Tutorial 4 and Assignment 2

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Question 1

For each of the following functions determine whether it is convex, concave, quasiconvex, or quasiconcave (you should learn the defs of quasiconvex, and quasiconcave by yourself).

(a) $f(x)=e^x-1$ on $\mathbf{R}$.

(b) $f(x_1,x_2)=x_1x_2$ on $\mathbf{R}^2_{++}$.

(c) $f(x_1,x_2)=1/(x_1x_2)$ on $\mathbf{R}^2_{++}$.

(d) $f(x_1,x_2)=x_1/x_2$ on $\mathbf{R}^2_{++}$.

(e) $f(x_1,x_2)=x_1^2/x_2$ on $\mathbf{R}\times\mathbf{R}_{++}$.

(f) $f(x_1,x_2)=x_1^\alpha x_2^{1-\alpha}$, where $0\le\alpha\le 1$, on $\mathbf{R}^2_{++}$.

Question 2

Prove that $f(X)=\mathbf{tr}\left(X^{-1}\right)$ is convex on $\mathbf{dom}f=\mathbf{S}^n_{++}$.

Solution:

Define $g(t)=f(Z+tV)$, where $Z>0$ and $V\in S^n$.

$$\begin{aligned}
g(t) &= tr((Z+tV)^{-1}) = tr\{Z^{-1}[I+tZ^{-1/2}VZ^{-1/2}]^{-1}\} \\
&= tr\{Z^{-1}Q(I+t\Lambda)^{-1}Q^T\} = tr\{Q^TZ^{-1}Q(I+t\Lambda)^{-1}\} \\
&= \sum_{i=1}^{n}(Q^TZ^{-1}Q)_{ii}(1+t\lambda_i)^{-1},
\end{aligned}$$

where we used the eigenvalue decomposition $Z^{-1/2}VZ^{-1/2}=Q\Lambda Q^T$. Noting that $(1+t\lambda_i)^{-1}$ is convex, and $(Q^TZ^{-1}Q)_{ii}>0$, hence $g(t)$ is convex.

Question 3

Show that for $p>1$,

$$f(x,t)=\frac{|x_1|^p+\cdots+|x_n|^p}{t^{p-1}}=\frac{\|x\|_p^p}{t^{p-1}}$$

is convex on $\{(x,t)|t>0\}$.

Question 4

Derive the conjugates of the following functions.

(a) *Max function.* $f(x) = \max_{i=1,\ldots,n} x_i$ on $\mathbf{R}^n$.
(try $n = 2, 3$ first, then prove it).

(b) *Piecewise-linear function on* $\mathbf{R}$. $f(x) = \max_{i=1,\ldots,m}(a_i x + b_i)$ on $\mathbf{R}$. You can assume that the a_i are sorted in increasing order, *i.e.*, $a_1 \le \cdots \le a_m$, and that none of the functions $a_i x + b_i$ is redundant, *i.e.*, for each k there is at least one x with $f(x) = a_k x + b_k$.

Question 5

Show that the conjugate of $f(X) = \mathbf{tr}\,(X^{-1})$ with $\mathbf{dom} f = \mathbf{S}^n_{++}$ is given by

$$f^*(Y) = -2\mathbf{tr}(-Y)^{1/2}, \quad \mathbf{dom} f^* = -\mathbf{S}^n_+.$$

Hint. The gradient of f is $\nabla f(X) = -X^{-2}$.

Pf:

- Suppose $Y \notin -S^n_+$ (i.e. the largest eigenvalue > 0. Do eigenvalue decomposition

$$Y = Q\Lambda Q^T = \sum \lambda_i q_i q_i^T$$

where $\lambda_1 > 0$. Let

$$X = Qdiag(t, 1, \ldots, 1)Q^T = tq_1q_1^T + \sum_{i=2}^n q_i q_i^T$$

Then

$$tr(XY) - tr(X^{-1}) = [t\lambda_1 + \sum_{i=2}^n \lambda_i] - [1/t + (n-1)]$$

it $\to +\infty$ as $t \to +\infty$. So when $y \notin -S^n_+$, $f^*(Y) = +\infty$.

- When $Y \in -S^n_{++}$. Noting that

$$\nabla_X tr(XY) = Y, \nabla f(X) = -X^{-2}$$

To find the maximum of

$$trXY - trX^{-1}$$

by setting the gradient to zero, we obtain $X = (-Y)^{-1/2}$, and then

$$f^*(Y) = -2tr[(-Y)^{1/2}]$$

- When $Y \in -S^n_+$. Using the closeness of epigraph to handle it (not required).

Question 6

Conjugate of negative normalized entropy. Show that the conjugate of the negative normalized entropy

$$f(x) = \sum_{i=1}^n x_i \log\left(x_i/\mathbf{1}^T x\right),$$

with $\mathbf{dom} f = \mathbf{R}^n_{++}$ is given by

$$f^*(y) = \begin{cases} 0 & \sum_{i=1}^n e^{y_i} \le 1 \\ +\infty & \text{otherwise.} \end{cases}$$