First Name: Zhihong
Last Name: Wang
Student ID: 1002095207

First Name: Aofei
Last Name: Liu
Student ID: 1002291334

First Name: Heng
Last Name: Ye
Student ID: 1001846363

We declare that this assignment is solely our own work, and is in accordance with the University of Toronto Code of Behaviour on Academic Matters.

This submission has been prepared using LaTeX.

Problem 1.

(6 Marks)
Define the sequence $\{a_k\}$ as follows: $a_0 = 0, a_1 = 0, a_2 = 2, a_k = 3a_{\lfloor \frac{k}{2} \rfloor} + 2$ for all $k \geq 3$.

1. (2 Marks) Find the first 8 terms of the sequence.
2. (4 Marks) Prove that
$$\forall n \in \mathbb{N} : E(a_n)$$
where $E(n)$ is the usual predicate "n is even".

Solution

1. $$\begin{aligned}
a_0 &= 0 \\
a_1 &= 0 \\
a_2 &= 2 \\
a_3 &= 3a_{\lfloor \frac{3}{2} \rfloor} + 2 \\
&= 3a_{\lfloor 1.5 \rfloor} + 2 \\
&= 3a_1 + 2 \\
&= 3 \times 0 + 2 \\
&= 2 \\
a_4 &= 3a_{\lfloor \frac{4}{2} \rfloor} + 2 \\
&= 3a_{\lfloor 2 \rfloor} + 2 \\
&= 3a_2 + 3 \\
&= 3 \times 2 + 2 \\
&= 8 \\
a_5 &= 3a_{\lfloor \frac{5}{2} \rfloor} + 2 \\
&= 3a_{\lfloor 2.5 \rfloor} + 2 \\
&= 3a_2 + 3 \\
&= 3 \times 2 + 2 \\
&= 8 \\
a_6 &= 3a_{\lfloor \frac{6}{2} \rfloor} + 2 \\
&= 3a_{\lfloor 3 \rfloor} + 2 \\
&= 3a_3 + 2 \\
&= 3 \times 2 + 2 \\
&= 8 \\
a_7 &= 3a_{\lfloor \frac{7}{2} \rfloor} + 2 \\
&= 3a_{\lfloor 3.5 \rfloor} + 2
\end{aligned}$$

$= 3a_3 + 2$
$= 3 \times 2 + 2$
$= 8$

2. Let $n \in \mathbb{N}$
   Let $P(n)$ represent $E(a_n)$
   $\sharp$ Prove by strong induction

   Basis step: prove P(0), P(1), P(2), P(3)
   $a_0 = 0$
   Then $P(0)$
   $a_1 = 0$
   Then $P(1)$
   $a_2 = 2$
   Then $P(2)$
   $a_3 = 3a_{\lfloor \frac{3}{2} \rfloor} + 2 = 3a_1 + 2 = 3 \times 0 + 2 = 2$
   Then $P(3)$

   Inductive step: prove $\forall k \in \mathbb{N} : [k \geq 3 \rightarrow ((\forall i \in \{0, ..., k\} : P(i)) \rightarrow P(k+1))]$
   Let $k \in \mathbb{N}$
   Assume $k \geq 3$
   Assume $\forall i \in \{0, ..., k\} : P(i)$
   Then $a_{k+1} = 3a_{\lfloor \frac{k+1}{2} \rfloor} + 2$
   Then $0 < \lfloor \frac{k+1}{2} \rfloor \leq \frac{k+1}{2}$ and $\frac{k+1}{2} < k+1$ $\sharp$ $k \geq 3$
   Then $\lfloor \frac{k+1}{2} \rfloor \leq \frac{k+1}{2} < k+1$
   Then $\lfloor \frac{k+1}{2} \rfloor < k+1$
   Then $\lfloor \frac{k+1}{2} \rfloor \leq k$ $\sharp$ $\lfloor \frac{k+1}{2} \rfloor$ and k are integers
   Then $\lfloor \frac{k+1}{2} \rfloor \in \{0, ..., k\}$
   Then $a_{\lfloor \frac{k+1}{2} \rfloor}$ is even $\sharp$ By assumption
   Then $\exists j \in \mathbb{Z} : a_{\lfloor \frac{k+1}{2} \rfloor} = 2j$
   Let $j_0 \in \mathbb{Z}$ such that $a_{\lfloor \frac{k+1}{2} \rfloor} = 2j_0$
   Then $3 \times a_{\lfloor \frac{k+1}{2} \rfloor} = 6j_0$
   Then $3 \times a_{\lfloor \frac{k+1}{2} \rfloor} + 2 = 6j_0 + 2$
   $= 2(3j_0 + 1)$
   Let $j_1 = 3j_0 + 1$
   Then $j_1 \in \mathbb{Z}$
   Then $3 \times a_{\lfloor \frac{k+1}{2} \rfloor} + 2 = 2j_1$

Then $\exists j \in \mathbb{Z} : 3 \times a_{\lfloor \frac{k+1}{2} \rfloor} + 2 = 2j$

Then $3a_{\lfloor \frac{k+1}{2} \rfloor} + 2$ is even

Then a_{k+1} is even

Then $P(k+1)$

Therefore $(\forall i \in \{0, ...k\} : P(i)) \rightarrow P(k+1))$

Therefore $k \geq 3 \rightarrow ((\forall i \in \{0, ..., k\} : P(i)) \rightarrow P(k+1))$

Therefore $\forall k \in \mathbb{N} : [k \geq 3 \rightarrow ((\forall i \in \{0, ..., k\} : P(i)) \rightarrow P(k+1))]$

Therefore $\forall n \in \mathbb{N} : E(a_n)$ $\sharp$ By strong induction

Problem 2.

(6 Marks)
Prove or disprove:

1. (3 Marks) $\forall x \in \mathbb{R} : \forall n \in \mathbb{N} : \lfloor x + n \rfloor = \lfloor x \rfloor + n$
2. (3 Marks) $\forall x \in \mathbb{R} : \forall n \in \mathbb{N} : \lfloor nx \rfloor = n\lfloor x \rfloor$

Solution

1. True statement. Prove.
   Let x $\in \mathbb{R}$
   Let n $\in \mathbb{N}$

   $\sharp$ Definition of floor: $\forall x \in \mathbb{R} : (\lfloor x \rfloor \in \mathbb{Z}) \wedge (\lfloor x \rfloor \leq x) \wedge (\forall z \in \mathbb{Z} : (z \leq x) \rightarrow (z \leq \lfloor x \rfloor))$
   $\sharp$ Prove $\lfloor x + n \rfloor \leq n + \lfloor x \rfloor$
   Then $\lfloor x + n \rfloor \leq x + n$ $\sharp$ By definition of $\lfloor x + n \rfloor$
   Then $\lfloor x + n \rfloor - n \leq x$ and $\lfloor x + n \rfloor - n \in \mathbb{Z}$ $\sharp$ $\lfloor x + n \rfloor \in \mathbb{Z}$ and $n \in \mathbb{N}$
   Then $\lfloor x + n \rfloor - n \leq \lfloor x \rfloor$ $\sharp$ By definition of $\lfloor x \rfloor$
   Then $\lfloor x + n \rfloor \leq n + \lfloor x \rfloor$

   $\sharp$ Prove $\lfloor x \rfloor + n \leq \lfloor x + n \rfloor$
   Then $\lfloor x \rfloor \leq x$ $\sharp$ By definition of $\lfloor x \rfloor$
   Then $\lfloor x \rfloor + n \leq x + n$ and $\lfloor x \rfloor + n \in \mathbb{Z}$ $\sharp$ $\lfloor x \rfloor \in \mathbb{Z}$ and $n \in \mathbb{N}$
   Then $\lfloor x \rfloor + n \leq \lfloor x + n \rfloor$ $\sharp$ By definition of $\lfloor x + n \rfloor$

   Then $(\lfloor x + n \rfloor \leq n + \lfloor x \rfloor) \wedge (\lfloor x + n \rfloor \geq \lfloor x \rfloor + n)$
   Then $\lfloor x + n \rfloor = \lfloor x \rfloor + n$
   Therefore $\forall n \in \mathbb{N} : \lfloor x + n \rfloor = \lfloor x \rfloor + n$
   Therefore $\forall x \in \mathbb{R} : \forall n \in \mathbb{N} : \lfloor x + n \rfloor = \lfloor x \rfloor + n$

2. False statement. Disprove by proving negation is true.
   Prove $\neg(\forall x \in \mathbb{R} : \forall n \in \mathbb{N} : \lfloor nx \rfloor = n\lfloor x \rfloor)$
   Prove $\exists x \in \mathbb{R} : [\exists n \in \mathbb{N} : [\lfloor nx \rfloor \neq n\lfloor x \rfloor]]$
   Let $x = 1.5, n = 2$
   Then $x \in \mathbb{R}, n \in \mathbb{N}$
   Then $nx = 3$

Then $\lfloor nx \rfloor = 3$ $\sharp$ By definition of $\lfloor nx \rfloor$
Then $\lfloor x \rfloor = 1$ $\sharp$ By definition of $\lfloor x \rfloor$
Then $n\lfloor x \rfloor = 2$
Then $\lfloor nx \rfloor \neq n\lfloor x \rfloor$ $\sharp$ $3 \neq 2$
Therefore $\exists n \in \mathbb{N} : [\lfloor nx \rfloor \neq n\lfloor x \rfloor]]$
Therefore $\exists x \in \mathbb{R} : [\exists n \in \mathbb{N} : [\lfloor nx \rfloor \neq n\lfloor x \rfloor]]$
Therefore the negation is true.
Therefore the original statement "$(\forall x \in \mathbb{R} : \forall n \in \mathbb{N} : \lfloor nx \rfloor = n\lfloor x \rfloor)$" is false.

Problem 3.

(5 Marks)
Prove the following claims:

1. (3 Marks) $\forall x \in [0, \pi/2] : \sin x + \cos x \geq 1$.

2. (2 Marks) Prove that $\log_2 3$ is irrational.

Solution

1. Prove by contradiction
   Assume $\neg(\forall x \in [0, \frac{\pi}{2}] : \sin x + \cos x \geq 1)$
   Then $\exists x \in [0, \frac{\pi}{2}] : sinx + cosx < 1$
   Let $x_0 \in [0, \frac{\pi}{2}]$ such that $sinx_0 + cosx_0 < 1$
   Then $(1 \geq sinx_0 \geq 0) \wedge (1 \geq cosx_0 \geq 0)$ ♯ $x \in [0, \frac{\pi}{2}]$
   Then $sinx_0 cosx_0 \geq 0$
   Then $sinx_0 + cosx_0 \geq 0$
   Then $0 \leq sinx_0 + cosx_0 < 1$ ♯ By assumption
   Then $(sinx_0 + cosx_0)^2 < 1$
   Then $sin^2 x_0 + cos^2 x_0 + 2sinx_0 cosx_0 < 1$
   Then $1 + 2sinx_0 cosx_0 < 1$ ♯ $sin^2 x_0 + cos^2 x_0 = 1$
   Then $2sinx_0 cosx_0 < 0$
   Then $sinx_0 cosx_0 < 0$ ♯ Contradiction with $sinx_0 cosx_0 \geq 0$
   Then $sinx + cosx \geq 1$
   Therefore $\forall x \in [0, \frac{\pi}{2}] : sinx + cosx \geq 1$

2. Prove by contradiction
   Assume $log_2 3$ is rational
   Then $\exists p, q \in \mathbb{Z} : log_2 3 = \frac{p}{q} \wedge q \neq 0$
   Let $p_0, q_0 \in \mathbb{Z}$ such that $log_2 3 = \frac{p_0}{q_0} \wedge q_0 \neq 0$
   Then $log_2 3 > 0$ ♯ By definition of logarithm
   Then $\frac{p_0}{q_0} > 0$
   Then $(p_0 > 0 \wedge q_0 > 0) \ \vee \ (p_0 < 0 \wedge q_0 < 0)$
   Then $p_0 > 0 \wedge q_0 > 0$
   ♯ If p_0 and q_0 are both less than 0, we can erase their negative sign at once
   ♯ then they become the same case of p_0 and q_0 are both greater than 0
   Then p_0, q_0 are integers which greater or equal to 1 ♯ By assumption
   Then $\forall p_0, q_0 \in \mathbb{Z} : (p_0 \geq 1) \wedge (q_0 \geq 1)$

Then $\forall p_0, q_0 \in \mathbb{N} : (p_0 \geq 1) \wedge (q_0 \geq 1)$ $\sharp$ Natural numbers are integers begin with 0
Then $2^{\frac{p_0}{q_0}} = 3$ $\sharp$ By definition of logarithm
Then $2^{p_0} = 3^{q_0}$

$\sharp$ Prove 2^{p_0} is even by induction
Basis step: Prove 2^1
$2^1 = 2$
Then 2^1 is even
Inductive step: Prove $\forall k \in \mathbb{N} : [(k \geq 1) \rightarrow (2^k \text{ is even} \rightarrow 2^{k+1} \text{ is even})]$
Let $k \in \mathbb{N}$
    Assume $k \geq 1$
        Assume 2^k is even
        Then $\exists j \in \mathbb{Z} : 2^k = 2j$
            Let $j_0 \in \mathbb{Z}$ such that $2^k = 2j_0$
            Then $2^{k+1} = 2^k \times 2$
            Then $2^{k+1} = 2j_0 \times 2$
            Let $j_1 = 2j_0$
            Then $j_1 \in \mathbb{Z}$
            Then $2^{k+1} = 2j_1$
            Then $\exists j \in \mathbb{Z} : 2^{k+1} = 2j$
            Then 2^{k+1} is even
        Therefore 2^k is even $\rightarrow 2^{k+1}$ is even
    Therefore $k \geq 1 \rightarrow (2^k \text{ is even} \rightarrow 2^{k+1} \text{ is even})$
Therefore $\forall k \in \mathbb{N} : [(k \geq 1) \rightarrow (2^k \text{ is even} \rightarrow 2^{k+1} \text{ is even})]$
Therefore 2^{p_0} is even $\sharp$ By induction

$\sharp$ Prove 3^{q_0} is odd by induction
Basis step: Prove 3^1
$3^1 = 3$
Then 3^1 is odd
Inductive step: Prove $\forall k \in \mathbb{N} : [(k \geq 1) \rightarrow (3^k \text{ is odd} \rightarrow 3^{k+1} \text{ is odd})]$
Let $k \in \mathbb{N}$
    Assume $k \geq 1$
        Assume 3^k is odd
        Then $\exists j \in \mathbb{Z} : 3^k = 2j + 1$
            Let $j_0 \in \mathbb{Z}$ such that $3^k = 2j_0 + 1$
            Then $3^{k+1} = 3^k \times 3$
            Then $3^{k+1} = (2j_0 + 1) \times 3$

Then $3^{k+1} = 6j_0 + 3$
Then $3^{k+1} = 6j_0 + 2 + 1$
Then $3^{k+1} = 2(3j_0 + 1) + 1$
Let $j_1 = 3j_0 + 1$
Then $j_1 \in \mathbb{Z}$
Then $3^{k+1} = 2j_1 + 1$
Then $\exists j \in \mathbb{Z} : 3^{k+1} = 2j + 1$
Then 3^{k+1} is odd
Therefore 3^k is odd $\rightarrow 3^{k+1}$ is odd
Therefore $k \geq 1 \ \rightarrow (3^k$ is odd $\rightarrow 3^{k+1}$ is odd)
Therefore $\forall k \in \mathbb{N} : [(k \geq 1) \rightarrow (3^k$ is odd $\rightarrow 3^{k+1}$ is odd)]
Therefore 3^{q_0} is odd $\sharp$ By induction

Then even number = odd number $\sharp$ Contradiction with even $\neq$ odd
Then $\forall p, q \in \mathbb{Z} : log_2 3 \neq \frac{p}{q}$
Therefore $log_2 3$ is irrational

Problem 4.

(6 Marks) Prove the following:

1. (3 Marks) $\forall x, y \in \mathbb{R} : x^2 + y^2 = (x+y)^2 \Leftrightarrow x = 0 \vee y = 0$.
2. (3 Marks) $\forall x, y \in \mathbb{R} : x^3 + x^2y = y^2 + xy \Leftrightarrow y = x^2 \vee y = -x$.

Solution

1. Let $x, y \in \mathbb{R}$
   First, prove $x^2 + y^2 = (x+y)^2 \rightarrow x = 0 \vee y = 0$

   Assume $x^2 + y^2 = (x+y)^2$
   Then $(x+y)^2 = x^2 + 2xy + y^2$
   Then $x^2 + y^2 = x^2 + 2xy + y^2$
   Then $2xy = 0$
   Then $xy = 0$
   Case 1
   Assume $x = 0$
   Then $x = 0$
   Case 2
   Assume $x \neq 0$
   Then $\frac{xy}{x} = \frac{0}{x}$
   Then $y = 0$
   Then $(x = 0) \vee (y = 0)$
   Therefore $x^2 + y^2 = (x+y)^2 \rightarrow x = 0 \vee y = 0$

   Second, prove $x = 0 \vee y = 0 \rightarrow x^2 + y^2 = (x+y)^2$

   Assume $x = 0 \vee y = 0$

   Case 1
   Assume $x = 0$
   Then $xy = 0$
   Then $2xy = 0$
   Then $x^2 + y^2 = x^2 + y^2 + 0$
   Then $x^2 + y^2 = x^2 + y^2 + 2xy$
   Then $x^2 + y^2 = (x+y)^2$
   Therefore $x = 0 \rightarrow x^2 + y^2 = (x+y)^2$

Case 2
Assume $y = 0$
Then $xy = 0$
Then $2xy = 0$
Then $x^2 + y^2 = x^2 + y^2 + 0$
Then $x^2 + y^2 = x^2 + y^2 + 2xy$
Then $x^2 + y^2 = (x + y)^2$
Therefore $y = 0 \to x^2 + y^2 = (x + y)^2$

Therefore $x = 0 \vee y = 0 \to x^2 + y^2 = (x + y)^2$

Therefore $\forall x, y \in \mathbb{R} : x^2 + y^2 = (x + y)^2 \Leftrightarrow x = 0 \vee y = 0$

2. Let $x, y \in \mathbb{R}$
First, prove $x^3 + x^2y = y^2 + xy \to y = x^2 \vee y = -x$

Assume $x^3 + x^2y = y^2 + xy$
Then $x^2(x + y) = y(x + y)$

Case 1
Assume $x + y \neq 0$
Then $x^2 = y$
Therefore $x^3 + x^2y = y^2 + xy \to y = x^2$

Case 2
Assume $x + y = 0$
Then $y = -x$
Therefore $x^3 + x^2y = y^2 + xy \to y = -x$

Therefore $x^3 + x^2y = y^2 + xy \to y = x^2 \vee y = -x$

Second, prove $y = x^2 \vee y = -x \to x^3 + x^2y = y^2 + xy$

Assume $y = x^2 \vee y = -x$

Case 1

Assume $y = x^2$
Then $y(x + y) = x^2(x + y)$
Then $xy + y^2 = x^3 + x^2y$
Therefore $y = x^2 \to x^3 + x^2y = xy + y^2$

Case 2
Assume $y = -x$
Then $x + y = 0$
Then $y(x + y) = y \times 0 = 0$
Then $x^2(x + y) = x^2 \times 0 = 0$
Then $y(x + y) = x^2(x + y) = 0$
Then $xy + y^2 = x^3 + x^2y = 0$
Therefore $y = -x \to x^3 + x^2y = xy + y^2$

Therefore $y = x^2 \lor y = -x \to x^3 + x^2y = y^2 + xy$

Therefore $\forall x, y \in \mathbb{R} : x^3 + x^2y = y^2 + xy \Leftrightarrow y = x^2 \lor y = -x$