Tutorial 1

January 5, 2025

Question 1

Let $C \subseteq \mathbb{R}^n$ be a convex set, with $x_1, ..., x_n \in C$, and let $\theta_1, ..., \theta_n \in \mathbb{R}$ satisfy $\theta_i \geq 0$, $\theta_1 + ... + \theta_k = 1$. Show that $\theta_1 x_1 + ... + \theta_k x_k \in C$. (The definition of convexity is that this holds for $k = 2$; you must show it for arbitrary k.) *Hint.* Use induction on k.

Question 2

Show that a set is convex if and only if its intersection with any line is convex.

Question 3

Midpoint convexity A set C is midpoint convex if whenever two points a, b are in C, the average or midpoint $(a+b)/2$ is in C. Obviously a convex set is midpoint convex. It can be proved that under mild conditions midpoint convexity implies convexity. As a simple case, prove that if C is closed and midpoint convex, then C is convex.

Question 4

What is the distance between two parallel hyperplanes $\{x \in \mathbb{R}^n | a^T x = b_1\}$ and $\{x \in \mathbb{R}^n | a^T x = b_2\}$?

Question 5

Which of the following sets S are polyhedra? If possible, express S in the form $S = \{x | Ax \preceq b, Fx = g\}$.

(a) $S = \{y_1 a_1 + y_2 a_2 \big| -1 \leq y_1 \leq 1, -1 \leq y_2 \leq 1\}$, where $a_1, a_2 \in \mathbb{R}^n$.

(b) $S = \{x \in \mathbb{R}^n \big| x \succeq 0, \mathbf{1}^T x = 1, \sum_{i=1}^n x_i a_i = b_1, \sum_{i=1}^n x_i a_i^2 = b_2\}$, where $a_1, ..., a_n \in \mathbb{R}$ and $b_1, b_2 \in \mathbb{R}$.

(c) $S = \{x \in \mathbb{R} \big| x \succeq 0, x^T y \leq 1$ for all y with $\|y\|_2 = 1\}$.

(d) $S = \{x \in \mathbb{R}^n \big| x \succeq 0, x^T y \leq 1$ for all y with $\sum_{i=1}^n |y_i| = 1\}$.

Question 6

Solution set of a quadratic inequality. Let $C \subseteq \mathbb{R}^n$ be the solution set of a quadratic inequality,

$$C = \{x \in \mathbb{R}^n \big| x^T A x + b^T x + c \leq 0\},$$

with $A \in \mathbb{S}^n$, $b \in \mathbb{R}^n$, and $c \in \mathbb{R}$.

(a) Show that C is convex if $A \succeq 0$.

(b) Show that the intersection of C and the hyperplane defined by $g^T x + h = 0$ (where $g \neq 0$) is convex if $A + \lambda g g^T \succeq 0$ for some $\lambda \in \mathbb{R}$.

Are the converses of these statements true?

Question 7

Which of the following sets are convex?

(a) A *slab, i.e.*, a set of the form $\{x \in \mathbb{R}^n \big| \alpha \leq a^T x \leq \beta\}$.

(b) A *rectangle, i.e.*, a set of the form $\{x \in \mathbb{R}^n \big| \alpha_i \leq x_i \leq \beta_i, i = 1, ..., n\}$. A rectangle is sometimes called a *hyperrectangle* when $n > 2$.

(c) A *wedge, i.e.*, a set of the form $\{x \in \mathbb{R}^n \big| a_1^T x \leq b_1, a_2^T x \leq b_2\}$.

(d) The set of points closer to a given point than a given set, *i.e.*,

$$\left\{x \middle| \|x - x_0\|_2 \leq \|x - y\|_0, \forall y \in S\right\}, \text{ where } S \subseteq \mathbb{R}^n.$$

(e) The set of points closer to one set than another, *i.e.*,

$$\left\{x \middle| \operatorname{dist}(x, S) \leq \operatorname{dist}(x, T)\right\},$$

where $S, T \subseteq \mathbb{R}^n$, and

$$\operatorname{dist}(x, S) = \inf \left\{\|x - z\|_2 \middle| z \in S\right\}$$

(f) The set $\{x \big| x + S_2 \subseteq S_1\}$, where $S_1, S_2 \subseteq \mathbb{R}^n$ with S_1 convex.

(g) The set of points whose distance to a does not exceed a fixed fraction θ of the distance to b, *i.e.*, the set $\{x \big| \|x - a\|_2 \leq \theta \|x - b\|_2\}$. You can assume $a \neq b$ and $0 \leq \theta \leq 1$.