Analytic Solution

Given that,

$$\frac{dy}{dt} = \cos t - y, \quad y(0) = 1$$

Or, It can be written as

$$\frac{dy}{dt} + y = \cos t, \quad y(0) = 1 \tag{1}$$

It is a first order linear differential equation with integrating factor (I.F.):

$$\text{I.F.} = e^{\int 1\,dt} = e^t$$

Now multiplying both sides of equation (1) by the integrating factor(I.F)= e^t, we get:

$$e^t \frac{dy}{dt} + e^t y = e^t \cos t$$

$$\Rightarrow \frac{d}{dt}\left(e^t y\right) = e^t \cos t$$

Integrating with respect to t, we get:

$$e^t y = \int e^t \cos t\,dt$$

$$\Rightarrow e^t y = \cos t \cdot e^t + \int \sin t \cdot e^t dt$$

$$\Rightarrow e^t y = \cos t \cdot e^t + \sin t \int e^t dt - \int \left(\frac{d}{dt}(\sin t) \cdot e^t\right) dt, \text{ Using Integration by Parts formula}$$

$$\Rightarrow e^t y = \cos t \cdot e^t + \sin t \cdot e^t - \int \cos t \cdot e^t dt$$

$$\Rightarrow 2e^t y = \cos t \cdot e^t + \sin t \cdot e^t + C$$

$$\Rightarrow y = \frac{1}{2}(\cos t + \sin t) + \frac{1}{2}e^{-t}C, \text{ where C is a arbitrary constant.}$$

Since, $y(0) = 1$ Therefore,

$$y(0) = \frac{1}{2}(cos 0 + sin 0) + \frac{1}{2}e^0 C$$

$$\Rightarrow 1 = \frac{1}{2}(1+0) + \frac{1}{2}C$$

$$\Rightarrow C = 1$$

Therefore the solution is

$$y = \frac{1}{2}\left(\cos t + \sin t\right) + \frac{1}{2}e^{-t}$$