Problem 2

Taylor Series

$$f(x+h) = f(x) + hf'(x) + \tfrac{1}{2}h^2 f''(x) + \dots$$
$$= f(x) + h\hat{D}[f(x)] + \tfrac{1}{2}h^2 \hat{D}^2[f(x)] + \dots$$
$$= (1 + h\hat{D} + \tfrac{1}{2}h^2\hat{D}^2 + \dots) f(x) \quad = e^{h\hat{D}} f(x)$$

$$\frac{df(x)}{dx} = Af(x+\tfrac{h}{2}) + Bf(x-\tfrac{h}{2}) + Err[f(x)]$$
$$\hat{D}[f(x)] = Ae^{\frac{1}{2}h\hat{D}}f(x) + Be^{-\frac{1}{2}h\hat{D}}f(x) + Err[f(x)]$$
$$\hat{D} = Ae^{\frac{1}{2}h\hat{D}} + Be^{-\frac{1}{2}h\hat{D}} + Err$$

$$\hat{D} = A(1 + \tfrac{1}{2}h\hat{D} + \tfrac{1}{2}(\tfrac{1}{2}h)^2\hat{D}^2 + \tfrac{1}{3!}(\tfrac{1}{2}h)^3\hat{D}^3 + \dots) + B(1 - \tfrac{1}{2}h\hat{D} + \tfrac{1}{2}(\tfrac{1}{2}h)^2 D^2 - \tfrac{1}{3!}(\tfrac{1}{2}h)^3\hat{D} + \dots) + Err$$

$$\left.\begin{aligned} \hat{D}^0 &: \ A\cdot 1 + B\cdot 1 = A+B = 0 \\ \hat{D}^1 &: \ \tfrac{1}{2}hA - \tfrac{1}{2}hB = 1 \end{aligned}\right\} \quad \begin{aligned} A &= \tfrac{1}{h} \\ B &= -\tfrac{1}{h} \end{aligned}$$

$$f'(x) = Af(x+\tfrac{h}{2}) + Bf(x-\tfrac{h}{2}) + Err[f(x)]$$
$$= \tfrac{1}{h}f(x+\tfrac{h}{2}) - \tfrac{1}{h}f(x-\tfrac{h}{2}) + Err[f(x)]$$
$$= \frac{f(x+\frac{h}{2}) - f(x-\frac{h}{2})}{h} + Err[f(x)]$$

$$\hat{D} = \tfrac{1}{h}(1 + \tfrac{1}{2}h\hat{D} + \tfrac{1}{2}(\tfrac{1}{2}h)^2\hat{D}^2 + \tfrac{1}{3!}(\tfrac{1}{2}h)^3\hat{D}^3 + \dots) - \tfrac{1}{h}(1 - \tfrac{1}{2}h\hat{D} + \tfrac{1}{2}(\tfrac{1}{2}h)^2 D^2 - \tfrac{1}{3!}(\tfrac{1}{2}h)^3\hat{D}^3 + \dots) + Err$$
$$\hat{D} = \cancel{\tfrac{1}{h}} + \tfrac{1}{2}\hat{D} + \tfrac{1}{2h}\cancel{(\tfrac{1}{2}h)^2\hat{D}^2} + \tfrac{1}{6h}(\tfrac{1}{2}h)^3\hat{D}^3 + \dots - \cancel{\tfrac{1}{h}} + \tfrac{1}{2}\hat{D} - \tfrac{1}{2h}\cancel{(\tfrac{1}{2}h)^2 D^2} + \tfrac{1}{6h}(\tfrac{1}{2}h)^3\hat{D}^3 + \dots) + Err$$
$$\cancel{\hat{D}} = \cancel{\hat{D}} + 2\cdot\tfrac{1}{6h}(\tfrac{1}{8}h^3)\hat{D}^3 + \dots + Er$$
$$0 = \tfrac{1}{24}h^2\hat{D}^3 + \dots + Err \Rightarrow \quad Err = \tfrac{1}{24}h^2\hat{D}^3 + \dots = \tfrac{1}{24}h^2\hat{D}^3 + O(h^4)$$

the odd terms of h cancel out and the even terms of h remains, starting at the order of 4^{th}

$$f'(x) = \frac{f(x+\frac{h}{2}) - f(x-\frac{h}{2})}{h} - [\tfrac{1}{24}h^2\hat{D}^3 + O(h^4)]f(x)$$
$$= \frac{f(x+\frac{h}{2}) - f(x-\frac{h}{2})}{h} - \tfrac{1}{24}h^2 f'''(x) + O(h^4) \quad \checkmark$$