In[4]:=
```
(*Find F2,F1 and F0, numerators of 1/rho^4,1/rho^2, log(rho2) for Q11
use expansion of F(k) and E(k) about k=1
call log(rho^2)=logm, rho^2=|x-x0|^2
m3=rho^2 / rho2^2 where rho2^2=(x+x0)^2+(y-y0)^2

Version2: much simpler!
(1)when r(=x) is small, it is important to expand O(r)/O(rho2)
 together. so break up each numerator into pieces, each one of them is
 factored into "Small" times "fact". small is expanded to highest order,
fact to smaller order. In code,we then expand fact/rho2 or fact/(xrho2) first,
then multiplied by expansion of small.
(2)we dont need to expand "rest"! instead,
evaluate exactly in code using known values of x,x0,xi
(3)we no longer expand in d. that was a detour that lead to errors. high
 derivatives are large and need to be accurate to all powers in d.

For values of a1,a2,b1,b2,etc see Abramowitz and Stegun AND
https://functions.wolfram.com/EllipticIntegrals/EllipticK/introductions/
      CompleteEllipticIntegrals/05/

a0=Log[4], a1=(Log[4]-1)/4, a2=(6*Log[4]-7)*3/128
b1=1/8, b2=9/128, b3 not used
c1=(4log2-1)/4, c2=(24log2-13)/64 (not used), c3=3(5log2-3)/64 (not used)
d1=1/4, d2=3/32
*)

f = 2 * log2 - logm / 2 + a1 * m3 + a2 * m3 ^ 2 +
    a3 * m3 ^ 3 - (b1 * m3 + b2 * m3 ^ 2 + b3 * m3 ^ 3 + b4 * m3 ^ 4) * logm;
e = 1 + c1 * m3 + c2 * m3 ^ 2 + c3 * m3 ^ 3 - (d1 * m3 + d2 * m3 ^ 2 + d3 * m3 ^ 3 + d4 * m3 ^ 4) * logm;

eth = Series[m3 * e / m3, {m3, 0, 3}] / m3;
efh = Series[m3 ^ 2 * (2 * (1 + m3) * e / m3 ^ 2 - f / m3) / 3, {m3, 0, 3}] / m3 ^ 2;

c = Sqrt[a + b];
temp = Series[m3 ^ 2 * efh / c ^ 5, {m3, 0, 3}] / m3 ^ 2;
i50 = 4 * temp;
i51 = Series[m3 ^ 2 * 4 / b * (a * temp - eth / c ^ 3), {m3, 0, 3}] / m3 ^ 2;
i52 = Series[4 * m3 ^ 2 / b ^ 2 * (a ^ 2 * temp - 2 * a * eth / c ^ 3 + f / c), {m3, 0, 3}] / m3 ^ 2;
i53 = Series[4 * m3 ^ 2 / b ^ 3 * (a ^ 3 * temp - 3 * a ^ 2 * eth / c ^ 3 + 3 * a * f / c - c * e), {m3, 0, 3}] / m3 ^ 2;
```

In[14]:=
```
qu11 = Series[
    -m3^2*6*x*(x^3*i51 - x^2*x0*(i50 + 2*i52) + x*x0^2*(i53 + 2*i51) - x0^3*i52),
    {m3, 0, 3}] / m3^2;
qu12 = Series[-m3^2*6*x*xi*((x^2 + x0^2)*i51 - x*x0*(i50 + i52)), {m3, 0, 3}] / m3^2;
qu22 = Series[-m3^2*6*x*xi^2*(x*i51 - x0*i50), {m3, 0, 6}] / m3^2;
qv11 = Series[-m3^2*6*x*xi*(x0^2*i52 + x^2*i50 - 2*x*x0*i51), {m3, 0, 6}] / m3^2;
qv12 = Series[-m3^2*6*x*xi^2*(x*i50 - x0*i51), {m3, 0, 6}] / m3^2;
qv22 = Series[-m3^2 * 6 * x * xi^3 * i50, {m3, 0, 6}] / m3^2;
```

In[20]:=
```
term = qu12;
term = Simplify[term /. {d1 → 1/4, d2 → 3/32, d3 → 15/256,
      b1 → 1/8, b2 → 9/128, c1 → (4*log2 - 1)/4, c2 → (24*log2 - 13)/64,
      c3 → 3*(5*log2 - 3)/64, a1 → (2*log2 - 1)/4, a2 → (12*log2 - 7)*3/128}];
term = Simplify[term /. {b → 2*x*x0, a → x0^2 + x^2 + xi^2}]
```

Out[22]=
$$-\frac{4\,(xi\,(x^4 - 2\,x^2\,x0^2 + x0^4 - xi^4))}{\left(x0\,(x^2 + 2\,x\,x0 + x0^2 + xi^2)^{5/2}\right)m3^2} -$$

$$\frac{3\,(xi\,(x^4 + x0^4 + 8\,x\,x0\,xi^2 + 4\,x0^2\,xi^2 + 3\,xi^4 - 2\,x^2\,(x0^2 - 2\,xi^2)))}{\left(x0\,(x^2 + 2\,x\,x0 + x0^2 + xi^2)^{5/2}\right)m3} +$$

$$\Big(3\,xi\,\big((7 + 40\,\log2 - 10\,\log m)\,x^4 + 64\,(4\,\log2 - \log m)\,x^3\,x0 + (x0^2 + xi^2)\,\big((7 + 40\,\log2 - 10\,\log m)\,x0^2 + 3\,(3 + 8\,\log2 - 2\,\log m)\,xi^2\big) + 2\,x^2\,\big((-7 + 216\,\log2 - 54\,\log m)\,x0^2 + 8\,(1 + 4\,\log2 - \log m)\,xi^2\big) + 32\,x\,x0\,\big(xi^2 + 4\,\log2\,(2\,x0^2 + xi^2) - \log m\,(2\,x0^2 + xi^2)\big)\big)\Big) \Big/ \left(16\,x0\,(x^2 + 2\,x\,x0 + x0^2 + xi^2)^{5/2}\right) +$$

$$\Big(xi\,\big((-29 + 72\,\log2 - 18\,\log m)\,x^4 + 192\,(-2 + 4\,\log2 - \log m)\,x^3\,x0 + (x0^2 + xi^2)\,\big((-29 + 72\,\log2 - 18\,\log m)\,x0^2 + (-7 + 24\,\log2 - 6\,\log m)\,xi^2\big) + 2\,x^2\,\big((-355 + 696\,\log2 - 174\,\log m)\,x0^2 + 6\,(-3 + 8\,\log2 - 2\,\log m)\,xi^2\big) + 24\,x\,\big(8\,(-2 + 4\,\log2 - \log m)\,x0^3 + (-3 + 8\,\log2 - 2\,\log m)\,x0\,xi^2\big)\big)\,m3\Big) \Big/ \left(64\,x0\,(x^2 + 2\,x\,x0 + x0^2 + xi^2)^{5/2}\right) + O[m3]^2$$

In[23]:=
```
msqcoeff = Simplify[term * m3^2 /. m3 → 0]
mnum = Simplify[Numerator[msqcoeff]/(-4)]
(* Lead =
 -4/x0  The whole term is multiplied by rho2^4/rho1^4 leaving 1/rho2 in denom

mnum is numerator without the lead. we now
  break up mnum into pieces consisting of O(r)*O(small) factors
we need to expand O(r)/O(r2) separately, to avoid errors when r2 is small
that term is then multiplied by remaining O(small) term*)
```

Out[23]= $$-\frac{4\,xi\left(x^4-2\,x^2\,x0^2+x0^4-xi^4\right)}{x0\left(x^2+2\,x\,x0+x0^2+xi^2\right)^{5/2}}$$

Out[24]= $$xi\left(x^4-2\,x^2\,x0^2+x0^4-xi^4\right)$$

In[25]:=
```
piece2 = -xi^5
piece1 = Simplify[mnum - piece2]
```

Out[25]= $$-xi^5$$

Out[26]= $$\left(x^2-x0^2\right)^2\,xi$$

In[27]:=
```
small1 = (x - x0)^2 * xi
fact1 = (x + x0)^2
Expand[%];
FortranForm[%];
```

Out[27]= $$(x-x0)^2\,xi$$

Out[28]= $$(x+x0)^2$$

In[31]:=
```
small2 = -xi^5
fact2 = 1
Simplify[mnum - small1 * fact1 - small2 * fact2]
```

Out[31]= $$-xi^5$$

Out[32]= $$1$$

Out[33]= $$0$$

In[34]:=
```
mcoeff = Simplify[Coefficient[term * m3^2, m3]]
mnum = Simplify[Numerator[mcoeff]/(-3)]
(* Lead =
 -3/x0  The whole term is multiplied by rho2^2/rho1^2 leaving 1/rho2^3 in denom*)
```

Out[34]= $$-\frac{3\,xi\left(x^4+x0^4+8\,x\,x0\,xi^2+4\,x0^2\,xi^2+3\,xi^4-2\,x^2\left(x0^2-2\,xi^2\right)\right)}{x0\left(x^2+2\,x\,x0+x0^2+xi^2\right)^{5/2}}$$

Out[35]= $$xi\left(x^4+x0^4+8\,x\,x0\,xi^2+4\,x0^2\,xi^2+3\,xi^4-2\,x^2\left(x0^2-2\,xi^2\right)\right)$$

In[36]:=
```
piece3 = 3 * xi ^ 5
piece1 = xi * (x ^ 2 - x0 ^ 2) ^ 2
piece2 = Simplify[mnum - piece3 - piece1]
```

Out[36]= $3\ xi^5$

Out[37]= $\left(x^2 - x0^2\right)^2\ xi$

Out[38]= $4\ (x + x0)^2\ xi^3$

In[39]:=
```
small1 = xi * (x - x0) ^ 2
fact1 = Simplify[piece1 / small1]
Expand[%];
FortranForm[%];
```

Out[39]= $(x - x0)^2\ xi$

Out[40]= $(x + x0)^2$

In[43]:=
```
small2 = 4 * xi ^ 3
fact2 = Simplify[piece2 / small2]
```

Out[43]= $4\ xi^3$

Out[44]= $(x + x0)^2$

In[45]:=
```
small3 = 3 * xi ^ 5
Simplify[mnum - small1 * fact1 - small2 * fact2 - small3]
```

Out[45]= $3\ xi^5$

Out[46]= 0

In[47]:=
```
logcoeff=Simplify[Coefficient[term,logm]/.m3→0]
mnum=Simplify[Numerator[logcoeff]/(-3)]
FortranForm[%];
(* Lead = -3/(8*x0)  This terms is multiplied by log(rho1^2) with a remainder of F0*log
```

Out[47]= $-\dfrac{3\ xi\left(5\ x^4 + 32\ x^3\ x0 + 5\ x0^4 + 8\ x0^2\ xi^2 + 3\ xi^4 + 16\ x\ x0\left(2\ x0^2 + xi^2\right) + x^2\left(54\ x0^2 + 8\ xi^2\right)\right)}{8\ x0\left(x^2 + 2\ x\ x0 + x0^2 + xi^2\right)^{5/2}}$

Out[48]= $xi\left(5\ x^4 + 32\ x^3\ x0 + 5\ x0^4 + 8\ x0^2\ xi^2 + 3\ xi^4 + 16\ x\ x0\left(2\ x0^2 + xi^2\right) + x^2\left(54\ x0^2 + 8\ xi^2\right)\right)$

In[50]:=
```
piece3 = 3 * xi ^ 5
piece2 = (x + x0) ^ 2 * 8 * xi ^ 3
piece1 = Simplify[mnum - piece3 - piece2]
```

Out[50]= $3\ xi^5$

Out[51]= $8\ (x + x0)^2\ xi^3$

Out[52]= $(x + x0)^2\left(5\ x^2 + 22\ x\ x0 + 5\ x0^2\right)\ xi$

In[53]:=
```
small1 = xi
fact1 = Simplify[piece1 / small1]
Expand[%]
FortranForm[%];
```

Out[53]= xi

Out[54]= $(x + x0)^2 \left(5\, x^2 + 22\, x\, x0 + 5\, x0^2\right)$

Out[55]= $5\, x^4 + 32\, x^3\, x0 + 54\, x^2\, x0^2 + 32\, x\, x0^3 + 5\, x0^4$

In[57]:=
```
small2 = 8 * xi ^ 3
fact2 = Simplify[piece2 / small2]
Expand[%];
FortranForm[%];
```

Out[57]= $8\, xi^3$

Out[58]= $(x + x0)^2$

In[61]:=
```
small3 = 3 * xi ^ 5
Simplify[mnum - small1 * fact1 - small2 * fact2 - small3]
```

Out[61]= $3\, xi^5$

Out[62]= 0

In[63]:=
```
logmcoeff = Simplify[Coefficient[term, m3]];
logmcoeff = Simplify[Coefficient[%, logm]]
mnum = Simplify[Numerator[logmcoeff]]
denom = Simplify[Denominator[logmcoeff]]
mnum0 = Simplify[mnum /. xi → 0]
Simplify[Coefficient[mnum, xi]]
FortranForm[%];
Simplify[Coefficient[mnum, xi^2]]
Simplify[Coefficient[mnum, xi^3]]
FortranForm[%];
Simplify[Coefficient[mnum, xi^4]]
```

Out[64]= $$-\frac{3\,xi\left(3\,x^4+32\,x^3\,x0+3\,x0^4+4\,x0^2\,xi^2+xi^4+8\,x\,x0\left(4\,x0^2+xi^2\right)+x^2\left(58\,x0^2+4\,xi^2\right)\right)}{32\,x0\left(x^2+2\,x\,x0+x0^2+xi^2\right)^{5/2}}$$

Out[65]= $$-3\,xi\left(3\,x^4+32\,x^3\,x0+3\,x0^4+4\,x0^2\,xi^2+xi^4+8\,x\,x0\left(4\,x0^2+xi^2\right)+x^2\left(58\,x0^2+4\,xi^2\right)\right)$$

Out[66]= $$32\,x0\left(x^2+2\,x\,x0+x0^2+xi^2\right)^{5/2}$$

Out[67]= 0

Out[68]= $$-3\,(x+x0)^2\left(3\,x^2+26\,x\,x0+3\,x0^2\right)$$

Out[70]= 0

Out[71]= $$-12\,(x+x0)^2$$

Out[73]= 0

In[74]:=
```
rest =
  Simplify[term - msqcoeff / m3^2 - mcoeff / m3 - logcoeff * logm - mnum0 * m3 * logm / denom];
```

```
rest0 = Simplify[rest /. m3 → 0];
rest0num = Simplify[Numerator[rest0]]
Simplify[rest0num /. xi → 0]
FortranForm[%];
Simplify[Coefficient[rest0num, xi]]
FortranForm[%];
Simplify[Coefficient[rest0num, xi^2]]
Simplify[Coefficient[rest0num, xi^3]]
FortranForm[%];
Simplify[Coefficient[rest0num, xi^4]]
Simplify[Coefficient[rest0num, xi^5]]
FortranForm[%];
Simplify[Coefficient[rest0num, xi^6]]
Simplify[Coefficient[rest0num, xi^7]]
Denominator[rest0]
```

Out[76]= $3\,xi\left((7+40\,\log 2)\,x^4+256\,\log 2\,x^3\,x0+2\,x^2\left((-7+216\,\log 2)\,x0^2+8\,(1+4\,\log 2)\,xi^2\right)+\left(x0^2+xi^2\right)\left((7+40\,\log 2)\,x0^2+3\,(3+8\,\log 2)\,xi^2\right)+32\,x\,x0\left(xi^2+4\,\log 2\left(2\,x0^2+xi^2\right)\right)\right)$

Out[77]= 0

Out[79]= $3\,(x+x0)^2\left((7+40\,\log 2)\,x^2+2\,(-7+88\,\log 2)\,x\,x0+(7+40\,\log 2)\,x0^2\right)$

Out[81]= 0

Out[82]= $48\,(1+4\,\log 2)\,(x+x0)^2$

Out[84]= 0

Out[85]= $9\,(3+8\,\log 2)$

Out[87]= 0

Out[88]= 0

Out[89]= $16\,x0\left(x^2+2\,x\,x0+x0^2+xi^2\right)^{5/2}$

In[90]:=
```
rest1 = Simplify[Coefficient[rest, m3]];
rest1num = Simplify[Numerator[rest1]];
Simplify[rest1num /. xi → 0]
FortranForm[%];
Simplify[Coefficient[rest1num, xi]]
FortranForm[%];
Simplify[Coefficient[rest1num, xi^4]]
Simplify[Coefficient[rest1num, xi^5]]
Simplify[Coefficient[rest1num, xi^7]]
Denominator[rest1]
```

Out[92]= 0

Out[94]= $(x + x0)^2$
$\left((-29 + 72\ \text{log2} - 18\ \text{logm})\ x^2 + 2\ (-163 + 312\ \text{log2} - 78\ \text{logm})\ x\ x0 + (-29 + 72\ \text{log2} - 18\ \text{logm})\ x0^2\right)$

Out[96]= 0

Out[97]= $-7 + 24\ \text{log2} - 6\ \text{logm}$

Out[98]= 0

Out[99]= $64\ x0\left(x^2 + 2\ x\ x0 + x0^2 + xi^2\right)^{5/2}$