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This submission has been prepared using LaTeX.

1. Consider the Fibonacci-esque function g:

$$g(n) = \begin{cases} 1, & \text{if } n = 0 \\ 3, & \text{if } n = 1 \\ g(n-2) + g(n-1) & \text{if } n > 1 \end{cases}$$

Use complete induction to prove that if n is a natural number greater than 1, then $2^{n/2} \leq g(n) \leq 2^n$. You may **not** derive or use a closed-form for $g(n)$ in your proof.

Proof:

Let $g(n) = \begin{cases} 1, & \text{if } n = 0 \\ 3, & \text{if } n = 1 \\ g(n-2) + g(n-1) & \text{if } n > 1 \end{cases}$

Let $P(n) : 2^{\frac{n}{2}} \leq g(n) \leq 2^n$

Prove by complete induction:

Base Case:

1.

$n = 2$

then $g(2) = g(0) + g(1) = 1 + 3 = 4$

then $2^1 \leq 4 \leq 2^2$

then $P(2)$

2.

$n = 3$

then $g(3) = g(1) + g(2) = 3 + 4 = 7$

then $2^{\frac{3}{2}} \leq 7 \leq 2^3$

then $P(3)$

Inductive Step:

I.H: Assume $\forall k \in \mathbb{N} : 1 < k < n, P(k)$

Want To Prove: $P(n)$

(i) Prove $2^{\frac{n}{2}} \leq g(n)$

Assume $n > 3$ $\sharp$ $n \leq 3$ was discussed in base case

then $g(n) = g(n-2) + g(n-1)$

$\geq 2^{\frac{n-2}{2}} + 2^{\frac{n-1}{2}}$ $\sharp$ By I.H. $2^{\frac{n-2}{2}} \leq g(n-2)$, $2^{\frac{n-1}{2}} \leq g(n-1)$

$= 2^{\frac{n}{2}-1} + 2^{\frac{n}{2}-\frac{1}{2}}$

$= 2^{\frac{n}{2}} \times 2^{-1} + 2^{\frac{n}{2}} \times 2^{-\frac{1}{2}}$

$= 2^{\frac{n}{2}} \times 2^{-\frac{3}{2}}$
then $g(n) \geq 2^{\frac{n}{2}}$ $\sharp$ $2^{-\frac{3}{2}} \geq 1$

(ii) Prove $g(n) \leq 2^n$
Assume $n > 3$ $\sharp$ $n \leq 3$ was discussed in base case
then $g(n) = g(n-2) + g(n-1)$
$\leq 2^{n-2} + 2^{n-1}$ $\sharp$ By I.H. $2^{n-2} \geq g(n-2)$,
$2^{n-1} \geq g(n-1)$
$= 2^n \times 2^{-2} + 2^n \times 2^{-1}$
$= 2^n \times 2^{-3}$
then $g(n) \leq 2^n$ $\sharp$ $0 < 2^{-3} \leq 1$

then $2^{\frac{n}{2}} \leq g(n) \leq 2^n$ $\sharp$ by (i) and (ii)
then $P(n)$
Therefore If $n > 1$, then $2^{\frac{n}{2}} \leq g(n) \leq 2^n$

2. Suppose B is a set of binary strings where each binary string is of length n. n is positive (greater than 0), and no two strings in B differ in fewer than 2 positions. Use simple induction to prove that B has no more than 2^{n-1} elements.

Proof:
Let $|B|$ = number of elements in B that begin with "0" or "1"
Let $P(n)$: If B is a set of binary strings of length n, n is positive, and no two strings in B differ in fewer than 2 positions, then $|B| \leq 2^{n-1}$
Prove by simple induction:

Base Case:
Let $n = 1$
then B = $\{0\}$ or $\{1\}$
then $|B| = 1 \leq 2^{1-1} = 1$
then $P(1)$

Inductive Step:
I.H: Assume $\forall k \geq 1, P(k)$
WTP: $P(k+1)$

Let B be such a set of binary strings of length k+1
then $|B|$ = number of elements in B that begin with "0"
+ number of elements in B that begin with "1"
then $|B| \leq 2^{k-1}+2^{k-1}$ $\sharp$ Because no two strings in B differ in fewer than 2 positions,
let $B_0 = \{S \mid (\text{"0"} + S) \in B\}$
then $|B_0|$ is number of elements in B begins with "0"
then the length of elements of B_0 is k
then no two strings in B_0 differ in fewer than 2 positions
By I.H. $|B_0| \leq 2^{k-1}$
let $B_1 = \{S \mid (\text{"1"} + S) \in B\}$
then $|B_1|$ is number of elements in B begins with "1"
then the length of elements of B_1 is k
then no two strings in B_1 differ in fewer than 2 positions
By I.H. $|B_1| \leq 2^{k-1}$
then $|B| \leq 2 \times 2^{k-1}$
then $|B| \leq 2^k$
then P(k + 1)
Therefore If B is a set of binary strings of length n, then $|B| \leq 2^{n-1}$

3. Define T as the smallest set of strings such that:

   (a) $"b" \in T$

   (b) If $t_1, t_2 \in T$, then $t_1 + "ene" + t_2 \in T$, where the + operator is string concatenation.

   Use structural induction to prove that if $t \in T$ has n "b" characters, then t has $2n - 2$ "e" characters.

   Proof:
   For $t \in T$,
   Let $P(t)$: If $t \in T$ has n "b" characters, then t has $2n - 2$ "e" characters
   Prove by structural induction:

   Base Case:
   Let $t = "b" \in T$
   then $n = 1, 2n - 2 = 0$
   then $P(t)$

   Inductive Step:
   Let $t_1, t_2 \in T$ and $t = t_1 + "ene" + t_2$
   I.H: Assume $P(t_1)$ and $P(t_2)$
   WTP: $P(t)$

   Suppose t_1 has n_1 "b" characters, t_2 has n_2 "b" characters
   then t_1 has $2n_1 - 2$ "e" characters, t_2 has $2n_2 - 2$ "e" characters ♯By I.H.

   then t has $(n_1 + n_2)$ "b" characters, $(2n_1 - 2) + (2n_2 - 2) + 2$ "e" characters
   then number of "e" characters in t $= (2n_1 - 2) + (2n_2 - 2) + 2$
   $= 2n_1 + 2n_2 - 2$
   $= 2(n_1 + n_2) - 2$
   Let $n = n_1 + n_2$
   then t has n "b" characters, $2n - 2$ "e" characters
   then $P(t)$
   Therefore if $t \in T$ has n "b" characters, then t has $2n - 2$ "e" characters

4. On page 79 of the Course Notes the quantity $\phi = (1+\sqrt{5})/2$ is shown to be closely related to the Fibonacci function. You may assume that $1.61803 < \phi < 1.61804$. Complete the steps below to show that ϕ is irrational.

   (a) Show that $\phi(\phi - 1) = 1$.

   $\phi(\phi-1) = \frac{1+\sqrt{5}}{2} \times (\frac{1+\sqrt{5}}{2} - 1)$
   $= \frac{1+\sqrt{5}}{2} \times \frac{-1+\sqrt{5}}{2}$
   $= \frac{(1+\sqrt{5})\times(-1+\sqrt{5})}{4}$
   $= \frac{4}{4}$
   $= 1$

   (b) Rewrite the equation in the previous step so that you have ϕ on the left-hand side, and on the right-hand side a fraction whose numerator and denominator are expressions that may only have integers, + or −, and ϕ. There are two different fractions, corresponding to the two different factors in the original equation's left-hand side. Keep both fractions around for future consideration.

   Solution:
   Because $\phi(\phi - 1) = 1$

   Case 1:
   $\phi = \frac{1}{\phi-1}$

   Case 2:
   $\phi - 1 = \frac{1}{\phi}$
   then $\phi = \frac{1}{\phi} + 1 = \frac{1+\phi}{\phi}$

   (c) Assume, for a moment, that there are natural numbers m and n such that $\phi = n/m$. Re-write the right-hand side of both equations in the previous step so that you end up with fractions whose numerators and denominators are expressions that may only have integers, + or −, m and n.

   Solution:
   Because $\phi = \frac{n}{m}$

   Case 1:
   $\phi = \frac{1}{\phi-1} = \frac{n}{m}$
   then $n\phi - n = m$
   then $n\phi = m + n$
   then $\phi = \frac{m+n}{n}$

   Case 2:
   $\phi = \frac{1+\phi}{\phi} = \frac{n}{m}$

then $n\phi = m(1 + \phi)$
then $n\phi = m + m\phi$
then $(n - m)\phi = m$
then $\phi = \frac{m}{n-m}$

(d) Use your assumption from the previous part to construct a non-empty subset of the natural numbers that contains m. Use the Principle of Well-Ordering, plus one of the two expressions for ϕ from the previous step to derive a contradiction.

Proof:
Prove by contradiction:

Let $S = \{m \in \mathbb{N} \mid \exists n \in \mathbb{N}, \frac{n}{m} = \phi\}$, S is a subset of natural number
Because S has at least one element,
By well-ordering, $\exists m_0$ as the smallest but not 0 natural number in S

then $\exists n_0 \in \mathbb{N}$ such that $\frac{n_0}{m_0} = \phi$ $\sharp$ By Definition of S
then $\frac{n_0}{m_0} = \frac{m_0}{n_0 - m_0} = \phi$ $\sharp$ By (c)
then $(n_0 - m_0) > 0$ and $(n_0 - m_0) \in S$ $\sharp$ $m_0 > 0, \phi > 0$
Because $\phi = \frac{n_0}{m_0} < 1.61804$ $\sharp$ $1.61803 < \phi < 1.61804$
then $n_0 < 1.61804 m_0$
then $n_0 - m_0 < 0.61804 m_0$
then $n_0 - m_0 < m_0$ $\sharp$ Contradiction!
Because m_0 is the smallest but not 0 natural number in S, and $(n_0 - m_0) \in \mathbb{N}$. $(n_0 - m_0)$ smaller then the smallest element m_0, which leads a contradiction.

(e) Combine your assumption and contradiction from the previous step into a proof that ϕ cannot be the ratio of two natural numbers. Extend this to a proof that ϕ is irrational.

Assume ϕ is rational
then $\exists m, n \in \mathbb{N} : \phi = \frac{n}{m}$
But Part(d), we reach a contradiction given the assumption that $\phi = \frac{n}{m}$
Therefore ϕ is irrational

5. Consider the function f, where $3 \div 2 = 1$ (integer division, like 3//2 in Python):

$$f(n) = \begin{cases} 1 & \text{if } n = 0 \\ f^2(n \div 3) + 3f(n \div 3) & \text{if } n > 0 \end{cases}$$

Use complete induction to prove that for every natural number n greater than 2, $f(n)$ is a multiple of 7. **NB:** Think carefully about which natural numbers you are justified in using the inductive hypothesis for.

Proof:
$f(1) = f^2(1 \div 3) + 3f(1 \div 3) = f^2(0) + 3f(0) = 1^2 + 3 \times 1 = 4$
$f(2) = f^2(2 \div 3) + 3f(2 \div 3) = f^2(0) + 3f(0) = 1^2 + 3 \times 1 = 4$
Let $P(n)$: then $\exists x \in \mathbb{N},\ f(n) = 7x$
Prove by complete induction:

Base Case: Let $n \in \mathbb{N},\ 2 < n \leq 8$
$f(3) = f^2(3 \div 3) + 3f(3 \div 3) = f^2(1) + 3f(1) = 4^2 + 3 \times 4 = 28$
$f(4) = f^2(4 \div 3) + 3f(4 \div 3) = f^2(1) + 3f(1) = 4^2 + 3 \times 4 = 28$
$f(5) = f^2(5 \div 3) + 3f(5 \div 3) = f^2(1) + 3f(1) = 4^2 + 3 \times 4 = 28$
$f(6) = f^2(6 \div 3) + 3f(6 \div 3) = f^2(2) + 3f(2) = 4^2 + 3 \times 4 = 28$
$f(7) = f^2(7 \div 3) + 3f(7 \div 3) = f^2(2) + 3f(2) = 4^2 + 3 \times 4 = 28$
$f(8) = f^2(8 \div 3) + 3f(8 \div 3) = f^2(2) + 3f(2) = 4^2 + 3 \times 4 = 28$
then $P(n)$ holds for $2 < n \leq 8$

Inductive Step: Let $n, i \in \mathbb{N}$
I.H. Assume $2 < i < n$, $P(i)$ holds
WTP: $P(n)$

Case 1:
Let $2 < n \leq 8$, $P(n)$ holds $\sharp$ Already shown in Base Case

Case 2:
Let $n \geq 9$, $i = n \div 3$
then $2 < i < n$
then $\exists x \in \mathbb{N}, f(i) = 7x$ $\sharp$ By I.H. $P(i)$ holds
then $f(n) = f^2(n \div 3) + 3f(n \div 3)$
$= f^2(i) + 3f(i)$
$= (7x)^2 + 3 \times 7x$
$= 7(7x^2 + 3x)$
Let x' $= (7x^2 + 3x)$ $\sharp$ $x \in \mathbb{N}, x' \in \mathbb{N}$
then $f(n) = 7x'$
then $\exists x \in \mathbb{N}, f(n) = 7x$

then $P(n)$ $\sharp$ By Case1 and Case2
Therefore for every natural number n greater than 2, $f(n)$is a multiple of 7.