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Elements of Statistics

Chapter 5:
Probability theory

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Example 5.1: Some experiments

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a) Rolling a dice
b) Playing the lottery
c) Duration between two computer malfunctions
d) Kilometer reading of a car
e) Burning life of a bulb
f) Turnover of a pharmacy on a Friday
g) Life span of a male live birth
h) Time to solve a problem

- What are the outcomes of these experiments?
- Are these experiments reproducible?

Experiment and sample space

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Definition 5.1: Experiment

Procedures, which can actually or at least theoretically be repeated under a constant set of conditions and the outcomes of which cannot be forecast precisely, are called experiments.

Definition 5.2: Sample space

The set of all possible, mutually exclusive outcomes of an experiment is called sample space Ω.

Sample spaces may be:

- Finite
- Infinite
  - (Un)countable

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Example 5.2:

One roll of a dice:

a) $\Omega = \{i \,|i \; \in \mathbb{N}; \quad 1 \leq i \leq 6\}$
i is number of pips

b) $\Omega = \{\omega_g \,, \omega_u\}$
Even / uneven number of pips

c) $\Omega = \{\{1, 2, 3\}, \{4, 5\}, \{6\}\}$

Example 5.3:

Burning life of a bulb:

a) $\Omega = \{x|x \; \in \mathbb{R}; 0 \leq x \leq 10{,}000\}$
Infinite accuracy of measurement
$\rightarrow$ continuous variable

b) $\Omega = \{x|x \; \in \mathbb{N}_0, x \leq 10{,}000\}$
Measurement in hours
$\rightarrow$ discrete variable

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Definition 5.3:

Each subset of a sample space is called event.

We write:

$A, B, C \ldots$

We say:

An experiment with sample space Ω yielded event A (B, C, ...).

Example 5.4:

One roll of a dice $\Omega = \{1, 2, 3, 4, 5, 6\}$:

$$A = \{1, 3, 5\} \qquad B = \{2, 4, 6\} \qquad C = \{3\}$$

Application in R:

```
Omega <- 1:6

A <- Omega[c(1,3,5)] ; B <- Omega[c(2,4,6)] ; C <- Omega[3]
```

Bear in mind: Set theory and its calculation rules

Example 5.5: see Ex. 5.4 (1)

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$B = \overline{A}$

```
A_Bar <- Omega[-A]
setequal(x = B, y = A_Bar)
[1] TRUE
```

$\overline{C} = \{1, 2, 4, 5, 6\}$

```
C_Bar <- Omega[-C]
C_Bar
[1] 1 2 4 5 6
```

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Example 5.5: see Ex. 5.4 (2)

$A \cap B = \emptyset$

```
intersect(x = A, y = B)
integer(0)
```

$B \cup C = \{2, 3, 4, 6\}$

```
sort(union(x = B, y = C))
[1] 2 3 4 6
```

Events may be characterised using *admissible questions*.

$\rightarrow$ System of events (*set of admissible questions*)

Event space

Definition 5.4:

Let Ω be a sample space and let $\mathcal{K}$ be a non-empty subset of the power set of the sample space $\mathcal{P}(\Omega)$. $\mathcal{K}$ is called an event space if the following properties hold:

I: $\emptyset \in \mathcal{K}$

II: If $A \in \mathcal{K}$, then $\overline{A} \in \mathcal{K}$

III: If $A \in \mathcal{K}$ and $B \in \mathcal{K}$, then $A \cup B \in \mathcal{K}$

We may call $\mathcal{K}$ an algebra over Ω as well.

Example 6.6:

$\mathcal{K}_1 = \{\emptyset, \{1,3,5\}, \{2,4,6\}, \Omega\}$
is an event space.

$\mathcal{K}_2 = \{\emptyset, \{1,2\}, \{3,4\}, \{5,6\}, \Omega\}$
is not an event space.

$\mathcal{P}(\Omega)$ is always an event space (containing 2^n elements).

Sigma algebra

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Definition 5.5:

Let Ω be a sample space. A non-empty subset $\mathcal{S}$ of $\mathcal{P}(\Omega)$ is called sigma algebra if the following holds:

I: $A \in \mathcal{S} \Rightarrow \overline{A} \in \mathcal{S}$ (closed under complementation)

II: $A_i \in \mathcal{S}$ for all $i = 1, 2, \ldots$

$$\Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{S} \text{ (closed under countable union)}$$

- Every countable union of events itself is part of $\mathcal{S}$.
- All set operations can be derived from these two set operations

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Definition 5.6:

Let $\mathcal{K}$ be an event space over Ω. The (set) function $P : \mathcal{K} \to \mathbb{R}$ is called a *probability content*, if the following holds:

1. If $A \in \mathcal{K}$, then $P(A) \geq 0$
2. If $A, B \in \mathcal{K}$ and $A \cap B = \emptyset$, then
$$P(A \cup B) = P(A) + P(B)$$
3. $P(\Omega) = 1$

Implications

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1. $P(\overline{A}) = 1 - P(A)$
2. $P(A \cap B)$ follows from Definition 6.6.
3. Finite unions follow from Definition 6.6.
4. Countable unions do not follow from Definition 6.6 (e. g. even numbers within the natural numbers).

Definition 5.7:

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Let $\mathcal{S}$ be a sigma algebra over a sample space Ω. A (set) function $P : \mathcal{S} \to \mathbb{R}$ is called *probability measure*, if the following holds:

1. If $A \in \mathcal{S}$, then $P(A) \geq 0$ **(Non-negativity)**
2. If $A_i \in \mathcal{S}\,, \quad i = 1, 2, \ldots$ and all A_i are pairwise disjoint, we have
$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$
(σ-**additivity**)
3. $P(\Omega) = 1$ **(Standardisation)**

Bear in mind:
Pairwise disjoint and *overall disjoint* have to be distinguished!

Probability measure and implications

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Definition 5.8:

A triple $(\Omega; \mathcal{S}; P)$ consisting of a sample space Ω, a sigma algebra $\mathcal{S}$ over Ω and a probability measure P over $\mathcal{S}$ is called *probability space*.

Further implications

1. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
2. $P(A) \leq 1$
3. $P(\emptyset) = 0$
4. $A \subset B \qquad \Rightarrow \qquad P(A) \leq P(B)$
5. $$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) \\ -P(B \cap C) + P(A \cap B \cap C)$$

Classical or Laplace's concept - Principle of symmetry

- Finite sample space Ω
- Elementary events (outcomes) have equal probability of occurrence

$\nu(\Omega)$ is the number of elementary events and $\nu(A)$ is the number of *favourable* cases.

Then

$$P(A) = \frac{\nu(A)}{\nu(\Omega)}$$

is the probability that event A occurs (number of favourable cases divided by number of possible cases).

Example 5.7: Two rolls of a dice (1)

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Two rolls of a dice: $\Omega = \{(i,j) | i,j \in \mathbb{N} \quad \text{mit} \quad 1 \leq i,j \leq 6\}$

There are 36 possible events.

Application in R:

```
Omega <- expand.grid(Dice1 = 1:6, Dice2 = 1:6)
head(Omega, n = 9)
```

```
  Dice1 Dice2
1     1     1
2     2     1
3     3     1
4     4     1
5     5     1
6     6     1
7     1     2
8     2     2
9     3     2
```

```
length(Omega[,1])
```

```
[1] 36
```

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Example 5.7: Two rolls of a dice (2)

a) Sum of pips is **at least** 10:

$$A = \{(4;6);(5;5);(5;6);(6;4);(6;5);(6;6)\}$$

$$W(A) = \frac{\nu(A)}{36} = \frac{6}{36} = \frac{1}{6}$$

Calculations in R:

```
Sum_of_pips <- apply(Omega, 1, sum)
Omega <- cbind(Omega, Sum_of_pips)
A <- Omega[Omega$Sum_of_pips >= 10, ]
A
```

```
      Dice1    Dice2 Sum_of_pips
24        6        4          10
29        5        5          10
30        6        5          11
34        4        6          10
35        5        6          11
36        6        6          12
```

```
length(A[,1])/length(Omega[,1])
```

```
[1] 0.1666667
```

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Example 5.7: Two rolls of a dice (3)

b) Sum of pips is **exactly** 4:

$$B = \{(1;3);(2;2);(3;1)\}$$

$$W(B) = \frac{3}{36} = \frac{1}{12}$$

Calculations in R:

```
B <- Omega[Omega$Sum_of_pips == 4, ]
B
```

```
   Dice1 Dice2 Sum_of_pips
3      3     1           4
8      2     2           4
13     1     3           4
```

```
length(B[,1])/length(Omega[,1])
```

```
[1] 0.08333333
```

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Statistical concept of probability - Principle of frequency

- Experiments can be arbitrarily repeated
- A as an event in the experiment
- n repetitions (independent trials)

$p_n(A)$ is the relative frequency of occurrences of event A in n repeated experiments.

Properties: Non-negativity, additivity and standardisation!

Then $P(A) = \lim_{n \to \infty} p_n(A)$.

Law of large numbers (see later).

Subjectivistic concept of probability

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- Probability of rain
- Investment in shares
- Risk of an accident for a certain new car

Subjectivistic determination:

- Expert knowledge
- Experience
- Intuition

Verifiability is a problem here.

Example 5.8: Probability of survival

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We determine the *probability* for a 50 year old man to survive the next year using the life table 2008/2010.

$p_{50} = 0.995868.$

Contrary to this: $\frac{l_{51}}{l_0} = 0.95169.$

Segmentation possible regarding further information:

- Person smokes
- Person has not been ill since 20 years
- Person has had a heart attack
- ...

Example 5.9

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We want to determine the probability of obtaining a 6 in a roll of a dice.

We are informed that the experiment yielded an even number of pips.
Does that change the probability?

We have: $P(\{6\}|\{2,4,6\}) = \dfrac{P(\{6\})}{P(\{2,4,6\})} = \dfrac{\frac{1}{6}}{\frac{1}{2}} = \dfrac{1}{3}$.

Calculations in R:

```
Prob <- length(6)/length(c(2,4,6))
Prob
```

```
[1] 0.3333333
```

Definition 5.9:

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Let A and B be two events and let $P(B) \neq 0$. Then

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

is called the probability of A conditional on the occurrence of event B.

Analogously: $P(B|A)$ with $P(A) \neq 0$.

The multiplication theorem follows:

$$P(A \cap B) = P(B) \cdot P(A|B) = P(A) \cdot P(B|A)$$

Example 5.10: Evaluation of a course (1)

The evaluation of a course yielded the following frequency table:

	C male	D female	$\sum$
A good	0.45	0.35	0.8
B bad	0.15	0.05	0.2
$\sum$	0.6	0.4	1

Calculations in R:

```
load("Example5-10.RData")
p_j_k <- addmargins(p_j_k)
p_j_k
```

```
       C    D Sum
A   0.45 0.35 0.8
B   0.15 0.05 0.2
Sum 0.60 0.40 1.0
```

Example 5.10: Evaluation of a course (2)

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Then:

$$P(A) = 0.8$$

$$P(A|C) = \frac{P(A \cap C)}{P(C)} = \frac{0.45}{0.6} = 0.75$$

$$P(A|D) = \frac{0.35}{0.4} = \frac{7}{8} = 0.875$$

Calculations in R:

```
Prob_A   <- p_j_k[1,3]
Prob_A_C <- p_j_k[1,1]/p_j_k[3,1]
Prob_A_D <- p_j_k[1,2]/p_j_k[3,2]
```

```
Prob_A
```

```
[1] 0.8
```

```
Prob_A_C
```

```
[1] 0.75
```

```
Prob_A_D
```

```
[1] 0.875
```

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Definition 5.10:

Let A and B be two random events. A and B are called stochastically independent if and only if:

$$P(A \cap B) = P(A) \cdot P(B)$$

Then we have:

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \overset{\text{st. ind.}}{=} \frac{P(A) \cdot P(B)}{P(B)} = P(A)$$

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Example 5.11:

A coin is tossed twice (heads or tails). Consider the following two events:

A: 1st toss yields heads

B: 2nd toss yields tails

Then:

$$P(A \cap B) = P(A) \cdot P(B) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

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Example 5.12: see Ex. 5.10

We want to check if rating and gender are stochastically independent. Inter alia we should have:

$$P(A \cap C) = P(A) \cdot P(C).$$

But we actually have:

$$P(A) \cdot P(C) = 0.8 \cdot 0.6 = 0.48 \neq 0.45 = P(A \cap C)\,.$$

It follows that A and C are stochastically dependent..

Calculations in R:

```
Prob_C <- p_j_k[3,1]
Prob_A * Prob_C == 0.45
```

```
[1] FALSE
```

Addendum: What would the joint probabilities be, if the variables were stochastically independent and the marginal distributions were unchanged?

Example 5.13: Doorknobs (1)

A supplier to the automobile industry produces 10,000 door handles per day on 4 different machines. Production is distributed as follows:

M_1	1000 pieces with 8% scrap,
M_2	2000 pieces with 5% scrap,
M_3	3000 pieces with 3% scrap,
M_4	4000 pieces with 2% scrap.

One door handle is randomly chosen from the daily production. What is the probability that the item is defective?

Data input in R:

```
Doorknobs_per_machine <- c(1000,2000,3000,4000)
Number_doorknobs
Rejects_per_machine <- c(0.08,0.05,0.03,0.02)
```

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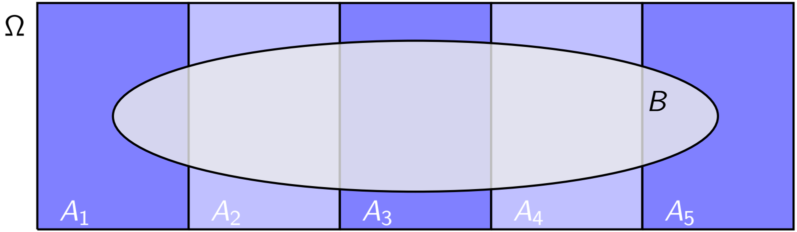
Theorem 5.1:

Law of total probability:

Let $A_1, \ldots, A_m$ be a disjoint decomposition of Ω. Then, for $B \subset \Omega$ we have:

$$P(B) = \sum_{i=1}^{m} P(B \cap A_i)$$
$$= \sum_{i=1}^{m} P(B|A_i) \cdot P(A_i)$$



Example 5.13: Doorknobs (2)

Let A_i for $i = 1, \ldots, 4$ be the event that the door handle has been produced on machine M_i. Let F be the event that the door handle is faulty. Furthermore, we have:

$$
\begin{aligned}
P(A_1) &= \frac{1000}{10000} = 0.1, & P(F|A_1) &= 0.08 \\
P(A_2) &= \frac{2000}{10000} = 0.2, & P(F|A_2) &= 0.05 \\
P(A_3) &= \frac{3000}{10000} = 0.3, & P(F|A_3) &= 0.03 \\
P(A_4) &= \frac{4000}{10000} = 0.4, & P(F|A_4) &= 0.02
\end{aligned}
$$

Calculations in R:

```
P_Ai   <- Doorknobs_per_machine/Number_doorknobs
P_F_Ai <- Rejects_per_machine
```

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Example 5.13: Doorknobs (3)

The probability for event F is composed as follows:

$$
\begin{aligned}
P(F) = & P(A_1) \cdot P(F|A_1) + P(A_2) \cdot P(F|A_2) + P(A_3) \cdot P(F|A_3) + \\
& P(A_4) \cdot P(F|A_4) = \\
& \sum_{i=1}^{4} P(A_i) \cdot P(F|A_i).
\end{aligned}
$$

Therefore, we have:
$P(F) = 0.1 \cdot 0.08 + 0.2 \cdot 0.05 + 0.3 \cdot 0.03 + 0.4 \cdot 0.02 = 0.035.$

Calculations in R:

```
P_F <- sum(P_Ai * P_F_Ai)
P_F
```

```
[1] 0.035
```

Example 5.14: Spam mails (1)

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We want to discuss the nuisance of spam mails. Let the following two events be given:

A: The mail is spam.

B: The mail client marks the mail as spam.

Past experience taught us that roughly 85% of mails are spam. 95% of spam is marked as such by the mail client. But 8% of mails are wrongly marked as spam.

What percentage of mails, which have been marked as spam, is indeed spam?

Tree diagrams and probabilities

Multiplication of probabilities
In a multi-stage experiment we get the probability of a single event by multiplying the probabilities on the respective path.

Addition of probabilities
Probabilities from different paths of a multi-stage experiment are added.

Total probability
The sum of probabilities at the *leaves* of a tree diagram is 1.

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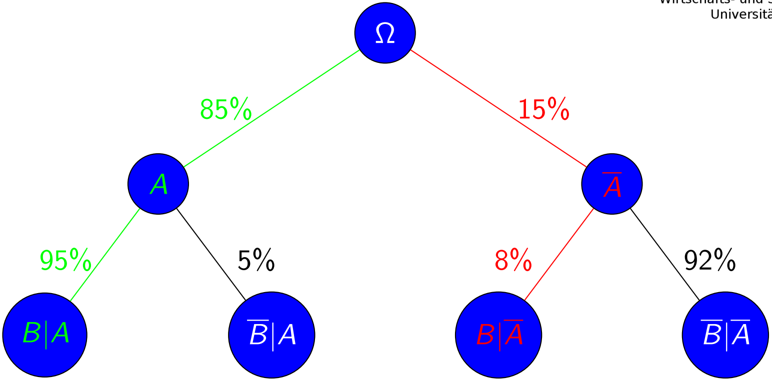
Example 5.14: Spam mails (2)



We have: $P(A \cap B) = P(A) \cdot P(B|A)$
$(P(\overline{A} \cap B)$, $P(A \cap \overline{B})$ and $P(\overline{A} \cap \overline{B})$ analogously). Overall we have

$$P(A|B) = \frac{0.85 \cdot 0.95}{0.85 \cdot 0.95 + 0.15 \cdot 0.08} = 0.985\,.$$

Theorem 5.2: Bayes' theorem

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Let $A_1, \ldots, A_m$ m be events of a sample space Ω. Furthermore, let the events A_i $(i = 1, \ldots, m)$ be pairwise disjoint and let

$$\Omega = \bigcup_{i=1}^{m} A_i. \qquad \text{(Decomposition of } \Omega\text{)}$$

Now, let B be another event and the probability for all positive events. Then we have for all $k = 1, \ldots, m$:

$$P(A_k|B) = \frac{P(A_k) \cdot P(B|A_k)}{\sum\limits_{i=1}^{m} P(A_i) \cdot P(B|A_i)}\,.$$

We call $P(A_i)$ the a-priori-probability and $P(A_i|B)$ the a-posteriori-probability.

Example 5.15: see Ex. 5.13 (1)

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A quality control inspector of the supplier to the automobile industry has randomly sampled and inspected a doorknob where he detects a flaw. Now, he wants to know the probabilities for the flawed doorknob to be produced by machine M_1, M_2, M_3 or M_4. According to *Bayes' theorem* we obtain the respective probabilities as follows:

$$P(A_1|F) = \frac{P(A_1) \cdot P(F|A_1)}{\sum_{i=1}^{4} P(A_i) \cdot P(F|A_i)} = \frac{0.1 \cdot 0.08}{0.035} = 0.229$$

$$P(A_2|F) = \frac{P(A_2) \cdot P(F|A_2)}{\sum_{i=1}^{4} P(A_i) \cdot P(F|A_i)} = \frac{0.2 \cdot 0.05}{0.035} = 0.286$$

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Example 5.15: see Ex. 5.13 (2)

According to *Bayes' theorem* we obtain the other searched-for probabilities:

$$P(A_3|F) = \frac{0.3 \cdot 0.03}{0.035} = 0.257$$
$$P(A_4|F) = \frac{0.4 \cdot 0.02}{0.035} = 0.229$$

Calculations in R:

```
P_Ai_F <- P_Ai * P_F_Ai / P_F
round(P_Ai_F, 3)
```

```
[1] 0.229 0.286 0.257 0.229
```

Example 5.16: Missing probabilities (1)

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A_1, A_2 and A_3 are three disjoint events that unite to Ω. Let $P(A_1) = 0.3$, $P(A_2) = 0.5$, $P(B|A_1) = 0.6$, $P(B|A_2) = 0.5$ and $P(B|A_3) = 0.1$.

Data input in R:

```
P_Ai    <- c(0.3,0.5,NA)
P_B_Ai  <- c(0.6,0.5,0.1)
```

We calculate $P(A_1|B)$.

First we have $P(A_3) = 1 - 0.3 - 0.5 = 0.2$.

Calculations in R:

```
P_Ai[3] <- 1 - P_Ai[1] - P_Ai[2]
P_Ai
```

```
[1] 0.3 0.5 0.2
```

Example 5.16: Missing probabilities (2)

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$$P(A_1|B) = \frac{0.3 \cdot 0.6}{0.3 \cdot 0.6 + 0.5 \cdot 0.5 + 0.2 \cdot 0.1} = \frac{0.18}{0.45} = \frac{2}{5} = 0.4$$

Calculations in R:

```
P_A1_B <- P_Ai[1] * P_B_Ai[1] / sum(P_Ai * P_B_Ai)
P_A1_B
```

```
[1] 0.4
```

Example 5.17: Accidents

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We have the following notation:

W: Woman

M: Man

A: Accident

The probability for an accident is $P(A) = 1/100,000$. Furthermore $P(W|A) = 1/8$ and $P(M|A) = 7/8$.

Can we conclude from this that women are better drivers because they cause less accidents?

Urn model with replacement (WR)

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In an urn there are N different balls.
From there we take n balls WR.

- Multiple draws possible
- **Independence** of draws

There are N^n variations.

Calculation of the number of variations in R:

```
# Attention: N and n must be defined
Number_of_variations <- N^n
```

Random draws following an urn model WR in R:

```
set.seed(123) # starting point for (pseudo-) random number
# generator
draws <- sample(x = 1:100, size = 3, replace = TRUE)
draws
```

```
[1] 31 79 51
```

Example 5.18: WR

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In an urn there are 100 balls with the numbers 1 to 100. We draw three times one ball, note the number and return it into the urn.

$$P(K_1 = 31; K_2 = 79; K_3 = 51) = \frac{1}{100} \cdot \frac{1}{100} \cdot \frac{1}{100} = 10^{-6}$$

Urn model without replacement (WoR)

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In an urn there are N different balls.
From this we draw n balls WoR.

- Multiple draws not possible (number of balls decreases)
- **Dependence** of the draws

There are $N \cdot (N-1) \cdot \cdots \cdot (N-n+1) = \dfrac{N!}{(N-n)!}$ variations.

Calculation of the number of variations in R:

```
# Attention: N and n must be defined
Number_of_variantions <- factorial(N)/factorial(N-n)
```

Random draws following the urn model WoR in R:

```
set.seed(321) #  new starting point
draws <- sample(x = 1:100, size = 3, replace = FALSE)
draws
[1] 54 77 88
```

Example 5.19: WoR

In an urn there are 100 balls with the numbers 1 to 100. We draw three times one ball and note the number.

$$P(K_1 = 54; K_2 = 77; K_3 = 88) =$$
$$W(K_1 = 54) \cdot P(K_2 = 77 | K_1 = 54) \cdot P(K_3 = 88 | K_1 = 54, K_2 = 77) =$$
$$\frac{1}{100} \cdot \frac{1}{99} \cdot \frac{1}{98}$$

For $n/N < 0{,}05$ there is only a small numerical difference.

More urn models

Wirtschafts- und Sozialstatistik
Universität Trier

- WoR without ordering:
  As before, but here $n!$ orders of the balls are identical.

  There are $\dfrac{N!}{(N-n)! \cdot n!} = \binom{N}{n}$ combinations.

  Lottery as example: $\binom{49}{6} = 13{,}983{,}816$

  Determination in R:

```
Number_of_variantions <- choose(49,6)
```

- WR without ordering:

  There are $\binom{N+n-1}{n}$ combinations

  Determination in R:

```
# Attention: N and n must be defined
Number_of_variantions <- choose(N+n-1,n)
```