Chapter 7 part 2

Properties of Expectations

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MATH 241

Outline

Moment generating functions

Definition

The moment generating function $M(t)$ of the random variable X is defined for all real values of t by

$$M(t) = E[e^{tX}] = \begin{cases} \sum_x e^{tx} p(x) & \text{if } X \text{ is discrete with pmf } p(x) \\ \int_{-\infty}^{\infty} e^{tx} f(x) dx & \text{if } X \text{ is continuous with pdf } f(x) \end{cases}$$

All the moments of X can be obtained by successively differentiating $M(t)$ and then evaluating the result at $t = 0$

$$M'(t) = \frac{d}{dt} E[e^{tX}] = E[\frac{d}{dt}(e^{tX})] = E[Xe^{tX}], eM'(0) = E[X]$$

$$M''(t) = \frac{d}{dt} E[Xe^{tX}] = E[\frac{d}{dt}(Xe^{tX})] = E[X^2 e^{tX}], M''(0) = E[X^2]$$

In general

$$M^n(t) = E[X^n e^{tX}], M^n(0) = E[X^n], n \geq 1$$

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$$E[X] = M'(0) = \lambda$$

$$Var(X) = E[X^2] - (E[X])^2 = M''(0) - (M'(0))^2 = \lambda^2 + \lambda - \lambda^2 = \lambda$$

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Properties of MGFs

- Property 1: The MGF of the sum of independent random variables equals to the product of the individual MGFs.

$$M_{X+Y}(t) = E[e^{t(X+Y)}] = E[e^{tX}e^{tY}] = E[e^{tX}]E[e^{tY}] = M_X(t)M_Y(t)$$

- Property 2: The MGF uniquely determines the distribution. Refer to the tables of MGF of some discrete and continuous distributions. Textbook pages 339 and 340 for lists of MGFs of distributions.

MGF of sums of independent normal random variables

Question

Show that if X and Y are independent normal random variables with respective parameters (μ_1, σ_1^2) and (μ_2, σ_2^2), then $X + Y$ is normal with mean $\mu_1 + \mu_2$ and variance $\sigma_1^2 + \sigma_2^2$.

For normal (μ, σ^2), the MGF is $e^{\{\frac{\sigma^2 t^2}{2} + \mu t\}}$. Textbook pages 339 and 340 for lists of MGFs of distributions.

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$$
\begin{aligned}
M_{X+Y}(t) &= M_X(t) M_Y(t) \\
&= e^{\{\frac{\sigma_1^2 t^2}{2} + \mu_1 t\}} e^{\{\frac{\sigma_2^2 t^2}{2} + \mu_2 t\}}
\end{aligned}
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This is the MGF of a normal random variable with mean $\mu_1 + \mu_2$ and variance $\sigma_1^2 + \sigma_2^2$. This result follows because the MGF uniquely determins the distribution.

Joint moment generating functions

Definition

For any n random variables $X_1, ..., X_n$, the joint MGF $M(t_1, ..., t_n)$ is defined for all real values of $t_1, ..., t_n$ by

$$M(t_1, ..., t_n) = E[e^{t_1 X_1 + ... t_n X_n}]$$

- The individual MGF can be obtained from $M(t_1, ..., t_n)$ by letting all but one of the t_j's be 0

$$M_{X_i}(t) = E[e^{tX_i}] = M(0, ..., t, 0, ..., 0)$$

- The joint MGF $M(t_1, ..., t_n)$ uniquely determines the joint distribution of $X_1, ..., X_n$
- Then we have the n random variables $X_1, ..., X_n$ are independent **if and only if**

$$M(t_1, ..., t_n) = M_{X_1}(t_1) ... M_{X_n}(t_n)$$