7. Operador de crossvarianza C_{xy}

Mide la relación entre dos variables aleatorias

$$C_{x\tilde{y}} = \mathcal{E}(xy^T) \qquad f^T \hat{C}_{xy}\, g = \mathcal{E}_{xy}[(f^T x)(g^T y)]$$

operador sin centrar

$$C_{xy} = C_{x\tilde{y}} - \mu_x \mu_y^T \qquad \mu_x = \mathcal{E}(x) \quad \mu_y = \mathcal{E}(y)$$

$$C_{xy} = \frac{1}{n}\sum_{i=1}^{\hat{}} \phi(x_i) \otimes \psi(y_i) - \mu_x \otimes \mu_y$$

Haciendo $H = I_n - n^{-1} 1_n \in \mathbb{R}^{n\times n}$ entonces

$$C_{x\hat{y}} = \frac{1}{n} X H Y^T$$

donde

$$X = [\phi(x_1) \ldots \phi(x_n)]$$
$$Y = [\psi(y_1) \ldots \psi(y_n)]$$

A partir de los cuales se obtienen matrices kernel

$$K_{ij} = (X^T X)_{ij} = K(x_i, x_j) =$$

$$L_{ij} = (Y^T Y)_{ij} = L(y_i, y_j)$$

$$\tilde{K} = HKH \quad \text{y} \quad \tilde{L} = HLH$$

matrices kernel entre variables centradas

$$C_{\hat{xy}} = \frac{1}{n} X H Y^T$$

2. Constrained Covariance

és el valor singular más grande del operador de covarianza

Se debe maximizar $\langle g, \hat{C}_{xy} f \rangle_G$

Con las restricciones $\|f\|_F = 1$ y $\|g\|_G = 1$

De acuerdo al kernel reproductor

$$f = \sum_{i=1}^{n} \alpha_i [\phi(x_i) - \mu_x] = XH\alpha$$

$$g = \sum_{j=1}^{n} \beta_i [\psi(y_i) - \mu_y] = \underset{n\times 1}{Y} \underset{n\times n}{H} \underset{n\times 1}{\beta} \qquad \underset{1\times n}{Y^T} \underset{n\times 1}{H\beta}$$

y la Funcion de costos

$$\mathcal{L} = -f^T \hat{C}_{xy}\, g + \frac{\lambda}{2}(\|f\|^2 - 1) + \frac{\gamma}{2}(\|g\|^2 - 1)$$

$$\mathcal{L} = -\frac{1}{n}\alpha^T \tilde{K}\tilde{L}\beta + \frac{\lambda}{2}(\alpha^T \tilde{K}\alpha - 1) + \frac{\gamma}{2}(\beta^T \tilde{L}\beta - 1)$$

Derivando la anterior expresión respecto a α y β obtenemos un conjunto de ecuaciones 2x2

$$-\frac{1}{n}\tilde{K}\tilde{L}\beta + \lambda\tilde{K}\alpha = 0 \qquad (1)$$

$$-\frac{1}{n}\tilde{L}\tilde{K}\alpha + \gamma\tilde{L}\beta = 0 \qquad (2)$$

Multiplicando (1) por α^T y (2) por β^T

$$\frac{1}{n}\alpha^T \tilde{K}\tilde{L}\beta = \lambda \alpha^T \tilde{K}\alpha \quad (3)$$

$$\frac{1}{n}\beta^T \tilde{L}\tilde{K}\alpha = \gamma \beta^T L \beta \quad (4)$$

Restando 3 y 4

$$\lambda \alpha^T \tilde{K}\alpha = \gamma \beta^T \tilde{L}\beta$$

De donde se concluye que $\lambda = \gamma$. Haciendo este reemplazo en (1) y (2)

$$\frac{1}{n}\tilde{K}\tilde{L}\beta = \gamma \tilde{K}\alpha \quad (5)$$

$$\frac{1}{n}\tilde{L}\tilde{K}\alpha = \gamma \tilde{L}\beta \quad (6)$$

De (5) $\quad \frac{1}{n}L\beta = \gamma\alpha \qquad \beta = \frac{\gamma n}{L}\alpha$

De (6) $\quad \frac{1}{n}K\alpha = \gamma\beta \qquad \beta = \frac{K}{\gamma n}\alpha$

De forma que $\quad \frac{\gamma n}{L} = \frac{K}{\gamma n} \Rightarrow \gamma^2 n^2 = KL$

$$\boxed{\gamma = \frac{\sqrt{KL}}{n}}$$

$$\boxed{\beta = \frac{\sqrt{KL}}{L}\alpha}$$

Relación entre β y α

Multiplicando (1) por α^T y (2) por β^T

$$\frac{1}{n}\alpha^T \tilde{K}\tilde{L}\beta = \lambda\alpha^T\tilde{K}\alpha \quad (3)$$

$$\frac{1}{n}\beta^T \tilde{L}\tilde{K}\alpha = \gamma\beta^T L\beta \quad (4)$$

Restando 3 y 4

$$\lambda\alpha^T\tilde{K}\alpha = \gamma\beta^T\tilde{L}\beta$$

De donde se concluye que $\lambda = \gamma$. Haciendo este reemplazo en (1) y (2)

$$\frac{1}{n}\tilde{K}\tilde{L}\beta = \gamma\tilde{K}\alpha \quad (5)$$

$$\frac{1}{n}\tilde{L}\tilde{K}\alpha = \gamma\tilde{L}\beta \quad (6)$$

De (5) $\quad \frac{1}{n}L\beta = \gamma\alpha \qquad \beta = \frac{\gamma n}{L}\alpha$

De (6). $\quad \frac{1}{n}K\alpha = \gamma\beta \qquad \beta = \frac{K}{\gamma n}\alpha$

De forma que $\frac{\gamma n}{L} = \frac{K}{\gamma n} \Rightarrow \gamma^2 n^2 = KL$ $\quad \boxed{\gamma = \frac{\sqrt{KL}}{n}}$

$$\boxed{\beta = \frac{\sqrt{KL}}{L}\alpha}$$

Relación entre β y α