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We declare that this assignment is solely our own work, and is in accordance with the University of Toronto Code of Behaviour on Academic Matters.

This submission has been prepared using LaTeX.

Problem 1.

(6 Marks)
Consider the following Python code:

```
def mystery(L):
    '''
    :param L: List of size n
    :return: A mystery number
    '''
    sum1 = 0
    sum2 = 0
    bound = 1
    while bound <= len(L):
        i = 0
        while i < bound:
            j = 0
            while j < len(L):
                if L[j] > L[i]:
                    sum1 = sum1 + L[j]
                j = j + 2
            j = 1
            while j < len(L):
                sum2 = sum2 + L[j]
                j = j*2
            i = i + 1
        bound = bound * 2
    return sum1 + sum2
```

1. (3 Marks) Denote the time complexity of the given code $T(n)$ as a function of n where n is the size of the list L. Compute $T(n)$. Justify all steps.
2. (3 Marks) Prove that $T(n) \in O(n^{\frac{5}{2}})$.
   Hint: You can use without proof the following: $\forall \alpha \in \mathbb{R}^+ : \log_2 n \in O(n^\alpha)$.

Solution

1. Let $A1$ represent the times of loop of:

```
while j < len(L):
        if L[j] > L[i]:
            sum1 = sum1 + L[j]
        j = j + 2
```

So $A1 = 4\lceil \frac{n}{2} \rceil + 1$

Let $A2$ represent the times of loop of:

```
while j < len(L):
        sum2 = sum2 + L[j]
        j = j*2
```

So $A2 = 3\lceil \log_2 n \rceil + 1$

$T(n) = \sum_{k=1}^{\lfloor \log_2 n \rfloor + 1} (\sum_{i=1}^{2^{k-1}} (4 + A1 + A2) + 4) + 5$
$= \sum_{k=1}^{\lfloor \log_2 n \rfloor + 1} (\sum_{i=1}^{2^{k-1}} (4 + 4\lceil \frac{n}{2} \rceil + 1 + 3\lceil \log_2 n \rceil + 1) + 4) + 5$
$= \sum_{k=1}^{\lfloor \log_2 n \rfloor + 1} (2^{k-1}(6 + 4\lceil \frac{n}{2} \rceil + 3\lceil \log_2 n \rceil) + 4) + 5$
$= (6 + 4\lceil \frac{n}{2} \rceil + 3\lceil \log_2 n \rceil) \sum_{k=1}^{\lfloor \log_2 n \rfloor + 1} 2^{k-1} + \sum_{k=1}^{\lfloor \log_2 n \rfloor + 1} 4 + 5$
$= (6 + 4\lceil \frac{n}{2} \rceil + 3\lceil \log_2 n \rceil)(2^{\lfloor \log_2 n \rfloor + 1} - 1) \ + 4\lfloor log_2 n \rfloor + 9$

2. Prove $T(n) \in O(n^{\frac{5}{2}})$
   Prove $\exists C \in \mathbb{R}^+ : [\exists B \in \mathbb{N} : [\forall n \in \mathbb{N} : (n \geq B) \rightarrow T(n) \leq Cn^{\frac{5}{2}}]]$
   Let $C = 49$
   Then $C \in \mathbb{R}^+$
   Let $B = 1$
   Then $B \in \mathbb{N}$
   Let $n \in \mathbb{N}$
   Assume $n \geq B$
   Then $n \geq 1$
   Then $49n^{\frac{5}{2}} = (6 + 4 + 4 + 26 + 9)n^{\frac{5}{2}}$
   $= 6n^{\frac{5}{2}} + 4n^{\frac{5}{2}} + 4n^{\frac{5}{2}} + 26n^{\frac{5}{2}} + 9n^{\frac{5}{2}}$

   $\geq 6n^{\frac{5}{2}} + 4n^2 + 4n^{\frac{3}{2}} + 26n + 9$
   $\sharp\ n \geq 1, n^{\frac{5}{2}} > n^2, n^{\frac{5}{2}} > n^{\frac{3}{2}}, n^{\frac{5}{2}} > n, n^{\frac{5}{2}} > 1$

$= 4n^2 + 6n \times n^{\frac{3}{2}} + 26n + 4n^{\frac{3}{2}} + 9$

$\geq 4n^2 + 6n\log_2 n + 26n + 4\log_2 n + 9$
$\sharp$ By hint:$\forall \alpha \in \mathbb{R}^+ : \log_2 n \in O(n^\alpha)$ when C=1, B=1, $\alpha = \frac{3}{2},\ \log_2 n \leq n^{\frac{3}{2}}$

$= (2n + 3\log_2 n + 13)2n + 4\log_2 n + 9$
$= (2n + 4 + 3\log_2 n + 3 + 6)2n + 4\log_2 n + 9$
$= (4(\frac{n}{2} + 1) + 3(\log_2 n + 1) + 6)2n + 4\log_2 n + 9$

$\geq (4\lceil \frac{n}{2} \rceil + 3\lceil \log_2 n \rceil + 6)2n + 4\log_2 n + 9$
$\sharp$ By definition of $\lceil \frac{n}{2} \rceil$ and $\lceil \log_2 n \rceil$

$= (4\lceil \frac{n}{2} \rceil + 3\lceil \log_2 n \rceil + 6)2^{\log_2 n + 1} + 4\log_2 n + 9$
$\sharp\ 2^{\log_2 n} = n, 2^{\log_2 n + 1} = 2^{\log_2 n} \times 2 = n \times 2 = 2n$

$\geq (4\lceil \frac{n}{2} \rceil + 3\lceil \log_2 n \rceil + 6)2^{\lfloor \log_2 n \rfloor + 1} + 4\lfloor \log_2 n \rfloor + 9$
$\sharp$ By the definition of $\lfloor \log_2 n \rfloor$

$\geq (4\lceil \frac{n}{2} \rceil + 3\lceil \log_2 n \rceil + 6)(2^{\lfloor \log_2 n \rfloor + 1} - 1) + 4\lfloor \log_2 n \rfloor + 9$
$\sharp\ 2^{\lfloor \log_2 n \rfloor + 1} > 2^{\lfloor \log_2 n \rfloor + 1} - 1$

$= T(n)$

Therefore $49n^{\frac{5}{2}} \geq T(n)$

Therefore $\forall n \in \mathbb{N} : (n \geq 1) \rightarrow T(n) \leq 49n^{\frac{5}{2}}$

Therefore $\exists B \in \mathbb{N} : [\forall n \in \mathbb{N} : (n \geq B) \rightarrow T(n) \leq 49n^{\frac{5}{2}}]$

Therefore $\exists C \in \mathbb{R}^+ : [\exists B \in \mathbb{N} : [\forall n \in \mathbb{N} : (n \geq B) \rightarrow T(n) \leq Cn^{\frac{5}{2}}]]$

Therefore $T(n) \in O(n^{\frac{5}{2}})$

Problem 2.

(6 Marks) Using the appropriate definitions, prove the following:

1. (3 Marks)
$$7n^2 + 77n + 1 \in \Theta(n^2 + n + 165).$$

2. (3 Marks)
$$n\log(n^7) + n^{\frac{7}{2}} \in O(n^{\frac{7}{2}}).$$

Solution

1. Prove $7n^2 + 77n + 1 \in \Theta(n^2 + n + 165)$
   Prove $7n^2 + 77n + 1 \in O(n^2 + n + 165) \wedge 7n^2 + 77n + 1 \in \Omega(n^2 + n + 165)$

   First, prove $7n^2 + 77n + 1 \in O(n^2 + n + 165)$
   prove $\exists C \in \mathbb{R}^+ : [\exists B \in \mathbb{N} : [\forall n \in \mathbb{N} : (n \geq B) \rightarrow 7n^2 + 77n + 1 \leq C(n^2 + N + 165)]]$
   Let $C = 77$
   Then $C \in \mathbb{R}^+$
   Let $B = 1$
   Then $B \in \mathbb{N}$
   Let $n \in \mathbb{N}$
   Assume $n \geq B$
   Then $n \geq 1$

   Then $n^2 \geq n$
   $\sharp$ multiple n in both sides

   Then $77n^2 > 7n^2$
   $\sharp$ $n \geq 1$

   Then $77n^2 + 77n > 7n^2 + 77n$
   $\sharp$ add 77n in both sides

   Then $77n^2 + 77n + 1 > 7n^2 + 77n + 1$
   $\sharp$ add 1 in both sides

Then $77n^2 + 77n + 165 \times 77 > 7n^2 + 77n + 1$
$\sharp$ $165 \times 7 > 1$

Then $77(n^2 + n + 165) > 7n^2 + 77n + 1$
Then $7n^2 + 77n + 1 < 77(n^2 + n + 165)$
Therefore $\forall n \in \mathbb{N} : (n \geq 1) \to 7n^2 + 77n + 1 \leq 77(n^2 + n + 165)$
Therefore $\exists B \in \mathbb{N} : [\forall n \in \mathbb{N} : (n \geq B) \to 7n^2 + 77n + 1 \leq 77(n^2 + n + 165)$
Therefore $\exists C \in \mathbb{R}^+ : [\exists B \in \mathbb{N} : [\forall n \in \mathbb{N} : (n \geq B) \to 7n^2 + 77n + 1 \leq C(n^2 + n + 165)]]$
Therefore $7n^2 + 77n + 1 \in O(n^2 + n + 165)$

Second, prove $7n^2 + 77n + 1 \in \Omega(n^2 + n + 165)$
prove $\exists C \in \mathbb{R}^+ : [\exists B \in \mathbb{N} : [\forall n \in \mathbb{N} : (n \geq B) \to 7n^2 + 77n + 1 \geq C(n^2 + n + 165)]]$
Let $c = \frac{1}{165}$
Then $c \in \mathbb{R}^+$
Let $B = 1$
Then $B \in \mathbb{N}$
Let $n \in \mathbb{N}$
Assume $n \geq B$
Then $n \geq 1$

Then $7n^2 \geq \frac{1}{165}n^2$
$\sharp$ $7 > \frac{1}{165}$ and multiple n^2 in both sides

Then $77n \geq \frac{1}{165}n$
$\sharp$ $77 > \frac{1}{165}$ and multiple n in both sides

Then $7n^2 + 77n \geq \frac{1}{165}n^2 + \frac{1}{165}n$

Then $7n^2 + 77n + 1 \geq \frac{1}{165}n^2 + \frac{1}{165}n + 1$
$\sharp$ Add 1 in both sides

Then $7n^2 + 77n + 1 \geq \frac{1}{165}(n^2 + n + 165)$
Therefore $\forall n \in \mathbb{N} : (n \geq 1) \to 7n^2 + 77n + 1 \geq \frac{1}{165}(n^2 + n + 165)$
Therefore $\exists B \in \mathbb{N} : [\forall n \in \mathbb{N} : (n \geq B) \to 7n^2 + 77n + 1 \geq \frac{1}{165}(n^2 + n + 165)]$

Therefore $\exists C \in \mathbb{R}^+ : [\exists B \in \mathbb{N} : [\forall n \in \mathbb{N} : (n \geq B) \rightarrow 7n^2 + 77n + 1 \geq C(n^2 + n + 165)]]$
Therefore $7n^2 + 77n + 1 \in \Omega(n^2 + n + 165)$

Therefore $7n^2 + 77n + 1 \in O(n^2 + n + 165) \wedge 7n^2 + 77n + 1 \in \Omega(n^2 + n + 165)$
Therefore $7n^2 + 77n + 1 \in \Theta(n^2 + n + 165)$

2. Prove $n \log(n^7) + n^{\frac{7}{2}} \in O(n^{\frac{7}{2}})$
Prove $\exists C \in \mathbb{R}^+ : [\exists B \in \mathbb{N} : [\forall n \in \mathbb{N} : n \geq B \rightarrow n \log(n^7) + n^{\frac{7}{2}} \leq Cn^{\frac{7}{2}}]]$
Let $C = 8$
Then $C \in \mathbb{R}^+$
Let $B = 1$
Then $B \in \mathbb{N}$
Let $n \in \mathbb{N}$
Assume $n \geq B$
Then $n \geq 1$

Then $\log n \leq n^{\frac{5}{2}}$
$\sharp$ By hint:$\forall \alpha \in \mathbb{R}^+ : \log_2 n \in O(n^\alpha)$ when C=1, B=1, $\alpha = \frac{5}{2}$, $\log_2 n \leq n^{\frac{5}{2}}$

Then $\log n \leq \frac{7}{7} n^{\frac{5}{2}}$
Then $\log n \leq \frac{7+1-1}{7} n^{\frac{5}{2}}$
Then $7 \log n \leq (7 + 1 - 1)n^{\frac{5}{2}}$
Then $7n \log n \leq (7 + 1 - 1)n^{\frac{7}{2}}$ $\sharp$ Multiply n in both sides
Then $7n \log n \leq (7 + 1)n^{\frac{7}{2}} - n^{\frac{7}{2}}$
Then $7n \log n + n^{\frac{7}{2}} \leq 8n^{\frac{7}{2}}$
Then $n \log(n^7) + n^{\frac{7}{2}} \leq 8n^{\frac{7}{2}}$ $\sharp$ Definition of log
Therefore $\forall n \in \mathbb{N} : n \geq 1 \rightarrow n \log(n^7) + n^{\frac{7}{2}} \leq 8n^{\frac{7}{2}}$
Therefore $\exists B \in \mathbb{N} : [\forall n \in \mathbb{N} : n \geq B \rightarrow n \log(n^7) + n^{\frac{7}{2}} \leq 8n^{\frac{7}{2}}]$
Therefore $\exists C \in \mathbb{R}^+ : [\exists B \in \mathbb{N} : [\forall n \in \mathbb{N} : n \geq B \rightarrow n \log(n^7) + n^{\frac{7}{2}} \leq Cn^{\frac{7}{2}}]]$
Therefore $n \log(n^7) + n^{\frac{7}{2}} \in O(n^{\frac{7}{2}})$

Problem 3.

(6 Marks) Let $\mathcal{F} = \{f | f : \mathbb{N} \to \mathbb{R}^+\}$. Using the appropriate definitions, prove or disprove the following:

1. (3 Marks)

$$\forall f, g \in \mathcal{F} : \log f(n) \in O(g(n)) \implies f(n) \in O(3^{g(n)}).$$

2. (3 Marks)

$$\forall f \in \mathcal{F} : \lfloor \sqrt{\lfloor f(n) \rfloor} \rfloor \in O(\sqrt{f(n)}).$$

Solution

1. (disprove)
   Prove by negation
   Prove $\neg(\forall f, g \in \mathcal{F} : \log f(n) \in O(g(n)) \implies f(n) \in O(3^{g(n)}))$
   Prove $\exists f, g \in \mathcal{F} : \log f(n) \in O(g(n)) \wedge f(n) \notin O(3^{g(n)})$

   Let $f(n) = 3^n, g(n) = \frac{n}{2}$
   Then $f \in \mathcal{F} \wedge g \in \mathcal{F}$
   Then $\log f(n) = \log 3^n$
   Then $\log f(n) = nlog3$

   First, prove $\log f(n) \in O(g(n))$
   Prove $\exists C \in \mathbb{R}^+ : [\exists B \in \mathbb{N} : [\forall n \in \mathbb{N} : (n \geq B) \implies nlog3 \leq C \cdot \frac{n}{2}]]$
   Let $C = 4\log 3$
   Then $C \in \mathbb{R}^+$
      Let $B = 1$
      Then $B \in \mathbb{N}$
         Let $n \in \mathbb{N}$
         Assume $n \geq B$
         Then $n \geq 1$
         Then $2n > n$
         Then $\frac{4n}{2} \geq n$
         Then $4 \cdot \frac{n}{2} \cdot \log 3 \geq n \cdot \log 3$ ♯ Multiply log3 in both sides
         Then $4\log 3 \cdot \frac{n}{2} \geq n \log 3$
         Therefore $(n \geq 1) \implies n\log 3 \leq 4log3 \cdot \frac{n}{2}$
         Therefore $\forall n \in \mathbb{N} : (n \geq 1) \implies n\log 3 \leq 4log3 \cdot \frac{n}{2}$

Therefore $\exists B \in \mathbb{N} : [\forall n \in \mathbb{N} : (n \geq B) \implies n \log 3 \leq 4log3 \cdot \frac{n}{2}]$
Therefore $\exists C \in \mathbb{R}^+ : [\exists B \in \mathbb{N} : [\forall n \in \mathbb{N} : (n \geq B) \implies n \log 3 \leq C \cdot \frac{n}{2}]]$
Therefore $n \log 3 \in O(\frac{n}{2})$
Therefore $\log f(n) \in O(g(n))$

Second, prove $f(n) \notin O(3^{g(n)})$
Prove $\neg[\exists C \in \mathbb{R}^+ : [\exists B \in \mathbb{N} : [\forall n \in \mathbb{N} : (n \geq B) \implies (3^n \leq C(3^{\frac{n}{2}}))]]]$
Prove $\forall C \in \mathbb{R}^+ : [\forall B \in \mathbb{N} : [\exists n \in \mathbb{N} : (n \geq B) \wedge (3^n > C(3^{\frac{n}{2}}))]]$
Let $C \in \mathbb{R}^+$
Let $B \in \mathbb{N}$
Let $n = max(B, \lceil 2\log_3 C + 1 \rceil)$
Then $n \in N$
Then $n \geq B$
Then $n > 2\log_3 C$
Then $\frac{n}{2} > \log_3 C$
Then $\log_3 3^{\frac{n}{2}} > log_3 C \; \sharp \; \log_3 3^{\frac{n}{2}} = \frac{n}{2}$
Then $3^{\frac{n}{2}} > C$
Then $3^{n-\frac{n}{2}} > C$
Then $3^n \div 3^{\frac{n}{2}} > C$
Then $3^n > C \cdot 3^{\frac{n}{2}}$
Therefore $(n \geq B) \wedge (3^n > C3^{\frac{n}{2}})$
Therefore $\exists n \in \mathbb{N} : (n \geq B) \wedge (3^n > C3^{\frac{n}{2}})$
Therefore $\forall B \in \mathbb{N} : [\exists n \in \mathbb{N} : (n \geq B) \wedge (3^n > C(3^{\frac{n}{2}})]$
Therefore $\forall C \in \mathbb{R}^+ : [\forall B \in \mathbb{N} : [\exists n \in \mathbb{N} : (n \geq B) \wedge (3^n > C(3^{\frac{n}{2}})]]$
Therefore $\neg[\exists C \in \mathbb{R}^+ : [\exists B \in \mathbb{N} : [\forall n \in \mathbb{N} : (n \geq B) \implies (3^n \leq C(3^{\frac{n}{2}}))]]]$
Therefore $f(n) \notin O(3^{g(n)})$

Therefore $\exists f, g \in \mathcal{F} : \log f(n) \in O(g(n)) \wedge f(n) \notin O(3^{g(n)})$
Therefore $\neg(\forall f, g \in \mathcal{F} : \log f(n) \in O(g(n)) \implies f(n) \in O(3^{g(n)}))$
Therefore the original claim is false.

2. Prove $\forall f \in \mathcal{F} : \lfloor \sqrt{\lfloor f(n) \rfloor} \rfloor \in O(\sqrt{f(n)})$
Let $f(n) \in \mathcal{F}$
Prove $\exists C \in \mathbb{R}^+ : [\exists B \in \mathbb{N} : [\forall n \in \mathbb{N} : (n \geq B) \implies \lfloor \sqrt{\lfloor f(n) \rfloor} \rfloor \leq C\sqrt{f(n)}]]$
Let $C = 1$
Then $C \in \mathbb{R}^+$
Let $B = 1$

Then $B \in \mathbb{N}$

Let $n \in \mathbb{N}$

Assume $n \geq B$

Then $n \geq 1$

Then $\lfloor f(n) \rfloor \leq f(n)$ $\sharp$ Definition of $\lfloor f(n) \rfloor$

Then $\sqrt{\lfloor f(n) \rfloor} \leq \sqrt{f(n)}$ $\sharp$ Power 1/2 in both side

Then $\lfloor \sqrt{\lfloor f(n) \rfloor} \rfloor \leq \sqrt{\lfloor f(n) \rfloor}$ $\sharp$ Definition of $\lfloor \sqrt{\lfloor f(n) \rfloor} \rfloor$

Then $\lfloor \sqrt{\lfloor f(n) \rfloor} \rfloor \leq \sqrt{f(n)}$

Therefore $\forall n \in \mathbb{N} : n \geq 1 \implies \lfloor \sqrt{\lfloor f(n) \rfloor} \rfloor \leq \sqrt{f(n)}$

Therefore $\exists B \in \mathbb{N} : [\forall n \in \mathbb{N} : n \geq B \implies \lfloor \sqrt{\lfloor f(n) \rfloor} \rfloor \leq \sqrt{f(n)}]$

Therefore $\exists C \in \mathbb{R}^+ : [\exists B \in \mathbb{N} : [\forall n \in \mathbb{N} : \forall n \in \mathbb{N} : n \geq B \implies \lfloor \sqrt{\lfloor f(n) \rfloor} \rfloor \leq C\sqrt{f(n)}]]$

Therefore $\forall f \in \mathcal{F} : \lfloor \sqrt{\lfloor f(n) \rfloor} \rfloor \in O(\sqrt{f(n)})$

Problem 4.

(6 Marks) Recall that $n! = 1 \cdot 2 \ldots n$. Also, by convention, $0! = 1$. Using the method of mathematical induction, prove the following:

1. (3 Marks)
$$\forall n \in \mathbb{N} : \sum_{i=0}^{n} i \cdot i! = (n+1)! - 1.$$

2. (3 Marks)
$$\forall n \in \mathbb{N} : n \geq 1 \to 2^n \leq 2^{n+1} - 2^{n-1} - 1.$$

Solution

1. Let P(n) be $\sum_{i=0}^{n} i \cdot i! = (n+1)! - 1$

    Base case:
    $P(0) : 0 = (0+1)! - 1 = 0$
    So P(0) is true

    Inductive steps:
    Prove $\forall k \in \mathbb{N} : [(k \geq 0) \implies (P(k) \implies P(k+1))]$
    Let $k \in \mathbb{N}$
    Assume $k \geq 0$
    Assume P(k)
    Then $P(k) = \sum_{i=0}^{k} i \cdot i! = (k+1)! - 1$
    Then $\sum_{i=0}^{k+1} i \cdot i! = \sum_{i=0}^{k} i \cdot i! + (k+1) \cdot (k+1)!$ ♯ Definition of $\sum_{i=0}^{k+1} i \cdot i!$
    $= (k+1)! - 1 + (k+1)(k+1)!$ ♯ $\sum_{i=0}^{k} i \cdot i! = (k+1)! - 1$
    $= (k+1+1)(k+1)! - 1$
    $= (k+2)(k+1)! - 1$
    $= (k+2)! - 1$ ♯ Definition of !
    Then $\sum_{i=0}^{k+1} i \cdot i! = (k+2)! - 1$
    Then $P(k+1)$ is True
    Therefore $P(k) \implies P(k+1)$
    Therefore $(k \geq 0) \implies (P(k) \implies P(k+1))$

Therefore $\forall k \in \mathbb{N} : [(k \geq 0) \implies (P(k) \implies P(k+1))]$

Therefore $\forall n \in \mathbb{N} : \sum_{i=0}^{n} i \cdot i! = (n+1)! - 1$ $\sharp$ By induction

2. Prove $\forall n \in \mathbb{N} : n \geq 1 \rightarrow 2^n \leq 2^{n+1} - 2^{n-1} - 1$
Let P(n) be $n \geq 1 \rightarrow 2^n \leq 2^{n+1} - 2^{n-1} - 1$

Vacuously ture case:
$P(0)$ is true. $\sharp$ $0 \geq 1$ is false

Base case:
$P(1) : 1 \geq 1 \rightarrow 2^1 \leq 2^2 - 2^0 - 1 = 4 - 2 = 2$
So $P(1)$ is True

Inductive step:
Prove $\forall k \in \mathbb{N} : [(k \geq 1) \implies (P(k) \implies P(k+1))]$
Let $k \in \mathbb{N}$
Assume $k \geq 1$
Assume $P(k)$
Then $k \geq 1 \rightarrow 2^k \leq 2^{k+1} - 2^{k-1} - 1$
Then $k + 1 \geq 1$
Then $2^{k+1} = 2 \cdot 2^k$
Then $2^{k+1} \leq 2 \cdot (2^{k+1} - 2^{k-1} - 1)$ $\sharp$ $2^k \leq 2^{k+1} - 2^{k-1} - 1$
Then $2^{k+1} \leq 2^{k+2} - 2^k - 2$
Then $2^{k+1} \leq 2^{k+2} - 2^k - 1$ $\sharp$ $-1 > -2$
Then $(k + 1 \geq 1) \rightarrow (2^{k+1} \leq 2^{k+2} - 2^k - 1)$
Then $P(k+1)$ is True
Therefore $P(k) \implies P(k+1)$
Therefore $(k \geq 1) \implies (P(k) \implies P(k+1))$
Therefore $\forall k \in \mathbb{N} : [(k \geq 1) \implies (P(k) \implies P(k+1))]$

Therefore $\forall n \in \mathbb{N} : n \geq 1 \implies 2^n \leq 2^{n+1} - 2^{n-1} - 1$ $\sharp$ By induction