Chapter 5 part 1

Continuous Random Variables

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MATH 241

Outline

Outline

Continuous random variables

☞ A *continuous random variable* X can take any real value in $(-\infty, \infty)$.

Definition

X is a continuous random variable if there exists a **nonnegative** *function f defined for any $x \in (-\infty, \infty)$, such that for any set B of real numbers,*

$$P(X \in B) = \int_B f(x)dx$$

This function f is called the probability density function (pdf) of the random variable X.

Examples of continuous random variable

- Rainfall amount for a year.
- Lifetime of your first car.
- Amount of beer consumed on a game day.

Pdf and cdf

- For pdf to be valid, in additional to being non-negative,

$$\int_{-\infty}^{\infty} f(x)dx = P(-\infty < X < \infty) = 1$$

- Cdf function of continuous random variable

$$F(a) = P(X \leq a) = \int_{-\infty}^{a} f(x)dx$$

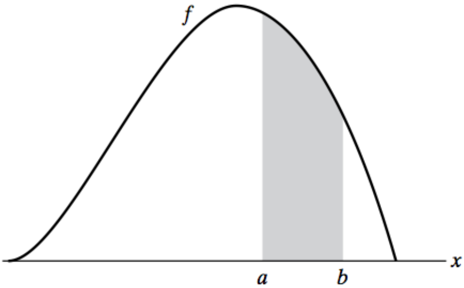
- For continuous random variable, probability to be a single point is zero.

$$P(X = a) = \int_{a}^{a} f(x)dx = 0$$

$$P(X < a) = P(X \leq a) - P(X = a) = F(a)$$

- Probability on an interval

$$P(a \leq X \leq b) = F(b) - F(a) = P(a < X < b)$$



$P(a \leq X \leq b)$ = area of shaded region

Connection between pdf and cdf of continuous random variable. If we know the pdf,

$$F(a) = P(X \leq a) = \int_{-\infty}^{a} f(x)dx$$

How to find the pdf if we know the cdf?

$$f(x) = \frac{d}{dx}F(x)$$

Question

Suppose that X is a continuous random variable whose pdf is

$$f(x) = \begin{cases} c(8x - 4x^3) & \text{if } 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

- Find $P(X > 0.5)$.

Question

Suppose that X is a continuous random variable whose pdf is

$$f(x) = \begin{cases} c(8x - 4x^3) & \text{if } 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

- Find $P(X > 0.5)$.

$$\begin{aligned} P(X > 0.5) &= \int_{0.5}^{\infty} f(x)dx \\ &= \int_{0.5}^{1} \frac{8x - 4x^3}{3} dx \\ &= \frac{1}{3}\left(4x^2 - x^4\Big|_{0.5}^{1}\right) \\ &= \frac{1}{3}\left[3 - 4 \times \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^4\right] = \frac{1}{3} \times \frac{33}{16} = \frac{11}{16} \end{aligned}$$

Question

Suppose that X is a continuous random variable whose pdf is

$$f(x) = \begin{cases} c(8x - 4x^3) & \text{if } 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

- Find the cdf function $F(x)$.

$$\int_{-\infty}^{\infty} f(x)dx = 1 \Longrightarrow c = 1/3$$

$$F(x) = \int_{-\infty}^{x} f(t)dt = \begin{cases} 0 & x < 0 \\ \frac{4x^2 - x^4}{3} & 0 \leq x \leq 1 \\ 1 & x > 1 \end{cases}$$

Note that $F(0) = 0$ and $F(1) = 1$.

Interpretation of the pdf

For some small value $h > 0$,

$$
\begin{aligned}
P\left(x - \frac{h}{2} \leq X \leq x + \frac{h}{2}\right) &= \int_{x-\frac{h}{2}}^{x+\frac{h}{2}} f(t)dt \\
&\approx \left[(x + \frac{h}{2}) - (x - \frac{h}{2})\right] f(x) \\
&= h \cdot f(x)
\end{aligned}
$$

The larger $f(x)$ is, the more likely X is to be “near” x.

Recap

A continuous random variable X can take more than countable number of values in $\mathbb{R}$.

- We defined continuous random variable using pdf

$$P(X \in B) = \int_B f(x)dx$$

- Cdf function of continuous random variable

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)dt$$

$$f(x) = \frac{d}{dx}F(x)$$

Question

Comparing discrete random variables and continuous random variables.

	discrete	continuous
values can take		
pmf / pdf		
validness (i.e. the sum of probability is 1)		
cdf		
from cdf to pmf / pdf		
probability at a single point		

	discrete	continuous
values can take	a countable number	any real value in $(-\infty, \infty)$
pmf / pdf	pmf, $p(x)$	pdf, $f(x)$
validness (i.e. the sum of probability is 1)	$\sum_{x:p(x)>0} p(x) = 1$	$\int_{-\infty}^{\infty} f(x)dx = P(-\infty < X < \infty) = 1$
cdf	$F(a) = \sum_{x \leq a} p(x)$	$F(a) = P(X \leq a) = \int_{-\infty}^{a} f(x)dx$
from cdf to pmf / pdf	use a graph and pay attention to start / end points	take derivative with respect to the random variable
probability at a single point	$p(x)$	0

Outline

Expectation and variance of continuous random variable

- Recall that expected value of the discrete random variable X with pmf $p(x)$,

$$E[X] = \sum_{\text{all } x} xp(x)$$

- We define expected value of a continuous random variable X with pdf $f(x)$ as

$$E[X] = \int_{-\infty}^{\infty} xf(x)dx$$

- Definitions of variance and standard deviation are the same.

$$Var(X) = E[X^2] - (E[X])^2, \quad SD(X) = \sqrt{Var(X)}$$

Definition:

$$Var(X) = E[(X - \mu)^2]$$

Properties of $E[X]$ for continuous random variable

- Recall that expected value of the discrete random variable X with pmf $p(x)$,

$$E[g(X)] = \sum_{\text{all } x} g(x)p(x)$$

- Similarly, expected value of a continuous random variable X with pdf $f(x)$ as

$$E[g(X)] = \int_{-\infty}^{\infty} g(x)f(x)dx$$

Properties of $E[X]$ for continuous random variable

Similarly as discrete random variable, for continuous random variable X and Y

- Sum of two random variable's

$$E[X+Y] = E[X] + E[Y]$$

- If a and b are constants, then

$$E[aX+b] = aE[X] + b$$

- If a and b are constants, then

$$Var(aX+b) = a^2 Var(X)$$

Question

Find $E[e^X]$, if the pdf of X is given by

$$f(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

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$$E[e^X] = \int_0^1 e^x dx = e - 1$$

Question

X is a continuous random variable. Its pdf $f(x)$ is an even function, i.e.,

$$f(-x) = f(x), \quad \text{for any } x > 0$$

What is $E[X]$?

a. Cannot decide. Need more information.
b. 0
c. 1
d. e

Question

X is a continuous random variable. Its pdf $f(x)$ is an even function, i.e.,

$$f(-x) = f(x), \quad \text{for any } x > 0$$

What is $E[X]$?

(a) Cannot decide. Need more information.
(b) 0
(c) 1
(d) e

$$E[X] = \int_{-\infty}^{0} xf(x)dx + \int_{0}^{\infty} xf(x)dx \quad (\text{let } y = -x)$$

$$= \int_{-\infty}^{0} (-y)f(-y)d(-y) + \int_{0}^{\infty} xf(x)dx$$

$$= \int_{\infty}^{0} yf(-y)dy + \int_{0}^{\infty} xf(x)dx = -\int_{0}^{\infty} yf(y)dy + \int_{0}^{\infty} xf(x)dx$$

Question

The density function of X is given by

$$f(x) = \begin{cases} a + bx^2 & \text{if } 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

If $E[X] = \frac{3}{5}$, find a and b.

Question

The density function of X is given by

$$f(x) = \begin{cases} a + bx^2 & \text{if } 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

If $E[X] = \frac{3}{5}$, find a and b.

$$\begin{aligned} E[X] &= \int_0^1 xf(x)dx = \int_0^1 x(a + bx^2)dx = \int_0^1 ax + bx^3 dx \\ &= \frac{1}{2}ax^2 + \frac{1}{4}bx^4|_0^1 = \frac{a}{2} + \frac{b}{4} = \frac{3}{5} \end{aligned}$$

Moreover,

$$\int_0^1 f(x)dx = \int_0^1 a + bx^2 dx = ax + \frac{1}{3}bx^3|_0^1 = a + \frac{b}{3} = 1$$

Then we have $a = 3/5$ and $b = 6/5$.

Question

Comparing discrete random variables and continuous random variables.

	discrete	continuous
E[X]		
E[g(X)]		
Var(X)		
σ		
E[aX+b]		
Var(aX+b)		
E[X+Y]		

	discrete	continuous
E[X]	$\sum_{x:p(x)>0} xp(x)$	$\int_{-\infty}^{\infty} xf(x)dx$
E[g(X)]	$\sum_{x:p(x)>0} g(x)p(x)$	$\int_{-\infty}^{\infty} g(x)f(x)dx$
Var(X)	$E[X^2] - E[X]^2$	$E[X^2] - E[X]^2$
σ	$\sqrt{E[X^2] - E[X]^2}$	$\sqrt{E[X^2] - E[X]^2}$
E[aX+b]	$aE[X] + b$	$aE[X] + b$
Var(aX+b)	$a^2Var(X)$	$a^2Var(X)$
E[X+Y]	$E[X] + E[Y]$	$E[X] + E[Y]$

Outline

Uniform Distribution

Let's define a continuous probability distribution that has some constant value c between α and β where $\alpha < \beta$. What is the pdf?

$$f(x) = \begin{cases} c & \text{if } \alpha < x < \beta \\ 0 & \text{otherwise} \end{cases}$$

$$1 = \int_{-\infty}^{\infty} f(x)dx = \int_{\alpha}^{\beta} c dx = c(\beta - \alpha) \implies c = \frac{1}{\beta - \alpha}$$

Definition

A continuous random variable X has a Uniform distribution on the interval (α, β) if its pdf is

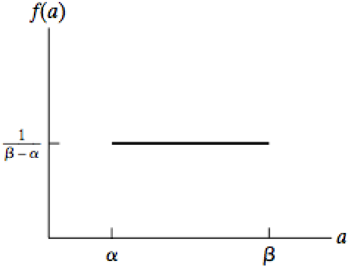
$$X \sim \textit{Unif}(\alpha, \beta) \iff f(x) = \frac{1}{\beta - \alpha} \cdot \mathbf{1}_{(\alpha,\beta)}(x) = \begin{cases} \frac{1}{\beta-\alpha} & \textit{if } \alpha < x < \beta \\ 0 & \textit{otherwise} \end{cases}$$

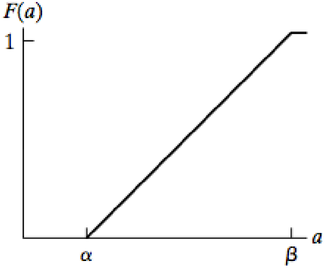
Cdf of Uniform distribution

For any $x \in (\alpha, \beta)$,

$$F(x) = \int_{-\infty}^{x} f(t)\ dt = \int_{\alpha}^{x} \frac{1}{\beta - \alpha}\ dt = \frac{x - \alpha}{\beta - \alpha}$$

- $X \sim \text{Unif}(\alpha, \beta)$, then its cdf is $F(x) = \begin{cases} 0 & \text{if } x \leq \alpha \\ \frac{x-\alpha}{\beta-\alpha} & \text{if } \alpha < x < \beta \\ 1 & \text{if } x \geq \beta \end{cases}$





Mean of a Uniform distribution

- (required) Expected value of $X \sim \text{Unif}(\alpha, \beta)$ is

$$E[X] = \frac{\alpha + \beta}{2}$$

$$\begin{aligned} E[X] &= \int_a^\beta \frac{x}{\beta - \alpha} dx = \left. \frac{x^2}{2(\beta - \alpha)} \right|_\alpha^\beta \\ &= \frac{\beta^2 - \alpha^2}{2(\beta - \alpha)} = \frac{\alpha + \beta}{2} \end{aligned}$$

Variance of a Uniform distribution

- (required) Variance of $X \sim \text{Unif}(\alpha, \beta)$ is

$$Var(X) = \frac{(\beta - \alpha)^2}{12}$$

$$E[X^2] = \int_a^\beta \frac{x^2}{\beta - \alpha} dx = \frac{x^3}{3(\beta - \alpha)}\Big|_\alpha^\beta$$

$$= \frac{\beta^3 - \alpha^3}{3(\beta - \alpha)} = \frac{\beta^2 + \beta\alpha + \alpha^2}{3}$$

$$Var(X) = E[X^2] - E[X]^2 = \frac{\beta^2 + \beta\alpha + \alpha^2}{3} - \frac{(\beta + \alpha)^2}{4}$$

$$= \frac{4b^2 + 4\beta\alpha + 4\alpha^2}{12} - \frac{3b^2 + 6\alpha\beta + 3\alpha^2}{12}$$

$$= \frac{\beta^2 - 2\beta\alpha + \alpha^2}{12} = \frac{(\beta - \alpha)^2}{12}$$

Uniform distribution: probability calculation

If $X \sim \text{Unif}(\alpha, \beta)$, then $$P(X \in B) = \frac{\text{length}(B)}{\beta - \alpha}$$

Question

If X is uniformly distributed over (0, 10), calculate the probability that (a) $X < 3$, (b) $X > 6$, and (c) $3 < X < 8$.

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Question

If X is uniformly distributed over (0, 10), calculate the probability that (a) $X < 3$, (b) $X > 6$, and (c) $3 < X < 8$.

You can solve this using definitions...

- Part (a): $P(X < 3) = \int_0^3 \frac{1}{10} dx = \frac{3}{10}$
- Part (b): $P(X > 6) = \int_6^1 0 \frac{1}{10} dx = \frac{4}{10}$
- Part (c): $P(3 < X < 8) = \int_3^8 \frac{1}{10} dx = \frac{1}{2}$

Uniform distribution: probability calculation

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- Part (a): $P(X < 3) = \int_0^3 \frac{1}{10} dx = \frac{3}{10}$
- Part (b): $P(X > 6) = \int_6^1 0 \frac{1}{10} dx = \frac{4}{10}$
- Part (c): $P(3 < X < 8) = \int_3^8 \frac{1}{10} dx = \frac{1}{2}$

Or recognize the result above to get $\frac{3}{10}$ directly by $\frac{3-0}{10}$. Same for (b) and (c).

Recap

Expectation for continuous random variable X and a function of it $g(X)$

$$E[X] = \int_{-\infty}^{\infty} x f(x) dx$$

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f(x) dx$$

Uniform distribution $X \sim \text{Unif}(\alpha, \beta)$

- Pdf
$$f(x) = \begin{cases} \frac{1}{\beta-\alpha} & \text{if } \alpha < x < \beta \\ 0 & \text{otherwise} \end{cases}$$
- Cdf
$$F(x) = \begin{cases} 0 & \text{if } x \leq \alpha \\ \frac{x-\alpha}{\beta-\alpha} & \text{if } \alpha < x < \beta \\ 1 & \text{if } x \geq \beta \end{cases}$$
- Mean and variance
$$E[X] = \frac{\alpha+\beta}{2}, \quad Var(X) = \frac{(\beta-\alpha)^2}{12}$$