Chapter 8

Limit Theorems

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MATH 241

Outline

The central limit theorem

Let $X_1, X_2, ...$ be a sequence of independent and identically distributed random variables, each having mean μ and variance σ^2. Then the distribution of

$$\frac{X_1 + X_2 + ... + X_n - n\mu}{\sigma\sqrt{n}}$$

tends to the standard normal as $n \to \infty$.

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tends to the standard normal as $n \to \infty$.

In words, if the random variables have a finite mean μ and a finite variance σ^2, then the distribution of the sum of the first n of them is, for large n, approximately that of a normal random variable with mean $n\mu$ and variance $n\sigma^2$.

Applications of the CLT

A person has 100 light bulbs whose lifetimes are independent exponentials with mean 5 hours. If the bulbs are used one at a time, with a failed bulb being replaced immediately by a new one, approximate the probability that there is still a working bulb after 525 hours. Note that if $X \sim \text{Exponential}(\lambda)$, then $E[X] = \frac{1}{\lambda}$ and $Var(X) = \frac{1}{\lambda^2}$.

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$$
\begin{aligned}
P\{\sum_{i=1}^{100} X_i > 525\} \approx P\{N(0,1) > \frac{525 - 500}{\sqrt{100 \times 25}}\} &= P\{N(0,1) > .5\} \\
&= .3085
\end{aligned}
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- Note that because exponential distribution is also continuous, we do not apply the continuity correction $(.5)$; for discrete distributions (e.g. binomial, poisson), the continuity correction is necessary