Chapter 2
Axioms of Probability

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MATH 241

Outline

Sample space

Definition

A sample space S is the set of all possible outcomes of an experiment.

Question

Chapter 2 Problem 5: A system is composed of 5 components, each of which is either working or failed. Consider an experiment that consists of observing the status of each component, and let the outcome of the experiment be given by the vector $(x_1, x_2, x_3, x_4, x_5)$, where x_i is equal to 1 if component i is working and is equal to 0 if component i is failed.

(a) How many outcomes are in the sample space of this experiment?

$$2^5$$

An event

Definition

An event E is any subset of the sample space S.

$$E \subset S$$

- Impossible event: empty set $\emptyset \subset S$
- $S \subset S$

Question

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(b) Suppose that the system will work if components 1 and 2 are both working, or if components 3 and 4 are both working, or if components 1, 3, and 5 are all working. Let W be the event that the system will work. Specify all the outcomes in W.

$$W = \{(1, 1, ?, ?, ?), (?, ?, 1, 1, ?), (1, ?, 1, ?, 1)\}$$

Watch out for duplicate outcomes!

Question

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(c) Let A be the event that components 4 and 5 are both failed. How many outcomes are contained in the event A?

$$A = \{(?, ?, ?, 0, 0)\}$$

The number of outcomes in event A is 2^3.

Set theory

Let A, B be two events.

Definition

1. **Intersection** $A \cap B$: *implies the event that both A and B occur*
2. **Union** $A \cup B$: *implies the event that at least one of A or B occur*
3. *The* **complement** *of an event A denoted A^c (also notated A' or $\bar{A}$):* $A^c = S \backslash A$ *- the event that A does not occur*
4. $A \subset B$ *implies that the occurrence of A implies the occurrence of B*

Venn diagram

Question

Chapter 2 Problem 5: A system is composed of 5 components, each of which is either working or failed. Consider an experiment that consists of observing the status of each component, and let the outcome of the experiment be given by the vector $(x_1, x_2, x_3, x_4, x_5)$, where x_i is equal to 1 if component i is working and is equal to 0 if component i is failed. Suppose that the system will work if components 1 and 2 are both working, or if components 3 and 4 are both working, or if components 1, 3, and 5 are all working. Let W be the event that the system will work. Let A be the event that components 4 and 5 are both failed.

(d) Write out all the outcomes in the event $A \cap W$.

$$W = \{(1,1,?,?,?), (?,?,1,1,?), (1,?,1,?,1)\}, \;\; A = \{(?,?,?,0,0)\}$$

$$A \cap W = \{(1,1,0,0,0), (1,1,1,0,0)\}$$

More set theory

Definition

Two events A and B are **mutually exclusive** *or* **disjoint** *if they have no outcomes in common, i.e. $A \cap B = \emptyset$.*

Some rules

1. Commutative laws

$$A \cup B = B \cup A, \quad A \cap B = B \cap A$$

2. Associative laws

$$(A \cup B) \cup C = A \cup (B \cup C)$$

$$(A \cap B) \cap C = A \cap (B \cap C)$$

3. Distributive laws

$$(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$$

$$(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$$

DeMorgan's Laws

$$(A \cup B)^c = A^c \cap B^c, \quad (A \cap B)^c = A^c \cup B^c$$

- Outline of proof (to the first equation): two steps
  Left $\subset$ Right $\Longleftrightarrow$ For any $x \in (A \cup B)^c$, then $x \in A^c \cap B^c$.
  Right $\subset$ Left $\Longleftrightarrow$ For any $x \in A^c \cap B^c$, then $x \in (A \cup B)^c$.

- DeMorgan's laws can be generalized to n events $A_1, \ldots, A_n$:

$$\left(\bigcup_{i=1}^{n} A_i\right)^c = \bigcap_{i=1}^{n} A_i^c, \quad \left(\bigcap_{i=1}^{n} A_i\right)^c = \bigcup_{i=1}^{n} A_i^c$$

https://www.youtube.com/watch?v=oOx7FSzSav4

Recap

- Sample space S, event E
- Set operations: intersection, union, complement, subset, disjoint/mutually exclusive
- Rules: commutative, associative, distributive, DeMorgan's laws
- Hint: Venn Diagram is helpful

Outline

Axioms of probability

Definition

Let P be a function that assigns a nonnegative real number to each event E of a sample space S. We call P a probability if

1. *Axiom 1: non-negative*
$$0 \leq P(E) \leq 1$$
2. *Axiom 2: total one*
$$P(S) = 1$$
3. *Axiom 3: countable addition*
$$P\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i), \text{ if } E_i \cap E_j = \emptyset \text{ for } i \neq j$$

In particular, for k **disjoint** events $E_1, \ldots, E_k$,

$$P\left(\bigcup_{i=1}^{k} E_i\right) = \sum_{i=1}^{k} P(E_i)$$

Outline

Propositions

- Complement Rule:

$$P(A^c) = 1 - P(A)$$

  - $P(\emptyset) = 0$
- Difference Rule:

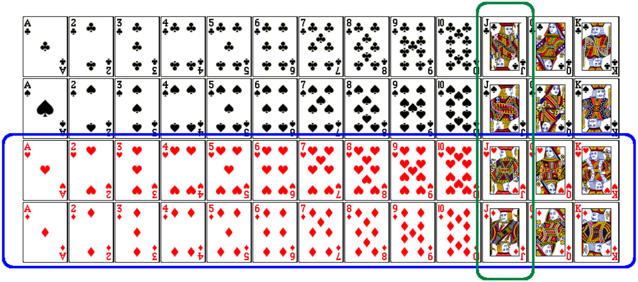
$$P(B \cap A^c) = P(B) - P(A), \text{ if } A \subseteq B$$

  - $P(B) \geq P(A)$, if $A \subseteq B$
- Inclusion-Exclusion: two events A, B (not necessarily disjoint)

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Question

What is the probability of drawing a jack or a red card from a well shuffled full deck?



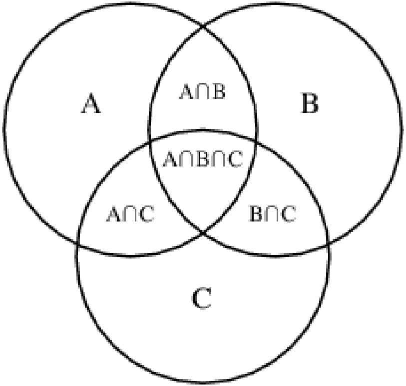
$$P(jack\ or\ red) = P(jack) + P(red) - P(jack\ and\ red)$$
$$= \frac{4}{52} + \frac{26}{52} - \frac{2}{52} = \frac{28}{52}$$

Figure from *http://www.milefoot.com/math/discrete/counting/cardfreq.htm* .

Propositions

☞ Inclusion-Exclusion: three events A, B, C (not necessarily disjoint)

$$\begin{aligned} P(A \cup B \cup C) = & P(A) + P(B) + P(C) - P(A \cap B) \\ & - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C) \end{aligned}$$



Question

Suppose that for a randomly selected student in a probability class,

- $P(\text{live Eastern Time Zone at home}) = 63\%$, $P(\text{senior}) = 41\%$, $P(\text{brown eye color}) = 55\%$.
- $P(\text{live Eastern Time Zone at home and senior}) = 31\%$, $P(\text{live Eastern Time Zone at home and brown eye color}) = 33\%$, $P(\text{senior and brown eye color}) = 24\%$
- $P(\text{live Eastern Time Zone at home and senior and brown eye color}) = 18\%$

Find the probability that a student is either live Eastern Time Zone at home, a senior or having brown eye color.

1. Conditions: Event $A = \{\text{live Eastern Time Zone at home}\}$, $B = \{\text{senior}\}$, $C = \{\text{brown eye color}\}$,
2. Question: find $P(A \cup B \cup C)$.
3. Formula: inclusion-exclusion

$$\begin{aligned} P(A \cup B \cup C) =& P(A) + P(B) + P(C) - P(A \cap B) \\ & - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C) \\ & = 0.63 + 0.41 + 0.55 - 0.31 - 0.33 - 0.24 + 0.18 = 0.89 \end{aligned}$$

Recap

Three axioms of probability P

1. $0 \leq P(E) \leq 1$
2. $P(S) = 1$
3. $P\left(\bigcup\limits_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i)$, if $E_i \cap E_j = \emptyset$ for $i \neq j$

Propositions of probability

- $P(A^c) = 1 - P(A)$
- $P(B \cap A^c) = P(B) - P(A)$, if $A \subseteq B$
- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
- $P(A \cup B \cup C) =$
  $P(A)+P(B)+P(C)-P(A\cap B)-P(A\cap C)-P(B\cap C)+P(A\cap B\cap C)$

Outline

Sample spaces with equally likely outcomes

- ☞ Suppose a sample space has N equally likely outcomes $\{1\}, \ldots, \{N\}$, then

  $$S = \{1\} \cup \ldots \cup \{N\}$$

  Disjointness gives

  $$1 = P(S) = P(\{1\}) + \ldots + P(\{N\}) = NP(\{i\}),$$

  so for each $1 \leq i \leq N$,

  $$P(\{i\}) = \frac{1}{N}$$

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- For event E in a sample space S with equally likely outcomes,

  $$P(E) = \frac{\#(E)}{\#(S)}$$

  Notation:

  Cardinality - $\#(E) =$ number of elements in set E

Question

Two fair four-sided dice are rolled. Two events:

$$A = \{\text{sum of two rolls is 5}\}$$

$$B = \{\text{minimum roll is 2}\}$$

1. Compute $P(A)$ and $P(B)$
2. Compute $P(A \cup B)$

Hint: sample space

1,1	1,2	1,3	1,4
2,1	2,2	2,3	2,4
3,1	3,2	3,3	3,4
4,1	4,2	4,3	4,4

1. $P(A) = \frac{4}{16}$, $P(B) = \frac{5}{16}$
2. $$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
   $$= \frac{4}{16} + \frac{5}{16} - \frac{2}{16} = \frac{7}{16}$$

Birthday problem (cont.)

Question

For a class of 30 students (i.e. $n = 30$), $P(\text{no match}) = 29.4\%$. What is the chance of at least a tie in birthdays among these 30 students?

$$P(\text{at least a tie}) = 1 - P(\text{no tie}) = 1 - 0.294 = 0.706$$

Question

Randomly pair 4 keys {a, b, c, d} with 3 locks {a, b, c}.
Compute $P(\text{at least one match})$.

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Let A denote the event that lock a and key a matches. Similarly, B, C.

- $P(A) =$
- $P(A \cap B) =$
- $P(A \cap B \cap C) =$
- $P(\text{at least one match}) = P(A \cup B \cup C) =$

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Randomly pair 4 keys {a, b, c, d} with 3 locks {a, b, c}.
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Let A denote the event that lock a and key a matches. Similarly, B, C.

- $P(A) = \frac{3\times 2}{4\times 3\times 2}$
- $P(A \cap B) =$
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- $P(\text{at least one match}) = P(A \cup B \cup C) =$

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- $P(A \cap B) = \frac{2}{4\times 3\times 2}$
- $P(A \cap B \cap C) =$
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- $P(\text{at least one match}) = P(A \cup B \cup C) =$

$$P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$
$$= \frac{6}{24} \times 3 - \frac{2}{24} \times 3 + \frac{1}{24} = \frac{13}{24}$$

Recap

For event E in a sample space S with equally likely outcomes,

$$P(E) = \frac{\#(E)}{\#(S)}$$

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For event E in a sample space S with equally likely outcomes,

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Question

In the game of bridge, the entire deck of 52 cards is dealt out to 4 players. What is the probability that one of the players receives all face cards?

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$$\#(\text{Player A gets all face cards}) = \binom{12}{12}\binom{52-12}{1} = 40$$

$$P(\text{Player A gets all face cards}) = \frac{40}{\binom{52}{13}}$$

$$P(\text{one player gets all face cards}) = 4 \times \frac{40}{\binom{52}{13}}$$