Taylor Series

$$f(x+h) = f(x) + hf'(x) + \frac{1}{2}h^2 f''(x) + \dots$$
$$= f(x) + h\hat{D}[f(x)] + \frac{1}{2}h^2 \hat{D}^2[f(x)] + \dots$$
$$= (1 + h\hat{D} + \frac{1}{2}h^2\hat{D}^2 + \dots) f(x) \quad = e^{h\hat{D}} f(x)$$

$$\int_0^{2h} f(x)\,dx \qquad f(0) \quad f(h) \quad f'(h) \quad f''(h) \quad f(2h)$$

$$\int f(h) = \frac{1}{D} f(h)$$

$$f(x+h) = e^{hD} f(x) \quad \begin{cases} f(2h) = f(h+h) = e^{hD} f(h) \\ f(0) = f(-h+h) = e^{-hD} f(h) \end{cases}$$

$$\int_0^{2h} f(h) = \frac{1}{D} f(2h) - \frac{1}{D} f(0) = \frac{1}{D} e^{hD} f(h) - \frac{1}{D} e^{-hD} f(h) = (e^{hD} - e^{-hD}) \frac{1}{D} f(h)$$

$$\int_0^{2h} f(x)dx = (e^{hD} - e^{-hD}) \frac{1}{D} f(h)$$
$$= Ae^{-hD} f(h) + B f(h) + CD f(h) + ED^2 f(h) + Fe^{hD} f(h) + Err\, f(h)$$

$$(e^{hD} - e^{-hD})\frac{1}{D} = \frac{2}{D}\sinh(h\hat{D}) = Ae^{-hD} + B + CD + ED^2 + Fe^{hD} + Err$$

$$\frac{2}{D}[hD + \frac{1}{3!}h^3D^3 + \frac{1}{5!}h^5D^5 + \dots] = A(1 - hD + \frac{1}{2}h^2D^2 - \frac{1}{3!}h^3D^3 + \frac{1}{4!}h^4D^4 + \dots) + B$$
$$+ CD + ED^2 + F(1 + hD + \frac{1}{2}h^2D^2 + \frac{1}{3!}h^3D^3 + \frac{1}{4!}h^4D^4 + \dots) + Err$$

$$2h + \frac{2}{3!}h^3D^2 + \frac{2}{5!}h^5D^4 \dots$$

$$D^0 \rightarrow 2h = A + B + F$$
$$D^1 \rightarrow 0 = -hA + C + hF$$
$$D^2 \rightarrow \frac{1}{3}h^3 = \frac{1}{2}h^2A + E + \frac{1}{2}h^2F$$
$$D^3 \rightarrow 0 = -\frac{1}{6}h^3A + \frac{1}{6}h^3F$$
$$D^4 \rightarrow \frac{1}{60}h^5 = \frac{1}{24}h^4A + \frac{1}{24}h^4F$$

for Mathematica, $(A, B, C, E, F) \rightarrow (a1, a2, a3, a4, a5)$

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In[10]:= Solve[{a1 + a2 + a5 == 2*h,
    -h*a1 + a3 + h*a5 == 0,
    1/2*h^2*a1 + a4 + 1/2*h^2*a5 == 1/3*h^3,
    -1/6*h^3*a1 + 1/6*h^3*a5 == 0,
    1/24*h^4*a1 + 1/24*h^4*a5 == 1/60*h^5}, {a1, a2, a3, a4, a5}]
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Out[10]= $\left\{\left\{a1 \rightarrow \frac{h}{5},\ a2 \rightarrow \frac{8h}{5},\ a3 \rightarrow 0,\ a4 \rightarrow \frac{2h^3}{15},\ a5 \rightarrow \frac{h}{5}\right\}\right\}$

$$\begin{array}{ccccc} A & B & C & E & F \\ a_1 = \frac{1}{5}h & a_2 = \frac{8}{5}h & a_3 = 0 & a_4 = \frac{2}{15}h^3 & a_5 = \frac{1}{5}h \end{array}$$

$$\frac{2}{D}[hD + \frac{1}{3!}h^3D^3 + \frac{1}{5!}h^5D^5 + \dots] = \frac{h}{5}(1 - hD + \frac{1}{2}h^2D^2 - \frac{1}{3!}h^3D^3 + \frac{1}{4!}h^4D^4 + \dots) + \frac{8}{5}h$$
$$+ \frac{2}{15}h^3D^2 + \frac{h}{5}(1 + hD + \frac{1}{2}h^2D^2 + \frac{1}{3!}h^3D^3 + \frac{1}{4!}h^4D^4 + \dots) + Err$$

$$2h + \frac{2}{3!}h^3D^2 + \frac{2}{5!}h^5D^4 \dots = 2h + \frac{1}{3}h^3D^2 + \frac{1}{60}h^5D^4 \dots$$

$$\frac{2}{D}[hD + \frac{1}{3!}h^3D^3 + \frac{1}{5!}h^5D^5 + \frac{1}{7!}h^7D^7 + \frac{1}{9!}h^9D^9]$$
$$= 2h + \frac{1}{3}h^3D^2 + \frac{1}{60}h^5D^4 + \frac{1}{2520}h^7D^6 + \dots$$

$$= (\frac{h}{5} - \cancel{\frac{h^2}{5}D} + \frac{h^3}{10}D^2 - \cancel{\frac{h^4}{30}D^3} + \frac{h^5}{120}D^4 - \cancel{\frac{h}{5}\cdot\frac{1}{5!}h^5D^5} + \frac{h}{5}\cdot\frac{1}{6!}h^6D^6 + \dots)$$
$$+ \frac{8}{5}h + \frac{2}{15}h^3D^2 + (\frac{h}{5} + \cancel{\frac{h^2}{5}D} + \frac{h^3}{10}D^2 + \cancel{\frac{h^4}{30}D^3} + \frac{h^5}{120}D^4 + \cancel{\frac{h^6}{600}D^5} + \frac{h^7}{3600}D^6) + Err$$

$$= \frac{2h}{5} + \frac{8h}{5} + \frac{2h^3}{10}D^2 + \frac{2}{15}h^3D^2 + \frac{2h^6}{120}D^4 + \frac{2h^7}{3600}D^6 + Err$$
$$= 2h + \frac{1}{3}h^3D^2 + \frac{1}{60}h^5D^4 + \frac{h^7}{1800}D^6 + Err$$
$$Err = \frac{h^7}{2520}D^6 - \frac{h^7}{1800}D^6 = \boxed{-\frac{h^7}{6300}D^6 + \dots}$$

$$\int_0^{2h} f(x)dx = Ae^{-hD}f(h) + B\,f(h) + CD\,f(h) + ED^2f(h) + Fe^{hD}f(h) + Err\,f(h)$$
$$= \frac{h}{5}e^{-hD}f(h) + \frac{8h}{5}f(h) + \frac{2h^3}{15}D^2f(h) + \frac{h}{5}e^{hD}f(h) - \frac{h^7}{6300}D^6(h) + O(h^9)$$
$$= \frac{h}{5}f(0) + \frac{8h}{5}f(h) + \frac{2h^3}{15}f''(h) + \frac{h}{5}f(2h) \underbrace{- \frac{h^7}{6300}f^{(6)}(h)}_{\text{leading order term}} + O(h^9)$$

? $f^{(6)}(x)$ or $f^{(6)}(h)$

Simpson's rule:

(4.14) $$\int_{x_0}^{x_2} f(x)\,dx = h\left[\frac{1}{3}f_0 + \frac{4}{3}f_1 + \frac{1}{3}f_2\right] + O(h^5 f^{(4)})$$

our formula:

$$\frac{h}{5}f(0) + \frac{8h}{5}f(h) + \frac{2h^3}{15}f''(h) + \frac{h}{5}f(2h) - \frac{h^7}{6300}f^{(6)}(h) + O(h^9)$$

Simpson's rule uses 3 points and our formula uses 5 points so our formula is probably going to be more accurate. As we can also see from the error leading term, our error has an higher order.