Chapter 2

Axioms of Probability

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MATH 241

Outline

Sample space

Definition

A sample space S is the set of all possible outcomes of an experiment.

Examples:	3 coin tosses	S $= \{$HHH, HHT, HTH, HTT, THH, THT, TTH, TTT$\}$
	One die roll	S $= \{$1,2,3,4,5,6$\}$
	Sum of two rolls	S $= \{$2,3,...,11,12$\}$
	Seconds waiting for bus	S $= [0, \infty)$

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$$4 \times 4 \times 4 = 64$$

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- Impossible event: empty set $\emptyset \subset S$
- $S \subset S$

Set theory

Let A, B be two events.

Definition

1. **Intersection** $A \cap B$: *implies the event that both A and B occur*
2. **Union** $A \cup B$: *implies the event that at least one of A or B occur*
3. *The* **complement** *of an event A denoted A^c (also notated A' or $\bar{A}$):* $A^c = S \backslash A$ *- the event that A does not occur*
4. $A \subset B$ *implies that the occurrence of A implies the occurrence of B*

Venn diagram

More set theory

Definition

Two events A and B are **mutually exclusive** *or* **disjoint** *if they have no outcomes in common, i.e. $A \cap B = \emptyset$.*

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Examples of exclusive events?

Some rules

1. Commutative laws

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3. Distributive laws

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DeMorgan's Laws

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  Left $\subset$ Right $\Longleftrightarrow$ For any $x \in (A \cup B)^c$, then $x \in A^c \cap B^c$.
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- DeMorgan's laws can be generalized to n events $A_1, \ldots, A_n$:

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https://www.youtube.com/watch?v=oOx7FSzSav4

Recap

- Sample space S, event E
- Set operations: intersection, union, complement, subset, disjoint/mutually exclusive
- Rules: commutative, associative, distributive, DeMorgan's laws
- Hint: Venn Diagram is helpful

Outline

What is probability?

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What is $P(A)$ of the following events (A's)?

1. $A =$ Someone in this class room wins the MegaMillion.
2. $A =$ You spin a quarter and it comes up heads.
3. $A =$ You spin a quarter and it stands up.
4. $A =$ At least two people in this class room have the same birthday.

Probability: Frequentist interpretation

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- Often associated with Jerzy Neyman and Egon Pearson who described the logic of statistical hypothesis testing.

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- Bayesian interpretations of probabilities avoid some of the philosophical difficulties of frequency interpretations.
- Named after the 18^{th} century Presbyterian minister and mathematician Thomas Bayes.
- Largely popularized by revolutionary advance in computational technology and methods during the last twenty years.

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- Kolmogorov (1903-1987) was one of the greatest mathematicians of the 20th century. This axiomatization was one of his "trivial" accomplishment.

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3. *Axiom 3: countable addition*

$$P\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i), \text{ if } E_i \cap E_j = \emptyset \text{ for } i \neq j$$

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In particular, for k **disjoint** events $E_1, \ldots, E_k$,

$$P\left(\bigcup_{i=1}^{k} E_i\right) = \sum_{i=1}^{k} P(E_i)$$

Outline

Propositions

☞ Complement Rule:

$$P(A^c) = 1 - P(A)$$

Proof (hint: use Axiom 2 and Axiom 3)

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- $P(B) \geq P(A)$, if $A \subseteq B$

Proof (hint: use Axiom 1)

☞ Inclusion-Exclusion: two events A, B (not necessarily disjoint)

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Proof (hint: use Axiom 3 and Venn-diagram)

Inclusion-Exclusion example

Suppose that for a randomly selected student in a probability class,

- $P(\text{live Eastern Time Zone at home}) = 63\%$.
- $P(\text{senior}) = 41\%$.
- $P(\text{live Eastern Time Zone at home and senior}) = 31\%$.

Find the probability that a student is either live Eastern Time Zone at home or a senior.

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1. Conditions: event $A = \{$live Eastern Time Zone at home$\}$, $B = \{$senior$\}$,

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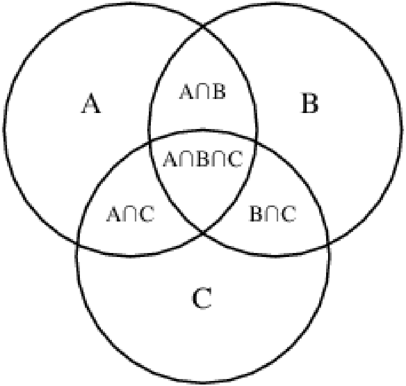
2. Question: find $P(A \cup B)$.
3. Formula: inclusion-exclusion

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.63 + 0.41 - 0.31 = 0.73$$

Propositions

☞ Inclusion-Exclusion: three events A, B, C (not necessarily disjoint)

$$\begin{aligned} P(A \cup B \cup C) =& P(A) + P(B) + P(C) - P(A \cap B) \\ &- P(A \cap C) - P(B \cap C) + P(A \cap B \cap C) \end{aligned}$$



Recap

Three axioms of probability P

1. $0 \leq P(E) \leq 1$
2. $P(S) = 1$
3. $P\left(\bigcup\limits_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i)$, if $E_i \cap E_j = \emptyset$ for $i \neq j$

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Propositions of probability

- $P(A^c) = 1 - P(A)$
- $P(B \cap A^c) = P(B) - P(A)$, if $A \subseteq B$
- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
- $P(A \cup B \cup C) =$ $P(A)+P(B)+P(C)-P(A\cap B)-P(A\cap C)-P(B\cap C)+P(A\cap B\cap C)$

Outline

Sample spaces with equally likely outcomes

- ☞ Suppose a sample space has N equally likely outcomes $\{1\}, \ldots, \{N\}$, then

  $$S = \{1\} \cup \ldots \cup \{N\}$$

  Disjointness gives

  $$1 = P(S) = P(\{1\}) + \ldots + P(\{N\}) = NP(\{i\}),$$

  so for each $1 \leq i \leq N$,

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- For event E in a sample space S with equally likely outcomes,

  $$P(E) = \frac{\#(E)}{\#(S)}$$

  Notation:

  Cardinality - $\#(E) =$ number of elements in set E

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$$E = \{BB, GB, BG\} \text{ and } S = \{BB, GG, GB, BG\}$$
$$P(E) = 3/4$$

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$$\begin{aligned}
\#(\text{birthdays of } n \text{ people}) &= 365^n \\
\#(\text{no match}) &= 365 \times 364 \times \cdots \times (365 - n + 1) \\
P(\text{no match}) &= \frac{365!}{(365 - n)!\, 365^n}
\end{aligned}$$

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