Physics 335 March 2, 2023
Physics 635 **Due**: March 21, 2023

Homework #8

You are encouraged to work together for this homework.

Problem 1

Solve the same problem as in Homework #7 by relaxation. Solve Mathieu's equation in second-order form and use a block-tridiagonal solver following the examples given in class. An algebraic form of Mathieu's equation is obtained by setting $x = \cos t$:

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + \left(a + 2q - 4qx^2\right)y = 0.$$

You should use this version of Mathieu's equation when solving via relaxation.

As in Homework #7, find the eigenvalues a_n and b_n of Mathieu's equation for $n = 0$, 1, 2, 10, 15 and $q = 5$, 25, and plot the solutions in the range $0 \leq x \leq 1$.

Equations and Boundary Conditions: When using the block-tridiagonal solver, it is essential to use the algebraic form of Mathieu's equation

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + \left(a + 2q - 4qx^2\right)y = 0.$$

Using the method of Frobenius, we expand $y(x)$ as

$$y(x) = \sum_{k=0}^{\infty} f_k (1-x^2)^{k+\alpha},$$

where $x = \pm 1$ are regular singular points. We find that there are two solutions, $\alpha = 0$ which is regular at $x = \pm 1$ and corresponds to the EVEN solutions, and $\alpha = \frac{1}{2}$ which is irregular at $x = \pm 1$ and corresponds to the ODD solutions.

The power series expansion for the regular solution leads to a boundary condition:

$$\left.\frac{dy}{dx}\right|_{\pm 1} = \pm(a - 2q)y(\pm 1).$$

We can rewrite the irregular solution as $y(x) = \sqrt{1-x^2}\,z(x)$, and rewriting Mathieu's equation in terms of $z(x)$, we find

$$(1-x^2)\frac{d^2z}{dx^2} - 3x\frac{dz}{dx} + \left(a - 1 + 2q - 4qx^2\right)z = 0.$$

Using the method of Frobenius again, we find that the regular solution for $z(x)$ leads to the following boundary condition:

$$\left.\frac{dz}{dx}\right|_{\pm 1} = \pm\frac{1}{3}(a - 2q - 1)z(\pm 1).$$

We can solve the eigenvalue problem over the interval $0 \leq x \leq 1$ (where $x = 0$ corresponds to $t = \frac{\pi}{2}$) using the following boundary conditions:

Periodic Even	$y'(0) = 0$	$y'(1) = (a - 2q)y(1)$
Antiperiodic Even	$y(0) = 0$	$y'(1) = (a - 2q)y(1)$
Periodic Odd	$z(0) = 0$	$z'(1) = \frac{1}{3}(a - 2q - 1)z(1)$
Antiperiodic Odd	$z'(0) = 0$	$z'(1) = \frac{1}{3}(a - 2q - 1)z(1)$