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We declare that this assignment is solely our own work, and is in accordance with the University of Toronto Code of Behaviour on Academic Matters.

This submission has been prepared using L^AT_EX.

Problem 1.

(6 MARKS)

Define the sequence $\{a_k\}$ as follows: $a_0 = 0, a_1 = 0, a_2 = 2, a_k = 3a_{\lfloor \frac{k}{2} \rfloor} + 2$ for all $k \geq 3$.

1. (2 MARKS) Find the first 8 terms of the sequence.
2. (4 MARKS) Prove that

$$\forall n \in \mathbb{N} : E(a_n)$$

where $E(n)$ is the usual predicate "n is even".

Solution

1. $a_0 = 0, a_1 = 0, a_2 = 2, a_3 = 3a_1 + 2 = 2, a_4 = 3a_2 + 2 = 8, a_5 = 3a_2 + 2 = 8, a_6 = 3a_3 + 2 = 8, a_7 = 3a_3 + 2 = 8$.
2. # Proof by induction.
Base Step.
 $a_0 = 0$.
Then $E(a_0)$.
#Inductive Step.
Let $n \in \mathbb{N}$.
Assume $\forall k \in \mathbb{N} : k \leq n \implies E(a_k)$. # Strong induction
Let $k = \lfloor \frac{n+1}{2} \rfloor$.
Then $k \in \mathbb{N}$.
Also $k \leq \frac{n+1}{2} \leq n$.
Then $E(a_k)$.
Then $\exists q \in \mathbb{N} : a_k = 2q$.
Let $q_0 \in \mathbb{N}$ such that $a_k = 2q_0$.
Then $a_{n+1} = 3a_{\lfloor \frac{n+1}{2} \rfloor} + 2 = 6q_0 + 2 = 2(3q_0 + 1)$.
Let $q_1 = 3q_0 + 1$.
Then $q_1 \in \mathbb{N}$ and $a_{n+1} = 2q_1$.
Then $\exists q \in \mathbb{N} : a_{n+1} = 2q$.
Then $E(a_{n+1})$.
Then $[\forall k \in \mathbb{N} : k \leq n \implies E(a_k)] \implies E(a_{n+1})$.
Then $\forall n \in \mathbb{N} : E(a_n)$.

Problem 2.

(6 MARKS)

Prove or disprove:

1. (3 MARKS) $\forall x \in \mathbb{R} : \forall n \in \mathbb{N} : \lfloor x + n \rfloor = \lfloor x \rfloor + n$
2. (3 MARKS) $\forall x \in \mathbb{R} : \forall n \in \mathbb{N} : \lfloor nx \rfloor = n \lfloor x \rfloor$

Solution

1. Let $x \in \mathbb{R}, n \in \mathbb{N}$.
Then $\lfloor x + n \rfloor \leq x + n$. # By definition of floor(x+n).
Then $\lfloor x + n \rfloor - n \leq x$.
Then $\lfloor x + n \rfloor - n \leq \lfloor x \rfloor$. # By definition of floor(x).
Then $\lfloor x + n \rfloor \leq \lfloor x \rfloor + n$. (*)
Also $\lfloor x \rfloor + n \leq x + n$. # By definition of floor(x).
Then $\lfloor x \rfloor + n \leq \lfloor x + n \rfloor$. # By definition of floor(x+n).
Then $\lfloor x + n \rfloor = \lfloor x \rfloor + n$. # Combine the above statement with (*).
Then $\forall x \in \mathbb{R} : \forall n \in \mathbb{N} : \lfloor x + n \rfloor = \lfloor x \rfloor + n$.
2. # We disprove it.
Let $n = 4, x = 0.5$.
Then $\lfloor nx \rfloor = \lfloor 2 \rfloor = 2 \neq 0 = 4 \cdot 0 = 4 \lfloor 0.5 \rfloor$.
Then $\exists x \in \mathbb{R}, \exists n \in \mathbb{N} : \lfloor nx \rfloor \neq n \lfloor x \rfloor$.

Problem 3.

(5 MARKS)

Prove the following claims:

1. (3 MARKS) $\forall x \in [0, \pi/2] : \sin x + \cos x \geq 1$.
2. (2 MARKS) Prove that $\log_2 3$ is irrational.

Solution

1. # Proof by contradiction.

Assume $\exists x \in [0, \pi/2] : \sin x + \cos x < 1$.

Then $\sin x \geq 0$. # For $x \in [0, \pi/2]$.

Also $\cos x \geq 0$. # For $x \in [0, \pi/2]$.

Then $0 \leq \sin x + \cos x < 1$.

Then $(\sin x + \cos x)^2 < 1$.

Then $\sin^2 x + \cos^2 x + 2 \sin x \cos x < 1$.

Then $1 + 2 \sin x \cos x < 1$. # By fundamental identity of trigonometry.

Then $\sin x \cos x < 0$.

Also $\sin x \cos x \geq 0$ # See above - so we have contradiction.

Then $\forall x \in [0, \pi/2] : \sin x + \cos x \geq 1$.

2. # Proof by contradiction.

Assume $\log_2 3$ is rational.

Then $\exists p, q \in \mathbb{Z} : q \neq 0 \wedge \log_2 3 = \frac{p}{q}$.

Let $p', q' \in \mathbb{Z} : q' \neq 0 \wedge \log_2 3 = \frac{p'}{q'}$.

Then $\frac{p'}{q'} > 0$ # Log of a number greater than the base is positive. Then $q' \log_2 3 = p'$.

Then $\log_2 3^{q'} = p'$.

Then $3^{q'} = 2^{p'}$. # By def. of logarithms. Also, if $p' < 0$, then also $q' < 0$.
so by cross multiplying the fraction, we get same result.

Then $p' > 0$.

Then $3^{q'}$ is odd # Product of odd numbers

Then $2^{p'}$ is even # Product of even numbers

Then $\exists k \in \mathbb{N} : 3^{q'} = 2k + 1$.

Let $k_0 \in \mathbb{N} : 3^{q'} = 2k_0 + 1$.

Then $2^{p'} = 2 \cdot 2^{p'-1}$.

Let $l_0 = 2^{p'-1}$.

Then $l_0 \in \mathbb{N}$.

Then $2l_0 = 2k_0 + 1$.

Then $l_0 - k_0 = \frac{1}{2}$.

Then $l_0 - k_0 \in \mathbb{Z}$. $\#$ Difference of two naturals is integer.

Also $l_0 - k_0 = \frac{1}{2} \notin \mathbb{Z}$ $\#$ $1/2$ is not integer - Contradiction.

Then $\log_2 3$ is irrational.

Then $\log_2 3$ is irrational.

Problem 4.

(6 MARKS) Prove the following:

1. (3 MARKS) $\forall x, y \in \mathbb{R} : x^2 + y^2 = (x + y)^2 \Leftrightarrow x = 0 \vee y = 0$.
2. (3 MARKS) $\forall x, y \in \mathbb{R} : x^3 + x^2y = y^2 + xy \Leftrightarrow y = x^2 \vee y = -x$.

Solution

1. # Prove \Rightarrow .

Let $x, y \in \mathbb{R}$.

Assume $x^2 + y^2 = (x + y)^2$.

Then $x^2 + y^2 = x^2 + 2xy + y^2$.

Then $2xy = 0$.

Then $x = 0 \vee y = 0$.

Then $x^2 + y^2 = (x + y)^2 \implies x = 0 \vee y = 0$.

Then $\forall x, y \in \mathbb{R} : x^2 + y^2 = (x + y)^2 \implies x = 0 \vee y = 0$.

#Prove \Leftarrow .

Let $x, y \in \mathbb{R}$.

Assume $x = 0 \vee y = 0$.

#Case 1: $x = 0$.

. Then $x^2 + y^2 = y^2$.

. Also $(x + y)^2 = y^2$.

. Then $x^2 + y^2 = (x + y)^2$.

#Case 2: $y = 0$.

. Then $x^2 + y^2 = x^2$.

. Also $(x + y)^2 = x^2$.

. Then $x^2 + y^2 = (x + y)^2$.

Then $x^2 + y^2 = (x + y)^2$.

Then $x = 0 \vee y = 0 \implies x^2 + y^2 = (x + y)^2$.

Then $\forall x, y \in \mathbb{R} : x = 0 \vee y = 0 \implies x^2 + y^2 = (x + y)^2$.

Then $\forall x, y \in \mathbb{R} : x^2 + y^2 = (x + y)^2 \Leftrightarrow x = 0 \vee y = 0$.

2. # Prove \Rightarrow .

Let $x, y \in \mathbb{R}$.

Assume $x^3 + x^2y = y^2 + xy$.

Then $x^3 - xy + x^2y - y^2 = 0$.

Then $x(x^2 - y) + y(x^2 - y) = 0$.

Then $(x^2 - y)(x + y) = 0$.

Then $x^2 - y = 0 \vee x + y = 0$.

Then $y = x^2 \vee y = -x$.

Then $x^3 + x^2y = y^2 + xy \implies y = x^2 \vee y = -x$.

Then $\forall x, y \in \mathbb{R} : x^3 + x^2y = y^2 + xy \implies y = x^2 \vee y = -x$.

#Prove \Leftarrow .

Let $x, y \in \mathbb{R}$.

Assume $y = x^2 \vee y = -x$.

#Case 1: $y = x^2$.

. Then $y - x^2 = 0$.

. Then $0 = 0 \cdot (x + y) = (y - x^2)(x + y)$.

. Then $x^3 + x^2y = y^2 + xy$. # After some algebra

#Case 2: $y = -x$.

. Then $x + y = 0$.

. Then $0 = 0 \cdot (y - x^2) = (x + y)(y - x^2)$.

. Then $x^3 + x^2y = y^2 + xy$. # After some algebra

Then $x^3 + x^2y = y^2 + xy$.

Then $y = x^2 \vee y = -x \implies x^3 + x^2y = y^2 + xy$.

Then $\forall x, y \in \mathbb{R} : y = x^2 \vee y = -x \implies x^3 + x^2y = y^2 + xy$.

Then $\forall x, y \in \mathbb{R} : x^3 + x^2y = y^2 + xy \Leftrightarrow y = x^2 \vee y = -x$.