Chapter 5 part 2

Continuous Random Variables

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Vassar College

MATH 241

Outline

1. Normal distribution
2. Distribution of a function of a continuous random variable

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Normal Distribution

Definition

A continuous random variable X has a Normal distribution distribution with mean μ and variance σ^2 if its pdf is

$$X \sim \mathsf{N}(\mu, \sigma^2) \iff f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \textit{ where } x \in \mathbb{R}$$

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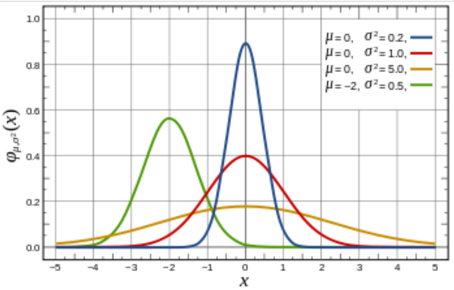
Pdf: unimodal and symmetric, bell shaped continuous random variable



The normal pdf is well-defined

(not required)

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \; e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = 1$$

Use the fact that normal pdf is well define, calculate the integral

$$\int_{-\infty}^{\infty} e^{-\frac{(x-1)^2}{10}} dx = ?$$

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$$1 = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}}\; e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi \cdot 5}}\; e^{-\frac{(x-1)^2}{10}} dx$$

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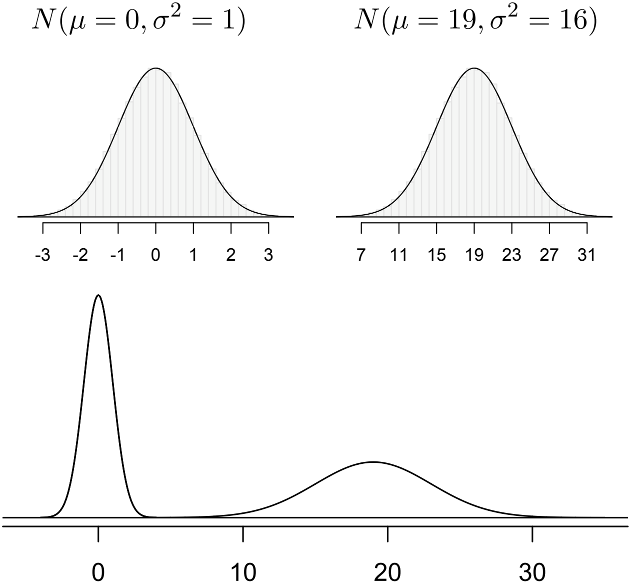
$$\Longrightarrow \int_{-\infty}^{\infty} e^{-\frac{(x-1)^2}{10}} dx = \sqrt{10\pi}$$

Normal distributions with different parameters



Mean and variance of Normal random variable

If $X \sim \mathsf{N}(\mu, \sigma^2)$, and $Z = \frac{X-\mu}{\sigma}$, then Z has a standard normal distribution (more later in Chapter 5.7).

$$Z \sim \mathsf{N}(0, 1)$$

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$$\begin{aligned} E[Z] &= \int_{-\infty}^{\infty} x f_Z(x) dx = \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x \; e^{-\frac{x^2}{2}} dx \\ &= -\frac{1}{\sqrt{2\pi}} \; e^{-\frac{x^2}{2}} \Big|_{-\infty}^{\infty} = 0 \end{aligned}$$

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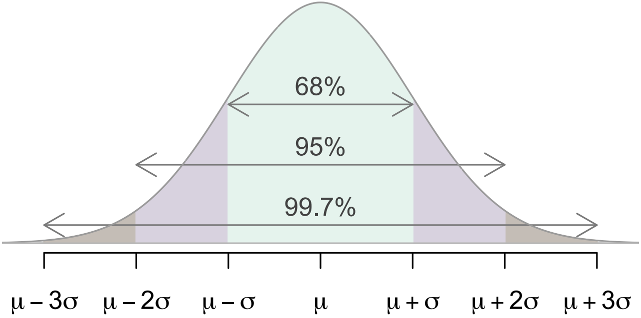
$$\begin{aligned}
Var[Z] = E[Z^2] - (E[Z])^2 = E[Z^2] &= \int_{-\infty}^{\infty} x^2 f_Z(x)dx = \int_{-\infty}^{\infty} x^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 \ e^{-\frac{x^2}{2}} dx \ \ (\text{integration by parts} : u = x, dv = xe^{-x^2/2}dx) \\
&= \frac{1}{\sqrt{2\pi}} \left\{ - xe^{-\frac{x^2}{2}} \Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx \right\} \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = 1
\end{aligned}$$

68-95-99.7 Rule

- A random variable X has a normal distribution,
  - about 68% probability X falls within 1 SD of the mean,
  - about 95% probability X falls within 2 SD of the mean,
  - about 99.7% probability X falls within 3 SD of the mean.
- The probability of X falls 4, 5, or more standard deviations away from the mean is very low.



Normal probability calculation

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- Probability calculations for X in terms of Z:

$$\begin{aligned} P(a < X < b) &= P\left(\frac{a-\mu}{\sigma} < Z < \frac{b-\mu}{\sigma}\right) \\ &= \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right) \end{aligned}$$

At Heinz ketchup factory the amounts which go into bottles of ketchup are supposed to be normally distributed with mean 36 oz. and standard deviation 0.11 oz. Once every 30 minutes a bottle is selected from the production line, and its contents are noted precisely. If the amount of ketchup in the bottle is below 35.8 oz. or above 36.2 oz., then the bottle will fails the quality control inspection. What's the probability that the amount of ketchup in a randomly selected bottle is less than 35.8 ounces?

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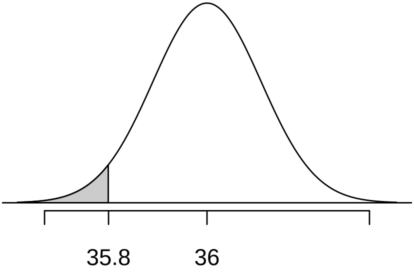

$$z = \frac{x - \mu}{\sigma} = \frac{35.8 - 36}{0.11} = -1.82$$

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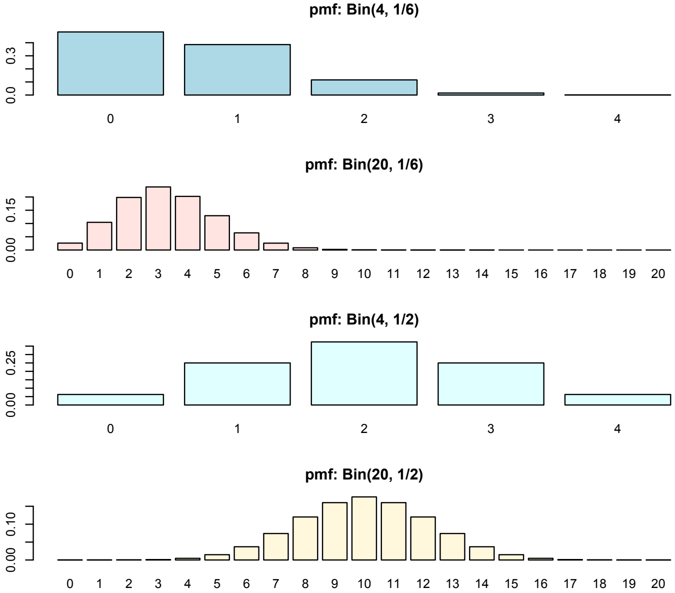
0.09	0.08	0.07	0.06	0.05	0.04	0.03	0.02	0.01	0.00	Z
0.0143	0.0146	0.0150	0.0154	0.0158	0.0162	0.0166	0.0170	0.0174	0.0179	-2.1
0.0183	0.0188	0.0192	0.0197	0.0202	0.0207	0.0212	0.0217	0.0222	0.0228	-2.0
0.0233	0.0239	0.0244	0.0250	0.0256	0.0262	0.0268	0.0274	0.0281	0.0287	-1.9
0.0294	0.0301	0.0307	0.0314	0.0322	0.0329	0.0336	*0.0344*	0.0351	0.0359	-1.8
0.0367	0.0375	0.0384	0.0392	0.0401	0.0409	0.0418	0.0427	0.0436	0.0446	-1.7
0.0455	0.0465	0.0475	0.0485	0.0495	0.0505	0.0516	0.0526	0.0537	0.0548	-1.6
0.0559	0.0571	0.0582	0.0594	0.0606	0.0618	0.0630	0.0643	0.0655	0.0668	-1.5

Normal approximation to the Binomial distribution



Normal approximation to the Binomial distribution

Let $X \sim \text{Bin}(n, p)$. When n is large enough, or more specifically, $np(1-p) \geq 10$, the binomial distribution can be approximated by the normal distribution

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with parameters $\mu = np$ and $\sigma^2 = np(1-p)$.

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- Probability calculation (actually, approximation)

$$\begin{aligned} P(a \leq X \leq b) &\approx P(a - 0.5 < Y < b + 0.5) \\ &= P\left(\frac{a - 0.5 - np}{\sqrt{np(1-p)}} < \frac{Y - \mu}{\sigma} < \frac{b + 0.5 - np}{\sqrt{np(1-p)}}\right) \\ &= \Phi\left(\frac{b + 0.5 - np}{\sqrt{np(1-p)}}\right) - \Phi\left(\frac{a - 0.5 - np}{\sqrt{np(1-p)}}\right) \end{aligned}$$

The ideal size of a first-year class at a particular college is 150 students. The college, knowing from past experience that, on the average, only 30 percent of those accepted for admission will actually attend, uses a policy of approving the applications of 450 students. Compute the probability that more than 150 first-year students attend this college.

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- We are given that $X \sim \text{Bin}(n = 450, p = 0.3)$, and we are asked for the probability

$$P(X > 150) = p(151) + \cdots + p(450) = 1 - p(0) - p(1) - \cdots - p(151)$$

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- To use normal approximation, first check conditions

$$np(1-p) = 450 \times 0.3 \times 0.7 = 94.5 \geq 10$$

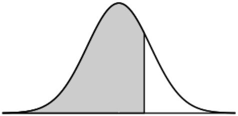
Use Normal approximation with continuity correction,

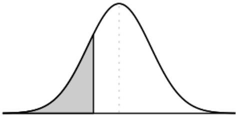
$$\begin{aligned} P(X > 150) &\approx P(X > 150 + 0.5) \\ &= P\left(Z > \frac{150.5 - 450(0.3)}{\sqrt{450(0.3)(0.7)}}\right) \\ &= P(Z > 1.59) \\ &= 1 - 0.9441 = 0.0559 \end{aligned}$$

Recap: Normal distribution $X \sim \mathrm{N}(\mu, \sigma^2)$

- Mean μ, variance σ^2
- Symmetry

$$f(\mu - x) = f(\mu + x), F(\mu - x) = 1 - F(\mu + x)$$





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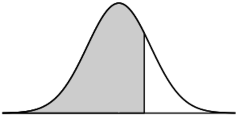

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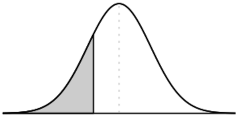

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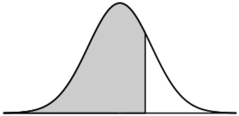

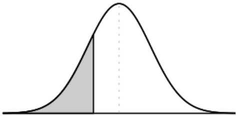

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- Normal approximation to Binomial $X \sim \mathsf{Bin}(n, p)$

$$Y \sim \mathsf{N}(\mu = np, \sigma^2 = np(1 - p))$$

$$P(X = i) \approx P(i - 0.5 < Y < i + 0.5)$$

Review: continuous distributions

Name	Range	pdf $f(x)$	mean	variance
Unif(α, β)	$[\alpha, \beta]$	$\frac{1}{\beta-\alpha}$	$\frac{\alpha+\beta}{2}$	$\frac{(\beta-\alpha)^2}{12}$
N(μ, σ^2)	$(-\infty, \infty)$	$\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}}$	μ	σ^2

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A (important!) theorem on finding pdf of $g(X)$

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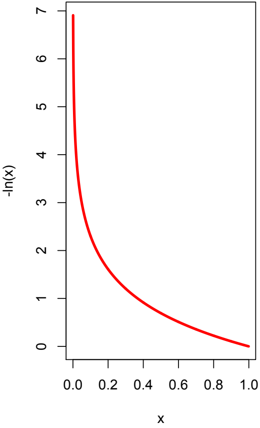
Or more rigorously,

$$f_Y(y) = \begin{cases} f_X\left[g^{-1}(y)\right] \cdot \left| \frac{d}{dy} g^{-1}(y) \right| & \text{if } y = g(x) \text{ for some } x \\ 0 & \text{if } y \neq g(x) \text{ for all } x \end{cases}$$

Example: let $X \sim \text{Unif}(0,1)$, what distribution does $Y = -\ln(X)$ have?



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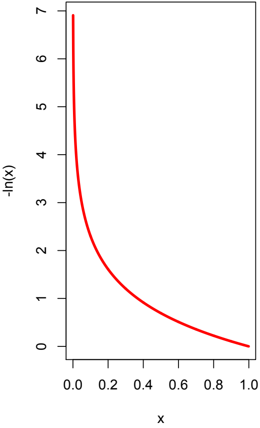

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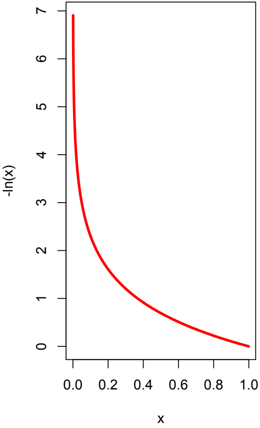

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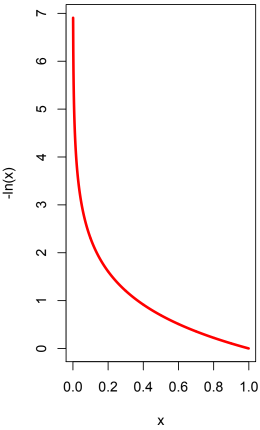

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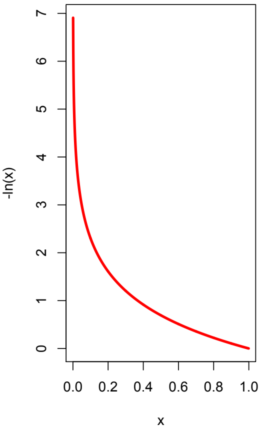

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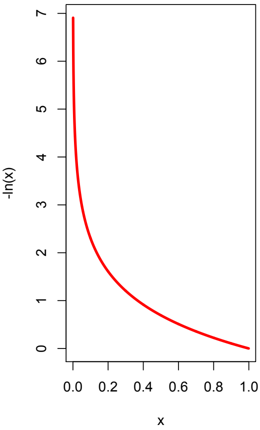

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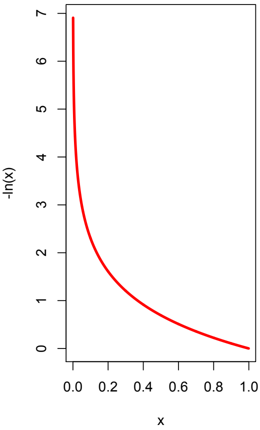

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Range of Y: $y \in [0, \infty)$.

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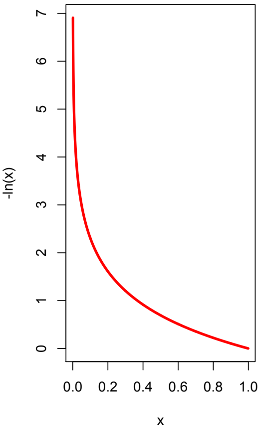

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$$f_Y(y) = f_X(x) \left| \frac{dx}{dy} \right| = 1 \cdot \left| -e^{-y} \right| = e^{-y}$$

Therefor, Y has an exponential distribution,

$$Y \sim \text{Exp}(1)$$

Let X have pdf $f_X(x)$, where $-\infty < x < \infty$. We want to find the pdf for $Y = |X|$. Can we use the formula $f_Y(y) = f_X(x) \left|\frac{dx}{dy}\right|$?

a. Yes
b. No

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(a) Yes
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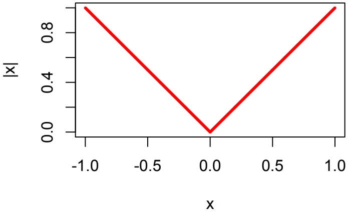
$g(x) = |x|$ is not monotonic; also not differentiable at 0.



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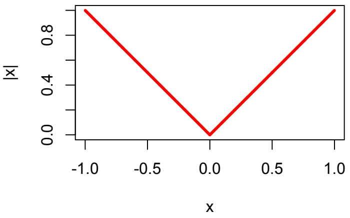

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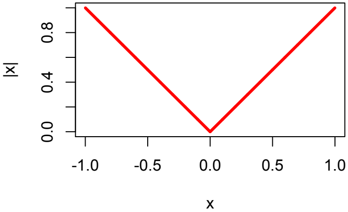

$$
\begin{aligned}
F_Y(y) &= P(Y < y) \\
&= P(|X| < y) \\
&= P(-y < X < y) \\
&= F_X(y) - F_X(-y)
\end{aligned}
$$

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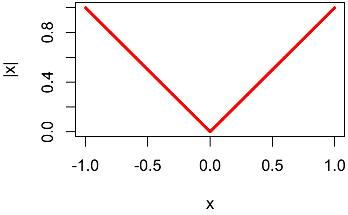

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\begin{aligned}
f_Y(y) &= \frac{d}{dy}[F_X(y) - F_X(-y)] \\
&= f_X(y) + f_X(-y)
\end{aligned}
$$