y(5) \approx -0.3913$$

The Euler method provides a straightforward yet powerful approach to numerically solving ordinary differential equations. While the accuracy depends on the step size h, this method is especially useful for gaining quick insights into the behavior of solutions.

Note: All cosine values are rounded to four decimal places for clarity. Calculations can be further refined using more decimal places or computational tools.