1. Consider the Fibonacci-esque function g:

$$g(n) = \begin{cases} 1, & \text{if } n = 0 \\ 3, & \text{if } n = 1 \\ g(n-2) + g(n-1) & \text{if } n > 1 \end{cases}$$

Use complete induction to prove that if n is a natural number greater than 1, then $2^{n/2} \leq g(n) \leq 2^n$. You may **not** derive or use a closed-form for $g(n)$ in your proof.

Prove $1 + 2 + 3 + ... + n = \frac{n(n+1)}{2}$

Proof:

Let $g(n) = \begin{cases} 1, & \text{if } n = 0 \\ 3, & \text{if } n = 1 \\ g(n-2) + g(n-1) & \text{if } n > 1 \end{cases}$

Let $P(n)$: If $\{n \in \mathbb{N} \mid n > 1\}$, then $2^{\frac{n}{2}} \leq g(n) \leq 2^n$

Prove by complete induction:

Base Case:

1.

$n = 2$

then $g(2) = g(0) + g(1) = 1 + 3 = 4$

then $2^1 \leq 4 \leq 2^2$

then $P(2)$

2.

$n = 3$

then $g(3) = g(1) + g(2) = 3 + 4 = 7$

then $2^{\frac{3}{2}} \leq 7 \leq 2^3$

then $P(3)$

Inductive Step:

I.H: Assume $\forall k \in \mathbb{N} : 1 < k < n, P(k)$

Want To Prove: $P(n)$

(i) Prove $2^{\frac{n}{2}} \leq g(n)$

Assume $n > 3$ $\sharp$ $n \leq 3$ was discussed in base case

then $g(n) = g(n-2) + g(n-1)$

$\geq 2^{\frac{n-2}{2}} + 2^{\frac{n-1}{2}}$ $\sharp$ By I.H. $2^{\frac{n-2}{2}} \leq g(n-2)$,

$2^{\frac{n-1}{2}} \leq g(n-1)$

$= 2^{\frac{n}{2}-1} + 2^{\frac{n}{2}-\frac{1}{2}}$
$= 2^{\frac{n}{2}} \times 2^{-1} + 2^{\frac{n}{2}} \times 2^{-\frac{1}{2}}$
$= 2^{\frac{n}{2}} \times 2^{-\frac{3}{2}}$
then $g(n) \geq 2^{\frac{n}{2}}$ $\sharp$ $2^{-\frac{3}{2}} > 0$

(ii) Prove $g(n) \leq 2^n$
Assume $n > 3$ $\sharp$ $n \leq 3$ was discussed in base case
then $g(n) = g(n-2) + g(n-1)$
$\leq 2^{n-2} + 2^{n-1}$ $\sharp$ By I.H. $2^{n-2} \geq g(n-2)$,
$2^{n-1} \geq g(n-1)$
$= 2^n \times 2^{-2} + 2^n \times 2^{-1}$
$= 2^n \times 2^{-3}$
then $g(n) \leq 2^n$ $\sharp$ $2^{-3} > 0$

then $2^{\frac{n}{2}} \leq g(n) \leq 2^n \sharp$ by (i) and (ii)
then $P(n)$
Therefore If $\{n \in \mathbb{N} \mid n > 1\}$, then $2^{\frac{n}{2}} \leq g(n) \leq 2^n$

2. Suppose B is a set of binary strings where each binary string is of length n. n is positive (greater than 0), and no two strings in B differ in fewer than 2 positions. Use simple induction to prove that B has no more than 2^{n-1} elements.

   Proof:
   Let $|B|$ = number of elements in B that begin with "0" or "1"
   Let $P(n)$: If B is a set of binary strings of length n, then $|B| \leq 2^{n-1}$
   Prove by simple induction:

   Base Case:
   Let $n = 1$
   then B = $\{0\}$ or $\{1\}$
   then $|B| = 1 \leq 2^{1-1} = 1$
   then $P(1)$

   Inductive Step:
   I.H: Assume $\forall k \geq 1, P(k)$
   WTP: $P(k+1)$

   Let B be such a set of binary strings of length k+1
   then $|B|$ = number of elements in B that begin with "0" + number of elements in B that begin with "1"
   then $|B| \leq 2^{k-1}+2^{k-1}$ $\sharp$ By I.H. and no two strings in B differ in fewer than 2 positions,
   let b be such a set of binary strings of length k,
   then $|b| \leq 2^{k-1}$
   Because there exist at least one string begins with "0" or "1".
   So for b's elements, if we maximum the number of strings which begin with "0", this number must smaller than 2^{k-1}
   if we maximum the number of strings which begin with "1", this number must also smaller than 2^{k-1}
   then $|B| \leq 2 \times 2^{k-1}$
   then $|B| \leq 2^k$
   then P(k + 1)
   Therefore If B is a set of binary strings of length n, then $|B| \leq 2^{n-1}$

3. Define T as the smallest set of strings such that:

   (a) $"b" \in T$

   (b) If $t_1, t_2 \in T$, then $t_1 + "ene" + t_2 \in T$, where the $+$ operator is string concatenation.

   Use structural induction to prove that if $t \in T$ has n "b" characters, then t has $2n - 2$ "e" characters.

   Proof:
   For $t \in T$,
   Let $P(t)$: If $t \in T$ has n "b" characters, then t has $2n - 2$ "e" characters
   Prove by structural induction:

   Base Case:
   Let $t = \{b\} \in T$
   then $n = 1, 2n - 2 = 0$
   then $P(t)$

   Inductive Step:
   Let $t_1, t_2 \in T$ and $t = t_1 + "ene" + t_2$
   I.H: Assume $P(t_1)$ and $P(t_2)$
   WTP: $P(t)$

   Suppose t_1 has n_1 "b" characters, t_2 has n_2 "b" characters
   then t_1 has $2n_1 - 2$ "e" characters, t_2 has $2n_2 - 2$ "e" characters $\sharp$By I.H.

   then t has $(n_1 + n_2)$ "b" characters, $(2n_1 - 2) + (2n_2 - 2) + 2$ "e" characters
   then number of "e" characters in t $= (2n_1 - 2) + (2n_2 - 2) + 2$
   $= 2n_1 + 2n_2 - 2$
   $= 2(n_1 + n_2) - 2$
   Let $n = n_1 + n_2$
   then t has n "b" characters, $2n - 2$ "e" characters
   then $P(t)$
   Therefore if $t \in T$ has n "b" characters, then t has $2n - 2$ "e" characters

4. On page 79 of the Course Notes the quantity $\phi = (1+\sqrt{5})/2$ is shown to be closely related to the Fibonacci function. You may assume that $1.61803 < \phi < 1.61804$. Complete the steps below to show that ϕ is irrational.

   (a) Show that $\phi(\phi - 1) = 1$.
   $$\begin{aligned}\phi(\phi-1) &= \tfrac{1+\sqrt{5}}{2} \times (\tfrac{1+\sqrt{5}}{2} - 1)\\ &= \tfrac{1+\sqrt{5}}{2} \times \tfrac{-1+\sqrt{5}}{2}\\ &= \tfrac{(1+\sqrt{5})\times(-1+\sqrt{5})}{4}\\ &= \tfrac{4}{4}\\ &= 1\end{aligned}$$

   (b) Rewrite the equation in the previous step so that you have ϕ on the left-hand side, and on the right-hand side a fraction whose numerator and denominator are expressions that may only have integers, $+$ or $-$, and ϕ. There are two different fractions, corresponding to the two different factors in the original equation's left-hand side. Keep both fractions around for future consideration.
   Solution:
   Because $\phi(\phi - 1) = 1$

   Case 1:
   $\phi = \frac{1}{\phi-1}$

   Case 2:
   $\phi - 1 = \frac{1}{\phi}$
   then $\phi = \frac{1}{\phi} + 1 = \frac{1+\phi}{\phi}$

   (c) Assume, for a moment, that there are natural numbers m and n such that $\phi = n/m$. Re-write the right-hand side of both equations in the previous step so that you end up with fractions whose numerators and denominators are expressions that may only have integers, $+$ or $-$, m and n.
   Solution:
   Because $\phi = \frac{n}{m}$

   Case 1:
   $\phi = \frac{1}{\phi-1} = \frac{n}{m}$
   then $n\phi - n = m$
   then $n\phi = m + n$
   then $\phi = \frac{m+n}{n}$

   Case 2:
   $\phi = \frac{1+\phi}{\phi} = \frac{n}{m}$

then $n\phi = m(1+\phi)$
then $n\phi = m + m\phi$
then $(n-m)\phi = m$
then $\phi = \frac{m}{n-m}$

(d) Use your assumption from the previous part to construct a non-empty subset of the natural numbers that contains m. Use the Principle of Well-Ordering, plus one of the two expressions for ϕ from the previous step to derive a contradiction.

Proof:
Prove by contradiction:

Assume ϕ is rational
Let $S = \{m \in \mathbb{N} \mid \exists n \in \mathbb{N}, \frac{n}{m} = \phi\}$, S is a subset of natural number
Because S has at least one element,
By well-ordering, $\exists m_0$ as the smallest but not 0 natural number in S

then $\exists n_0 \in \mathbb{N}$ such that $\frac{n_0}{m_0} = \phi$ $\sharp$ By assumption, ϕ is rational
then $\frac{n_0}{m_0} = \frac{m_0}{n_0 - m_0} = \phi$ $\sharp$ By (c)
Because $\phi = \frac{n_0}{m_0} < 1.61804$ $\sharp$ $1.61803 < \phi < 1.61804$
then $n_0 < 1.61804m_0$
then $n_0 - m_0 < 0.61804m_0$
then $n_0 - m_0 < m_0$ $\sharp$ Contradiction!
Because m_0 is the smallest but not 0 natural number in S, and $(n_0 - m_0) \in \mathbb{N}$. $(n_0 - m_0)$ smaller then the smallest element m_0, which leads a contradiction.

Therefore ϕ is irrational

(e) Combine your assumption and contradiction from the previous step into a proof that ϕ cannot be the ratio of two natural numbers. Extend this to a proof that ϕ is irrational.

5. Consider the function f, where $3 \div 2 = 1$ (integer division, like 3//2 in Python):

$$f(n) = \begin{cases} 1 & \text{if } n = 0 \\ f^2(n \div 3) + 3f(n \div 3) & \text{if } n > 0 \end{cases}$$

Use complete induction to prove that for every natural number n greater than 2, $f(n)$ is a multiple of 7. **NB:** Think carefully about which natural numbers you are justified in using the inductive hypothesis for.

Before proof:
$f(1) = f^2(1 \div 3) + 3f(1 \div 3) = f^2(0) + 3f(0) = 1^2 + 3 \times 1 = 4$
$f(2) = f^2(2 \div 3) + 3f(2 \div 3) = f^2(0) + 3f(0) = 1^2 + 3 \times 1 = 4$

Proof:
Let $P(n)$: If $\{n \in \mathbb{N} \mid n > 2\}$, then $\exists x \in \mathbb{N},\ f(n) = 7x$
Prove by complete induction:

Base Case: Let $n \in \mathbb{N},\ 2 < n \leq 8$
$f(3) = f^2(3 \div 3) + 3f(3 \div 3) = f^2(1) + 3f(1) = 4^2 + 3 \times 4 = 28$
$f(4) = f^2(4 \div 3) + 3f(4 \div 3) = f^2(1) + 3f(1) = 4^2 + 3 \times 4 = 28$
$f(5) = f^2(5 \div 3) + 3f(5 \div 3) = f^2(1) + 3f(1) = 4^2 + 3 \times 4 = 28$
$f(6) = f^2(6 \div 3) + 3f(6 \div 3) = f^2(2) + 3f(2) = 4^2 + 3 \times 4 = 28$
$f(7) = f^2(7 \div 3) + 3f(7 \div 3) = f^2(2) + 3f(2) = 4^2 + 3 \times 4 = 28$
$f(8) = f^2(8 \div 3) + 3f(8 \div 3) = f^2(2) + 3f(2) = 4^2 + 3 \times 4 = 28$
then $\forall\{n \in \mathbb{N} \mid n \in (2, 8]\}, P(n)$ holds

Inductive Step: Let $n, i \in \mathbb{N}$
I.H. Assume $2 < i < n$, $P(i)$ holds
WTP: $P(n)$

Case 1:
Let $2 < n \leq 8$, $P(n)$ holds $\sharp$ Already shown in Base Case

Case 2:
Let $n \geq 9$, $i = n \div 3$
then $\exists x \in \mathbb{N}, f(i) = 7x$ $\sharp$ By I.H. $P(i)$ holds
then $f(n) = f^2(n \div 3) + 3f(n \div 3)$
$= f^2(i) + 3f(i)$
$= (7x)^2 + 3 \times 7x$
$= 7(7x^2 + 3x)$
Let x' $= (7x^2 + 3x)$ $\sharp$ $x \in \mathbb{N}, x' \in \mathbb{N}$
then $f(n) = 7x'$
then $\exists x \in \mathbb{N}, f(n) = 7x$

then $P(n)$ $\sharp$ By Case1 and Case2
Therefore for every natural number n greater than 2, $f(n)$ is a multiple of 7.