CS5285
Information Security for eCommerce

Lecture 3

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Reminder of last week

- Symmetric Encryption
  - Substitution ciphers and frequency analysis
  - One time pad (perfectly secure/impractical)
  - Stream and block ciphers (RC4/DES/AES)
  - Block cipher modes of operation
    - Error propagation

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Today's Lecture

- Number theory
  - Background maths to public key crypto
- CILO5

  (properties/design of security mechanisms)

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Number Theory

We work on integers only

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Slides 5-23

This is background information. See this as a reference section for terminology. You do not need to know every single slide in detail but you must be familiar enough with the material to apply it to subsequent cryptography.

For example, if I ask you to show how a message is encrypted/decrypted using RSA you must be able to do the calculation (so it will help you to understand if you know what a prime number, what is Eulers totient is, etc.)

Divisors

Two integers: a and b (b is non-zero)

- b divides a if there exists some integer m such that $a = m \cdot b$
- Notation: $b|a$
- eg. 1,2,3,4,6,8,12,24 divide 24
- b is a **divisor** of a

Relations

1. If $b|1$ $\Rightarrow$ $b = \pm 1$
2. If $b|a$ and $a|b$ $\Rightarrow$ $b = \pm a$
3. If $b|0$ $\Rightarrow$ any $b \neq 0$
4. If $b|g$ and $b|h$ then $b \mid (mg + nh)$ for any integers m and n.

Congruence

a is **congruent** to b modulo n if n | a-b.

Notation: $a \equiv b \pmod{n}$

Examples

1. $23 \equiv 8 \pmod{5}$ because $5 \mid 23-8$
2. $-11 \equiv 5 \pmod{8}$ because $8 \mid -11-5$
3. $81 \equiv 0 \pmod{27}$ because $27 \mid 81-0$

Properties

1. $a \equiv b \pmod{n}$ implies $b \equiv a \pmod{n}$
2. $a \equiv b \pmod{n}$ and $b \equiv c \pmod{n}$ imply $a \equiv c \pmod{n}$

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Examples

1. m=3 (5|15)
2. m=-2 (8|-16
3. m=3 (27|81)

Modular Arithmetic

- **modular reduction:** a mod n = r
  r is the remainder when a is divided by a natural number n
- r is also called the residue of a mod n
  - it can be represented as: $a = qn + r$ where $0 \le r < n$, $q = \lfloor a/n \rfloor$ where $\lfloor x \rfloor$ is the largest integer less than or equal to x
  - q is called the quotient
- 18 mod 7 = ?
- 29345723547 mod 2 = ?
- Relation between modular reduction and congruence
  - $-12 \equiv -5 \equiv 2 \equiv 9 \pmod 7$
  - -12 mod 7 = 2 (what's the quotient?)
  - -12 = q*n+r= -2*7+2

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-12 mod 7 =2

2 -2*7 mod 7, so n is 7 and q is -2

Modular Arithmetic Operations

- can do modular reduction at any point,
  - a + b mod n = [a mod n + b mod n] mod n
  - E.g. 97 + 23 mod 7 = [97 mod 7 + 23 mod 7] mod 7 = [6 + 2] mod 7 = 1
  - E.g. 11 – 14 mod 8 = ?
    3-6 mod 8 = 5
  - E.g. 11 x 14 mod 8 = ?
    3 x 6 mod 8 = 2

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When reducing, we "usually" want to find the **positive** remainder after dividing by the modulus. For positive numbers, this is simply the normal remainder. For negative numbers we have to "overshoot" (ie find the next multiple larger than the number) and "come back" (ie add a positive remainder to get the number); rather than have a "negative remainder".

Prime and Composite Numbers

- An integer p is prime if its only divisors are ±1 and ±p only.
- Otherwise, it is a composite number.
- E.g. 2,3,5,7 are prime; 4,6,8,9,10 are not
- List of prime numbers less than 200:

  2 3 5 7 11 13 17 19 23 29 31 37 41 43 47 53 59 61 67 71 73 79 83 89 97 101 103 107 109 113 127 131 137 139 149 151 157 163 167 173 179 181 191 193 197 199

- Prime Factorization: If a is a composite number, then a can be factored in a unique way as

$$a = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_t^{\alpha_t}$$

where $p_1 > p_2 > \dots > p_t$ are prime numbers and each α_i is a natural number (i.e. a positive nonzero integer).

e.g. $12{,}250 = 7^2 \cdot 5^3 \cdot 2$

Prime Factorization

- It is generally hard to do (prime) factorization when the number is large
- E.g. factorize
    1. 24070280312179
    2. 10893002480924910251
    3. 938740932174981739832107481234871432497617
    4. 93874093217498173983210748123487143249761717

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Greatest Common Divisor (GCD)

- GCD (a,b) of a and b is the largest number that divides both a and b
  - E.g. GCD(60,24) = 12
- If GCD(a, b) = 1, then a and b are said to be **relatively prime**
  - E.g. GCD(8,15) = 1
  - 8 and 15 are relatively prime (co-prime)

Question: How to compute gcd(a,b)?

Naive method: factorize a and b and compute the product of all their common factors.

e.g. $540 = 2^2 \times 3^3 \times 5$
$144 = 2^4 \times 3^2$
$\gcd(540, 144) = 2^2 \times 3^2 = 36$

Problem of this naive method: factorization becomes very difficult when integers become large.

Better method: Euclidean Algorithm (a.k.a. Euclid's GCD algorithm)

Euclidean Algorithm

Euclid's Algorithm:

```
A=a, B=b
while B>0
        R = A mod B
        A = B, B = R
return A
```

Rationale
Theorem gcd(a, b) = gcd(a, b mod a)

Compute gcd(911, 999) :

A	= q x B + R
999	= 1 x 911 + 88
911	= 10 x 88 + 31
88	= 2 x 31 + 26
31	= 1 x 26 + 5
26	= 5 x 5 + 1
5	= 5 x 1 + 0

Value returned

Hence gcd(911, 999) = 1

Hence gcd(911, 999) = gcd(911, 999 mod 911) = gcd(911 mod 88, 88)
= gcd(31, 88 mod 31) = gcd(31 mod 26, 26) = gcd(5, 26 mod 5)
= gcd(5, 1) = 1.

Modular Inverse

A is the modular inverse of B mod n if

AB mod n = 1.

A is denoted as B^{-1} mod n.

e.g.
- 3 is the modular inverse of 5 mod 7. In other words, 5^{-1} mod 7 = 3.
- 7 is the modular inverse of 7 mod 16. In other words, 7^{-1} mod 16 = 7.

However, there is no modular inverse for 8 mod 14.

There exists a modular inverse for B mod n if B is relatively prime to n.

Question:
What's the modular inverse of 911 mod 999?

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This not a fraction!!! A is not 1/B (remember that A and B and integers)

What can we do?

We use the extended euclidean algorithm, we know to have a modular inverse 911 and 999 must be relative prime. So what is the GCD?

Extended Euclidean Algorithm

The extended Euclidean algorithm can be used to solve the integer equation

$$ax + by = \gcd(a, b)$$

For any given integers a and b.

Example

Let a = 911 and b = 999. From the Euclidean algorithm,

$$999 = 1 \times 911 + 88$$
$$911 = 10 \times 88 + 31$$
$$88 = 2 \times 31 + 26$$
$$31 = 1 \times 26 + 5$$
$$26 = 5 \times 5 + 1 \qquad \Rightarrow \gcd(a, b) = 1$$

Tracing backward, we get

$$\begin{aligned} 1 &= 26 - 5 \times 5 \\ &= 26 - 5 \times (31 - 1 \times 26) = -5 \times 31 + 6 \times 26 \\ &= -5 \times 31 + 6 \times (88 - 2 \times 31) = 6 \times 88 - 17 \times 31 \\ &= 6 \times 88 - 17 \times (911 - 10 \times 88) = -17 \times 911 + 176 \times 88 \\ &= -17 \times 911 + 176 \times (999 - 1 \times 911) = \mathbf{176 \times 999 - 193 \times 911} \end{aligned}$$

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Extended Euclidean Algorithm solves for combination of x and y.

Calculating the Modular Inverse

we now have

$\gcd(911, 999) = 1 = -193 \times 911 + 176 \times 999.$

If we do a modular reduction of 999 to this equation, we have

$1 \pmod{999} = -193 \times 911 + 176 \times 999 \pmod{999}$

$\Rightarrow 1 = -193 \times 911 \bmod 999$

$\Rightarrow 1 = (-193 \bmod 999) \times 911 \bmod 999$

$\Rightarrow 1 = 806 \times 911 \bmod 999$

$1 \equiv 806 \times 911 \pmod{999}$.

Hence 806 is the **modular inverse** of 911 modulo 999.

The Euler phi Function

For $n \geq 1$, $\phi(n)$ denotes the number of integers in the interval $[1, n]$ which are relatively prime to n. The function ϕ is called the **Euler phi function** (or the **Euler totient function**).

Fact 1. The Euler phi function is multiplicative. I.e. if $\gcd(m, n) = 1$, then $\phi(mn) = \phi(m) \times \phi(n)$.

Fact 2. For a prime p and an integer $e \geq 1$, $\phi(p^e) = p^{e-1}(p-1)$.

- From these two facts, we can find ϕ for any composite n if the prime factorization of n is known.
- Let $n = p_1^{e_1} p_2^{e_2} \dots p_k^{e_k}$ where $p_1, \dots, p_k$ are prime and each e_i is a nonzero positive integer.
- Then

$$\phi(n) = p_1^{e_1-1} (p_1-1) \,.\, p_2^{e_2-1} (p_2-1) \dots p_k^{e_k-1} (p_k-1)$$

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The Euler phi Function

$$\phi(n) = \left|\{x : 1 \le x \le n \quad and \quad \gcd(x,n) = 1\}\right|$$

- $\phi(2) = |\{1\}| = 1$
- $\phi(3) = |\{1,2\}| = 2$
- $\phi(4) = |\{1,3\}| = 2$
- $\phi(5) = |\{1,2,3,4\}| = 4$
- $\phi(6) = |\{1,5\}| = 2$

- $\phi(37) = 36$
- $\phi(21) = (3-1)\times(7-1) = 2\times6 = 12$

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Magnitude of all numbers between 1 and n wher GCD (x,n) =1.

Fermat's Little Theorem

Let p be a prime. Any integer a not divisible by p satisfies $a^{p-1} \equiv 1 \pmod{p}$.

- We can generalize the Fermat's Little Theorem as follows. This is due to Euler.

  Euler's Generalization Let n be a composite. Then $a^{\phi(n)} \equiv 1 \pmod{n}$ for any integer a which is relatively prime to n.

- E.g. a=3;n=10; $\varphi(10)=4 \Rightarrow 3^4 \equiv 81 \equiv 1 \pmod{10}$
- E.g. a=2;n=11; $\varphi(11)=10 \Rightarrow 2^{10} \equiv 1024 \equiv 1 \pmod{11}$

Exercise: Compute $11^{1,073,741,823}$ mod 13.
Compute $11^{12}.11^{12}.11^{12}.11^{12}.....11^{3}$ mod 13 $\equiv 5 \pmod{13}$

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What is your strategy?

(11^12)^89478485 .(11^3) mod 13 = 11^3 mod 13= 5 mod 13

Modular Exponentiation

Let Z = { …, -2, -1, 0, 1, 2, … } be the set of integers.
Let $a, e, n \in Z$.

Modular exponentiation a^e mod n is defined as repeated multiplications of a for e times modulo n.

Method 1 : Repeated Modular Multiplication (as defined)

e.g. $11^{15} \bmod 13 = \underline{11 \times 11} \times 11 \times 11 \times \ldots \times 11 \bmod 13$
$= \underline{4 \times 11} \times 11 \times \ldots \times 11 \bmod 13$
$= \underline{5 \times 11} \times \ldots \times 11 \bmod 13$
:
$= 5$

- performed 14 modular multiplications
- Complexity = O(e)
- What if the exponent is large?

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Things do not always work with Fermat's theorem – and we cannot do repeated modular multiplication….need another method…square and multiply.

Modular Exponentiation

Method 2 : Square-and-Multiply Algorithm

e.g. $11^{15} \bmod 13 = 11^{8+4+2+1} \bmod 13 = 11^8 \times 11^4 \times 11^2 \times 11 \bmod 13$ — (1)

- $11^2 = 121 \equiv 4 \pmod{13}$ — (2)
- $11^4 = (11^2)^2 \equiv (4)^2 \equiv 3 \pmod{13}$ — (3)
- $11^8 = (11^4)^2 \equiv (3)^2 \equiv 9 \pmod{13}$ — (4)

Put (2), (3) and (4) into (1) and get
$11^{15} \equiv 9 \times 3 \times 4 \times 11 \equiv 5 \pmod{13}$

- performed at most $2\lfloor \log_2 15 \rfloor$ modular multiplications
- Complexity = $O(\lg(e))$

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Every time we just square the previous result.

This means we are working with square of less than n, rather than larger exponentiation.

Modular Exponentiation

Pseudo-code of Square-and-Multiply Algorithm to compute $a^e \bmod n$:

Let the binary representation of e be ($e_{t-1}\ e_{t-2}\ \ldots\ e_1\ e_0$).
Hence t is the number of bits in the binary representation of e.

1. z = 1
2. for i = t-1 downto 0 do
3. $\quad z = z^2 \bmod n$
4. $\quad$ if $e_i = 1$ then $z = z \times a \bmod n$

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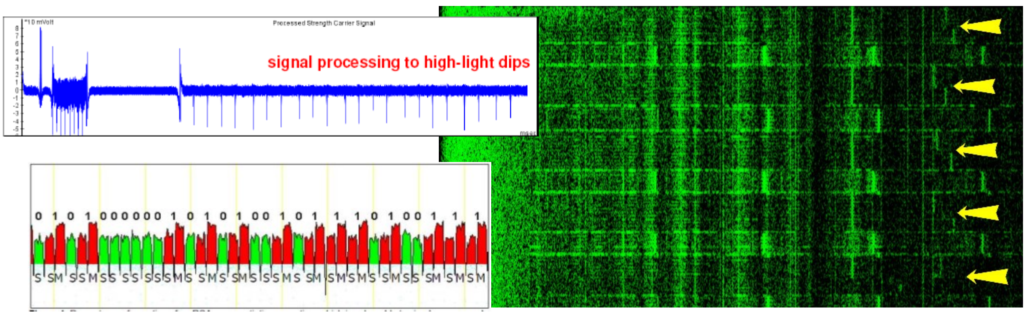
If we wanted to do this on a binary number? How would it work?

Here is a good time to think – ok so this is why I need to understand the underlying maths even if I just design and implement systems…

Great = what if e is a key? Is there a problem? What if someone can see time taken for each for loop iteration?

Side Channel

- Platform on which software runs leaks information
- Power usage, electromagnetic…acoustic
  - Consider again (square multiply) – timing?
  - Power (embedded hardware) and acoustic (PC, GNU RSA)



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For interest only.

Two strips on acoustic is exponentiation modulo P and the exponentiation modulo Q, for each key slightly different positions. Once again choose ciphertext and you can distinguish specific key bits.

http://www.cs.tau.ac.il/~tromer/acoustic/

http://www.ecs.umass.edu/~tbashir/timing_attack_rsa_theory.htm

The end!

?

Any questions…

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Exercise (Inverse)

e=79 and *e.d* mod 3220 ≡ 1 mod 3220 – find d
d ≡ 79^{-1} mod 3220

Euclidean Algorithm
3220 = 40.79+60
79=1.60+19
60=3.19+3
19=6.3+1

Extended Euclidean Algorithm
1= 19-6.3
1= 19-6 (60-3.19) = -6.60+19.19
1= -6.60+19(79-1.60) = -25.60+19.79
1= -25(3220-40.79)+19.79 = 1019.79 -25.3220

1019.79 -25.3220 mod 3220 ≡ 1019.79 mod 3220 ≡ 1 mod 3220

Hence d = 1019 is the **modular inverse** of 79 modulo 3220.

Exercise 2 (Inverse)

Calculate 2084^{-1} mod 2357

Euclidean Algorithm

- 2357 = 1.2084 + 273
- 2084 = 7.273 + 173
- 273 = 1.173 + 100
- 173 = 1. 100 + 73
- 100 = 1.73+27
- 73=2.27+19
- 27=19+8
- 19=2.8+3
- 8=2.3+2
- 3=2+1

Exercise 2 (Inverse) ctd

- 1= 3-1.2=3-(8-2.3)= 3.3-8
- 3.(19-2.8)-8=3.19-7.8 = 3.19-7(27-19)=10.19-7.27
- 10(73-2.27)-7.27 = 10.73-27.27 = 10.73 – 27(100-1.73) = 37.73-27.100
- 37.73-27.100 = 37.(173-100)-27.100 = -64.100+37.173 = -64. (273-173)+37.173 = -64.273 +101.173
- -64.273 +101.173 = -64.273 +101.(2084-7.273) = -771.273+101.2084 = -771(2357-2084)+101.2084
- -771(2357-2084)+101.2084 = 872.2084-771.2357
- 872.2084-771.2357mod 2357 ≡ 872.2084 mod 2357 ≡ 1 mod 2357
- So 872 must be modular inverse of 2084 mod 2357.

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Exercise (Square/Mult)

Calculate 17^{130} mod 11

Powers of two? 1,2,4,8,16,32,64,128,256…
130 dec = 10000010 binary

$17^{130} = 17^{128+2} \bmod 11 = 17^{128} \times 17^2 \bmod 11$

- $17^2 = 289 \equiv 3 \pmod{11}$ — (1)
- $17^4 = (17^2)^2 \equiv (3)^2 \equiv 9 \pmod{11}$ — (2)
- $17^8 = (17^4)^2 \equiv (9)^2 \equiv 4 \pmod{11}$ — (3)
- $17^{16} = (17^8)^2 \equiv (4)^2 \equiv 5 \pmod{11}$ — (4)
- $17^{32} = (17^{16})^2 \equiv (5)^2 \equiv 3 \pmod{11}$ — (5)
- $17^{64} = (17^{32})^2 \equiv (3)^2 \equiv 9 \pmod{11}$ — (6)
- $17^{128} = (17^{64})^2 \equiv (9)^2 \equiv 4 \pmod{11}$ — (7)

Use (7), (1) and get
$17^{130} \equiv 4 \times 3 \bmod 11 \equiv 1 \bmod 11$

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Every time we just square the previous result.

This means we are working with square of less than n, rather than larger exponentiation.

Exercise 2 (Square/Mult)

Calculate 17^{170} mod 13

Powers of two? 1,2,4,8,16,32,64,128,256…

$17^{170} = 17^{128+32+8+2}$ mod 13 = $17^{128} \times 17^{32} \times 17^{8}\ 17^{2}$mod 13

- $17^2 = 289 \equiv 3 \pmod{13}$ — (1)
- $17^4 = (17^2)^2 \equiv (3)^2 \equiv 9 \pmod{13}$ — (2)
- $17^8 = (17^4)^2 \equiv (9)^2 \equiv 3 \pmod{13}$ — (3)
- $17^{16} = (17^8)^2 \equiv (3)^2 \equiv 9 \pmod{13}$ — (4)
- $17^{32} = (17^{16})^2 \equiv (9)^2 \equiv 3 \pmod{13}$ — (5)
- $17^{64} = (17^{32})^2 \equiv (3)^2 \equiv 9 \pmod{13}$ — (6)
- $17^{128} = (17^{64})^2 \equiv (9)^2 \equiv 3 \pmod{13}$ — (7)

Use (7), (5), (3), (1) and get
17^{170} mod 13 $\equiv$ 3 x 3 x 3 x 3 mod 13 $\equiv$ 3 mod 13

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Every time we just square the previous result.

This means we are working with square of less than n, rather than larger exponentiation.