1. Another Approximation for Set Cover

Given a set of ground elements $U := \{ e_1, \ldots, e_n \}$, there is a collection $\mathcal{S} := \{ S_1, \ldots, S_m \}$ of subsets defined over U. The set cover problem seeks a minimum number of subsets from $\mathcal{S}$ such that they cover all elements in the ground set.

In the lecture, we have seen that a greedy algorithm can achieve $O(\log n)$-approximation and this is the best possible unless P=NP. Now, let us consider another greedy algorithm (Algorithm 1). We define the *frequency* of an element to be the number of sets that contain that element. Let f be the largest frequency among all elements.

$U = \{e_1\ e_2\}$ $\quad$ $\mathcal{S} = \{S_1, S_2, S_3\}$

Algorithm 1 Greedy Algorithm for Set Cover

$\mathcal{T} \leftarrow \emptyset$.
while there is an element not covered by $\mathcal{T}$ **do**
    Pick an arbitrary uncovered element e.
    Add all sets containing e to $\mathcal{T}$.
end while
return $\mathcal{T}$.

e_1

$S_1 = \{e_1\}$ $\quad$ $S_2 = \{e_2\}$ $\quad$ $S_3 = \{e_1, e_2\}$

$f = 2$ $\quad$ $\frac{2^0}{1} = f$

$T = \{S_1\ S_3\}$ $\quad$ S_3

Q1. (5 marks) Prove that the approximation ratio of Algorithm 1 is at least f. Namely, construct a set cover instance such that if we run Algorithm 1 on that instance, the ratio between the output and the instance's optimal solution is exactly f.

Q2. (10 marks) Prove that Algorithm 1 is f-approximate.

Q3. (10 marks) Now, let us consider the following LP rounding algorithm for the set cover problem. For each set S, we have a variable x_S: $x_S = 1$ indicates that we select set S; otherwise, we do not select the set S. Then, we have the following LP formulation (SC-LP). The second constraint ensures that for each element, one of the set includes the element is selected.

$x_S = 0.5$

$$\min \quad \sum_{S \in \mathcal{S}} x_S \tag{SC-LP}$$

s.t.

$$\sum_{S : e \in S} x_S \geq 1 \qquad \forall e \in U \tag{1}$$

$$x_S \geq 0, \qquad \forall S \in \mathcal{S} \tag{2}$$

Now, we solve (SC-LP) above and obtain the optimal fractional solution $(x^*_S)_{S \in \mathcal{S}}$. Consider the following simple rounding algorithm:

$$\mathcal{T} := \{ S \mid x^*_S \geq \frac{1}{f} \}.$$

Show that $\mathcal{T}$ is a f-approximate solution.

$x^*_{S_1}$ $\quad$ $x^*_{S_2}$ $\ldots$ $x^*_{S_t}$

$\frac{1}{f} = 0.3$ $\quad$ 0.2 $\quad$ 0.5 $\quad$ 0.8

2. Another LP Rounding for Vertex Cover

Given an undirected graph $G := (V, E)$ and find a min-size subset of vertices $S \subseteq V$ such that every edge is incident to at least one vertex in S.

In the lecture, we have seen that the weighted version of the problem admits a 2-approximate algorithm via a simple LP rounding. For each vertex i, we have a variable x_i indicating whether we pick i or not; see (VC-LP).

$$\begin{aligned} \min \quad & \sum_{i \in V} x_i & & \text{(VC-LP)} \\ \text{s.t.} \quad & & & \\ & x_i + x_j \geq 1 & \forall (i,j) \in E \quad & (3) \\ & x_i \geq 0, & \forall i \in V \quad & (4) \end{aligned}$$

Let $(x_i^*)_{i \in V}$ be the optimal fractional solution to (VC-LP). Now, we slightly change the parameter used in the lecture; recall that we used $\theta = 2$ in the lecture.

$$S = \left\{ i \in V \mid x_i^* \geq \frac{1}{\theta} \right\}, \quad \theta \geq 2.$$

4 2

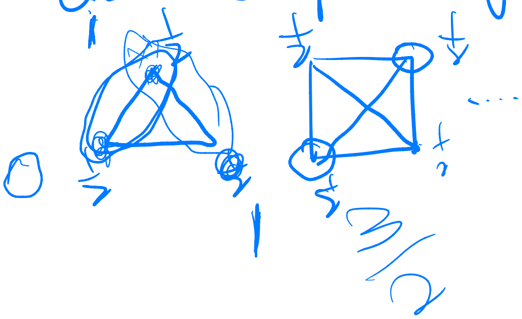
Q1. (6 marks) Prove that the new rounding algorithm is a θ-approximate algorithm, and explain why θ must be some value larger than 2.

Q2. (9 marks) Show that the *integrality gap* of (VC-LP) is at least $2 - \epsilon$ for any $\epsilon > 0$. Namely, construct a vertex cover instance such that the ratio between the optimal solution to that instance and a fractional solution to (VC-LP) is $2 - \epsilon$.

$$|S| = \sum_{i \in S} 1 \leq \sum_{i \in S} \theta \cdot x_i^* \leq \sum_{i \in V} \theta \cdot x_i^* = \theta \cdot \sum_{i \in V} x_i^* \leq \theta \cdot |OPT|$$

$\theta = \frac{3}{2} < 2 \qquad S = \{ i \in V \mid x_i^* > \frac{2}{3} \} \qquad x_i^* = \frac{1}{2}, x_j^* = \frac{1}{2} \quad (i, j)$

Q2: complete graph with n vertices $\leq (n-2)$



OPT (integral): $n-1$

fraction: $\frac{n}{2}$

$$\frac{n-1}{\frac{n}{2}} = 2 - \frac{2}{n} \to 2 \quad @\, n \to \infty$$

$\varepsilon > 0 \quad n = \frac{2}{\varepsilon} \quad \dots \; 2 - \varepsilon$

3. Core and Shapley Value

Consider the following cooperative game:

$$v(C) = \begin{cases} 2, & \text{if } C = \{1\} \\ 3, & \text{if } C = \{2\} \\ 4, & \text{if } C = \{3\} \\ 6, & \text{if } C = \{1,2\} \\ 8, & \text{if } C = \{1,3\} \\ 5, & \text{if } C = \{2,3\} \\ 16, & \text{if } C = \{1,2,3\} \end{cases}$$

$V(C) \le V(C')$ for $\forall\ C \subseteq C'$

123
132
213
231
312
321

Q1. (4 marks) Is the game monotonic? Is the game superadditive?

Q2. (12 marks) Derive the Shapley value and show the steps.

Q3. (4 marks) Suppose the Shapley value is ϕ. Is $(\{1,2,3\}, \phi)$ in the core?

4. Boolean Games

In Boolean games, there are n players $N = \{1, \dots, n\}$ and a set of boolean variables Φ. Each player i has a subset Φ_i indicating the set of variables whose values are under player i's control. No two players control one common variable and each variable in Φ is controlled by one player, i.e., $(\Phi_1, \dots, \Phi_n)$ is a partition of Φ. Each player will choose a value (0 or 1) for each variable under her control. After each player makes her choices, each variable in Φ will have a value and ξ denotes a value profile of Φ where each entry of ξ specifies the value of each variable.

Each player i also has a goal θ_i which is a boolean formula over Φ. Player i gets utility 1 if θ_i is True under ξ and gets 0 otherwise. Note that a player's goal formula may include variables controlled by other players, and thus what utility a player can obtain depends not just on the choices she herself makes.

Definition 1. *Given a value profile ξ, if there exist a coalition $S \subseteq N$ and a value profile $\xi' \neq \xi$ such that*

1. *$\xi = \xi'$ on those variables controlled by $i \in N \setminus S$, meaning that the values for those variables not controlled by players in S are the same in both ξ and ξ';*

2. *When deviating from ξ to ξ', each player $i \in S$'s utility will change from 0 to 1, meaning that each player in the coalition S strictly prefers ξ' to ξ,*

*then we say the coalition S **strictly prefers** ξ' to ξ.*

Definition 2. *A value profile ξ is in the **core**, if for any coalition $S \subseteq N$, there exists no value profile ξ' such that S strictly prefers ξ' to ξ.*

goal

Example 1. *Suppose $N = \{1, 2\}$ with $\Phi = \{p, q\}$, $\Phi_1 = \{p\}$, $\Phi_2 = \{q\}$. θ_1 is $(p \wedge \neg q) \vee (\neg p \wedge q)$, θ_2 is $(\neg p \vee q) \wedge (p \vee \neg q)$. Then player 1 gets utility 1 by any value profile that gives p and q different values, while player 2 gets utility 1 by any value profile that gives p and q the same value.*

Definition 3. *A **stable set** X is a set of value profiles, which have the property that they are both internally and externally stable,*

1. *Internal stability: If a value profile $\xi \in X$, for any coalition $S \subseteq N$, there exists no $\xi' \in X$ such that S strictly prefers ξ' to ξ;*

2. *External stability: For any $\xi \notin X$, there exist a coalition S and a value profile $\xi' \in X$, such that S strictly prefers ξ' to ξ.*

$S = \{1\}$ $\xi' = (p = 1, q = 0)$

Q1. (6 marks) Given a value profile $\xi = (p = 0, q = 0)$, do there exist a coalition S and a value profile ξ', such that S strictly prefers ξ' to ξ? If yes, find the S and ξ'; if no, explain why.

$(\cancel{p=0 \; q=1})$

Q2. (6 marks) For the given example above, does there exist value profile(s) in the core? If yes, list all of them; if no, for each possible value profile ξ, find the corresponding coalition S and another value profile ξ' such that S strictly prefers ξ' to ξ.

Q3. (8 marks) Use the given example to show that there may exist multiple stable sets.

p	q
0	0
0	1
1	0
1	1

$\to$ $\{(0,0), (1,1)\}$

$\{(1,0), (0,1)\}$

5. Hedonic Games:

In the context of **hedonic games**, a set of players form coalitions, and each player has preferences over which coalition they belong to, based solely on the members of the coalition. Consider the following scenario:

There are 5 players, denoted by $P = \{1, 2, 3, 4, 5\}$. Let $\succ_i$ denote the **strict preference relation** of player i over coalitions. That is, for any two coalitions S and T where $i \in S$ and $i \in T$, we write $S \succ_i T$ if player i strictly prefers to be in coalition S rather than in coalition T. Each player has preferences over coalitions as follows:

- **Player 1**:
$$\{1, 2, 4\} \succ_1 \{1, 2, 3\} \succ_1 \{1\} \succ_1 \ldots$$
- **Player 2**:
$$\{2, 3\} \succ_2 \{1, 2, 3\} \succ_2 \{2\} \succ_2 \ldots$$
- **Player 3**:
$$\{2, 3\} \succ_3 \{3\} \succ_3 \ldots$$
- **Player 4**:
$$\{4, 5\} \succ_4 \{4\} \succ_4 \ldots$$
- **Player 5**:
$$\{4, 5\} \succ_5 \{5\} \succ_5 \ldots$$

(2 3) (1 4 5)

Based on the preferences above, answer the following questions:

No

Q1. (5 marks) A coalition structure is **individually rational** if no player strictly prefers being alone (in a singleton coalition) over being in their current coalition. Is the coalition structure $\{\{1, 2\}, \{3, 4, 5\}\}$ individually rational for all players? Why or why not? 3

Q2. (5 marks) A coalition structure is in the **core** if there is no coalition S such that every player in S strictly prefers being in S to their current coalition. Is the coalition structure $\{\{1, 2, 3\}, \{4, 5\}\}$ core-stable? Why or why not? No

$\{2,3\}$

Q3. (10 marks) A coalition structure is **Nash-stable** if no player can benefit by unilaterally moving to a different coalition, including forming a singleton coalition. Find a Nash-stable partition. Justify your answer by explaining why no player has an incentive to deviate.

No

1 $C = \{\{2,3\}, \ldots\}$ player 1

2 $C = \{\{3\}, \ldots\}$ player 2

3 $C = \{$ other ~~than~~ 1 2 $\}$ player 3