# Solutions to Quantum Mechanics: The Theoretical Minimum

# Lecture 1

## Exercise 1: (1.1)

a) Using the axioms for inner products, prove

$$\{\langle A| + \langle B|\}|C\rangle = \langle A|C\rangle + \langle B|C\rangle.$$

b) Prove $\langle A|A\rangle$ is a real number.

## Solution

a)
Let $\langle U| = \langle A| + \langle B|$, then $\{\langle A| + \langle B|\}|C\rangle = \langle C|U\rangle^*$ according to the second *inner-product* axiom. From the first *inner-product* axiom the inner-product can be rewritten to:

$$\langle C|U\rangle^* \iff [\langle C|\{|A\rangle + |B\rangle\}]^* \iff [\langle C|A\rangle + \langle C|B\rangle]^* \iff \langle C|A\rangle^* + \langle C|B\rangle^*$$

By using the second inner-product axiom again, this gives us $\langle A|C\rangle + \langle B|C\rangle$. **Q.E.D**!

b)
From p.30 we know that $\langle A|$ and $|A\rangle$ can be represented as a column and a row vector respectively:

$$\langle A| = \left(a_1^* a_2^* \cdots a_n^*\right)$$
$$|A\rangle = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$$

Where $a_i^*$ denotes the complex conjugate of the scalar. Then $\langle A|A\rangle$ is a simple "dot product" as demonstrated on p.31:

$$\langle A|A\rangle = \left(a_1^* a_2^* \cdots a_n^*\right)\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = a_1^* a_1 + a_2^* a_2 + \cdots + a_n^* a_n$$

Let $a_j = \alpha_j + \beta_j i$, then $a_j^* = \alpha_j - \beta_j i$, where $i = \sqrt{-1}$, $a_j$ is a component of $|A\rangle$, and $\alpha_j,\ \beta_j \in \mathbb{R}$.
It follows that $a_j a_j^* = \alpha_j^2 + \alpha_j\beta_j i - \alpha_j\beta_j i - \beta_j^2(-1)^2 = \alpha_j^2 + \beta_j^2$. By this we show that a multiplication of a complex scalar with its conjugate gives a **real** number. Thus the sum, $a_1^* a_1 + a_2^* a_2 + \cdots + a_n^* a_n$ is also a **real** number. **Q.E.D**!