Tutorial 6 + Assignment 3

February 27, 2025

Question 1

Consider the following optimization problem,

$$\begin{array}{ll} \text{minimize} & x^2+1 \\ \text{subject to} & (x-2)(x-4) \leq 0 \end{array}$$

with variable $x \in \mathbf{R}$.

1. *Analysis of primal problem.* Give the feasible set, the optimal value, and the optimal solution.

2. *Lagrangian and dual function.* Plot the objective x^2+1 versus x. On the same plot, show the feasible set, optimal point and value, and plot the Lagrangian $L(x, \lambda)$ versus x for a few positive values of λ. Verify the lower bound property ($p^* \geq \inf_x L(x, \lambda)$ for $\lambda \geq 0$). Derive and sketch the Lagrange dual function g.

3. *Lagrange dual problem.* State the dual problem, and verify that it is a concave maximization problem. Find the dual optimal value and dual optimal solution λ^* .Does strong duality hold?

4. *Sensitivity analysis.* Let $p^*(u)$ denote the optimal value of the problem

$$\begin{array}{ll} \text{minimize} & x^2+1 \\ \text{subject to} & (x-2)(x-4) \leq u \end{array}$$

as a function of parameter u. Plot $p^*(u)$. Verify that $dp^*(0)/du = -\lambda^*$.

Question 2

Express the dual problem of

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & f(x) \leq 0, \end{array}$$

with $c \neq 0$, in terms of the conjugate f^*. Explain why the problem you give is convex. We do not assume f is convex.

Question 3

Consider the following LP.

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Gx \preceq h \\ & Ax = b \end{array}$$

Give the dual problem, and make the implicit equality constraints explicit.

Question 4

Consider the convex piecewise-linear minimization problem,

$$\text{minimize } \max_{i=1,\ldots,m} \left(a_i^T x + b_i\right) \tag{1}$$

with variable $x \in \mathbf{R}^n$.

1. Derive a dual form based on the Lagrange dual of the equivalent problem

$$\begin{array}{ll} \text{minimize} & \max_{i=1,\ldots,m} y_i \\ \text{subject to} & a_i^T x + b_i = y_i, \quad i = 1,\ldots,m \end{array}$$

   with variable $x \in \mathbf{R}^n$ and $y \in \mathbf{R}^m$.

2. Formulate the piecewise-linear minimization problem (1) as an LP, and form the dual of the LP. Relate the LP dual to the dual obtained in part (1).

Question 5

Analytical centering. Derive a dual form for

$$\text{minimize} \quad -\sum_{i=1}^{m} \log\left(b_i - a_i^T x\right)$$

with domain $\left\{x \mid a_i^T x < b_i, i = 1,\ldots,m\right\}$. First introduce new variables y_i and equality constraints $y_i = b_i - a_i^T x$.

Question 6

Consider the the convex optimization problem,

$$\begin{array}{ll} \text{minimize} & e^{-x} \\ \text{subject to} & x^2/y \leq 0 \end{array}$$

with variable x and y, and domain $\mathcal{D} = \{(x, y) \mid y > 0\}$.

1. Verify that this problem is a convex problem and find the optimal solution and value.

2. Give the Lagrange dual problem, and find the optimal solution λ^* and optimal value d^* of the dual problem. What is the optimal duality gap?

3. Does Slater's condition hold for this problem?

4. What is the optimal value $p(u)$ of the perturbed problem

$$\begin{array}{ll} \text{minimize} & e^{-x} \\ \text{subject to} & x^2/y \leq u \end{array}$$

   as a function of u? Verify that the global sensitivity inequality

$$p^\star(u) \geq p^\star(0) - \lambda^\star u$$

   does not hold.

Question 7

Consider the QCQP,

$$\begin{array}{ll}\text{minimize} & x_1^2 + x_2^2 \\ \text{subject to} & (x_1 - 1)^2 + (x_2 - 1)^2 \leq 1 \\ & (x_1 - 1)^2 + (x_2 + 1)^2 \leq 1\end{array}$$

with variable $x \in \mathbf{R}^2$.

1. Sketch the feasible set and level sets of the objective. Find the optimal point x^* and optimal value p^*.

2. Give the KKT conditions. Do there exist Lagrange multipliers λ_1^* and λ_2^* that prove x^* is optimal?

3. Derive and solve the Lagrange dual problem. Does strong duality hold?