Elements of Mathematics

Exercise Sheet 7

Submission due date: **14.12.2021, 10:15h**

THEORY

1 Compute the SVD

Consider the matrix

$$A = \begin{pmatrix} 3 & 0 \\ 4 & 5 \end{pmatrix}.$$

i) Compute its SVD $A = U\Sigma V^T$.

ii) Write A as a sum of rank-1 matrices.

iii) Is A invertible?

Hint: For i) follow this recipe:

1. Compute the eigenvalues λ_i with eigenvectors v_i of A^TA. Number the eigenvalues so that $\lambda_1 \geq \cdots \geq \lambda_r > 0$, where $r :=$"number of positive eigenvalues". Normalize the eigenvectors v_i.
2. For $i = 1, \ldots, r$: Set $\sigma_i := \sqrt{\lambda_i}$ and $u_i := \frac{1}{\sigma_i} Av_i$.
   [Until here we will already have the reduced SVD]
3. Extend the bases:
   - If $r < n$: Find orthonormal $v_{r+1}, \ldots, v_n \in \ker(A)$ by solving $Av_i = 0$ and orthogonalizing.
   - If $r < m$: Find orthonormal $u_{r+1}, \ldots, u_m \in \ker(A^T)$ by solving $A^Tu_i = 0$ and orthogonalizing.

(8 Points)

Solution:

$$A = \begin{pmatrix} 3 & 0 \\ 4 & 5 \end{pmatrix}, \quad A^TA = \begin{pmatrix} 25 & 20 \\ 20 & 25 \end{pmatrix}$$

i) <u>COMPUTE SVD:</u>

1. Compute $\sigma(A^TA)$ and corresponding eigenvectors.
   - Eigenvalues:

$$0 \overset{!}{=} \det(A^TA - \lambda I) = \det \begin{pmatrix} 25-\lambda & 20 \\ 20 & 25-\lambda \end{pmatrix} = (25-\lambda)^2 - 400$$
$$\Leftrightarrow \quad 25 - \lambda = \pm\sqrt{400} = \pm 20$$
$$\Leftrightarrow \quad \lambda_1 = 45, \quad \lambda_2 = 5.$$

   - Eigenvectors v_1 and v_2 are solutions of $(A^TA - \lambda_i I)v_i = 0$.

a)

$$(A^TA - \lambda_1 I) = \begin{pmatrix} -20 & 20 \\ 20 & -20 \end{pmatrix}$$
$$\left(\begin{array}{cc|c} -20 & 20 & 0 \\ 20 & -20 & 0 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c} -20 & 20 & 0 \\ 0 & 0 & 0 \end{array}\right)$$
$$\Rightarrow \quad -x_1 + x_2 = 0 \quad \Rightarrow \quad x_1 = x_2$$

$\Rightarrow \quad v_1 \in \{s \begin{pmatrix} 1 \\ 1 \end{pmatrix} : s \in \mathbb{R}\}$

Choose $s = \frac{1}{\sqrt{2}}$ and $v_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

b)

$$(A^TA - \lambda_2 I) = \begin{pmatrix} 20 & 20 \\ 20 & 20 \end{pmatrix}$$
$$\left(\begin{array}{cc|c} 20 & 20 & 0 \\ 20 & 20 & 0 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c} 20 & 20 & 0 \\ 0 & 0 & 0 \end{array}\right)$$
$$\Rightarrow \quad x_1 + x_2 = 0 \quad \Rightarrow \quad x_1 = -x_2.$$

$\Rightarrow \quad v_2 \in \{s \begin{pmatrix} -1 \\ 1 \end{pmatrix} : s \in \mathbb{R}\}$

Choose $s = \frac{1}{\sqrt{2}}$ and $v_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

2. Set $\sigma_1 := \sqrt{45} = 3\sqrt{5}$, $\sigma_2 := \sqrt{5}$.
   Compute u_i:

$$u_1 = \frac{1}{\sigma_1} A v_1 = \frac{1}{3\sqrt{5}} \frac{1}{\sqrt{2}} \begin{pmatrix} 3 & 0 \\ 4 & 5 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{3\sqrt{1}} \begin{pmatrix} 3 \\ 9 \end{pmatrix} = \frac{1}{\sqrt{10}} \begin{pmatrix} 1 \\ 3 \end{pmatrix},$$
$$u_2 = \frac{1}{\sigma_2} A v_2 = \frac{1}{\sqrt{5}} \frac{1}{\sqrt{2}} \begin{pmatrix} 3 & 0 \\ 4 & 5 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{10}} \begin{pmatrix} -3 \\ 1 \end{pmatrix}.$$

3. Since $2 = r = m = n$, we are finished.

ii) A as sum of rank-1 matrices:

$$A = \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T = \underbrace{\frac{3\sqrt{5}}{\sqrt{10}\sqrt{2}}}_{\frac{3}{2}} \begin{pmatrix} 1 \\ 3 \end{pmatrix} (1 \ 1) + \underbrace{\frac{\sqrt{5}}{\sqrt{10}\sqrt{2}}}_{\frac{1}{2}} \begin{pmatrix} -3 \\ 1 \end{pmatrix} (-1 \ 1)$$
$$= \frac{3}{2} \begin{pmatrix} 1 & 1 \\ 3 & 3 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 3 & -3 \\ -1 & 1 \end{pmatrix}.$$

iii) A invertible?
Yes, since $\sigma_1 \neq 0 \neq \sigma_2$ and thus Σ is invertible implying that the product $U\Sigma V^T = A$ is invertible with inverse $A^{-1} = V\Sigma^{-1}U^T$.

2 The SVD of Rank-1-Matrices

Let $u \in \mathbb{R}^m \setminus \{0\}$ and $v \in \mathbb{R}^n \setminus \{0\}$ be nonzero vectors and define $A := uv^T$. Find a reduced SVD of A and shortly explain why $\text{rank}(A) = 1$. *(4 Points)*

Solution:

We will in all detail derive the full and reduced SVD by following our recipe (your answer may be shorter!):

We have

$$A^T A = \|u\|^2 v v^T.$$

Following the recipe for computing the SVD of A we determine the eigenpairs of $A^T A$. We find

$$A^T A v = \|u\|^2 \|v\|^2 v,$$

which implies that v is an eigenvector to the positive (note: $u, v \neq 0$) eigenvalue $\|u\|^2\|v\|^2$.

- Since $A^T A$ is symmetric we know all eigenvectors are mutually orthogonal.
- However, any vector x orthogonal to v is eigenvector to the eigenvalue 0, since

  $$A^T A x = \|u\|^2 v \underbrace{v^T x}_{=0} = 0 \cdot x.$$

  Therefore the eigenvalues of $A^T A$ are given by

  $$\lambda_1 = \|u\|^2\|v\|^2, \ \lambda_2 = \cdots = \lambda_n = 0,$$

  with precisely $r = 1$ positive one ($\Rightarrow \text{rank}(A) = 1$). Thus we find the singular values and right-singular vector by

  $$v_1 := \frac{v}{\|v\|}, \ \sigma_1 = \|u\|\|v\| > 0.$$

  Extend v_1 to the orthogonal matrix $V = \begin{pmatrix} | & \\ v_1 & \ldots \\ | & \end{pmatrix} \in \mathbb{R}^{n \times n}$ with orthonormal columns, where $v_2, \ldots, v_n \in \ker(A)$. Also set

  $$\Sigma = \text{diag}(\sigma_1, 0, \ldots, 0) \in \mathbb{R}^{m \times n}.$$

- Following the recipe, the corresponding left-singular vector is given by

  $$u_1 := \frac{Av_1}{\sigma_1} = \frac{1}{\|u\|\|v\|} u v^T \frac{v}{\|v\|} = \frac{u}{\|u\|} \underbrace{\frac{v^T v}{\|v\|^2}}_{=1} = \frac{u}{\|u\|}.$$

  Extend u_1 to the orthogonal matrix $U = \begin{pmatrix} | & \\ u_1 & \ldots \\ | & \end{pmatrix} \in \mathbb{R}^{m \times m}$ with orthonormal columns, where $u_2, \ldots, u_m \in \ker(A^T)$.

- All in all we then obtain the full and truncated SVD

  $$\Rightarrow \quad A = U \Sigma V^T = \|u\|\|v\| u_1 v_1^T.$$

3 The SVD of Symmetric Matrices

Let $A \in \mathbb{R}^{n \times n}$ be symmetric with eigenvalues $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$.

1. What are the singular values of A?
2. How does the SVD look like? What can you say about the SVD if in addition A is positive definite?
3. Give a representation of the condition and rank of A in terms of its eigenvalues.

(6 Points)

Solution:

$$\underline{\text{REMARK:}}\ A \in \mathbb{R}^{n\times n},\ (\lambda, v) \text{ eigenpair of } A, \text{ then}$$
$$A^k v \;=\; A^{k-1}\underbrace{(Av)}_{=\lambda v} \;=\cdots=\lambda^k v \quad \forall k \in \mathbb{N}$$
$$\Rightarrow (\lambda^k, v) \text{ is eigenpair of } A^k.$$

1. A symmetric $\Rightarrow$ $A^TA = A^2$ $\Rightarrow$ $\sigma(A^TA) = \{\lambda^2 : \lambda \in \sigma(A)\}$,
   $\Rightarrow$ singular values are $\sigma = \sqrt{\lambda^2} = |\lambda|$ for $\lambda \in \sigma(A)$ nonzero

   ("Singular values of a symmetric matrix are the absolute values of its nonzero eigenvalues!")

2. How does the SVD look like?
   Let (λ_i, v_i) be eigenpairs of A, so that the v_i's are orthonormal. We set

$$\sigma_i = |\lambda_i|, \quad (i = 1,\dots,r)$$
$$u_i = \begin{cases} \frac{1}{\sigma_i} A v_i = \frac{\lambda_i}{|\lambda_i|} v_i & : \text{ for } i = 1,\dots,r \ \ (\lambda_i \neq 0), \\ v_i & : \text{ for } i = r+1,\dots,n \ (\lambda_i = 0). \end{cases}$$

   (note: we have shown in the lecture that the u_i as defined here are orthonormal). Then

$$A = V\Lambda V^\top \qquad \text{[eigendecomposition]}$$
$$= \underbrace{\underbrace{\begin{pmatrix} | & & | & | & & | \\ \frac{\lambda_1}{|\lambda_1|} v_1 & \dots & \frac{\lambda_r}{|\lambda_r|} v_r & v_{r+1} & \dots & v_n \\ | & & | & | & & | \end{pmatrix}}_{=:U} \underbrace{\begin{pmatrix} |\lambda_1| & & & & 0 \\ & \ddots & & & \\ & & |\lambda_r| & & \\ & & & \ddots & \\ 0 & & & & 0 \end{pmatrix}}_{=:\Sigma = \mathrm{diag}(\sigma_1,\dots,\sigma_r)}}_{=V\Lambda} \underbrace{\begin{pmatrix} - & v_1^\top & - \\ & \vdots & \\ - & v_n^\top & - \end{pmatrix}}_{V^\top} \quad [SVD]$$

   A symmetric and positive definite $\Rightarrow$ $|\lambda_i| = \lambda_i > 0$ $\Rightarrow$ $\frac{\lambda_i}{|\lambda_i|} = 1$. Thus:

   ("For spd matrices: SVD = Eigendecomposition!")

3. For the condition and the rank we obtain (by inserting $\sigma_i = |\lambda_i|$)

$$\mathrm{cond}(A) = \frac{\max(\sigma_i)}{\min(\sigma_i)} = \frac{\max(|\lambda_i|)}{\min(|\lambda_i|)}$$

   and

$$\mathrm{rank}(A) = |\{i : \sigma_i > 0\}| = |\{i : |\lambda_i| > 0\}| = |\{i : \lambda_i \neq 0\}|.$$

PROGRAMMING

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4 Using the SVD for Image Compression

Use the code snippet below to transform an image of your choice (extension `.png` or `.jpg`) into gray scale and to load it as an array $A \in \mathbb{R}^{m\times n}$ into Python.

1. Find a `scipy` routine to compute the SVD $U\Sigma V^T = A$.

2. Plot the singular values.
3. For several $1 \le k \le \text{rank}(A)$:
   (a) Compute the *truncated SVD*: Use only the first k columns of U, the first k singular values $\sigma_1, \ldots, \sigma_k$ from Σ and the first k rows of V^T to reconstruct A and plot the resulting image A_k using `plt.imshow(A_k, cmap='gray')`.
   (b) For each k, compute the total number of floats that need to be stored for the truncated SVD A_k and compare it to the total number of floats that need to be stored for the full image A.

```
path_to_image = 'spider.jpg'


def load_image_as_gray(path_to_image):
    import matplotlib
    img = matplotlib.image.imread(path_to_image)
    print(np.shape(img))
    # ITU-R 601-2 luma transform (rgb to gray)
```

(8 Points)

Solution:

```
import numpy as np
import matplotlib.pyplot as plt
import scipy.linalg as linalg

# load image as matrix
path_to_image = 'spider.jpg'


def load_image_as_gray(path_to_image):
    import matplotlib
    img = matplotlib.image.imread(path_to_image)
    print(np.shape(img))
    # ITU-R 601-2 luma transform (rgb to gray)
    img = np.dot(img, [0.2989, 0.5870, 0.1140])
    return img


A = load_image_as_gray(path_to_image)

# SVD
U, sigma, Vt = linalg.svd(A)
m, n = np.shape(A)

# plot sigma
# note: the subplot routines should be fixed in the future
plt.figure()
plt.subplot(3, 3, 1)
plt.title(r'$\sigma$')
x = np.linspace(1, sigma.size, sigma.size)
plt.semilogy(x, sigma, 'o-', label='sigma', markersize=3)
plt.semilogy([100, 100, 0], [0, sigma[100],sigma[100]], '-', lw=2, color='red')
plt.grid(True)

# plot truncated svd images
K = [1, 3, 5, 10, 20, 50, 100]
for k in K:
    aux = U[:, :k] @ np.diag(sigma[:k]) @Vt[:k, :]
    print("\nrank =", k, "\nrelative storage (truncated-SVD/A):",
```

```
        np.round(100 * k * (1. + m + n) / (m * n), 2), "%")
    plt.subplot(3, 3, K.index(k)+2)
    plt.imshow(aux, cmap='gray')
    plt.title('k='+'{:d}'.format(k))
    plt.tight_layout(pad=0.4, w_pad=0.05, h_pad=0.5)

# plot original
plt.subplot(3, 3, 9)
plt.imshow(A, cmap='gray')
plt.title("original")


# ===================== #
#    generate a video
# ===================== #
#from matplotlib import animation
#dpi = 250
#idlist = range(120)
#frames = [] # for storing the generated images
#fig = plt.figure()
#for k in range(len(idlist)):
#    i=idlist[k]
#    aux =  np.zeros((m, 2*n+50))
#    aux[:,:n] = U[:, :i]@np.diag(sigma[:i])@Vt[:i, :]
#    aux[:,-n:] = A
#    storage = np.round(100* i * (1. + m + n)/(m*n), 2)
#    frames.append([plt.imshow(aux, cmap='gray'), plt.text(1,1, "k="+str(idlist[i])+
     ", storage="+str(storage)+"%",horizontalalignment='left',  verticalalignment='
    top' ,color = "white" ),plt.title("Truncated SVD A_k          VS
    Original A         ")])
#    #plt.title("Truncated SVD A_k          VS          Original        \n"+" k =  "+
    str(idlist[i])+ ",  storage = "+str(storage[i])+"%
          " )])#, vmin = Amin, vmax = Amax, animated=True,cmap=cm.Greys_r), plt.text
    (8, 0.99, 'A_'+str(i), color='white')])
#ani = animation.ArtistAnimation(fig, frames, interval=180, blit=True,  repeat_delay
    =1000)
#ani.save('movie.mp4',dpi=dpi)
#plt.show()
```

Total Number of Points = 26 (T:18, P:8)