Chapter 1

Combinatorial Analysis

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MATH 241

Outline

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The basic rule of counting

- ☞ Suppose an experiment consists outcome 1 and outcome 2. If there are n_1 possibilities for outcome 1, and n_2 possibilities for outcome 2, then together there are

$$n_1 \times n_2$$

possibilities for the experiment.

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(Team A is different from Team B if at least one player is different.)

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$$\begin{matrix} G_1B_1 & G_2B_1 & G_3B_1 & G_4B_1 & G_5B_1 \\ G_1B_2 & G_2B_2 & G_3B_2 & G_4B_2 & G_5B_2 \end{matrix}$$

$$5 \times 2 = 10$$

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☞ Suppose an experiment consists r different outcomes, with the i-th outcome having n_i possibilities, then together there are

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Example: how many different license plates?

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letter letter letter number number number

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_____	_____	_____	_____	_____	_____
letter	letter	letter	number	number	number

$$26 \times 26 \times 26 \times 10 \times 10 \times 10 = 17,576,000$$

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Permutations

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$$3 \quad\quad \times \quad\quad 2 \quad\quad \times \quad\quad 1 \quad = 3! = 6$$

- ☞ Each of these arrangements is a *permutation*
- ☞ The order matters!
- ☞ Number of permutations of n different objects

$$n \times (n-1) \times \cdots 2 \times 1 = n!$$

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Permutation of r groups of n objects

Number of permutations of the letters in the word "Facebook"?

a. $7!$
b. $8!$
c. $8!/2$

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Notice that there are two "o" in the word "Facebook".

$$\begin{array}{cccccccc} \mathsf{c} & \mathsf{e} & \mathsf{F} & \mathsf{a} & \mathsf{b} & \mathsf{o}_1 & \mathsf{o}_2 & \mathsf{k} \\ \mathsf{c} & \mathsf{e} & \mathsf{F} & \mathsf{a} & \mathsf{b} & \mathsf{o}_2 & \mathsf{o}_1 & \mathsf{k} \end{array}$$

The same words! Different orders between the two "o" do not matter.

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$$\frac{\#(\text{permutations if all different})}{\#(\text{permutations among the 2 "o"})} = \frac{8!}{2!}$$

Permutation of r groups of n objects

☞ Among n objects, if n_1 are alike, n_2 are alike, ..., n_r are alike, then there are

$$\frac{n!}{n_1!n_2!\cdots n_r!}$$

different permutations.

Permutation of selecting r items from n objects

How many permutations of 2 numbers among $\{1, 2, 3, 4, 5, 6\}$ are there?

(a) $6 \times 6 = 36$

(b) $2^6 = 64$

(c) $6 \times 5 = 30$

(d) $6!$

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☞ If we have n items and want to select r of them,

$$\#(\text{permutations}) = n \times (n-1) \times \cdots \times (n-r+1) = \frac{n!}{(n-r)!}$$

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What if the order doesn't matter? e.g. handshakes.

Recap

The basic rule of counting

- r different outcomes; the i-th outcome having n_i possibilities, then the number of possibilities is

$$\prod_{i=1}^{r} n_i$$

Permutations

- Number of permutations of n different objects is $n!$.
- Number of permutations of n objects, if n_1 are alike, n_2 are alike, ..., n_r are alike, is

$$\frac{n!}{n_1! n_2! \cdots n_r!}$$

- Number of permutations of selecting r items from n objects

$$\frac{n!}{(n-r)!}$$

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When order matters, there are $r!$ different orderings of the r items selected.

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  - $0 \leq r \leq n$, otherwise 0

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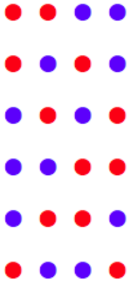
- $0! = 1$

For example, the number of ways to arrange two red marbles and two blue marbles is



$$\binom{4}{2} = \frac{4!}{2! \times 2!} = \frac{24}{2 \times 2} = 6.$$

Example: how many handshakes take place between a group of 6 people if everyone need to shake hands with everyone else?

Same questions: how many combinations of 2 numbers among $\{1, 2, 3, 4, 5, 6\}$ are there?

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{2,3} {2,4} {2,5} {2,6}
{3,4} {3,5} {3,6}
{4,5} {4,6}
{5,6}

$$\binom{6}{2} = \frac{6!}{4!2!} = \frac{6 \times 5}{2 \times 1} = 15$$

Example: Poker hand. A standard poker deck has 52 cards, in four suits (clubs, diamonds, hearts, spades) of thirteen cards each (2, 3, ..., 10, Jack, Queen, King, Ace).

How many poker hands (5 cards) can be dealt from a deck of 52 cards?

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Combinations, choose 5 out of 52 different cards.

$$\binom{52}{5} = \frac{52!}{47!5!} = \frac{52 \times 51 \times 50 \times 49 \times 48}{5 \times 4 \times 3 \times 2 \times 1} = 2,598,960$$

Properties of combinations $\binom{n}{r} = \frac{n!}{(n-r)!r!}$

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1. $$\binom{n}{1} = n \qquad \binom{n}{n} = 1$$

2. $$\binom{n}{r} = \binom{n}{n-r}$$

3. $$\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}, \quad 1 \leq r \leq n$$

Binomial theorem

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

Proof: 1) mathematical induction or 2) combinatorial consideration.

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Total number of distinct plates:

$$N = (a+b)^n = N_0 + N_1 + \cdots + N_n,$$

where N_k is the number of distinct plates that contains exactly k number of letters.

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$$N_k = \binom{n}{k} \times a^k \times b^{n-k}$$

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Multinomial coefficients

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☞ *Multinomial coefficient*: a set of n distinct items is to be divided into r distinct groups of respective sizes $n_1, \ldots, n_r$, where $n_1 + n_2 + \cdots + n_r = n$. Number of possible divisions is

$$\binom{n}{n_1, n_2, \ldots, n_r} \stackrel{\text{def}}{=} \frac{n!}{n_1! n_2! \cdots n_r!}$$

- $$\binom{n}{n_1, n_2, \ldots, n_r} = \binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3}\cdots\binom{n_r}{n_r}$$

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- When $r = 2$, becomes binomial coefficient (choose function)

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Note that $n_1 + n_2 = n$

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- Multinomial Theorem

$$(a_1 + a_2 + \cdots + a_r)^n = \sum_{n_1+\cdots+n_r=n} \binom{n}{n_1, n_2, \ldots, n_r} a_1^{n_1} a_2^{n_2} \cdots a_r^{n_r}$$

- $$\binom{n}{n_1, n_2, \ldots, n_r} = \binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3}\cdots\binom{n_r}{n_r}$$

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- The Binomial theorem is a special case when $r = 2$.