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A Fast High Order Iterative Solver for the Electromagnetic Scattering by Open Cavities Filled with the Inhomogeneous Media

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Abstract. The scattering of the open cavity filled with the inhomogeneous media is studied. The problem is discretized with a fourth order finite difference scheme and the immersed interface method, resulting in a linear system of equations with the high order accurate solutions in the whole computational domain. To solve the system of equations, we design an efficient iterative solver, which is based on the fast Fourier transformation, and provides an ideal preconditioner for Krylov subspace method. Numerical experiments demonstrate the capability of the proposed fast high order iterative solver.

AMS subject classifications: 65N06, 78M20

Key words: Helmholtz equation, compact finite difference scheme, discontinuous wave numbers, immerse interface method, fast iterative solver.

1 Introduction

The scattering properties of open cavities are of high interest to the engineering community, with a number of applications including the design of jet engine inlet ducts and cavity-backed antenna for military and civil use. In this paper we mainly develop a fast high order iterative solver concerning with the electromagnetic scattering from a two-dimensional open cavity filled with inhomogeneous media for large wave number, which is shown in Fig. 1. The ground plane and the walls of the open cavity are assumed as perfect electric conductors (PEC), and the interior of the open cavity is filled with non-magnetic materials which may be inhomogeneous. The half space above the ground plane is filled with a homogenous and isotropic medium with its permittivity ε_0 and permeability μ_0. In this setting, the electromagnetic scattering by the cavity is governed by the Helmholtz equation along with Sommerfeld's radiation conditions imposed at infinity.

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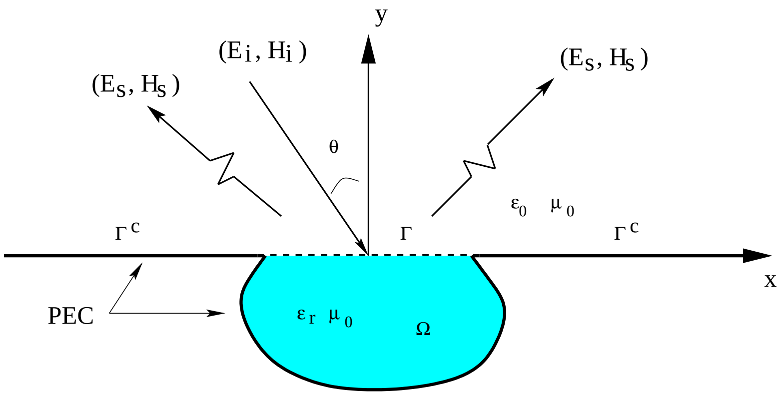

Figure 1: The geometry of the cavity.

Since the solutions to the Helmholtz equation are waves, it is evident that mesh size h must follow the wave number k in order to achieve a given accuracy. For a large wave number, the phase error (pollution) of the computed solution obtained with low order discretization is large unless fine meshes are used per wavelength. See [1] for detailed information. A fine mesh would lead to a large system of equations which may be computationally prohibitive. The memory requirement might be a bottle-neck. Many numerical approaches have been proposed to reduce the phase error. For example, the high order finite element method was proposed in [2]; the h-version and h-p-version finite element methods were proposed in [3, 4]. In [5], a standard bilinear finite element together with a modified quadrature rule was used, which led to fourth order phase accuracy on orthogonal uniform meshes. The high order spectral method and compact high order finite difference method have been presented to solve the Helmholtz equation in [6–12]. In [13], a fully high-order finite element with curvilinear tetrahedral elements was developed to simulate the scattering by cavities. High order methods are attractive for solving the Helmholtz problem with large wave number since they can offer relatively higher accurate solution by utilizing fewer mesh points and spending less computational costs than the low order approaches.

In this paper we first construct a high order finite difference discretization for the scattering of electromagnetic plane waves by a two-dimensional (2-D) rectangular cavity filled with inhomogeneous media. In the cavity domain, the compact fourth order finite difference scheme is used for the discretization of the equation, and at the aperture, a fourth order approximation is also designed by a special technique. In the discontinuous interface of various medium, in terms of the immersed interface method (IIM), see [14, 15], high order accuracy can be obtained. In [16, 17], the related fourth order method for the scattering by open cavity filled with the homogeneous medium for large wave number was considered, and the method with eight points per wavelength sufficiently well achieves the resolution we need and thus leads to a linear system with much smaller numbers of unknowns.

The resulting system of linear equations for large wave number is too large to be solved using a direct method. Many preconditioners have been developed for solving Helmholtz problem. One class is called as the operator-based precondtioning, such as the precondtioner based on separation of variables in [18] and Laplace preconditoner in [19]. They is built on an operator for the spectrum of preconditoned system is favorably clustered. Later work in [20], a complex perturbation to the Laplace operator is introduced to improve the convergence rate, which is so-called shifted-Laplace preconditioner. Incomplete LU factorization, such as [21], [24], can be seen as matrix-based preconditioner. In [25] by perturbing the diagonal of entries of an indefinite system with a small complex perturbation one can significantly improve the quality of the incomplete LU factorization of the coefficient matrix. We refer to [22, 23] for detailed discussions. These preconditioners are efficient to solve the Helmholtz equations with local boundary conditions, but not suitable to be directly used for the solution of the cavity scattering problem, which has a nonlocal boundary condition. In this work, we propose an iterative solver with a preconditioner which coincides with the system matrix constructed by high order scheme without special treatment to the rows corresponding to unknowns near the interfaces. Then we reduce iterations on a small sparse subspace as has been shown in [26], which leads to much reduced storage and computational requirements in GMRES iterations.

The remainder of the paper is organized as follows. In the next section, the scattering model from open cavity is stated and further is reduced to a bounded domain problem. In Section 3, the high order immersed interface method (IIM) for cavity model filled with the inhomogeneous media is constructed in detail. Then the fast iterative solver based on high order discretization is presented in Section 4. Numerical experiments are given to illustrate the competitive behavior of the method in section 5, including the accuracy observation for the presented method and the efficiency study of the fast iterative solver. The paper ends with some conclusions in the last section.

2 The Scattering Problem of Open Cavity Model

We consider the plane wave scattering problem by an open cavity embedded in an infinite ground plane as in Fig. 1. Assume that the cavity and the medium is z-invariant. The ground plane and the cavity wall are perfect electric conductors. The cavity may be filled with inhomogeneous media, which has the relative electric permittivity $\varepsilon_r(x,y)$. And the medium is assumed to be nonmagnetic, having the relative permeability $\mu_r = 1$. Let us denote $\Omega \in R^2$ as the cavity embedded in the ground plane with boundary $\partial\Omega$, which consists of the cavity aperture Γ and the cavity wall $\partial\Omega \setminus \Gamma$. Let R^2_+ be the region above the ground plane $\left\{(x,y) \in R^2 : y > 0\right\}$. $\partial R^2_+ \setminus \Gamma$ is the ground plane without the aperture.

In the transverse magnetic (TM) case, the magnetic field is transverse to the invariant direction. The incident electric field and the electric field are parallel to the longitudinal z-axis, i.e., $E_I = (0,0,u^i)$ and $E_{tol} = (0,0,u)$. By the electric continuity conditions, u vanishes on the cavity walls and on the ground plane except over the aperture Γ. The time-harmonic Maxwell equation is reduced to

$$\begin{aligned} &\Delta u + k^2 u = f(x,y) \qquad (x,y) \in \Omega \cup R_2^+, \\ &u = 0 \qquad\qquad\qquad \text{on} \quad \partial\Gamma^c \cup (\partial\Omega \setminus \Gamma), \end{aligned} \tag{2.1}$$

together with the radiation boundary condition

$$\lim_{r\to\infty} \sqrt{r}\,\Big(\frac{\partial u^s}{\partial r} - \mathrm{i}k_0 u^s\Big) = 0. \tag{2.2}$$

where u^s is the tangential component of the scattered field $E_S = (0,0,u^s)$, and $k^2 = \omega^2\varepsilon\mu = k_0^2\varepsilon_r(x,y)\mu_r$. The fields are said to be source free if the source term $f = 0$.

Assume that a plane wave $u^i = e^{\mathrm{i}(\alpha x - \beta y)}$ is incident on the cavity from above, where $\alpha = k_0 \sin\theta$, $\beta = k_0 \cos\theta$, and $-\pi/2 < \theta < \pi/2$ is the angle of incident with respect to the positive y-axis. The scattered field u^s can be expressed by $u^s = u - u^i + e^{\mathrm{i}(\alpha x + \beta y)}$. Clearly, u^s satisfies

$$\begin{aligned} &\Delta u^s + k^2 u^s = f(x,y) \quad (x,y) \in R_2^+, \\ &u^s = u(x,0) \qquad\qquad\quad \text{on} \quad \Gamma, \\ &u^s = 0 \qquad\qquad\qquad\quad \text{on} \quad \Gamma^c. \end{aligned} \tag{2.3}$$

By using the Green's theorem, we have

$$u^s(x,y) = \int_\Gamma \frac{\partial G_d(\mathbf{x},\mathbf{x}')}{\partial n'} u^s(x',0)dx', \quad x \in \Gamma. \tag{2.4}$$

In (2.4), $G_d(\mathbf{x},\mathbf{x}')$ is the upper half-plane Dirichlet Green's function for the Helmholtz equation

$$G_d(\mathbf{x},\mathbf{x}') = \frac{\mathrm{i}}{4}\left[H_0^1(k_0|\mathbf{x}-\mathbf{x}'|) - H_0^1(k_0|\mathbf{x}-\bar{\mathbf{x}}'|)\right],$$

where $\mathbf{x}$ and $\mathbf{x}'$ denote source point and field point separately, and $\bar{\mathbf{x}}'$ is the image of $\mathbf{x}'$ with respect to the real axis. By the boundary conditions and the field continuity, the total field u satisfies the condition on Γ

$$\frac{\partial u}{\partial y}|_{y=0^+} = I(u) + g(x) \quad x \in \Gamma,$$

where $I(u) = \frac{ik_0}{2}\int_\Gamma \frac{1}{|x-x'|} H_1^{(1)}(k_0|x-x'|)u(x',0)dx'$ is called the nonlocal boundary condition or the transparent boundary condition, and $g(x) = -2\mathrm{i}\beta e^{\mathrm{i}\alpha x}$, $x \in \Gamma$. Consequently, the scattering problem can be reduced to a bounded problem:

$$\begin{aligned} &\Delta u + k^2 u = f(x,y) \qquad (x,y) \in \Omega, \\ &u = 0 \qquad\qquad\qquad\quad \text{on} \quad \partial\Omega \setminus \Gamma, \\ &\frac{\partial u}{\partial n} = I(u) + g(x) \qquad\quad \text{on} \quad \Gamma. \end{aligned} \tag{2.5}$$

In the TE case, the formulation process can be similarly deduced (see [32]), and the total field satisfies

$$
\begin{aligned}
&\nabla \cdot \left(\frac{1}{\varepsilon_r}\nabla u\right) + k_0^2\mu_r u = f(x,y) \qquad (x,y)\in\Omega,\\
&\frac{\partial u}{\partial n} = 0 \qquad\qquad\qquad\qquad\qquad\quad \text{on} \quad \partial\Omega\setminus\Gamma,\\
&u = \tilde{I}(u) + \tilde{g}(x) \qquad\qquad\qquad\qquad \text{on} \quad \Gamma,
\end{aligned}
\tag{2.6}
$$

where $\tilde{I}(u) = -\frac{1}{2}\int_\Gamma \frac{1}{\varepsilon_r(x')} H_0^1(k_0|x'-x|)\frac{\partial u(x',y')}{\partial y'}|_{y'=0^-}dx'$, $\tilde{g}(x) = 2e^{\mathrm{i}\alpha x}$.

3 High Order Immersed Interface Method for the Scattering of Cavity Filled with the Layered Media

3.1 High order immersed interface method

When the cavity is filled with the inhomogeneous media, the usual high order method will fail because of the discontinuity across the interface of various media. In this section, we take the cavity filled with the layered media as example, and use the immersed interface method to develop a compact fourth order scheme in the whole domain.

Let $\{x_i, y_j\}_{i,j=0}^{M+1,N+1}$ define a uniform partition of $\Omega = [0,a]\times[-b,0]$. For ease of notations, we only consider $\Delta x = \Delta y = h$, and the main ideas in this work can be extended to rectangular cavities with $\Delta x \neq \Delta y$. Using the notation

$$
\delta_x^2 u_{i,j} = \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2}, \qquad \delta_y^2 u_{i,j} = \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{h^2}, \tag{3.1}
$$

the discrete fourth order finite difference system in the TM case can be given by

$$
\begin{aligned}
(1+\frac{k^2(x,y)h^2}{12})(\delta_x^2+\delta_y^2)u_{i,j} + \frac{h^2}{6}\delta_x^2\delta_y^2 u_{i,j} + k^2(x,y)u_{i,j} = f_{i,j} + \frac{h^2}{12}(\delta_x^2+\delta_y^2)f_{i,j},\\
i=1,2,...,M,\ j=1,2,...,N+1,
\end{aligned}
\tag{3.2}
$$

where $k(x,y) = k_0^2\varepsilon_r(x,y)\mu_0$.

If k is a constant in the computational domain, in another form, (3.2) can be written as

$$
\begin{aligned}
&a_0u_{i-1,j-1} + a_1u_{i-1,j} + a_2u_{i-1,j+1} + a_3u_{i,j-1} + a_4u_{i,j}\\
&+a_5u_{i,j+1} + a_6u_{i+1,j-1} + a_7u_{i+1,j} + a_8u_{i+1,j+1} = F_{ij},\\
&\qquad\qquad i=1,2,...,M, j=1,2,...,N+1,
\end{aligned}
\tag{3.3}
$$

where $a_0 = a_2 = a_6 = a_8 = \frac{1}{6}$, $a_1 = a_3 = a_5 = a_7 = \frac{h^2k^2}{12} + \frac{2}{3}$, $a_4 = \frac{8h^2k^2}{12} - \frac{10}{3}$ and the right hand side $F_{ij} = \frac{h^2}{12}(f_{i+1,j} + f_{i-1,j} + f_{i,j+1} + f_{i,j-1} + 8f_{ij})$.

If the cavity is filled with the layered media, (3.3) can not be used directly. Below exploiting the immersed interface method, we modify (3.3) at the irregular grid points in the discontinuous interface to get the global high order scheme. Any point on the interface is set as in the domain of Ω^-, that is, $\Gamma^I \in \Omega^-$. Since the discontinuity is across the interface that is parallel to the $x-$axis, we set the interface as one of grid lines. Across the interface, we assume that the following natural jump conditions

$$[u] = u^+ - u^- = 0, \qquad [u_x] = u_x^+ - u_x^- = 0, \qquad [u_y] = u_y^+ - u_y^- = 0, \tag{3.4}$$

where u^+ reflects solutions in Ω^+ and u^- are those in Ω^-.

Let (x_i, y_j) be an irregular grid point on the interface Γ^I. As noted before, it belongs to the Ω^- domain. Thus the grid points $(x_l, y_{j+1}), l = i-1, i, i+1$ are three grid points from the Ω^+ side. Based on the continuity from (3.4) between the Ω^- and Ω^+ domain, the finite difference scheme can be written as

$$\begin{aligned}
&\frac{1}{6}(u_{i+1,j+1} + u_{i-1,j+1} + u_{i-1,j-1} + u_{i+1,j-1}) \\
&+\left(\frac{h^2k^2}{12} + \frac{2}{3}\right)(u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1}) + \left(\frac{8h^2k^2}{12} - \frac{10}{3}\right) u_{ij} \\
&= \frac{h^2}{12}(f^-_{i+1,j} + f^-_{i-1,j} + f^-_{i,j+1} + f^-_{i,j-1} + 8f^-_{i,j}) \\
&+ C_1 \left(u_{i-1,j-1}, ..., u_{i+1,j+1}\right) + C_2 \left([k^2], [f]_{i,j-1}, ..., [f_x]_{i,j-1}\right),
\end{aligned} \tag{3.5}$$

where C_1 and C_2 are the correction terms, and they are zero at regular points.

Lemma 3.1. *Assume $f(x,y)$ is piecewise $C^2(\Omega^\pm)$ which leads to the fact that $u(x,y)$ is piecewise $C^4(\Omega^\pm)$. In addition to the condition (3.4), We have the following jump conditions*

$$\begin{aligned}
&[u_{xx}] = 0, \qquad [u_{xy}] = 0, \qquad [u_{yy}] = -[k^2]u^- + [f], \\
&[u_{yyy}] = -[k^2]u_y^- + [f_y], \qquad [u_{yyx}] = -[k^2]u_x^- + [f_x] \\
&[u_{xxy}] = 0, \qquad [u_{xxx}] = 0,
\end{aligned} \tag{3.6}$$

$$\begin{aligned}
&[u_{xxxy}] = 0, \qquad [u_{xxxx}] = 0, \qquad [u_{yyxx}] = -[k^2]u_{xx}^- + [f_{xx}], \\
&[u_{yyyy}] = 2[k^2]u_{xx}^- + [k^4]u^- - [k^2 f] + [f_{yy}] - [f_{xx}],
\end{aligned} \tag{3.7}$$

Proof: The proof for (3.6) is straightforward. We show the proof for (3.7). Differentiating the original differential equation twice with respect to x and then taking the jump, we get

$$[u_{xxxx}] + [u_{yyxx}] + [k^2 u_{xx}] = [f_{xx}].$$

Since u is continuous with respect to x, we get

$$[u_{yyxx}] = -[k^2 u_{xx}] + [f_{xx}].$$

Note that u_{xx} is continuous, so we can write $u_{xx} = u^-_{xx}$. Differentiating the original equation twice with respect to y and then taking the jump, we get

$$[u_{xxyy}] + [u_{yyyy}] + [k^2 u_{yy}] = [f_{yy}].$$

Multiplying k^2 to the original differential equation and then taking the jump, we have

$$[k^2 u_{yy}] = -[k^2]\, u^-_{xx} - [k^4]\, u^- + [k^2 f].$$

Thus using $[u_{xxyy}]$ and $[k^2 u_{yy}]$, we get

$$\begin{aligned}
[u_{yyyy}] &= -[u_{xxyy}] - [k^2 u_{yy}] + [f_{yy}] \\
&= [k^2]\, u^-_{xx} - [f_{xx}] + [k^2]\, u^-_{xx} + [k^4]\, u^- - [k^2 f] + [f_{yy}] \\
&= 2[k^2]\, u^-_{xx} + [k^4]\, u^- - [k^2 f] + [f_{yy}] - [f]_{xx}.
\end{aligned}$$

This completes the proof. □

The idea of deriving the correction term C_1 and C_2 comes from the immersed interface method. See in [15,27]. Use almost the same procedure, we replace the values of $u^+(x_{i+1}, y_{j+1})$, $u^+(x_i, y_{j+1})$ and $u^+(x_{i-1}, y_{j+1})$ in terms of those from $'-'$ side, then we can get their contribution to C_1 and C_2.

$$\begin{aligned}
u^+(x_{i+1}, y_{j+1}) = {}& u^-(x_{i+1}, y_{j+1}) + \left(-\frac{h^2}{6}[k^2] + \frac{h^4}{24}[k^4] \right) u^-(x_i, y_j) \\
& - \frac{h^2}{2}[k^2]u^-(x_{i+1}, y_j) + \frac{h^2}{6}[k^2]u^-(x_{i-1}, y_j) - \frac{h^2}{12}[k^2]u^-(x_i, y_{j+1}) \\
& + \frac{h^2}{12}[k^2]u^-(x_i, y_{j-1}) + \frac{h^2}{12}[k^2]u^-(x_{i+1}, y_{j-1}) - \frac{h^2}{12}[k^2]u^-(x_{i-1}, y_{j-1}) \\
& + \left(\frac{h^2}{2}\, [f] + \frac{h^3}{2}\, [f_x] + \frac{h^3}{6}\, [f_y] - \frac{h^4}{24}\, [k^2 f] + \frac{5h^4}{24}[f_{xx}] + \frac{h^4}{24}[f_{yy}] + \frac{h^4}{6}[f_{xy}] \right) \Big|_{(x_i, y_j)} \\
& + \mathcal{O}(h^5).
\end{aligned} \tag{3.8}$$

$$\begin{aligned}
u^+(x_i, y_{j+1}) = {}& \left(1 - \frac{h^2}{12}[k^2] \right) u^-(x_i, y_{j+1}) + \left(-\frac{2h^2}{3}[k^2] + \frac{h^4}{24}[k^4] \right) u^-(x_i, y_j) \\
& + \frac{h^2}{12}[k^2]u^-(x_i, y_{j-1}) + \frac{h^2}{12}[k^2]u^-(x_{i+1}, y_j) + \frac{h^2}{12}[k^2]u^-(x_{i-1}, y_j) \\
& + \left(\frac{h^2}{2}\, [f] + \frac{h^3}{6}\, [f_x] - \frac{h^4}{24}\, [k^2 f] + \frac{h^4}{24}[f_{yy}] - \frac{h^4}{24}[f_{xx}] \right) \Big|_{(x_i, y_j)} \\
& + \mathcal{O}(h^5).
\end{aligned} \tag{3.9}$$

$$
\begin{aligned}
u^+(x_{i-1},y_{j+1}) = & \ u^-(x_{i-1},y_{j+1}) + \left(-\frac{h^2}{6}[k^2] + \frac{h^4}{24}[k^4]\right) u^-(x_i,y_j) \\
& + \frac{h^2}{6}[k^2]u^-(x_{i+1},y_j) - \frac{h^2}{2}[k^2]u^-(x_{i-1},y_j) - \frac{h^2}{12}[k^2]u^-(x_i,y_{j+1}) \\
& + \frac{h^2}{12}[k^2]u^-(x_i,y_{j-1}) - \frac{h^2}{12}[k^2]u^-(x_{i+1},y_{j-1}) + \frac{h^2}{12}[k^2]u^-(x_{i-1},y_{j-1}) \\
& + \left(\frac{h^2}{2}[f] - \frac{h^3}{2}[f_x] + \frac{h^3}{6}[f_y] - \frac{h^4}{24}[k^2 f] + \frac{5h^4}{24}[f_{xx}] + \frac{h^4}{24}[f_{yy}] - \frac{h^4}{6}[f_{xy}]\right)\Bigg|_{(x_i,y_j)} \\
& + \mathcal{O}(h^5).
\end{aligned} \tag{3.10}
$$

Let R_1, R_2, and R_3 denote the following formulation:

$$
\begin{aligned}
R_1 &= \left(\frac{h^2}{2}[f] + \frac{h^3}{2}[f_x] + \frac{h^3}{6}[f_y] - \frac{h^4}{24}[k^2 f] + \frac{5h^4}{24}[f]_{xx} + \frac{h^4}{24}[f_{yy}] + \frac{h^4}{6}[f_{xy}]\right)\Bigg|_{(x_i,y_j)}, \\
R_2 &= \left(\frac{h^2}{2}[f] + \frac{h^3}{6}[f_y] - \frac{h^4}{24}[k^2 f] - \frac{h^4}{24}[f]_{xx} + \frac{h^4}{24}[f_{yy}]\right)\Bigg|_{(x_i,y_j)}, \\
R_3 &= \left(\frac{h^2}{2}[f] - \frac{h^3}{2}[f_x] + \frac{h^3}{6}[f_y] - \frac{h^4}{24}[k^2 f] + \frac{5h^4}{24}[f]_{xx} + \frac{h^4}{24}[f_{yy}] - \frac{h^4}{6}[f_{xy}]\right)\Bigg|_{(x_i,y_j)}.
\end{aligned}
$$

Using the above derivation into (3.5), we can obtain the high order immersed method for scattering of cavity filled with the layered media.

Lemma 3.2. *The compact fourth order scheme for the scattering of the cavity filled with layered media is given as follows*

$$
\begin{aligned}
& \tilde{a}_0 u_{i-1,j-1} + \tilde{a}_1 u_{i-1,j} + \tilde{a}_2 u_{i-1,j+1} + \tilde{a}_3 u_{i,j-1} + \tilde{a}_4 u_{i,j} \\
& + \tilde{a}_5 u_{i,j+1} + \tilde{a}_6 u_{i+1,j-1} + \tilde{a}_7 u_{i+1,j} + \tilde{a}_8 u_{i+1,j+1} = \tilde{F}_{ij}, \\
& \qquad i = 1,2,...,M; j = 1,2,...,N+1
\end{aligned} \tag{3.11}
$$

where

$$
\begin{aligned}
\tilde{a}_0 &= \tilde{a}_2 = \tilde{a}_6 = \tilde{a}_8 = \frac{1}{6}, \\
\tilde{a}_1 &= \tilde{a}_7 = \beta\left(1 - \frac{\alpha}{1-\alpha}\right) + \frac{2\alpha}{3}\left(1 - \frac{\alpha}{2(1-\alpha)}\right), \\
\tilde{a}_3 &= \beta\left(1 - \frac{\alpha}{1-\alpha}\right) - \frac{\alpha}{3}\left(1 + \frac{\alpha}{1-\alpha}\right), \\
\tilde{a}_4 &= -\frac{26}{3} + 8\beta + \frac{2\alpha}{3} - \frac{\gamma}{3} - \frac{1}{1-\alpha}\left(\beta + \frac{\alpha}{3}\right)(-8\alpha + \gamma), \\
\tilde{a}_5 &= \frac{1}{1-\alpha}\left(\beta + \frac{\alpha}{3}\right),
\end{aligned}
$$

and

$$\tilde{F}_{ij} = \frac{1}{6}R_1 + \frac{1}{1-\alpha}\left(\beta + \frac{\alpha}{3}\right)R_2 + \frac{1}{6}R_3 + \frac{h^2}{12}\left(f^-_{i+1,j} + f^-_{i-1,j} + f^-_{i,j+1} + f^-_{i,j-1} + 8f^-_{i,j}\right)$$

and

$$\alpha = \frac{h^2}{12}[k^2], \quad \beta = \frac{2}{3} + \frac{h^2}{12}k^2_-, \quad \gamma = \frac{h^4}{24}[k^4].$$

3.2 The treatment of boundary conditions

Next we introduce the fourth order approximation of the nonlocal boundary condition in (2.5). Using the Taylor expansion at (x_i, y_{N+1}), we can obtain

$$\frac{u_{i,N+2} - u_{iN}}{2h} = (u_y)_{i,N+1} + O(h^2) = \sum_{l=1}^{M} G_{il}u_{l,N+1} + g(x_i) + O(h^2), i = 1,2,...,M,$$

where $\sum_{l=1}^{M} G_{il}u_{l,N+1}$ is the approximation of the hypersingular integral in (2.5), and it can be calculated numerically (see in [28]).

Assuming f is sufficiently smooth in $\overline{\Omega}$, and $k(x,y)$ is a constant $k(y_{N+1})$ in the aperture Γ. Add the (3.2) on the boundary Γ, we can express

$$\begin{aligned}&\frac{u_{i,N+2} - u_{iN}}{2h}\\ =&(u_y)_{i,N+1} + \frac{h^2}{6}(f_y - k^2(y_{N+1})u_y - u_{xxy})_{i,N+1} + O(h^4)\\ =&Gu_{i,N+1} + g_i + \frac{h^2}{6}(f_y)_{i,N+1} - \frac{k^2(y_{N+1})h^2}{6}\frac{u_{i,N+2} - u_{i,N}}{2h} - \frac{h^2}{6}\frac{\delta_x^2 u_{i,N+2} - \delta_x^2 u_{i,N}}{2h} + O(h^4),\end{aligned}$$

which leads to a fourth order approximation expression for the boundary condition $\frac{\partial u}{\partial n} = I(u) + g(x)$,

$$\left(1 + \frac{k^2(y_{N+1})h^2}{6}\right)\frac{u_{i,N+2} - u_{iN}}{2h} + \frac{h^2}{6}\frac{\delta_x^2 u_{i,N+2} - \delta_x^2 u_{i,N}}{2h} - 2hGu_{i,N+1} = 2hg_i + \frac{h^2}{6}(f_y)_{i,N+1}, \tag{3.12}$$

Combining (3.12) and (3.11), the ghost points $u_{i,N+2}$ will be eliminated, then the global system is formed.

4 Fast Iterative Solver For the Scattering of Cavity Filled with the Inhomogeneous Media

4.1 Linear System

Without employing the immersed interface method, (3.2) can also be written

$$
\begin{aligned}
&\left((I_{MN}+\frac{h^2}{12}(I_M\otimes D))(A_M\otimes I_N+I_M\otimes A_N)+\frac{h^2}{6}(A_M\otimes A_N)+I_M\otimes D\right)U_1\\
&+\left((I_{MN}+\frac{h^2}{12}(I_M\otimes D))(I_M\otimes a_N)+\frac{h^2}{6}(A_M\otimes a_N)\right)u_{:,N+1}\\
=&\frac{h^2}{12}\left((A_M\otimes I_N+I_M\otimes A_N)\right)F_1+\frac{1}{12}\begin{pmatrix} I_N\\ \mathbf{0}\end{pmatrix}f_{0,:}+\frac{1}{12}\begin{pmatrix} \mathbf{0}\\ I_N\end{pmatrix}f_{M+1,:}\\
&+\frac{1}{12}(I_M\otimes b_N)f_{:,0}+\frac{1}{12}(I_M\otimes a_N)f_{:,N+1}+I_{MN}F_1,
\end{aligned}
$$

where $\otimes$ denotes the tensor product (Kronecker product), I_{MN} is the $MN\times MN$ identity matrix, and I_M is the $M\times M$ identity matrix,

$$
A_M=\frac{1}{h^2}\begin{pmatrix} -2 & 1 & & \\ 1 & -2 & 1 & \\ & \ddots & \ddots & \ddots \\ & & 1 & -2\end{pmatrix},\qquad A_N=\frac{1}{h^2}\begin{pmatrix} -2 & 1 & & \\ 1 & -2 & 1 & \\ & \ddots & \ddots & \ddots \\ & & 1 & -2\end{pmatrix}
$$

$$
D=\omega^2\mu_0\,\mathrm{diag}\left(\varepsilon(y_1),\varepsilon(y_2),...,\varepsilon(y_N)\right),
$$

$$
a_N=\frac{1}{h^2}\left(0,0,...,0,1\right)^T,\quad b_N=\frac{1}{h^2}\left(1,0,...,0,0\right)^T
$$

and

$$
U_1=(u_{11},\cdots,u_{1N},u_{21},\cdots,u_{2N},\cdots,u_{M1},\cdots,u_{MN})^T,
$$

$$
u_{:,N+1}=(u_{1,N+1},u_{2,N+1},\cdots,u_{M,N+1}),
$$

$$
F_1=(f_{11},\cdots,f_{1N},f_{21},\cdots,f_{2N},\cdots,f_{M1},\cdots,f_{MN})^T,
$$

A_M is an $M\times M$ matrix, and A_N is an $N\times N$ matrix. a_N and b_N are also N dimension vectors. $f_{0,:}$, $f_{M+1,:}$ and $f_{:,0}$, $f_{:,N+1}$ denote the vectors when $x=0,1$ and $y=0,1$ respectively.

In the same time, (3.12) can be expressed by matrix form

$$
\left(I_M+\frac{h^2}{6}D_0+\frac{h^2}{6}A_M\right)u_{:,N+2}-\left(I_M+\frac{h^2}{6}D_0+\frac{h^2}{6}A_M\right)u_{:,N}=2hGu_{:,N+1}+2hg+\frac{h^3}{3}(f_y)_{:,N+1},\tag{4.1}
$$

where $D_0 = \omega^2\mu_0\,\varepsilon(y_{N+1})\,I_M$, and D_0 is M dimensional diagonal matrix. To eliminate the values of u at the ghost points, we add the difference equation on (3.2) at the boundary points $(x_i, y_{N+1})(1 \le i \le M)$,

$$\begin{aligned}&\left(I_M + \frac{h^2}{12}D_0 + \frac{h^2}{6}A_M\right)u_{:,N+2} + \left(I_M + \frac{h^2}{12}D_0 + \frac{h^2}{6}A_M\right)u_{:,N}\\&+\left((I_M + \frac{h^2}{12}D_0)(h^2A_M - 2I_M) - \frac{h^2}{3}A_M + h^2D_0\right)u_{:,N+1}\\=&h^2 f_{:,N+1} + \frac{h^4}{12}\Delta f_{:,N+1}.\end{aligned}\tag{4.2}$$

By eliminating the terms $u_{:,N+2}$ from (4.2) with (4.1), we obtain a fourth order approximation of the transparent boundary Γ as follows,

$$\begin{aligned}&\left((I_M + \frac{h^2}{12}D_0)(h^2A_M - 2I_M) - \frac{h^2}{3}A_M + h^2D_0 + 2hJ_2J_1^{-1}G\right)\overline{u}_{:,N+1} + 2J_2\overline{u}_{:,N}\\=&-2hJ_2J_1^{-1}g - \frac{h^3}{3}J_2J_1^{-1}(f_y)_{:,N+1} + h^2 f_{:,N+1} + \frac{h^4}{12}\Delta f_{:,N+1},\end{aligned}\tag{4.3}$$

where $J_1 = \left(I_M + \frac{h^2}{6}D_0 + \frac{h^2}{6}A_M\right)$, $J_2 = \left(I_M + \frac{h^2}{12}D_0 + \frac{h^2}{6}A_M\right)$.

Finally, combining (4.1), (3.11) and (4.3) yields the global system for the cavity model filled with the layered media

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22}\end{pmatrix}\begin{pmatrix} U_1 \\ u_{:,N+1}\end{pmatrix} = \begin{pmatrix} F_I \\ f_b\end{pmatrix},\tag{4.4}$$

where

$$A_{12} = \left((I_{MN} + \frac{h^2}{12}(I_M\otimes D))(I_M\otimes a_N) + \frac{h^2}{6}(A_M\otimes a_N)\right),$$

$$A_{21} = 2h^2 J_2\otimes a_N^T,$$

$$A_{22} = \left((I_M + \frac{h^2}{12}D_0)(h^2A_M - 2I_M) - \frac{h^2}{3}A_M + h^2D_0 + 2hJ_2J_1^{-1}G\right),$$

$$\begin{aligned}F_I &= \frac{h^2}{12}\left(A_M\otimes I_N + I_M\otimes A_N\right)F_1 + \frac{1}{12}\begin{pmatrix} I_N \\ \mathbf{0}\end{pmatrix}f_{0,:} + \frac{1}{12}\begin{pmatrix} \mathbf{0} \\ I_N\end{pmatrix}f_{M+1,:}\\&+\frac{1}{12}(I_M\otimes b_N)f_{:,0} + \frac{1}{12}(I_M\otimes a_N)f_{:,N+1} + I_{MN}F_1 + F_T,\end{aligned}$$

$$f_b = -2hJ_2J_1^{-1}g - \frac{h^3}{3}J_2J_1^{-1}(f_y)_{:,N+1} + h^2 f_{:,N+1} + \frac{h^4}{12}\Delta f_{:,N+1},$$

and

$$\begin{aligned}A_{11} &= \left((I_{MN} + \frac{h^2}{12}(I_M\otimes D))(A_M\otimes I_N + I_M\otimes A_N) + \frac{h^2}{6}(A_M\otimes A_N) + I_M\otimes D\right) + T\\&= B_{11} + T.\end{aligned}\tag{4.5}$$

Note the matrix F_T and T depend on the interface situation, in the simple case that the cavity is filled with two kinds of vertically layered media, $TU_1 = F_T$ has the corresponding expression, that is, the difference of (3.11) and (3.3)

$$\begin{aligned}&(\tilde{a}_1 - a_1)u_{i-1,j_0} + (\tilde{a}_3 - a_3)u_{i,j_0-1} + (\tilde{a}_4 - a_4)u_{i,j_0}\\ &+(\tilde{a}_5 - a_5)u_{i,j_0+1} + (\tilde{a}_7 - a_7)u_{i+1,j_0} = \tilde{F}_{ij_0} - F_{ij_0},\\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad i = 1,2,...,M,\end{aligned} \tag{4.6}$$

where j_0 is the index indicating the discontinuous interface. Thus MN-dimensional matrix T is very sparse, which has only M nonzero rows.

4.2 Preconditioning Iteration

Now we use the following block triangular form as our right preconditioner

$$B = \begin{pmatrix} B_{11} & \mathbf{0} \\ A_{21} & A_{22} \end{pmatrix}, \tag{4.7}$$

where A_{21} and A_{22} come from the coefficient matrix of (4.4), $B_{11} = A_{11} - T$. If (4.4) is simply denoted by $Au = f$, using the right preconditoning B yields the system like

$$AB^{-1}v = f. \tag{4.8}$$

After solving this system the solution of the original problem (4.4) is $u = B^{-1}v$.

The choice of the preconditioner is motivated by the Neumann-Dirichlet domain decomposition preconditoner, see [5, 26], for example. For a Poisson equation this preconditioner leads to a well conditioned matrix AB^{-1} and a rapid convergence of a preconditioned iteration, see [29]. Based on this we can expect the conditioning to be good for the cavity model filled with inhomogeneous media, which is also tested by the numerical experiment in Section 5.

4.3 Reduction to sparse subspace

We consider the structure of the vectors needed during (4.8). Particularly, it is worth mentioning that the vectors are very sparse. This reduces memory usage by orders of magnitude. We recommend [26,30] for a more detailed description of iterations on sparse subspace.

We solve (4.8) on the Krylov subspace

$$\mathrm{span}\{f, AB^{-1}f, ..., (AB^{-1})^{k-1}f\}. \tag{4.9}$$

Find a vector v^k from the subspace by minimizes the norm of the residual $AB^{-1}v^k - f$, and v^k is a linear combination of the vectors f, $f^1 = AB^{-1}f$,...,$f^k = (AB^{-1})^{k-1}f$. Here

$$f^1 = AB^{-1}f = f + (A - B)B^{-1}f \tag{4.10}$$

and similarly

$$f^j = (AB^{-1})^j f = AB^{-1} f^{j-1} = f + (A - B)B^{-1} f^{j-1}, j = 2, ..., k-1. \tag{4.11}$$

From the above we can see $f^j \in X = \text{span}\{f\} + \text{range}(A - B)$, and Krylov subspace (4.9) is a subspace of X. Then any iterative method based on this Krylov subspace for the solution of (4.8) generates a sequence of approximate solution v^k on X if the initial guess v^0 belongs to X. Hence all required operations are carried out on the sparse subspace X.

We consider the sparsity of X. Recall that

$$A - B = \begin{pmatrix} T & A_{12} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}. \tag{4.12}$$

As mentioned that T is very sparse, which reflects the discontinuous interface of different medium. In addition, A_{12} has only M nonzero rows from (4.4). Thus, for the simple case of vertically two layered medium, the vectors in the sparse subspace X have $2M$ nonzero components, and this can greatly reduce computational costs. Moreover it is predicted that the complexity of the medium filled in open cavity, reflected by T, will influence the sparsity of X and the computational spent.

4.4 Technique in Each Iteration

When given a vector $y \in X$, at each iteration, the multiplication $AB^{-1}z$ will be performed, which includes the linear system

$$By = \begin{pmatrix} B_{11} & \mathbf{0} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = z. \tag{4.13}$$

The fast Poisson solver can be applied for solving

$$B_{11} y_1 = z_1, \tag{4.14}$$

since $S_M A_M S_M = \Lambda = \text{diag}\,(\lambda_1, \lambda_2, \cdots, \lambda_M)$, where S_M denotes the discrete Fourier-sine transformation,

$$S_M = \sqrt{\frac{2}{M+1}} \left(\sin \frac{lm\pi}{M+1} \right)_{l,m=1}^{M}, \qquad \lambda_l = -\frac{4(M+1)^2}{a^2} \sin^2 \frac{l\pi}{2(M+1)}$$

and $S_M^2 = I_M$. Using the discrete Fourier sine transformation, the discrete system (4.14) can be written as

$$\left((I_{MN} + \frac{h^2}{12}(I_M \otimes D))(\Lambda \otimes I_N + I_M \otimes A_N) + \frac{h^2}{6}(\Lambda \otimes A_N) + I_M \otimes D \right) \overline{y}_1 = \overline{z}_1, \tag{4.15}$$

where $\overline{y}_1 = (S_M \otimes I_N) y_1$, $\overline{z}_1 = (S_M \otimes I_N) z_1$. By the fast algorithm the linear system (4.14) can be solved with $O\,(NM\log M)$ operations.

Summing up the above discussion each iteration can be done by three steps:

1. Solve
$$B_{11}y_1 = z_1$$
using the fast discrete Fourier transformation.

2. Solve
$$A_{22}y_2 = z_2 - A_{21}y_1.$$
Because G is a Toeplitz matrix, the system can be computed by the fast Toeplitz solvers. (See in [31]).

3. Compute
$$(A - B)y = Ty_1 + A_{12}y_2.$$

5 Numerical experiments

We have solved many cavity models filled with layered media or inhomogeneous media and confirmed the expected order of accuracy and efficiency of the proposed high order fast iterative solver. Several typical examples in TM case will be presented in this section. All computations are performed on a PC with an Intel Core2 Duo 2.25GHz processor with 1.72GBytes of memory.

5.1 Example 1

We consider an artificial example defined by (2.5) with the cavity $a = b = 1.0$ which is filled with layered media, and the discontinuous interface is the line $y = -b/2$. In this example, the $f(x,y)$ and $g(x)$ are chosen such that the exact solution is

$$u(x,y) = \begin{cases} (y - \dfrac{1}{2})^2 \sin \pi x \ \cos \pi y, & (x,y) \in \Omega^+ = (0,a) \times (-\frac{b}{2}, 0), \\ (y - \dfrac{1}{2})^2 \sin \pi x \ \sin \pi y, & (x,y) \in \Omega^- = (0,a) \times (-b, -\frac{b}{2}]. \end{cases}$$

First, for testing the accuracy of the proposed method, we compute the cavity model filled with the following layered media

$$\varepsilon_r(x,y) = \begin{cases} 4.0 + 2.0\mathrm{i}, & -b/2 \le y \le 0, \\ 2.39 + 1.84\mathrm{i}, & -b \le y < -b/2. \end{cases} \tag{5.1}$$

Error measures in L_2 norm and L_∞ norm in the domain Ω are defined by

$$e_M(\Omega) = \max|u^h_{i,j} - u(x_i, y_j)|, \quad e_2(\Omega) = \left(\frac{ab}{M(N+1)} \Sigma_{i,j=1}^{M,N+1} |u^h_{ij} - u(x_i, y_j)|^2 \right)^{1/2}$$

respectively, where u^h_{ij} denotes the numerical solution at the point (x_i, y_j). The solution at the aperture of cavity is more interesting for the scattering calculation, so we also define the following error measures on Γ,

$$e_M(\Gamma) = \max_i |u^h_{i,N+1} - u(x_i,0)|, \quad e_2(\Gamma) = \Big(\frac{a}{M}\Sigma_{i=1}^M |u^h_{i,N+1} - u(x_i,0)|^2\Big)^{1/2}.$$

The errors in the domain and aperture with a grid refinement analysis for different wave number are listed in Table 1 and 2.

Table 1: Errors for Example 1 with layered media (5.1) by the proposed method with $k_0 = 2\pi$.

Meshes	16×16	Order	32×32	Order	64×64	Order	128×128
$e_M(\Gamma)$	$3.7506e-05$	**4.0352**	$2.2877e-06$	**4.0094**	$1.4205e-07$	**4.0040**	$8.8533e-09$
$e_2(\Gamma)$	$2.6915e-05$	**4.0413**	$1.6347e-06$	**4.0133**	$1.0123e-07$	**4.0064**	$6.2989e-09$
$e_M(\Omega)$	$9.4050e-05$	**3.8591**	$6.4813e-06$	**3.9292**	$4.2545e-07$	**3.9644**	$2.7254e-08$
$e_2(\Omega)$	$2.3941e-05$	**4.0067**	$1.4894e-06$	**4.0073**	$9.2619e-08$	**4.0045**	$5.7708e-09$

Table 2: Errors for Example 1 with layered media (5.1) by the proposed method with $k_0 = 32\pi$ and $k_0 = 36\pi$.

k_0	Error	64×64	Order	128×128	Order	256×256
32π	$e_M(\Gamma)$	$5.0819e-06$	**4.8294**	$1.7874e-07$	**4.3635**	$8.6831e-09$
	$e_2(\Gamma)$	$3.6222e-06$	**4.8350**	$1.2691e-07$	**4.3663**	$6.1531e-09$
36π	$e_M(\Gamma)$	$4.9861e-06$	**4.4246**	$2.3218e-07$	**4.5547**	$9.8792e-09$
	$e_2(\Gamma)$	$3.5537e-06$	**4.4301**	$1.6485e-07$	**4.5576**	$7.0003e-09$

From Table 1 and 2, one can see that although the second order derivative u_{yy} is discontinuous at the interface $y = -b/2$, we use the compact high order scheme and immersed interface method to obtain the fourth order convergence in the whole computational domain and at the aperture for different wave number. In Table 2, for relatively large wave number, such as 30π, 36π, quite high accuracy can be obtained only with 4 points per wavelength, which can not be achieved by low order method [32,33].

Next we illustrate the power of the preconditioning. The spectrum of the original and preconditioned coefficient matrices of the model filled with (5.1) are shown in (a) and (b) of Fig. 2 separately when $k_0 = 8\pi$. For the original coefficient matrix in (a), the real parts of almost all eigenvalues are negative, and a few even equal zero, while, in (b), all eigenvalues are on the right side of the complex plane, and in fact most of eigenvalues equal 1. The preconditioned spectrum exhibits a high degree of clustering around one, which is favorable for Krylov subspace methods [34]. Moreover, the ratio of the absolute value of the imaginary part and the real part of eigenvalues is also an important factor for most iterative algorithms. The eigenvalues with a large ratio will lead to ill-conditioning systems. For most eigenvalues shown in (b) of Fig. 2, the absolute value of the imaginary part is small than the real part.

Then we apply our preconditioning iterative method for solving the cavity model

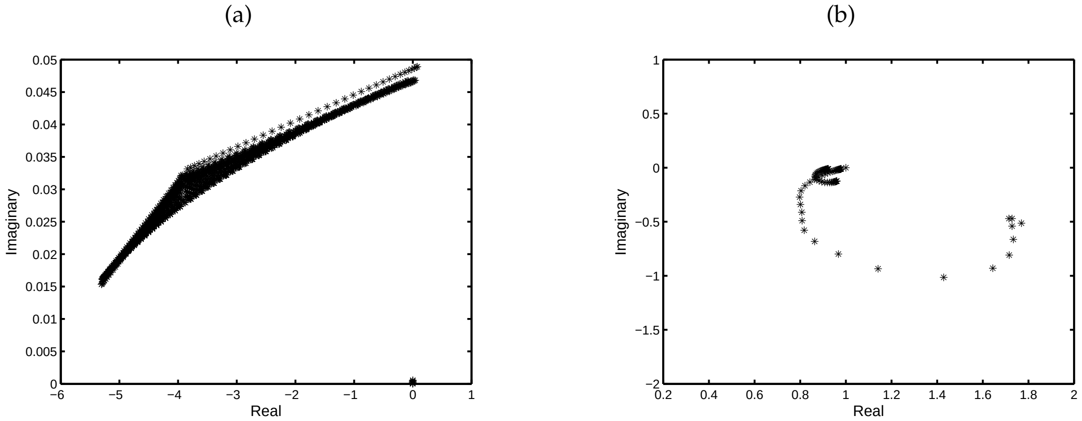

Figure 2: The eigenvalues distribution of the unpreconditioned A and preconditioned AB^{-1} for the cavity filled with layered media (5.1) with 40×40 meshes when $k_0 = 8\pi$ in the TM case.

filled with different layered media. Other layered media

$$\varepsilon_r(x,y) = \begin{cases} 3.78 + 1.6\mathrm{i}, & -b/2 \le y \le 0, \\ 10.27 + 2.56\mathrm{i}, & -b \le y < -b/2, \end{cases} \tag{5.2}$$

is also considered. The Krylov subspace method, GMRES with no restarts is used to solve the final system. The stopping criterion is $\|r_m\|/\|r_0\| \le 10^{-6}$, where r_m is the residual at the mth iteration.

Table 3: Iteration counts (Iter) and CPU times (in sec.) for computing the cavity filled with different layered media (5.1) and (5.2) with the proposed precondtioning solver and full GMRES.

k_0	Meshes	Iter with (5.1)	Time with (5.1)	Iter with (5.2)	Time with (5.2)
8π	8×8	5	0.087	6	0.085
	16×16	6	0.278	8	0.358
	32×32	7	0.762	7	0.862
16π	16×16	3	0.285	5	0.295
	32×32	6	1.790	6	1.731
	64×64	5	5.341	6	6.341
32π	32×32	3	1.937	3	1.957
	64×64	5	11.24	5	11.28
64π	64×64	3	14.36	3	14.34
	128×128	5	86.45	4	74.46
100π	100×100	3	43.55	4	43.29
	200×200	4	90.86	4	98.21

The iteration number and CPU time are obtained in Table 3 with different wave number for the open cavity filled with the layered media (5.1) and (5.2). For each wave number, we observe that the convergence rate of GMRES is almost less sensitive to the mesh size h and also is independent of the layered media parameter ε_r, confirming the robustness of the method. It is unexpected that the wave number seems to have minimal impact on the iterations number, and this is probably caused by the low

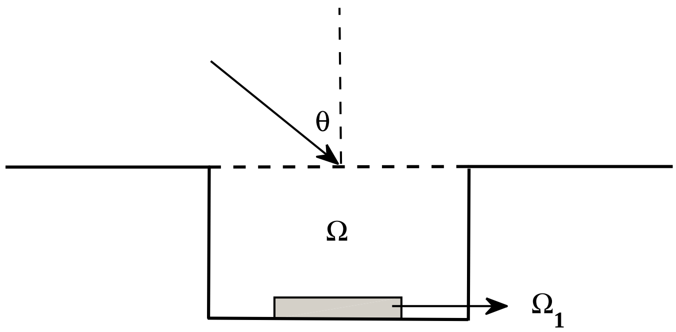

Figure 3: The cavity geometry of Example 2.

complexity of the media filled in the cavity, which is verified by the later numerical examples.

5.2 Example 2

We consider the cavity with $a = b = 1.0$ which is filled with the general inhomogeneous media as in Fig. 3, where

$$\varepsilon_r = \begin{cases} \varepsilon_{r_0} & (x,y) \in \Omega \setminus \Omega_1, \\ \varepsilon_{r_1} & (x,y) \in \Omega_1. \end{cases}$$

In this cavity, a small rectangular penetrable object Ω_1 with $a/2$ wide and $b/8$ deep is placed on the bottom, which is filled with a dielectric having $\varepsilon_{r_1} \neq \varepsilon_{r_0}$. Two different cases are considered as follows.

First we verify the accuracy of the proposed high order method when the field of the aperture is given as

$$u(x,y) = e^{xy}\sin\left(\frac{k_0 x}{2}\right)\sin\left((\frac{k_0}{2}+\frac{\pi}{4})y\right).$$

The solutions are converged to a tolerance of 10^{-10} in the relative residual. The maximum number of GMRES iterations to 300. Table. 4 and 5 show that the numerical approximation has fourth order accuracy when the cavity is filled with inhomogeneous media, which is an good agreement with the theoretical analysis.

Then we study the case that the analytic solution of the field of the aperture do not exist, which demonstrate the accuracy comparison of the fourth order and second order method. We take the cavity model filled with the inhomogeneous media $\varepsilon_{r_0} = 1.0$, $\varepsilon_{r_1} = 4.0 + \mathrm{i}$ when $k_0 = 18\pi$ as example. Since the analytic solution is not available, numerical solutions with 1024×1024 meshes are calculated to confirm the convergence. We conduct a grid convergence description in Fig. 4. The coarsest mesh we have taken has 14 points per wavelength , and the finest mesh is 1024×1024, which

Table 4: Errors for the cavity filled with the inhomogeneous media $\varepsilon_{r_0} = 1.0$, $\varepsilon_{r_1} = 4.0 + \mathrm{i}$ when $k_0 = 2\pi$.

Meshes	16×16	Order	32×32	Order	64×64	Order	128×128
$e_M(\Gamma)$	$5.0760e-04$	**4.2963**	$2.5835e-05$	**4.0762**	$1.5316e-06$	**4.0128**	$9.4882e-08$
$e_2(\Gamma)$	$3.6647e-04$	**4.3099**	$1.8477e-05$	**4.0853**	$1.0885e-06$	**4.0175**	$6.7212e-08$
$e_M(\Omega)$	$9.0237e-04$	**4.7837**	$3.2761e-05$	**4.0662**	$1.9557e-06$	**4.0070**	$1.2164e-07$
$e_2(\Omega)$	$3.3342e-04$	**4.4941**	$1.4796e-05$	**4.0928**	$8.6713e-07$	**4.0168**	$5.3569e-08$

Table 5: Errors for the cavity filled with the inhomogeneous media $\varepsilon_{r_0} = 2.0$, $\varepsilon_{r_1} = 1.0 + 4.0\mathrm{i}$ when $k_0 = 8\pi$.

Meshes	32×32	Order	64×64	Order	128×128	Order	256×256
$e_M(\Gamma)$	$8.8756e-03$	**5.3437**	$2.1856e-04$	**5.0037**	$6.8127e-06$	**4.3446**	$3.3533e-07$
$e_2(\Gamma)$	$4.1663e-03$	**5.4625**	$9.4490e-05$	**4.9704**	$3.0140e-06$	**4.2519**	$1.5820e-07$
$e_M(\Omega)$	$1.3287e-02$	**4.7751**	$4.8525e-04$	**4.2527**	$2.5455e-05$	**4.0832**	$1.5018e-06$
$e_2(\Omega)$	$3.0499e-03$	**5.2829**	$7.8336e-05$	**4.6023**	$3.2249e-06$	**4.1095**	$1.8683e-07$

corresponds to 114 points per wavelength. It can be seen that the numerical solutions have good convergence, and the solutions with 128×128 meshes is close to that with 1024×1024 meshes.

In Fig. 5, (a) illustrates the comparison between the present method and the second order method, and (b) focuses on the field of the aperture around $x = 0.5$. We note that the present method and 128×128 meshes yield the comparable accuracy of the solutions as those of the second order method with 1024×1024 meshes. This means the proposed method can drastically reduce the number of grid nodes needed while retains the desired accuracy.

5.3 Example 3

We apply the proposed method to solve plane wave scattering from a rectangular groove with 1.0 meter wide and 0.25 meter deep, i.e., the computational domain Ω is $[0.0m, 1.0m] \times [-0.25m, 0.0m]$. There are third penetrable objects with relative permit-

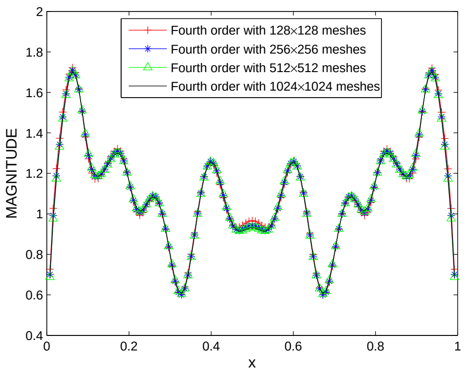

Figure 4: The magnitude of the aperture field at $\theta = 0$ by the fourth order method with $k_0 = 18\pi$.

tivity ε_{r_1}, which is denoted by $\Omega_1 = [0.0m, 0.2m] \times [-0.15m, -0.1m] \cup [0.4m, 0.6m] \times [-0.15m, -0.1m] \cup [0.8m, 1.0m] \times [-0.15m, -0.1m]$. The domain $\Omega \setminus \Omega_1$ has relative permittivity ε_{r_0}, which $\varepsilon_{r_0} \neq \varepsilon_{r_1}$.

We take into account the performance of the present preconditioning solver. The full GMRES method is used to solve the model with various wave number. The iteration is terminated when then norm of the residual vector was reduced by the factor 10^{-6}.

Table 6: Iteration counts by the proposed preconditioning solver with full GMRES with 20 mesh points per wavelength for Example 3.

k_0	Meshes	Iter with $\theta = 0$	Iter with $\theta = \frac{\pi}{4}$
2π	20×5	8	8
4π	40×10	19	18
8π	80×20	36	34
10π	100×25	41	45
16π	160×40	72	73
30π	300×75	177	180
36π	360×90	235	232
40π	400×100	268	269

Table 6 shows the computational performance in terms of number of iterations and computational time to reach the specified convergence with $\varepsilon_{r_0} = 1.0$ and $\varepsilon_{r_1} = 4 + \mathrm{i}$. For $k_0 = 2\pi, 4\pi, 8\pi, 16\pi, 30\pi, 36\pi$ and 40π, we use 20×5, 40×10, 80×20, 160×40, 300×75, 360×90 and 400×100 meshes leading to 20 points per wavelength. It is indicated that the GMRES iterations number grows as the wave number increases, approximately proportionally to k_0.

We notice that the iteration counts for this model is more than Example 1, especially for relatively large wave number. This coincides with our discussion in Section 4.3, that is, the complexity of the media in the cavity influences the iteration number of the fast iterative solver. Nevertheless, thanks to reducing the GMRES iterations

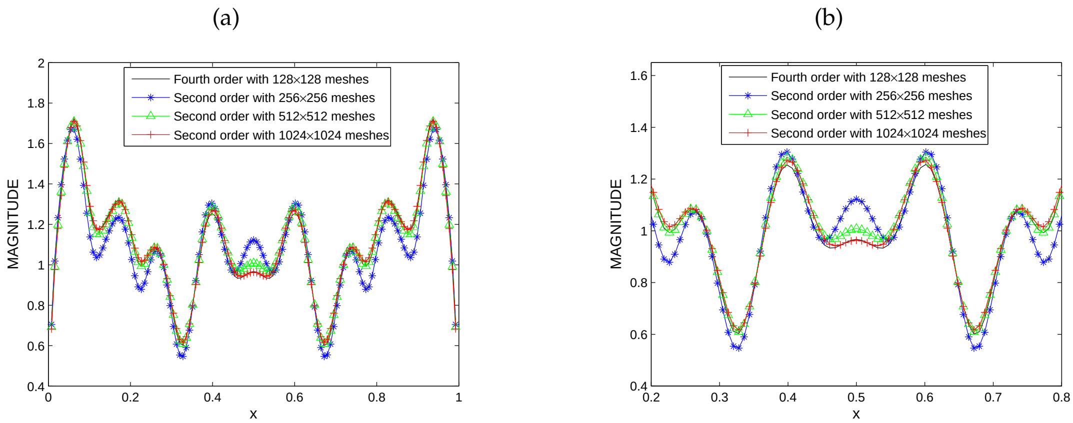

Figure 5: The comparison of the magnitude of the aperture field at $\theta = 0$ by the fourth order method and second order method with $k_0 = 18\pi$.

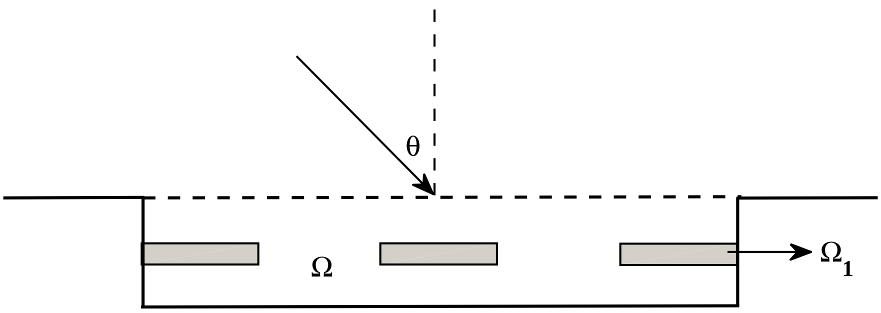

Figure 6: The cavity geometry of Example 3

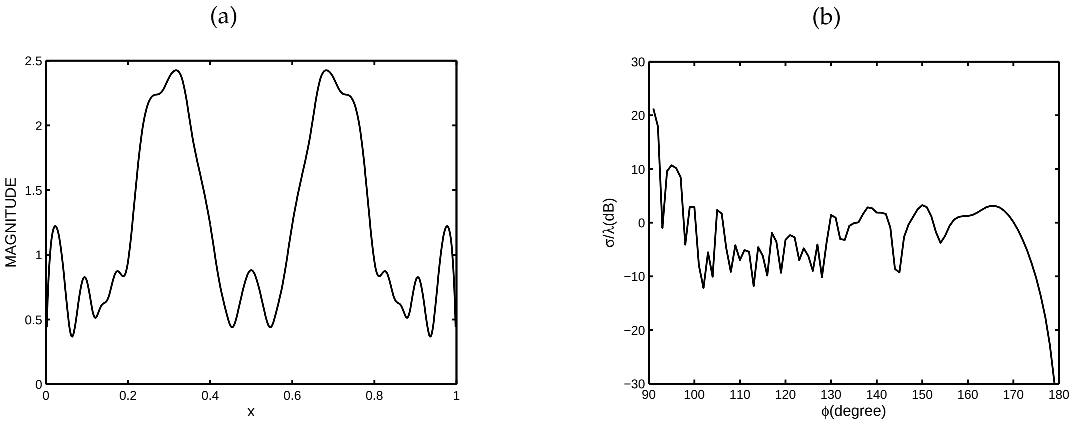

Figure 7: The magnitude of the aperture field at $\theta = 0$ (in (a)) and backscatter RCS (in (b)) with 400×100 meshes when $k_0 = 30\pi$.

onto a small sparse subspace associated with the discontinuous interface of different medium, we can use the GMRES without restart to solve this model on the PC, which can not usually done in the scattering of complex model.

In addition, we give the magnitude of aperture field and scattering RCS of the cavity filled with the inhomogeneous media when $k_0 = 30\pi$ and 40π on a 400×100 mesh in Fig. 7 and Fig. 8 respectively. The discretized system has the size of $40,299$ elements. If using the conventional method, the computation burden is considerable resulting in low efficiency in person computer.

6 Conclusions

We have developed an efficient fast high order preconditioning solver for the scattering of open cavity filled with the inhomogeneous media. The fourth order finite scheme and the immersed interface method (IIM) are applied in the interior domain of cavity, and the fourth order discretization is simultaneously enforced on the trun-

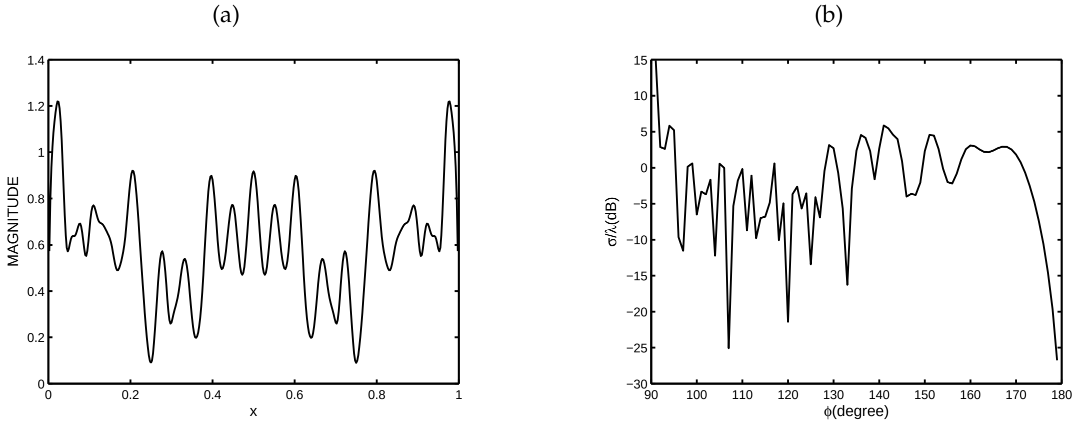

Figure 8: The magnitude of the aperture field at $\theta = 0$ (in (a)) and backscatter RCS (in (b)) with 400×100 meshes when $k_0 = 40\pi$.

cation boundary at the aperture of open cavity, which leads to high order accuracy solutions for practical application.

More complex is the media filled in the open cavity becomes, more fine mesh subdivision needs to be carried, which causes a large system of equations. The proposed preconditioning iterative solver makes coefficient matrices with favorable spectral distributions and ideal candidates for Krylov subspace algorithms. The numerical experiments show the high efficiency of the preconditioner, which is independent of problem parameter, such as the mesh size h, the incidence angle of the electromagnetic wave, etc. However the complexity of media filled in the open cavity has shown influence on iterations to a certain extent. Nevertheless, it is satisfying that by reducing the GMRES iterations onto the sparse subspace we can use the GMRES method without restarts for the case of relatively complex inhomogeneous media with large wave number, which usually needs restarts to avoid the degradation of convergence in complicated scattering problem.

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