

```

ln[4]:= (*Find F2,F1 and F0, numerators of 1/rho^4,1/rho^2, log(rho2) for Q11
use expansion of F(k) and E(k) about k=1
call log(rho^2)=logm, rho^2=|x-x0|^2
m3=rho^2 / rho2^2 where rho2^2=(x+x0)^2+(y-y0)^2

```

Version2: much simpler!

- (1)when $r(=x)$ is small, it is important to expand $O(r)/O(\rho^2)$ together. so break up each numerator into pieces, each one of them is factored into "Small" times "fact". small is expanded to highest order, fact to smaller order. In code,we then expand fact/ ρ^2 or fact/($x\rho^2$) first, then multiplied by expansion of small.
- (2)we dont need to expand "rest"! instead, evaluate exactly in code using known values of x, x_0, x_i
- (3)we no longer expand in d. unnecessary detour. that lead to errors. high derivatives are large and need to be accurate to all powers in d.

For values of a_1, a_2, b_1, b_2 , etc see Abramowitz and Stegun AND
<https://functions.wolfram.com/EllipticIntegrals/EllipticK/introductions/CompleteEllipticIntegrals/05/>

```

a0=Log[4], a1=(Log[4]-1)/4, a2=(6*Log[4]-7)*3/128
b1=1/8, b2=9/128, b3 not used
c1=(4log2-1)/4, c2=(24log2-13)/64, c3=3(5log2-3)/64
d1=1/4, d2=3/32, d3=15/256
*)

```

```

f = 2 * log2 - logm / 2 + a1 * m3 + a2 * m3 ^ 2 +
    a3 * m3 ^ 3 - (b1 * m3 + b2 * m3 ^ 2 + b3 * m3 ^ 3 + b4 * m3 ^ 4) * logm;
e = 1 + c1 * m3 + c2 * m3 ^ 2 + c3 * m3 ^ 3 - (d1 * m3 + d2 * m3 ^ 2 + d3 * m3 ^ 3 + d4 * m3 ^ 4) * logm;

```

```

eth = Series[m3 * e / m3, {m3, 0, 3}] / m3;
efh = Series[m3 ^ 2 * (2 * (1 + m3) * e / m3 ^ 2 - f / m3) / 3, {m3, 0, 3}] / m3 ^ 2;

```

```

c = Sqrt[a + b];
temp = Series[m3 ^ 2 * efh / c ^ 5, {m3, 0, 3}] / m3 ^ 2;
i50 = 4 * temp;
i51 = Series[m3 ^ 2 * 4 / b * (a * temp - eth / c ^ 3), {m3, 0, 3}] / m3 ^ 2;
i52 = Series[4 * m3 ^ 2 / b ^ 2 * (a ^ 2 * temp - 2 * a * eth / c ^ 3 + f / c), {m3, 0, 3}] / m3 ^ 2;
i53 = Series[4 * m3 ^ 2 / b ^ 3 * (a ^ 3 * temp - 3 * a ^ 2 * eth / c ^ 3 + 3 * a * f / c - c * e), {m3, 0, 3}] / m3 ^ 2;

```

```
In[14]:= qu11 = Series[
    -m3^2*6*x*(x^3*i51-x^2*x0*(i50+2*i52)+x*x0^2*(i53+2*i51)-x0^3*i52),
    {m3, 0, 3}]/m3^2;
qu12 = Series[-m3^2*6*x*x*i*((x^2+x0^2)*i51-x*x0*(i50+i52)), {m3, 0, 3}]/m3^2;
qu22 = Series[-m3^2*6*x*x*i^2*(x*i51-x0*i50), {m3, 0, 6}]/m3^2;
qv11 = Series[-m3^2*6*x*x*i*(x0^2*i52+x^2*i50-2*x*x0*i51), {m3, 0, 6}]/m3^2;
qv12 = Series[-m3^2*6*x*x*i^2*(x*i50-x0*i51), {m3, 0, 6}]/m3^2;
qv22 = Series[-m3^2*6*x*x*i^3*i50, {m3, 0, 6}]/m3^2;
```

```
In[20]:= term = qu11;
term = Simplify[term /. {d1 -> 1/4, d2 -> 3/32, d3 -> 15/256,
    b1 -> 1/8, b2 -> 9/128, c1 -> (4*log2-1)/4, c2 -> (24*log2-13)/64,
    c3 -> 3*(5*log2-3)/64, a1 -> (2*log2-1)/4, a2 -> (12*log2-7)*3/128}];
term = Simplify[term /. {b -> 2*x*x0, a -> x0^2+x^2+x*i^2}]
```

```
Out[22]= - 
$$\frac{2 \left( (x^2 - x0^2 + xi^2) (-x^2 + x0^2 + xi^2)^2 \right)}{(x x0 (x^2 + 2 x x0 + x0^2 + xi^2)^{5/2}) m3^2} -$$


$$\frac{\left( 3 \left( 3 x^6 + 4 x^5 x0 - 5 x^4 (x0^2 - xi^2) + (x0^2 - 5 xi^2) (x0^2 + xi^2)^2 - 8 x^3 (x0^3 - x0 xi^2) + \right. \right.}{x^2 (x0^4 - 2 x0^2 xi^2 - 3 xi^4) + 4 x (x0^5 - 2 x0^3 xi^2 - 3 x0 xi^4) \Big) \Big/}{\left( 2 \left( x x0 (x^2 + 2 x x0 + x0^2 + xi^2)^{5/2} \right) m3 \right) +}$$


$$\frac{\left( 3 \left( 8 x^2 x0 (8 (-1 + 4 \log 2 - \log m) x^3 + (-7 + 152 \log 2 - 38 \log m) x^2 x0 + 16 (1 + 4 \log 2 - \log m) x x0^2 + \right. \right.}{16 (4 \log 2 - \log m) x0^3) + 4 x ((-1 + 8 \log 2 - 2 \log m) x^3 + 32 (1 + 4 \log 2 - \log m) x^2 x0 +$$


$$2 (47 - 88 \log 2 + 22 \log m) x x0^2 + 16 (1 + 4 \log 2 - \log m) x0^3) (x^2 + x0^2 + xi^2) +$$


$$6 (2 (3 + 8 \log 2 - 2 \log m) x^2 + 24 (1 - 4 \log 2 + \log m) x x0 + (3 + 8 \log 2 - 2 \log m) x0^2)$$


$$(x^2 + x0^2 + xi^2)^2 - 15 (-1 + 8 \log 2 - 2 \log m) (x^2 + x0^2 + xi^2)^3 \Big) \Big/}{\left( 32 x x0 (x^2 + 2 x x0 + x0^2 + xi^2)^{5/2} \right) +}$$


$$\frac{\left( \left( 8 x^2 x0 (6 (-13 + 24 \log 2 - 6 \log m) x^3 + 7 (-37 + 72 \log 2 - 18 \log m) x^2 x0 + \right. \right.}{84 (-3 + 8 \log 2 - 2 \log m) x x0^2 + 48 (-2 + 4 \log 2 - \log m) x0^3) +$$


$$4 x ((-11 + 24 \log 2 - 6 \log m) x^3 + 24 (-3 + 8 \log 2 - 2 \log m) x^2 x0 +$$


$$2 (-11 + 312 \log 2 - 78 \log m) x x0^2 + 12 (-3 + 8 \log 2 - 2 \log m) x0^3) (x^2 + x0^2 + xi^2) +$$


$$2 (2 (-7 + 24 \log 2 - 6 \log m) x^2 + 54 (1 + 8 \log 2 - 2 \log m) x x0 + (-7 + 24 \log 2 - 6 \log m) x0^2)$$


$$(x^2 + x0^2 + xi^2)^2 + 5 (5 + 24 \log 2 - 6 \log m) (x^2 + x0^2 + xi^2)^3 \Big) m3 \Big/}{\left( 128 x x0 (x^2 + 2 x x0 + x0^2 + xi^2)^{5/2} \right) + O[m3]^2}$$

```

```

In[23]:= msqcoeff = Simplify[term*m3^2 /. m3 -> 0]
mnum = Simplify[Numerator[msqcoeff]/(-2)]
(* Lead =
-2/x0 The whole term is multiplied by rho2^4/rho1^4 leaving 1/rho2 in denom

mnum is numerator without the lead. we now
break up mnum into pieces consisting of 0(r)*0(small) factors
we need to expand 0(r)/0(r2) separately, to avoid errors when r2 is small
that term is then multiplied by remaining 0(small) term*)

Out[23]= 
$$-\frac{2(x^2 - x0^2 + xi^2)(-x^2 + x0^2 + xi^2)^2}{x x0 (x^2 + 2 x x0 + x0^2 + xi^2)^{5/2}}$$


Out[24]= 
$$(x^2 - x0^2 + xi^2)(-x^2 + x0^2 + xi^2)^2$$


In[25]:= piece4 = xi^6
piece3 = Simplify[xi^4 * Coefficient[mnum - piece4, xi^4]]
piece2 = Simplify[xi^2 * Coefficient[mnum - piece4 - piece3, xi^2]]
piece1 = Simplify[mnum - piece4 - piece3 - piece2]

Out[25]= xi^6
Out[26]= 
$$(-x^2 + x0^2) xi^4$$

Out[27]= 
$$-(x^2 - x0^2)^2 xi^2$$

Out[28]= 
$$(x^2 - x0^2)^3$$


In[29]:= small1 = (x - x0)^3
fact1 = Simplify[piece1 / small1]
FortranForm[Expand[%]];

Out[29]= 
$$(x - x0)^3$$

Out[30]= 
$$(x + x0)^3$$


In[32]:= small2 = -xi^2 * (x - x0)^2
fact2 = Simplify[piece2 / small2]
FortranForm[Expand[%]];

Out[32]= 
$$-(x - x0)^2 xi^2$$

Out[33]= 
$$(x + x0)^2$$


In[35]:= small3 = xi^4 * (x - x0)
fact3 = Simplify[piece3 / small3]

Out[35]= 
$$(x - x0) xi^4$$

Out[36]= 
$$-x - x0$$


```

```

In[37]:= small4 = xi^6
Simplify[mnum - small1*fact1 - small2*fact2 - small3*fact3 - small4]

Out[37]= xi^6

Out[38]= 0

In[39]:= mcoeff = Simplify[Coefficient[term*m3^2, m3]]
mnum = Simplify[Numerator[mcoeff]/(-3)]
(* Lead =
-3/2x0 The whole term is multiplied by rho2^2/rho1^2 leaving 1/rho2^3 in denom*)

Out[39]= -((3 (3 x^6 + 4 x^5 x0 - 5 x^4 (x0^2 - xi^2) + (x0^2 - 5 xi^2) (x0^2 + xi^2)^2 - 8 x^3 (x0^3 - x0 xi^2) + x^2
(x0^4 - 2 x0^2 xi^2 - 3 xi^4) + 4 x (x0^5 - 2 x0^3 xi^2 - 3 x0 xi^4))) / (2 x x0 (x^2 + 2 x x0 + x0^2 + xi^2)^{5/2}))

Out[40]= 3 x^6 + 4 x^5 x0 - 5 x^4 (x0^2 - xi^2) + (x0^2 - 5 xi^2) (x0^2 + xi^2)^2 -
8 x^3 (x0^3 - x0 xi^2) + x^2 (x0^4 - 2 x0^2 xi^2 - 3 xi^4) + 4 x (x0^5 - 2 x0^3 xi^2 - 3 x0 xi^4)

In[41]:= piece4 = -5 xi^6
piece3 = Simplify[xi^4 * Coefficient[mnum - piece4, xi^4]]
piece2 = Simplify[xi^2 * Coefficient[mnum - piece4 - piece3, xi^2]]
piece1 = Simplify[mnum - piece4 - piece3 - piece2]

Out[41]= -5 xi^6

Out[42]= -3 (x^2 + 4 x x0 + 3 x0^2) xi^4

Out[43]= (x + x0)^2 (5 x^2 - 2 x x0 - 3 x0^2) xi^2

Out[44]= (x - x0)^2 (x + x0)^3 (3 x + x0)

In[45]:= small1 = (x - x0)^2
fact1 = Simplify[piece1 / small1]
(*FortranForm[Expand[%]];*)
Expand[%]

Out[45]= (x - x0)^2

Out[46]= (x + x0)^3 (3 x + x0)

Out[47]= 3 x^4 + 10 x^3 x0 + 12 x^2 x0^2 + 6 x x0^3 + x0^4

In[48]:= small2 = xi^2 * (x - x0)
fact2 = Simplify[piece2 / small2]
(*FortranForm[Expand[%]];*)
Expand[%]

Out[48]= (x - x0) xi^2

Out[49]= (x + x0)^2 (5 x + 3 x0)

Out[50]= 5 x^3 + 13 x^2 x0 + 11 x x0^2 + 3 x0^3

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```
In[51]:= small3 = -3 * xi^4
fact3 = Simplify[piece3 / small3]
```

```
Out[51]= -3 xi^4
```

```
Out[52]= x^2 + 4 x x0 + 3 x0^2
```

```
In[53]:= small4 = -5 * xi^6
Simplify[mnum - small1 * fact1 - small2 * fact2 - small3 * fact3 - small4]
```

```
Out[53]= -5 xi^6
```

```
Out[54]= 0
```

```
In[55]:= logcoeff=Simplify[Coefficient[term, logm]/.m3->0]
mnum=Simplify[Numerator[logcoeff]/(-3)]
FortranForm[%];
(* Lead = -3/(16*x0) This terms is multiplied by log(rho1^2/dsq)-log(rho^2/d^2) *)
```

```
Out[55]= (3 (-x^6 - 24 x^5 x0 + 3 (x0^2 + xi^2)^2 (3 x0^2 + 5 xi^2) +
x^4 (-53 x0^2 + 17 xi^2) - 16 x^3 (x0^3 - 5 x0 xi^2) + x^2 (45 x0^4 + 142 x0^2 xi^2 + 33 xi^4) +
8 x (5 x0^5 + 14 x0^3 xi^2 + 9 x0 xi^4))) / (16 x x0 (x^2 + 2 x x0 + x0^2 + xi^2)^(5/2))
```

```
Out[56]= x^6 + 24 x^5 x0 + x^4 (53 x0^2 - 17 xi^2) - 3 (x0^2 + xi^2)^2 (3 x0^2 + 5 xi^2) +
16 x^3 (x0^3 - 5 x0 xi^2) - x^2 (45 x0^4 + 142 x0^2 xi^2 + 33 xi^4) - 8 x (5 x0^5 + 14 x0^3 xi^2 + 9 x0 xi^4)
```

```
In[58]:= piece4 = Simplify[xi^6 * Coefficient[mnum, xi^6]]
piece3 = Simplify[xi^4 * Coefficient[mnum - piece4, xi^4]]
piece2 = Simplify[xi^2 * Coefficient[mnum - piece4 - piece3, xi^2]]
piece1 = Simplify[mnum - piece4 - piece3 - piece2]
```

```
Out[58]= -15 xi^6
```

```
Out[59]= -3 (11 x^2 + 24 x x0 + 13 x0^2) xi^4
```

```
Out[60]= -(x + x0)^2 (17 x^2 + 46 x x0 + 33 x0^2) xi^2
```

```
Out[61]= (x + x0)^3 (x^3 + 21 x^2 x0 - 13 x x0^2 - 9 x0^3)
```

```
In[62]:= small1 = x - x0
fact1 = Simplify[piece1 / small1]
(*FortranForm[Expand[%]];*)
Expand[%]
```

```
Out[62]= x - x0
```

```
Out[63]= (x + x0)^3 (x^2 + 22 x x0 + 9 x0^2)
```

```
Out[64]= x^5 + 25 x^4 x0 + 78 x^3 x0^2 + 94 x^2 x0^3 + 49 x x0^4 + 9 x0^5
```

```

In[65]:= small2 = -xi ^ 2
         fact2 = Simplify[piece2 / small2]
         (*FortranForm[Expand[%]];*)
         Expand[%]
Out[65]= -xi^2
Out[66]= (x + x0)^2 (17 x^2 + 46 x x0 + 33 x0^2)
Out[67]= 17 x^4 + 80 x^3 x0 + 142 x^2 x0^2 + 112 x x0^3 + 33 x0^4

In[68]:= small3 = -3 * xi ^ 4
         fact3 = Simplify[piece3 / small3]
         FortranForm[%];
Out[68]= -3 xi^4
Out[69]= 11 x^2 + 24 x x0 + 13 x0^2

In[71]:= small4 = piece4
         Simplify[mnum - small1 * fact1 - small2 * fact2 - small3 * fact3 - small4]
Out[71]= -15 xi^6
Out[72]= 0

```

```

In[127]:= logmcoeff = Simplify[Coefficient[term, m3]];
logmcoeff = Simplify[Coefficient[%, logm]]
mnum = Simplify[Numerator[logmcoeff]]
Simplify[Coefficient[mnum, xi]]
Simplify[Coefficient[mnum, xi^3]]
Simplify[Coefficient[mnum, xi^5]]
denom = Simplify[Denominator[logmcoeff]]
mnum0 = Simplify[mnum /. xi -> 0]
FortranForm[%];
Simplify[Coefficient[mnum, xi^2]]
Simplify[Coefficient[mnum, xi^4]]
Simplify[Coefficient[mnum, xi^6]]

```

```

Out[128]=

$$-\left(\left(3\left(13x^6 + 116x^5x_0 + (x_0^2 + xi^2)^2(7x_0^2 + 5xi^2) + x^4(301x_0^2 + 27xi^2) + 8x^3(43x_0^3 + 13x_0xi^2) + x^2(191x_0^4 + 146x_0^2xi^2 + 19xi^4) + 4x(13x_0^5 + 22x_0^3xi^2 + 9x_0xi^4)\right)\right) / \left(64x_0(x^2 + 2xx_0 + x_0^2 + xi^2)^{5/2}\right)\right)$$


```

```

Out[129]=

$$-3\left(13x^6 + 116x^5x_0 + (x_0^2 + xi^2)^2(7x_0^2 + 5xi^2) + x^4(301x_0^2 + 27xi^2) + 8x^3(43x_0^3 + 13x_0xi^2) + x^2(191x_0^4 + 146x_0^2xi^2 + 19xi^4) + 4x(13x_0^5 + 22x_0^3xi^2 + 9x_0xi^4)\right)$$


```

```

Out[130]=
0

```

```

Out[131]=
0

```

```

Out[132]=
0

```

```

Out[133]=

$$64x_0(x^2 + 2xx_0 + x_0^2 + xi^2)^{5/2}$$


```

```

Out[134]=

$$-3(x + x_0)^3(13x^3 + 77x^2x_0 + 31xx_0^2 + 7x_0^3)$$


```

```

Out[136]=

$$-3(x + x_0)^2(27x^2 + 50xx_0 + 19x_0^2)$$


```

```

Out[137]=

$$-3(19x^2 + 36xx_0 + 17x_0^2)$$


```

```

Out[138]=
-15

```

```
In[86]:= rest =
      Simplify[term - msqcoeff / m3 ^ 2 - mcoeff / m3 - logcoeff * logm - mnum0 * m3 * logm / denom];
```

```
rest0 = Simplify[rest /. m3 -> 0];
rest0num = Simplify[Numerator[rest0]];
Simplify[Coefficient[rest0num, xi]]
Simplify[Coefficient[rest0num, xi ^ 3]]
Simplify[Coefficient[rest0num, xi ^ 5]]
Denominator[rest0]
Simplify[rest0num /. xi -> 0]
FortranForm[%];
Simplify[Coefficient[rest0num, xi ^ 2]]
FortranForm[%];
Simplify[Coefficient[rest0num, xi ^ 4]]
FortranForm[%];
Simplify[Coefficient[rest0num, xi ^ 6]]
Simplify[Coefficient[rest0num, xi ^ 8]]
```

```
Out[89]= 0
```

```
Out[90]= 0
```

```
Out[91]= 0
```

```
Out[92]= 32 x x0 (x^2 + 2 x x0 + x0^2 + xi^2)^{5/2}
```

```
Out[93]= 3 (x + x0)^3 ((47 + 8 log2) x^3 + (67 + 168 log2) x^2 x0 + (109 - 104 log2) x x0^2 + 3 (11 - 24 log2) x0^3)
```

```
Out[95]= -3 (x + x0)^2 ((-113 + 136 log2) x^2 + 2 (-95 + 184 log2) x x0 + 3 (-27 + 88 log2) x0^2)
```

```
Out[97]= -9 (x + x0) ((-27 + 88 log2) x + (-21 + 104 log2) x0)
```

```
Out[99]= 45 - 360 log2
```

```
Out[100]=
0
```



```

In[114]:= rest1 = Simplify[Coefficient[rest, m3]];
rest1num = Simplify[Numerator[rest1]];
Simplify[Coefficient[rest1num, xi]]
Simplify[Coefficient[rest1num, xi^3]]
Simplify[Coefficient[rest1num, xi^5]]
Denominator[rest1]
Simplify[rest1num /. xi -> 0]
FortranForm[%];
Simplify[Coefficient[rest1num, xi^2]]
FortranForm[%];
Simplify[Coefficient[rest1num, xi^4]]
Simplify[Coefficient[rest1num, xi^6]]
Simplify[Coefficient[rest1num, xi^8]]

```

Out[116]=

0

Out[117]=

0

Out[118]=

0

Out[119]=

$128 x x_0 (x^2 + 2 x x_0 + x_0^2 + x i^2)^{5/2}$

Out[120]=

$(x + x_0)^3 ((-47 + 312 \log 2) x^3 + 3 (-221 + 616 \log 2) x^2 x_0 + 3 (-23 + 248 \log 2) x x_0^2 + (11 + 168 \log 2) x_0^3)$

Out[122]=

$(x + x_0)^2 ((-25 + 648 \log 2 - 162 \log m) x^2 + 2 (-11 + 600 \log 2 - 150 \log m) x x_0 + (47 + 456 \log 2 - 114 \log m) x_0^2)$

Out[124]=

$(x + x_0) ((47 + 456 \log 2 - 114 \log m) x + (61 + 408 \log 2 - 102 \log m) x_0)$

Out[125]=

$5 (5 + 24 \log 2 - 6 \log m)$

Out[126]=

0