In[4]:=
```
(*Find F2,F1 and F0, numerators of 1/rho^4,1/rho^2, log(rho2) for Q11
use expansion of F(k) and E(k) about k=1
call log(rho^2)=logm, rho^2=|x-x0|^2
m3=rho^2 / rho2^2 where rho2^2=(x+x0)^2+(y-y0)^2

Version2: much simpler!
(1)when r(=x) is small, it is important to expand O(r)/O(rho2)
 together. so break up each numerator into pieces, each one of them is
 factored into "Small" times "fact". small is expanded to highest order,
fact to smaller order. In code,we then expand fact/rho2 or fact/(xrho2) first,
then multiplied by expansion of small.
(2)we dont need to expand "rest"! instead,
evaluate exactly in code using known values of x,x0,xi
(3)we no longer expand in d. that was a detour that lead to errors. high
 derivatives are large and need to be accurate to all powers in d.

For values of a1,a2,b1,b2,etc see Abramowitz and Stegun AND
https://functions.wolfram.com/EllipticIntegrals/EllipticK/introductions/
      CompleteEllipticIntegrals/05/

a0=Log[4], a1=(Log[4]-1)/4, a2=(6*Log[4]-7)*3/128
b1=1/8, b2=9/128, b3 not used
c1=(4log2-1)/4, c2=(24log2-13)/64 (not used), c3=3(5log2-3)/64 (not used)
d1=1/4, d2=3/32
*)

f = 2 * log2 - logm / 2 + a1 * m3 + a2 * m3 ^ 2 +
    a3 * m3 ^ 3 - (b1 * m3 + b2 * m3 ^ 2 + b3 * m3 ^ 3 + b4 * m3 ^ 4) * logm;
e = 1 + c1 * m3 + c2 * m3 ^ 2 + c3 * m3 ^ 3 - (d1 * m3 + d2 * m3 ^ 2 + d3 * m3 ^ 3 + d4 * m3 ^ 4) * logm;

eth = Series[m3 * e / m3, {m3, 0, 3}] / m3;
efh = Series[m3 ^ 2 * (2 * (1 + m3) * e / m3 ^ 2 - f / m3) / 3, {m3, 0, 3}] / m3 ^ 2;

c = Sqrt[a + b];
temp = Series[m3 ^ 2 * efh / c ^ 5, {m3, 0, 3}] / m3 ^ 2;
i50 = 4 * temp;
i51 = Series[m3 ^ 2 * 4 / b * (a * temp - eth / c ^ 3), {m3, 0, 3}] / m3 ^ 2;
i52 = Series[4 * m3 ^ 2 / b ^ 2 * (a ^ 2 * temp - 2 * a * eth / c ^ 3 + f / c), {m3, 0, 3}] / m3 ^ 2;
i53 = Series[4 * m3 ^ 2 / b ^ 3 * (a ^ 3 * temp - 3 * a ^ 2 * eth / c ^ 3 + 3 * a * f / c - c * e), {m3, 0, 3}] / m3 ^ 2;
```

In[14]:=
```
qu11 = Series[
    -m3^2*6*x*(x^3*i51 - x^2*x0*(i50 + 2*i52) + x*x0^2*(i53 + 2*i51) - x0^3*i52),
    {m3, 0, 3}]/m3^2;
qu12 = Series[-m3^2*6*x*xi*((x^2 + x0^2)*i51 - x*x0*(i50 + i52)), {m3, 0, 3}]/m3^2;
qu22 = Series[-m3^2*6*x*xi^2*(x*i51 - x0*i50), {m3, 0, 6}]/m3^2;
qv11 = Series[-m3^2*6*x*xi*(x0^2*i52 + x^2*i50 - 2*x*x0*i51), {m3, 0, 6}]/m3^2;
qv12 = Series[-m3^2*6*x*xi^2*(x*i50 - x0*i51), {m3, 0, 6}]/m3^2;
qv22 = Series[-m3^2*6*x*xi^3*i50, {m3, 0, 6}]/m3^2;
```

In[20]:=
```
term = qv11;
term = Simplify[term /. {d1 → 1/4, d2 → 3/32, d3 → 15/256,
      b1 → 1/8, b2 → 9/128, c1 → (4*log2 - 1)/4, c2 → (24*log2 - 13)/64,
      c3 → 3*(5*log2 - 3)/64, a1 → (2*log2 - 1)/4, a2 → (12*log2 - 7)*3/128}];
term = Simplify[term /. {b → 2*x*x0, a → x0^2 + x^2 + xi^2}]
```

Out[22]=
$$-\frac{4\left(xi\left(-x^2+x0^2+xi^2\right)^2\right)}{\left(x\left(x^2+2\,x\,x0+x0^2+xi^2\right)^{5/2}\right)m3^2}+$$

$$\frac{3\,xi\left(-5\,x^4-8\,x^3\,x0+2\,x^2\left(x0^2+xi^2\right)+8\,x\,x0\left(x0^2+xi^2\right)+3\left(x0^2+xi^2\right)^2\right)}{x\left(x^2+2\,x\,x0+x0^2+xi^2\right)^{5/2}\,m3}-$$

$$\left(3\left(xi\left((-23+152\,\log2-38\,\log m)\,x^4+32\,(-1+12\,\log2-3\,\log m)\,x^3\,x0+32\,(1+4\,\log2-\log m)\,x\,x0\right.\right.\right.$$
$$\left(x0^2+xi^2\right)+3\,(3+8\,\log2-2\,\log m)\left(x0^2+xi^2\right)^2+2\,x^2\left(7\,(1+24\,\log2-6\,\log m)\,x0^2+\right.$$
$$\left.\left.\left.(7+40\,\log2-10\,\log m)\,xi^2\right)\right)\right)\Big/\left(16\left(x\left(x^2+2\,x\,x0+x0^2+xi^2\right)^{5/2}\right)\right)-$$

$$\left(\left(xi\left((-319+600\,\log2-150\,\log m)\,x^4+24\,(-29+56\,\log2-14\,\log m)\,x^3\,x0+\right.\right.\right.$$
$$24\,(-3+8\,\log2-2\,\log m)\,x\,x0\left(x0^2+xi^2\right)+(-7+24\,\log2-6\,\log m)\left(x0^2+xi^2\right)^2+$$
$$\left.\left.\left.2\,x^2\left((-221+456\,\log2-114\,\log m)\,x0^2+(-29+72\,\log2-18\,\log m)\,xi^2\right)\right)\right)\right.$$
$$\left.m3\right)\Big/\left(64\left(x\left(x^2+2\,x\,x0+x0^2+xi^2\right)^{5/2}\right)\right)+O[m3]^2$$

In[23]:=
```
msqcoeff = Simplify[term * m3^2 /. m3 → 0]
mnum = Simplify[Numerator[msqcoeff]/(-4)]
(* Lead = -4  The whole term is multiplied by rho2^4/rho1^4 leaving 1/rho2 in denom

mnum is numerator without the lead. we now
  break up mnum into pieces consisting of O(r)*O(small) factors
we need to expand O(r)/O(r2) separately, to avoid errors when r2 is small
that term is then multiplied by remaining O(small) term*)
```

Out[23]=
$$-\frac{4\,xi\left(-x^2+x0^2+xi^2\right)^2}{x\left(x^2+2\,x\,x0+x0^2+xi^2\right)^{5/2}}$$

Out[24]=
$$xi\left(-x^2+x0^2+xi^2\right)^2$$

In[25]:=
```
piece3 = xi^5
piece2 = Simplify[xi^3 * Coefficient[mnum - piece3, xi^3]]
piece1 = Simplify[xi * Coefficient[mnum - piece2 - piece3, xi]]
Simplify[mnum - piece1 - piece3 - piece2]
```

Out[25]=
$$xi^5$$

Out[26]=
$$-2\left(x^2-x0^2\right)xi^3$$

Out[27]=
$$\left(x^2-x0^2\right)^2\,xi$$

Out[28]=
$$0$$

In[29]:=
```
small1 = (x - x0)^2 * xi
fact1 = Simplify[piece1/small1]
Expand[%]
FortranForm[Expand[%]];
```

Out[29]=
$$(x-x0)^2\,xi$$

Out[30]=
$$(x+x0)^2$$

Out[31]=
$$x^2+2\,x\,x0+x0^2$$

In[33]:=
```
small2 = xi^3 * (x - x0)
fact2 = Simplify[piece2/small2]
Expand[%]
FortranForm[Expand[%]];
```

Out[33]=
$$(x-x0)\,xi^3$$

Out[34]=
$$-2\,(x+x0)$$

Out[35]=
$$-2\,x-2\,x0$$

In[37]:=
```
small3 = xi^5
Simplify[mnum - small1 * fact1 - small2 * fact2 - small3]
```

Out[37]= xi^5

Out[38]= 0

In[39]:=
```
mcoeff = Simplify[Coefficient[term * m3^2, m3]]
mnum = Simplify[Numerator[mcoeff]/(3)]
(* Lead =
 3 The whole term is multiplied by rho2^2/rho1^2 leaving 1/rho2^3 in denom*)
```

Out[39]= $$\frac{3\,xi\left(-5\,x^4-8\,x^3\,x0+2\,x^2\left(x0^2+xi^2\right)+8\,x\,x0\left(x0^2+xi^2\right)+3\left(x0^2+xi^2\right)^2\right)}{x\left(x^2+2\,x\,x0+x0^2+xi^2\right)^{5/2}}$$

Out[40]= $xi\left(-5\,x^4-8\,x^3\,x0+2\,x^2\left(x0^2+xi^2\right)+8\,x\,x0\left(x0^2+xi^2\right)+3\left(x0^2+xi^2\right)^2\right)$

In[41]:=
```
piece3 = 3 * xi^5
piece2 = Simplify[xi^3 * Coefficient[mnum - piece3, xi^3]]
piece1 = Simplify[xi * Coefficient[mnum - piece2 - piece3, xi]]
Simplify[mnum - piece1 - piece3 - piece2]
```

Out[41]= $3\,xi^5$

Out[42]= $2\left(x^2+4\,x\,x0+3\,x0^2\right)xi^3$

Out[43]= $-(x+x0)^2\left(5\,x^2-2\,x\,x0-3\,x0^2\right)xi$

Out[44]= 0

In[45]:=
```
small1 = -xi
fact1 = Simplify[piece1/small1]
FortranForm[Expand[%]];
```

Out[45]= $-xi$

Out[46]= $(x+x0)^2\left(5\,x^2-2\,x\,x0-3\,x0^2\right)$

In[48]:=
```
small2 = 2 * xi^3
fact2 = Simplify[piece2/small2]
FortranForm[Expand[%]];
```

Out[48]= $2\,xi^3$

Out[49]= $x^2+4\,x\,x0+3\,x0^2$

In[51]:=
```
small3 = 3 * xi^5
fact3 = Simplify[piece3/small3]
```

Out[51]= $3\,xi^5$

Out[52]= 1

In[53]:=
```
Simplify[mnum - small1 * fact1 - small2 * fact2 - small3]
```

Out[53]= 0

In[103]:=
```
logcoeff=Simplify[Coefficient[term,logm]/.m3→0]
mnum=Simplify[Numerator[logcoeff]/(3)]
FortranForm[%];
(* Lead = 3/8  This terms is multiplied by log(rho1^2/dsq)-log(rho^2/d^2)  *)
```

Out[103]=

$$\frac{3\,xi\left(19\,x^4+48\,x^3\,x0+16\,x\,x0\left(x0^2+xi^2\right)+3\left(x0^2+xi^2\right)^2+2\,x^2\left(21\,x0^2+5\,xi^2\right)\right)}{8\,x\left(x^2+2\,x\,x0+x0^2+xi^2\right)^{5/2}}$$

Out[104]=

$$xi\left(19\,x^4+48\,x^3\,x0+16\,x\,x0\left(x0^2+xi^2\right)+3\left(x0^2+xi^2\right)^2+2\,x^2\left(21\,x0^2+5\,xi^2\right)\right)$$

In[57]:=
```
piece3 = Simplify[xi^5 * Coefficient[mnum, xi^5]]
piece2 = Simplify[xi^3 * Coefficient[mnum - piece3, xi^3]]
piece1 = Simplify[xi * Coefficient[mnum - piece2 - piece3, xi]]
Simplify[mnum - piece1 - piece3 - piece2]
```

Out[57]= $3\,xi^5$

Out[58]= $2\left(5\,x^2+8\,x\,x0+3\,x0^2\right)xi^3$

Out[59]= $(x+x0)^2\left(19\,x^2+10\,x\,x0+3\,x0^2\right)xi$

Out[60]= 0

In[61]:=
```
small1 = xi
fact1 = Simplify[piece1 / small1]
FortranForm[Expand[%]];
```

Out[61]= xi

Out[62]= $(x+x0)^2\left(19\,x^2+10\,x\,x0+3\,x0^2\right)$

In[64]:=
```
small2 = 2 * xi^3
fact2 = Simplify[piece2 / small2]
FortranForm[Expand[%]];
```

Out[64]= $2\,xi^3$

Out[65]= $5\,x^2+8\,x\,x0+3\,x0^2$

In[67]:=
```
small3 = 3 * xi ^ 5
fact3 = Simplify[piece3 / small3]
FortranForm[%]
Simplify[mnum - small1 * fact1 - small2 * fact2 - small3]
```

Out[67]= $3\ xi^5$

Out[68]= 1

Out[69]//FortranForm=
```
"1"
```

Out[70]= 0

In[106]:=
```
logmcoeff = Simplify[Coefficient[term, m3]];
logmcoeff = Simplify[Coefficient[%, logm]]
mnum = Simplify[Numerator[logmcoeff] /. d3 → 15 / 256]
denom = Simplify[Denominator[logmcoeff]]
mnum0 = Simplify[mnum /. xi → 0]
Simplify[Coefficient[mnum, xi]]
Simplify[Coefficient[mnum, xi^2]]
Simplify[Coefficient[mnum, xi^3]]
Simplify[Coefficient[mnum, xi^4]]
Simplify[Coefficient[mnum, xi^5]]
Simplify[Coefficient[mnum, xi^6]]
```

Out[107]=
$$\frac{3\,xi\left(25\,x^4+56\,x^3\,x0+8\,x\,x0\left(x0^2+xi^2\right)+\left(x0^2+xi^2\right)^2+x^2\left(38\,x0^2+6\,xi^2\right)\right)}{32\,x\left(x^2+2\,x\,x0+x0^2+xi^2\right)^{5/2}}$$

Out[108]=
$$3\,xi\left(25\,x^4+56\,x^3\,x0+8\,x\,x0\left(x0^2+xi^2\right)+\left(x0^2+xi^2\right)^2+x^2\left(38\,x0^2+6\,xi^2\right)\right)$$

Out[109]=
$$32\,x\left(x^2+2\,x\,x0+x0^2+xi^2\right)^{5/2}$$

Out[110]=
$$0$$

Out[111]=
$$3\,(x+x0)^2\left(25\,x^2+6\,x\,x0+x0^2\right)$$

Out[112]=
$$0$$

Out[113]=
$$6\left(3\,x^2+4\,x\,x0+x0^2\right)$$

Out[114]=
$$0$$

Out[115]=
$$3$$

Out[116]=
$$0$$

In[82]:=
```
rest =
  Simplify[term - msqcoeff / m3^2 - mcoeff / m3 - logcoeff * logm - mnum0 * m3 * logm / denom];
```

```
rest0 = Simplify[rest /. m3 → 0];
rest0num = Simplify[Numerator[rest0]]
Denominator[rest0]
Simplify[rest0num /. xi → 0]
Simplify[Coefficient[rest0num, xi]]
Simplify[Coefficient[rest0num, xi^2]]
Simplify[Coefficient[rest0num, xi^4]]
Simplify[Coefficient[rest0num, xi^6]]
Simplify[Coefficient[rest0num, xi^3]]
Simplify[Coefficient[rest0num, xi^5]]
```

Out[84]=
$$-3\,xi\left((-23+152\log 2)\,x^4+32\,(-1+12\log 2)\,x^3\,x0+32\,(1+4\log 2)\,x\,x0\left(x0^2+xi^2\right)+3\,(3+8\log 2)\left(x0^2+xi^2\right)^2+2\,x^2\left(7\,(1+24\log 2)\,x0^2+(7+40\log 2)\,xi^2\right)\right)$$

Out[85]=
$$16\,x\left(x^2+2\,x\,x0+x0^2+xi^2\right)^{5/2}$$

Out[86]= 0

Out[87]=
$$-3\,(x+x0)^2\left((-23+152\log 2)\,x^2+2\,(7+40\log 2)\,x\,x0+3\,(3+8\log 2)\,x0^2\right)$$

Out[88]= 0

Out[89]= 0

Out[90]= 0

Out[91]=
$$-6\,(x+x0)\left((7+40\log 2)\,x+3\,(3+8\log 2)\,x0\right)$$

Out[92]=
$$-9\,(3+8\log 2)$$

In[127]:=
```
rest1 = Simplify[Coefficient[rest, m3]];
rest1num = Simplify[Numerator[rest1]];
Denominator[rest1]
Simplify[rest1num /. xi → 0]
Simplify[Coefficient[rest1num, xi]]
Simplify[Coefficient[rest1num, xi^2]]
Simplify[Coefficient[rest1num, xi^3]]
Simplify[Coefficient[rest1num, xi^4]]
Simplify[Coefficient[rest1num, xi^5]]
Simplify[Coefficient[rest1num, xi^6]]
```

Out[129]=
$$64\, x \left(x^2 + 2\, x\, x0 + x0^2 + xi^2\right)^{5/2}$$

Out[130]=
$$0$$

Out[131]=
$$-(x + x0)^2 \left((-319 + 600\, \log 2 - 150\, \log m)\, x^2 + 2\, (-29 + 72\, \log 2 - 18\, \log m)\, x\, x0 + (-7 + 24\, \log 2 - 6\, \log m)\, x0^2\right)$$

Out[132]=
$$0$$

Out[133]=
$$-2\, (x + x0)\, ((-29 + 72\, \log 2 - 18\, \log m)\, x + (-7 + 24\, \log 2 - 6\, \log m)\, x0)$$

Out[134]=
$$0$$

Out[135]=
$$7 - 24\, \log 2 + 6\, \log m$$

Out[136]=
$$0$$