Optimization Lecture 11: Interior-point methods

Qingfu Zhang

Dept of CS , CityU

2024

Outline

Inequality constrained minimization

Inequality constrained minimization

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\ & Ax = b \end{array}$$

we assume

- f_i convex, twice continuously differentiable
- $A \in \mathbf{R}^{p \times n}$ with $\mathbf{rank}\, A = p$
- $p^\star$ is finite and attained
- problem is strictly feasible: there exists $\tilde{x}$ with

$$\tilde{x} \in \mathbf{dom}\, f_0, \quad f_i(\tilde{x}) < 0, \quad i = 1, \ldots, m, \quad A\tilde{x} = b$$

hence, strong duality holds and dual optimum is attained

Examples

- LP, QP, QCQP, GP
- entropy maximization with linear inequality constraints

$$\begin{array}{ll} \text{minimize} & \sum_{i=1}^{n} x_i \log x_i \\ \text{subject to} & Fx \preceq g, \quad Ax = b \end{array}$$

  with $\mathbf{dom}\, f_0 = \mathbf{R}^n_{++}$

- differentiability may require reformulating the problem, e.g., piecewise-linear minimization or ℓ_∞-norm approximation via LP

Logarithmic barrier and central path

Logarithmic barrier

- reformulation via **indicator function**:

$$\begin{array}{ll} \text{minimize} & f_0(x) + \sum_{i=1}^m I_-(f_i(x)) \\ \text{subject to} & Ax = b \end{array}$$

  where $I_-(u) = 0$ if $u \leq 0$, $I_-(u) = \infty$ otherwise

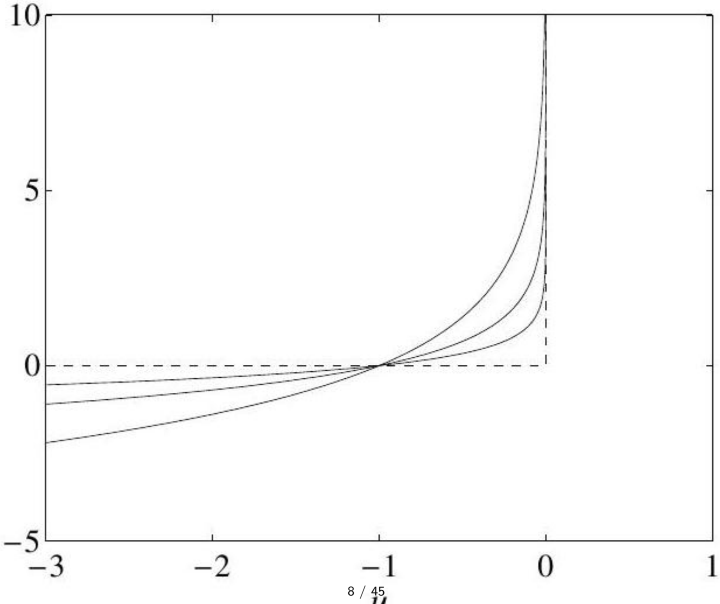
- **approximation via logarithmic barrier**:

$$\begin{array}{ll} \text{minimize} & f_0(x) - (1/t)\sum_{i=1}^m \log(-f_i(x)) \\ \text{subject to} & Ax = b \end{array}$$

- an equality constrained problem
- for $t > 0$, $-(1/t)\log(-u)$ is a smooth approximation of I_-
- approximation improves as $t \to \infty$

Logarithmic barrier

- $-(1/t) \log u$ for three values of t, and $I_-(u)$



Logarithmic barrier function

- log barrier function for constraints $f_1(x) \le 0, \dots, f_m(x) \le 0$

$$\phi(x) = -\sum_{i=1}^{m} \log\left(-f_i(x)\right), \quad \mathbf{dom}\,\phi = \{x \mid f_1(x) < 0, \dots, f_m(x) < 0\}$$

- convex (from composition rules)
- twice continuously differentiable, with derivatives

$$\nabla \phi(x) = \sum_{i=1}^{m} \frac{1}{-f_i(x)} \nabla f_i(x)$$

$$\nabla^2 \phi(x) = \sum_{i=1}^{m} \frac{1}{f_i(x)^2} \nabla f_i(x) \nabla f_i(x)^T + \sum_{i=1}^{m} \frac{1}{-f_i(x)} \nabla^2 f_i(x)$$

Central path

- for $t > 0$, define $x^\star(t)$ as the solution of

$$\begin{array}{ll} \text{minimize} & tf_0(x) + \phi(x) \\ \text{subject to} & Ax = b \end{array}$$

  (for now, assume $x^\star(t)$ exists and is unique for each $t > 0$)

- central path is $\{x^\star(t) \mid t > 0\}$

example: central path for an LP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & a_i^T x \leq b_i, \quad i = 1, \dots, 6 \end{array}$$

hyperplane $c^T x = c^T x^\star(t)$ is tangent to level curve of ϕ through $x^\star(t)$

Dual points on central path

- $x = x^\star(t)$ if there exists a w such that

$$t\nabla f_0(x) + \sum_{i=1}^{m} \frac{1}{-f_i(x)} \nabla f_i(x) + A^T w = 0, \quad Ax = b$$

- therefore, $x^\star(t)$ minimizes the Lagrangian

$$L\left(x, \lambda^\star(t), v^\star(t)\right) = f_0(x) + \sum_{i=1}^{m} \lambda_i^\star(t) f_i(x) + v^\star(t)^T (Ax - b)$$

where we define $\lambda_i^\star(t) = 1/\left(-tf_i\left(x^\star(t)\right)\right.$ and $v^\star(t) = w/t$

- this confirms the intuitive idea that $f_0\left(x^\star(t)\right) \to p^\star$ if $t \to \infty$:

$$p^\star \geq g\left(\lambda^\star(t), v^\star(t)\right) = L\left(x^\star(t), \lambda^\star(t), v^\star(t)\right) = f_0\left(x^\star(t)\right) - m/t$$

Interpretation via KKT conditions

$x = x^\star(t), \lambda = \lambda^\star(t), v = v^\star(t)$ satisfy

1. primal constraints: $f_i(x) \leq 0, i = 1, \dots, m, Ax = b$
2. dual constraints: $\lambda \geq 0$
3. approximate complementary slackness: $-\lambda_i f_i(x) = 1/t, i = 1, \dots, m$
4. gradient of Lagrangian with respect to x vanishes:

$$\nabla f_0(x) + \sum_{i=1}^{m} \lambda_i \nabla f_i(x) + A^T v = 0$$

difference with KKT is that condition 3 replaces $\lambda_i f_i(x) = 0$

Force field interpretation

- centering problem (for problem with no equality constraints)

$$\text{minimize } tf_0(x) - \sum_{i=1}^{m} \log\left(-f_i(x)\right)$$

- force field interpretation
  - $tf_0(x)$ is potential of force field $F_0(x) = -t\nabla f_0(x)$
  - $-\log\left(-f_i(x)\right)$ is potential of force field $F_i(x) = (1/f_i(x))\,\nabla f_i(x)$
- forces balance at $x^\star(t)$:

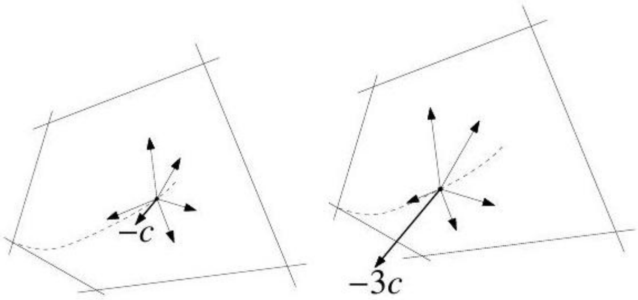
$$F_0\left(x^\star(t)\right) + \sum_{i=1}^{m} F_i\left(x^\star(t)\right) = 0$$

Example: LP

- minimize $c^T x$ subject to $a_i^T x \leq b_i$, $i = 1, \ldots, m$, with $x \in \mathbf{R}^n$
- objective force field is constant: $F_0(x) = -tc$
- constraint force field decays as inverse distance to constraint hyperplane:

$$F_i(x) = \frac{-a_i}{b_i - a_i^T x}, \quad \|F_i(x)\|_2 = \frac{1}{\mathrm{dist}\,(x, \mathcal{H}_i)}$$

where $\mathcal{H}_i = \{x \mid a_i^T x = b_i\}$



(a) $t = 1$ (b) $t = 3$

Barrier method

Barrier method

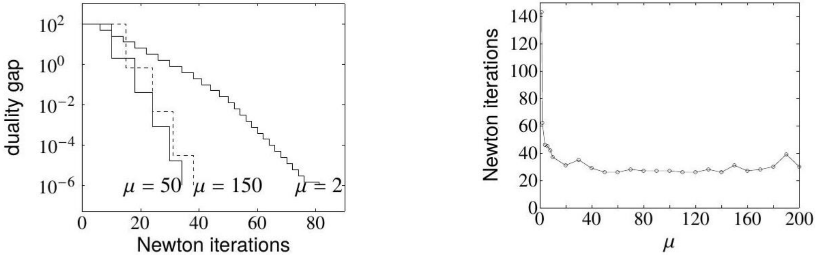
given strictly feasible $x, t := t^{(0)} > 0, \mu > 1$, tolerance $\epsilon > 0$.
repeat

1. Centering step. Compute $x^\star(t)$ by minimizing $tf_0 + \phi$, subject to $Ax = b$.
2. Update. $x := x^\star(t)$.
3. Stopping criterion. **quit** if $m/t < \epsilon$.
4. Increase $t . t := \mu t$.

- terminates with $f_0(x) - p^\star \leq \epsilon$ (stopping criterion follows from $f_0(x^\star(t)) - p^\star \leq m/t$)
- centering usually done using Newton's method, starting at current x
- choice of μ involves a trade-off: large μ means fewer outer iterations, more inner (Newton) iterations; typical values: $\mu = 10$ or 20
- several heuristics for choice of $t^{(0)}$

Example: Inequality form LP

($m = 100$ inequalities, $n = 50$ variables)

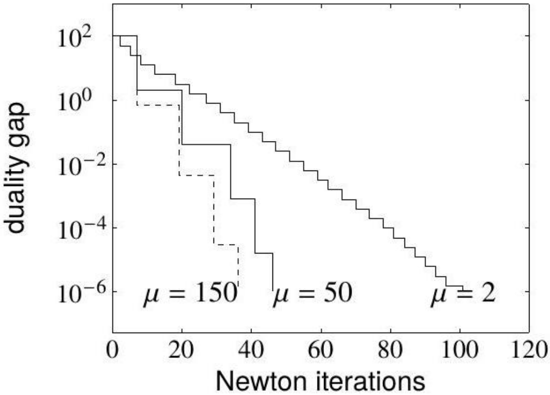


- starts with x on central path ($t^{(0)} = 1$, duality gap 100)
- terminates when $t = 10^8$ (gap 10^{-6})
- total number of Newton iterations not very sensitive for $\mu \geq 10$

Example: Geometric program in convex form

($m = 100$ inequalities and $n = 50$ variables)

$$\begin{array}{ll} \text{minimize} & \log\left(\sum_{k=1}^{5} \exp\left(a_{0k}^T x + b_{0k}\right)\right) \\ \text{subject to} & \log\left(\sum_{k=1}^{5} \exp\left(a_{ik}^T x + b_{ik}\right)\right) \leq 0, \quad i = 1, \ldots, m \end{array}$$

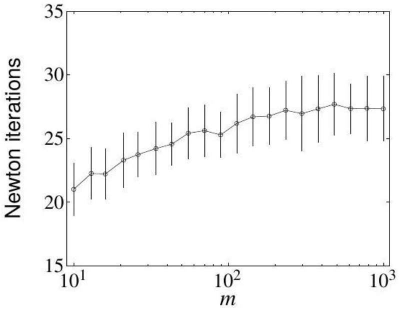


Family of standard LPs

$(A \in \mathbf{R}^{m \times 2m})$

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b, \quad x \succeq 0 \end{array}$$

$m = 10, \ldots, 1000$; for each m, solve 100 randomly generated instances



number of iterations grows very slowly as m ranges over a 100 : 1 ratio

Phase I methods

Phase I methods

- barrier method needs strictly feasible starting point, i.e., x with

$$f_i(x) < 0, \quad i = 1, \ldots, m, \quad Ax = b$$

- (like the infeasible start Newton method, more sophisticated interior-point methods do not require a feasible starting point)
- **phase I** method forms an optimization problem that
  - is itself strictly feasible
  - finds a strictly feasible point for original problem, if one exists
  - certifies original problem as infeasible otherwise
- **phase II** uses barrier method starting from strictly feasible point found in phase I

Basic phase I method

- introduce slack variable s in **phase I problem**

$$\begin{array}{ll} \text{minimize(over } x, s) & s \\ \text{subject to} & f_i(x) \leq s, \quad i = 1, \ldots, m \\ & Ax = b \end{array}$$

  with optimal value $\bar{p}^\star$
  - if $\bar{p}^\star < 0$, original inequalities are strictly feasible
  - if $\bar{p}^\star > 0$, original inequalities are infeasible
  - $\bar{p}^\star = 0$ is an ambiguous case
- start phase I problem with
  - any $\tilde{x}$ in problem domain with $A\tilde{x} = b$
  - $s = 1 + \max_i f_i(\tilde{x})$

Sum of infeasibilities phase I method

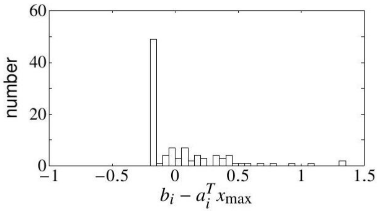
- minimize **sum** of slacks, not max:

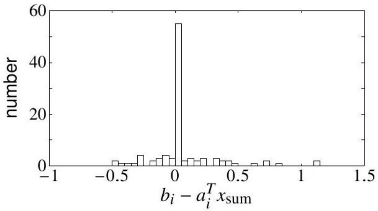
$$\begin{array}{ll} \text{minimize} & \mathbf{1}^T s \\ \text{subject to} & s \succeq 0, \quad f_i(x) \leq s_i, \quad i = 1, \ldots, m \\ & Ax = b \end{array}$$

- for infeasible problems, produces a solution that satisfies many (but not all) inequalities
- can weight slacks to set **priorities** (in satifying constraints)

Example

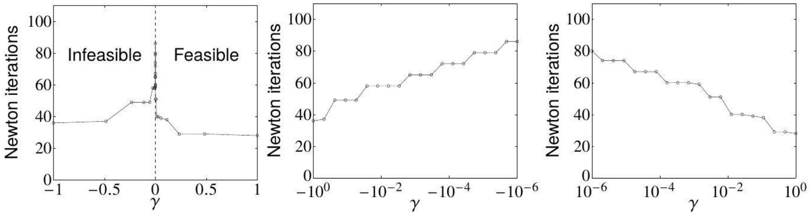
- infeasible set of 100 linear inequalities in 50 variables
- left: basic phase I solution; satisfies 39 inequalities
- right: sum of infeasibilities phase I solution; satisfies 79 inequalities





Example: Family of linear inequalities

- $Ax \leq b + \gamma \Delta b$; strictly feasible for $\gamma > 0$, infeasible for $\gamma < 0$
- use basic phase I, terminate when $s < 0$ or dual objective is positive
- number of iterations roughly proportional to $\log(1/|\gamma|)$



Complexity analysis

Number of outer iterations

- in each iteration duality gap is reduced by exactly the factor μ
- **number of outer (centering) iterations** is exactly

$$\left\lceil \frac{\log\left(m/\left(\epsilon t^{(0)}\right)\right)}{\log \mu} \right\rceil$$

  plus the initial centering step (to compute $x^\star\left(t^{(0)}\right)$)
- we will bound **number of Newton steps per centering iteration** using self-concordance analysis

Complexity analysis via self-concordance

same assumptions as on slide 4, plus:

- sublevel sets (of f_0, on the feasible set) are bounded
- $tf_0 + \phi$ is self-concordant with closed sublevel sets

Second condition

- holds for LP, QP, QCQP
- may require reformulating the problem, e.g.,

$$\begin{array}{ll} \text{minimize} & \sum_{i=1}^{n} x_i \log x_i \\ \text{subject to} & Fx \leq g \end{array} \quad \longrightarrow \quad \begin{array}{ll} \text{minimize} & \sum_{i=1}^{n} x_i \log x_i \\ \text{subject to} & Fx \leq g, \quad x \geq 0 \end{array}$$

- needed for complexity analysis; barrier method works even when self-concordance assumption does not apply

Newton iterations per centering step

- we compute $x^+ = x^\star(\mu t)$, by minimizing $\mu t f_0(x) + \phi(x)$ starting from $x = x^\star(t)$
- from self-concordance theory,

$$\#\text{Newton iterations} \;\leq \frac{\mu t f_0(x) + \phi(x) - \mu t f_0\left(x^+\right) - \phi\left(x^+\right)}{\gamma} + c$$

- γ, c are constants (that depend only on Newton algorithm parameters)

Newton iterations per centering step

- we will bound numerator $\mu t f_0(x) + \phi(x) - \mu t f_0\left(x^+\right) - \phi\left(x^+\right)$
- with $\lambda_i = \lambda_i^\star(t) = -1/\left(t f_i(x)\right)$, we have $-f_i(x) = 1/\left(t\lambda_i\right)$, so

$$\phi(x) = \sum_{i=1}^{m} -\log\left(-f_i(x)\right) = \sum_{i=1}^{m} \log\left(t\lambda_i\right)$$

so

$$\begin{aligned} \phi(x) - \phi\left(x^+\right) &= \sum_{i=1}^{m} \left(\log\left(t\lambda_i\right) + \log\left(-f_i\left(x^+\right)\right)\right) \\ &= \sum_{i=1}^{m} \log\left(-\mu t \lambda_i f_i\left(x^+\right)\right) - m \log \mu \end{aligned}$$

Newton iterations per centering step

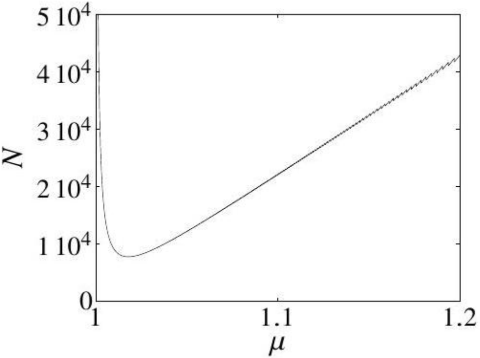
using $\log u \leq u - 1$ we have
$\phi(x) - \phi\left(x^{+}\right) \leq -\mu t \sum_{i=1}^{m} \lambda_i f_i\left(x^{+}\right) - m - m \log \mu$, so

$$
\begin{aligned}
\mu t f_0(x) &+ \phi(x) - \mu t f_0\left(x^{+}\right) - \phi\left(x^{+}\right) \\
&\leq \mu t f_0(x) - \mu t f_0\left(x^{+}\right) - \mu t \sum_{i=1}^{m} \lambda_i f_i\left(x^{+}\right) - m - m \log \mu \\
&= \mu t f_0(x) - \mu t \left(f_0\left(x^{+}\right) + \sum_{i=1}^{m} \lambda_i f_i\left(x^{+}\right) + v^T\left(A x^{+} - b\right) \right) - m - m \log \mu \\
&= \mu t f_0(x) - \mu t L\left(x^{+}, \lambda, v\right) - m - m \log \mu \\
&\leq \mu t f_0(x) - \mu t g(\lambda, v) - m - m \log \mu \\
&= m(\mu - 1 - \log \mu)
\end{aligned}
$$

using $L\left(x^{+}, \lambda, nu\right) \geq g(\lambda, v)$ in second last line and
$f_0(x) - g(\lambda, v) = m/t$ in last line

Total number of Newton iterations

$$\#\text{Newton iterations} \le N = \left\lceil \frac{\log\left(m/\left(t^{(0)}\epsilon\right)\right)}{\log \mu} \right\rceil \left(\frac{m(\mu - 1 - \log \mu)}{\gamma} + c \right)$$



N versus μ for typical values of γ, c; $m = 100$, initial duality gap $\frac{m}{t^{(0)}\epsilon} = 10^5$

- confirms trade-off in choice of μ
- in practice, #iterations is in the tens; not very sensitive for $\mu \ge 10$

Polynomial-time complexity of barrier method

- for $\mu = 1 + 1/\sqrt{m}$:

$$N = O\left(\sqrt{m}\log\left(\frac{m/t^{(0)}}{\epsilon}\right)\right)$$

- number of Newton iterations for fixed gap reduction is $O(\sqrt{m})$
- multiply with cost of one Newton iteration (a polynomial function of problem dimensions), to get bound on number of flops
- this choice of μ optimizes worst-case complexity; in practice we choose μ fixed and larger

Generalized inequalities

Generalized inequalities

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \preceq_{K_i} 0, \quad i = 1, \ldots, m \\ & Ax = b \end{array}$$

- f_0 convex, $f_i : \mathbf{R}^n \to \mathbf{R}^{k_i}$, $i = 1, \ldots, m$, convex with respect to proper cones $K_i \in \mathbf{R}^{k_i}$
- we assume
  - f_i twice continuously differentiable
  - $A \in \mathbf{R}^{p \times n}$ with $\mathbf{rank}\, A = p$
  - $p^\star$ is finite and attained
  - problem is strictly feasible; hence strong duality holds and dual optimum is attained
- examples of greatest interest: SOCP, SDP

Generalized logarithm for proper cone

$\psi : \mathbf{R}^q \to \mathbf{R}$ is **generalized logarithm** for proper cone $K \subseteq \mathbf{R}^q$ if:

- $\operatorname{dom} \psi = \operatorname{int} K$ and $\nabla^2 \psi(y) \prec 0$ for $y >_K 0$
- $\psi(sy) = \psi(y) + \theta \log s$ for $y >_K 0, s > 0$ (θ is the degree of ψ)

examples

- nonnegative orthant $K = \mathbf{R}^n_+ : \psi(y) = \sum_{i=1}^n \log y_i$, with degree $\theta = n$
- positive semidefinite cone $K = \mathbf{S}^n_+ : \psi(Y) = \log \det Y$, with degree $\theta = n$
- second-order cone $K = \left\{ y \in \mathbf{R}^{n+1} \mid \left(y_1^2 + \cdots + y_n^2\right)^{1/2} \leq y_{n+1} \right\}$:

$$\psi(y) = \log \left(y_{n+1}^2 - y_1^2 - \cdots - y_n^2\right) \qquad \text{with degree } (\theta = 2)$$

Properties

- (without proof): for $y >_K 0$,

$$\nabla\psi(y) \succeq_{K^*} 0, \quad y^T\nabla\psi(y) = \theta$$

- nonnegative orthant $\mathbf{R}^n_+$: $\psi(y) = \sum_{i=1}^n \log y_i$

$$\nabla\psi(y) = (1/y_1, \ldots, 1/y_n), \quad y^T\nabla\psi(y) = n$$

- positive semidefinite cone $\mathbf{S}^n_+$: $\psi(Y) = \log\det Y$

$$\nabla\psi(Y) = Y^{-1}, \quad \mathrm{tr}(Y\nabla\psi(Y)) = n$$

- second-order cone $K = \left\{y \in \mathbf{R}^{n+1} \mid \left(y_1^2 + \cdots + y_n^2\right)^{1/2} \leq y_{n+1}\right\}$:

$$\nabla\psi(y) = \frac{2}{y_{n+1}^2 - y_1^2 - \cdots - y_n^2}\begin{bmatrix} -y_1 \\ \vdots \\ -y_n \\ y_{n+1} \end{bmatrix}, \quad y^T\nabla\psi(y) = 2$$

Logarithmic barrier and central path

logarithmic barrier for $f_1(x) \preceq_{K_1} 0, \ldots, f_m(x) \preceq_{K_m} 0$:

$$\phi(x) = -\sum_{i=1}^{m} \psi_i\left(-f_i(x)\right), \quad \mathbf{dom}\,\phi = \{x \mid f_i(x) \prec_{K_i} 0, i = 1, \ldots, m\}$$

- ψ_i is generalized logarithm for K_i, with degree θ_i
- ϕ is convex, twice continuously differentiable

central path: $\{x^\star(t) \mid t > 0\}$ where $x^\star(t)$ is solution of

$$\begin{array}{ll} \text{minimize} & tf_0(x) + \phi(x) \\ \text{subject to} & Ax = b \end{array}$$

Dual points on central path

$x = x^\star(t)$ if there exists $w \in \mathbf{R}^p$,

$$t\nabla f_0(x) + \sum_{i=1}^m Df_i(x)^T \nabla\psi_i\left(-f_i(x)\right) + A^T w = 0$$

$\left(Df_i(x) \in \mathbf{R}^{k_i \times n}\right.$ is derivative matrix of $\left.f_i\right)$

- therefore, $x^\star(t)$ minimizes Lagrangian $L\left(x, \lambda^\star(t), v^\star(t)\right)$, where

$$\lambda_i^\star(t) = \frac{1}{t}\nabla\psi_i\left(-f_i\left(x^\star(t)\right)\right), \quad v^\star(t) = \frac{w}{t}$$

- from properties of ψ_i : $\lambda_i^\star(t) >_{K_i^*} 0$, with duality gap

$$f_0\left(x^\star(t)\right) - g\left(\lambda^\star(t), v^\star(t)\right) = (1/t)\sum_{i=1}^m \theta_i$$

Example: Semidefinite programming

(with $F_i \in \mathbf{S}^p$)

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & F(x) = \sum_{i=1}^n x_i F_i + G \preceq 0 \end{array}$$

- logarithmic barrier: $\phi(x) = \log\det\left(-F(x)^{-1}\right)$
- central path: $x^\star(t)$ minimizes $tc^T x - \log\det(-F(x))$; hence

$$tc_i - \operatorname{tr}\left(F_i F\left(x^\star(t)\right)^{-1}\right) = 0, \quad i = 1, \ldots, n$$

- dual point on central path: $Z^\star(t) = -(1/t) F\left(x^\star(t)\right)^{-1}$ is feasible for

$$\begin{array}{ll} \text{maximize} & \operatorname{tr}(GZ) \\ \text{subject to} & \operatorname{tr}\left(F_i Z\right) + c_i = 0, \quad i = 1, \ldots, n \\ & Z \geq 0 \end{array}$$

- duality gap on central path: $c^T x^\star(t) - \operatorname{tr}\left(GZ^\star(t)\right) = p/t$

Barrier method

---

given strictly feasible $x, t := t^{(0)} > 0, \mu > 1$, tolerance $\epsilon > 0$.
repeat

1. Centering step. Compute $x^\star(t)$ by minimizing $tf_0 + \phi$, subject to $Ax = b$.
2. Update. $x := x^\star(t)$.
3. Stopping criterion. quit if $\left(\sum_i \theta_i\right)/t < \epsilon$.
4. Increase $t . t := \mu t$.

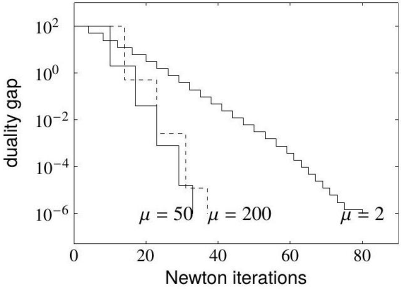
- only difference is duality gap m/t on central path is replaced by $\sum_i \theta_i / t$
- number of outer iterations:

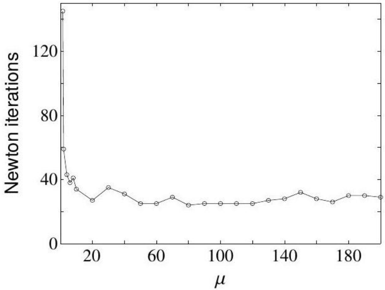
$$\left\lceil \frac{\log\left(\left(\sum_i \theta_i\right) / \left(\epsilon t^{(0)}\right)\right)}{\log \mu} \right\rceil$$

- complexity analysis via self-concordance applies to SDP, SOCP

Example: SOCP

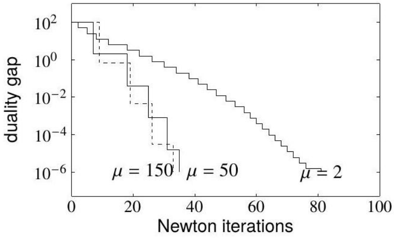
(50 variables, 50 SOC constraints in $\mathbf{R}^6$)

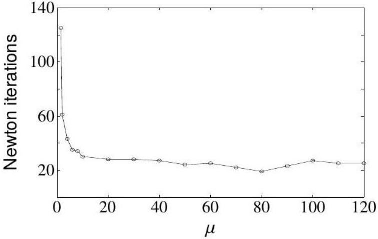




Example: SDP

(100 variables, LMI constraint in $\mathbf{S}^{100}$)



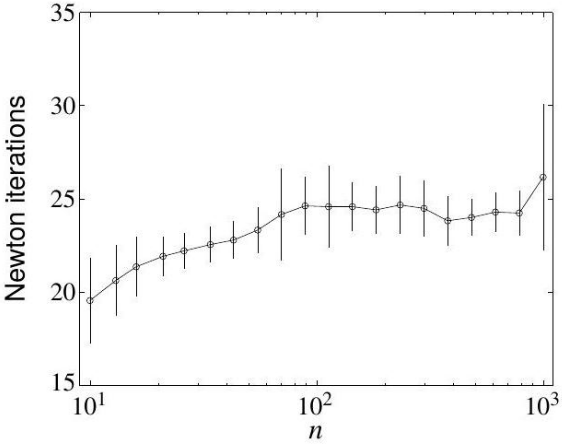


Example: Family of SDPs

$(A \in \mathbf{S}^n, x \in \mathbf{R}^n)$

$$\begin{array}{ll} \text{minimize} & \mathbf{1}^T x \\ \text{subject to} & A + \mathbf{diag}(x) \succeq 0 \end{array}$$

$n = 10, \ldots, 1000$; for each n solve 100 randomly generated instances



Primal-dual interior-point methods

- more efficient than barrier method when high accuracy is needed
- update primal and dual variables, and κ, at each iteration; no distinction between inner and outer iterations
- often exhibit superlinear asymptotic convergence
- search directions can be interpreted as Newton directions for modified KKT conditions
- can start at infeasible points
- cost per iteration same as barrier method