Part (a)

$$\frac{d^2y}{dt^2} + (a - 2q\cos 2t)y = 0 \qquad y = \frac{1}{2}\sum_{k=-\infty}^{\infty} c_k e^{ikt}$$

$\hookrightarrow q c_{k-2} + k^2 c_k + q c_{k+2} = a c_k$

even $c_{-k} = c_k$ odd $c_{-k} = -c_k$

periodic π $y_{ep} = \frac{1}{2}c_0 + \sum_{k=1}^{K/2} c_{2k}\cos(2kt)$ $y_{op} = \sum_{k=1}^{K/2} c_{2k}\sin(2kt)$

anti-periodic 2π $y_{ea} = \sum_{k=1}^{K/2} c_{2k-1}\cos((2k-1)t)$ $y_{oa} = \sum_{k=1}^{K/2} c_{2k-1}\sin((2k-1)t)$

$k = 0, 1, 2$ even periodic $y_{ep} = \frac{1}{2}c_0 + \sum_{k=1}^{K/2} c_{2k}\cos(2kt)$

$k = 0, 2, 4, \ldots$ $c_{-k} = c_k$

$q c_{k-2} + k^2 c_k + q c_{k+2} = a c_k$

$k=0$ $q c_{-2} + 0^2 c_0 + q c_2 = a c_0 \Rightarrow 2q c_2 = a c_0$

$k=2$ $q c_0 + 2^2 c_2 + q c_4 = a c_2$

$k=4$ $q c_2 + 4^2 c_4 + q c_6 = a c_4$

$$\begin{pmatrix} 0 & 2q & 0 & 0 & \cdots \\ q & 2^2 & q & 0 & \ddots \\ 0 & q & 4^2 & q & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix} \cdot \begin{pmatrix} c_0 \\ c_2 \\ c_4 \\ \vdots \end{pmatrix} = a \cdot \begin{pmatrix} c_0 \\ c_2 \\ c_4 \\ \vdots \end{pmatrix}$$

even antiperiodic $y_{ea} = \sum_{k=1}^{K/2} c_{2k-1}\cos((2k-1)t)$

$k = 1, 3, 5$. $c_{-k} = c_k$

$q c_{k-2} + k^2 c_k + q c_{k+2} = a c_k$

$k=1$ $q c_{-1} + c_1 + q c_3 = a c_1 \Rightarrow (q+1)c_1 + q c_3 = a c_1$

$k=3$ $q c_1 + 3^2 c_3 + q c_5 = a c_3$

$k=5$ $q c_3 + 5^2 c_5 + q c_7 = a c_5$

$$\begin{pmatrix} q+1 & q & 0 & 0 & \cdots \\ q & 3^2 & q & 0 & \ddots \\ 0 & q & 5^2 & q & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix} \cdot \begin{pmatrix} c_1 \\ c_3 \\ c_5 \\ \vdots \end{pmatrix} = a \cdot \begin{pmatrix} c_1 \\ c_3 \\ c_5 \\ \vdots \end{pmatrix}$$

odd periodic

$$y_{op} = \sum_{k=1}^{K/2} C_{2k} \sin(2kt)$$

$k = 2, 4, 6, \ldots$

$$C_{-k} = -C_k$$

$$qC_{k-2} + k^2 C_k + qC_{k+2} = aC_k \qquad C_0 = 0$$

$k=2$ $\quad qC_0 + 2^2 C_2 + qC_4 = aC_2$

$k=4$ $\quad qC_2 + 4^2 C_4 + qC_6 = aC_4$

$k=6$ $\quad qC_4 + 6^2 C_6 + qC_8 = aC_6$

$$\begin{pmatrix} 2^2 & q & 0 & 0 & \cdots \\ q & 4^2 & q & 0 & \ddots \\ 0 & q & 6^2 & q & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix} \cdot \begin{pmatrix} C_2 \\ C_4 \\ C_6 \\ \vdots \end{pmatrix} = a \cdot \begin{pmatrix} C_2 \\ C_4 \\ C_6 \\ \vdots \end{pmatrix}$$

odd antiperiodic

$$y_{oa} = \sum_{k=1}^{K/2} C_{2k-1} \sin((2k-1)t)$$

$k = 1, 3, 5, \ldots$

$$C_{-k} = -C_k$$

$$qC_{k-2} + k^2 C_k + qC_{k+2} = aC_k$$

$k=1$ $\quad qC_{-1} + 1^2 C_1 + qC_3 = aC_1 \quad \Rightarrow \quad (1-q)\, C_1 + qC_3 = aC_1$

$k=3$ $\quad qC_1 + 3^2 C_3 + qC_5 = aC_3$

$k=5$ $\quad qC_3 + 5^2 C_5 + qC_7 = aC_5$

$$\begin{pmatrix} 1-q & q & 0 & 0 & \cdots \\ q & 3^2 & q & 0 & \ddots \\ 0 & q & 5^2 & q & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix} \cdot \begin{pmatrix} C_1 \\ C_3 \\ C_5 \\ \vdots \end{pmatrix} = a \cdot \begin{pmatrix} C_1 \\ C_3 \\ C_5 \\ \vdots \end{pmatrix}$$

Part (b)

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 2q \\ q & 2^2 \end{pmatrix} \begin{pmatrix} \sqrt{2} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & \sqrt{2}q \\ \sqrt{2}q & 2^2 \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \cdots \\ \vdots & 0 & 0 & 1 \cdots \\ & & \vdots & \end{pmatrix} \begin{pmatrix} 0 & 2q & 0 & 0 & \cdots \\ q & 2^2 & q & 0 & \ddots \\ 0 & q & 4^2 & q & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} \sqrt{2} & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \cdots \\ \vdots & \cdots & 0 & 1 \cdots \\ & & & \vdots \end{pmatrix}$$

$$\begin{pmatrix} 0 & \sqrt{2}q & 0 & 0 \cdots \\ \sqrt{2}q & 2^2 & q & 0 \cdots \\ 0 & q & 4^2 & 0 \cdots \\ \vdots & \ddots & & \end{pmatrix}$$