SICP Exercise 1.13

Prove: $\mathrm{Fib}(n) = \dfrac{\varphi^n - \psi^n}{\sqrt{5}}$ where $\varphi = (1+\sqrt{5})/2$

$\psi = (1-\sqrt{5})/2$

Base case $n = 0$ and $n = 1$:

$$\mathrm{Fib}(0) = \frac{\varphi^0 - \psi^0}{\sqrt{5}} = \left(\frac{1-1}{\sqrt{5}}\right) = 0$$

$$\mathrm{Fib}(1) = \frac{\varphi^1 - \psi^1}{\sqrt{5}} = \frac{(1+\sqrt{5})/2 - (1-\sqrt{5})/2}{\sqrt{5}}$$

$$= \frac{\sqrt{5}}{\sqrt{5}} = 1$$

Assumption: Statement is true for 0 to $n-1$

Induction:

$$\mathrm{Fib}(n) = \mathrm{Fib}(n-1) + \mathrm{Fib}(n-2)$$

$$\frac{\varphi^n - \psi^n}{\sqrt{5}} = \frac{\varphi^{n-1} - \psi^{n-1}}{\sqrt{5}} + \frac{\varphi^{n-2} - \psi^{n-2}}{\sqrt{5}}$$

$$\varphi^n - \psi^n = \varphi^{n-1} - \psi^{n-1} + \varphi^{n-2} - \psi^{n-2}$$

$$\varphi^n - \psi^n = \frac{\varphi^n}{\varphi} + \frac{\varphi^n}{\varphi^2} - \frac{\psi^n}{\psi} - \frac{\psi^n}{\psi^2}$$

$$\varphi^n - \psi^n = \varphi^n\left(\frac{1}{\varphi} + \frac{1}{\varphi^2}\right) - \psi^n\left(\frac{1}{\psi} + \frac{1}{\psi^2}\right)$$

(Statement is true if both factors are equal to 1)

$$\varphi^n - \psi^n = \varphi^n\left(\frac{1}{\left(\frac{1+\sqrt{5}}{2}\right)} + \frac{1}{\left(\frac{1+\sqrt{5}}{2}\right)^2}\right) - \psi^n\left(\frac{1}{\left(\frac{1-\sqrt{5}}{2}\right)} + \frac{1}{\left(\frac{1-\sqrt{5}}{2}\right)^2}\right)$$

$$\varphi^n - \psi^n = \varphi^n\left(\frac{2}{1+\sqrt{5}} + \frac{4}{(1+\sqrt{5})^2}\right) - \psi^n\left(\frac{2}{1-\sqrt{5}} + \frac{4}{(1-\sqrt{5})^2}\right)$$

$$\varphi^n - \psi^n = \varphi^n\left(\frac{2 + 2\sqrt{5} + 4}{(1+\sqrt{5})^2}\right) - \psi^n\left(\frac{2 - 2\sqrt{5}) + 4}{(1-\sqrt{5})^2}\right)$$

$$\varphi^n - \psi^n = \varphi^n\left(\frac{6+2\sqrt{5}}{6+2\sqrt{5}}\right) - \psi^n\left(\frac{6-2\sqrt{5}}{6-2\sqrt{5}}\right)$$

$$\varphi^n - \psi^n = \varphi^n - \psi^n \quad \square$$

Prove: Fib(n) is the closest integer to $\varphi^n / \sqrt{5}$.

$$\Leftrightarrow \quad Fib(n) - \frac{\varphi^n}{\sqrt{5}} \leq \frac{1}{2}$$

From previous proof:

$$Fib(n) = \frac{\varphi^n - \psi^n}{\sqrt{5}} = \frac{\varphi^n}{\sqrt{5}} - \frac{\psi^n}{\sqrt{5}}$$

$$\Leftrightarrow Fib(n) - \frac{\varphi^n}{\sqrt{5}} = -\frac{\psi^n}{\sqrt{5}}$$

$$\Rightarrow Fib(n) - \frac{\varphi^n}{\sqrt{5}} \leq \frac{1}{2} \quad \Leftrightarrow \quad \frac{\psi^n}{\sqrt{5}} \leq \frac{1}{2}$$

$$\Leftrightarrow \quad \psi^n \leq \frac{\sqrt{5}}{2}$$

Since $\psi = \dfrac{(1-\sqrt{5})}{2} \approx -0{,}62$:

$$-1 < \psi < 0 \quad \Rightarrow \quad -1 < \psi^2 < 1$$

$$\Rightarrow \quad \psi^n < \frac{\sqrt{5}}{2} \quad \square$$

$$\Leftrightarrow \quad Fib(n) - \frac{\varphi^n}{\sqrt{5}} \leq \frac{1}{2} \text{ holds.}$$