Chapter 6 part 2

Jointly Distributed Random Variables

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MATH 241

Outline

Outline

Sums of continuous random variables

If X, Y have a joint density $f(x, y)$, then $X + Y$ has the following density

$$\begin{aligned} f_{X+Y}(z) &= \int_{-\infty}^{\infty} f(x, z - x)\; dx \\ &= \int_{-\infty}^{\infty} f(z - y, y)\; dy \end{aligned}$$

If X and Y are independent, we can use the convolution formula

$$\begin{aligned} f_{X+Y}(z) &= \int_{-\infty}^{\infty} f_X(x) f_Y(z - x)\; dx \\ &= \int_{-\infty}^{\infty} f_X(z - y) f_Y(y)\; dy \end{aligned}$$

Cdf F_{X+Y} is called the *convolution* of the distributions F_X and F_Y.

Why?

$$
\begin{aligned}
F_{X+Y}(z) &= \iint_{x+y\leq z} f_{X,Y}(x,y)\ dy\ dx \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{z-x} f_{X,Y}(x,y)\ dy\ dx \\
f_{X+Y}(z) &= \frac{d}{dz}F_{X+Y}(z) \\
&= \int_{-\infty}^{\infty}\left\{\frac{d}{dz}\int_{-\infty}^{z-x} f_{X,Y}(x,y)\ dy\right\}\ dx \\
&= \int_{-\infty}^{\infty} f_{X,Y}(x,z-x)\ dx
\end{aligned}
$$

Textbook page 239 gives a derivation when X and Y are independent.

Question

Suppose $X, Y \overset{\text{ind}}{\sim} \text{Exp}(\lambda)$. What distribution does $X + Y$ have? Note that the pdf of exponential of $X \sim \text{Exp}(\lambda)$ is: $f(x) = \lambda e^{-\lambda x}, x > 0$.

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&= \lambda^2 x e^{-\lambda z} \Big|_0^z
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$$f_{X+Y}(z) = \begin{cases} \lambda^2 z e^{-\lambda z} & \text{if } z > 0 \\ 0 & \text{otherwise} \end{cases}$$

Sum of independent random variables

Random variables X and Y are independent, then

X	Y	$X+Y$
$\mathsf{N}(\mu_1, \sigma_1^2)$	$\mathsf{N}(\mu_2, \sigma_2^2)$	$\mathsf{N}(\mu_1+\mu_2, \sigma_1^2+\sigma_2^2)$
$\mathsf{Poi}(\lambda_1)$	$\mathsf{Poi}(\lambda_2)$	$\mathsf{Poi}(\lambda_1+\lambda_2)$
$\mathsf{Bin}(n_1, p)$	$\mathsf{Bin}(n_2, p)$	$\mathsf{Bin}(n_1+n_2, p)$

Question

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Hint: find the distribution of $W = -Y$ first.

Since $g(y) = -y$ is monotonic and differentiable on $\mathbb{R}$,

$$f_W(w) = f_Y(y) \left| \frac{dy}{dw} \right|$$

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Hint: find the distribution of $W = -Y$ first.
Since $g(y) = -y$ is monotonic and differentiable on $\mathbb{R}$,

$$f_W(w) = f_Y(y)\left|\frac{dy}{dw}\right| = \frac{1}{\sqrt{2\pi\sigma_2^2}}e^{-\frac{(-w-\mu_2)^2}{2\sigma_2^2}}$$

$$W \sim \mathsf{N}(-\mu_2, \sigma_2^2)$$

Since X and W are also independent,

$$X - Y = X + W \sim \mathsf{N}\left(\mu_1 - \mu_2, \sigma_1^2 + \sigma_2^2\right)$$

Recap

Random variables X and Y are independent if any real sets $A, B \subset \mathbb{R}$,

$$P(X \in A, Y \in B) = P(X \in A)P(Y \in B)$$

Random variables X and Y are independent **if and only if**

- Cdf: for any $x, y \in \mathbb{R}$

$$F_{X,Y}(x, y) = F_X(x)F_Y(y)$$

- For any $x, y \in \mathbb{R}$, the pmf / pdf

$$f_{X,Y}(x, y) = f_X(x)f_Y(y)$$

If X and Y are independent continuous random variables, then

$$f_{X+Y}(z) = \int_{-\infty}^{\infty} f_X(x)f_Y(z-x)\, dx$$

Outline

Discrete conditional distributions

- Recall that for any two events E and F, the conditional probability of E given F is defined by

$$P(E|F) = \frac{P(E \cap F)}{P(F)}$$

provided that $P(F) > 0$.

Discrete conditional distributions

- Recall that for any two events E and F, the conditional probability of E given F is defined by

$$P(E|F) = \frac{P(E \cap F)}{P(F)}$$

provided that $P(F) > 0$.

- If X and Y are discrete random variables, we define the conditional probability mass function (pmf) of X given $Y = y$ by

$$p_{X|Y}(x|y) = P\{X = x|Y = y\} = \frac{P\{X = x, Y = y\}}{P\{Y = y\}} = \frac{p(x,y)}{p_Y(y)}$$

for all values of y such that $p_Y(y) > 0$.

Discrete conditional distributions

- Similarly for the conditional probability density function (cdf), we define it by

$$F_{X|Y}(x|y) = P\{X \leq x|Y \leq y\} = \sum_{x \leq a} p_{X|Y}(a|y)$$

- If X is independent of Y, then

$$\begin{aligned} p_{X|Y}(x|y) &= P\{X = x|Y = y\} \\ &= \frac{P\{X = x, Y = y\}}{P\{Y = y\}} \\ &= \frac{P\{X = x\}P\{Y = y\}}{P\{Y = y\}} \\ &= P\{X = x\} \end{aligned}$$

Question

If X and Y are independent Poisson random variables with respective parameters λ_1 and λ_2, calculate the conditional distribution of X given that $X + Y = n$.

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- $P\{X = k | X + Y = n\} = \frac{P\{X=k, X+Y=n\}}{P\{X+Y=n\}} = \frac{P\{X=k, Y=n-k\}}{P\{X+Y=n\}} \underset{ind}{=} \frac{P\{X=k\}P\{Y=n-k\}}{P\{X+Y=n\}}$

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- We know that $X + Y \sim P(\lambda_1 + \lambda_2)$ (sum of Poisson)

$$
\begin{aligned}
P\{X = k|X + Y = n\} &= \frac{e^{-\lambda_1}\lambda_1^k}{k!}\frac{e^{-\lambda_2}\lambda_2^{n-k}}{(n-k)!}\left[\frac{e^{-(\lambda_1+\lambda_2)}(\lambda_1+\lambda_2)^n}{n!}\right]^{-1} \\
&= \frac{n!}{(n-k)!k!}\frac{\lambda_1^k\lambda_2^{(n-k)}}{(\lambda_1+\lambda_2)^n} \\
&= \binom{n}{k}\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^k\left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{(n-k)}
\end{aligned}
$$

- That is, the conditional distribution of X given $X + Y = n$ is a $Bin(n, \lambda_1/(\lambda_1 + \lambda_2))$

Outline

Continuous conditional distributions

- With joint probability function of X and Y as $f(x, y)$, the conditional pdf of X given that $Y = y$ (for $f_Y(y) > 0$) is defined by

$$f_{X|Y}(x|y) = \frac{f(x, y)}{f_Y(y)}$$

  note that the event $\{Y = y\}$ has probability 0; we just use it for conditioning.

- If X and Y are jointly continuous, then for any set A

$$P\{X \in A|Y = y\} = \int_A f_{X|Y}(x|y)dx$$

  This is defining conditional probabilities of events associated with one random variable when we are given the value of a second random variable.

Continuous conditional distributions

- By setting $A = (-\infty, a)$, we can define the cdf of X given that $Y = y$ by

$$F_{X|Y}(a|y) = P\{X \leq a|Y = y\} = \int_{-\infty}^{a} f_{X|Y}(x|y)dx$$

- And if X and Y are independent continuous random variables,

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{f_X(x)f_Y(y)}{f_Y(y)} = f_X(x)$$

This is the unconditional density of X.

Question

Suppose the joint density of X and Y is given by

$$f(x,y) = \begin{cases} \frac{e^{-x/y}e^{-y}}{y} & \text{for } 0 < x < \infty, 0 < y < \infty \\ 0 & \text{otherwise} \end{cases}$$

Find $P\{X > 1 \mid Y = y\}$.

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Find $P\{X > 1 \mid Y = y\}$.

First, obtain the conditional density of X given $Y = y$,

$$\begin{aligned} f_{X|Y}(x \mid y) &= \frac{f(x,y)}{f_Y(y)} \\ &= \frac{e^{-x/y}e^{-y}/y}{e^{-y}\int_0^\infty (1/y)e^{-x/y}dx} \\ &= \frac{1}{y}e^{-x/y} \end{aligned}$$

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Hence,

$$\begin{aligned} P\{X > 1 \mid Y = y\} &= \int_1^\infty \frac{1}{y}e^{-x/y}dx \\ &= -e^{-x/y}\big|_1^\infty \\ &= e^{-1/y} \end{aligned}$$