# Chapter 3

## Conditional Probability and Independence

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MATH 241

# Outline

# Conditional probability: motivation

The probability of getting a one when rolling a fair 6-sided die is $1/6$

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*Conditional on this new information*, the probability of a one is now $1/3$

# Conditional probability

## Definition

*Given two events $A$ and $B$ with $P(B) > 0$, the conditional probability of $A$ given $B$ has occurred is defined as*

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

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- When $B$ is the sample space: $P(A \mid B) = P(A)$
- Intuition: in a sample space with equally likely outcomes,

$$P(A \mid B) = \frac{\#(A \cap B)}{\#(B)}$$

# Example

Consider the die roll example: $B = \{1 \text{ or } 3 \text{ or } 5\}, A = \{1\}$

$$
\begin{aligned}
P(\text{get 1 given that roll is odd}) =& P(A \mid B) \\
=& \frac{P(A \cap B)}{P(B)} \\
=& \frac{P(\text{get 1})}{P(\text{get 1 or 3 or 5})} \\
=& \frac{1/6}{3/6} = \frac{1}{3}
\end{aligned}
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In context,

$$P(A \cap B) = P\left(\{(4,3)\}\right) = 1/36, \quad P(B) = 1/6$$

$$P(A \mid B) = \frac{1/36}{1/6} = \frac{1}{6}$$

# Propositions of conditional probability

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2. $P(A^c \mid A) = 0$

3. $P(A^c \mid B) = 1 - P(A \mid B)$

# Multiplication rule

☞ By the definition of conditional probability, the *joint probability* of $A$ and $B$ is

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- Usually, $P(A)$ and $P(B)$ are called *marginal probabilities*.

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- ☞ Generalize to $n$ events: chaining of probabilities

$$P\left(\bigcap_{i=1}^{n} A_i\right) = P(A_1)P(A_2 \mid A_1)P(A_3 \mid A_1, A_2) \cdots P(A_n \mid A_1, \ldots, A_{n-1})$$

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$$\begin{aligned} P(W_1B_2B_3) &= P(W_1)P(B_2 \mid W_1)P(B_3 \mid W_1B_2) \\ &= \frac{8}{12} \cdot \frac{4}{11} \cdot \frac{3}{10} \end{aligned}$$

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2. If we want to consider draw 1, which can be $W_1$ or $B_1$ (disjoint!)

$$\begin{aligned} P(B_2) &= P(W_1B_2) + P(B_1B_2) \\ &= \frac{8}{12} \cdot \frac{4}{11} + \frac{4}{12} \cdot \frac{3}{11} = \frac{44}{132} = \frac{4}{12} \end{aligned}$$

# Law of total probability

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$$P(B) = P(B \mid A_1)P(A_1) + \ldots + P(B \mid A_n)P(A_n)$$

- Such collection of sets $A_1, \ldots, A_n$ is also called a partition of sample space.

Example: a firm is considering a drug-testing program for its employees, but before it begins it wants to know the scope of the problem if any exists. Realizing the sensitivity of this issue, the personnel director decides to use a randomized response survey. It is believed that respondents are more likely to be honest when such forms are used. Each employee is asked to flip a fair coin. If the coin comes up heads, answer the question "Do you carpool to work?". If the coin comes up tails, answer the question "Have you used illegal drugs within the last month?". Assume that all employees answer the survey honestly. Out of 8000 responses, 1420 answered "YES". Suppose the firm knows that 35% of its employees carpool to work. What is the probability that an employee chosen at random used illegal drugs within the last month?

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\begin{aligned}
P(B) =& P(B \mid A_1)P(A_1) + P(B \mid A_2)P(A_2) \\
P(\text{YES}) =& P(\text{carpool})P(\text{head}) + P(\text{drug})P(\text{tail}) \\
\frac{1420}{8000} =& 0.35 \times 0.5 + P(\text{drug}) \times 0.5 \\
P(\text{drug}) &= 0.005
\end{aligned}
$$

# Recap

- Marginal probability: $P(A), P(B)$
- Joint probability: $P(A \cap B)$
- Conditional probability

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

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- Multiplication rule:

$$P(A \cap B) = P(A \mid B) \times P(B)$$

- Law of total probability: for a partition $\{A_1, A_2, \ldots, A_n\}$ of $S$,

$$P(B) = \sum_{j=1}^{n} P(B \mid A_j) P(A_j)$$

# Outline

# Bayes theorem (also called Bayes rule)

Suppose events $A_1, \ldots, A_n$ are disjoint, and $\bigcup_{i=1}^n A_i = S$, with $P(A_i) > 0$, $i = 1, 2, \ldots, n$. Then for any event $B$ with $P(B) > 0$,

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\begin{aligned}
P(A_i \mid B) &= \frac{P(B \mid A_i)P(A_i)}{\sum\limits_{j=1}^{n} P(B \mid A_j)P(A_j)}, \quad i = 1, \ldots, n \\
&= \frac{P(B \mid A_i)P(A_i)}{P(B \mid A_1)P(A_1) + \cdots + P(B \mid A_n)P(A_n)}
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- $P(A_i)$ is often called *prior probability*
- $P(A_i \mid B)$ is called *posterior probability*.

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- Note that the majors define a finite partition, and the campus folklore gives us the conditional probabilities $\Pr(B \mid A_i)$.
- The point of Bayes' rule is to reverse the conditioning to get $\Pr(A_i \mid B)$.

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Events $A_1 = \{\text{math}\}, A_2 = \{\text{music}\}, A_3 = \{\text{econ}\}$ define a partition.
Let $B = \{\text{date}\}$. In order to calculate $P(A_1 \mid B)$, use the Bayes theorem:

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=& \frac{0.9 \times 0.25}{0.9 \times 0.25 + 0.5 \times 0.55 + 0.1 \times 0.20} = 0.43
\end{aligned}
$$

# Recap

Bayes theorem

$$P(A_i \mid B) = \frac{P(B \mid A_i)P(A_i)}{\sum_{j=1}^{n} P(B \mid A_j)P(A_j)}, \quad i = 1, \ldots, n$$

# Outline

# Product rule for independent events

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- If $A$ and $B$ are independent, then

$$P(A \mid B) = \frac{P(A)P(B)}{P(B)} = P(A).$$

Knowing $B$ doesn't affect the odds of $A$.

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  Since one out of four possible outcomes match this definition, the probability is $\frac{1}{4}$.
  *Inefficient way if the number of trials was much higher*.

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  *Inefficient way if the number of trials was much higher*.
- Method 2: use independence.

$$P(\text{T on the first toss}) \times P(\text{T on the second toss}) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$$

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- $B$ and $A$ are independent.
- $A$ and $B^c$ are independent. And so are $A^c$ and $B^c$.
- Independent is not disjoint.

# Mutually independent

☞ Three events $A, B, C$ are called *mutually independent* if

1. $$P(A \cap B) = P(A)P(B)$$
   $$P(B \cap C) = P(B)P(C)$$
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- If only (1) holds but not (2), then $A, B, C$ are called *pairwise independent*.

# Example: pairwise indpt but not mutually indpt

Roll two fair 6-sided dice. Set

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- $B = \{\text{First roll is 4}\}$
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NOT mutually indpt

☞ Events $A_1, A_2, \ldots, A_n$ are called *independent* if for any $1 \leq r \leq n$ of them

$$P(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_r}) = P(A_{i_1})P(A_{i_2}) \cdots P(A_{i_r})$$

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- The key to compute the probability of independent events is to just multiply the probability of the individual events.

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A fair coin is tossed 5 times. Compute $P(5 \text{ tails})$.

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$$P(T_1\ T_2\ T_3\ T_4\ T_5) = P(T_1)P(T_2)P(T_3)P(T_4)P(T_5) = \frac{1}{2^5}$$

# Outline

# The Linda problem

Linda is 31 years old, single, outspoken, and very bright. She majored in philosophy. As a student she was deeply concerned with issues of discrimination and social justice, and she also participated in anti-nuclear demonstrations. Please rank the following statements by their probability, using 1 for the most probable and 8 for the least probable.

1. Linda is a teacher in an elementary school.
2. Linda works in a bookstore and takes yoga classes.
3. Linda is active in the feminist movement.
4. Linda is a psychiatric social worker.
5. Linda is a member of the League of Women Voters.
6. Linda is a bank teller.
7. Linda sells insurance.
8. Linda is a bank teller and is active in the feminist movement.

# The Linda problem cont'd

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# Set theory - intersection

- Event $A = \{$Linda is active in the feminist movement$\}$
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- Event $B = \{$Linda is a bank teller$\}$
- Event $A \cap B = \{$Linda is a bank teller and is active in the feminist movement$\}$.
- $P(A \cap B) \leq P(A)$ and $P(A \cap B) \leq P(B)$!
- What did our class do? 21 of you responded.
  - Ranking: 1 is the most probable choice, and 8 is the least probable choice
  - Avg(Ranking of event $A$) $= 3.38$
  - Avg(Ranking of event $B$) $= 4.76$
  - Avg(Ranking of event $A \cap B$) $= 4.62$
  - Percentage of people ranking event $A \cap B$ higher than event $B \approx 62\%$ (13 out of 21 responses)

# The Linda problem - historical results

The Linda problem was proposed by Amos Tversky and Daniel Kahneman in 1983.

- Ranking: 1 is the most probable choice, and 8 is the least probable choice
- Avg(Ranking of event $A$) $= 2.1$
- Avg(Ranking of event $B$) $= 6.2$
- Avg(Ranking of event $A \cap B$) $= 4.1$
- Percentage of people ranking event $A \cap B$ higher than event $B =$ 90% (even among those who have had several advanced courses in probability and statistics)

# Representativeness and Conjunction Bias

- Representativeness heuristic:
  - people determine the probability of an event by the degree to which the event is similar in essential characteristics to its parent population.
  - it leads people to inflate the believed probability of events that resembles the data available.

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- Interested in these types of topics? ECON 333 Behavioral Economics!