Chapter 6 part 1

Jointly Distributed Random Variables

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MATH 241

Outline

1. Joint distribution

2. Independent random variables

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Joint cdf

Definition

We have a pair of random variables (either discrete or continuous) X *and* Y. *The joint cumulative probability distribution function of* X *and* Y *is defined by*

$$F_{X,Y}(x, y) = P[X \leq x, Y \leq y]$$

Joint cdf

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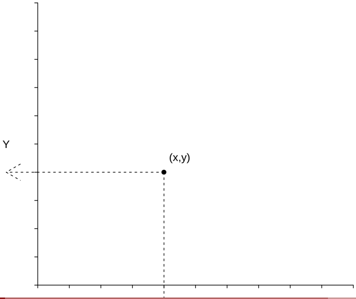
We have a pair of random variables (either discrete or continuous) X and Y. The joint cumulative probability distribution function of X and Y is defined by

$$\begin{aligned} F_{X,Y}(x,y) &= P[X \leq x, Y \leq y] \\ &= P[(X,Y) \text{ lies south-west of the point } (x,y)] \end{aligned}$$



Properties of joint cdf

- For one random variable: marginal cdf

$$F_X(x) = F_{X,Y}(x, \infty)$$

$$F_X(x) = P(X \leq x) = P(X \leq x, Y \leq \infty) = F_{X,Y}(x, \infty)$$

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- Joint probabilities

$$P(X > x, Y > y) = 1 - F_X(x) - F_Y(y) + F_{X,Y}(x, y)$$

Example: two discrete random variables

Draw two socks at random, without replacement, from a drawer full of twelve colored socks:

6 black, 4 white, 2 purple

Let B be the number of Black socks, W the number of White socks drawn.

Then the distributions of B and W are given by:

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P(B=k)			
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Note - $P(B=k) = \frac{\binom{6}{k}\binom{6}{2-k}}{\binom{12}{2}}$ and $P(W=k) = \frac{\binom{4}{k}\binom{8}{2-k}}{\binom{12}{2}}$

Draw two socks at random, without replacement, from a drawer full of twelve colored socks: 6 black, 4 white, 2 purple. Let B be the number of Black socks, W the number of White socks drawn.

The *joint distribution* is given by: $p_{B,W}(b, w) = P(B = b, W = w)$

W

		0	1	2	
	0				
B	1				
	2				

$$P(B = b, W = w) = \begin{cases} \\ \\ \end{cases}$$

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	0	$\frac{1}{66}$			
B	1				
	2				

$$P(B = b, W = w) = \begin{cases} 1/66 & \text{If b=0,w=0} \\ & \\ & \\ & \end{cases}$$

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		0	1	2	
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	2				

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		0	1	2	
	0	$\frac{1}{66}$	$\frac{8}{66}$	$\frac{6}{66}$	
B	1				
	2				

$$P(B=b, W=w) = \begin{cases} 1/66 & \text{If b=0,w=0} \\ 8/66 & \text{If b=0,w=1} \\ 6/66 & \text{If b=0,w=2} \\ & \\ & \\ & \end{cases}$$

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W

		0	1	2	
	0	$\frac{1}{66}$	$\frac{8}{66}$	$\frac{6}{66}$	
B	1	$\frac{12}{66}$			
	2				

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W

B		0	1	2	
	0	$\frac{1}{66}$	$\frac{8}{66}$	$\frac{6}{66}$	
	1	$\frac{12}{66}$	$\frac{24}{66}$		
	2				

$$P(B=b, W=w) = \begin{cases} 1/66 & \text{If b=0,w=0} \\ 8/66 & \text{If b=0,w=1} \\ 6/66 & \text{If b=0,w=2} \\ 12/66 & \text{If b=1,w=0} \\ 24/66 & \text{If b=1,w=1} \end{cases}$$

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		0	1	2	
	0	$\frac{1}{66}$	$\frac{8}{66}$	$\frac{6}{66}$	
B	1	$\frac{12}{66}$	$\frac{24}{66}$	0	
	2				

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		W			
		0	1	2	
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		W: 0	1	2	
B	0	$\frac{1}{66}$	$\frac{8}{66}$	$\frac{6}{66}$	$\frac{15}{66}$
	1	$\frac{12}{66}$	$\frac{24}{66}$	0	$\frac{36}{66}$
	2	$\frac{15}{66}$	0	0	$\frac{15}{66}$
		$\frac{28}{66}$	$\frac{32}{66}$	$\frac{6}{66}$	

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Marginal Distributions

Note that the column and row sums are the distributions of B and W respectively.

$$P(B = b) = P(B = b, W = 0) + P(B = b, W = 1) + P(B = b, W = 2)$$
$$P(W = w) = P(B = 0, W = w) + P(B = 1, W = w) + P(B = 2, W = w)$$

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These are the <u>marginal</u> distributions of B and W. In general,

$$P(X=x) = \sum_y P(X=x, Y=y) = \sum_y P(X=x \mid Y=y) P(Y=y)$$

Conditional Distribution

Conditional distributions are defined as we have seen previously with

$$P(X = x \mid Y = y) = \frac{P(X = x, Y = y)}{P(Y = y)} = \frac{\text{joint pmf}}{\text{marginal pmf}}$$

Joint distribution of two continuous random variables

Definition

Random variables X *and* Y *are* *jointly continuous* *if there exists a function* $f(x, y)$ *such that*

1. *Non-negative* $f(x, y) \geq 0$*, for any* $x, y \in \mathbb{R}$*, and*
2. $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)\, dx\, dy = 1.$

$f_{X,Y}(x, y)$ *is called the* *joint probability density function* *of* X *and* Y*.*

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$$P[(X, Y) \in C] = \iint_{(x,y)\in C} f(x, y)\, dx\, dy$$

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- Connection between joint pdf and joint cdf

$$F(a, b) = P(X \leq a, Y \leq b) = \int_{-\infty}^{b} \int_{-\infty}^{a} f(x, y)\, dx\, dy$$

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$$f(x,y) = \frac{\partial^2}{\partial x \partial y} F(x,y)$$

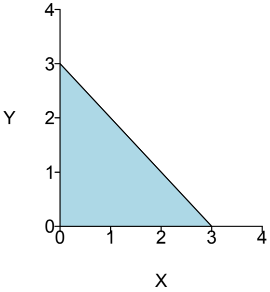
Marginal pdfs

Marginal probability density functions are defined in terms of "integrating out" one of the random variables.

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y) \; dy$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x, y) \; dx$$

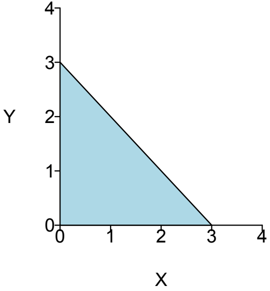
Let X and Y be drawn uniformly from the triangle below. Find the joint pdf $f_{X,Y}(x, y)$.



Let X and Y be drawn uniformly from the triangle below. Find the joint pdf $f_{X,Y}(x, y)$.

Since the joint density is constant, then

$$f(x, y) = \begin{cases} c & \text{for } x \geq 0, y \geq 0 \text{ and } x + y \leq 3 \\ 0 & \text{otherwise} \end{cases}$$



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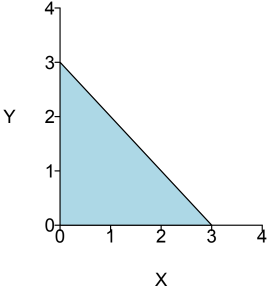

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Because

$$1 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x,y)\; dx\; dy$$

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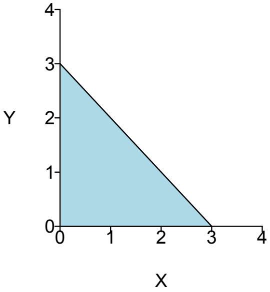

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Because

$$\begin{aligned} 1 &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x,y)\ dx\ dy \\ &= \iint\limits_{x \geq 0, y \geq 0, x+y \leq 3} c\ dx\ dy \end{aligned}$$

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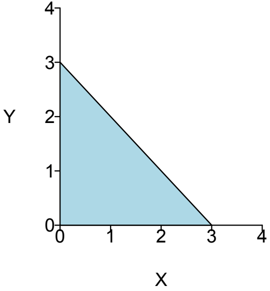

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Because

$$\begin{aligned} 1 &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x,y)\ dx\ dy \\ &= \iint\limits_{x\geq 0, y \geq 0, x+y\leq 3} c\ dx\ dy \\ &= c \times \text{area of the triangle} = c \times \frac{3 \times 3}{2} \end{aligned}$$

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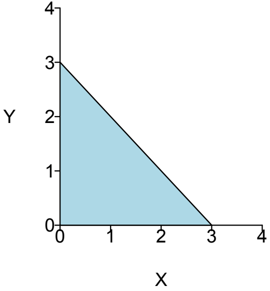

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$$f(x,y) = \begin{cases} c & \text{for } x \geq 0, y \geq 0 \text{ and } x + y \leq 3 \\ 0 & \text{otherwise} \end{cases}$$

Because

$$\begin{aligned} 1 &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\ dx\ dy \\ &= \iint\limits_{x\geq 0, y\geq 0, x+y\leq 3} c\ dx\ dy \\ &= c \times \text{area of the triangle} = c \times \frac{3\times 3}{2} \end{aligned}$$

Therefore, $c = \frac{2}{9}$.

Recap

Joint cdf of two random variables X and Y:

$$F_{X,Y}(x,y) = P[X \leq x, Y \leq y], -\infty < x, y < \infty$$

- Probability of (X,Y) in a rectangle

$$P(x_1 < X \leq x_2, y_1 < Y \leq y_2)$$

$$= F(x_2,y_2) + F(x_1,y_1) - F(x_1,y_2) - F(x_2,y_1)$$

- Marginal cdfs

$$F_X(x) = F_{X,Y}(x,\infty), \quad F_Y(y) = F_{X,Y}(\infty,y)$$

Joint distribution of two discrete random variables

- Joint pmf

$$p_{X,Y}(x, y) = P(X = x, Y = y)$$

- Marginal pmfs

$$p_X(x) = \sum_{y:p(x,y)>0} p_{X,Y}(x, y), \quad p_Y(y) = \sum_{x:p(x,y)>0} p_{X,Y}(x, y)$$

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Joint distribution of two continuous random variables

- Joint pdf
  - Non-negative $f_{X,Y}(x,y) \geq 0$, for any $x, y \in \mathbb{R}$
  - $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x,y)\ dx\ dy = 1$
  - For any set $C \subset \mathbb{R}^2$,

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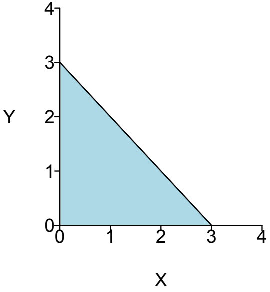
$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\ dy, \quad f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\ dx$$

Let X and Y have the following joint pdf

$$f(x, y) = \begin{cases} \frac{2}{9} & \text{for } x \geq 0, y \geq 0 \text{ and } x + y \leq 3 \\ 0 & \text{otherwise} \end{cases}$$

Find the marginal pdf $f_X(x)$.



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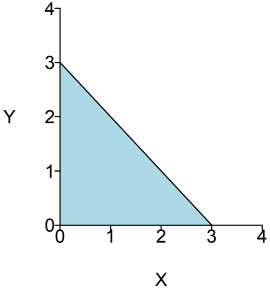

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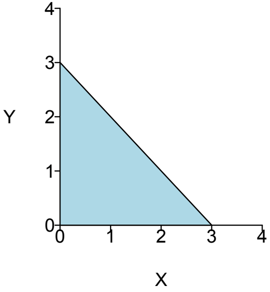

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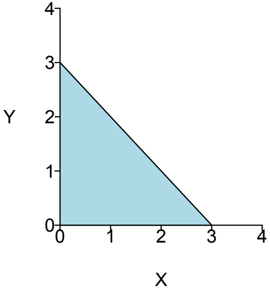

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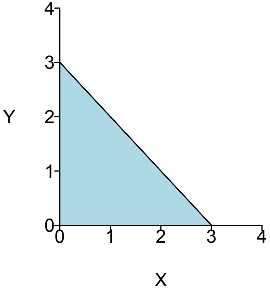

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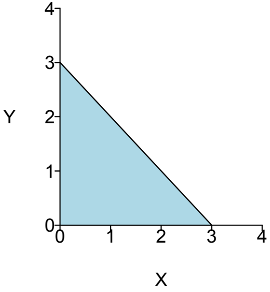

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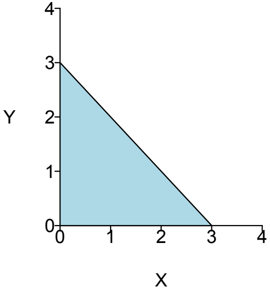

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$$f_X(x)=\begin{cases}\frac{2}{9}(3-x) & \text{for } x\in[0,3]\\ 0 & \text{otherwise}\end{cases}$$

In the previous example, find $P(X < Y)$.

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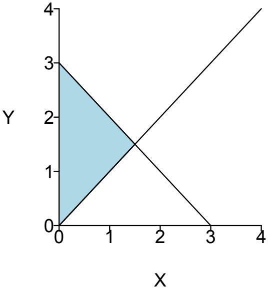

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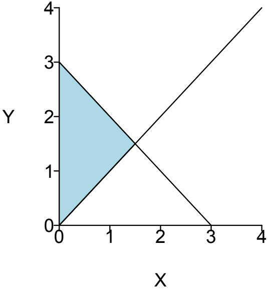
Identify the region
$C = \{(x,y) : x \geq 0, y \geq 0 \text{ and } x + y \leq 3, x < y\}$.



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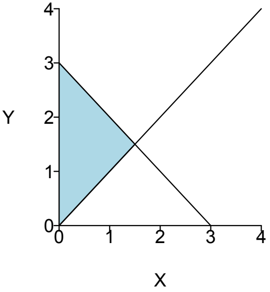

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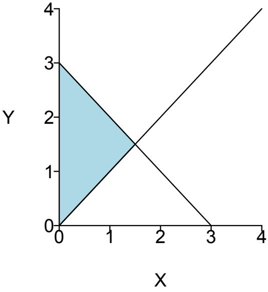

$$P(X < Y) = \iint_{(x,y)\in C} f(x,y)\, dx\, dy$$
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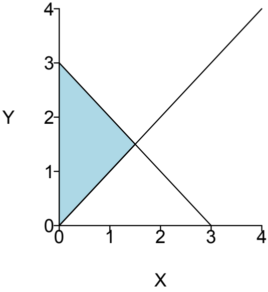

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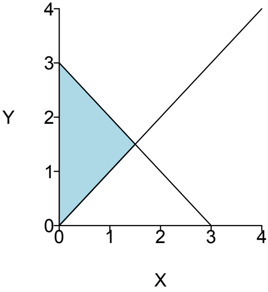

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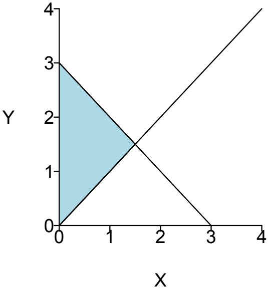

$$\begin{aligned} P(X < Y) &= \iint_{(x,y)\in C} f(x,y)\, dx\, dy \\ &= \int_0^{\frac{3}{2}} \left[\int_x^{3-x} \frac{2}{9}\, dy \right] dx \\ &= \int_0^{\frac{3}{2}} \frac{2}{9}(3 - 2x)\, dx = \frac{2}{9} \times \left[3x - x^2 \Big|_0^{\frac{3}{2}} \right] \\ &= \frac{2}{9}\left(\frac{9}{2} - \frac{9}{4} \right) \end{aligned}$$

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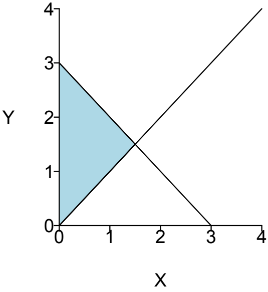

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Joint distribution of two continuous random variables

- Joint pdf
  - Non-negative $f_{X,Y}(x,y) \geq 0$, for any $x, y \in \mathbb{R}$
  - $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x,y)\ dx\ dy = 1$
  - For any set $C \subset \mathbb{R}^2$,

$$P[(X,Y) \in C] = \iint_{(x,y)\in C} f_{X,Y}(x,y)\ dx\ dy$$

- Between joint cdf and joint pdf

$$F_{X,Y}(a,b) = \int_{-\infty}^{b} \int_{-\infty}^{a} f_{X,Y}(x,y)\ dx\ dy$$

$$f_{X,Y}(x,y) = \frac{\partial^2}{\partial x \partial y} F_{X,Y}(x,y)$$

- Marginal pdfs

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\ dy, \quad f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\ dx$$

Outline

1 Joint distribution

2 Independent random variables

Independent random variables

Definition

Random variables X and Y are independent if any real sets $A, B \subset \mathbb{R}$,

$$P(X \in A, Y \in B) = P(X \in A)P(Y \in B)$$

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- If both are continuous, pdf: for any $x, y \in \mathbb{R}$

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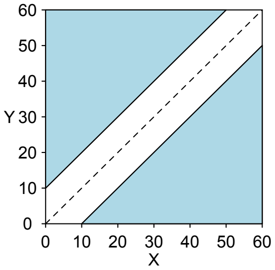
Let random variables X, Y be the time they arrive (uniform between 0 to 60 minutes).



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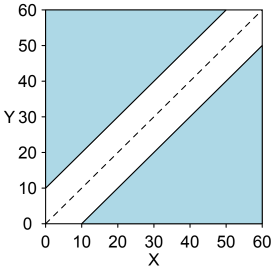
$$P(|X - Y| > 10)$$



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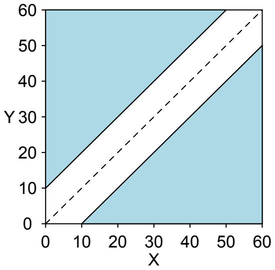

$$
\begin{aligned}
& P(|X - Y| > 10) \\
= & P(Y > X + 10) + P(Y < X - 10)
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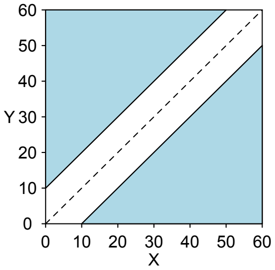

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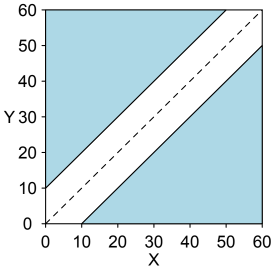

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& P(|X-Y|>10) \\
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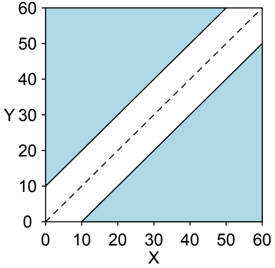

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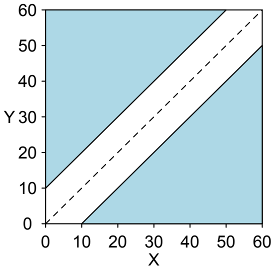

$$
\begin{aligned}
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&= 2P(Y > X+10) \\
&= 2\iint_{y>x+10} f_{X,Y}(x,y)\, dx\, dy \\
&= 2\iint_{y>x+10} f_X(x) f_Y(y)\, dx\, dy \\
&= 2\int_{10}^{60}\int_{0}^{y-10} (1/60)^2\, dx\, dy
\end{aligned}
$$

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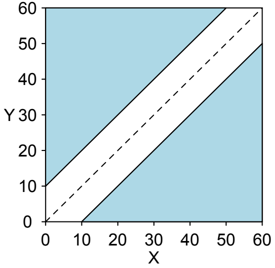

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= & 2\iint_{y>x+10} f_X(x) f_Y(y)\,dx\,dy \\
= & 2\int_{10}^{60}\int_0^{y-10} (1/60)^2\,dx\,dy = 25/36
\end{aligned}
$$

Independent random variables

The continuous (discrete) random variables X and Y are independent **if and only if** their joint probability density (mass) function can be expressed as

$$f_{X,Y}(x, y) = g(x)h(y), \quad -\infty < x, y < \infty$$

More than two random variables

- ☞ Random variables $X_1, X_2, \ldots, X_n$ are *independent* if any real sets $A_1, A_2, \ldots, A_n \subset \mathbb{R}$,

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More than two random variables

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- If both are continuous, pdf: for any $x_1, x_2, \ldots, x_n \in \mathbb{R}$

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