Chapter 6 part 2

Jointly Distributed Random Variables

Jingchen (Monika) Hu

Vassar College

MATH 241

Outline

1. Sums of independent random variables
2. Conditional distributions: discrete case
3. Conditional distributions: continuous case

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3. Conditional distributions: continuous case

Sums of continuous random variables

If X, Y have a joint density $f(x, y)$, then $X + Y$ has the following density

$$\begin{aligned} f_{X+Y}(z) &= \int_{-\infty}^{\infty} f(x, z - x)\ dx \\ &= \int_{-\infty}^{\infty} f(z - y, y)\ dy \end{aligned}$$

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If X and Y are independent, we can use the convolution formula

$$\begin{aligned} f_{X+Y}(z) &= \int_{-\infty}^{\infty} f_X(x) f_Y(z-x)\ dx \\ &= \int_{-\infty}^{\infty} f_X(z-y) f_Y(y)\ dy \end{aligned}$$

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Cdf F_{X+Y} is called the *convolution* of the distributions F_X and F_Y.

Why?

$$F_{X+Y}(z) = \iint_{x+y \leq z} f_{X,Y}(x,y)\; dy\; dx$$

Textbook page 239 gives a derivation when X and Y are independent.

Why?

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F_{X+Y}(z) &= \iint_{x+y\leq z} f_{X,Y}(x,y)\ dy\ dx \\
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Sum of two independent Uniforms:

Let X and Y have independent Unif$(0,1)$ distribution. Find pdf of $X+Y$.

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Range of $X + Y$ is $(0, 2)$.

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$$f_{X+Y}(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z - x)\ dx$$
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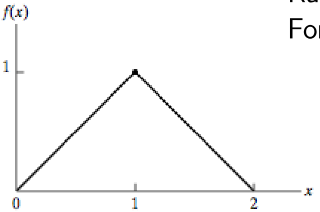
Sum of two independent Uniforms: Triangular distribution

Let X and Y have independent Unif$(0,1)$ distribution. Find pdf of $X+Y$.



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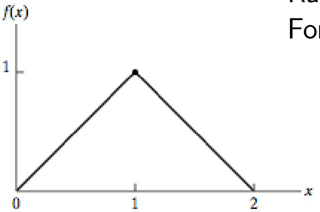

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$$f_{X+Y}(z) = \begin{cases} z & \text{if } 0 < z \leq 1 \\ 2-z & \text{if } 1 < z < 2 \\ 0 & \text{otherwise} \end{cases}$$

Sum of independent random variables

Random variables X and Y are independent, then

X	Y	$X+Y$
$\mathsf{N}(\mu_1, \sigma_1^2)$	$\mathsf{N}(\mu_2, \sigma_2^2)$	
$\mathsf{Poi}(\lambda_1)$	$\mathsf{Poi}(\lambda_2)$	
$\mathsf{Bin}(n_1, p)$	$\mathsf{Bin}(n_2, p)$	

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Recap

Random variables X and Y are independent if any real sets $A, B \subset \mathbb{R}$,

$$P(X \in A, Y \in B) = P(X \in A)P(Y \in B)$$

Random variables X and Y are independent **if and only if**

- Cdf: for any $x, y \in \mathbb{R}$

$$F_{X,Y}(x, y) = F_X(x)F_Y(y)$$

- For any $x, y \in \mathbb{R}$, the pmf / pdf

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$$f_{X,Y}(x, y) = f_X(x)f_Y(y)$$

If X and Y are independent continuous random variables, then

$$f_{X+Y}(z) = \int_{-\infty}^{\infty} f_X(x)f_Y(z - x)\, dx$$

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2. Conditional distributions: discrete case
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Discrete conditional distributions

- Recall that for any two events E and F, the conditional probability of E given F is defined by

$$P(E|F) = \frac{P(E \cap F)}{P(F)}$$

provided that $P(F) > 0$.

Discrete conditional distributions

- Recall that for any two events E and F, the conditional probability of E given F is defined by

$$P(E|F) = \frac{P(E \cap F)}{P(F)}$$

provided that $P(F) > 0$.

- If X and Y are discrete random variables, we define the conditional probability mass function (pmf) of X given $Y = y$ by

$$p_{X|Y}(x|y) = P\{X = x|Y = y\} = \frac{P\{X = x, Y = y\}}{P\{Y = y\}} = \frac{p(x,y)}{p_Y(y)}$$

for all values of y such that $p_Y(y) > 0$.

Discrete conditional distributions

- Similarly for the conditional probability density function (cdf), we define it by

$$F_{X|Y}(x|y) = P\{X \leq x|Y \leq y\} = \sum_{x \leq a} p_{X|Y}(a|y)$$

Discrete conditional distributions

- Similarly for the conditional probability density function (cdf), we define it by

$$F_{X|Y}(x|y) = P\{X \leq x|Y \leq y\} = \sum_{x \leq a} p_{X|Y}(a|y)$$

- If X is independent of Y, then

$$\begin{aligned} p_{X|Y}(x|y) &= P\{X = x|Y = y\} \\ &= \frac{P\{X = x, Y = y\}}{P\{Y = y\}} \\ &= \frac{P\{X = x\}P\{Y = y\}}{P\{Y = y\}} \\ &= P\{X = x\} \end{aligned}$$

Suppose that 3 balls are chosen without replacement from an urn consisting 5 white and 8 red balls. Suppose the white balls are numbered, and let X_i equal 1 if the ith white ball is selected and 0 otherwise. Calculate the conditional probability mass function of X_1 given that (1)$X_2 = 1$; (2) $X_2 = 0$.

Suppose that 3 balls are chosen without replacement from an urn consisting 5 white and 8 red balls. Suppose the white balls are numbered, and let X_i equal 1 if the ith white ball is selected and 0 otherwise. Calculate the conditional probability mass function of X_1 given that (1)$X_2 = 1$; (2) $X_2 = 0$.

Suppose $p(x_1, x_2)$ is the joint pmf of X_1 and X_2, then

- $p(0,0) = \frac{\binom{11}{3}\binom{2}{0}}{\binom{13}{3}} = \frac{15}{26}$, $p(1,1) = \frac{\binom{11}{1}\binom{2}{2}}{\binom{13}{3}} = \frac{1}{26}$,
  $p(0,1) = p(1,0) = \frac{\binom{11}{2}\binom{1}{1}}{\binom{13}{3}} = \frac{5}{26}$

Suppose that 3 balls are chosen without replacement from an urn consisting 5 white and 8 red balls. Suppose the white balls are numbered, and let X_i equal 1 if the ith white ball is selected and 0 otherwise. Calculate the conditional probability mass function of X_1 given that (1)$X_2 = 1$; (2) $X_2 = 0$.

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- Part (1): $p_{X_2}(1) = \sum_{x_1} p(x_1, 1) = p(0,1) + p(1,1) = \frac{6}{26}$
  $p_{X_1|X_2}(0,1) = \frac{p(0,1)}{p_{X_2}(1)} = \frac{5}{6}, p_{X_1|X_2}(1,1) = \frac{p(1,1)}{p_{X_2}(1)} = \frac{1}{6}$

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Suppose $p(x_1, x_2)$ is the joint pmf of X_1 and X_2, then

- $p(0,0) = \frac{\binom{11}{3}\binom{2}{0}}{\binom{13}{3}} = \frac{15}{26}$, $p(1,1) = \frac{\binom{11}{1}\binom{2}{2}}{\binom{13}{3}} = \frac{1}{26}$,
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- Part (2): $p_{X_2}(0) = \sum_{x_1} p(x_1, 0) = p(0,0) + p(1,0) = \frac{20}{26}$
  $p_{X_1|X_2}(0,0) = \frac{p(0,0)}{p_{X_2}(0)} = \frac{15}{20} = \frac{3}{4}, p_{X_1|X_2}(1,0) = \frac{p(1,0)}{p_{X_2}(0)} = \frac{5}{20} = \frac{1}{4}$

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Continuous conditional distributions

- With joint probability function of X and Y as $f(x, y)$, the conditional pdf of X given that $Y = y$ (for $f_Y(y) > 0$) is defined by

$$f_{X|Y}(x|y) = \frac{f(x, y)}{f_Y(y)}$$

  note that the event $\{Y = y\}$ has probability 0; we just use it for conditioning.

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- With joint probability function of X and Y as $f(x, y)$, the conditional pdf of X given that $Y = y$ (for $f_Y(y) > 0$) is defined by

$$f_{X|Y}(x|y) = \frac{f(x, y)}{f_Y(y)}$$

  note that the event $\{Y = y\}$ has probability 0; we just use it for conditioning.

- If X and Y are jointly continuous, then for any set A

$$P\{X \in A|Y = y\} = \int_A f_{X|Y}(x|y)dx$$

  This is defining conditional probabilities of events associated with one random variable when we are given the value of a second random variable.

Continuous conditional distributions

- By setting $A = (-\infty, a)$, we can define the cdf of X given that $Y = y$ by

$$F_{X|Y}(a|y) = P\{X \leq a|Y = y\} = \int_{-\infty}^{a} f_{X|Y}(x|y)dx$$

Continuous conditional distributions

- By setting $A = (-\infty, a)$, we can define the cdf of X given that $Y = y$ by

$$F_{X|Y}(a|y) = P\{X \leq a|Y = y\} = \int_{-\infty}^{a} f_{X|Y}(x|y)dx$$

- And if X and Y are independent continuous random variables,

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{f_X(x)f_Y(y)}{f_Y(y)} = f_X(x)$$

  This is the unconditional density of X.

The joint density of X and Y is given by

$$f(x,y) = \begin{cases} \frac{12}{5}x(2-x-y) & \text{for } 0 < x < 1, 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

Compute the conditional density of X given that $Y = y$, where $0 < y < 1$.

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For $0<x<1, 0<y<1$, we have

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Compute the conditional density of X given that $Y = y$, where $0 < y < 1$.

For $0 < x < 1, 0 < y < 1$, we have

$$\begin{aligned} f_{X|Y}(x|y) &= \frac{f(x,y)}{f_Y(y)} \\ &= \frac{f(x,y)}{\int_{-\infty}^{\infty} f(x,y)dx} \\ &= \frac{x(2-x-y)}{\int_0^1 x(2-x-y)dx} \\ &= \frac{x(2-x-y)}{2/3 - y/2} \\ &= \frac{6x(2-x-y)}{4-3y} \end{aligned}$$