Chapter 1

Combinatorial Analysis

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MATH 241

Outline

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The generalized rule of counting

☞ Suppose an experiment consists r different outcomes, with the i-th outcome having n_i possibilities, then together there are

$$n_1 \times n_2 \times \cdots \times n_r = \prod_{i=1}^{r} n_i$$

possibilities for the experiment.

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How many different 4-digit pins?

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$$(26 + 10) \times (26 + 10) \times (26 + 10) \times (26 + 10) = 1,679,616$$

Outline

Permutations

Our example: how many different arrangements of the letters a, b, c?

- Each of these arrangements is a *permutation*
- The order matters!
- Number of permutations of n different objects

$$n \times (n-1) \times \cdots 2 \times 1 = n!$$

Permutations of r groups of n objects

- Among n objects, if n_1 are alike, n_2 are alike, ..., n_r are alike, then there are
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different permutations.

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a. $6!/2$
b. $5!/2$
c. $6!/4$
d. $6!$

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Permutation of selecting r items from n objects

☞ If we have n items and want to select r of them,

$$\#(\text{permutations}) = n \times (n-1) \times \cdots \times (n-r+1) = \frac{n!}{(n-r)!}$$

Question

Suppose you have 3 distinctive gifts to give to 8 friends. How many permutations of gift giving strategy do you have?

a. $3!$
b. 3^3
c. $8!$
d. $8 \times 7 \times 6$

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What if the order doesn't matter? e.g. handshakes.

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(b) What if couples must stand together?

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Another way to look at the problem:
For the first position, there are 8 choices. For the next position, only one, because the first person's spouse needs to take that position. The next one has 6, next 1 again, etc.. So the answer is

$$8 \times 1 \times 6 \times 1 \times 4 \times 1 \times 2 \times 1 = 384.$$

Recap

The basic rule of counting

- r different outcomes; the i-th outcome having n_i possibilities, then the number of possibilities is

$$\prod_{i=1}^{r} n_i$$

Permutations

- Number of permutations of n different objects is $n!$.
- Number of permutations of n objects, if n_1 are alike, n_2 are alike, $\ldots$, n_r are alike, is

$$\frac{n!}{n_1! n_2! \cdots n_r!}$$

- Number of permutations of selecting r items from n objects

$$\frac{n!}{(n-r)!}$$

Outline

Combinations: order doesn't matter!

When order matters, there are $r!$ different orderings of the r items selected.

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  - The number $\binom{n}{r}$ is pronounced as n choose r, it is the number of ways to pick r objects from a set of n distinct objects.
  - $0 \le r \le n$, otherwise 0

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- $0 \leq r \leq n$, otherwise 0
- $0! = 1$

Example: Poker hand. A standard poker deck has 52 cards, in four suits (clubs, diamonds, hearts, spades) of thirteen cards each (2, 3, ..., 10, Jack, Queen, King, Ace).

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How many distinct hands of "four of a kind" (four of the five cards are of the same rank)?

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$$\underset{\text{the same rank}}{\binom{13}{1}} \times \underset{\text{suits for the same rank}}{\binom{4}{4}} \times \underset{\text{the rest 1 card}}{\binom{48}{1}}$$

Related to the previous question...

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Related to the previous question...

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Note that the number of possible ways to get "four of a kind" $\binom{13}{1} \times \binom{4}{4} \times \binom{48}{1}$ is smaller than the number of possible ways to get "four of a suit" $\binom{4}{1} \times \binom{13}{4} \times \binom{39}{1}$, that means it is less possible to get "four of a kind". In a game, "four of a kind" wins over "four of a suit" because of its smaller probability.

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$$N_1 = \binom{8}{3} = \frac{8!}{3! \times 5!} = 56$$

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Alternatively: there are $8 \times 6 \times 4$ ways of permuting 3 people where no married couple is contained. However, the order plays a role in this calculation, which we do not want. Therefore, there are $\frac{8\times6\times4}{3!} = 32$ number of choices that the group does not have a couple.

Properties of combinations $\binom{n}{r} = \frac{n!}{(n-r)!r!}$

1. $$\binom{n}{1} = n \qquad \binom{n}{n} = 1$$

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$$\binom{n}{1} = n \qquad \binom{n}{n} = 1$$

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3.
$$\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}, \quad 1 \leq r \leq n$$

Binomial theorem

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

Proof: 1) mathematical induction or 2) combinatorial consideration.

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Think about license plates that are formed by n digits, where each digit can be a letter or a number. $a = 26, b = 10$.
Total number of distinct plates:

$$N = (a+b)^n = N_0 + N_1 + \cdots + N_n,$$

where N_k is the number of distinct plates that contains exactly k number of letters.

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Use the Binomial Theorem to simplify

$$\sum_{k=0}^{n} \binom{n}{k} = \sum_{k=0}^{n} \binom{n}{k} 1^k 1^{n-k} = (1+1)^n = 2^n$$

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Each element can be either in the subset or out of the subset (2 choices); there are n elements; therefore the number of subsets is 2^n (basic rule of counting).

Outline

Multinomial coefficients

- ☞ *Multinomial coefficient*: a set of n distinct items is to be divided into r distinct groups of respective sizes $n_1, \ldots, n_r$, where $n_1 + n_2 + \cdots + n_r = n$. Number of possible divisions is

$$\binom{n}{n_1, n_2, \ldots, n_r} \stackrel{\text{def}}{=} \frac{n!}{n_1! n_2! \cdots n_r!}$$

- $$\binom{n}{n_1, n_2, \ldots, n_r} = \binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3}\cdots\binom{n_r}{n_r}$$

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- When $r = 2$, becomes binomial coefficient (choose function)

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Note that $n_1 + n_2 = n$

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- Multinomial Theorem

$$(a_1 + a_2 + \cdots + a_r)^n = \sum_{n_1+\cdots+n_r=n} \binom{n}{n_1, n_2, \ldots, n_r} a_1^{n_1} a_2^{n_2} \cdots a_r^{n_r}$$

- $$\binom{n}{n_1, n_2, \ldots, n_r} = \binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3}\cdots\binom{n_r}{n_r}$$

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- The Binomial theorem is a special case when $r = 2$.

Outline

Recap

- From n distinct items, number of ways to draw r of them

	without replacement	with replacement
order matters	$n!/(n-r)!$	n^r
order doesn't matter	$\binom{n}{r} = n!/(n-r)!r!$	see Ch1.6*

Recap

Order matters or not?

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- Textbook Example 5b: Ten children are to be divided into an A team and a B team of 5 each. The A team will play in one league and the B team in another. How many different divisions are possible?

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$$\binom{10}{5} = \frac{10!}{5! \times 5!} = 252.$$

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- Textbook Example 5c: In order to play a game of basketball, 10 children at a playground divide themselves into two teams of 5 each. How many different divisions are possible?

$$\frac{\binom{10}{5}}{2!} = \frac{10!}{5! \times 5! \times 2!} = 126.$$

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$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

License plate problem?

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  License plate problem?

- Multinomial coefficient

$$\binom{n}{n_1, n_2, \ldots, n_r} = \frac{n!}{n_1! n_2! \cdots n_r!}$$

  Connection?

$$1, 1, \cdots, 1, 2, 2, \cdots 2, \cdots, r, r, \cdots r$$

  (n_1 of 1, n_2 of 2, ..., n_r of r.)

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Each permutation yields a division of the items $\rightarrow$ multinomial.

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Also consider the number of permutations among the four aces: $4!$

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$$N = 49! \times 4!$$