Home Assignment №4 Solutions

November 27, 2024

Exercise 1

[2 points]. Prove that the eigenvalues of a real symmetric matrix $A \in \mathbb{R}^{N\times N}$, where $A = A^T$, are real-valued (*i.e.*, not complex-valued). Prove also that the eigenvectors of A corresponding to different eigenvalues are orthogonal to each other.

Proof. a) let (λ, v) be any eigenpair of A. Let

$$A = A^T = A^*,$$

we have

$$\begin{aligned} < Av, Av > &= v^* A^* Av \\ &= v^* A^* \lambda v \\ &= \lambda v^* Av \\ &= \lambda^2 ||v||_2^2, \end{aligned}$$

thus

$$\lambda^2 = \frac{< Av, Av >}{||v||_2^2}.$$

Since λ^2 is real and non-negtive number, λ must be real.

b) For any real matrix A and any vectors x and y, we have

$$< Ax, y >=< x, A^T y > .$$

We assume x and y are eigenvectors of A corresponding to distinct eigenvalue λ and μ

$$\begin{aligned}
\lambda < x, y > &= < \lambda x, y > \\
&= < Ax, y > \\
&= < x, A^T y > \\
&= < x, Ay > \\
&= < x, \mu y > \\
&= \mu < x, y > .
\end{aligned}$$

Therefore,

$$(\lambda - \mu) < x, y >= 0.$$

Since $\lambda \neq \mu$, we must have

$$< x, y >= 0,$$

that is, eigenvectors of real symmetry matrix are orthogonal.

$\square$

Exercise 2

[5 points]. Derive the K-th largest direction of variance in principal component analysis (PCA).

Solution: Finding the K-th largest direction of variance v_K in PCA corresponding to solving the optimization problem:

$$\begin{aligned}
\max_{v} \quad & v^T \Sigma v \\
\text{s.t.} \quad & ||v||_2^2 = 1 \\
& v^T v_i = 0, for\ i = 1, 2, \ldots, K-1,
\end{aligned}$$

where $\Sigma = \frac{1}{M} X^T X$. When $K = 1$, finding v_1 turns out to be

$$\begin{aligned}
\max_{v} \quad & v^T \Sigma v \\
\text{s.t.} \quad & ||v||_2^2 = 1.
\end{aligned}$$

We write down its Lagrangian

$$L(v,\lambda) = v^T\Sigma v + \lambda(1 - ||v||_2^2), \tag{1}$$

$$\frac{\partial L}{\partial v} = 2\Sigma v - 2\lambda v. \tag{2}$$

Hence $\Sigma v_1 = \lambda v_1$, i.e. v_1 is an eigenvector of Σ with eigenvalue λ. Since we constrain $v_1^T v_1 = 1$, the value of the objective is precisely

$$v_1^T \Sigma v_1 = \lambda, \tag{3}$$

so the optimal value for λ is the largest eigenvalue of Σ, which is achieved when v_1 is a unit eigenvector of Σ corresponding to its largest eigenvalue. For v_K, the Lagrangian is

$$L(v,\lambda,v_1,...,v_{K-1},\eta_1,...,\eta_{K-1}) = v^T\Sigma v + \lambda(1 - ||v||_2^2) + \sum_{i=1}^{K-1} \eta_i v^T v_i. \tag{4}$$

$$\frac{\partial L}{\partial v} = 2\Sigma v - 2\lambda v + \sum_{i=1}^{K-1} \eta_i v_i = 0, \tag{5}$$

which can be written as

$$2\Sigma v_K - 2\lambda v_K + \sum_{i=1}^{K-1} \eta_i v_i = 0. \tag{6}$$

Multiply both sides of the Eq. (6) by $v_j, j = 1, \ldots, K-1$, we have

$$2\Sigma v_K v_j - 2\lambda v_K v_j + \sum_{i=1}^{K-1} \eta_i v_i v_j = 0 \cdot v_j = 0. \tag{7}$$

Note that

$$v_i v_j \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}. \tag{8}$$

From the orthogonal constraints above we can conclude that

$$\eta_j = 0, j = 1, \ldots, K-1.$$

Plugging these values back into the optimality equation above, we see that v_K must satisfy $\Sigma v_K = \lambda v_K$, i.e. v_K is an eigenvector of Σ with eigenvalue λ. As before, the value of the objective function is then λ. To maximize, we want the largest eigenvalue, but we must respect the constraints that v_K is orthogonal to $v_1, \ldots, v_{K-1}$. Thus to maximize the expression, v_K should be a unit eigenvector of Σ corresponding to its K-th largest eigenvalue.

Exercise 3

[5 points]. 1) Suppose that a discrete-time linear system has outputs $y[n]$ for the given inputs $x[n]$, as shown in Fig. 1. Determine the response $y_4[n]$ when the input is as shown in Fig. 2.

a) [1 point]. Express $x_4[n]$ as a linear combination of $x_1[n]$, $x_2[n]$, and $x_3[n]$.

b) [1 point]. Using the fact that the system is linear, determine $y_4[n]$, the response to $x_4[n]$.

c) [1 point]. From the input-output pairs in Fig. 1, determine whether the system is time-invariant.

2) Determine the discrete-time convolution of $x[n]$ and $h[n]$ for the following two cases.

a) [1 point]. As shown in Fig. 3.

b) [1 point]. As shown in Fig. 4.

Solution: 1)

a)

$$x_4[n] = 2x_1[n] - 2x_2[n] + x_3[n]$$

b)

$$y_4[n] = 2y_1[n] - 2y_2[n] + y_3[n]$$

c) The system is not time-invariant because an input $x_i[n] + x_i[n-1]$ does not produce an output $y_i[n] + y_i[n-1]$. The input $x_1[n] + x_1[n-1]$ is $x_1[n] + x_1[n-1] = x_2[n]$, which we are told produces $y_2[n]$. Since $y_2[n] \neq y_1[n] + y_1[n-1]$, this system is not time-invariant.

2)

a)

$$
\begin{aligned}
y[0] &= 2 \\
y[1] &= 4 \\
y[2] &= 6 \\
y[3] &= 8 \\
y[4] &= 6 \\
y[5] &= 4 \\
y[6] &= 2 \\
y[7] &= 0
\end{aligned}
$$

b)

$$
\begin{aligned}
y[0] &= 0 \\
y[1] &= 0 \\
y[2] &= \frac{1}{2} \\
y[3] &= 1 \\
y[4] &= \frac{3}{2} \\
y[5] &= 1 \\
y[6] &= \frac{1}{2} \\
y[7] &= 0
\end{aligned}
$$

Exercise 4

[3 points] From the lecture slides, we know that the convolution of discrete-time signals $x[n]$ and $y[n]$, for $n \in [-\infty, +\infty]$, is defined as

$$z[n] = (x * y)[n] = \sum_{m=-\infty}^{\infty} x[m]y[n-m] = \sum_{m=-\infty}^{\infty} x[n-m]y[m]. \tag{9}$$

Here we introduce the Discrete-time Fourier Transform (DTFT) of a discrete-time signal $x[n]$:

$$X(\omega) = \mathrm{DTFT}(x) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}. \tag{10}$$

Try to prove the Convolution Theorem:

$$Z(\omega) = \mathrm{DTFT}(x * y) = \mathrm{DTFT}(x) \cdot \mathrm{DTFT}(y) = X(\omega) \cdot Y(\omega). \tag{11}$$

Proof.

$$\begin{aligned}
\mathrm{DTFT}(x * y) &= \sum_{n=-\infty}^{\infty} (x * y)[n] e^{-j\omega n} \\
&= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} x[m] y[n-m] e^{-j\omega n} \\
&= \sum_{m=-\infty}^{\infty} x[m] \sum_{n=-\infty}^{\infty} y[n-m] e^{-j\omega n} \\
&= \sum_{m=-\infty}^{\infty} x[m] \sum_{t=-\infty}^{\infty} y[t] e^{-j\omega(t+m)} \\
&= \sum_{m=-\infty}^{\infty} x[m] e^{-j\omega m} \sum_{t=-\infty}^{\infty} y[t] e^{-j\omega t} \\
&= X(\omega) Y(\omega).
\end{aligned}$$

$\square$

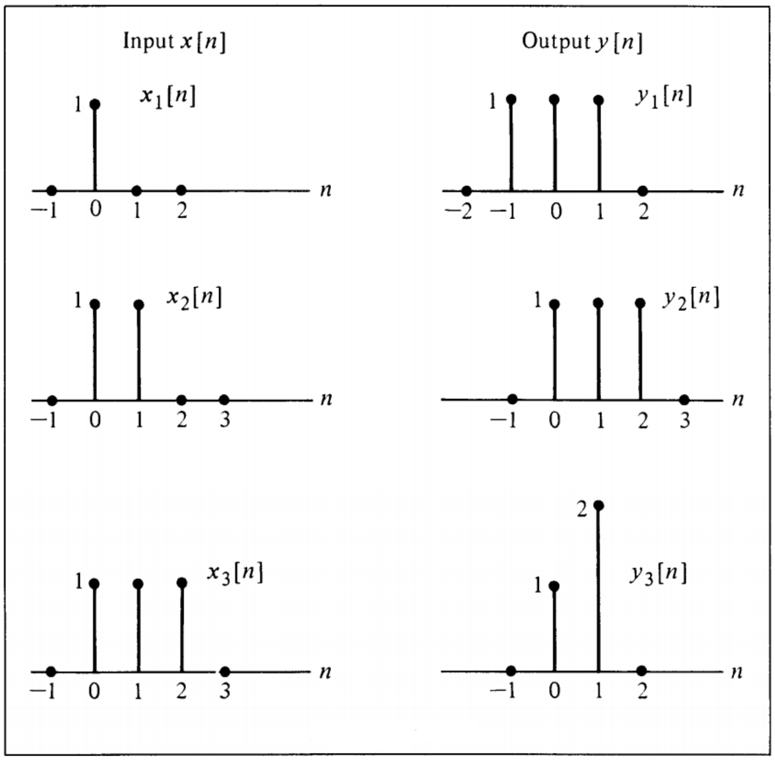

Figure 1:

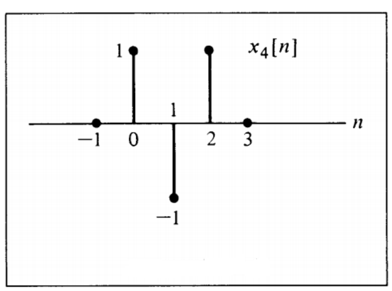

Figure 2:

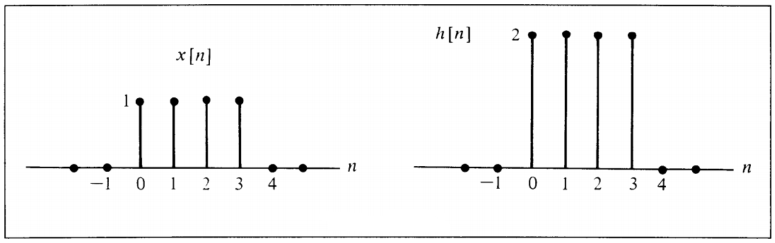

Figure 3:

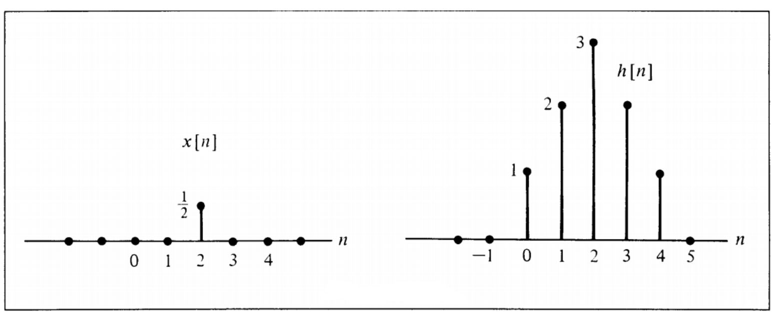

Figure 4: