In[89]:=
```
(*Find F2,F1 and F0, numerators of 1/rho^4,1/rho^2, log(rho2) for Q11
use expansion of F(k) and E(k) about k=1
call log(rho^2)=logm, rho^2=|x-x0|^2
m3=rho^2 / rho2^2 where rho2^2=(x+x0)^2+(y-y0)^2

Version2: much simpler!
(1)when r(=x) is small, it is important to expand O(r)/O(rho2)
 together. so break up each numerator into pieces, each one of them is
 factored into "Small" times "fact". small is expanded to highest order,
fact to smaller order. In code,we then expand fact/rho2 or fact/(xrho2) first,
then multiplied by expansion of small.
(2)we dont need to expand "rest"! instead,
evaluate exactly in code using known values of x,x0,xi
(3)we no longer expand in d. that was a detour that lead to errors. high
 derivatives are large and need to be accurate to all powers in d.

For values of a1,a2,b1,b2,etc see Abramowitz and Stegun AND
https://functions.wolfram.com/EllipticIntegrals/EllipticK/introductions/
      CompleteEllipticIntegrals/05/

a0=Log[4], a1=(Log[4]-1)/4, a2=(6*Log[4]-7)*3/128
b1=1/8, b2=9/128, b3 not used
c1=(4log2-1)/4, c2=(24log2-13)/64 (not used), c3=3(5log2-3)/64 (not used)
d1=1/4, d2=3/32
*)

f = 2 * log2 - logm / 2 + a1 * m3 + a2 * m3 ^ 2 +
    a3 * m3 ^ 3 - (b1 * m3 + b2 * m3 ^ 2 + b3 * m3 ^ 3 + b4 * m3 ^ 4) * logm;
e = 1 + c1 * m3 + c2 * m3 ^ 2 + c3 * m3 ^ 3 - (d1 * m3 + d2 * m3 ^ 2 + d3 * m3 ^ 3 + d4 * m3 ^ 4) * logm;

eth = Series[m3 * e / m3, {m3, 0, 3}] / m3;
efh = Series[m3 ^ 2 * (2 * (1 + m3) * e / m3 ^ 2 - f / m3) / 3, {m3, 0, 3}] / m3 ^ 2;

c = Sqrt[a + b];
temp = Series[m3 ^ 2 * efh / c ^ 5, {m3, 0, 3}] / m3 ^ 2;
i50 = 4 * temp;
i51 = Series[m3 ^ 2 * 4 / b * (a * temp - eth / c ^ 3), {m3, 0, 3}] / m3 ^ 2;
i52 = Series[4 * m3 ^ 2 / b ^ 2 * (a ^ 2 * temp - 2 * a * eth / c ^ 3 + f / c), {m3, 0, 3}] / m3 ^ 2;
i53 = Series[4 * m3 ^ 2 / b ^ 3 * (a ^ 3 * temp - 3 * a ^ 2 * eth / c ^ 3 + 3 * a * f / c - c * e), {m3, 0, 3}] / m3 ^ 2;
```

In[99]:=
```
qu11 = Series[
     -m3^2*6*x*(x^3*i51 - x^2*x0*(i50 + 2*i52) + x*x0^2*(i53 + 2*i51) - x0^3*i52),
     {m3, 0, 3}] / m3^2;
qu12 = Series[-m3^2*6*x*xi* ((x^2 + x0^2) *i51 - x*x0* (i50 + i52)), {m3, 0, 3}] / m3^2;
qu22 = Series[-m3^2*6*x*xi^2*(x*i51 - x0*i50), {m3, 0, 6}]/m3^2;
qv11 = Series[-m3^2*6*x*xi*(x0^2*i52 + x^2*i50 - 2*x*x0*i51), {m3, 0, 6}]/m3^2;
qv12 = Series[-m3^2*6*x*xi^2*(x*i50 - x0*i51), {m3, 0, 6}]/m3^2;
qv22 = Series[-m3^2 * 6 * x * xi^3 * i50, {m3, 0, 6}] / m3^2;
```

In[105]:=
```
term = qv12;
term = Simplify[term /. {d1 → 1/4, d2 → 3/32, d3 → 15/256,
      b1 → 1/8, b2 → 9/128, c1 → (4*log2 - 1)/4, c2 → (24*log2 - 13)/64,
      c3 → 3*(5*log2 - 3)/64, a1 → (2*log2 - 1)/4, a2 → (12*log2 - 7)*3/128}];
term = Simplify[term /. {b → 2*x*x0, a → x0^2 + x^2 + xi^2}]
```

Out[107]=

$$\frac{8\,xi^2\left(-x^2+x0^2+xi^2\right)}{\left(x^2+2\,x\,x0+x0^2+xi^2\right)^{5/2}\,m3^2}-\frac{6\left(xi^2\left(3\,x^2+4\,x\,x0+x0^2+xi^2\right)\right)}{\left(x^2+2\,x\,x0+x0^2+xi^2\right)^{5/2}\,m3}-$$
$$\frac{3\left(xi^2\left((-15+56\,\text{log2}-14\,\text{logm})\,x^2+16\,(-1+4\,\text{log2}-\text{logm})\,x\,x0+(-1+8\,\text{log2}-2\,\text{logm})\left(x0^2+xi^2\right)\right)\right)}{8\left(x^2+2\,x\,x0+x0^2+xi^2\right)^{5/2}}$$
$$-\left(\left(xi^2\left((-145+264\,\text{log2}-66\,\text{logm})\,x^2+12\,(-13+24\,\text{log2}-6\,\text{logm})\,x\,x0+\right.\right.\right.$$
$$\left.\left.\left.(-11+24\,\text{log2}-6\,\text{logm})\left(x0^2+xi^2\right)\right)\right)m3\right)\Big/\left(32\left(x^2+2\,x\,x0+x0^2+xi^2\right)^{5/2}\right)+O[m3]^2$$

In[108]:=
```
msqcoeff = Simplify[term * m3^2 /. m3 → 0]
mnum = Simplify[Numerator[msqcoeff]/(8)]
(* Lead = 8  The whole term is multiplied by rho2^4/rho1^4 leaving 1/rho2 in denom

mnum is numerator without the lead. we now
  break up mnum into pieces consisting of O(r)*O(small) factors
we need to expand O(r)/O(r2) separately, to avoid errors when r2 is small
that term is then multiplied by remaining O(small) term*)
```

Out[108]=

$$\frac{8\,xi^2\left(-x^2+x0^2+xi^2\right)}{\left(x^2+2\,x\,x0+x0^2+xi^2\right)^{5/2}}$$

Out[109]=

$$xi^2\left(-x^2+x0^2+xi^2\right)$$

In[110]:=
```
piece2 = xi^4
piece1 = Simplify[mnum - piece2]
Simplify[mnum - piece1 - piece2]
```

Out[110]=
$$xi^4$$

Out[111]=
$$\left(-x^2 + x0^2\right) xi^2$$

Out[112]=
$$0$$

In[113]:=
```
small1 = (x - x0) * xi^2
fact1 = Simplify[piece1 / small1]
Expand[%]
FortranForm[Expand[%]];
```

Out[113]=
$$(x - x0)\, xi^2$$

Out[114]=
$$-x - x0$$

Out[115]=
$$-x - x0$$

In[117]:=
```
small2 = xi^4
Simplify[mnum - small1 * fact1 - small2]
```

Out[117]=
$$xi^4$$

Out[118]=
$$0$$

In[119]:=
```
mcoeff = Simplify[Coefficient[term * m3^2, m3]]
mnum = Simplify[Numerator[mcoeff]/(-6)]
(* Lead =
 -6  The whole term is multiplied by rho2^2/rho1^2 leaving 1/rho2^3 in denom*)
```

Out[119]=
$$-\frac{6\, xi^2 \left(3\, x^2 + 4\, x\, x0 + x0^2 + xi^2\right)}{\left(x^2 + 2\, x\, x0 + x0^2 + xi^2\right)^{5/2}}$$

Out[120]=
$$xi^2 \left(3\, x^2 + 4\, x\, x0 + x0^2 + xi^2\right)$$

In[121]:=
```
piece2 = xi^4
piece1 = Simplify[mnum - piece2]
Simplify[mnum - piece1 - piece2]
```

Out[121]=
$$xi^4$$

Out[122]=
$$\left(3\,x^2 + 4\,x\,x0 + x0^2\right) xi^2$$

Out[123]=
$$0$$

In[124]:=
```
small1 = xi^2
fact1 = Simplify[piece1 / small1]
FortranForm[Expand[%]];
```

Out[124]=
$$xi^2$$

Out[125]=
$$3\,x^2 + 4\,x\,x0 + x0^2$$

In[127]:=
```
small2 = xi^4
Simplify[mnum - small1 * fact1 - small2]
```

Out[127]=
$$xi^4$$

Out[128]=
$$0$$

In[129]:=
```
logcoeff=Simplify[Coefficient[term,logm]/.m3→0]
mnum=Simplify[Numerator[logcoeff]/(3)]
FortranForm[%];
(* Lead = 3/4  This terms is multiplied by log(rho1^2)-log(rho^2)  *)
```

Out[129]=
$$\frac{3\,xi^2\left(7\,x^2 + 8\,x\,x0 + x0^2 + xi^2\right)}{4\left(x^2 + 2\,x\,x0 + x0^2 + xi^2\right)^{5/2}}$$

Out[130]=
$$xi^2\left(7\,x^2 + 8\,x\,x0 + x0^2 + xi^2\right)$$

In[132]:=
```
piece2 = Simplify[xi^4 * Coefficient[mnum, xi^4]]
piece1 = Simplify[mnum - piece2]
Simplify[mnum - piece1 - piece2]
```

Out[132]=
$$xi^4$$

Out[133]=
$$\left(7\,x^2 + 8\,x\,x0 + x0^2\right) xi^2$$

Out[134]=
$$0$$

In[135]:=
```
small1 = xi^2
fact1 = Simplify[piece1 / small1]
FortranForm[Expand[%]];
```

Out[135]=
$$xi^2$$

Out[136]=
$$7\,x^2 + 8\,x\,x0 + x0^2$$

In[138]:=
```
small2 = xi^4
Simplify[mnum - small1 * fact1 - small2]
```

Out[138]=
$$xi^4$$

Out[139]=
$$0$$

In[140]:=
```
logmcoeff = Simplify[Coefficient[term, m3]]
logmcoeff = Simplify[Coefficient[%, logm]]
mnum = Simplify[Numerator[logmcoeff] /. d3 → 15 / 256]
Simplify[Coefficient[mnum, xi]]
Simplify[Coefficient[mnum, xi^3]]
Simplify[Coefficient[mnum, xi^5]]
denom = Simplify[Denominator[logmcoeff]]
mnum0 = Simplify[mnum /. xi → 0]
Simplify[Coefficient[mnum, xi^2]]
Simplify[Coefficient[mnum, xi^4]]
Simplify[Coefficient[mnum, xi^6]]
Simplify[Coefficient[mnum, xi^8]]
```

Out[140]=
$$-\left(\left(xi^2\left((-145+264\,\text{log2}-66\,\text{logm})\,x^2+12\,(-13+24\,\text{log2}-6\,\text{logm})\,x\,x0+(-11+24\,\text{log2}-6\,\text{logm})\left(x0^2+xi^2\right)\right)\right)\Big/\left(32\left(x^2+2\,x\,x0+x0^2+xi^2\right)^{5/2}\right)\right)$$

Out[141]=
$$\frac{3\,xi^2\left(11\,x^2+12\,x\,x0+x0^2+xi^2\right)}{16\left(x^2+2\,x\,x0+x0^2+xi^2\right)^{5/2}}$$

Out[142]=
$$3\,xi^2\left(11\,x^2+12\,x\,x0+x0^2+xi^2\right)$$

Out[143]=
$$0$$

Out[144]=
$$0$$

Out[145]=
$$0$$

Out[146]=
$$16\left(x^2+2\,x\,x0+x0^2+xi^2\right)^{5/2}$$

Out[147]=
$$0$$

Out[148]=
$$3\left(11\,x^2+12\,x\,x0+x0^2\right)$$

Out[149]=
$$3$$

Out[150]=
$$0$$

Out[151]=
$$0$$

In[152]:=
```
rest =
  Simplify[term - msqcoeff / m3^2 - mcoeff / m3 - logcoeff * logm - mnum0 * m3 * logm / denom];
rest = Simplify[rest /. {a1 → (2 * log2 - 1) / 4, c2 → (24 * log2 - 13) / 64}];

rest0 = Simplify[rest /. m3 → 0];
rest0num = Simplify[Numerator[rest0]]
Denominator[rest0]
Simplify[rest0num /. xi → 0]
Simplify[Coefficient[rest0num, xi]]
Simplify[Coefficient[rest0num, xi^2]]
Simplify[Coefficient[rest0num, xi^3]]
Simplify[Coefficient[rest0num, xi^4]]
Simplify[Coefficient[rest0num, xi^5]]
Simplify[Coefficient[rest0num, xi^6]]
```

Out[155]=
$$-3\,xi^2\left((-15+56\log 2)\,x^2+16\,(-1+4\log 2)\,x\,x0+(-1+8\log 2)\left(x0^2+xi^2\right)\right)$$

Out[156]=
$$8\left(x^2+2\,x\,x0+x0^2+xi^2\right)^{5/2}$$

Out[157]=
$$0$$

Out[158]=
$$0$$

Out[159]=
$$-3\,(x+x0)\,((-15+56\log 2)\,x+(-1+8\log 2)\,x0)$$

Out[160]=
$$0$$

Out[161]=
$$3-24\log 2$$

Out[162]=
$$0$$

Out[163]=
$$0$$

In[164]:=
```
rest1 = Simplify[Coefficient[rest, m3]];
(*rest1num=
 Simplify[Numerator[rest1]/.{a2→(6*Log[4]-7)*3/128,d3→15/256,c3→3*(5*Log[2]-3)/64}]*)
rest1num = Simplify[Numerator[rest1]];
Denominator[rest1]
Simplify[rest1num /. xi → 0]
Simplify[Coefficient[rest1num, xi]]
Simplify[Coefficient[rest1num, xi^2]]
Simplify[Coefficient[rest1num, xi^3]]
Simplify[Coefficient[rest1num, xi^4]]
Simplify[Coefficient[rest1num, xi^5]]
Simplify[Coefficient[rest1num, xi^6]]
```

Out[166]=
$$32\left(x^2+2\,x\,x0+x0^2+xi^2\right)^{5/2}$$

Out[167]=
$$0$$

Out[168]=
$$0$$

Out[169]=
$$-((x+x0)\,((-145+264\,\text{log2}-66\,\text{logm})\,x+(-11+24\,\text{log2}-6\,\text{logm})\,x0))$$

Out[170]=
$$0$$

Out[171]=
$$11-24\,\text{log2}+6\,\text{logm}$$

Out[172]=
$$0$$

Out[173]=
$$0$$