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We declare that this assignment is solely our own work, and is in accordance with the University of Toronto Code of Behaviour on Academic Matters.

This submission has been prepared using LaTeX.

Problem 1.

(8 MARKS) Let U denote the set of **computer programmers**, **software testers** and **software projects** of Great Software Company, Inc.
Let `SSP (Super Software Project)` and `USP (Ultra Software Project)` be specific software projects in U. Let `Bugs Bunny` be a specific individual in U.
Finally, consider the following predicates, along with their meaning:

$S(x)$: "x is a software tester";

$P(x)$: "x is a computer programmer";

$C(x)$: "x is a software project";

$N(x)$: "x is new" (software project or software tester);

$W(x, y)$: "x writes code for y";

$A(x, y)$: "x is more advanced than y".

$T(x, y)$: "x tests y"

Using only the domain U, the given constants `SSP` and `USP` , and predicates above (in addition to appropriate connectives and quantifiers), translate each sentence below: give a proper English sentence that corresponds to each symbolic sentence, and give a symbolic sentence that corresponds to each English sentence. State clearly any assumptions you need to make.

1. (1 MARK) `Bugs Bunny` writes code for `SSP` and `USP` and neither software project is more advanced than the other.
2. (1 MARK) $\forall x \in U : [C(x) \rightarrow [\exists y \in U : [P(y) \text{ and } W(y, x)]]]$
3. (1 MARK) Some software project is more advanced than every software project that Bugs Bunny writes code for.
4. (1 MARK) $S(x)$ and $\big(W(x, \texttt{SSP})$ or $T(x, \texttt{USP})\big)$
5. (1 MARK) No software tester has tested every software project.
6. (1 MARK) If any computer programmer has coded every new software project, then that computer programmer has coded every software project.

7. (1 Mark) $\exists x \in U : [S(x) \text{ and } (\forall y \in U : [[C(y) \text{ and } (W(\texttt{Bugs Bunny}, y) \text{ or } N(y))] \rightarrow W(x,y)]])]$

8. (1 Mark) `Bugs Bunny` writes code for `SSP` unless there is some new software project more advanced than `SSP`.

Solution

1. $[W(\texttt{Bugs Bunny}, \texttt{SSP})] \text{ and } [W(\texttt{Bugs Bunny}, \texttt{USP})] \text{ and } [\forall x \in U, y = u - x : [C(x) \text{ and } C(y)] \rightarrow \neg A(x,y)]$

2. Every software project requires some computer programmers and the programmer writes code for projects.

3. $[\exists x \in U] \text{ and } [\forall y \in U] : [C(x) \text{ and } [C(y) : W(\texttt{Bugs Bunny}, y)]] \rightarrow A(x,y)$

4. A software tester writes code for `SSP` or tests `USP`.

5. $\forall x, y \in U : [S(x) \text{ and } C(y) \text{ and } \neg T(x,y)]$

6. $\forall x, y \in U : [P(x) \text{ and } C(y) \text{ and } N(y) \text{ and } W(x,y)] \rightarrow \forall z \in U : [C(z) \text{ and } W(x,z)]$

7. If Bugs Bunny weites code for every project or every project is new, then some software testers write code for projects.

8. $\neg W(\texttt{Bugs Bunny}, \texttt{SSP}) \rightarrow \forall x \in U : [C(x) \text{ and } N(x) \text{ and } A(x, \texttt{SSP})]$

Problem 2.

(4 MARKS)

Consider the quadratic equation $x^2 + bx + c = 0$. Let

$Q(x)$ be the predicate "x is a rational number"

$S(x)$ be the predicate "x is a solution of the equation $x^2 + bx + c = 0$"

Assuming the domain is the set $\mathbb{Q}$ of rational numbers, do the following:

1. (2 MARKS) Write a symbolic sequence that corresponds to English statement: "for all rational numbers b,c if there is a rational number r such that r is a solution to the equation $x^2 + bx + c = 0$ then any other solution s of this equation is a rational number".
2. (2 MARKS) Formally prove the above claim (i.e., prove the statement in the above bullet). (HINT: If the equation $x^2 + bx + c = 0$ has a real solution r, then the other solution s is also real and the following is true: $x^2 + bx + c = (x - r)(x - s)$).

Solution

1. $\forall b, c \in \mathbb{Q} : [[(\exists r : Q(r)) \text{ } and \text{ } S(r)] \rightarrow [(\forall s : S(s)) \rightarrow Q(s)]]$

2. proof:
   Let $b, c \in \mathbb{Q}$

   Assume $[(\exists r : Q(r)) \text{ } and \text{ } S(r)]$

   then r is rational and r is a solution of $x^2 + bx + c = 0$

   then $\exists p \in \mathbb{Z} : [\exists q \in \mathbb{Z} : r = \frac{p}{q} \wedge q \neq 0]$

   Let every s is a solution of $x^2 + bx + c$

   then $x^2 + bx + c = (x - r)(x - s) = 0$ is true $\sharp$ by hint

   then $x^2 - sx - rx + sr = 0$

   then $x^2 - (s + r)x + sr = 0$

   then $b = -s - r, c = sr$

   then $s = -b - r = \frac{c}{r}$

then s is rational $\sharp$ basic math

then Q(s)

Therefore $[(\forall s : S(s)) \to Q(s)]$

Therefore $[(\exists r : Q(r))\ and\ S(r)] \to [(\forall s : S(s)) \to Q(s)]$

Therefore $\forall b, c \in \mathbb{Q} : [[(\exists r : Q(r))\ and\ S(r)] \to [(\forall s : S(s)) \to Q(s)]]$

Problem 3.

(4 Marks)

Prove the following claims:

1. (2 Marks) $\forall x, y \in \mathbb{R} : x \geq 0$ and $y \geq 0 \implies \frac{x+y}{2} \geq \sqrt{xy}$
2. (2 Marks) $\forall n \in \mathbb{Z} : n \geq 1 \implies \frac{2n+1}{2n+2} \geq \frac{\sqrt{n}}{\sqrt{n+1}}$ (Hint: you could use the claim in above bullet as an aid).

Solution

1. proof:

Let $x, y \in \mathbb{R}$

Assume $x \geq 0$ and $y \geq 0$

then $\sqrt{x} \geq 0$ and $\sqrt{y} \geq 0$

then $(\sqrt{x} - \sqrt{y})^2 \geq 0$

then $x + y - 2\sqrt{xy} \geq 0$

then $x + y \geq 2\sqrt{xy}$

then $\frac{x+y}{2} \geq \sqrt{xy}$

Therefore $x \geq 0$ and $y \geq 0 \implies \frac{x+y}{2} \geq \sqrt{xy}$

Therefore $\forall x, y \in \mathbb{R} : x \geq 0$ and $y \geq 0 \implies \frac{x+y}{2} \geq \sqrt{xy}$

2. proof:

Let $n \in \mathbb{Z}$

Assume $n \geq 1$

then $n + 1 \geq 2$

then $\frac{n}{n+1} > 0$ and $\frac{n+1}{n+1} > 0$

then $\sqrt{\frac{n}{n+1}} > 0$ and $\sqrt{\frac{n+1}{n+1}} > 0$

then $\left(\sqrt{\frac{n}{n+1}} - \sqrt{\frac{n+1}{n+1}}\right)^2 \geq 0$

then $\frac{n}{n+1} + \frac{n+1}{n+1} - 2\sqrt{\frac{n}{n+1}} \geq 0$

$$\text{then } \frac{2n+1}{n+1} \geq 2\sqrt{\frac{n}{n+1}}$$

$$\text{then } \frac{2n+1}{2n+2} \geq \frac{\sqrt{n}}{\sqrt{n+1}}$$

$$\text{Therefore } n \geq 1 \implies \frac{2n+1}{2n+2} \geq \frac{\sqrt{n}}{\sqrt{n+1}}$$

$$\text{Therefore } \forall n \in \mathbb{Z} : n \geq 1 \implies \frac{2n+1}{2n+2} \geq \frac{\sqrt{n}}{\sqrt{n+1}}$$

Problem 4.

(8 Marks)

Any natural number can be either an odd or an even number, but not both. For each natural number n, let $E(n)$ be defined as "$\exists k \in \mathbb{N} : n = 2k$". Also for each natural number n, let $O(n)$ be defined as "$\exists k \in \mathbb{N} : n = 2k + 1$". That is, for any natural number n, $E(n)$ means "n is even" and $O(n)$ means "n is odd".

Prove or disprove the following:

1. (2 Marks) $\forall n \in \mathbb{N} : O(n) \implies E(n^2 + n)$

2. (2 Marks) $\forall n \in \mathbb{N} : [\neg E(n)] \implies [\neg E(n+2)]$

3. (2 Marks) $\forall n, m \in \mathbb{N} : O(n)$ and $E(m) \implies O(m^2 + 3n)$

4. (2 Marks) $\forall n, m \in \mathbb{N} : [\neg E(mn)] \implies [\neg E(m)$ and $\neg E(n)]$

Solution

1. proof:
   Let $n \in \mathbb{N}$

   Assume $O(n)$

   then n is odd number

   then $\exists k \in \mathbb{N} : n = 2k + 1$

   Let $k_0 \in \mathbb{N}$ be such that $n = 2k_0 + 1$

   then $n^2 + n = (2k_0 + 1)^2 + (2k_0 + 1)$

   then $n^2 + n = 4k_0^2 + 6k_0 + 2$

   then $n^2 + n = 2(2k_0^2 + 3k_0 + 1)$

   Let $k_1 = 2k_0^2 + 3k_0 + 1$

   then $k_1 \in \mathbb{N}$ $\sharp$ basic math

   then $n^2 + n = 2k_1$

   then $\exists k \in \mathbb{N}, n^2 + n = 2k$

then $n^2 + n$ is even

Therefore $O(n) \Rightarrow E(n^2 + n)$

Therefore $\forall n \in \mathbb{N} : O(n) \Rightarrow E(n^2 + n)$

2. proof: This claim is equal to $\forall n \in \mathbb{N} : E(n+2) \Rightarrow E(n)$
   Let $n \in \mathbb{N}$

   Assume $E(n+2)$

   then $n+2$ is even

   then $\exists k \in \mathbb{N} : n = 2k$

   Let $k_0 \geq 1$ and $k_0 \in \mathbb{Z}$

   then $k_0 \in \mathbb{N}$ be such that $n + 2 = 2k_0$

   then $n = 2k_0 - 2$

   then $n = 2(k_0 - 1)$

   Let $k_1 = k_0 - 1$

   then $k_1 \in \mathbb{N}$ $\sharp$ basic math

   then $n = 2k_1$

   then $\exists k \in \mathbb{N}, n = 2k$

   then n is even

   Therefore $E(n+2) \Rightarrow E(n)$

   Therefore $\forall n \in \mathbb{N} : E(n+2) \Rightarrow E(n)$

   Therefore $\forall n \in \mathbb{N} : [\neg E(n)] \implies [\neg E(n+2)]$ $\sharp$ by contrapositive

3. proof:
   Let $n, m \in \mathbb{N}$

   Assume $O(n)$ and $E(m)$

   then n is odd and m is even

   then $\exists k \in \mathbb{N} : n = 2k + 1$ and $\exists k \in \mathbb{N} : m = 2k$

   Let $k_0 \in \mathbb{N}$ be such that $n = 2k_0 + 1$ and $m = 2k_0$

then $m^2 + 3n = (2k_0)^2 + 3(2k_0 + 1)$

then $m^2 + 3n = 4k_0^2 + 6k_0 + 2 + 1$

then $m^2 + 3n = 2(2k_0^2 + 3k_0 + 1) + 1$

Let $k_1 = 2k_0^2 + 3k_0 + 1$

then $k_1 \in \mathbb{N}$ $\sharp$ basic math

then $m^2 + 3n = 2k_1 + 1$

then $\exists k \in \mathbb{N} : m^2 + 3n = 2k + 1$

then $m^2 + 3n$ is odd

Therefore $O(n)$ and $E(m) \Rightarrow O(m^2 + 3n)$

Therefore $\forall n, m \in \mathbb{N} : O(n)$ and $E(m) \Rightarrow O(m^2 + 3n)$

4. proof: This claim is equal to $\forall n, m \in \mathbb{N} : E(m) or E(n) \Rightarrow E(mn)$
Let $n, m \in \mathbb{N}$

Assume $E(m)$ or $E(n)$

then m is even or n is even

then $\exists k \in \mathbb{N} : m = 2k$ or $\exists k \in \mathbb{N} : n = 2k$

Let $k_0 \in \mathbb{N}$ be such that $m = 2k_0$ or $n = 2k_0$

then $mn = 2k_0m$ or $2k_0n$

Let $k_1 = k_0m$ or k_0n

then $k_1 \in \mathbb{N}$ $\sharp$ basic math

then $mn = 2k_1$

then $\exists k \in \mathbb{N} : mn = 2k$

then mn is even

Therefore $E(m)$ or $E(n) \Rightarrow E(mn)$

Therefore $\forall n, m \in \mathbb{N} : E(m)$ or $E(n) \Rightarrow E(mn)$

Therefore $\forall n, m \in \mathbb{N} : [\neg E(mn)] \implies [\neg E(m)$ and $\neg E(n)]$ $\sharp$ by contrapositive

Problem 5.

(6 Marks)

Let $a, b, c \in \mathbb{R}^+$ be three sides of a triangle. Then the following statement is true:

$\forall a, b, c \in \mathbb{R}^+$: If a, b, c are sides of some triangle, then $a < b+c$ and $b < a+c$ and $c < a+b$.

This statement is know as the *triangle inequality*.

Let's denote $T(a, b, c)$ the following predicate : "a, b, c are sides of a triangle".

1. (2 Marks) Prove the following claim:

$$\forall a, b, c \in \mathbb{R}^+ : T(a, b, c) \implies [|a - b| < c \text{ and } |b - c| < a \text{ and } |a - c| < b].$$

2. (2 Marks) Prove or disprove :

$$\forall a, b, c \in \mathbb{R}^+ : T(a, b, c) \implies a^2 + 2bc > b^2 + c^2.$$

3. (2 Marks) Prove or disprove:

$$\forall a, b, c \in \mathbb{R}^+ : T(a, b, c) \implies \frac{a^2 + 2bc}{b^2 + c^2} + \frac{b^2 + 2ca}{c^2 + a^2} + \frac{c^2 + 2ab}{a^2 + b^2} > 3.$$

Solution

1. proof:
   Let $a, b, c \in \mathbb{R}^+$

   Assume $T(a, b, c)$

   then $a < b + c$ and $b < a + c$ and $c < a + b$

   then $a - b < c$ and $b - a < c$ and $b - c < a$ and $c - b < a$ and $a - c < b$ and $c - a < b$

   then $|a - b| < c$ and $|b - c| < a$ and $|a - c| < b$ $\sharp$ basic math

   Therefore $T(a, b, c) \Longrightarrow [|a - b| < c$ and $|b - c| < a$ and $|a - c| < b]$

   Therefore $\forall a, b, c \in \mathbb{R}^+ : T(a, b, c) \Longrightarrow [|a-b| < c$ and $|b-c| < a$ and $|a-c| < b]$

2. proof:
   Let $a, b, c \in \mathbb{R}^+$

   Assume $T(a, b, c)$.

then $|a-b| < c$ and $|b-c| < a$ and $|a-c| < b$ $\sharp$ by Problem 5.1.

then $|a-b|^2 < c^2$ and $|b-c|^2 < a^2$ and $|a-c|^2 < b^2$

then $a^2+b^2-2ab < c^2$ and $b^2+c^2-2bc < a^2$ and $a^2+c^2-2ac < b^2$

then $a^2+2bc > b^2+c^2$ and $b^2+2ac > a^2+c^2$ and $c^2+2ab > a^2+b^2$

Therefore $T(a,b,c) \implies a^2+2bc > b^2+c^2$ and $b^2+2ac > a^2+c^2$ and $c^2+2ab > a^2+b^2$

Therefore $\forall a,b,c \in \mathbb{R}^+ : T(a,b,c) \implies a^2+2bc > b^2+c^2, b^2+2ac > a^2+c^2, c^2+2ab > a^2+b^2$

Therefore $\forall a,b,c \in \mathbb{R}^+ : T(a,b,c) \implies a^2+2bc > b^2+c^2$

3. proof:

Let $a,b,c \in \mathbb{R}^+$

Assume $T(a,b,c)$

then $a^2+2bc > b^2+c^2$ and $b^2+2ac > a^2+c^2$ and $c^2+2ab > a^2+b^2$ $\sharp$ by Problem 5.2.

then $\frac{a^2+2bc}{b^2+c^2} > 1$ and $\frac{b^2+2ac}{a^2+c^2} > 1$ and $\frac{c^2+2ab}{a^2+b^2} > 1$ $\sharp$ basic math

then $\frac{a^2+2bc}{b^2+c^2} + \frac{b^2+2ac}{a^2+c^2} + \frac{c^2+2ab}{a^2+b^2} > 3$ $\sharp$ basic math

Therefore $T(a,b,c) \implies \frac{a^2+2bc}{b^2+c^2} + \frac{b^2+2ac}{a^2+c^2} + \frac{c^2+2ab}{a^2+b^2} > 3$

Therefore $\forall a,b,c \in \mathbb{R}^+ : T(a,b,c) \implies \frac{a^2+2bc}{b^2+c^2} + \frac{b^2+2ac}{a^2+c^2} + \frac{c^2+2ab}{a^2+b^2} > 3$