Bayesian Synthesis Models part 2

Jingchen (Monika) Hu

Vassar College

Statistical Data Privacy

Outline

Outline

Recap

- Lecture 4
  - Synthesizing continuous outcome variables: Bayesian simple / multiple linear regressions
  - Synthesizing binary outcome variables: Bayesian logistic regression
  - In both cases
    - ★ Prior (default priors in `brms`)
    - ★ MCMC estimation and diagnostics
    - ★ Posterior predictions for synthetic data generation (writing R functions)
    - ★ Simple utility checks

The CE sample

- Our sample is from the 1st quarter of 2019, containing five variables on 5133 CUs

Variable Name	Variable information
UrbanRural	Binary; the urban / rural status of CU: 1 = Urban, 2 = Rural.
Income	Continuous; the amount of CU income before taxes in past 12 months (in *USD*).
Race	Categorical; the race category of the reference person: 1 = White, 2 = Black, 3 = Native American, 4 = Asian, 5 = Pacific Islander, 6 = Multi-race.
Expenditure	Continuous; CU's total expenditures in last quarter (in *USD*).
KidsCount	Count; the number of CU members under age 16.

Plan for this lecture

- Synthesis models for unordered / nominal categorical outcome variables (e.g., Race)
- Synthesis models for count outcome variables (e.g., KidsCount) within **sequential synthesis**
- A joint synthesis model for multivariate categorical data (the DPMPM model (Hu, Reiter, and Wang (2014))) for an ACS sample

Outline

A Bayesian multinomial logistic regression: model specification

- Let Y_i be the Race variable for CU i, taking values from 1 to C (C refers to the number of levels and $C = 6$ in our CE sample)
- We work with a vectorized version of Y_i, denoted as $\tilde{\boldsymbol{Y}}_i$, where $\tilde{\boldsymbol{Y}}_i = (\tilde{Y}_{i1}, \cdots, \tilde{Y}_{iC})$ with $(C-1)$ 0's and one 1 at the occupied category

A Bayesian multinomial logistic regression: model specification

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- A Bayesian multinomial logistic regression model:

$$\{\tilde{\boldsymbol{Y}}_i\} \overset{ind}{\sim} \text{Multinomial}(1; p_{i1}, \cdots, p_{iC}), \tag{1}$$

where p_{ic} is the probability of observation i taking $\tilde{Y}_{ic} = 1$ (i.e. $Y_i = c$), and the 1 indicates 1 trial of this multionmial experiment; $\sum_{c'=1}^{C} p_{ic'} = 1$ for each i

A Bayesian multinomial logistic regression: model specification

- The multinomial logistic regression is a generalization of binary logistic regression to the multi-categorical outcome case

A Bayesian multinomial logistic regression: model specification

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- Assume r predictor variables, $X_{i1}, \cdots, X_{ir}$, we can define the **log odds ratio** for category c relative to category 1 as:

$$\log\left(\frac{p_{ic}}{p_{i1}}\right) = \beta_{0c} + \beta_{1c}X_{i1} + \cdots + \beta_{rc}X_{ir}, \tag{2}$$

with $r+1$ parameters, $\{\beta_{0c}, \cdots, \beta_{rc}\}$ for each category $c > 1$

A Bayesian multinomial logistic regression: model specification

- The multinomial logistic regression is a generalization of binary logistic regression to the multi-categorical outcome case
- Assume r predictor variables, $X_{i1}, \cdots, X_{ir}$, we can define the **log odds ratio** for category c relative to category 1 as:

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  with $r+1$ parameters, $\{\beta_{0c}, \cdots, \beta_{rc}\}$ for each category $c > 1$
- Note that the first category, $c = 1$ of Equation (2), is used as a **baseline**; i.e., Equation (2) $= 0$ when $c = 1$

Discussion question: How many parameters do we have in this model?

A Bayesian multinomial logistic regression: synthesis details

- After some algebra transformation, we can express the probability p_{ic} of record i falling into category c as:

$$p_{ic} = \frac{\exp(\beta_{0c} + \beta_{1c}X_{i1} + \cdots + \beta_{rc}X_{ir})}{\sum_{c'=1}^{C} \exp(\beta_{0c'} + \beta_{1c'}X_{i1} + \cdots + \beta_{rc'}X_{ir})}, \tag{3}$$

  with the constraint that $\exp(\beta_{01} + \beta_{11}X_{i1} + \cdots + \beta_{r1}X_{ir}) = 1$ for the baseline level $c = 1$
- This model is also more specifically referred to as a **baseline-category logit model** (see details in Chapter 8 of Agresti (2012))
- This transformation will be used in our synthesis step later

A Bayesian multinomial logistic regression: synthesis details

- MCMC estimation produces a set of posterior parameter draws, denoted as $\{\beta^*_{0c}, \cdots, \beta^*_{rc}\}$ for $c > 1$
- For notation simplicity, let $p^*_{ic} = h(\beta^*_{0c}, \cdots, \beta^*_{rc}, \boldsymbol{X}_i)$

$$
\begin{aligned}
\text{compute } p^*_{1c} &= h(\beta^*_{0c}, \cdots, \beta^*_{rc}, \boldsymbol{X}_1), \forall c > 1 \rightarrow \\
&\quad \text{sample } \tilde{Y}^*_1 \sim \text{Multinomial}(1; p^*_{11}, \cdots, p^*_{1C}) \\
&\vdots \\
\text{compute } p^*_{ic} &= h(\beta^*_{0c}, \cdots, \beta^*_{rc}, \boldsymbol{X}_i), \forall c > 1 \rightarrow \\
&\quad \text{sample } \tilde{Y}^*_i \sim \text{Multinomial}(1; p^*_{i1}, \cdots, p^*_{iC}) \\
&\vdots \\
\text{compute } p^*_{nc} &= h(\beta^*_{0c}, \cdots, \beta^*_{rc}, \boldsymbol{X}_n), \forall c > 1 \rightarrow \\
&\quad \text{sample } \tilde{Y}^*_n \sim \text{Multinomial}(1; p^*_{n1}, \cdots, p^*_{nC}).
\end{aligned}
$$

A Bayesian multinomial logistic regression: synthesis details

- Here $p_{i1}^* = 1$ for every i
- From the vectorized $\tilde{Y}_i^*$, we can obtain scalar Y_i^*, and therefore we have simulated one synthetic vector $(Y_i^*)_{i=1,\cdots,n}$

A Bayesian multinomial logistic regression: model estimation

- We wish to synthesize categorical variable Race with LogIncome as the single predictor
- The multinomial logistic regression synthesis model can then be set up as follows.

$$\tilde{Y}_i \overset{ind}{\sim} \text{Multinomial}(1; p_{i1}, \cdots, p_{iC}), \tag{4}$$

$$\log\left(\frac{p_{ic}}{p_{i1}}\right) = \beta_{0c} + \beta_{1c} X_i, \tag{5}$$

  where $\tilde{Y}_i$ is the vectorized Race outcome variable and X_i is the LogIncome predictor variable
- $\{\beta_{0c}, \beta_{1c}\}$ are two parameters in this multinomial logistic regression model for category $c > 1$

Discussion question: How many parameters do we have in this model?

A Bayesian multinomial logistic regression: model estimation

- Load the CE dataset and create LogIncome and LogExpenditure

```
CEdata <- readr::read_csv(file = "CEdata.csv")
CEdata$LogIncome <- log(CEdata$Income)
CEdata$LogExpenditure <- log(CEdata$Expenditure)
```

- To streamline our synthesis process, we create the **design matrix $\boldsymbol{X}$** based on the chosen model

```
ml_ff <- stats::as.formula(Race ~ 1 + LogIncome)
ml_model <- stats::model.frame(ml_ff, CEdata)
ml_X <- data.frame(stats::model.matrix(ml_ff, ml_model))
```

A Bayesian multinomial logistic regression: model estimation

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```

- We use the **default priors**

A Bayesian multinomial logistic regression: model estimation

- We use `family = categorical(link = "logit")` in the `brm()` function to fit the multinomial logistic regression model

```
ml_fit <- brms::brm(data = CEdata,
                    family = categorical(link = "logit"),
                    Race ~ 1 + LogIncome,
                    iter = 5000,
                    warmup = 3000,
                    thin = 1,
                    chains = 1,
                    seed = 823)
```

A Bayesian multinomial logistic regression: model estimation

- The key to applying Bayesian synthesis models is to save posterior parameter draws of estimated parameters
- These draws will be used to generate synthetic data given the posterior predictive distribution
- We use the `posterior_samples()` function to retrieve the posterior parameter draws in `post_ml`

A Bayesian multinomial logistic regression: model estimation

```
post_ml <- brms::posterior_samples(x = ml_fit)
post_ml[1:3, ]
```

```
##   b_mu2_Intercept b_mu3_Intercept b_mu4_Intercept b_mu5_Intercept
## 1        1.418636       -3.519348       -4.413770       -7.419548
## 2        1.513357       -3.375913       -4.443419       -7.537551
## 3        1.014357       -3.034133       -3.916273       -6.332770
##   b_mu6_Intercept b_mu2_LogIncome b_mu3_LogIncome b_mu4_LogIncome
## 1       -2.269950      -0.3389376      -0.1799465       0.1323971
## 2       -2.188688      -0.3490482      -0.1875901       0.1371984
## 3       -3.150873      -0.2951873      -0.1782070       0.0929380
##   b_mu5_LogIncome b_mu6_LogIncome      lp__
## 1       0.2182136     -0.14867769 -3423.668
## 2       0.2262599     -0.15525579 -3423.847
## 3       0.0928044     -0.06041435 -3417.185
```

A Bayesian multinomial logistic regression: MCMC diagnostics

- Don't forget to do them!

A Bayesian multinomial logistic regression: synthesis

- Suppose we use one set of posterior draws at the `index` iteration of the MCMC

```
ml_synthesize <- function(X, post_draws, index, n, C, seed){
  set.seed(seed)
  log_p_allC <- matrix(NA, nrow = n, ncol = C)
  for (c in 2:C){
    name_Intercept_c <- paste0("b_mu", c, "_Intercept")
    name_LogIncome_c <- paste0("b_mu", c, "_LogIncome")
    log_p_c <- as.matrix(X) %*%
      t(post_draws[index, c(name_Intercept_c, name_LogIncome_c)])
    log_p_allC[, c] <- log_p_c
  }
  log_p_allC[, 1] <- rep(0, n)
```

A Bayesian multinomial logistic regression: synthesis

```
  p_allC <- exp(log_p_allC) / (1 + exp(log_p_allC))

  synthetic_Y <- rep(NA, n)
  for (i in 1:n){
    synthetic_Y[i] <- which(stats::rmultinom(n = 1, size = 1,
                                            prob = p_allC[i, ]) == 1)
  }
  data.frame(X, synthetic_Y)
}
```

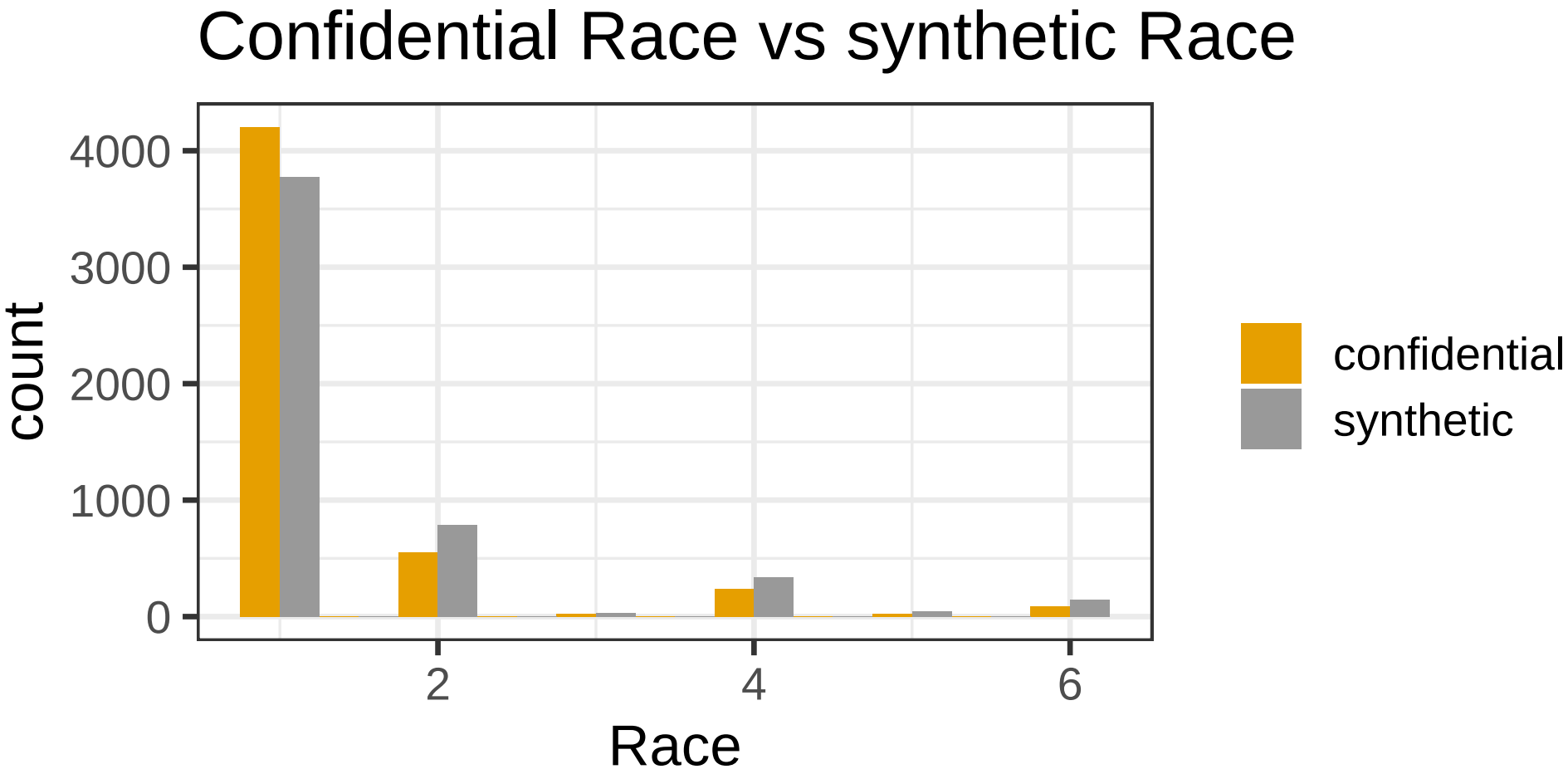
Dicussion question: Discuss the details of this function - why do we use `c in 2:C`?

A Bayesian multinomial logistic regression: synthesis

- Perform synthesis for the CE dataset

```
n <- nrow(CEdata)
ml_synthetic_one <- ml_synthesize(X = ml_X,
                                  post_draws = post_ml,
                                  index = 1,
                                  n = nrow(ml_X),
                                  C = 6,
                                  seed = 983)
names(ml_synthetic_one) <- c("Intercept", "LogIncome", "Race")
```

A Bayesian multinomial logistic regression: utility check



Additional Bayesian models for categorical variables

- Unordered / nominal categorical variables
  - Dirichlet-multinomial conjugate model

Additional Bayesian models for categorical variables

- Unordered / nominal categorical variables
  - Dirichlet-multinomial conjugate model
- Ordered categorical variables
  - Probit model
  - Cumulative logit model
  - Proportional odds model

Outline

Within a sequential synthesis process

- Recall Poisson regression for count variable with predictors for the bike sharing dataset from Lecture 3
- In this section, suppose we need to synthesize KidsCount for privacy protection and we wish to use LogIncome and LogExpenditure as predictors
- Moreover, LogExpenditure is also considered sensitive, which is to be synthesized with predictor LogIncome (as done in Lecture 4)
- We will therefore utilize a sequential synthesis approach

Dicussion question: Discuss the process of this sequential synthesis task. Pay attention to what data is used in what step.

Within a sequential synthesis process

1. We first build a Bayesian synthesis model for LogExpenditure using predictor LogIncome. We estimate this model on the **confidential data**, and generate synthetic LogExpenditure using confidential LogIncome via linear regression synthesis model
2. We next build a Bayesian synthesis model for KidsCount using predictors LogIncome and LogExpenditure. We estimate this model on the **confidential data**, and generate synthetic KidsCount using **synthetic** LogExpenditure from step 1 and **confidential** LogIncome.

A Bayesian Poisson regression: model specification

- Let Y_i be a count random variable, which follows a Poisson distribution with record-specific parameter λ_i:

$$Y_i \overset{ind}{\sim} \text{Poisson}(\lambda_i), \tag{6}$$

where $\lambda_i > 0$ is the rate parameter of the Poisson model for record i

A Bayesian Poisson regression: model specification

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$$Y_i \overset{ind}{\sim} \text{Poisson}(\lambda_i), \tag{6}$$

  where $\lambda_i > 0$ is the rate parameter of the Poisson model for record i

- We can specify the logarithm of the rate parameter, $\log(\lambda_i)$ as a linear function of the predictor variables
- Assume r predictor variables, $X_{i1}, \cdots, X_{ir}$

$$\log(\lambda_i) = \beta_0 + \beta_1 X_{i1} + \cdots + \beta_r X_{ir}. \tag{7}$$

A Bayesian Poisson regression: synthesis details

- MCMC estimation of the Poisson regression synthesis model produces posterior parameter draws, denoted as $\{\beta_0^*, \beta_1^*, \cdots, \beta_r^*\}$
- If none of $X_{i1}, \cdots, X_{ir}$ are synthesized for privacy protection, then to simulate synthetic values for all n observations, we perform the following.

$$\begin{aligned}
\text{compute } \log(\lambda_1^*) &= \beta_0^* + \beta_1^* X_{11} + \cdots + \beta_r^* X_{1r} \rightarrow \\
&\quad \text{sample } Y_1^* \sim \text{Poisson}(\lambda_1^*) \\
&\vdots \\
\text{compute } \log(\lambda_i^*) &= \beta_0^* + \beta_1^* X_{i1} + \cdots + \beta_r^* X_{ir} \rightarrow \\
&\quad \text{sample } Y_i^* \sim \text{Poisson}(\lambda_1^*) \\
&\vdots \\
\text{compute } \log(\lambda_n^*) &= \beta_0^* + \beta_1^* X_{n1} + \cdots + \beta_r^* X_{nr} \rightarrow \\
&\quad \text{sample } Y_n^* \sim \text{Poisson}(\lambda_n)
\end{aligned}$$

A Bayesian Poisson regression: synthesis details

- We will then have simulated one synthetic vector $(Y_i^*)_{i=1,\cdots,n}$.
- Discussion question: What happens if the last variable, X_{ir}, is also sensitive and synthesized beforehand?

A Bayesian Poisson regression: synthesis details

- In such case, we do the following to simulate synthetic Y_i^* for all n observations:

$$\begin{aligned}
\text{compute } \log(\lambda_1^*) &= \beta_0^* + \beta_1^* X_{11} + \cdots + \beta_{r-1}^* X_{1(r-1)} + \beta_r^* X_{1r}^* \rightarrow \\
&\quad \text{sample } Y_1^* \sim \text{Poisson}(\lambda_1^*) \\
&\vdots \\
\text{compute } \log(\lambda_i^*) &= \beta_0^* + \beta_1^* X_{i1} + \cdots + \beta_{r-1}^* X_{i(r-1)} + \beta_r^* X_{ir}^* \rightarrow \\
&\quad \text{sample } Y_1^* \sim \text{Poisson}(\lambda_1^*) \\
&\vdots \\
\text{compute } \log(\lambda_n^*) &= \beta_0^* + \beta_1^* X_{n1} + \cdots + \beta_{r-1}^* X_{n(r-1)} + \beta_r^* X_{nr}^* \rightarrow \\
&\quad \text{sample } Y_n^* \sim \text{Poisson}(\lambda_n^*)
\end{aligned}$$

A Bayesian Poisson regression: synthesis details

- Both **synthetic** $(X^*_{ir})_{i=1,\cdots,n}$ and **confidential** $(X_{i1}, \cdots, X_{i(r-1)})_{i=1,\cdots,n}$ are used, due to the process of sequential synthesis
- We will have simulated one synthetic vector $(Y^*_i)_{i=1,\cdots,n}$

A Bayesian Poisson regression: model estimation

- We wish to synthesize count variable KidsCount with LogIncome and LogExpenditure as two predictors
- Moreover, LogExpenditure is sensitive and synthesized given LogIncome

A Bayesian Poisson regression: model estimation

- We wish to synthesize count variable KidsCount with LogIncome and LogExpenditure as two predictors
- Moreover, LogExpenditure is sensitive and synthesized given LogIncome

- Let Y_i be the KidsCount of CU i, X_{i1} be the LogIncome and X_{i2} be the LogExpenditure of CU i

$$Y_i \overset{ind}{\sim} \text{Poisson}(\lambda_i), \tag{8}$$

$$\log(\lambda_i) = \beta_0 + \beta_1 X_{i1} + \beta_2 X_{i2}, \tag{9}$$

Discussion question: How many parameters do we have in this model? What are they?

A Bayesian Poisson regression: model estimation

- We need to first fit the SLR synthesis model from Lecture 4
- To streamline our synthesis process, we create the **synthetic design matrix $\boldsymbol{X}^*$** based on the chosen model

```
CEdata$LogExpenditure_synthetic <- SLR_synthetic_one$LogExpenditure
Poisson_ff <- stats::as.formula(KidsCount ~ 1 + LogIncome +
                                  LogExpenditure_synthetic)
Poisson_model <- stats::model.frame(Poisson_ff, CEdata)
Poisson_X_star <- data.frame(stats::model.matrix(Poisson_ff,
                                                 Poisson_model))
```

- We use the **default priors**

A Bayesian Poisson regression: model estimation

- We use `family = poisson(link = "log")` in the `brm()` function to fit the Poisson regression model

```
Poisson_fit <- brms::brm(data = CEdata,
                         family = poisson(link = "log"),
                         KidsCount ~ 1 + LogIncome + LogExpenditure,
                         iter = 5000,
                         warmup = 3000,
                         thin = 1,
                         chains = 1,
                         seed = 398)
```

- Note that here the model expression is `KidsCount ~ 1 + LogIncome + LogExpenditure` with `LogExpenditure`; previously for the synthetic design matrix, we use `LogExpenditure_synthetic`

A Bayesian Poisson regression: model estimation

- The key to applying Bayesian synthesis models is to save posterior parameter draws of estimated parameters
- These draws will be used to generate synthetic data given the posterior predictive distribution
- We use the `posterior_samples()` function to retrieve the posterior parameter draws in `post_Poisson`

```
post_Poisson <- brms::posterior_samples(x = Poisson_fit)
post_Poisson[1:3, ]
```

```
##   b_Intercept b_LogIncome b_LogExpenditure      lp__
## 1   -4.564058   0.1734750        0.2244817 -5216.570
## 2   -4.378532   0.1684752        0.2043459 -5215.492
## 3   -4.603462   0.1706977        0.2292895 -5214.753
```

A Bayesian Poisson regression: MCMC diagnostics

- Don't forget to do them!

A Bayesian Poisson regression: synthesis

- Suppose we use one set of posterior draws at the `index` iteration of the MCMC

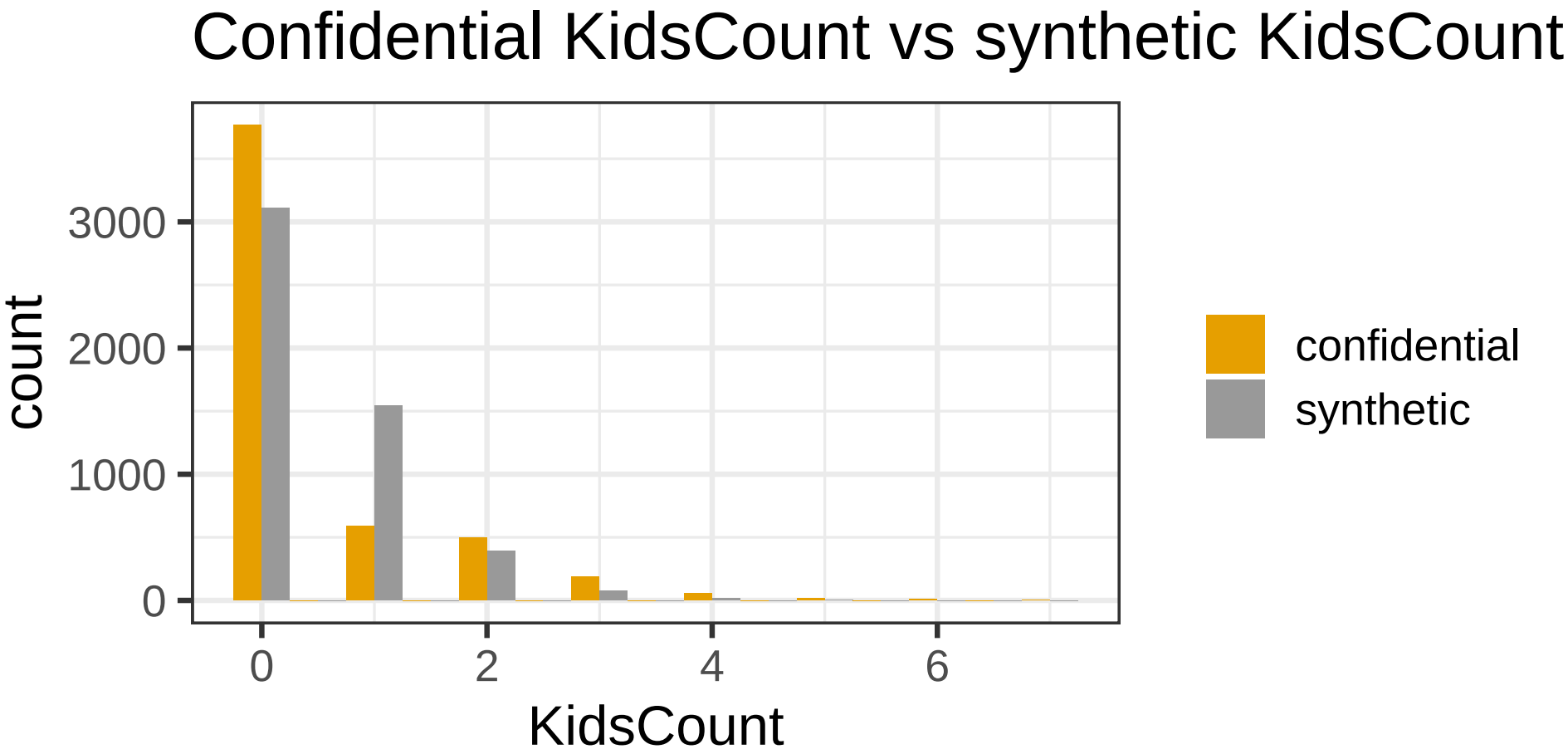
```
Poisson_synthesize <- function(X, post_draws, index, n, seed){
  set.seed(seed)
  lambda_log <- as.matrix(X) %*%
    t(data.matrix(post_draws[index, c("b_Intercept", "b_LogIncome",
                                      "b_LogExpenditure")]))
  synthetic_Y <- stats::rpois(n, exp(lambda_log))
  data.frame(X, synthetic_Y)
}
```

A Bayesian Poisson regression: synthesis

- Now to synthesize KidsCount with **synthetic** LogExpenditure and **confidential** LogIncome, we supply `X_star` which is the synthetic design matrix including the desired values: synthetic for LogExpenditure and confidential for LogIncome

```
n <- nrow(CEdata)
Poisson_ss_synthetic_one <- Poisson_synthesize(X = Poisson_X_star,
                                               post_draws = post_Poisson,
                                               index = 1,
                                               n = nrow(Poisson_X_star),
                                               seed = 653)
names(Poisson_ss_synthetic_one) <- c("Intercept", "LogIncome",
                                     "LogExpenditure", "KidsCount")
```

A Bayesian Poisson regression: utility check



A note on multiple synthetic datasets

- If multiple synthetic datasets are needed, for example $m = 20$, we can then repeat this process m times using m independent MCMC iterations
- A caveat in creating multiple synthetic datasets within sequential synthesis is that we need to create the synthetic design matrix for the second step modeling in the loop over $m = 20$ since that design matrix depends on the synthetic predictor simulated from the first step

Outline

The ACS sample

- Our sample is from 2012 ACS public use microdata, containing ten categorical variables on 10,000 observations

Variable name	Variable information
SEX	1 = male, 2 = female.
RACE	1 = White alone, 2 = Black or African American alone, 3 = American Indian alone, 4 = other, 5 = two or more races, 6 = Asian alone.
MAR	1 = married, 2 = widowed, 3 = divorced, 4 = separated, 5 = never married.
LANX	1 = speaks another language, 2 = speaks only English
WAOB	born in: 1 = US state, 2 = PR and US island areas, oceania and at sea, 3 = Latin America, 4 = Asia, 5 = Europe, 6 = Africa, 7 = Northern America.

The ACS sample

Variable name	Variable information
DIS	1 = has a disability, 2 = no disability.
HICOV	1 = has health insurance coverage, 2 = no coverage.
MIG	1 = live in the same house (non movers), 2 = move to outside US and PR, 3 = move to different house in US or PR.
SCH	1 = has not attended school in the last 3 months, 2 = in public school or college, 3 = in private school or college or home school.
HISP	1 = not Spanish, Hispanic, or Latino, 2 = Spanish, Hispanic, or Latino.

The DPMPM: overview

- A popular joint approach to synthesizing multivariate (unordered / nominal) categorical data is the Dirichlet Process mixture of products of multinomials (DPMPM)
- Hu, Reiter, and Wang (2014) first applied it to a sample of the American Community Survey (ACS) with 14 categorical variables and produced fully synthetic datasets
- Hu and Savitsky (2018) and Drechsler and Hu (2021) utilized DPMPM to synthesize geographic labels for privacy protection and produced partially synthetic data

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- The R package `NPBayesImputeCat` (Wang et al. (2021)) is available to implement the model estimation and synthesis steps

The DPMPM: model specification

- Consider a sample $\mathbf{Y}$ consisting of n records, where each ith record, with $i = 1, \ldots, n$, has r unordered categorical variables
- The basic assumption of the DPMPM is that every record $\mathbf{Y}_i = (Y_{i1}, \cdots, Y_{ir})$ belongs to one of K underlying unobserved **latent classes**
  - A latent class includes observations being classified into that class but the class assignment is unobserved and typically requires model to estimate

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  - A latent class includes observations being classified into that class but the class assignment is unobserved and typically requires model to estimate

- Given the latent class assignment z_i of record i each variable Y_{ij} independently follows a multinomial distribution, where d_j is the number of categories of variable j, and $j = 1, \ldots, r$

$$Y_{ij} \mid z_i, \theta \overset{ind}{\sim} \text{Multinomial}(\theta^{(j)}_{z_i 1}, \ldots, \theta^{(j)}_{z_i d_j}; 1) \quad \forall i, j, \tag{10}$$

$$z_i \mid \pi \sim \text{Multinomial}(\pi_1, \ldots, \pi_K; 1) \quad \forall i \tag{11}$$

The DPMPM: model specification

- The marginal probability of $Pr(Y_{i1} = y_{i1}, \cdots, Y_{ip} = y_{ip} \mid \pi, \theta)$ can be expressed as averaging over the latent classes:

$$Pr(Y_{i1} = y_{i1}, \cdots, Y_{ip} = y_{ir} \mid \pi, \theta) = \sum_{k=1}^{K} \pi_k \prod_{j=1}^{r} \theta_{ky_{ij}}^{(j)} \qquad (12)$$

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- While all variables are considered independent within each of the latent classes, such averaging over latent classes results in dependence among the variables
- The DPMPM effectively clusters records with similar characteristics based on all r variables
- Relationships among these r categorical variables are made explicit after integrating out the latent class assignment z_i

Discussion question: Can you identify the parameters in this model?

The DPMPM: model specification

- The parameters in the DPMPM model include:
  - A vector of probabilities $(\pi_1, \cdots, \pi_K)$
  - A collection of vectors of probabilities $(\theta_{k1}^{(j)}, \ldots, \theta_{kd_j}^{(j)})$, one for each combination of variable j and latent class k.
- We need to provide priors for all these parameters

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  - A collection of vectors of probabilities $(\theta_{k1}^{(j)}, \ldots, \theta_{kd_j}^{(j)})$, one for each combination of variable j and latent class k.
- We need to provide priors for all these parameters

- The choice of K
  - We select a large enough K, which serves as the upper bound of possible latent classes to be used
  - As long as K is set to a large enough value, we allow the DPMPM to fully explore the parameter space and determine the **effective number of occupied latent classes**

The DPMPM: model specification

- To empower the DPMPM to pick the effective number of occupied latent classes, the **truncated stick-breaking representation** (Sethuraman (1994)) of the Dirichlet Process prior is used as

$$\pi_k = V_k \prod_{l<k}(1-V_l) \quad \text{for } k = 1,\ldots,K, \tag{13}$$

$$V_k \overset{iid}{\sim} \text{Beta}(1,\alpha) \quad \text{for } k = 1,\ldots,K-1, \quad V_K = 1, \tag{14}$$

$$\alpha \sim \text{Gamma}(a_\alpha, b_\alpha), \tag{15}$$

$$\boldsymbol{\theta}_k^{(j)} = (\theta_{k1}^{(j)},\ldots,\theta_{kd_j}^{(j)}) \sim \text{Dirichlet}(a_1^{(j)},\ldots,a_{d_j}^{(j)}) \tag{16}$$

$$\text{for } j = 1,\ldots,r, \quad k = 1,\ldots,K. \tag{17}$$

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- Dirichlet prior is conjugate for multinomial data model
- A **blocked Gibbs sampler** is implemented for the MCMC sampling procedure (Ishwaran and James (2001))

The DPMPM: synthesis details

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- Sensitive variable values are synthesized as an extra step at chosen MCMC iteration

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- When used as a data synthesis model, the confidential dataset is used for model estimation through MCMC
- Sensitive variable values are synthesized as an extra step at chosen MCMC iteration

- For example, at MCMC iteration l, we first sample a value of the latent class indicator z_i; given the sampled z_i, we then sample synthetic values of sensitive variables using independent draws; this process is repeated for every record that has sensitive values to be synthesized, obtaining one synthetic dataset

The DPMPM: model estimation

- Make sure to install and load the `NPBayesImputeCat` library

```
library(NPBayesImputeCat)
```

- Load the ACS sample and make sure all variables are categorical with a known number of levels

```
ACSdata <- data.frame(readr::read_csv(file = "ACSdata.csv"))

dj <- c(2, 6, 5, 2, 7, 2, 2, 3, 3, 2)
r <- ncol(ACSdata)
# make sure all variables are categorical
for (j in 1:r){
  ACSdata[, j] <- factor(ACSdata[, j], levels = 1:dj[j])
}
```

The DPMPM: model estimation and synthesis

- Create the DPMPM model

```
model <- NPBayesImputeCat::CreateModel(X = ACSdata,
                                       MCZ = NULL,
                                       K = 80,
                                       Nmax = 0,
                                       aalpha = 0.25,
                                       balpha = 0.25,
                                       seed = 221)
```

The DPMPM: model estimation and synthesis

- Use the `DPMPM_nozeros_syn()` function to create `m = 5` partially synthetic datasets

```
m <- 5
ACSdata_syn <- NPBayesImputeCat::DPMPM_nozeros_syn(X = ACSdata,
                                                   dj = dj,
                                                   nrun = 10000,
                                                   burn = 5000,
                                                   thin = 10,
                                                   K = 80,
                                                   aalpha = 0.25,
                                                   balpha = 0.25,
                                                   m = m,
                                                   vars = c("DIS", "HICOV"),
                                                   seed = 221,
                                                   silent = TRUE)
```

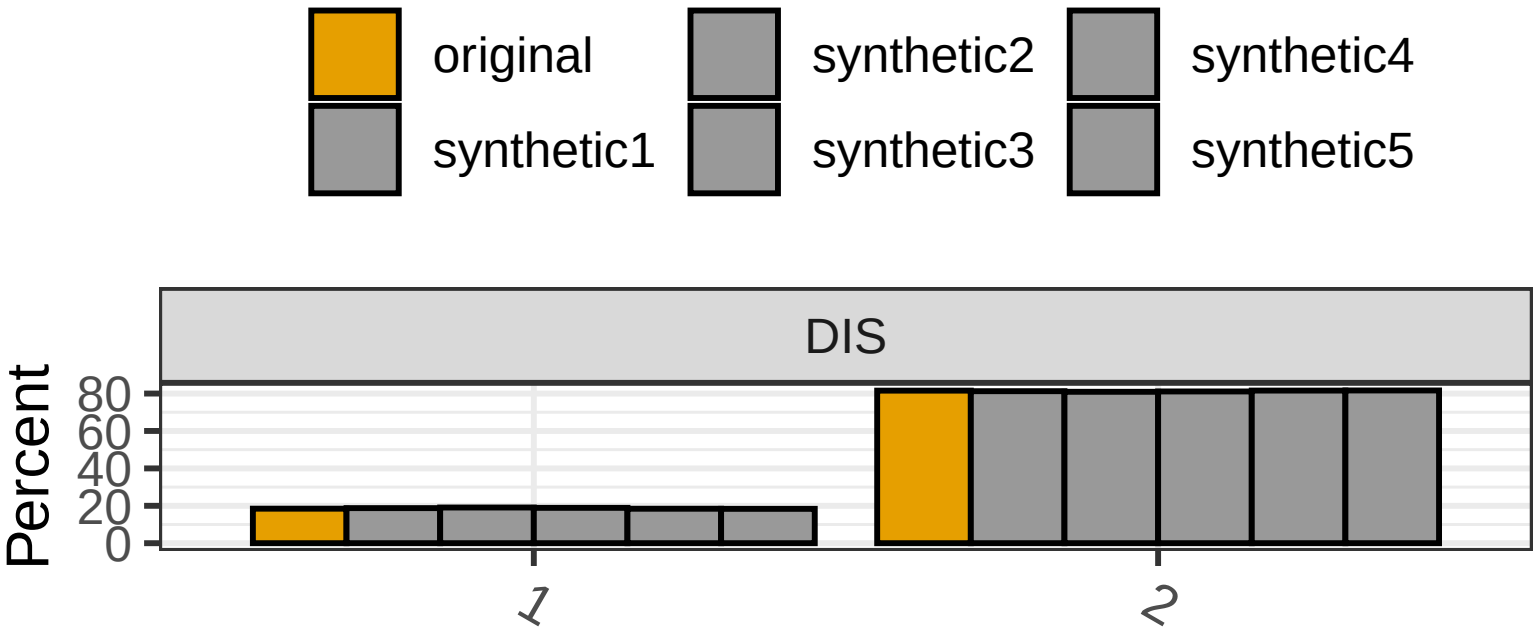
The DPMPM: MCMC diagnostics

- We can focus on diagnosis of `kstar`
- The code below produces a traceplot and an autocorrelation plot

```
library(bayesplot)
kstar_MCMCdiag(kstar = ACSdata_syn$kstar,
               nrun = 10000,
               burn = 5000,
               thin = 50)
```

The DPMPM: utility checks

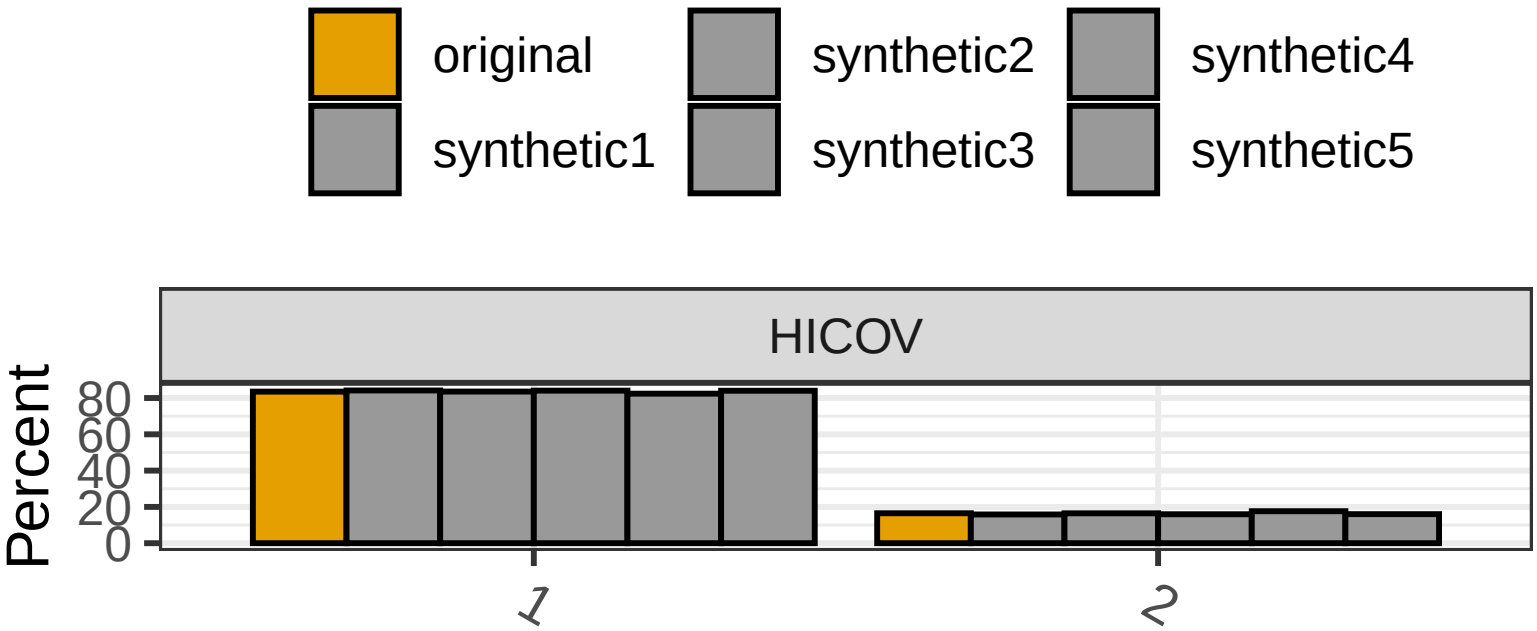
```
## $Plot
```




```
##
## $Comparison
##   level original synthetic1 synthetic2 synthetic3 synthetic
## 1     1    18.46       18.74       19.07       18.89       18.4
## 2     2    81.54       81.26       80.93       81.11       81.5
```

The DPMPM: utility checks

```
## $Plot
```




```
##
## $Comparison
##   level original synthetic1 synthetic2 synthetic3 synthetic
## 1     1     83.5       84.16       83.53       84.05      82.3
## 2     2     16.5       15.84       16.47       15.95      17.6
```

Outline

Summary

- Synthesizing categorical outcome variables
  - Bayesian multinomial logistic regressions
- Synthesizing count outcome variables
  - Bayesian Poisson regressions
  - Examples for sequential synthesis
- Synthesizing multivariate categorical data
  - The DPMPM
  - Use the `NPBayesImputCat` R package

Summary

- Synthesizing categorical outcome variables
  - Bayesian multinomial logistic regressions
- Synthesizing count outcome variables
  - Bayesian Poisson regressions
  - Examples for sequential synthesis
- Synthesizing multivariate categorical data
  - The DPMPM
  - Use the `NPBayesImputCat` R package
- Homework 5: a few R programming exercises
  - Submission on Moodle and prepare to discuss next time
- Lecture 6: Methods for utility evaluation part 1
  - All sections of Woo et al. (2009)

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