Chapter 4 part 2

Discrete Random Variables

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MATH 241

Outline

Variance

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- One common simplification:

$$Var(X) = E(X^2) - \mu^2$$

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$$SD(X) = \sqrt{Var[X]} = 1.12$$

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x	-1	1
$p_{X_1}(x)$	$1/2$	$1/2$

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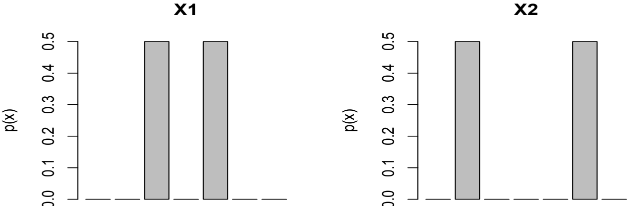
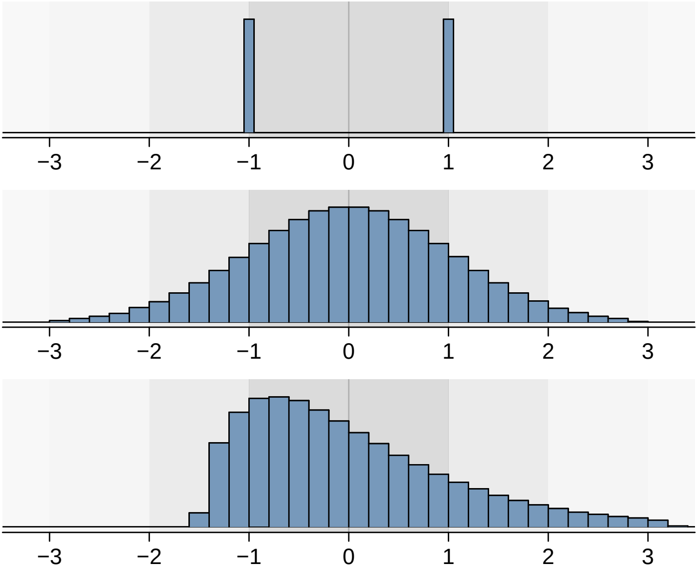
Increasing variance (or sd) reflects increasing spread.



Distributions with SD = 1



Property of variance

We know that $Var(X) = E[(X - \mu)^2]$, and for constants a, b,

$$E[aX + b] = aE[X] + b.$$

Write $Var(aX + b)$ as a function of $Var(X)$.

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$$\begin{aligned} Var(aX + b) &= E[(aX + b - E[aX + b])^2] \\ &= E[(aX + b - aE[X] - b)^2] \\ &= E[(aX - a\mu)^2] \\ &= E[a^2(X - \mu)^2] \\ &= a^2 Var(X) \end{aligned}$$

Properties of variance

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- $$Var(aX) = a^2 Var(X)$$
- $$Var(X + b) = Var(X)$$
- $$Var(b) = 0$$

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- Found by Swiss mathematician Jacob Bernoulli.

Examples of Bernoulli distributions

- Toss a fair coin and obtain a head. $p = 0.5$.
- Roll a fair 6-sided die and obtain a 6. $p = 1/6$.
- Earn an A for this class. $p \in [0, 1]$.

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Find the mean and variance of $X \sim \text{Ber}(p)$.

$$E(X) = 1 \times p + 0 \times (1 - p) = p$$

$$Var(X) = E(X^2) - (E[X])^2 = 1^2 \times p + 0^2 \times (1 - p) - p^2 = p - p^2 = p(1 - p)$$

Recap

Variance σ^2

- **For all random variable**

$$Var(X) = E[(X-\mu)^2] = E[X^2] - \mu^2$$

- **Linear function**

$$Var(aX+b) = a^2 Var(X)$$

- **Constants**

$$Var(c) = 0$$

Standard deviation

$$SD(X) = \sqrt{Var(X)}$$

Bernoulli distribution

$$p(1) = p, \quad p(0) = 1-p$$

$$\mu = p, \quad \sigma^2 = p(1-p)$$

Binomial distribution

Definition

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- The number of A's students will earn in this semester. $n =$ number of classes you're taking, p varies by classes... $\Longrightarrow$ not really a Binomial random variable!

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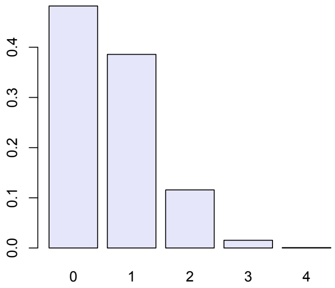

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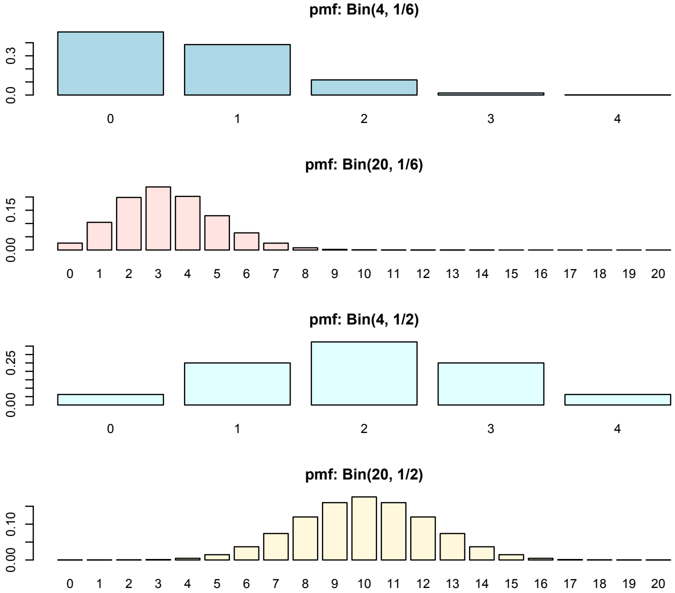
2. the number of trials, n, is fixed
3. each trial outcome can be classified as a *success* or a *failure*
4. the probability of success, p, is the same for each trial

Pmf of Binomial distribution: uni-modal



Binomial pmf is valid (or well-defined)

- Positive: for any $x \in \mathbb{R}$

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- Total one (required):

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Recall the Binomial Theorem $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$

$$\sum_{k=0}^{n} p(k) = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} = [p + (1-p)]^n = 1^n = 1$$

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$$E[X] = np$$

Check textbook page 131 for the derivation (not required).

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- If we have independent Bernoulli random variable's $X_1, X_2, \dots, X_n$ with the same probability of success p, then their sum has a Binomial distribution

$$X = X_1 + X_2 + \cdots + X_n \sim \text{Bin}(n, p)$$

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$$E[X] = np = 4 \times (1/6) = 2/3$$
$$Var(X) = np(1-p) = 4 \times (1/6) \times (5/6) = 5/9$$

Recap

Binomial distribution $X \sim \text{Bin}(n, p)$

$$p(k) = \binom{n}{k} p^k (1-p)^{n-k}$$

- mean $\mu = np$
- variance $\sigma^2 = np(1-p)$