Introduction to Bayesian Modeling part 1

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Statistical Data Privacy

Outline

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Frequentist / classical and Bayesian inference

- Statistical inference aims to use data $\boldsymbol{Y}$ to learn about some unknown parameters $\boldsymbol{\theta}$
- Frequentist / classical inference:
  - Data $\boldsymbol{Y}$ are a repeatable random sample from a statistical model or a finite population
  - Underlying parameters $\boldsymbol{\theta}$ remain constant during this hypothetical repeatable process; considered fixed

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- Bayesian inference:
  - Data $\boldsymbol{Y}$ are fixed; they are observed from the realized sample
  - Underlying parameters $\boldsymbol{\theta}$ are unknown, and can be described **probabilistically**

Bayesian inference

- We can use probability distributions to describe the unknown underlying parameters $\boldsymbol{\theta}$
- These distributions represent our belief / knowledge about the values of the parameters
- The collected data can further sharpen / update our belief about the parameters

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The setup

- We will use a toy example of a coin flipping trial to introduce the foundation
- The outcome is either a head or tail
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- The parameter of interest is the probability of flipping a head, denoted as θ

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- The foundation topics include:
  - Prior
  - Likelihood
  - Bayes' rule (discrete and continuous) and posterior

Prior

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- To describe our beliefs probabilistically
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  - Really no idea and θ can be anywhere in [0, 1] with equal probability: $\theta \sim \mathrm{Uniform}(0, 1)$
  - Half the times $\theta = 0.3$ and half the times $\theta = 0.7$: $\theta = 0.3$ with probability 0.5 and $\theta = 0.7$ with probability 0.5

Prior in Bayesian inference

- Bayesian inference combines our prior belief with the information contained in the collected data
- In our toy example:
  - Prior refers to our belief about θ before we start collecting data
  - Collecting data refers to the coin flipping trial
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- Discussion question: If we start with a fair coin prior belief, and we see an outcome of head, how would you use this collected data to update your belief? What if we see 5 heads in a row?

Expressing a prior

- Three examples of discrete prior for θ:

Value of θ	Case 1: $Pr(\theta)$	Case 2: $Pr(\theta)$	Case 3: $Pr(\theta)$
0.3	0.333...	0.25	0.5
0.5	0.333...	0.25	0.25
0.7	0.333...	0.5	0.25

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- Examples of continuous prior for θ:
  - $\theta \sim \mathrm{Uniform}(0, 1)$ (i.e., θ can be anywhere in [0, 1] with equal probability)

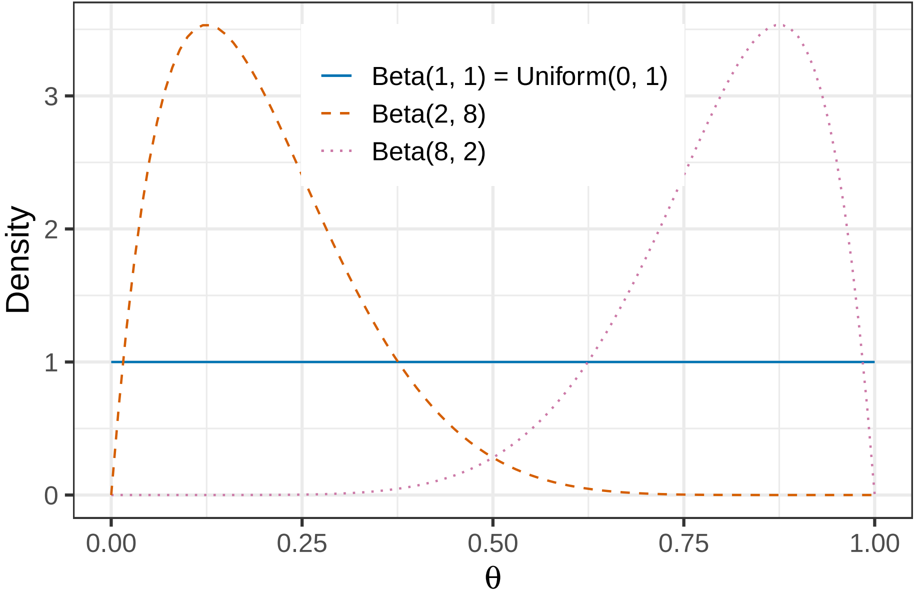
Beta prior for θ

- A common choice for continuous prior for $\theta \in [0, 1]$
- The probability density function of $\mathrm{Beta}(a, b)$ is:

$$f(\theta) = \frac{\theta^{a-1}(1-\theta)^{b-1}}{B(a, b)}, \theta \in [0, 1], a > 0, b > 0, \tag{1}$$

where $B(a, b) = \Gamma(a)\Gamma(b)/\Gamma(a + b)$ and $\Gamma(\cdot)$ is the gamma function. The mean and variance of $\theta \sim \mathrm{Beta}(a, b)$ are $a/(a + b)$ and $ab/((a + b)^2(a + b + 1))$ respectively.

Three beta priors



Comments on priors

- Bayesian inference is subjective
- Each person might have a different way to describe the underlying parameter θ probabilistically
- We can incorporate useful information about the unknown parameter θ, if we do have such useful information

Likelihood

- Bayesian inference combines our prior belief with the information contained in the collected data
- The collected data can be expressed as a model describing the distribution of the data as a function of the parameters
- We will refer to this data model as **likelihood**: a probability distribution of the data as a function of the parameters

Review: the Bernoulli distribution

The probability mass function of $\mathrm{Bernoulli}(\theta)$ is:

$$f(Y = y \mid \theta) = \theta^y (1-\theta)^{1-y}, y = \{0, 1\}, \theta \in [0, 1]. \tag{2}$$

The mean and variance of $Y \sim \mathrm{Bernoulli}(\theta)$ is θ and $\theta(1-\theta)$ respectively.

Bernoulli likelihood for Y

From this Bernoulli probability mass function, we can write our likelihood function as:

$$L(\theta) = \theta^y (1-\theta)^{1-y}. \tag{3}$$

- The likelihood is a function of θ (recall in Bayesian inference, data are considered fixed and parameters are considered random)
- Bayes' rule will help us to combine prior and data to posterior

Discrete Bayes' rule and posterior

- If we start with $\theta = \{0.3, 0.5, 0.7\}$ with equally probability of 0.333... for each, after seeing a flipped head, what are our updated probability of each of the possible values?

Discrete Bayes' rule and posterior

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- The discrete Bayes' rule:

$$\pi(\theta = \theta_c \mid y) = \frac{f(y \mid \theta_c)\pi(\theta_c)}{\sum_j f(y \mid \theta_j)\pi(\theta_j)} = \frac{L(\theta_c)\pi(\theta_c)}{\sum_j L(\theta_j)\pi(\theta_j)}, \tag{4}$$

  - The denominator is the marginal probability of y by the Law of Total Probability, i.e. $\sum_j L(\theta_j)\pi(\theta_j) = \sum_j f(y \mid \theta_j)\pi(\theta_j) = f(y)$
  - $L(\theta_c)$ is our likelihood function of observing y given θ_c

- Discussion question: Calculate the posterior probability of $\theta = 0.3$ after seeing one head (i.e., $y = 1$).

Use R for posterior calculation

```
theta <- c(0.3, 0.5, 0.7)
y <- 1
likelihood <- stats::dbinom(x = y, size = 1, prob = theta)
likelihood
```

```
## [1] 0.3 0.5 0.7
```

```
prior <- c(1/3, 1/3, 1/3)
product <- prior * likelihood
product
```

```
## [1] 0.1000000 0.1666667 0.2333333
```

```
product / sum(product)
```

```
## [1] 0.2000000 0.3333333 0.4666667
```

The posterior

Value of θ	$\pi(\theta)$	$f(y \mid \theta)$	$f(y \mid \theta)\pi(\theta)$	$\pi(\theta \mid y)$
0.3	0.333...	0.3	0.100	0.200
0.5	0.333...	0.5	0.167...	0.333...
0.7	0.333...	0.7	0.233...	0.467...

Continuous Bayes' rule and posterior

- If we start with a beta prior distribution for θ, what is our posterior distribution for θ after seeing a flipped head?
- The continuous Bayes' rule:

$$\pi(\theta \mid y) = \frac{f(y \mid \theta)\pi(\theta)}{\int_{\theta'} f(y \mid \theta')\pi(\theta')d\theta'} = \frac{L(\theta)\pi(\theta)}{\int_{\theta'} L(\theta')\pi(\theta')d\theta'}. \tag{5}$$

  - The denominator is the marginal probability of $f(y)$, i.e. $\int_{\theta'} L(\theta')\pi(\theta')d\theta' = \int_{\theta'} f(y \mid \theta')\pi(\theta')d\theta' = \int_{\theta'} f(y, \theta')d\theta' = f(y)$, which is a constant since the data is fixed

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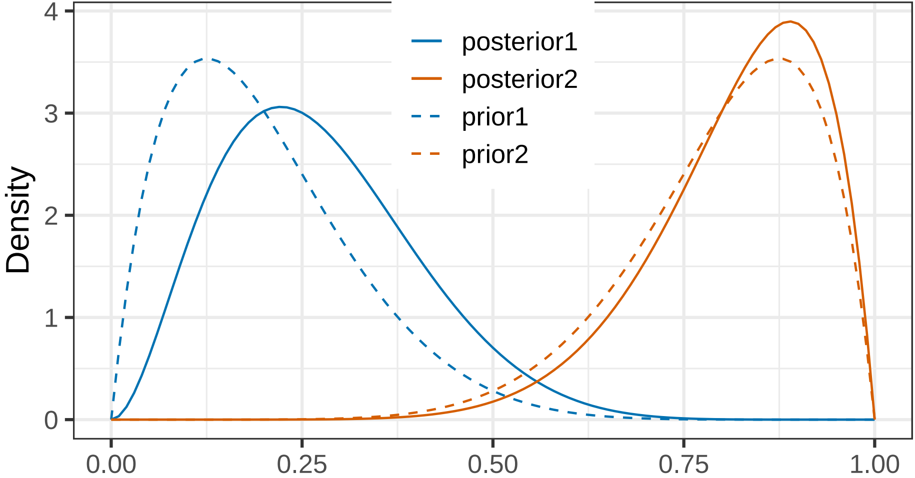
- Typically we use this following version of the continuous Bayes' rule:
$$\pi(\theta \mid y) \propto f(y \mid \theta)\pi(\theta) = L(\theta)\pi(\theta). \tag{6}$$

The beta-binomial conjugacy

For a binomial data model where $Y \sim \mathrm{Binomial}(n, p)$ and p is the model parameter (the success probability), a $\mathrm{Beta}(a, b)$ prior for p gives a $\mathrm{Beta}(a + y, b + n - y)$ posterior.

Two beta priors and two beta posteriors

- Data: $y = 1$ (i.e., one trial and the outcome is a head)
- Prior 1: $\theta \sim \mathrm{Beta}(2, 8)$; Posterior 1: $\theta \sim \mathrm{Beta}(3, 8)$
- Prior 2: $\theta \sim \mathrm{Beta}(8, 2)$; Posterior 2: $\theta \sim \mathrm{Beta}(9, 2)$



Recap: 3 general steps in Bayesian inference

1. **Prior** $\pi(\theta)$: Specify distributions for the parameters prior to seeing the data.
2. **Likelihood** $L(\theta) = f(y \mid \theta)$: Choose an appropriate data model for the data generation process and create the likelihood function expression.
3. **Posterior** $\pi(\theta \mid y)$: Use Bayes' rule to combine the prior and the data, and obtain the posterior distributions for the parameters.

Outline

MCMC overview

- Simple Bayesian models have analytical solution to the posterior (e.g., beta-binomial conjugacy)
- More sophisticated Bayesian models (e.g., more than one parameter) require more complicated posterior estimation methods
- The general class of posterior estimation methods is called Markov chain Monte Carlo (MCMC)

MCMC estimation

- General strategy: design a Markov chain whose stationary distribution is the posterior distribution of interest
    - We set up a stochastic process which moves randomly in the space of parameters of our Bayesian model such that after a certain time, the process will produce samples from the posterior distribution of interest
    - At each iteration, the move of the process only depends on the value of the process at the previous iteration; therefore it is a Markov chain

(more details available at Chapter 9 of Albert and Hu (2019): https://bayesball.github.io/BOOK/proportion.html)

Gibbs sampler

- To obtain a draw from a multivariate posterior distribution
- The Gibbs sampler works by simply repeatedly sampling from the posterior distribution of each parameter, conditional on the remaining parameters (i.e., the full conditional posterior distribution)

Gibbs sampler example

- A normal data model with mean μ and standard deviation σ unknown
- Start with an appropriate prior $\pi(\mu, \sigma)$, we derive the two full conditional posterior distributions: $f(\mu \mid \sigma, Y)$ and $f(\sigma \mid \mu, Y)$
- Iteratively sample draws of μ and σ in the following: at iteration $s + 1$
    1. Sample $\mu^{s+1} \sim f(\mu \mid \sigma^s, Y)$.
    2. Sample $\sigma^{s+1} \sim f(\sigma \mid \mu^{s+1}, Y)$.

Gibbs sampler example cont'd

- After a certain number S of iterative sampling is done, we can evaluate whether the chain has converged
- If so, the collective sampled draws of $\{\mu^1, \cdots, \mu^S\}$ are an approximation to the posterior distribution of μ, and those of $\{\sigma^1, \cdots, \sigma^S\}$ are an approximation to the posterior distribution of σ

Gibbs sampler example cont'd

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- If so, the collective sampled draws of $\{\mu^1, \cdots, \mu^S\}$ are an approximation to the posterior distribution of μ, and those of $\{\sigma^1, \cdots, \sigma^S\}$ are an approximation to the posterior distribution of σ
- Any posterior inferences on μ and σ can then be obtained by summarizing these posterior draws

Metropolis algorithm

- The Metropolis algorithm relies on making a proposal draw for the values of the parameters in the model, and evaluating whether we move to the proposal draw or stay at the current draw

Metropolis algorithm example

- A normal data model with mean μ and standard deviation σ unknown
- We illustrate the process for μ (updating of σ follows a similar approach)
- A Metropolis algorithm has three general steps at iteration $s+1$:
  1. Given the current simulated value μ^s, we propose a new value μ^*, selected at random from a uniform with width $2C$ (i.e. from the interval of $(\mu^s - C, \mu^s + C)$ or other symmetric distributions (e.g. normal centered at μ^s).
  2. Compute the ratio R of the posterior density (denoted as $\pi_n(\mu)$) at the proposed value μ^* and at the current value μ^s: $R = \pi_n(\mu^*)/\pi_n(\mu^s)$; from this we obtain an acceptance probability of $p = \min\{R, 1\}$.
  3. Simulate a uniform random variable from [0, 1], denoted as U. If $U < p$, we move to the proposed value μ^*, and otherwise we stay at the current value μ^s.

Hamiltonian Monte Carlo and Stan

- Not all models are amenable to Gibbs sampler or Metropolis algorithm, and in some cases these algorithms can take a long time to converge due to their random walk nature
- The Hamiltonian Monte Carlo (HMC) sampling algorithm is one of the more recent posterior sampling algorithms that is gaining popularity

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- In this course, we use the `brms` R package to estimate Bayesian models for statistical data privacy purposes, most often for generating synthetic data
- The `brms` package relies on HMC
- The estimation output will return the complete sequence of draws for each of the parameters in the model

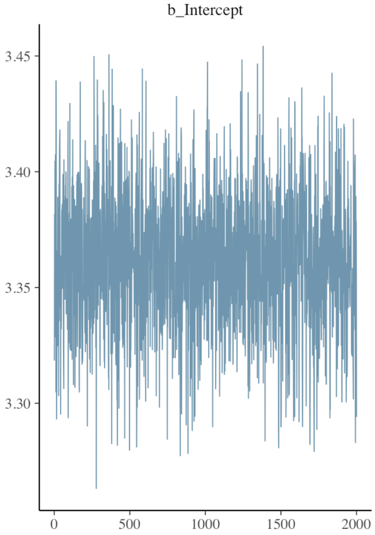
MCMC diagnostics for convergence

- The obtained posterior draws from MCMC estimation can be used for posterior inference only once we can confirm that they serve as a good approximation to the true posterior distribution (i.e., enough iterations have occurred for the MCMC to have converged to the true posterior)
- While no test can guarantee that the MCMC has indeed converged to its stationary distribution, we can use several diagnostic tools to identify obvious issues showing that the MCMC has **not** converged

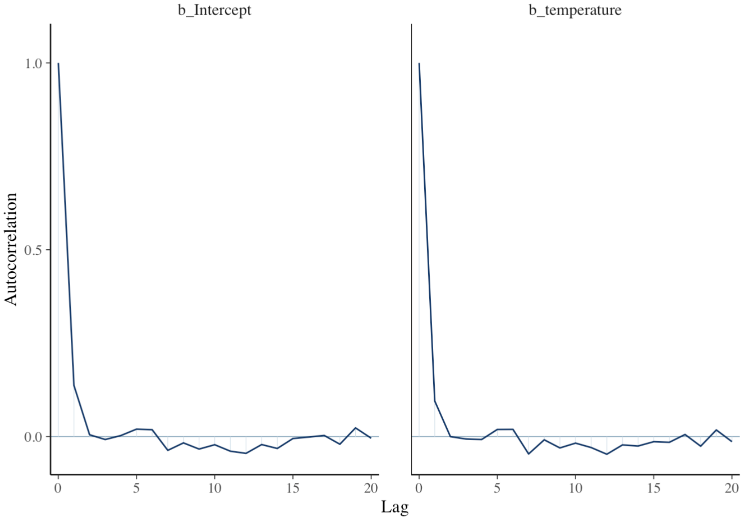
Traceplots and autocorrelation plots

- A converged MCMC should include posterior parameter draws that have extensively explored the parameter space and are relatively independent from each other
- We can check by using **traceplots**: plot the parameter values against each MCMC iteration number
- We can also check by using **autoccorelation plots** to see if the posterior draws are relatively independent from each other

Traceplots example



Autocorrelation plots example



Burn-in, thinning, and multiple MCMC chains

- Suppose the MCMC has been run for S iterations
- **Burn-in**: After iteration S^* (at which point we believe the chain has converged), we discard all draws before this iteration; this period is called the burn-in period; inference will only be using the remaining $S - S^*$ draws

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- Run multiple chains at different starting points to see if they arrive at a similar sample space

A brms example

- We will learn this example in the next lecture
- Discussion question: Identify S, S^*, k and the number of MCMC chains.

```
PoissonReg_fit <- brms::brm(data = bikesharing_data,
                            family = poisson(link = "log"),
                            count ~ 1 + temperature,
                            prior = c(prior(normal(0, 10),
                                            class = Intercept),
                                      prior(normal(0, 10), class = b)),
                            iter = 5000,
                            warmup = 3000,
                            thin = 1,
                            chains = 1,
                            seed = 123)
```

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- Homework 2: a few derivation and R programming exercises
  - Submission on Moodle and prepare to discuss next time
- Lecture 3: Introduction to Bayesian modeling part 2
  - Chapter 11 of Albert and Hu (2019): https://bayesball.github.io/BOOK/proportion.html (a different MCMC software is used)

References I

Albert, J., and J. Hu. 2019. Probability and Bayesian Modeling. Texts in Statistical Science, Chapman Hall CRC Press.