Chapter 8

Limit Theorems

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MATH 241

Outline

The central limit theorem

Let $X_1, X_2, ...$ be a sequence of independent and identically distributed random variables, each having mean μ and variance σ^2. Then the distribution of

$$\frac{X_1 + X_2 + ... + X_n - n\mu}{\sigma\sqrt{n}}$$

tends to the standard normal as $n \to \infty$.

In words, if the random variables have a finite mean μ and a finite variance σ^2, then the distribution of the sum of the first n of them is, for large n, approximately that of a normal random variable with mean $n\mu$ and variance $n\sigma^2$.

Applications of the CLT

Question

A person has 100 light bulbs whose lifetimes are independent exponentials with mean 5 hours. If the bulbs are used one at a time, with a failed bulb being replaced immediately by a new one. Suppose that it takes a random time, uniformly distributed over $(0, .5)$, to replace a failed bulb. Approximate the probability that all bulbs have failed by time 550. Note that if $X \sim \text{Exponential}(\lambda)$, then $E[X] = \frac{1}{\lambda}$ and $Var(X) = \frac{1}{\lambda^2}$.

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- Thus $\sum_{i=1}^{100} X_i + \sum_{i=1}^{99} R_i$ has mean $100 \times 5 + 99 \times .25 = 524.75$ and variance $2500 + 99/48 = 2502$
- Then the desired probability is approximately equal to

$$P\{N(0,1) \leq \frac{550 - 524.75}{\sqrt{2502}}\} = P\{N(0,1) \leq .505\} = .693$$