First Name:

Last Name:

Student ID:

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We declare that this assignment is solely our own work, and is in accordance with the University of Toronto Code of Behaviour on Academic Matters.

This submission has been prepared using LaTeX.

Problem 1.

(8 MARKS) Let U denote the set of **computer programmers**, **software testers** and **software projects** of Great Software Company, Inc.
Let `SSP (Super Software Project)` and `USP (Ultra Software Project)` be specific software projects in U. Let `Bugs Bunny` be a specific individual in U.
Finally, consider the following predicates, along with their meaning:

$S(x)$: "x is a software tester";

$P(x)$: "x is a computer programmer";

$C(x)$: "x is a software project";

$N(x)$: "x is new" (software project or software tester);

$W(x, y)$: "x writes code for y";

$A(x, y)$: "x is more advanced than y".

$T(x, y)$: "x tests y"

Using only the domain U, the given constants `SSP` and `USP` , and predicates above (in addition to appropriate connectives and quantifiers), translate each sentence below: give a proper English sentence that corresponds to each symbolic sentence, and give a symbolic sentence that corresponds to each English sentence. State clearly any assumptions you need to make.

1. (1 MARK) `Bugs Bunny` writes code for `SSP` and `USP` and neither software project is more advanced than the other.
2. (1 MARK) $\forall x \in U : [C(x) \rightarrow [\exists y \in U : [P(y) \text{ and } W(y, x)]]]$
3. (1 MARK) Some software project is more advanced than every software project that Bugs Bunny writes code for.
4. (1 MARK) $S(x)$ and $\big(W(x, \texttt{SSP}) \text{ or } T(x, \texttt{USP})\big)$
5. (1 MARK) No software tester has tested every software project.
6. (1 MARK) If any computer programmer has coded every new software project, then that computer programmer has coded every software project.

7. (1 Mark) $\exists x \in U : [S(x)$ and $(\forall y \in U : [[C(y)$ and $(W(\texttt{Bugs Bunny}, y)$ or $N(y))] \rightarrow W(x, y)])]$

8. (1 Mark) `Bugs Bunny` writes code for `SSP` unless there is some new software project more advanced than `SSP`.

Solution

1. W(Bugs Bunny,SSP) $\wedge$ W(Bugs Bunny,USP) $\wedge$[$\neg$ A(SSP,USP)]$\wedge$[$\neg$ A(USP,SSP)]

2. All software projects have been written by some computer programmer

3. $\exists x \in U : C(x) \wedge [\forall y \in U : [C(y) \wedge W(\text{Bugs Bunny}, y)] \implies A(x, y)]$

4. Tester writes code for SSP or tests USP.

5. $\forall x \in U : S(x) \implies [\exists y \in U : C(y) \wedge [\neg T(x, y)]]$

6. $[\forall x \in U : P(x) \wedge (\forall y \in U : C(y) \wedge N(y) \implies W(x, y))] \implies [\forall z \in U : C(z) \implies W(x, z)]$

7. There is a tester who has written code for all projects that are new or that have been written by Bugs Bunny.

8. $\forall x \in U : \neg N(x) \vee \neg A(x, \text{SSP}) \implies W(\text{Bugs Bunny,SSP})$

Problem 2.

(4 Marks)

Consider the quadratic equation $x^2 + bx + c = 0$. Let

$Q(x)$ be the predicate "x is a rational number"

$S(x)$ be the predicate "x is a solution of the equation $x^2 + bx + c = 0$"

Assuming the domain is the set $\mathbb{Q}$ of rational numbers, do the following:

1. (2 Marks) Write a symbolic sequence that corresponds to English statement: "for all rational numbers b,c if there is a rational number r such that r is a solution to the equation $x^2 + bx + c = 0$ then any other solution s of this equation is a rational number".
2. (2 Marks) Formally prove the above claim (i.e., prove the statement in the above bullet). (Hint: If the equation $x^2 + bx + c = 0$ has a real solution r, then the other solution s is also real and the following is true: $x^2 + bx + c = (x - r)(x - s)$).

Solution

1. $$\forall b, c \in \mathbb{Q} : \forall r, s \in \mathbb{R} : Q(r) \wedge S(r) \wedge S(s) \implies Q(s)$$

2. Let $b, c \in \mathbb{Q}$.

   Also, let $r, s \in \mathbb{R}$.

   Assume $Q(r) \wedge S(r) \wedge S(s)$.

   Then $x^2 + bx + c = (x - r)(x - s)$

   $= x^2 - (r + s) + rs.$

   Then $b = -(r + s)$ # Equate coefficents of x

   Then $s = -(r + b)$.

   Then $Q(s)$ # Sum of rationals is rational.

Then $Q(r) \wedge S(r) \wedge S(s) \implies Q(s)$.

Therefore $\forall b, c \in \mathbb{Q} : \forall r, s \in \mathbb{R} : Q(r) \wedge S(r) \wedge S(s) \implies Q(s)$.

Problem 3.

(4 Marks)

Prove the following claims:

1. (2 Marks) $\forall x, y \in \mathbb{R} : x \geq 0 \text{ and } y \geq 0 \implies \frac{x+y}{2} \geq \sqrt{xy}$
2. (2 Marks) $\forall n \in \mathbb{Z} : n \geq 1 \implies \frac{2n+1}{2n+2} \geq \frac{\sqrt{n}}{\sqrt{n+1}}$ (Hint: you could use the claim in above bullet as an aid).

Solution

1.

Let $x, y \in \mathbb{R}$.

Assume $x \geq 0$ and $y \geq 0$.

Then $\sqrt{x} \geq 0$.

Also $\sqrt{y} \geq 0$.

Then $(\sqrt{x} - \sqrt{y})^2 \geq 0$.

Then $x - 2\sqrt{xy} + y \geq 0$.

Then $x + y \geq 2\sqrt{xy}$.

Then $\frac{x+y}{2} \geq \sqrt{xy}$.

Then $x \geq 0$ and $y \geq 0 \implies \frac{x+y}{2} \geq \sqrt{xy}$.

Therefore $\forall x, y \in \mathbb{R} : x \geq 0 \text{ and } y \geq 0 \implies \frac{x+y}{2} \geq \sqrt{xy}$.

2.

Let $n \in \mathbb{N}$.

Assume $n \geq 1$.

Let $x = 1$.

Also let $y = \frac{n}{n+1}$.

Then $x, y \in \mathbb{R}$.

Then $\frac{x+y}{2} \geq \sqrt{xy}$. # Apply previous exercise.

Then $\frac{1}{2}\left(1 + \frac{n}{n+1}\right) \geq \sqrt{1 \cdot \frac{n}{n+1}}$.

Then $\frac{2n+1}{2n+2} \geq \frac{\sqrt{n}}{\sqrt{n+1}}$.

Then $n \geq 1 \implies \frac{2n+1}{2n+2} \geq \frac{\sqrt{n}}{\sqrt{n+1}}$.

Therefore $\forall n \in \mathbb{Z} : n \geq 1 \implies \frac{2n+1}{2n+2} \geq \frac{\sqrt{n}}{\sqrt{n+1}}$.

Problem 4.

(8 Marks)

Any natural number can be either an odd or an even number, but not both. For each natural number n, let $E(n)$ be defined as "$\exists k \in \mathbb{N} : n = 2k$". Also for each natural number n, let $O(n)$ be defined as "$\exists k \in \mathbb{N} : n = 2k + 1$". That is, for any natural number n, $E(n)$ means "n is even" and $O(n)$ means "n is odd".

Prove or disprove the following:

1. (2 Marks) $\forall n \in \mathbb{N} : O(n) \implies E(n^2 + n)$
2. (2 Marks) $\forall n \in \mathbb{N} : [\neg E(n)] \implies [\neg E(n + 2)]$
3. (2 Marks) $\forall n, m \in \mathbb{N} : O(n)$ and $E(m) \implies O(m^2 + 3n)$
4. (2 Marks) $\forall n, m \in \mathbb{N} : [\neg E(mn)] \implies [\neg E(m)$ and $\neg E(n)]$

Solution

1.

Let $n \in \mathbb{N}$.

Assume $O(n)$.

Then $\exists k \in \mathbb{N} : n = 2k + 1$.

Let $k_0 \in \mathbb{N} : n = 2k_0 + 1$.

Then $n^2 + n = 4k_0^2 + 4k_0 + 1 + 2k_0 + 1$.

$= 2(2k_0^2 + 3k_0 + 1)$.

Let $k_1 = 2k_0^2 + 3k_0 + 1$.

Then $k_1 \in \mathbb{N}$.

Then $n^2 + n = 2k_1$.

Then $\exists k \in \mathbb{N} : n^2 + n = 2k$.

Then $E(n^2 + n)$.

Then $O(n) \implies E(n^2 + n)$.

Therefore $\forall n \in \mathbb{N} : O(n) \implies E(n^2 + n)$.

2.

Let $n \in \mathbb{N}$.

Proof by contrapositive.

Assume $E(n + 2)$.

Then $\exists k \in \mathbb{N} : n + 2 = 2k$.

Let $k_0 \in \mathbb{N} : n + 2 = 2k_0$.

Then $n = 2(k_0 - 1)$.

Let $k_1 = k_0 - 1$. # $k_1 \in \mathbb{N}$ because $k_0 \geq 1$.

Then $k_1 \in \mathbb{N}$.

Then $n = 2k_1$.

Then $\exists k \in \mathbb{N} : n = 2k$.

Then $E(n)$.

Then $E(n + 2) \implies E(n)$.

Then $[\neg E(n)] \implies [\neg E(n + 2)]$.

Therefore $\forall n \in \mathbb{N} : [\neg E(n)] \implies [\neg E(n + 2)]$.

3.

Let $n, m \in \mathbb{N}$.

Assume $O(n)$ and $E(m)$.

Then $\exists k \in \mathbb{N} : n = 2k + 1$.

Also $\exists l \in \mathbb{N} : m = 2l$.

Then $m^2 + 3n = 2(2l^2 + k + 1) + 1$.

Let $k_1 = 2l^2 + k + 1$.

Then $k_1 \in \mathbb{N}$.

Then $m^2 + 3n = 2k_1 + 1$.

Then $\exists k \in \mathbb{N} : m^2 + 3n = 2k + 1$.

Then $O(m^2 + 3n)$.

Then $O(n)$ and $E(m) \implies O(m^2 + 3n)$.

Therefore $\forall n, m \in \mathbb{N} : O(n)$ and $E(m) \implies O(m^2 + 3n)$.

4.

Let $n \in \mathbb{N}$. Rewrite the claim as follows: $\forall n, m \in \mathbb{N} : [O(mn)] \implies [O(m)$ and $O(n)]$

Let $m, n \in \mathbb{N}$.

Assume $O(mn)$.

Then $\exists k \in \mathbb{N} : mn = 2k + 1$.

Let $k_0 \in \mathbb{N} : mn = 2k_0 + 1$.

Then $\exists q, r \in \mathbb{N} : m = 2q + r$ # Divide by 2 (with reminder)

Let $q_0, r_0 \in \mathbb{N} : m = 2q_0 + r_0$.

Similarly, $n = 2s_0 + t_0$.

$$\begin{aligned} \text{Then } mn &= 2(2q_0s_0 + q_0t_0 + r_0s_0) + r_0t_0 \\ &= 2k_0 + 1. \end{aligned}$$

Let $M = 2q_0s_0 + q_0t_0 + r_0s_0$.

Then $M \in \mathbb{N}$.

Then $2(M - k_0) = 1 - r_0t_0$.

Then 2 divides $1 - r_0t_0$.

Then $1 - r_0t_0 = 0$ # Since $r_0 < 2, t_0 < 2$.

Then $r_0 = 1, t_0 = 1$.

Then $m = 2q_0 + 1$ and $n = 2s_0 + 1$.

Then $\exists k \in \mathbb{N} : m = 2k + 1$ and $\exists l \in \mathbb{N} : n = 2l + 1$.

Then $O(m) \wedge O(n)$.

Then $[O(mn)] \implies [O(m)$ and $O(n)]$.

Therefore $\forall n, m \in \mathbb{N} : [O(mn)] \implies [O(m)$ and $O(n)]$.

Problem 5.

(6 Marks)

Let $a, b, c \in \mathbb{R}^+$ be three sides of a triangle. Then the following statement is true:

$\forall a, b, c \in \mathbb{R}^+:$ If a, b, c are sides of some triangle, then $a < b+c$ and $b < a+c$ and $c < a+b$.

This statement is know as the *triangle inequality*.

Let's denote $T(a, b, c)$ the following predicate : "a, b, c are sides of a triangle".

1. (2 Marks) Prove the following claim:

$$\forall a, b, c \in \mathbb{R}^+ : T(a, b, c) \implies [|a-b| < c \text{ and } |b-c| < a \text{ and } |a-c| < b].$$

2. (2 Marks) Prove or disprove :

$$\forall a, b, c \in \mathbb{R}^+ : T(a, b, c) \implies a^2 + 2bc > b^2 + c^2.$$

3. (2 Marks) Prove or disprove:

$$\forall a, b, c \in \mathbb{R}^+ : T(a, b, c) \implies \frac{a^2 + 2bc}{b^2 + c^2} + \frac{b^2 + 2ca}{c^2 + a^2} + \frac{c^2 + 2ab}{a^2 + b^2} > 3.$$

Solution

1.

Let $a, b, c \in \mathbb{R}$.

Assume $T(a, b, c)$.

Then $a < b + c \wedge b < a + c \wedge c < a + b$.

Then $a - b < c \wedge b - a < c$. # using ineq. $a < b+c$ and $b < a+c$.

Then $|a - b| < c$. # By the definition of abs.value

Then $|b - c| < a$. # Similarly - or, permuting a, b, c cyclicaly.

Then $|a - c| < b$. # same reason.

Then Then $T(a, b, c) \implies [|a-b| < c$ and $|b-c| < a$ and $|a-c| < b]$.

Therefore $\forall a, b, c \in \mathbb{R}^+ : T(a,b,c) \implies [|a-b| < c \text{ and } |b-c| < a \text{ and } |a-c| < b]$.

2.

Let $a, b, c \in \mathbb{R}$.

Assume $T(a,b,c)$.

Then $|b-c| < a$. # From previous.

Then $|b-c|^2 < a^2$. # Apply nondecreasing square function.

Then $b^2 - 2bc + c^2 < a^2$.

Then $a^2 + 2bc > b^2 + c^2$.

Then Then $T(a,b,c) \implies a^2 + 2bc > b^2 + c^2$.

Therefore $\forall a, b, c \in \mathbb{R}^+ : T(a,b,c) \implies a^2 + 2bc > b^2 + c^2$.

3.

Let $a, b, c \in \mathbb{R}$.

Assume $T(a,b,c)$.

Then $a^2 + 2bc > b^2 + c^2$. # From previous.

Then $b^2 + c^2 > 0$. # b, c are positive numbers

Then $\frac{a^2+2bc}{b^2+c^2} > 1$. # Divide by $b^2 + c^2$.

Also $\frac{b^2+2ca}{c^2+c^a} > 1$. # Cyclicly permuting the variables.

Also $\frac{c^2+2ab}{a^2+b^a} > 1$. # Cyclicly permuting the variables.

Then $T(a,b,c) \implies \frac{a^2+2bc}{b^2+c^2} + \frac{b^2+2ca}{c^2+a^2} + \frac{c^2+2ab}{a^2+b^2} > 3$.

Therefore $\forall a, b, c \in \mathbb{R}^+ : T(a,b,c) \implies \frac{a^2+2bc}{b^2+c^2} + \frac{b^2+2ca}{c^2+a^2} + \frac{c^2+2ab}{a^2+b^2} > 3$.