Physics 335 March 21, 2023
Physics 635 **Due**: March 30, 2023

Homework #9

You may work together for this homework.

Problem 1

Solve the same problem as in Homework #7 by a *spectral method.*

$$\frac{d^2y}{dt^2} + (a - 2q\cos 2t)\, y = 0$$

Since the eigenfunctions are all periodic, we will try expanding them as a Fourier series:

$$y = \frac{1}{2}\sum_{k=-\infty}^{\infty} c_k e^{ikt}.$$

Substitute this expansion for y into Mathieu's equation and replace the cosine by a sum of exponentials. The result will be a matrix equation of the form

$$\text{matrix} \times \begin{pmatrix} c_{-K} \\ \vdots \\ c_{-1} \\ c_0 \\ c_1 \\ \vdots \\ c_K \end{pmatrix} = a \begin{pmatrix} c_{-K} \\ \vdots \\ c_{-1} \\ c_0 \\ c_1 \\ \vdots \\ c_K \end{pmatrix}.$$

Here we have truncated the expansion at $k = K \gg 1$. We will find the eigenvalues of the differential equation by finding the eigenvalues of the matrix. First, we simplify the problem by noting that we only need work with c_k's with $k \geq 0$. This is because $c_k = \pm c_{-k}$ depending on whether y is even or odd. (In other words, the series is either a cosine or a sine series.) Next, we note that we only need k even or odd depending on whether we want solutions that have period π or 2π. Thus, the matrix will have a special first row corresponding to the smallest value of k appropriate for the particular symmetry we are imposing ($k = 0, 1,$ or 2). The remaining rows will have a general form corresponding to increasing k by 2 each time.

(a) Write down the form of the matrix for the 4 kinds of solutions.

(b) The 4 matrices you get will all be symmetric and tridiagonal, except for the case of even solutions of period π. Find a similarity transformation with a 2×2 diagonal matrix that symmetrizes the matrix for this case. (The transformation matrix is really $K \times K$, but it is the identity matrix except for a 2×2 submatrix.)

(c) Find the eigenvalues and investigate the accuracy and convergence of this method by varying K. Use $q = 5$ and $q = 25$ so you can also compare with Homeworks 7 and 8.

(d) Use the eigenvectors of the matrices to construct the eigenfunctions of the differential equation for $n = 0, 1, 2, 10, 15$ for $q = 5$ and $q = 25$, for all for 4 cases.

As a reminder from Homework #7

Mathieu's equation arises in several different ways, for example from separation of the wave equation in elliptical coordinates. It first appeared in celestial mechanics. A standard form is

$$\frac{d^2y}{dt^2} + (a - 2q\cos 2t)\, y = 0$$

where a and q are real constants. We are usually interested in solutions for which y is periodic, with period π or 2π. There exists a countably finite set of eigenvalues $a_n(q)$, $n = 0, 1, \ldots$, for which y is an even periodic solution, with n zeros in the interval $0 \leq t < \pi$. For n even, y has period π; while for n odd, y has period 2π. There is another set of eigenvalues $b_n(q)$, $n = 1, 2, \ldots$, for which y is an odd periodic solution, also with n zeros in the interval $0 \leq t < \pi$ and with the same periodicity properties as the even solutions. You can see roughly the behavior of the solutions by noting that for $q = 0$ the even solutions are $\cos nt$ and the odd solutions are $\sin nt$, with $a_n = b_n = n^2$.