Chapter 4 part 1

Discrete Random Variables

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MATH 241

Outline

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Random Variables

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Example: If $S = \{a = (a_1, a_2) : 1 \leq a_1, a_2 \leq 6\}$ is the 36 element space resulting from rolling two fair six-sided dice, then the following are all random variables

$$X(a) = a_1$$
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- Random variable is a number associated with a random experiment.
- Random variables are in essence a fancy way of describing an event, e.g.

$$P(X = 1) = 1/6$$

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$$
\begin{aligned}
P(X = 1) &= P(\mathsf{H}) = p \\
P(X = 2) &= P(\mathsf{TH}) = (1-p)p \\
P(X = 3) &= P(\mathsf{TTH}) = (1-p)^2 p \\
&\cdots\cdots \\
P(X = n) &= P(\mathsf{TT}\cdots\mathsf{TH}) = (1-p)^{n-1} p \\
&\cdots\cdots
\end{aligned}
$$

Cumulative distribution function (cdf)

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For a random variable X, the function F defined by

$$F(x) = P(X \leq x), \quad -\infty < x < \infty$$

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- Little x: a real-valued number
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In the previous coin flipping example,

$$F(n) = P(X \leq n) = \sum_{i=1}^{n} (1-p)^{i-1} p = \frac{[1-(1-p)^n]p}{1-(1-p)} = 1-(1-p)^n$$

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4. $F(x)$ is right continuous; i.e., for any decreasing sequence $\{x_n : n = 1, 2, \ldots\}$ that converges to x,
$$\lim_{n \to \infty} F(x_n) = F(x) \Longrightarrow \lim_{n \to \infty} P(X \leq x + \frac{1}{n}) = P(X \leq x)$$

The cdf tells us everything about a random variable X

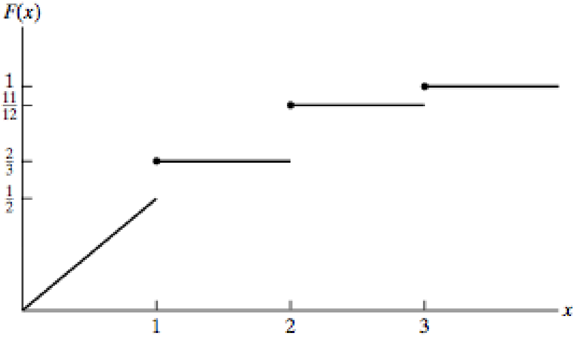

FIGURE 4.8: Graph of $F(x)$.

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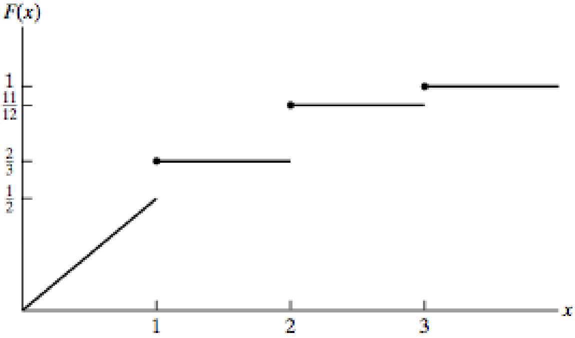

FIGURE 4.8: Graph of *F*(*x*).

- $P(X \leq 0.5) = 0.5 \times \frac{1/2}{1} = 0.25$

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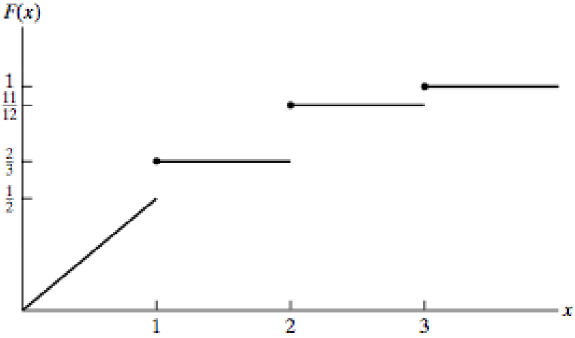

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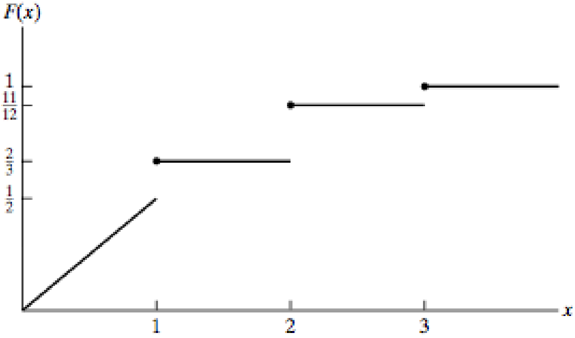

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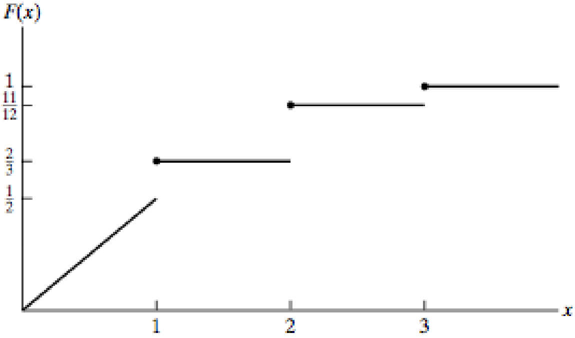

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- $P(X = 1) = P(X \leq 1)$
  $- P(X < 1) = \frac{2}{3} - \frac{1}{2} = \frac{1}{6}$

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  X can be $0, 1, 2, \ldots, \infty$, countable. $\Longrightarrow$ discrete random variable.
- $X =$ a random number in $[0, 1]$ generated by computer
  X can anything in $[0, 1]$, uncountable. $\Longrightarrow$ not discrete random variable.

pmf and cdf

- For a discrete random variable X, there exists a countable sequence $x_1, x_2, \ldots$, such that

$$\begin{aligned} p(x_i) &> 0 \quad \text{for } i = 1, 2, \ldots \\ p(x) &= 0 \quad \text{for all other values of } x \end{aligned}$$

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- Relationship between pmf and cdf (for discrete random variable)

$$F(a) = \sum_{\text{all } x \leq a} p(x)$$

- If we know pmf, we can compute cdf. And vice versa.

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$$F(a) = \begin{cases} 0 & a < 1 \\ \frac{1}{4} & 1 \leq a < 2 \\ \frac{3}{4} & 2 \leq a < 3 \\ \frac{7}{8} & 3 \leq a < 4 \\ 1 & 4 \leq a \end{cases}$$

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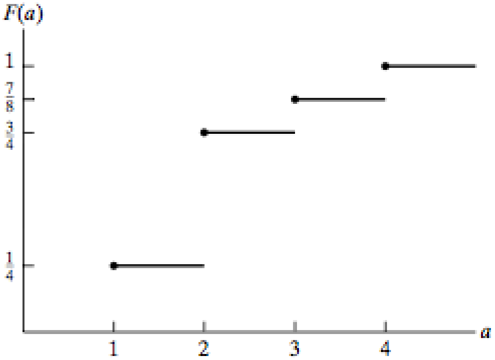

Recap

Random Variable

- Random Variable X is a real-valued function on the sample space S.

$$X : S \longrightarrow \mathbb{R}$$

- Cumulative distribution function (cdf)

$$F_X(x) = P(X \leq x), \quad \text{for any } x \in \mathbb{R}$$

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$$p_X(x) = P(X = x)$$

- $\sum_{i=1}^{\infty} p(x_i) = 1$
- $F(a) = \sum_{\text{all } x \leq a} p(x)$

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Expected value

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The expected value (or mean) of a discrete random variable is defined as

$$E[X] = \sum_{x:p(x)>0} x \cdot P(X = x) = \sum_{x:p(x)>0} xp(x)$$

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When she told me I was average,
she was just being mean.

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$$p(1) = p(2) = p(3) = p(4) = \frac{1}{4}$$

$$E[X] = 1 \times \frac{1}{4} + 2 \times \frac{1}{4} + 3 \times \frac{1}{4} + 4 \times \frac{1}{4} = 2.5$$

Expectation of a function of a random variable

- If X is a discrete random variable, and g is a real-valued function then the expectation (or expected value) of $Y = g(X)$ is

$$E[g(X)] = \sum_{x:p_X(x)>0} g(x) \cdot p_X(x)$$

X is a discrete random variable with pmf

x	-1	0	1
$p_X(x)$	$1/4$	$1/2$	$1/4$

$Y = X^2$. Compute $E[Y]$.

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$$
\begin{aligned}
E[Y] &= \sum_{\text{all } x} x^2 \cdot p_X(x) \\
&= (-1)^2 \times \frac{1}{4} + 0^2 \times \frac{1}{2} + 1^2 \times \frac{1}{4} \\
&= \frac{1}{2}
\end{aligned}
$$

Properties of expected values

If a and b are constants, then

$$E[aX + b] = aE[X] + b$$

- Holds for all random variable X (not necessary discrete random variable).

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- Holds for all random variable X (not necessary discrete random variable).
- Proof for discrete random variable: let $g(X) = aX + b$

Special cases of linear transformation $E[aX+b]=aE[X]+b$

- constant factor

$$E[aX]=aE[X]$$

- constant

$$E[b]=b$$

Recap

Expectation μ

- **For discrete random variable**: $E[X] = \sum_{\text{all } x} x \cdot p(x)$
- **Functions**: $E[g(X)] = \sum_{\text{all } x} g(x)\ p(x)$
- **Indicators**: $E[\delta_A] = P(A)$ where δ_A is an indicator function
- **Linear function**: $E[aX + b] = aE[X] + b$
- **Constants**: $E[c] = c$ if c is constant

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For two random variable's X and Y

$$E[X + Y] = E[X] + E[Y]$$