Chapter 4 part 1

Discrete Random Variables

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MATH 241

Outline

1. Discrete random variables

2. Expectation

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2. Expectation

Random Variables

Definition

Random Variable X *is a real-valued function on the sample space* S*.*

- Random variable is a number associated with a random experiment.
- Random variables are in essence a fancy way of describing an event.

Question

Two fair six-sided dice are rolled. Let the random variable X denote the product of the 2 dice. What are possible values of X and their associated probabilities? Just give a few examples and you do not need to calculate the associated probability for all possible values.

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$$
\begin{aligned}
P(X = 1) &= P(1, 1) = 1/6 \times 1/6 \\
P(X = 2) &= P(1, 2) + P(2, 1) = 1/6 \times 1/6 \times 2 \\
P(X = 3) &= P(1, 3) + P(3, 1) = 1/6 \times 1/6 \times 2 \\
&\cdots\cdots \\
P(X = 36) &= P(6, 6) = 1/6 \times 1/6
\end{aligned}
$$

Cumulative distribution function (cdf)

Definition

For a random variable X, the function F defined by

$$F(x) = P(X \leq x), \quad -\infty < x < \infty$$

is called the cumulative distribution function of X.

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Note that

- Capital X: random variable
- Little x: a real-valued number
- $\leq$: smaller than or equal to

Properties of the cdf (Ch 4.10)

- $P(a < X \leq b) = F(b) - F(a)$, for all $a < b$
- $P(X < b)$ does not necessary equal to $P(X \leq b)$.

1. $F(x)$ is a non-decreasing function; i.e., if $x_1 < x_2$, then
$$F(x_1) \leq F(x_2)$$
2. $$\lim_{x\to\infty} F(x) = 1 \iff P(X \leq \infty) = 1$$
3. $$\lim_{x\to-\infty} F(x) = 0 \iff P(X > -\infty) = 1$$
4. $F(x)$ is right continuous; i.e., for any decreasing sequence $\{x_n : n = 1, 2, \ldots\}$ that converges to x,
$$\lim_{n\to\infty} F(x_n) = F(x) \implies \lim_{n\to\infty} P(X \leq x + \frac{1}{n}) = P(X \leq x)$$

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Part (a)

$$\begin{aligned} P(X \leq 2) &= P(X = 1) + P(X = 2) \\ &= P(1, 1) + P(1, 2) + P(2, 1) = 1/6 \times 1/6 \times 3 \end{aligned}$$

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Part (a)

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Part (b)

$$\begin{aligned} P(X \leq 35) &= P(X = 1) + P(X = 2) + \cdots + P(X = 35) \\ &= 1 - P(X = 36) \\ &= 1 - P(6, 6) \\ &= 1 - 1/6 \times 1/6 \end{aligned}$$

Discrete random variables

Definition

- *A random variable X that can take on at most a countable number of possible values is a discrete random variable.*
- *For a discrete random variable X, we define the probability mass function (pmf) by*

$$p(x) = P(X = x)$$

Note: pmf is also written as $f(x)$.

pmf and cdf

- For a discrete random variable X, there exists a countable sequence $x_1, x_2, \ldots$, such that

$$\begin{aligned} p(x_i) > 0 &\quad \text{for } i = 1, 2, \ldots \\ p(x) = 0 &\quad \text{for all other values of } x \end{aligned}$$

  and

$$\sum_{i=1}^{\infty} p(x_i) = 1$$

- Relationship between pmf and cdf (for discrete random variable)

$$F(a) = \sum_{\text{all } x \leq a} p(x)$$

- If we know pmf, we can compute cdf. And vice versa.

Question

Three fair coins are tossed. Let the random variable X denote the number of heads. Write out the pmf and cdf of X.

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For pmf:

$$P(X = 0) = P(TTT) = (1/2)^3 = 1/8$$
$$P(X = 1) = P(HTT) + P(THT) + P(TTH) = (1/2)^3 \times 3 = 3/8$$
$$P(X = 2) = P(HHT) + P(HTH) + P(THH) = (1/2)^3 \times 3 = 3/8$$
$$P(X = 3) = P(HHH) = (1/2)^3 = 1/8$$

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$$P(X = 3) = P(HHH) = (1/2)^3 = 1/8$$

For pmf:

$$F(a) = \begin{cases} 1/8, & 0 \leq a < 1 \\ 1/2, & 1 \leq a < 2 \\ 7/8, & 2 \leq a < 3 \\ 1, & 3 \leq a \end{cases}$$

Question

Suppose that the distribution function of X is given by

$$F(b) = \begin{cases} 0, & b < 0 \\ \frac{b}{4}, & 0 \leq b < 1 \\ \frac{1}{2} + \frac{b-1}{4}, & 1 \leq b < 2 \\ \frac{11}{12}, & 2 \leq b < 3 \\ 1, & 3 \leq b \end{cases}$$

Find $P(X = i), i = 1, 2, 3.$

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Find $P(X = i), i = 1, 2, 3.$

- $P(X = 1) = P(X \leq 1) - P(X < 1) = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$

Question

Suppose that the distribution function of X is given by

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- $P(X = 1) = P(X \leq 1) - P(X < 1) = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$
- $P(X = 2) = P(X \leq 2) - P(X < 2) = \frac{11}{12} - (\frac{1}{2} + \frac{2-1}{4}) = \frac{11}{12} - \frac{3}{4} = \frac{1}{6}$

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- $P(X = 3) = P(X \leq 3) - P(X < 3) = 1 - \frac{11}{12} = \frac{1}{12}$

Recap

Random Variable

- Random Variable X is a real-valued function on the sample space S.

$$X : S \longrightarrow \mathbb{R}$$

- Cumulative distribution function (cdf)

$$F_X(x) = P(X \leq x), \quad \text{for any } x \in \mathbb{R}$$

Discrete random variable

- can only take at most a countable number of possible values.
- probability mass function (pmf)

$$p_X(x) = P(X = x)$$

- $\sum_{i=1}^{\infty} p(x_i) = 1$
- $F(a) = \sum_{\text{all } x \leq a} p(x)$

Outline

Expected value

Definition

The expected value *(or* mean*) of a* <u>*discrete random variable*</u> *is defined as*

$$E[X] = \sum_{x:p(x)>0} x \cdot P(X = x) = \sum_{x:p(x)>0} xp(x)$$

- $E[X]$ is a weighted average of the possible values x that X can take on, each value being weighted by the probability $p(x)$.

When she told me I was average,
she was just being mean.

Question

Toss a coin. Suppose the probability of a head is p. Let X be a 0-1 indicator random variable s.t.

$$X = \begin{cases} 1 & \text{if head is obtained} \\ 0 & \text{otherwise} \end{cases}$$

Compute $\mu = E[X]$.

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$$E[X] = 1 \cdot p + 0 \cdot (1 - p) = p$$

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- In general, for indicator variable $X = \delta_A$ or $\mathbf{1}_A$, denoted as

$$X = \begin{cases} 1 & \text{if event } A \text{ occurs} \\ 0 & \text{otherwise} \end{cases}$$

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- In general, for indicator variable $X = \delta_A$ or $\mathbf{1}_A$, denoted as

$$X = \begin{cases} 1 & \text{if event } A \text{ occurs} \\ 0 & \text{otherwise} \end{cases}$$

The expected value of X equals the probability that A occurs.

$$E[X] = 1 \cdot P(A) + 0 \cdot P(A^c) = P(A)$$

Question

Let the random variable X denote the GP a certain student will earn in this class. Suppose its pmf is

$$p(0) = 0.05, \quad p(1) = 0.05, \quad p(2) = 0.3, \quad p(3) = 0.4$$

Calculate their expected GP $E[X]$.

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Calculate their expected GP $E[X]$.

Since the domain of X (all possible values X can take) is $\{0, 1, 2, 3, 4\}$, first compute $p(4)$.

$$p(4) = 1 - p(0) - p(1) - p(2) - p(3) = 0.2$$

Then compute $E[X]$

$$E[X] = 0 \times 0.05 + 1 \times 0.05 + 2 \times 0.3 + 3 \times 0.4 + 4 \times 0.2 = 2.65$$

Expectation of a function of a random variable

- If X is a discrete random variable, and g is a real-valued function then the expectation (or expected value) of $Y = g(X)$ is

$$E[g(X)] = \sum_{x:p_X(x)>0} g(x) \cdot p_X(x)$$

Question

Let X denote a random variable that takes on any of the values -1, 0, and 2 with respective probability: $P(X = -1) = \frac{1}{5}, P(X = 0) = \frac{1}{5}, P(X = 2) = \frac{3}{5}$. Compute $E[X^3]$.

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Let $Y = X^3$, then

$$
\begin{aligned}
E[X^3] = E[Y] &= \sum_{\text{all } x} x^3 \cdot p_X(x) \\
&= (-1)^3 \times \frac{1}{5} + 0^3 \times \frac{1}{5} + 2^3 \times \frac{3}{5} \\
&= -\frac{1}{5} + 0 + \frac{24}{5} \\
&= \frac{24}{5}
\end{aligned}
$$

Properties of expected values

If a and b are constants, then

$$E[aX + b] = aE[X] + b$$

- Holds for all random variable X (not necessary discrete random variable).
- Special cases of linear transformation $E[aX + b] = aE[X] + b$
  - constant factor
    $$E[aX] = aE[X]$$
  - constant
    $$E[b] = b$$

Question

Let X denote a random variable that takes on any of the values -1, 0, and 2 with respective probability: $P(X=-1)=\frac{1}{5}, P(X=0)=\frac{1}{5}, P(X=2)=\frac{3}{5}$. Compute: (a) $E[2X^2]$, (b) $E[4X^2-1]$.

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Let X denote a random variable that takes on any of the values -1, 0, and 2 with respective probability: $P(X = -1) = \frac{1}{5}, P(X = 0) = \frac{1}{5}, P(X = 2) = \frac{3}{5}$. Compute: (a) $E[2X^2]$, (b) $E[4X^2 - 1]$.

Let $Y = X^2$, then

$$\begin{aligned} E[X^2] = E[Y] &= \sum_{\text{all } x} x^2 \cdot p_X(x) \\ &= (-1)^2 \times \frac{1}{5} + 0^2 \times \frac{1}{5} + 2^2 \times \frac{3}{5} \\ &= \frac{1}{5} + 0 + \frac{12}{5} \\ &= \frac{13}{5} \end{aligned}$$

Part (a): $E[2X^2] = 2E[X^2] = 2E[X^2] = 26/5$
Part (b): $E[4X^2 - 1] = 4E[X^2] - 1 = 52/5 - 1 = 47/5$

Recap

Expectation μ

- **For discrete random variable**: $E[X] = \sum_{\text{all } x} x \cdot p(x)$
- **Functions**: $E[g(X)] = \sum_{\text{all } x} g(x)\ p(x)$
- **Indicators**: $E[\delta_A] = P(A)$ where δ_A is an indicator function
- **Linear function**: $E[aX + b] = aE[X] + b$
- **Constants**: $E[c] = c$ if c is constant

For two random variable's X and Y

$$E[X + Y] = E[X] + E[Y]$$