Chapter 6 part 1

Jointly Distributed Random Variables

Jingchen (Monika) Hu

Vassar College

MATH 241

Outline

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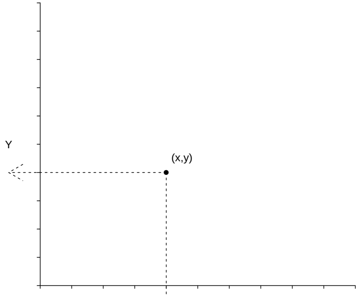
Joint cdf

Definition

We have a pair of random variables (either discrete or continuous) X *and* Y*. The* *joint cumulative probability distribution function* *of* X *and* Y *is defined by*

$$F_{X,Y}(x, y) = P[X \leq x, Y \leq y]$$
$$= P[(X, Y) \text{ lies south-west of the point } (x, y)]$$



Properties of joint cdf

- For one random variable: marginal cdf

$$F_X(x) = F_{X,Y}(x, \infty)$$

$$F_X(x) = P(X \leq x) = P(X \leq x, Y \leq \infty) = F_{X,Y}(x, \infty)$$

$$F_Y(y) = F_{X,Y}(\infty, y)$$

$$F_Y(y) = P(Y \leq y) = P(X \leq \infty, Y \leq y) = F_{X,Y}(\infty, y)$$

- Joint probabilities

$$P(X > x, Y > y) = 1 - F_X(x) - F_Y(y) + F_{X,Y}(x, y)$$

Question

Use joint cdf $F(x, y)$ to represent $P(x_1 < X \leq x_2, y_1 < Y \leq y_2)$.

a. $F(x_2, y_2) + F(x_1, y_1) - F(x_1, y_2) - F(x_2, y_1)$
b. $F(x_2, y_2) - F(x_1, y_1) - F(x_1, y_2) - F(x_2, y_1)$
c. $F(x_2, y_2) - F(x_1, y_1)$
d. none of the above

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c. $F(x_2, y_2) - F(x_1, y_1)$
d. none of the above

Marginal Distributions

Note that the column and row sums are the distributions of B and W respectively.

$$P(B = b) = P(B = b, W = 0) + P(B = b, W = 1) + P(B = b, W = 2)$$
$$P(W = w) = P(B = 0, W = w) + P(B = 1, W = w) + P(B = 2, W = w)$$

These are the marginal distributions of B and W. In general,

$$P(X = x) = \sum_y P(X = x, Y = y) = \sum_y P(X = x \mid Y = y) P(Y = y)$$

Conditional Distribution

Conditional distributions are defined as we have seen previously with

$$P(X = x \mid Y = y) = \frac{P(X = x, Y = y)}{P(Y = y)} = \frac{\text{joint pmf}}{\text{marginal pmf}}$$

Question

Draw two socks at random, without replacement, from a drawer full of twelve colored socks: 6 black, 4 white, 2 purple. Let B be the number of Black socks, W the number of White socks drawn. Find the pmf for white socks given no black socks were drawn.

		W = 0	W = 1	W = 2	
B	0	$\frac{1}{66}$	$\frac{8}{66}$	$\frac{6}{66}$	$\frac{15}{66}$
	1	$\frac{12}{66}$	$\frac{24}{66}$	0	$\frac{36}{66}$
	2	$\frac{15}{66}$	0	0	$\frac{15}{66}$
		$\frac{28}{66}$	$\frac{32}{66}$	$\frac{6}{66}$	$\frac{66}{66}$

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		$\frac{28}{66}$	$\frac{32}{66}$	$\frac{6}{66}$	$\frac{66}{66}$

$$
\begin{aligned}
& P(W = w \mid B = 0) \\
= & \frac{P(W = w, B = 0)}{P(B = 0)} \\
= & \begin{cases} \frac{1}{66} \Big/ \frac{15}{66} = \frac{1}{15} & \text{if } W = 0 \\ \frac{8}{66} \Big/ \frac{15}{66} = \frac{8}{15} & \text{if } W = 1 \\ \frac{6}{66} \Big/ \frac{15}{66} = \frac{6}{15} & \text{if } W = 2 \end{cases}
\end{aligned}
$$

Joint distribution of two continuous random variables

Definition

Random variables X and Y are jointly continuous if there exists a function $f(x, y)$ such that

1. *Non-negative $f(x, y) \geq 0$, for any $x, y \in \mathbb{R}$, and*
2. $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)\, dx\, dy = 1.$

$f_{X,Y}(x, y)$ is called the joint probability density function of X and Y.

- For any set $C \subset \mathbb{R}^2$,

$$P[(X, Y) \in C] = \iint_{(x,y)\in C} f(x, y)\, dx\, dy$$

- Connection between joint pdf and joint cdf

$$F(a, b) = P(X \leq a, Y \leq b) = \int_{-\infty}^{b} \int_{-\infty}^{a} f(x, y)\, dx\, dy$$

$$f(x, y) = \frac{\partial^2}{\partial x \partial y} F(x, y)$$

Marginal pdfs

Marginal probability density functions are defined in terms of "integrating out" one of the random variables.

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\ dy$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\ dx$$

Question

Which of the following can be obtained if the joint pdf $f_{X,Y}(x, y)$ is known?

a. Joint cdf $F_{X,Y}(x, y)$
b. Marginal cdfs $F_X(x), F_Y(y)$.
c. Expected values $E[X], E[Y]$.
d. all above

Question

Which of the following can be obtained if the joint pdf $f_{X,Y}(x, y)$ is known?

a. Joint cdf $F_{X,Y}(x, y)$
b. Marginal cdfs $F_X(x), F_Y(y)$.
c. Expected values $E[X], E[Y]$.
d. *all above*

Recap

Joint cdf of two random variables X and Y:

$$F_{X,Y}(x, y) = P[X \leq x, Y \leq y], -\infty < x, y < \infty$$

- Probability of (X, Y) in a rectangle

$$P(x_1 < X \leq x_2, y_1 < Y \leq y_2)$$

$$= F(x_2, y_2) + F(x_1, y_1) - F(x_1, y_2) - F(x_2, y_1)$$

- Marginal cdfs

$$F_X(x) = F_{X,Y}(x, \infty), \quad F_Y(y) = F_{X,Y}(\infty, y)$$

Joint distribution of two discrete random variables

- Joint pmf

$$p_{X,Y}(x,y) = P(X = x, Y = y)$$

- Marginal pmfs

$$p_X(x) = \sum_{y:p(x,y)>0} p_{X,Y}(x,y), \quad p_Y(y) = \sum_{x:p(x,y)>0} p_{X,Y}(x,y)$$

Joint distribution of two continuous random variables

- Joint pdf
  - Non-negative $f_{X,Y}(x,y) \geq 0$, for any $x, y \in \mathbb{R}$
  - $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x,y)\; dx\; dy = 1$
  - For any set $C \subset \mathbb{R}^2$,

$$P[(X,Y) \in C] = \iint_{(x,y)\in C} f_{X,Y}(x,y)\; dx\; dy$$

- Marginal pdfs

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\; dy, \quad f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\; dx$$

Question

Let X have a $\text{Bin}(n,p)$ distribution. What's the pmf of $Y = 2X$?

a. $f_Y(y) = \binom{2n}{y}(2p)^y(1-2p)^{2n-y}$ for any $y \in \{0, 2, 4, \ldots, 2n\}$
b. $f_Y(y) = \binom{2n}{y}p^y(1-p)^{2n-y}$ for any $y \in \{0, 1, 2, \ldots, 2n\}$
c. $f_Y(y) = \binom{n}{y/2}p^{\frac{y}{2}}(1-p)^{n-\frac{y}{2}}$ for any $y \in \{0, 2, 4, \ldots, 2n\}$
d. $f_Y(y) = \frac{1}{2}\binom{n}{y/2}p^{\frac{y}{2}}(1-p)^{n-\frac{y}{2}}$ for any $y \in \{0, 2, 4, \ldots, 2n\}$

Question

Let X have a $\text{Bin}(n, p)$ distribution. What's the pmf of $Y = 2X$?

1. $f_Y(y) = \binom{2n}{y}(2p)^y(1-2p)^{2n-y}$ for any $y \in \{0, 2, 4, \ldots, 2n\}$
2. $f_Y(y) = \binom{2n}{y}p^y(1-p)^{2n-y}$ for any $y \in \{0, 1, 2, \ldots, 2n\}$
3. $f_Y(y) = \binom{n}{y/2}p^{\frac{y}{2}}(1-p)^{n-\frac{y}{2}}$ *for any* $y \in \{0, 2, 4, \ldots, 2n\}$
4. $f_Y(y) = \frac{1}{2}\binom{n}{y/2}p^{\frac{y}{2}}(1-p)^{n-\frac{y}{2}}$ for any $y \in \{0, 2, 4, \ldots, 2n\}$

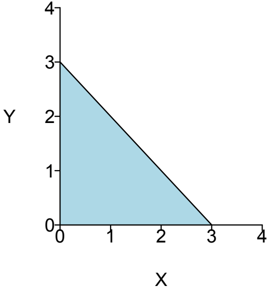
$$f_Y(y) = P(Y = y) = P\left(X = \frac{y}{2}\right) = f_X\left(\frac{y}{2}\right)$$

Question

Let X and Y have the following joint pdf

$$f(x,y) = \begin{cases} \frac{2}{9} & \text{for } x \geq 0, y \geq 0 \text{ and } x+y \leq 3 \\ 0 & \text{otherwise} \end{cases}$$

Find the marginal pdf of Y.



$$f_Y(y) = \begin{cases} \end{cases}$$

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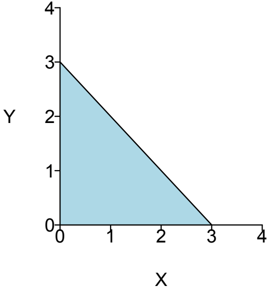


$$f_Y(y) = \begin{cases} \frac{2}{9}(3 - y) & \text{for } y \in [0,3] \\ 0 & \text{otherwise} \end{cases}$$

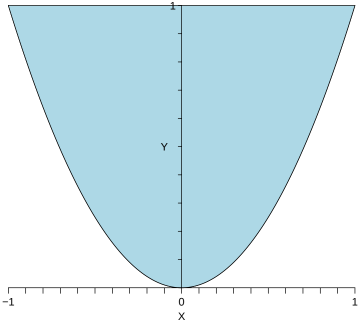
Question

Let $f(x, y) = cx^2 y$ for $x^2 \leq y \leq 1$. Find:

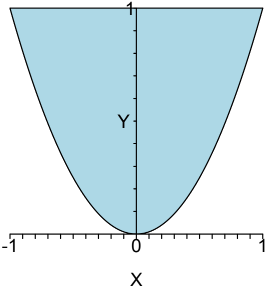
(a) c

(b) $P[X \geq Y]$

(c) $f_X(x)$ and $f_Y(y)$

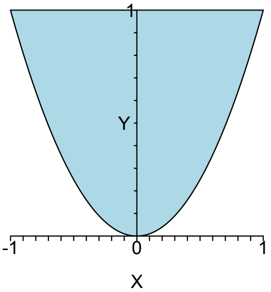


Let $f(x, y) = cx^2y$ for $x^2 \leq y \leq 1$, we can rewrite the bounds as



Let $f(x, y) = cx^2y$ for $x^2 \leq y \leq 1$, we can rewrite the bounds as

$$0 \leq y \leq 1, \quad -\sqrt{y} \leq x \leq \sqrt{y}$$



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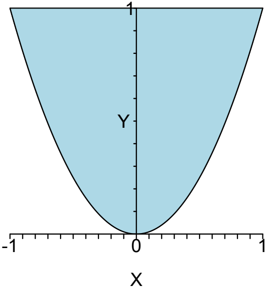
$$1 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y)\ dx\ dy$$



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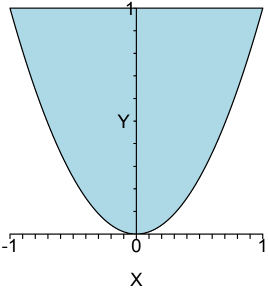

$$\begin{aligned} 1 &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)\ dx\ dy \\ &= \int_{0}^{1} \int_{-\sqrt{y}}^{\sqrt{y}} cx^2 y\ dx\ dy \end{aligned}$$

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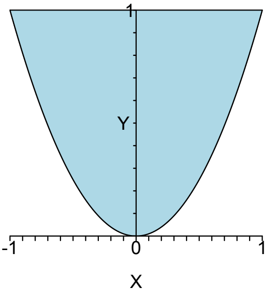

$$\begin{aligned} 1 &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)\ dx\ dy \\ &= \int_0^1 \int_{\color{red}{-\sqrt{y}}}^{\color{red}{\sqrt{y}}} cx^2 y\ \color{red}{dx}\ dy \\ &= \int_0^1 \left(cy \frac{x^3}{3} \Big|_{x=-\sqrt{y}}^{\sqrt{y}} \right) dy \end{aligned}$$

Let $f(x,y) = cx^2y$ for $x^2 \le y \le 1$, we can rewrite the bounds as

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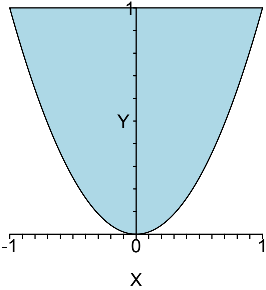

$$\begin{aligned} 1 &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy \\ &= \int_0^1 \int_{-\sqrt{y}}^{\sqrt{y}} cx^2y\,dx\,dy \\ &= \int_0^1 \left(cy\frac{x^3}{3}\Bigg|_{x=-\sqrt{y}}^{\sqrt{y}} \right) dy \\ &= \int_0^1 \left(cy^{5/2}/3 + cy^{5/2}/3 \right) dy \end{aligned}$$

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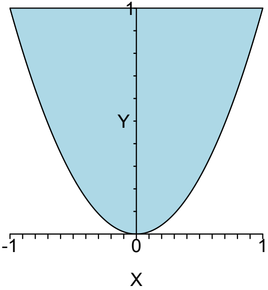

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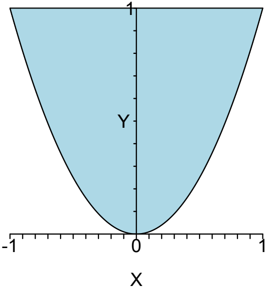
1 &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, dx\, dy \\
&= \int_0^1 \int_{-\sqrt{y}}^{\sqrt{y}} cx^2y\, dx\, dy \\
&= \int_0^1 \left(cy\frac{x^3}{3}\Big|_{x=-\sqrt{y}}^{\sqrt{y}} \right) dy \\
&= \int_0^1 \left(cy^{5/2}/3 + cy^{5/2}/3 \right) dy \\
&= \frac{4cy^{7/2}}{21}\Big|_{y=0}^{1}
\end{aligned}$$

Let $f(x,y) = cx^2y$ for $x^2 \le y \le 1$, we can rewrite the bounds as

$$0 \le y \le 1, \quad -\sqrt{y} \le x \le \sqrt{y}$$



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1 &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, dx\, dy \\
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&= \int_0^1 \left(cy\frac{x^3}{3}\Bigg|_{x=-\sqrt{y}}^{\sqrt{y}} \right) dy \\
&= \int_0^1 \left(cy^{5/2}/3 + cy^{5/2}/3 \right) dy \\
&= \frac{4cy^{7/2}}{21}\Bigg|_{y=0}^{1} = \frac{4}{21}c
\end{aligned}$$

Let $f(x, y) = cx^2 y$ for $x^2 \leq y \leq 1$, we can rewrite the bounds as

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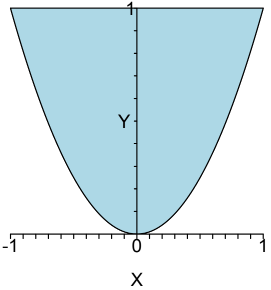

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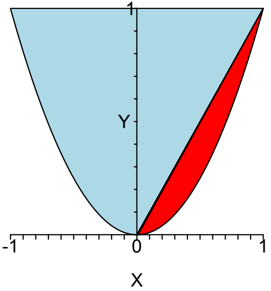
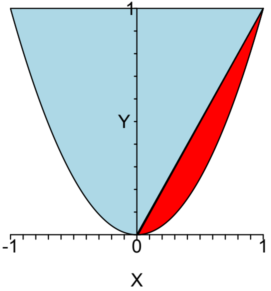
&= \int_0^1 \left(cy \frac{x^3}{3} \Bigg|_{x=-\sqrt{y}}^{\sqrt{y}} \right) dy \\
&= \int_0^1 \left(cy^{5/2}/3 + cy^{5/2}/3 \right) dy \\
&= \frac{4cy^{7/2}}{21} \Bigg|_{y=0}^{1} = \frac{4}{21} c \Longrightarrow c = \frac{21}{4}
\end{aligned}$$

We need to integrate over the region which is indicated in red below, where



We need to integrate over the region which is indicated in red below, where

$$x^2 \leq y \leq 1, \quad x \geq y$$



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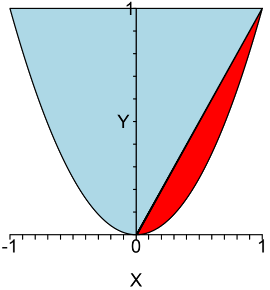
$$x^2 \leq y \leq 1, \quad x \geq y$$

$$P(X \geq Y) = \int_0^1 \int_{x^2}^{x} \frac{21}{4} x^2 y \, dy \, dx$$



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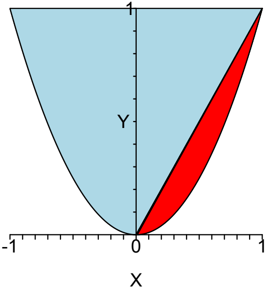

$$P(X \ge Y) = \int_0^1 \int_{x^2}^{x} \frac{21}{4} x^2 y \, dy \, dx$$
$$= \frac{21}{4} \int_0^1 \left(\frac{x^2 y^2}{2} \Big|_{x^2}^{x} \right) dx$$

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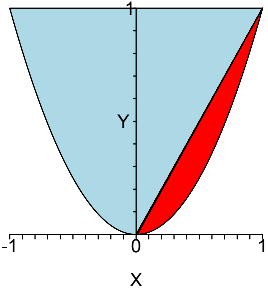
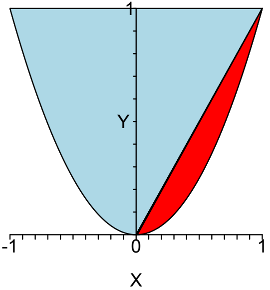


$$\begin{aligned} P(X \geq Y) &= \int_0^1 \int_{x^2}^{x} \frac{21}{4} x^2 y \, dy \, dx \\ &= \frac{21}{4} \int_0^1 \left(\frac{x^2 y^2}{2} \bigg|_{x^2}^{x} \right) dx \\ &= \frac{21}{4} \int_0^1 \left(\frac{x^4}{2} - \frac{x^6}{2} \right) dx \end{aligned}$$

We need to integrate over the region which is indicated in red below, where

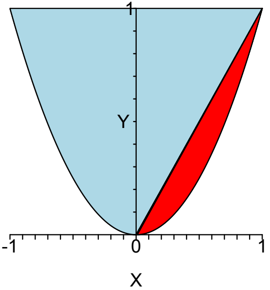
$$x^2 \le y \le 1, \quad x \ge y$$



$$\begin{aligned} P(X \ge Y) &= \int_0^1 \int_{x^2}^{x} \frac{21}{4} x^2 y \, dy \, dx \\ &= \frac{21}{4} \int_0^1 \left(\frac{x^2 y^2}{2} \Big|_{x^2}^{x} \right) dx \\ &= \frac{21}{4} \int_0^1 \left(\frac{x^4}{2} - \frac{x^6}{2} \right) dx \\ &= \frac{21}{4} \left(\frac{x^5}{10} - \frac{x^7}{14} \right) \Big|_0^1 \end{aligned}$$

We need to integrate over the region which is indicated in red below, where

$$x^2 \le y \le 1, \quad x \ge y$$



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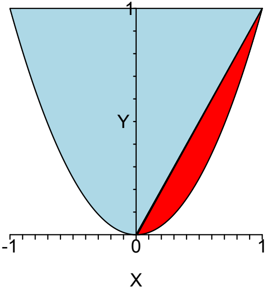
We need to integrate over the region which is indicated in red below, where

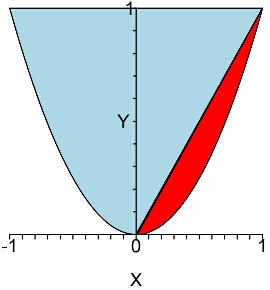
$$x^2 \leq y \leq 1, \quad x \geq y$$



$$\begin{aligned}
P(X \geq Y) &= \int_0^1 \int_{x^2}^{x} \frac{21}{4} x^2 y \, dy \, dx \\
&= \frac{21}{4} \int_0^1 \left(\frac{x^2 y^2}{2} \bigg|_{x^2}^{x} \right) dx \\
&= \frac{21}{4} \int_0^1 \left(\frac{x^4}{2} - \frac{x^6}{2} \right) dx \\
&= \frac{21}{4} \left(\frac{x^5}{10} - \frac{x^7}{14} \right) \bigg|_0^1 \\
&= \frac{21}{4} \left(\frac{1}{10} - \frac{1}{14} \right) \\
&= \frac{21}{4} \left(\frac{2}{70} \right)
\end{aligned}$$

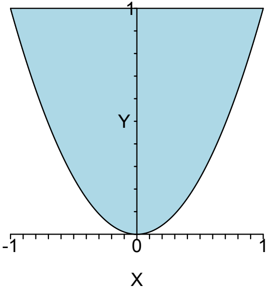
We need to integrate over the region which is indicated in red below, where

$$x^2 \leq y \leq 1, \quad x \geq y$$

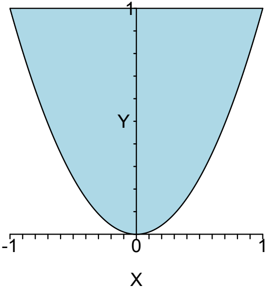


$$\begin{aligned}
P(X \geq Y) &= \int_0^1 \int_{x^2}^{x} \frac{21}{4} x^2 y \, dy \, dx \\
&= \frac{21}{4} \int_0^1 \left(\frac{x^2 y^2}{2} \Big|_{x^2}^{x} \right) dx \\
&= \frac{21}{4} \int_0^1 \left(\frac{x^4}{2} - \frac{x^6}{2} \right) dx \\
&= \frac{21}{4} \left(\frac{x^5}{10} - \frac{x^7}{14} \right) \Big|_0^1 \\
&= \frac{21}{4} \left(\frac{1}{10} - \frac{1}{14} \right) \\
&= \frac{21}{4} \left(\frac{2}{70} \right) = 0.15
\end{aligned}$$

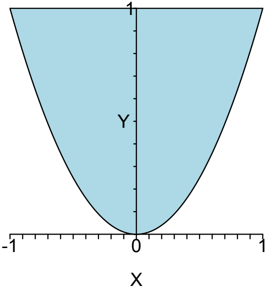
$$f_X(x) = \int_{x^2}^{1} \frac{21}{4} x^2 y \; dy$$

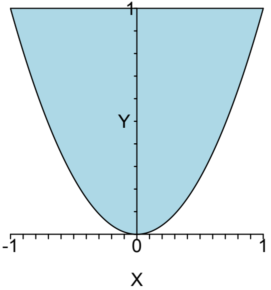


$$f_X(x) = \int_{x^2}^{1} \frac{21}{4} x^2 y \; dy$$
$$= \frac{21}{4} \left(\frac{x^2 y^2}{2} \bigg|_{x^2}^{1} \right)$$



$$
\begin{aligned}
f_X(x) &= \int_{x^2}^{1} \frac{21}{4} x^2 y \, dy \\
&= \frac{21}{4} \left(\frac{x^2 y^2}{2} \Bigg|_{x^2}^{1} \right) \\
&= \frac{21}{8} \left(x^2 - x^6 \right), \text{ for } x \in (-1, 1)
\end{aligned}
$$





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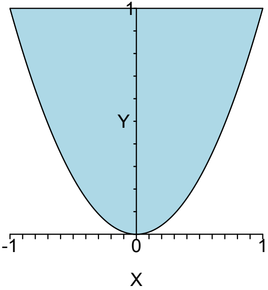

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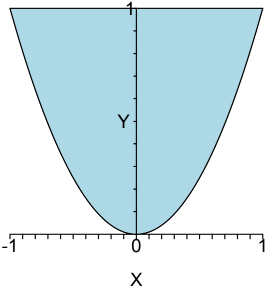

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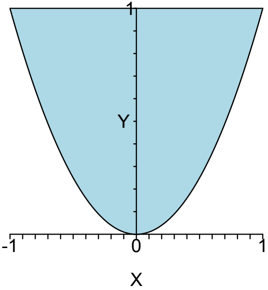

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$$= \frac{7}{2} y^{5/2}, \text{ for } y \in (0, 1)$$

Joint distribution of two continuous random variables

- Joint pdf
  - Non-negative $f_{X,Y}(x,y) \geq 0$, for any $x, y \in \mathbb{R}$
  - $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x,y)\ dx\ dy = 1$
  - For any set $C \subset \mathbb{R}^2$,

$$P[(X,Y) \in C] = \iint_{(x,y)\in C} f_{X,Y}(x,y)\ dx\ dy$$

- Between joint cdf and joint pdf

$$F_{X,Y}(a,b) = \int_{-\infty}^{b} \int_{-\infty}^{a} f_{X,Y}(x,y)\ dx\ dy$$

$$f_{X,Y}(x,y) = \frac{\partial^2}{\partial x \partial y} F_{X,Y}(x,y)$$

- Marginal pdfs

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\ dy, \quad f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\ dx$$

Outline

Independent random variables

Definition

Random variables X and Y are independent if any real sets $A, B \subset \mathbb{R}$,

$$P(X \in A, Y \in B) = P(X \in A)P(Y \in B)$$

Random variables X and Y are independent **if and only if**

- Cdf: for any $x, y \in \mathbb{R}$

$$F_{X,Y}(x, y) = F_X(x)F_Y(y)$$

- If both are discrete, pmf: for any $x, y \in \mathbb{R}$

$$p_{X,Y}(x, y) = p_X(x)p_Y(y)$$

- If both are continuous, pdf: for any $x, y \in \mathbb{R}$

$$f_{X,Y}(x, y) = f_X(x)f_Y(y)$$

Independent random variables

The continuous (discrete) random variables X and Y are independent **if and only if** their joint probability density (mass) function can be expressed as

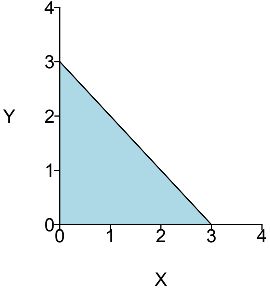
$$f_{X,Y}(x, y) = g(x)h(y), \quad -\infty < x, y < \infty$$

Question

Let X and Y be drawn uniformly from the triangle below, i.e., their joint pdf is

$$f(x,y) = \begin{cases} \frac{2}{9} & \text{for } x \geq 0, y \geq 0 \text{ and } x + y \leq 3 \\ 0 & \text{otherwise} \end{cases}$$

Are they independent?



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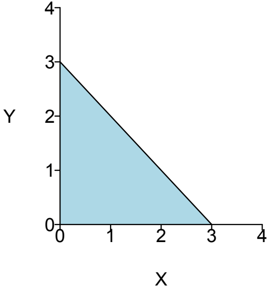

Denote indicator function,

$$I(x,y) = \begin{cases} 1 & \text{for } x \geq 0, y \geq 0 \text{ and } x+y \leq 3 \\ 0 & \text{otherwise} \end{cases}$$

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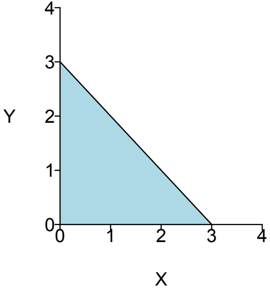

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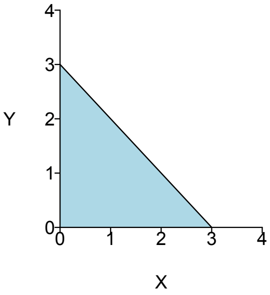

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So NOT independent.

More than two random variables

☞ Random variables $X_1, X_2, \ldots, X_n$ are *independent* if any real sets $A_1, A_2, \ldots, A_n \subset \mathbb{R}$,

$$P(X_1 \in A_1, \ldots, X_n \in A_n) = P(X_1 \in A_1) \cdots P(X_n \in A_n)$$

Random variables $X_1, X_2, \ldots, X_n$ are independent **if and only if**

- Cdf: for any $x_1, x_2, \ldots, x_n \in \mathbb{R}$

$$F(x_1, \ldots, x_n) = F_{X_1}(x_1) \cdots F_{X_n}(x_n)$$

- If both are discrete, pmf: for any $x_1, x_2, \ldots, x_n \in \mathbb{R}$

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- If both are continuous, pdf: for any $x_1, x_2, \ldots, x_n \in \mathbb{R}$

$$f(x_1, \ldots, x_n) = f_{X_1}(x_1) \cdots f_{X_n}(x_n)$$