Chapter 7 part 2

Properties of Expectations

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MATH 241

Outline

Moment generating functions

Definition

The moment generating function $M(t)$ of the random variable X is defined for all real values of t by

$$M(t) = E[e^{tX}] = \begin{cases} \sum_x e^{tx} p(x) & \text{if } X \text{ is discrete with pmf } p(x) \\ \int_{-\infty}^{\infty} e^{tx} f(x) dx & \text{if } X \text{ is continuous with pdf } f(x) \end{cases}$$

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In general

$$M^n(t) = E[X^n e^{tX}], M^n(0) = E[X^n], n \geq 1$$

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- The moment generating function is

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\begin{aligned}
M(t) = E[e^{tX}] &= \sum_{k=0}^{n} e^{tk} \binom{n}{k} p^k (1-p)^{n-k} \\
&= \sum_{k=0}^{n} \binom{n}{k} (pe^t)^k (1-p)^{n-k} \\
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M'(t) = n(pe^t + 1 - p)^{n-1} pe^t
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$$M'(t) = n(pe^t + 1 - p)^{n-1} pe^t$$

$$M''(t) = n(n-1)(pe^t + 1 - p)^{n-2}(pe^t)^2 + n(pe^t + 1 - p)^{n-1} pe^t$$

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Therefore

$$E[X] = M'(0) = np$$

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Therefore

$$E[X] = M'(0) = np$$

$$\begin{aligned} Var(X) = E[X^2] - (E[X])^2 &= M''(0) - (M'(0))^2 \\ &= n(n-1)p^2 + np - (np)^2 \\ &= np(1-p) \end{aligned}$$

Properties of MGFs

- Property 1: The MGF of the sum of independent random variables equals to the product of the individual MGFs.

$$M_{X+Y}(t) = E[e^{t(X+Y)}] = E[e^{tX}e^{tY}] = E[e^{tX}]E[e^{tY}] = M_X(t)M_Y(t)$$

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- Property 2: The MGF uniquely determines the distribution. Refer to the tables of MGF of some discrete and continuous distributions. Textbook pages 339 and 340 for lists of MGFs of distributions.

Joint moment generating functions

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For any n random variables $X_1, ..., X_n$, the joint MGF $M(t_1, ..., t_n)$ is defined for all real values of $t_1, ..., t_n$ by

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- The individual MGF can be obtained from $M(t_1, ..., t_n)$ by letting all but one of the t_j's be 0

$$M_{X_i}(t) = E[e^{tX_i}] = M(0, ..., t, 0, ..., 0)$$

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- The joint MGF $M(t_1, ..., t_n)$ uniquely determines the joint distribution of $X_1, ..., X_n$
- Then we have the n random variables $X_1, ..., X_n$ are independent **if and only if**

$$M(t_1, ..., t_n) = M_{X_1}(t_1) ... M_{X_n}(t_n)$$