Physics 335 February 23, 2023
Physics 635 **Due**: March 2, 2023

Homework #7

You are encouraged to work together for this homework.

Problem 1

Mathieu's equation arises in several different ways, for example from separation of the wave equation in elliptical coordinates. It first appeared in celestial mechanics. A standard form is

$$\frac{d^2y}{dt^2} + (a - 2q\cos 2t)\, y = 0, \tag{1}$$

where a and q are real constants. We are usually interested in solutions for which y is periodic, with period π or 2π. There exists a countably finite set of eigenvalues $a_n(q)$, $n = 0, 1, \ldots$, for which y is an even periodic solution, with n zeros in the interval $0 \le t < \pi$. For n even, y has period π; while for n odd, y has period 2π. There is another set of eigenvalues $b_n(q)$, $n = 1, 2, \ldots$, for which y is an odd periodic solution, also with n zeros in the interval $0 \le t < \pi$ and with the same periodicity properties as the even solutions. You can see roughly the behavior of the solutions by noting that for $q = 0$ the even solutions are $\cos nt$ and the odd solutions are $\sin nt$, with $a_n = b_n = n^2$.

Compute a_n and b_n for $n = 0, 1, 2, 10, 15$ with $q = 5$ and $q = 25$. Also, plot the eigenfunctions for each case. Your writeup should explain what your strategy is.

Boundary Conditions: First of all, note that all of the even and odd solutions that have period 2π but do not have period π are actually *antiperiodic* in π, that is, $y(t + \pi) = -y(t)$. For a proof of this, see the appendix. Because of this, we need only consider the region $[0, \pi]$. Here is one possible set of boundary conditions (to within an arbitrary multiplicative constant):

	$y(0)$	$y'(0)$	$y(\pi)$	$y'(\pi)$
Periodic Even	1	0	1	0
Antiperiodic Even	1	0	-1	0
Periodic Odd	0	1	0	1
Antiperiodic Odd	0	1	0	-1

Alternatively, the antiperiodic solutions can be found, if desired, using the boundary conditions

$$\begin{array}{lll} \text{Even Solutions:} & y(2\pi) = 1 & y'(2\pi) = 0 \\ \text{Odd Solutions:} & y(2\pi) = 0 & y'(2\pi) = 1, \end{array}$$

but these boundary conditions are common to both periodic and antiperiodic solutions.

In addition, there is yet another way to write the boundary conditions: Using evenness/oddness and periodicity/antiperiodicity, one can show that $y'(\pi/2) = 0$ for periodic even solutions and antiperiodic odd solutions, and $y(\pi/2) = 0$ for periodic odd solutions and antiperiodic even solutions. Therefore, it is only necessary to integrate up to $\pi/2$ instead of π. The full set of boundary conditions (up to an overall normalization factor) can therefore be written as follows:

Periodic Even	$y(0) = 1$	$y'(0) = 0$	$y'(\pi/2) = 0$
Antiperiodic Even	$y(0) = 1$	$y'(0) = 0$	$y(\pi/2) = 0$
Periodic Odd	$y(0) = 0$	$y'(0) = 1$	$y(\pi/2) = 0$
Antiperiodic Odd	$y(0) = 0$	$y'(0) = 1$	$y'(\pi/2) = 0$

Appendix

Let's show that any even solution of Mathieu's equation that is periodic in 2π must be either periodic or antiperiodic in π. (For an odd solution, the proof is similar.)

If $y(t)$ is an even function that has periodicity 2π, its derivative is an odd function with periodicity 2π. Therefore, $y'(0) = y'(2\pi) = 0$. In addition, from oddness and periodicity we have

$$y'(t+\pi) = y'(t-\pi) = -y'(\pi - t) = -y'(-\pi - t). \tag{2}$$

Plugging $t = 0$ into the above relation gives $y'(\pi) = 0$.

Therefore, if one integrates equation 1 from $t = 0$ to $t = \pi$ with a equal to an eigenvalue and initial conditions $y'(0) = 0$, $y(0) = A$, then one will end up with $y'(\pi) = 0$, and $y(\pi)$ will be equal to some number that I'll call B.

Now the crucial point is that if you pick any t, the factor $(a - 2q\cos 2t)$ appearing in Mathieu's equation has the *same value* for $t + \pi$ as it has for t. Because of this, if one integrates from from $t = 0$ to $t = \pi$ with initial conditions $y'(0) = 0$, $y(0) = A$, and if one integrates from $t = \pi$ to $t = 2\pi$ with initial conditions $y'(\pi) = 0$, $y(\pi) = B$, one must have

$$y(t+\pi)/B = y(t)/A \tag{3}$$

for t between zero and π. In other words, $y(t)$ and $y(t+\pi)$ are identical up to a normalization factor.

If we plug $t = \pi$ into this equation, we see that $By(\pi) = Ay(2\pi)$. Since we already know that $y(\pi) = B$ and $y(2\pi) = y(0) = A$, we end up with $A^2 = B^2$. Therefore, equation 3 reduces to

$$y(t) = \pm y(t+\pi), \tag{4}$$

which says that any solution that is periodic in 2π is either periodic or antiperiodic in π.