MONISH DEV SUDHAKHAR.
1220859588.

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Homework no: 2.

MAE: 598 Design Optimization.

Problem: 1

$$f = 2x_1^2 - 4x_1x_2 + 1.5x_2^2 + x_2$$

$$\text{Gradient} = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 4x_1 - 4x_2 \\ -4x_1 + 3x_2 + 1 \end{bmatrix}$$

P

$$H = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -4 \\ -4 & 3 \end{bmatrix}$$

At stationary point, gradient = 0

$$4x_1 - 4x_2 = 0$$

$$-4x_1 + 3x_2 + 1 = 0$$

$$x_1 = x_2$$

$$\Rightarrow -4x_1 + 3x_1 + 1 = 0$$

$$\Rightarrow -x_1 + 1 = 0$$

$$\boxed{x_1 = x_2 = 1}$$

The purpose of eigen vector is to prove that the point location.

$$HI - \lambda I = 0$$

$$\begin{bmatrix} 4 & -4 \\ -4 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 4-\lambda & -4 \\ -4 & 3-\lambda \end{bmatrix}$$

$$(4-\lambda)(3-\lambda) - 16 = 0$$

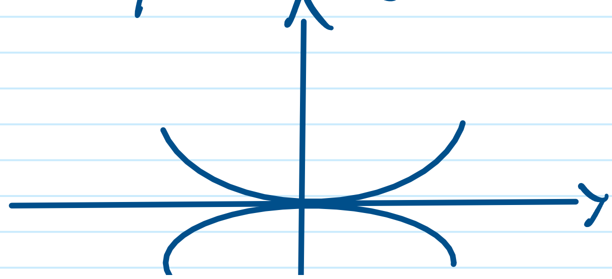
$$12 - 4\lambda - 3\lambda + \lambda^2 - 16 = 0$$

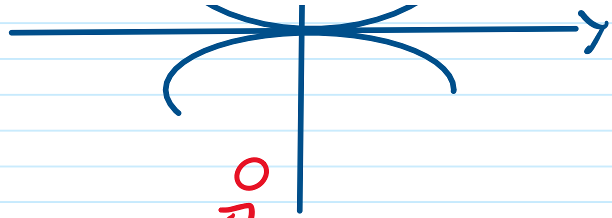
$$12 - 7\lambda + \lambda^2 - 16 = 0$$

$$\lambda^2 - 7\lambda - 4 = 0$$

$$\Rightarrow \lambda_1 = 7.53, \quad \lambda_2 = -0.53$$

The saddle point is $(7.53, -0.53)$.





(ii) $f(x) = f(x_0) + \cancel{g}^{T}_{x_0}(x - x_0) + \frac{1}{2}(x - x_0)^T H_x (x - x_0)$

$$\frac{1}{2}(x - x_0)^T H_{x_0}(x - x_0) = f(x) - f(0) < 0$$

$$\Rightarrow (ax_1 + bx_2)\cdot(cx_1 + dx_2) < 0$$

$$\Rightarrow ax_1 + bx_2 < 0 \;\&\; cx_1 + dx_2 > 0$$

(or)

$$ax_1 + bx_2 > 0 \;\&\; cx_1 + dx_2 < 0$$

Problem: 2

$f(x_1, x_2, x_3) \Rightarrow x_1 + 2x_2 + 3x_3 = 1.$

The nearest point to $(1, 0, -1)^T$ can be found by, $\sqrt{(x_1+1)^2 + x_2^2 + (x_3-1)^2}$

which is equivalent to $(x_1+1)^2 + x_2^2 + (x_3-1)^2$

Also $x_1 = 1 - 2x_2 - 3x_3$

$$\Rightarrow (1 - 2x_2 - 3x_3)^2 + x_2^2 + x_3^2 + 1 - 2x_3 + 2x_1$$

[illegible]

$= 1 + 4x_2^2 + 9x_3^2 - 4x_2 - 3x_3 + 6x_2x_3$
$+ x_2^2 + x_3^2 + 1 - 2x_3$
$= 5x_2^2 + 10x_3^2 - 4x_2 - 5x_3 + 6x_2x_3 + 2$.

$$\frac{\partial f}{\partial x_2} = 10x_2 - 4 + 6x_3$$

$$\frac{\partial f}{\partial x_3} = 20x_3 - 5 + 6x_2$$

$$\frac{\partial^2 f}{\partial x_2^2} = 10 \qquad \frac{\partial^2 f}{\partial x_2 \partial x_3} = 6$$

$$\frac{\partial^2 f}{\partial x_3^2} = 20 \qquad \frac{\partial^2 f}{\partial x_3 \partial x_2} = 6$$

$$\text{gradient} = \begin{bmatrix} \partial f/\partial x_2 \\ \partial f/\partial x_3 \end{bmatrix}$$

$$= \begin{bmatrix} 10x_2 - 4 + 6x_3 \\ 20x_3 - 5 + 6x_2 \end{bmatrix}$$

$$\text{Hessian} = \begin{bmatrix} \partial^2 f/\partial x_2^2 & \partial^2 f\, \partial x_2 \partial x_3 \\ \frac{\partial^2 f}{} & \frac{\partial^2 f}{} \end{bmatrix}$$

$$\begin{bmatrix} \frac{\partial^2 f}{\partial x_3 \partial x_2} & \frac{\partial^2 f}{\partial x_3^2} \end{bmatrix}$$

$$= \begin{bmatrix} 10 & 12 \\ 12 & 10 \end{bmatrix}$$

From which we understand that it has positive semi definite.

Problem: 3.

If the function is semidefinite everywhere and have a hessian, the second order derivative is zero.

$$\Rightarrow \quad a f(x) + b(g(x))$$

(a) The Condition to be a convex function,

So)

So $x_1 \propto x_2$,

$$f(\alpha_1 x_1 + \alpha_2 x_2) \le \alpha_1 f(x_1) + \alpha_2 f(x_2)$$

$\hookrightarrow ①$

Similarly,

$$g(\alpha_1 x_2 + \alpha_2 x_3) \le \alpha_1 g(x_1) + \alpha_2 g(x_2)$$

$g(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 g(\cdot) + \alpha_2 g(\cdot)$

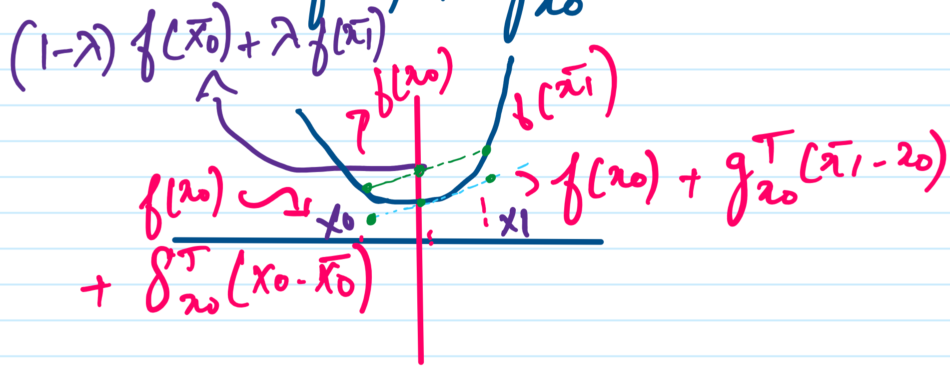
$\hookrightarrow$ ②

① + ② $\Rightarrow$

$$\alpha_1 f(x_1) + \alpha_2 f(x_2) + \alpha_1 g(x_1) + \alpha_2 g(x_2)$$
$$\leq \alpha\left[a f(x_1) + b\, g(x_1)\right] + \beta\left[a f(x_2) + b\, g(x_2)\right]$$
$$\Rightarrow a f(x) + b\, g(x) \text{ is convex.}$$

Problem : 4 .

Convex $\Leftrightarrow$ $f(x_0) + g_{x_0}^T (x - x_0)$



Method : 1

We know that in order to be a convex fn.

$$f(\alpha_1 x_1 + \alpha_2 x_2) \leq \alpha_1 f(x_1) + \alpha_2 f(x_2)$$

We know that the the the first point is the average of the 2 other points.

$$\Rightarrow f(x) \Rightarrow f(x_0) + f(x_1)$$

$\Rightarrow f(x) \geq f(x_0) + g^T(x-x_0)$

$\Rightarrow$ The function is convex

Method : 2

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}.$$

$$\Rightarrow g_{x_0}^T (x - x_0) \quad \frac{f(x) - f(x_0)}{x - x_0}.$$

$$\Rightarrow f(x) \geq f(x_0) + g_{x_0}^T (x - x_0), \forall x \in R$$

$\therefore$ The $f(x)$ is convex.

Problem: 5

$$\min_{P} \sum_{k} (a_k^T P - I)^2$$

$(a_k^T P - I)^2$

gradient $\Rightarrow 2(a_k^T P - I) \cdot a_k.$

$$\Rightarrow 2 \cdot (a_k^T \cdot P \cdot a_k - 2 I \cdot a_k)$$

$$= 2 \cdot a_k^T \cdot a_k \cdot P - 2 \cdot I \cdot a_k.$$

Hessian $\Rightarrow 2 a_k^T a_k$

Hessian for the entire objective function $\Big\} = \sum_k 2 \cdot a_k \cdot a_k^T$

Lemma: if $d^T H d \geq 0 \;\forall\; d \neq 0$
then H is positive semidefinite

$$d^T H d = \sum_k 2 \cdot \underline{d^T a_k}\; \underline{a_k^T d}.$$

Define $d^T a_k = u_k$

$$\Rightarrow \sum_k 2 u_k^2.$$

(b) As the Hessian matrix is PSD,
It is strictly convex.

It is when we have more mirrors as the lens.

$a_k <$ dimensionality.

d vector is perpendicular to A vectors

d vector is perpendicular to A vector

$H = PSD$

Convex, Infinity solution.

(c) Different Configuration can be used to give unique solution.

(d)

$$g = 2\left[a_k^T d \cdot (c - i)\, a\right].$$

$$H = 2\, a_k\, a_k^T \geq 0$$

$$\Sigma P_i \leq P^*$$

$$P_1 + P_2 < P^*$$

For the output power of the n lamps output of any of the n lamps to be less than p! the nature of the solution wont be changed and hence it is unique.

Same as if we require more than

Same as if we require more than half of the lamps to be switched ON, the nature of the output does not change, hence it is unique.