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Elements of Mathematics

Exercise Sheet 4

Submission due date: **23.11.2021, 10:15h**

THEORY

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1 The p-Norm

A mapping $\|\cdot\| : \mathbb{R}^n \to [0, +\infty)$ is called **norm** on $\mathbb{R}^n$ if it satisfies the three properties

i) $\|x\| = 0 \;\Rightarrow x = 0$ *(positive definite/ point separating)*

ii) $\|r \cdot x\| = |r| \cdot \|x\|\,,\ \forall x \in \mathbb{R}^n, r \in \mathbb{R}$ *(absolutely homogeneous)*

iii) $\|x + y\| \leq \|x\| + \|y\|,\ \forall x, y \in \mathbb{R}^n$ *(subadditive/ triangle inequality)*

The most common example is given by the p-**norm** for $p \in [1, +\infty)$, defined by

$$\|x\|_p := \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}}$$

and the **supremum (or maximum) norm** defined by

$$\|x\|_\infty := \max_{i=1,\dots,n} |x_i|.$$

Remark: For a fixed $x \in \mathbb{R}^n$ one can show $\lim_{p\to\infty} \|x\|_p = \|x\|_\infty$.

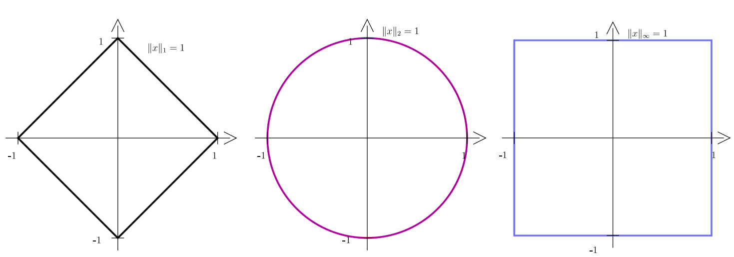
Tasks:

1. Draw the sets $\{x \in \mathbb{R}^2 : \|x\|_p = 1\}$ for $p = 1, 2, \infty$.
2. Show that the Euclidean norm $\|\cdot\|_2 : \mathbb{R}^n \to [0, +\infty),\ x \mapsto \sqrt{\sum_{i=1}^n x_i^2} = \sqrt{x^\top x}$ satisfies i)-iii).
   Hint: For iii) use the Cauchy-Schwarz inequality from the lecture and show $\|x+y\|^2 = (x+y)^T(x+y) \leq (x^Tx + y^Ty)^2 = (\|y\|_2 + \|x\|_2)^2$

(9 Points)

Solution:

1.



2. Recall: $\|x\|_2 := \sqrt{\sum_{i=1}^n x_i^2} = \sqrt{x^T x}$

i)

$$\|x\|_2 = 0 \Rightarrow \quad \|x\|_2^2 = 0 \quad \Rightarrow \quad \sum_{i=1}^n x_i^2 = 0 \quad \Rightarrow \quad x_i^2 = 0 \ \ \forall i \quad \Rightarrow \quad x_i = 0 \ \ \forall i$$
$$\Rightarrow \quad x = 0$$

ii) Let $\lambda \in \mathbb{R}$, $x \in \mathbb{R}^n$, then

$$\|\lambda x\|_2 = \sqrt{\sum_{i=1}^n \underbrace{(\lambda x_i)^2}_{\lambda^2 x_i^2}} = \sqrt{\lambda^2 \sum_{i=1}^n x_i^2} = \sqrt{\lambda^2}\sqrt{\sum_{i=1}^n x_i^2} = |\lambda| \cdot \|x\|_2.$$

iii) Let $x, y \in \mathbb{R}^n$ (to show: $\|x+y\| \leq \|x\| + \|y\|$, or equivalently $\|x+y\|^2 \leq (\|x\| + \|y\|)^2$ (note that $x \mapsto x^2$ is monotonically increasing for $x \geq 0$)). We find

$$\begin{aligned}
\|x+y\|^2 = |\underbrace{\|x+y\|^2}_{\geq 0}| &= \left|(x+y)^T(x+y)\right| = \left|(x+y)^T x + (x+y)^T y\right| = \left|x^T x + y^T x + x^T y + y^T y\right| \\
&= \left|x^T x + 2x^T y + y^T y\right| = \left|\|x\|_2^2 + 2x^T y + \|y\|_2^2\right| \\
&\leq \|x\|_2^2 + 2 \underbrace{|x^T y|}_{\leq \|x\|\|y\|} + \|y\|_2^2 \\
&\leq (\|x\|_2 + \|y\|_2)^2.
\end{aligned}$$

2 The Inner Product and SPD Matrices

A mapping $(\cdot,\cdot)\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is called **inner product** or **symmetric bilinear form** on $\mathbb{R}^n$ if it satisfies

i) $\forall x, y \in \mathbb{R}^n: \ (x,y) = (y,x)$ (symmetric)

ii) $\forall x, y_1, y_2 \in \mathbb{R}^n: \ (x, y_1 + y_2) = (x, y_1) + (x, y_2)$

$\forall x, y \in \mathbb{R}^n, r \in \mathbb{R}: \ (x, r \cdot y) = r \cdot (x,y)$ (linear in its second argument)

iii) $\forall x \in \mathbb{R}^n \setminus \{0\}: \ (x,x) > 0$ (positive definite)

All such inner products $(\cdot,\cdot)$ can be written as

$$(x,y) = x^T A y$$

for a **symmetric** (i.e., $A = A^T$) and **positive definite** (i.e., $x^\top A x > 0$ for all $x \neq 0$) matrix A.
In the lecture we mainly consider the *standard inner product*

$$(\cdot,\cdot)_2\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}, \ (x,y) \mapsto x^T I_n y = \sum_{i=1}^n x_i y_i$$

which corresponds to the symmetric and positive definite matrix I_n (identity matrix).

Tasks:

1. Find an example $A \in \mathbb{R}^{2\times 2}$ which is symmetric and positive definite.
2. Draw the set $\{Ax\colon \|x\|_2 = 1\}$.

(6 Points)

Solution:

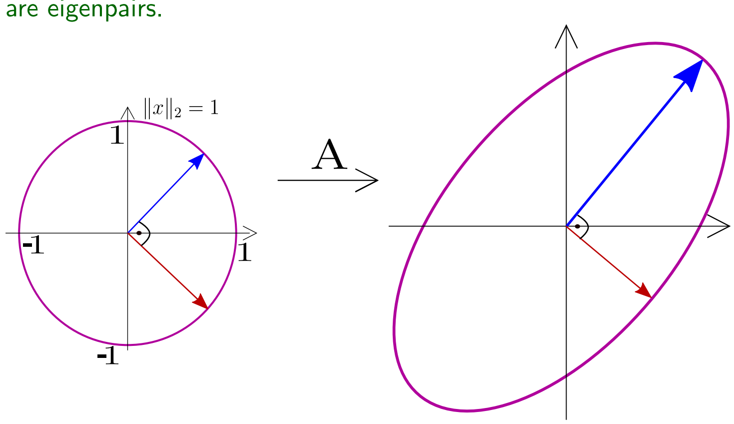
1. Let us consider $A := \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ with $a_{12} = a_{21}$ and first analye the property for being A being positive definite. We see

$$x^T A x = a_{11} x_1^2 + a_{12} x_1 x_2 + a_{12} x_2 x_1 + a_{22} x_2^2 = a_{11} x_1^2 + 2 a_{12} x_1 x_2 + a_{22} x_2^2 \overset{!}{>} 0 \quad \forall x \neq 0.$$

   Example:

$$\begin{aligned} A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix},\ x^T A x = & \overbrace{2x_1^2}^{x_1^2 + x_1^2} + 2 x_1 x_2 + 2 x_2^2 \\ &= x_1^2 + x_2^2 + \underbrace{(x_1 + x_2)^2}_{\geq 0} \\ &\geq x_1^2 + x_2^2 \\ &> 0 \qquad \forall \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{aligned}$$

2. Consider $\{Ax : \|x\|_2 = 1\}$ with $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$. Also recall from the lecture that $\left(3, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right), \left(1, \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)$ are eigenpairs.



3 Orthogonal Matrices

A matrix $Q \in \mathbb{R}^{n \times n}$ is called **isometric**, if

$$\|Qx\|_2 = \|x\|_2, \quad \forall x \in \mathbb{R}^n.$$

1. **Bonus** (You can get 4 bonus points for this subtask): Show the following equivalence:

$$Q \text{ is isometric} \quad \Leftrightarrow \quad Q \text{ is orthogonal.}$$

   Hint: To show "$\Rightarrow$", use the **polarization identity**

$$(x, y)_2 = \frac{1}{4}\left(\|x + y\|^2 - \|x - y\|^2\right), \quad x, y \in \mathbb{R}^n$$

   and the observation

$$a_{ij} = (e_i, A e_j)_2, \quad \text{for} \quad A = (a_{ij})_{ij} \in \mathbb{R}^{n \times n},$$

   where $e_j = (0, \ldots, 1, \ldots, 0)^T \in \mathbb{R}^n$ denote the standard basis vectors.

2. For an angle $\alpha \in [0, 2\pi]$, consider the *rotation matrix*

$$Q_\alpha = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}.$$

(a) Show that Q_α is orthogonal for all α.
Hint: Use the Pythagorian identity $\sin^2\alpha + \cos^2\alpha = 1$

(b) Draw the set $\{Q_\alpha x : \|x\|_1 = 1\}$ for $\alpha = \pi/4$.

(4 Points)

Solution:

1. "$\Rightarrow$" Let Q be isometric, i.e., $\|Qx\|_2 = \|x\|_2$ for all $x \in \mathbb{R}^n$. (to show: $Q^\top Q = I$ or equivalently $(Q^\top Q)_{ij} = \delta_{ij}$, where δ_{ij} denotes the Kronecker delta) Following the hint we first observe that

$$(Q^\top Q)_{ij} = (e_i, Q^\top Q e_j)_2 = (Qe_i, Qe_j)_2.$$

Second, by polarization formula and the isometry property we obtain

$$\begin{aligned}(Qx, Qy)_2 &= \frac{1}{4}\left(\|Qx + Qy\|^2 - \|Qx - Qy\|^2\right)\\ &= \frac{1}{4}\left(\|Q(x+y)\|^2 - \|Q(x-y)\|^2\right)\\ &= \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2\right)\end{aligned}$$

for all $x, y \in \mathbb{R}^n$. Now we insert the standard basis vectors $x = e_i, y = e_j$ and obtain

$$\begin{aligned}(Q^\top Q)_{ij} &= (Qe_i, Qe_j)_2\\ &= \frac{1}{4}\left(\|e_i + e_j\|^2 - \|e_i - e_j\|^2\right)\\ &= \delta_{ij},\end{aligned}$$

where δ_{ij} denotes the Kronecker delta. Thus, Q is orthogonal.

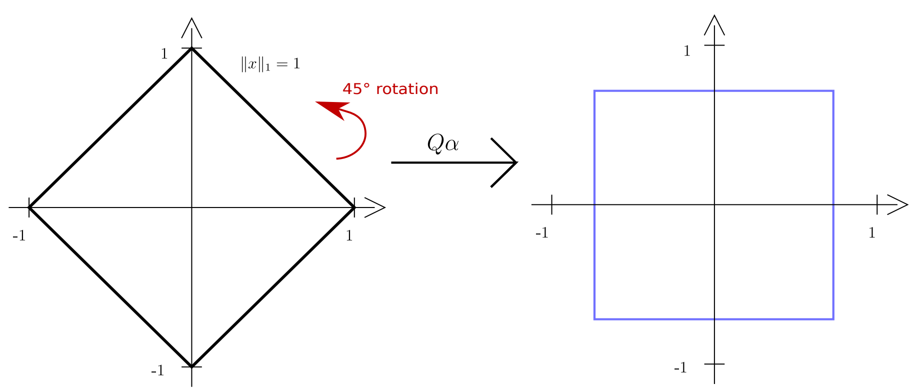
"$\Leftarrow$" Let Q be orthogonal, i.e., $Q^\top Q = I$. Then $\|Qx\|_2 = \sqrt{x^\top Q^\top Q x} = \|x\|_2$.

2. (a) By computing the matrix-matrix product we find

$$Q_\alpha^T Q_\alpha = \begin{pmatrix} \sin^2\alpha + \cos^2\alpha & -\cos\alpha\sin\alpha + \sin\alpha\cos\alpha \\ -\sin\alpha\cos\alpha + \cos\alpha\sin\alpha & \sin^2\alpha + \cos^2\alpha \end{pmatrix} = I.$$

(b) First note that

$$Q_{\pi/4} = \begin{pmatrix} \cos\pi/4 & -\sin\pi/4 \\ \sin\pi/4 & \cos\pi/4 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$



Remark: We have rotated the unit $\|\cdot\|_1$-norm ball to a $\|\cdot\|_\infty$-norm ball of radius $\frac{1}{\sqrt{2}}$.

4 Properties of Eigenvalues

Prove <u>three</u> (you get **2 bonus points** for a fourth solution) of the following statements (see also the corresponding Lemma from the lecture):

1. *The eigenvalues of the powers of a matrix:*
   Let $A \in \mathbb{F}^{n\times n}$, $\lambda \in \sigma(A)$ then $\lambda^k \in \sigma(A^k)$ for any $k \in \mathbb{N}$.

2. *Eigenvalues of invertible Matrices:*
   Let $A \in \mathbb{F}^{n\times n}$ be invertible and $\lambda \in \sigma(A)$, then $\lambda \neq 0$ and $\frac{1}{\lambda}$ is an eigenvalue of A^{-1}.

3. *Eigenvalues of a scaled matrix:*
   Let $A \in \mathbb{F}^{n\times n}$ and $\lambda \in \sigma(A)$, then $\alpha\lambda \in \sigma(\alpha A)$ for any $\alpha \in \mathbb{F}$.

4. *Real symmetric matrices have real eigenvalues:* (Not obvious without using properties of complex numbers!)
   $A \in \mathbb{R}^{n\times n},\ A = A^T \quad \Rightarrow \quad \sigma(A) \subset \mathbb{R}$.

5. *The eigenvalues of orthogonal matrices:* (Not obvious without <u>using properties</u> of complex numbers!)
   $Q \in \mathbb{R}^{n\times n}$ be orthogonal, $\lambda = a + ib \in \sigma(Q) \quad \Rightarrow \quad |\lambda| := \sqrt{a^2 + b^2} = 1$

6. *The eigenvalues of an upper (or lower) triangular matrix are sitting on its diagonal:*
   Let $U \in \mathbb{F}^{n\times n}$ with $u_{ij} = 0$ for $i > j$. Then $\sigma(U) = \{u_{11}, \dots, u_{nn}\}$.

7. *Similar matrices have the same eigenvalues:*
   Let $A \in \mathbb{F}^{n\times n}$ and $T \in GL_n(\mathbb{F})$, i.e., T is invertible. Then $\sigma(A) = \sigma(T^{-1}AT)$.

8. *Eigenvalues of a shifted matrix:*
   Let $A \in \mathbb{F}^{n\times n}$ and $\lambda \in \sigma(A)$, then $(\lambda - \alpha)$ is an eigenvalue of $(A - \alpha I)$ for any $\alpha \in \mathbb{F}$.

9. *Symmetric matrices have orthogonal eigenvectors:*
   Let $\lambda_1 \neq \lambda_2$ be two *distinct* eigenvalues of a real *symmetric* matrix $A \in \mathbb{R}^{n\times n}$ (i.e., $A = A^T$), and let $v_1, v_2 \in \mathbb{R}^n$ be corresponding eigenvectors. Proof that v_1 and v_2 are orthogonal, i.e., $v_1^\top v_2 = 0$.

(6 Points)

Solution:

1. ***The eigenvalues of the powers of a matrix:***
   For an eigenpair (λ, v) of A we find
   $$A^k v = A^{k-1}Av = A^{k-1}\lambda v.$$
   Iterating k times gives the desired result.

2. ***Eigenvalues of invertible Matrices:***
   Let $A \in GL_n(\mathbb{F})$ and $\lambda \in \sigma(A)$ with eigenvector $v \neq 0$.
   First, we show that $\lambda \neq 0$ holds. For this purpose let us assume $\lambda = 0$. Then $Av = \lambda v = 0$ implies $v = A^{-1} \cdot 0 = 0$, which contradicts that v is an eigenvector (and therefore nonzero).
   Second, we proof $\frac{1}{\lambda} \in \sigma(A^{-1})$. We find
   $$Av = \lambda v \quad \overset{A\in GL_n(\mathbb{F})}{\Leftrightarrow} \quad v = \lambda A^{-1} v \quad \overset{\lambda\neq 0}{\Leftrightarrow} \quad \frac{1}{\lambda} v = A^{-1}v \quad \Leftrightarrow \quad \frac{1}{\lambda} \in \sigma(A^{-1}) \quad \text{(with the same eigenvector } v\text{).}$$

3. ***Eigenvalues of a scaled matrix:***
   First, let $\alpha \neq 0$, then for an eigenpair (λ, v) of A we find
   $$Av = \lambda v \Leftrightarrow \alpha Av = \lambda \alpha v$$
   implying that $(\alpha\lambda, \alpha v)$ is an eigenpair of αA. Note that $\alpha v \neq 0$ since $\alpha \neq 0$.
   Now, for $\alpha = 0$ we have $\alpha A = 0$, so that any nonzero vector v satisfies $\alpha Av = 0 = \alpha\lambda v$.

4. ***Real symmetric matrices have real eigenvalues:***
   We first collect some observations:

- In general, for $x, y \in \mathbb{F}^n$ and a matrix $A \in \mathbb{F}^{n\times n}$ we easily find by the definition of the matrix product

$$x^\top Ay = \sum_{i,j} a_{ij}x_iy_j.$$

If the matrix is symmetric, i.e., $a_{ij} = a_{ji}$, we further find

$$x^\top Ay = \sum_i a_{ii}x_iy_i + \sum_{i\neq j} a_{ij}(x_iy_j + x_jy_i). \tag{1}$$

- For $z = x + iy \in \mathbb{C}$ let us define $\overline{z} := x - iy$ (the so-called *complex conjugate* of z). Then we easily find
  i) $z\overline{z} = a^2 + b^2 \in \mathbb{R}$ (real number!),
  ii) for $w = c + id$ we find $\overline{z}w + z\overline{w} = 2(ac + bd) \in \mathbb{R}$ (real number!) and also $\overline{z \cdot w} = \overline{z} \cdot \overline{w}$.

Now to the task: Let (λ, v) be an eigenpair of $A = A^\top \in \mathbb{R}^{n\times n}$, i.e.,

$$Av = \lambda v.$$

Multiplying $\overline{v} := (\overline{v}_1, \ldots, \overline{v}_n)$ from the left yields

$$\overline{v}^\top Av = \lambda\overline{v}^\top v.$$

Now, if we can show that $\overline{v}^\top Av$ and $\overline{v}^\top v$ are real, then we know that λ is real. First, we observe that

$$\overline{v}^\top v = \sum_{i=1}^n \overline{v}_iv_i,$$

which is real, since all summands $\overline{v}_iv_i$ are real by i) above. Secondly, since A is **symmetric** we can apply (1) to obtain

$$\overline{v}^\top Av = \sum_i a_{ii}\overline{v}_iv_i + \sum_{i\neq j} a_{ij}(\overline{v}_iv_j + \overline{v}_jv_i),$$

which is real, since all summands are real by i) and ii) above and the assumption that the matrix only has **real** coefficients.

Remarks:

- **We cannot relax symmetry assumption:** Matrices with just real coefficients can have complex eigenvalues. Take for example the (orthogonal) matrix

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

for which $\det(A - \lambda I) = \lambda^2 + 1$, so that $\sigma(A) = \{i, -i\}$ with eigenvectors $\{\begin{pmatrix} 1 \\ -i \end{pmatrix}, \begin{pmatrix} 1 \\ i \end{pmatrix}\}$. However, the additional property of symmetry is a sufficient condition for a real matrix A to have solely real eigenvalues!

- **We cannot relax assumption of real coefficients:** A symmetric matrix with complex coefficients can have complex eigenvalues. Consider for example

$$A = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$$

for which $\det(A - \lambda I) = \lambda^2 + 1$, so that $\sigma(A) = \{i, -i\}$.

- **The general result for complex matrices:** Like *symmetry* for real matrices we introduce for complex matrices: A matrix $A \in \mathbb{C}^{n \times n}$ is called *Hermitian* or *self-adjoint* if $A^\top = \overline{A}$. With other words, Hermitian matrices are invariant under transposition *and* (additionally) conjugation. With the same proof as above one can show that such matrices also have real eigenvalues. For example, consider the Hermitian matrix
$$A = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$
for which $\det(A - \lambda I) = \lambda^2 - 1$, so that $\sigma(A) = \{1, -1\}$.

5. ***The eigenvalues of orthogonal matrices:***
(Note that we have introduced the notion of *orthogonality* only for real matrices and vectors!)
We know $Q^\top Q = I$. Now let (λ, v) be an eigenpair of Q, i.e., $Qv = \lambda v$. Using the notation from the previous subtask, i.e., letting $\overline{v} = (\overline{v}_1, \ldots, \overline{v}_n)$ denote the complex conjugate of v, we find on the one hand that
$$(Q\overline{v})^\top (Qv) = \overline{v}^\top Q^\top Q v = \overline{v}^\top v \in \mathbb{R}.$$
On the other hand, since Q is assumed to be real we find
$$(Q\overline{v})^\top (Qv) = (\overline{Qv})^\top (Qv) = (\overline{\lambda v})^\top (\lambda v) = \overline{\lambda}\lambda(\overline{v}^\top v) = |\lambda|^2(\overline{v}^\top v) \in \mathbb{R}.$$
Thus combining these two equations gives
$$|\lambda| = 1.$$

6. ***The eigenvalues of an upper (or lower) triangular matrix are sitting on its diagonal:***
<u>Recall:</u>
1) The determinant of an (upper) triangular matrix is given by the product of its diagonal entries
$$\det(U) = u_{11} \cdot u_{22} \cdot \ldots \cdot u_{nn}.$$
2) The eigenvalues λ of a matrix $A \in \mathbb{R}^{n \times n}$ are the roots of the characteristic polynomial
$$\det(A - \lambda I) = 0.$$
Since $U - \lambda I = \begin{pmatrix} u_{11} - \lambda & & * \\ & \ddots & \\ 0 & & u_{nn} - \lambda \end{pmatrix}$ is also upper triangular we find
$$\det(U - \lambda I) = (u_{11} - \lambda) \cdot (u_{22} - \lambda) \cdot \ldots \cdot (u_{nn} - \lambda) \overset{!}{=} 0 \Leftrightarrow \lambda \in \{u_{11}, \ldots, u_{nn}\}.$$
Analogue proof for lower triangular matrices.

7. ***Similar matrices have the same eigenvalues:***
Let (λ, v) be an eigenpair of A, then since T is invertible we find
$$\underbrace{Av = \lambda v}_{\text{by definition of}(\lambda,v)} \overset{T^{-1}\cdot|}{\Leftrightarrow} \underbrace{T^{-1}Av}_{=T^{-1}AIv} = T^{-1}(\lambda v) \overset{I=TT^{-1}}{\Leftrightarrow} T^{-1}AT(T^{-1}v) = \lambda(T^{-1}v).$$
Thus, $(\lambda, T^{-1}v)$ is an eigenpair for the matrix $T^{-1}AT$ (note that $T^{-1}v \neq 0$, since $v \neq 0$).

Remark: We call two matrices A and B *similar* if there exists an invertible matrix T such that $B = T^{-1}AT$.

8. ***Eigenvalues of a shifted matrix:***
Let $\alpha \in \mathbb{F}$ and (λ, v) be an eigenpair of A. Then $((A - \alpha I) - (\lambda - \alpha)I)v = (A - \lambda I)v \overset{\lambda \in \sigma(A)}{=} 0$
Thus $(\lambda - \alpha)$ is an eigenvalue of $(A - \alpha I)$ with the same eigenvector v.

9. ***Symmetric matrices have orthogonal eigenvectors:***
(Note that we have introduced the notion of *orthogonality* only for real matrices and vectors!)
Remark upfront: Since we are considering a real symmetric matrix, we know that the eigenvalues are real. However, one may still find complex eigenvectors. For example consider the identity matrix I with spectrum $\sigma(I) = \{1\}$. Then obviously any nonzero vector v (complex or not) is an eigenvector to the eigenvalue 1. We now show that real eigenvalues enable us to choose real eigenvectors (as implicitly assumed in the task). To clarify this, let (λ, v) be an eigenpair, where $\lambda \in \mathbb{R}$ but v may potentially have complex coefficients. Let us split v according to the real and imaginary parts of its coefficient, more precisely

$$v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} x_1 + iy_1 \\ \vdots \\ x_n + iy_n \end{pmatrix} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} + i \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} =: x + iy.$$

Then we find

$$Av = \lambda v \Leftrightarrow A(x + iy) = \lambda(x + iy) \Leftrightarrow Ax + iAy = \lambda x + i\lambda y.$$

Note here that λ is real and thus we have the splitting into real λx and imaginary part λy. By comparing real parts in the last equation we obtain

$$Ax = \lambda x,$$

so that the real part of v is also an eigenvector to the eigenvalue λ.

Now to the task: Let A be symmetric, i.e., $A = A^T$ and let (λ_1, v_1) and (λ_2, v_2) be two (real) eigenpairs of A with $\lambda_1 \neq \lambda_2$, then

$$v_1^T \underbrace{Av_2}_{=\lambda_2 v_2} = \lambda_2 v_1^T v_2,$$

and also

$$v_1^T \underbrace{A}_{=A^T} v_2 = \underbrace{v_1^T A^T v_2}_{=(v_1^T A^T v_2)^T} = v_2^T \underbrace{Av_1}_{=\lambda_1 v_1} = \lambda_1 v_2^T v_1 = \lambda_1 v_1^T v_2.$$

Now we substract these terms and find

$$0 = v_1^T A v_2 - v_1^T A v_2 = \lambda_2 v_1^T v_2 - \lambda_1 v_1^T v_2 = \underbrace{\lambda_2 - \lambda_1}_{\neq 0, \text{ since } \lambda_2 \neq \lambda_1} v_1^T v_2$$

$$\overset{\lambda_1 \neq \lambda_2}{\Rightarrow} v_1^T v_2 = 0.$$

Total Number of Points = 25 (T:25, P:0)