Chapter 3

Conditional Probability and Independence

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MATH 241

Outline

Conditional probability

Definition

Given two events A *and* B *with* $P(B) > 0$, *the conditional probability of* A *given* B *has occurred is defined as*

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

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- When B is the sample space: $P(A \mid B) = P(A)$
- Intuition: in a sample space with equally likely outcomes,

$$P(A \mid B) = \frac{\#(A \cap B)}{\#(B)}$$

Question

A survey asked if whether voters who are familiar with the DREAM act support or oppose it.

- 32% of the respondents are Democrats,
- 51% of the respondents support the DREAM act, and
- 21% of the respondents are Democrats and support the DREAM act.

If we randomly select a respondent who supports the DREAM act, what is the probability that s/he is a Democrat?

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$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

$$P(\text{support}) = 0.51$$

$$P(\text{Democrat and support}) = 0.21$$

$$P(\text{Democrat} \mid \text{support}) = \frac{0.21}{0.51} = 0.41$$

Question

At an apartment complex, 58% of the units have a washer and dryer, 32% have double parking, and 20% have both washer & dryer and double parking.

1. What percent of apartments have neither double parking nor washer and dryer?
2. A unit with double parking just became available at this apartment complex, what is the probability that it also has washer and dryer?

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$$P(\text{w\&d} \cup \text{dbl prk})$$
$$= 0.58 + 0.32 - 0.20$$
$$= 0.70$$
$$P(\text{neither w\&d nor dbl prk})$$
$$= 1 - 0.70 = 0.30$$

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$$P(\text{neither w\&d nor dbl prk}) \\ = 1 - 0.70 = 0.30$$

$$P(\text{w\&d} \mid \text{dbl parking}) \\ = \frac{P(\text{w\&d} \cap \text{dbl prk})}{P(\text{dbl prk})} \\ = \frac{0.20}{0.32} = 0.625$$

Propositions of conditional probability

1. $P(A \mid A) = 1$

2. $P(A^c \mid A) = 0$

3. $P(A^c \mid B) = 1 - P(A \mid B)$

Multiplication rule

☞ By the definition of conditional probability, the *joint probability* of A and B is

$$P(A \cap B) = P(A \mid B)P(B)$$

- Usually, $P(A)$ and $P(B)$ are called *marginal probabilities*.

☞ Generalize to n events: chaining of probabilities

$$P\left(\bigcap_{i=1}^{n} A_i\right) = P(A_1)P(A_2 \mid A_1)P(A_3 \mid A_1, A_2)\cdots P(A_n \mid A_1, \ldots, A_{n-1})$$

Law of total probability

For events $A_1, \ldots, A_n$ are **disjoint**, and

$$\bigcup_{i=1}^{n} A_i = S,$$

then for any event B,

☞ law of total probability

$$P(B) = P(B \mid A_1)P(A_1) + \ldots + P(B \mid A_n)P(A_n)$$

- Such collection of sets $A_1, \ldots, A_n$ is also called a partition of sample space.

Question

Chapter 3 Problem 47. An urn contains 5 white and 10 black balls. A fair 6-sided die is rolled and that number of balls is randomly chosen from the urn.
(a) What is the probability that all of the balls selected are white?

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Let $A_i = \{\text{the outcome of the die roll is } i\}$. Let $B = \{\text{all white balls}\}$. Then $P(A_i) = 1/6$, and

$$P(B \mid A_i) = \frac{\binom{5}{i}}{\binom{15}{i}}.$$

Because $A_1, \cdots, A_n$ form a partition of the sample space (i.e. disjoint, and $\cup_{i=1}^{6} A_i = S$), by Law of Total Probability,

$$\begin{aligned} P(B) &= P(B \mid A_1)P(A_1) + \ldots + P(B \mid A_6)P(A_6) \\ &= \sum_{i=1}^{6} \frac{1}{6} \times \frac{\binom{5}{i}}{\binom{15}{i}}. \end{aligned}$$

Recap

- Marginal probability: $P(A), P(B)$
- Joint probability: $P(A \cap B)$
- Conditional probability

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

Recap

- Marginal probability: $P(A), P(B)$
- Joint probability: $P(A \cap B)$
- Conditional probability

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

- Multiplication rule:

$$P(A \cap B) = P(A \mid B) \times P(B)$$

- Law of total probability: for a partition $\{A_1, A_2, \ldots, A_n\}$ of S,

$$P(B) = \sum_{j=1}^{n} P(B \mid A_j) P(A_j)$$

Question

Which is the correct notation for the following probability?
"At a coffee shop you overhear a recent college graduate discussing that she doesn't believe that online courses provide the same educational value as one taken in person. What's the probability that she has taken an online course before?"

a. P(took online course | not valuable)
b. P(not valuable | took online course)
c. P(took online course and not valuable)
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$$P(\text{both boys} \mid \text{at least one boy}) = \frac{1/4}{3/4} = \frac{1}{3}$$

Outline

Bayes theorem (also called Bayes rule)

Suppose events $A_1, \ldots, A_n$ are disjoint, and $\bigcup_{i=1}^{n} A_i = S$, with $P(A_i) > 0$, $i = 1, 2, \ldots, n$. Then for any event B with $P(B) > 0$,

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\begin{aligned}
P(A_i \mid B) = & \frac{P(B \mid A_i)P(A_i)}{\sum_{j=1}^{n} P(B \mid A_j)P(A_j)}, \quad i = 1, \ldots, n \\
= & \frac{P(B \mid A_i)P(A_i)}{P(B \mid A_1)P(A_1) + \cdots + P(B \mid A_n)P(A_n)}
\end{aligned}
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\end{aligned}
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- $P(A_i)$ is often called *prior probability*
- $P(A_i \mid B)$ is called *posterior probability*.

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Let $A_i = \{\text{the outcome of the die roll is } i\}$. Let $B = \{\text{all white balls}\}$. Previously we have for part (a) that

$$\begin{aligned} P(B) &= P(B \mid A_1)P(A_1) + \ldots + P(B \mid A_6)P(A_6) \\ &= \sum_{i=1}^{6} \frac{1}{6} \times \frac{\binom{5}{i}}{\binom{15}{i}}. \end{aligned}$$

Part (b) asks $P(A_3 \mid B)$. By Bayes theorem,

$$P(A_3 \mid B) = \frac{P(B \mid A_3)P(A_3)}{\sum_{i=1}^{6} P(B \mid A_i)P(A_i)} = \frac{P(B \mid A_3)P(A_3)}{P(B)}.$$

Recap

Bayes theorem

$$P(A_i \mid B) = \frac{P(B \mid A_i)P(A_i)}{\sum_{j=1}^{n} P(B \mid A_j)P(A_j)}, \quad i = 1, \ldots, n$$

Outline

Product rule for independent events

Definition

Events A and B are independent if

$$P(A \cap B) = P(A) \times P(B)$$

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Events A and B are independent if

$$P(A \cap B) = P(A) \times P(B)$$

- If A and B are independent, then

$$P(A \mid B) = \frac{P(A)P(B)}{P(B)} = P(A).$$

Knowing B doesn't affect the odds of A.

Question

Roll two fair 6-sided dice. Set

- $A = \{\text{Sum is 7}\}$
- $B = \{\text{First roll is 5}\}$
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Are A and B independent? How about B and C?

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$$A \cap B = \{(5,2)\}, P(A \cap B) = 1/36$$
$$P(A) = 6/36 = 1/6, P(B) = 1/6$$

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$$P(A \cap B) = P(A) \times P(B) \Longrightarrow A \text{ and } B \text{ are independent.}$$

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$$B \cap C = \{(5,1),(5,2),(5,3),(5,4),(5,5)\}, P(B \cap C) = 5/36$$
$$P(B) = 1/6, P(C) = 9/36$$

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$$P(B) = 1/6, P(C) = 9/36$$
$$P(B \cap C) \neq P(B) \times P(C) \Longrightarrow B \text{ and } C \text{ are dependent.}$$

Properties of independence

If A and B are independent, then

- B and A are independent.
- A and B^c are independent. And so are A^c and B^c.
- Independent is not disjoint.

Mutually independent

☞ Three events A, B, C are called *mutually independent* if

1.
$$P(A \cap B) = P(A)P(B)$$
$$P(B \cap C) = P(B)P(C)$$
$$P(A \cap C) = P(A)P(C)$$

2.
$$P(A \cap B \cap C) = P(A)P(B)P(C)$$

☞ If only (1) holds but not (2), then A, B, C are called *pairwise independent*.

- Events $A_1, A_2, \ldots, A_n$ are called *mutually independent* (or just independent) if for any $1 \leq r \leq n$ of them

$$P(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_r}) = P(A_{i_1})P(A_{i_2}) \cdots P(A_{i_r})$$

- The key to compute the probability of independent events is to just multiply the probability of the individual events.

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Let S_i denote the i^{th} sum.

$$P(A_i) = P(\{S_1 \neq 5, 7\}) \cdots P(\{S_{i-1} \neq 5, 7\}) P(\{S_i = 5\})$$

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$$P(\{S_i = 5\}) = \frac{4}{36}, P(\{S_i = 7\}) = \frac{6}{36}, P(\{S_i \neq 5,7\}) = \frac{13}{18}$$

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$$P(\{S_i = 5\}) = \frac{4}{36}, P(\{S_i = 7\}) = \frac{6}{36}, P(\{S_i \neq 5, 7\}) = \frac{13}{18}$$

$$P(\{\text{Game ends in 5}\}) = \sum_{i=1}^{\infty} \left(\frac{13}{18}\right)^{i-1} \frac{1}{9} = \left(\frac{1}{1 - \frac{13}{18}}\right) \frac{1}{9} = 0.4$$

Recap

Independence

- $P(A \cap B) = P(A) \times P(B)$, $P(A|B) = P(A)$
- Events $A_1, A_2, \ldots, A_n$ are (mutually) independent if for any $1 \leq r \leq n$ of them

$$P(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_r}) = P(A_{i_1})P(A_{i_2}) \cdots P(A_{i_r})$$

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Which of the following statements is false?

a. Two disjoint events cannot occur at the same time.
b. Two independent events cannot occur at the same time.
c. Two complementary events cannot occur at the same time.

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