Assignment 3

Remark: There are two questions in this assignment. ONLY need to choose ONE of them to answer.

Question 1. Fair Allocation with Subsidy

Assume that there is a set of $M = \{e_1, \ldots, e_m\}$ indivisible goods and $N = \{1, \ldots, n\}$ agents, where each agent is endowed with an additive valuation function $v_i : 2^M \to \mathbb{R}_{\geq 0}$. An allocation $A = (A_1, \ldots, A_n)$, where A_i is the bundle allocated to agent i, is an partition of the set of m goods among n agents, i.e., $\bigcup_{i \in N} A_i = M$ and $A_i \cap A_j = \emptyset$ for any two agents $i \neq j$.

Two popular notions of fairness, envy-freeness (EF) and equitability (EQ), cannot be guaranteed all the time when items are indivisible. In the fair division literature, we can use some external money (subsidy) to circumvent it. Now, we would like to study whether an allocation can be EF and EQ simultaneously with the same amount of money. Here, let $\boldsymbol{p} = (p_1, \ldots, p_n)$ denote the payment vector, where $p_i \geq 0$ for any $i \in N$.

Definition 1 (EF) *An allocation $A = (A_1, \ldots, A_n)$ is envy-free (EF), if for any two agents $i, j \in N$, $v_i(A_i) \geq v_i(A_j)$ holds.*

Definition 2 (EQ) *An allocation $A = (A_1, \ldots, A_n)$ is equitable (EQ), if for any two agents $i, j \in N$, $v_i(A_i) = v_j(A_j)$ holds.*

Definition 3 (EF and EQ with subsidy) *An allocation $A = (A_1, \ldots, A_n)$ is EF and EQ with subsidy, if there exists a payment vector $\boldsymbol{p} = (p_1, \ldots, p_n)$, such that $v_i(A_i) + p_i \geq v_i(A_j) + p_j$ and $v_i(A_i) + p_i = v_j(A_j) + p_j$ hold simultaneously for any $i, j \in N$.*

Definition 4 (Utilitarian Social Welfare) *Given an allocation A, the utilitarian social welfare $SW(A)$ is defined as the sum of value that the agent derives from the allocation, i.e., $SW(A) = \sum_{i \in N} v_i(A_i)$.*

(a) Please give a sufficient and necessary condition to judge whether an allocation can be EF and EQ with subsidy.

(b) Based on the condition that you get, please argue that the maximum utilitarian social welfare allocation is EF and EQ with subsidy.

Answer

(a) The sufficient and necessary condition is $v_j(A_j) \geq v_i(A_j)$ for any $i, j \in N$, i.e., for any bundle A_j, the valuation that agent j has is no lower than that of any other agent.

Suppose an allocation A is EF and EQ with subsidy. In that case, it means that there exists a payment vector $\boldsymbol{p}$, such that for any two agents $i, j \in N$, we have $v_i(A_i) + p_i = v_j(A_j) + p_j$ and $v_i(A_i) + p_i \geq v_i(A_j) + p_j$. Combining the above two inequalities together, we get $v_j(A_j) \geq v_i(A_j)$.

Suppose that $v_j(A_j) \geq v_i(A_j)$ holds for any $i, j \in N$, it suffices to show that there exists a payment vector $\boldsymbol{p}$ to enable A to be EF and EQ. Let $p_i = \max_{k \in N} v_k(A_k) - v_i(A_i)$ for any agent $i \in N$. Note that for any agent $i \in N$, we have $v_i(A_i) + p_i = \max_{k \in N} v_k(A_k)$. Then, fix any two agents $i, j \in N$, we have $v_i(A_i) + p_i = \max_{k \in N} v_k(A_k) \geq \max_{k \in N} v_k(A_k) + v_i(A_j) - v_j(A_j) = v_i(A_j) + p_j$.

(b) Given any instance, consider a maximizing utilitarian social welfare allocation A^*. Now, pick an arbitrary agent $i \in N$. For any item $e \in A_i^*$, it must hold that $v_i(e) \geq v_j(e)$ for any agent $j \neq i$. Otherwise, we could find another allocation that has a higher utilitarian social welfare. Thus, we get $v_i(A_i^*) = \sum_{e \in A_i^*} v_i(e) \geq \sum_{e \in A_i^*} v_j(e) = v_j(A_i^*)$ for any agent $j \neq i$. Then, it is clear that A^* is EF and EQ with subsidy.