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Elements of Statistics

Chapter 6:
Random variables

Ralf Münnich, Jan Pablo Burgard and Florian Ertz

University of Trier
Faculty IV
Economic and Social Statistics Department

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Example 6.1: Triple coin toss (1)

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Triple coin toss (H: Heads, T: Tails):

Let $X =$ Number of heads be a random variable with $x \in \{0, 1, 2, 3\}$.

Toss	Result			Realisation			Number of heads
1	H	H	H	1	1	1	3
2	H	H	T	1	1	0	2
3	H	T	H	1	0	1	2
4	H	T	T	1	0	0	1
5	T	H	H	0	1	1	2
6	T	H	T	0	1	0	1
7	T	T	H	0	0	1	1
8	T	T	T	0	0	0	0

Example 6.1: Triple coin toss (2)

Construction of the table in R:

```
result <- expand.grid(lapply(
                          X = 1:3,
                          FUN = function(x) c("H", "T")))
domain <- expand.grid(lapply(
                             X = 1:3,
                             FUN = function(x) c(1, 0)))
number_of_hats <- rowSums(domain)

Example6_1<- data.frame(result,domain,number_of_hats)
names(Example6_1)<- c("result","","","domain","","",
                         "number_of_hats")

save(Example6_1, file="Example6-1.RData")
head(Example6_1, n = 4)
```

```
        result        domain     number_of_hats
1        H H H         1 1 1             3
2        T H H         0 1 1             2
3        H T H         1 0 1             2
4        T T H         0 0 1             1
```

Definition of a random variable

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Definition 6.1:

Let a probability space $(\Omega; \mathcal{S}; P)$ be given. A function

$$X : \Omega \to \mathbb{R}; \qquad \omega \mapsto X(\omega)$$

is called *random variable*, if the set

$$X^{-1}((-\infty, x]) = \{\omega \in \Omega | X(\omega) \leq x\}$$

belongs to the sigma algebra $\mathcal{S}$ over Ω for all $x \in \mathbb{R}$.

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Example 6.2: see Ex. 6.1 (1)

$$\begin{aligned} P(X=1) &= P(\{(H,T,T),(T,H,T),(T,T,H)\}) \\ &= \frac{3}{8} \end{aligned}$$

```
load("Example6-1.RData")
sum(Example6_1$number_of_hats == 1) /
length(Example6_1$number_of_hats)
```

```
[1] 0.375
```

$$\begin{aligned} P(X\leq 1) &= P(X=0)+P(X=1) \\ &= \frac{1}{8}+\frac{3}{8}=\frac{1}{2} \end{aligned}$$

```
sum(Example6_1$number_of_hats==0 |
    Example6_1$number_of_hats==1) /
    length(Example6_1$number_of_hats)
```

```
[1] 0.5
```

Example 6.2: see Ex. 6.1 (2)

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$$P(X > 1) = 1 - P(X \leq 1) = \frac{1}{2}$$

```
sum(Example6_1$number_of_hats > 1) /
length(Example6_1$number_of_hats)
```

```
[1] 0.5
```

$$P(0 < X \leq 2) = P(X = 1) + P(X = 2) = \frac{3}{8} + \frac{3}{8} = \frac{3}{4}$$

```
sum(Example6_1$number_of_hats>0&
Example6_1$number_of_hats<=2) /
length(Example6_1$number_of_hats)
```

```
[1] 0.75
```

Distribution function

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Definition 6.2:

The function $F(x) := P(\{X \leq x\})$, which assigns to each $x \in \mathbb{R}$ the probability that the random variable X is less than or equal to x, is called *distribution function* of X.

We use the short hand $P(X \leq x)$.

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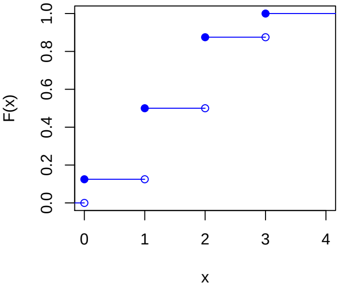
Example 6.3: see Ex. 6.2 (1)

For the random variable X we have:

1. For $x < 0$: $P(X \leq x) = 0$
2. For $0 \leq x < 1$: $P(X \leq x) = P(X = 0) = 1/8$
3. For $1 \leq x < 2$: $P(X \leq x) = P(X = 0) + P(X = 1) = 1/2$
4. For $2 \leq x < 3$: $P(X \leq x) = 1 - P(X = 3) = 7/8$
5. For $x \geq 3$: $P(X \leq x) = 1$

Therefore, we have:

$$F(x) = \begin{cases} 0 & \text{for } x < 0 \\ 1/8 & \text{for } 0 \leq x < 1 \\ 1/2 & \text{for } 1 \leq x < 2 \\ 7/8 & \text{for } 2 \leq x < 3 \\ 1 & \text{for } x \geq 3 \end{cases}$$



Example 6.3: see Ex. 6.2 (2)

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Calculation of $F(x)$ and construction of the graphic in R:

```
load("Example6-1.RData")

x6_3 <- Example6_1$number_of_hats
F_x6_3 <- cumsum(prop.table(x = table(x = x6_3)))

plot(ecdf(x6_3), col = "blue", xlab = "x", ylab = "F(x)",
     main = "", xlim = c(0, 4))
points(0:3, c(0, F_x6_3[-4]), col = "blue")

F_x6_3
```

```
    0     1     2     3
0.125 0.500 0.875 1.000
```

Properties of distribution functions

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We have:

1. $0 \leq F(x) \leq 1$ for all $x \in \mathbb{R}$
2. $\lim\limits_{x \to \infty} F(x) = 1$
3. $\lim\limits_{x \to -\infty} F(x) = 0$
4. F is monotonously increasing.
5. F has no more than a countable number of jump discontinuities.
6. F is right-continuous.

Discrete random variables

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Definition 6.3:

A random variable X is called *discrete* if it cannot take more than a countable number of values (realisations) with a positive probability. If $x_1, \ldots, x_i$ are the realisations of X, then the probabilities $P(X = x_1), \ldots, P(X = x_i)$ contain the complete information about this random variable.

Definition 6.4:

The function $f(x)$, which is defined for all real x and given by

$$f(x) = \begin{cases} P(X = x) & \text{for all possible realisations of } X \\ 0 & \text{else} \end{cases},$$

is called *probability function* of the (discrete) random variable X.

Example 6.4: Red and blue balls (1)

An urn contains 3 red and 7 black balls. 3 balls are drawn with replacement. Determine the probability table for the number of red balls drawn.

Furthermore, make suitable plots for the respective probability function and distribution function.

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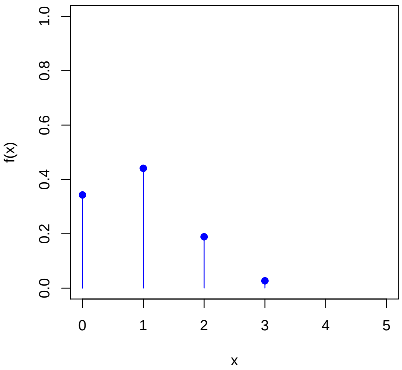
Example 6.4: Red and blue balls (2)

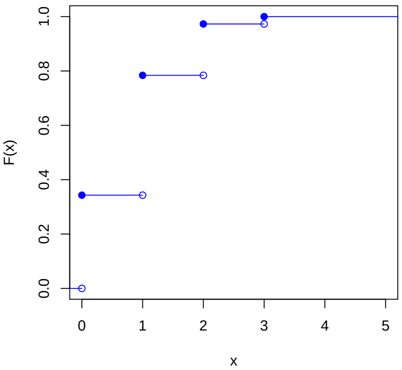
x_i	0	1	2	3
$f(x_i)$	0.343	0.441	0.189	0.027

```
x6_4 <- 0:3
f_x6_4 <- c(0.343,0.441,0.189,0.027)
F_x6_4 <- cumsum(f_x6_4)
```

Probability function



Distribution function



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Example 6.4: Red and blue balls (3)

Construction of the graphics in R:

```
plot(x = x6_4, y = f_x6_4, type = "h", lwd = 3,
     xlim = c(0,5), ylim = c(0,0.6), col = "blue", xlab="x",
     ylab = "f(x)", main = "Probability function")
points(x = x6_4, y = f_x6_4, col = "blue", pch = 19)

x_axis <- c(-1,sort(x6_4),4)
y_axis <- c(0,F_x6_4,1)

plot(x = c(0,4), y = c(0,1), main = "Distribution function",
     type = "n", col = "blue", xlab = "x", ylab = "F(x)")

lines(x = x_axis[1:2],y=rep(y_axis[1],2),col="blue",lwd=2)
lines(x = x_axis[2:3],y=rep(y_axis[2],2),col="blue",lwd=2)
lines(x = x_axis[3:4],y=rep(y_axis[3],2),col="blue",lwd=2)
lines(x = x_axis[4:5],y=rep(y_axis[4],2),col="blue",lwd=2)
lines(x = x_axis[5:6],y=rep(y_axis[5],2),col="blue",lwd=2)
points(x = x_axis[1:5],y=y_axis[1:5], col="blue", pch=19)
points(x = x_axis[2:5],y=y_axis[1:4], col="blue", pch=1)
```

Example 6.5: Another random variable (1)

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Let a discrete random variable have the following probability and distribution function, respectively:

$$f(x) = \begin{cases} 0.2 \cdot 0.8^x & \text{for } x = 0, 1, 2, \ldots \\ 0 & \text{else} \end{cases}$$

$$F(x) = \begin{cases} 1 - 0.8^{x+1} & \text{for } x \geq 0 \\ 0 & \text{else} \end{cases}$$

Definition of $f(x)$ and $F(x)$ in `R`:

```
x6_5 <- 0:10

# ATTENTION: here functions
f_x <- function(x) {0.2 * 0.8^x}

F_x <- function(x) {1 - 0.8^(x+1)}
```

Example 6.5: Another random variable (2)

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We get the following tabulated results:

x_i	0	1	2	3	4	5	6	7	8	9	10
$f(x_i)$	0.200	0.160	0.128	0.102	0.082	0.066	0.052	0.042	0.034	0.027	0.021
$F(x_i)$	0.200	0.360	0.488	0.590	0.672	0.738	0.790	0.832	0.866	0.893	0.914

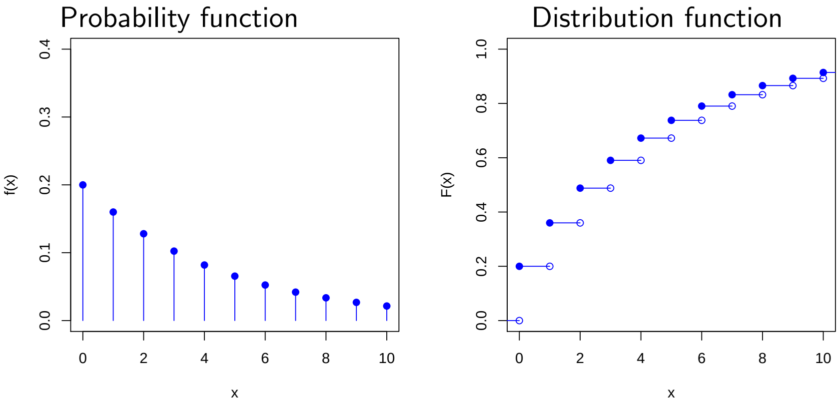
```r
round(f_x(x6_5), digits = 3)
```

```
[1] 0.200 0.160 0.128 0.102 0.082 0.066 0.052 0.042 0.034 0.027 0.021
```

```r
round(F_x(x6_5), digits = 3)
```

```
[1] 0.200 0.360 0.488 0.590 0.672 0.738 0.790 0.832 0.866 0.893 0.914
```

Example 6.5: Another random variable (3)



The random variable has a countably infinite number of realisations.
Here, we are dealing with a geometric distribution with parameter $p = 0.2$.

Continuous random variables

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Definition 6.5:

If there is a non-negative function $f(x)$ for a random variable X, in such a way that the distribution function for all x can be described by

$$F(x) = \int_{-\infty}^{x} f(y)\, dy,$$

we call X a *continuous random variable*.

Definition 6.6:

The function $f(x)$ of Definition 7.5 is called the *density function* of the continuous random variable X.

Properties of continuous random variables

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1. The area between the density curve and the abscissa has to sum up to 1. Notice the analogy to the empirical relative frequency distribution (histogram).

2. The probability $F(x)$ that X takes on a value which is less than or equal to x is expressed in terms of the measure of the area between the density curve and the abscissa on the interval $(-\infty, x]$. Notice the analogy to the empirical distribution function.

3. $P(x_1 < X \leq x_2) = F(x_2) - F(x_1)$

Properties of continuous random variables (ctd.)

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4. $P(X = x) = 0$
   Density values cannot be interpreted as probabilities!

5. X continuous $\Rightarrow$
   $P(x_1 < X \leq x_2) = P(x_1 \leq X < x_2) =$
   $P(x_1 \leq X \leq x_2) = P(x_1 < X < x_2)$

6. Interpretation of densities: $P(x_1 < X \leq x_2) \approx f(x) \cdot \underbrace{(x_2 - x_1)}_{\text{small}}$

7. $f(x) > 1$ is possible!

8. $F'(x) = f(x)$ for all x, for which F is differentiable.

Example 6.6: A continuous random variable

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Let the continuous random variable X have the following density function:

$$f(x) = \begin{cases} 0.5 & \text{for } 1 \leq x \leq 3 \\ 0 & \text{else} \end{cases} .$$

Then, we get the distribution function

$$F(x) = \begin{cases} 0 & \text{for } x < 1 \\ 0.5x - 0.5 & \text{for } 1 \leq x \leq 3 \\ 1 & \text{for } x > 3 \end{cases} .$$

Application of $f(x)$ and $F(x)$ in R:

```
# Distinction from the latest functions
f_x6_6 <- function(x) {0.5}

F_x6_6 <- function(x) {0.5 * x - 0.5}
```

Example 6.7: An exponentially distributed RV

Let the continuous random variable X have the following distribution function:

$$F(x) = \begin{cases} 1 - \mathrm{e}^{-\frac{1}{2}x} & \text{for } x \geq 0 \\ 0 & \text{else} \end{cases}$$

(exponential distribution with parameter $\lambda = \frac{1}{2}$). Then, differentiation yields the density function

$$f(x) = \begin{cases} \frac{1}{2}\mathrm{e}^{-\frac{1}{2}x} & \text{for } x \geq 0 \\ 0 & \text{else} \end{cases} .$$

Application of $F(x)$ and $f(x)$ in R:

```
F_x6_7 <- function(x) {1 - exp(-1/2 * x)}

f_x6_7 <- function(x) {1/2 * exp(-1/2 * x)}
```

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Definition 6.7:

Let the random variable X have the following probability or density function $f(x)$, respectively. If

$$\sum_i |f(x_i) \cdot x_i| < \infty \qquad \text{or} \quad \int_{-\infty}^{\infty} |f(x) \cdot x| \, dx < \infty$$

holds, then

$$\mathrm{E}\left(X\right) := \sum_i f(x_i) \cdot x_i \text{ or } \mathrm{E}\left(X\right) := \int_{-\infty}^{\infty} f(x) \cdot x \, dx$$

is called the *expected value* of the discrete or continuous random variable X, respectively.

Definition 6.8:

Let the random variable X have the following probability or density function $f(x)$, respectively. If

$$\sum_i |f(x_i) \cdot x_i^2| < \infty \qquad \text{or} \quad \int_{-\infty}^{\infty} |f(x) \cdot x^2| \, dx < \infty$$

holds, then

$$\operatorname{Var}(X) := \sum_i (x_i - \operatorname{E} X)^2 \cdot f(x_i) \text{ or}$$

$$\operatorname{Var}(X) := \int_{-\infty}^{\infty} (x - \operatorname{E} X)^2 \cdot f(x) \, dx$$

is called the *variance* of the discrete or continuous random variable X, respectively.

Example 6.8: see Ex. 6.4

$$
\begin{aligned}
\mathrm{E}(X) &= \sum_{x=0}^{3} x \cdot f(x) \\
&= 0 \cdot 0.343 + 1 \cdot 0.441 + 2 \cdot 0.189 + 3 \cdot 0.027 \\
&= 0.9
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Var}(X) &= \sum_{x=0}^{3} x^2 \cdot f(x) - \mathrm{E}(X)^2 \\
&= 0^2 \cdot 0.343 + 1^2 \cdot 0.441 + 2^2 \cdot 0.189 + 3^2 \cdot 0.027 - 0.9^2 \\
&= 1.44 - 0.9^2 = 0.63
\end{aligned}
$$

Calculation of $\mathrm{E}(X)$ and $\mathrm{Var}(X)$ in R:

```
Mean_X6_8 <- weighted.mean(x = x6_4, w = f_x6_4)

Var_X6_8 <- sum(f_x6_4*(x6_4 - Mean_X6_8)^2)
```

```
Mean_X6_8
```

```
[1] 0.9
```

```
Var_X6_8
```

```
[1] 0.63
```

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Example 6.9: see Ex. 6.6 (1)

$$
\begin{aligned}
\mathrm{E}(X) &= \int_{-\infty}^{\infty} x \cdot f(x) dx = \int_{1}^{3} x \cdot 0.5 dx \\
&= \left[\frac{1}{2} \cdot x^2 \cdot 0.5\right]_1^3 = \left[\frac{1}{4} \cdot x^2\right]_1^3 \\
&= \frac{9}{4} - \frac{1}{4} = 2
\end{aligned}
$$

Calculation of $\mathrm{E}(X)$ in R:

```r
Mean_X6_9 <- integrate(f = function(x){0.5*x}, lower = 1,
                       upper = 3)$value
Mean_X6_9
```

```
[1] 2
```

Example 6.9: see Ex. 6.6 (2)

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$$
\begin{aligned}
\operatorname{Var}(X) &= \int_{-\infty}^{\infty} x^2 \cdot f(x)dx - \operatorname{E}(X)^2 = \int_{1}^{3} x^2 \cdot 0.5dx - 2^2 \\
&= \left[\frac{1}{6}x^3\right]_1^3 - 4 \\
&= \frac{27}{6} - \frac{1}{6} - 4 = \frac{1}{3}
\end{aligned}
$$

Calculation of Var(X) in R:

```
Var_X6_9 <- integrate(f = function(x){0.5*x^2}, lower = 1,
                      upper = 3
                      )$value - Mean_X6_9^2
Var_X6_9
```

```
[1] 0.3333333
```

Linear transformation of a random variable

Let $Y = a + b \cdot X$, then

$$\mathrm{E}(Y) = a + b \cdot \mathrm{E}(X)$$

$$\mathrm{Var}(Y) = b^2 \cdot \mathrm{Var}(X)$$

Example 6.10:

X: Filling weight of a package of detergent in kg
Y: Deviation from targeted weight of 5 kg in g
Then $Y = (X - 5) \cdot 1000 = -5000 + 1000 \cdot X$ and therefore $a = -5000$ and $b = 1000$.

Example 6.11: (see Example 6.4)

Now we are interested in the share of black balls
(earlier: number of red balls):
$Y = \dfrac{n - X}{n} = 1 - \frac{1}{n} \cdot X$ with $n = 3$

Standard transformation

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The special linear transformation

$$Z = \frac{X - \mathrm{E}(X)}{\sqrt{\mathrm{Var}(X)}} = -\frac{\mathrm{E}(X)}{\sqrt{\mathrm{Var}(X)}} + \frac{1}{\sqrt{\mathrm{Var}(X)}} \cdot X$$

is called standard transformation of random variable X (see Chapter 4).

We have $\mathrm{E}(Z) = 0$ and $\mathrm{Var}(Z) = 1$.

Example 6.12: see Ex. 6.4 and 6.8

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X	0	1	2	3
Z	$-0.9/\sqrt{0.63}$	$0.1/\sqrt{0.63}$	$1.1/\sqrt{0.63}$	$2.1/\sqrt{0.63}$
$f(x)$	0.343	0.441	0.189	0.027

Calculation of Z in R:

```
Z6_12 <- (x6_4 - Mean_X6_8) / sqrt(Var_X6_8)
round(Z6_12, digits = 3)
```

```
[1] -1.134  0.126  1.386  2.646
```

Visualisation for Examples 6.11 and 6.12

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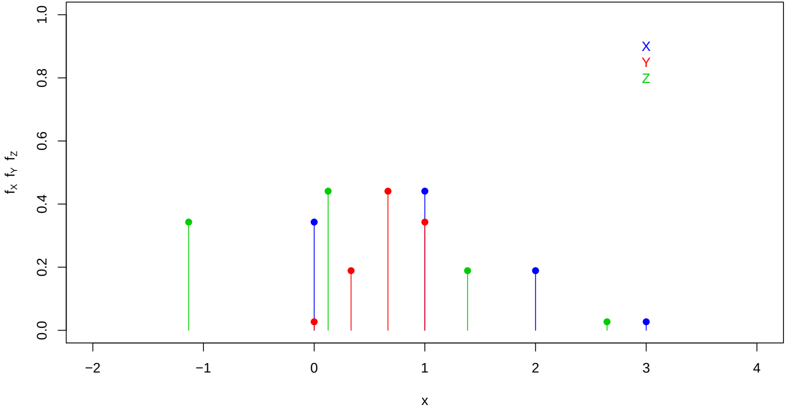
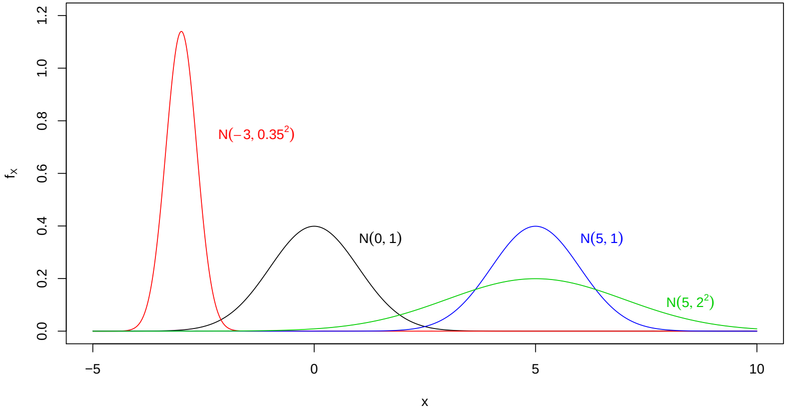


$$Y = \frac{3 - X}{3} \qquad Z = \frac{X - \mathrm{E}(X)}{\sqrt{\mathrm{Var}(X)}}$$

Different normal distributions $N(\mu;\sigma^2)$



$$f(x) = \varphi(x \mid \mu;\sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

Quantiles of distributions

Definition of a quantile (see Schaich and Münnich, 2001):

For a random variable X, a value x, which satisfies the inequalities

$$P(X \leq x) \geq p \qquad \text{and} \qquad P(X \geq x) \leq 1 - p$$

for $0 < p < 1$, is called its *quantile of order p* (p-quantile).

The median $x_{0.5}$ (also called the 0.5-quantile), as well as the first and third quartile ($p = 0.25$ and $p = 0.75$, respectively) are particularly interesting.

For continuous random variables (with a strictly monotonous distribution function) the p-quantile equals $x_p = F^{-1}(p)$.

Schaich, E. and Münnich, R. (2001): Mathematische Statistik für Ökonomen: Lehrbuch. Vahlen.

Example 6.13: Quantiles of exp. distr. (1)

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Let the random variable X follow an exponential distribution with parameter $\lambda = \frac{1}{2}$. The distribution function is:

$$F(x) = \begin{cases} 1 - e^{-\frac{1}{2}\cdot x} & \text{for } x \geq 0 \\ 0 & \text{else} \end{cases} \quad .$$

The p-quantile is derived as follows:

$$\begin{aligned} p &= 1 - e^{-\frac{1}{2}\cdot x} && \big| -p \quad \big| + e^{-\frac{1}{2}\cdot x} \\ e^{-\frac{1}{2}\cdot x} &= 1 - p && \big| \ln \\ -0.5 \cdot x &= \ln(1-p) && \big| : (-0.5) \\ x &= -2\ln(1-p) && \end{aligned}$$

Example 6.13: Quantiles of exp. distr. (2)

Therefore, the median is

$$x_{0.5} = -2\ln(1-0.5) = -2\ln\left(\frac{1}{2}\right) = -2(\ln 1 - \ln 2)$$
$$= -2\ln 1 + 2\ln 2 = 2\ln 2 \approx 1.3863$$

and the first quantile is
$x_{0.25} = -2\ln\frac{3}{4} \approx 0.5754.$

Calculation of $x_{0.5}$ and $x_{0.25}$ in R:

```
q_050 <- -2 * log(1 - 0.5)
q_025 <- -2 * log(1 - 0.25)
```

```
round(q_050, digits = 4)
```

```
[1] 1.3863
```

```
round(q_025, digits = 4)
```

```
[1] 0.5754
```

Markov's and Tchebysheff's inequality

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Theorem 6.1 (Markov's inequality): If a random variable X only takes on non-negative values and the expected value $\mathrm{E}(X)$ exists, the following approximation holds for every $x^* > 0$:

$$P(X \geq x^*) \leq \frac{\mathrm{E}(X)}{x^*} \qquad .$$

Theorem 6.2 (Tchebysheff's inequality): If the variance $\mathrm{Var}(X)$ of a random variable X exists, the following holds for $\varepsilon > 0$:

$$P(|X - \mathrm{E}(X)| \geq \varepsilon) \leq \frac{\mathrm{Var}(X)}{\varepsilon^2} \qquad .$$

Notice the special case where $\varepsilon = k \cdot \sqrt{\mathrm{Var}(X)}$.

Example 6.14: An inequality

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Let a non-negative random variable X have the expected value $\mathrm{E}(X) = 10$ (applies for discrete as well as continuous variables).

We can approximate:

$$P(X \geq 25) \leq \frac{10}{25} = 0.4$$

$$P(X \geq 40) \leq \frac{10}{40} = 0.25$$

$$P(X \geq 5) \leq \frac{10}{5} = 2\,.$$

The third row is a *trivial* approximation as probabilities are bounded by 0 and 1.

Example 6.15: Another inequality

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Let the expected value $\mathrm{E}(X) = 2$ and the variance $\mathrm{Var}(X) = 36$ of a random variable X be known.

Then we have:

$$P(-8 < X < 12) = P(|X - 2| < 10) \geq 1 - \frac{36}{100} = 0.64$$

$$P(|X - 2| \geq 10) \leq \frac{36}{100} = 0.36$$

$$P(X \leq -3 \vee X \geq 7) = P(|X - 2| \geq 5) \leq \frac{36}{25} = 1.44\,.$$

Example 6.16: More dimensions (1)

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We are looking at two- and multi-dimensional random variables.

a)
Random questioning of a person with replacement (income; age):
The resulting observation is $(1815; 25)$.

b)
Two rolls of a dice:
The resulting pair of number of pips is $(4; 6)$.
We could as well be interested in the overall sum of pips or the product of the number of pips $(10; 24)$.

Example 6.16: More dimensions (2)

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c) An urn contains $N = 100$ balls, of which 30 are red (r), 20 are white (w) and 50 are black (s). How does the sample space of this experiment look like, if we draw 3 balls? $X =$ (number r, number w)

ω	$X(\omega)$	ω	$X(\omega)$	ω	$X(\omega)$
(s,s,s)	(0,0)	(r,s,s)	(1,0)	(w,s,s)	(0,1)
(s,s,r)	(1,0)	(r,s,r)	(2,0)	(w,s,r)	(1,1)
(s,s,w)	(0,1)	(r,s,w)	(1,1)	(w,s,w)	(0,2)
(s,r,s)	(1,0)	(r,r,s)	(2,0)	(w,r,s)	(1,1)
(s,r,r)	(2,0)	(r,r,r)	(3,0)	(w,r,r)	(2,1)
(s,r,w)	(1,1)	(r,r,w)	(2,1)	(w,r,w)	(1,2)
(s,w,s)	(0,1)	(r,w,s)	(1,1)	(w,w,s)	(0,2)
(s,w,r)	(1,1)	(r,w,r)	(2,1)	(w,w,r)	(1,2)
(s,w,w)	(0,2)	(r,w,w)	(1,2)	(w,w,w)	(0,3)

Example 6.16: More dimensions (3)

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Construction of the table in R:

```
omega <- expand.grid(lapply(X = 1:3,
                            FUN = function(x) c("s", "r", "w")))
Number_of_r <- rowSums(omega == "r")
Number_of_w <- rowSums(omega == "w")
X6_16 <- cbind(Number_of_r, Number_of_w)

Example6_16 <- data.frame(omega, X6_16)
names(Example6_16) <- c("Omega","","",
                        "Number_of_r","Number_of_w")

head(Example6_16)
```

```
      Omega Number_of_r Number_of_w
1     s s s           0           0
2     r s s           1           0
3     w s s           0           1
4     s r s           1           0
5     r r s           2           0
6     w r s           1           1
```

Example 6.16: More dimensions (4)

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The set of all possible realisations is

$$\{(x_1, x_2) \,|\, x_1, x_2 \in \mathbb{N}_0, 0 \leq x_1 + x_2 \leq 3\}\,.$$

For instance, we could determine $P(X_1 = x_1; X_2 = x_2)$, $P(X_1 \leq x_1; X_2 \leq x_2)$, $P(X_1 \leq x_1 \vee X_2 \leq x_2)$ or $P(X_1 \leq x_1 | X_2 = x_2)$.

Multi-dimensional random variables

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Multi-dimensional random variables can be considered as a generalisation of one-dimensional random variables.

The inverse images of half-open n-intervals must again be part of the sigma algebra over Ω.

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Distribution function of multi-dimensional random variables

The function

$$F_X(x_1, \ldots, x_n) = P(X_1 \leq x_1, X_2 \leq x_2, \ldots, X_n \leq x_n) \quad ,$$

which gives the probability that X_1 is at most x_1 and X_n is at most x_n for all real n-tuple is called distribution function of the random vector $X_1, X_2, \ldots, X_n$.

Interval probabilities in the two-dimensional case:

$$P(x_1' < X_1 \leq x_1'', x_2' < X_2 \leq x_2'') = F(x_1'', x_2'') - F(x_1', x_2'') - F(x_1'', x_2') + F(x_1', x_2') \quad .$$

Discrete random vectors

Discrete random vector

A random vector $\mathbf{X}$ is called a multi-dimensional discrete random variable if each of its components can take on at most a countable number of values.

Probability function of a discrete random vector

The function $f(x_1, x_2)$, which is defined for all real pairs of numbers (x_1, x_2) and which is characterised by

$$f(x_1, x_2) = \begin{cases} P(X_1 = x_{1j}, X_2 = x_{2k}) & \text{for all } j, k \\ 0 & \text{else} \end{cases}$$

is called the probability function of the discrete random vector $\mathbf{X}$.

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Two-dimensional discrete random variables

$X_1 \backslash X_2$	x_{21}	$\cdots$	x_{2k}	$\cdots$	x_{2r}	$\sum$
x_{11}	$f(x_{11}, x_{21})$	$\cdots$	$f(x_{11}, x_{2k})$	$\cdots$	$f(x_{11}, x_{2r})$	$f_{X_1}(x_{11})$
$\vdots$	$\vdots$	$\ddots$	$\vdots$		$\vdots$	$\vdots$
x_{1j}	$f(x_{1j}, x_{21})$	$\cdots$	$f(x_{1j}, x_{2k})$	$\cdots$	$f(x_{1j}, x_{2r})$	$f_{X_1}(x_{1j})$
$\vdots$	$\vdots$		$\vdots$	$\ddots$	$\vdots$	$\vdots$
x_{1m}	$f(x_{1m}, x_{21})$	$\cdots$	$f(x_{1m}, x_{2k})$	$\cdots$	$f(x_{1m}, x_{2r})$	$f_{X_1}(x_{1m})$
$\sum$	$f_{X_2}(x_{21})$	$\cdots$	$f_{X_2}(x_{2k})$	$\cdots$	$f_{X_2}(x_{2r})$	1

Properties of discrete random vectors

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1. We have:

$$\sum_j \sum_k f(x_{1j}, x_{2k}) = 1\,.$$

2. Distribution function:

$$F(x_1, x_2) = \sum_{x_{1j} \le x_1} \sum_{x_{2k} \le x_2} f(x_{1j}, x_{2k})$$

3. Interval probabilities:

$$P(x_1' < X_1 \le x_1'', x_2' < X_2 \le x_2'') = \sum_{x_1' < x_{1j} \le x_1''} \sum_{x_2' < x_{2k} \le x_2''} f(x_{1j}, x_{2k})$$

Marginal distributions of bivariate distributions

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In addition to the joint distribution of the random vector (X_1, X_2) with the distribution function $F(x_1, x_2)$, the *marginal distributions*, ergo the univariate distributions of the random variables involved in the distribution functions $F_{X_1}(x_1)$ and $F_{X_2}(x_2)$, may be considered as well. We obtain those by

$$F_{X_1}(x_1) = \sum_{x_{1j} \leq x_1} \sum_{k} f(x_{1j}, x_{2k}) \qquad \text{or} \qquad F_{X_2}(x_2) = \sum_{j} \sum_{x_{2k} \leq x_2} f(x_{1j}, x_{2k})$$

and thus by adding up all probabilities of the variable which is not of interest.

The indexation of the marginal distribution functions is used for unique identification.

Example 6.17: see Ex. 6.16 c) (1)

We are interested in the random vector (number of red balls, number of white balls). For example, we have:

$$f(1,0) = 3 \cdot 0.3^1 \cdot 0.2^0 \cdot 0.5^2 = 0.225 \quad .$$

Calculation of $f(1,0)$ in R:

```
Number_of_s <- rowSums(omega == "s")
Example6_16 <- cbind(Example6_16, Number_of_s)
Probs <- 0.3^Example6_16$Number_of_r *
            0.2^Example6_16$Number_of_w *
            0.5^Example6_16$Number_of_s

Example6_16 <- cbind(Example6_16, Probs)

pos <- which(Example6_16$Number_of_r == 1 &
               Example6_16$Number_of_w == 0)
f_1_0 <- sum(Example6_16[pos, 7])
f_1_0
```

```
[1] 0.225
```

Example 6.17: see Ex. 6.16 c) (2)

Finally, we get the following probability table:

X_1 | X_2	$x_{21}=0$	$x_{22}=1$	$x_{23}=2$	$x_{24}=3$	$\sum$
$x_{11}=0$	0.125	0.150	0.060	0.008	0.343
$x_{12}=1$	0.225	0.180	0.036	0.000	0.441
$x_{13}=2$	0.135	0.054	0.000	0.000	0.189
$x_{14}=3$	0.027	0.000	0.000	0.000	0.027
$\sum$	0.512	0.384	0.096	0.008	1.000

Probability table in R:

```
X1_6_17 <- 0:3 ; X2_6_17 <- 0:3
ProbTable6_17 <- matrix(c(0.125, 0.150, 0.060, 0.008,
                          0.225, 0.180, 0.036, 0.000,
                          0.135, 0.054, 0.000, 0.000,
                          0.027, 0.000, 0.000, 0.000),
                        ncol = length(X2_6_17),
                        byrow = TRUE)
dimnames(ProbTable6_17) <- list(X1_6_17, X2_6_17)
ProbTable_new6_17 <- addmargins(ProbTable6_17)
```

Example 6.17: see Ex. 6.16 c) (3)

```
ProbTable_new6_17
```

	0	1	2	3	Sum
0	0.125	0.150	0.060	0.008	0.343
1	0.225	0.180	0.036	0.000	0.441
2	0.135	0.054	0.000	0.000	0.189
3	0.027	0.000	0.000	0.000	0.027
Sum	0.512	0.384	0.096	0.008	1.000

For $F(1,2)$ we have:

$X_1 \mid X_2$	$x_{21}=0$	$x_{22}=1$	$x_{23}=2$	$x_{24}=3$	$\sum$
$x_{11}=0$	0.125	0.150	0.060	0.008	0.343
$x_{12}=1$	0.225	0.180	0.036	0.000	0.441
$x_{13}=2$	0,135	0.054	0.000	0.000	0.189
$x_{14}=3$	0.027	0.000	0.000	0.000	0.027
$\sum$	0.512	0.384	0.096	0.008	1.000

$$F(1,2) = 0.125 + 0.150 + 0.060 + 0.225 + 0.180 + 0.036 = 0.776$$

Example 6.17: see Ex. 6.16 c) (4)

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Determination of the joint distribution function in R:

```
F_x1_x2 <- t(apply(X = apply(X = ProbTable6_17,
                             MARGIN = 2, FUN = cumsum),
                   MARGIN = 1, FUN = cumsum))
F_x1_x2
```

```
      0     1     2     3
0 0.125 0.275 0.335 0.343
1 0.350 0.680 0.776 0.784
2 0.485 0.869 0.965 0.973
3 0.512 0.896 0.992 1.000
```

```
F_x1_x2[rownames(F_x1_x2) == 1, colnames(F_x1_x2) == 2]
```

```
[1] 0.776
```

Example 6.17: see Ex. 6.16 c) (5)

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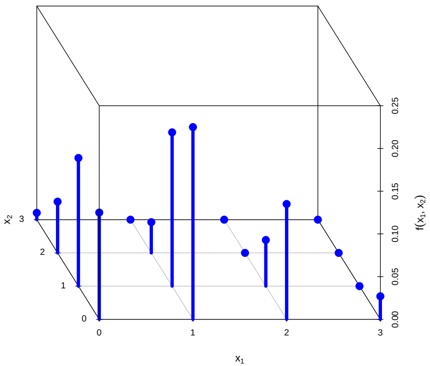
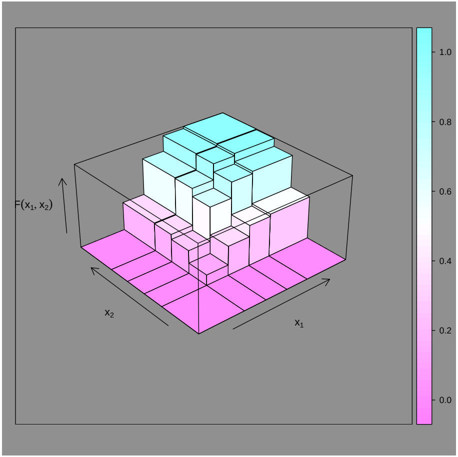




Continuous random vectors

Continuous random vectors are defined analogously to continuous random variables. We have:

1. $\int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} f(x_1, x_2)\, dx_2 dx_1 = 1$

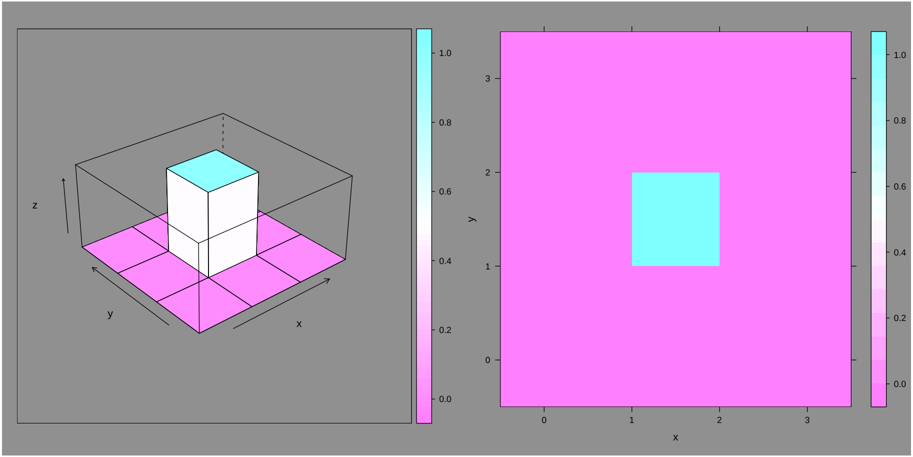
2. $F(x_1, x_2) = \int\limits_{-\infty}^{x_1} \int\limits_{-\infty}^{x_2} f(y_1, y_2)\, dy_2 dy_1$

3. $P(x_1' < X_1 \leq x_1'', x_2' < X_2 \leq x_2'') = \int\limits_{x_1'}^{x_1''} \int\limits_{x_2'}^{x_2''} f(x_1, x_2)\, dx_2 dx_1$

4. $f(x_1, x_2) = \dfrac{\partial^2 F(x_1, x_2)}{\partial x_1\, \partial x_2}$ (assuming differentiability)

5. Marg. distr.: $f(x_1) = \int\limits_{-\infty}^{\infty} f(x_1, x_2) dx_2$ and $f(x_2) = \int\limits_{-\infty}^{\infty} f(x_1, x_2) dx_1$

Rectangular distribution



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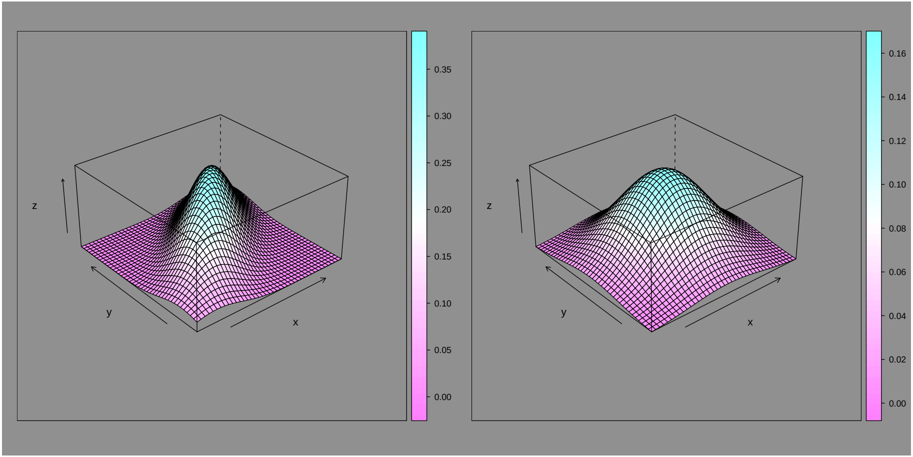
Bivariate normal distribution



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Example 6.18: A continuous random vector (1)

Let the density function of a continuous random vector be

$$f(x_1, x_2) = \begin{cases} 6 \cdot \exp(-2x_1) \cdot \exp(-3x_2) & \text{for } x_1, x_2 > 0 \\ 0 & \text{else} \end{cases} .$$

Using integration we get the following distribution function:

$$F(x_1, x_2) = \begin{cases} (1 - \exp(-2x_1)) \cdot (1 - \exp(-3x_2)) & \text{for } x_1, x_2 > 0 \\ 0 & \text{else} \end{cases} .$$

For $x_2 > 0$ we get the following marginal density function for random variable X_2:

$$f(x_2) = \int_{-\infty}^{\infty} f(x_1, x_2) dx_1 = \int_{0}^{\infty} 6 \cdot \exp(-2x_1 - 3x_2) dx_1 = 3 \cdot \exp(-3x_2) \quad .$$

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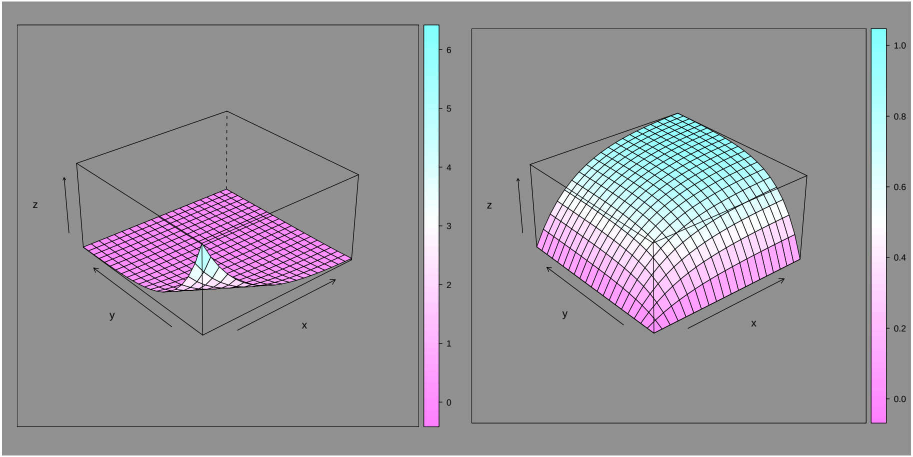
Example 6.18: A continuous random vector (2)



Stochastical independence

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Let $F(x_1, x_2)$ be the joint distribution function of the random vector $\mathbf{X}$ and let $F(x_1)$ and $F(x_2)$ be the marginal distribution functions. Two random variables X_1 and X_2 are called stochastically independent if and only if we have

$$F(x_1, x_2) = F(x_1) \cdot F(x_2)$$

for all $(x_1, x_2) \in \mathbb{R}$. Otherwise, they are called stochastically dependent. Stochastical independence may be proven using probabilities or probability functions and density functions as well (see Schaich and Münnich, 2001).

Example 6.19: see Ex. 6.17

$$f_{X_1}(0) \cdot f_{X_2}(0) = 0.343 \cdot 0.512 = 0.175616 \neq 0.125 = f(0,0)$$

X_1 and X_2 are stochastically dependent.

Checking in R:

```
(ProbTable_new6_17[1,5] * ProbTable_new6_17[5,1]) ==
ProbTable_new6_17[1,1]
```

```
[1] FALSE
```

Example 6.20 (1)

Let the random vector **X** have the following probability table:

X_1 | X_2	2	4	6	$\sum$
1	0.05	0.14	0.01	0.20
5	0.20	0.56	0.04	0.80
$\sum$	0.25	0.70	0.05	1.00

Example 6.20 (2)

Probability table in R:

```
ProbTable6_20 <- matrix(c(0.05,0.14,0.01,
                          0.20,0.56,0.04),
                        ncol = 3, byrow = TRUE)

X1_6_20 <- c(1, 5)
X2_6_20 <- seq(2, 6, 2)

rownames(ProbTable6_20) <- X1_6_20
colnames(ProbTable6_20) <- X2_6_20

ProbTable6_20 <- addmargins(ProbTable6_20)

ProbTable6_20
```

```
       2    4    6  Sum
1   0.05 0.14 0.01 0.2
5   0.20 0.56 0.04 0.8
Sum 0.25 0.70 0.05 1.0
```

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Example 6.20 (3)

We have $f_{X_1}(x_{1i}) \cdot f_{X_2}(x_{2j}) = f(x_{1i}, x_{2j})$ for all i, j. Therefore, X_1 and X_2 are stochastically independent.

Checking of stochastic independence in R:

```
round(ProbTable6_20[3,] * ProbTable6_20[1,4],4) ==
ProbTable6_20[1,]
```

```
   2    4    6  Sum
TRUE TRUE TRUE TRUE
```

```
round(ProbTable6_20[3,] * ProbTable6_20[2,4],4) ==
ProbTable6_20[2,]
```

```
   2    4    6  Sum
TRUE TRUE TRUE TRUE
```

Covariance of two random variables

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The covariance of two random variables X_1 and X_2 is defined as:

$$\operatorname{Cov}(X_1, X_2) = \operatorname{E}\left((x_1 - \operatorname{E} X_1) \cdot (x_2 - \operatorname{E} X_2)\right).$$

For discrete random variables we have:

$$\operatorname{Cov}(X_1, X_2) = \sum_i \sum_j (x_{1i} - \operatorname{E} X_1) \cdot (x_{2j} - \operatorname{E} X_2) \cdot f(x_{1i}, x_{2j}).$$

Analoguously, for continuous random variables we have:

$$\operatorname{Cov}(X_1, X_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x_1 - \operatorname{E} X_1) \cdot (x_2 - \operatorname{E} X_2) f(x_1, x_2) dx_1 dx_2 .$$

Furthermore, the displacement law holds:

$$\operatorname{Cov}(X_1, X_2) = \operatorname{E}(X_1 \cdot X_2) - \operatorname{E} X_1 \cdot \operatorname{E} X_2 .$$

Example 6.21: see Ex. 6.17 (1)

$$\mathrm{E}\, X_1 = 0 \cdot 0.343 + 1 \cdot 0.441 + 2 \cdot 0.189 + 3 \cdot 0.027 = 0.9$$
$$\mathrm{E}\, X_2 = 0 \cdot 0.512 + 1 \cdot 0.384 + 2 \cdot 0.096 + 3 \cdot 0.008 = 0.6$$

Calculation of $\mathrm{E}\, X_1$ and $\mathrm{E}\, X_2$ in R:

```
f_X1_6_17 <- rowSums(ProbTable6_17)
f_X2_6_17 <- colSums(ProbTable6_17)

Mean_X1 <- sum(f_X1_6_17 * X1_6_17)
Mean_X2 <- sum(f_X2_6_17 * X2_6_17)

Mean_X1_old <- Mean_X1
Mean_X2_old <- Mean_X2
```

```
Mean_X1
```

```
[1] 0.9
```

```
Mean_X2
```

```
[1] 0.6
```

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Example 6.21: see Ex. 6.17 (2)

$$
\begin{aligned}
\operatorname{Cov}(X_1, X_2) = {} & 0 \cdot 0 \cdot 0.125 + 0 \cdot 1 \cdot 0.150 + 0 \cdot 2 \cdot 0.060 + 0 \cdot 3 \cdot 0.008 \\
& + 1 \cdot 0 \cdot 0.225 + 1 \cdot 1 \cdot 0.180 + 1 \cdot 2 \cdot 0.036 + 1 \cdot 3 \cdot 0.000 \\
& + 2 \cdot 0 \cdot 0.135 + 2 \cdot 1 \cdot 0.054 + 2 \cdot 2 \cdot 0.000 + 2 \cdot 3 \cdot 0.000 \\
& + 3 \cdot 0 \cdot 0.027 + 3 \cdot 1 \cdot 0.000 + 3 \cdot 2 \cdot 0.000 + 3 \cdot 3 \cdot 0.000 \\
& - 0.9 \cdot 0.6 \\
= {} & -0.18
\end{aligned}
$$

Calculation of $\operatorname{Cov}(X_1, X_2)$ in R:

```
Intermed_matrix6_21 <- matrix(
               rep(x = X2_6_17, times = length(X1_6_17)),
               ncol = length(X2_6_17), byrow = TRUE)

Cov_X1_X2 <- sum(Intermed_matrix6_21 * X2_6_17 *
                 ProbTable6_17) - Mean_X1 * Mean_X2
Cov_X1_X2_old <- Cov_X1_X2
Cov_X1_X2
```

```
[1] -0.18
```

Example 6.22: see Ex. 6.20

$$\begin{aligned}\mathrm{Cov}\left(X_1, X_2\right) &= 1 \cdot 2 \cdot 0.05 + 1 \cdot 4 \cdot 0.14 + 1 \cdot 6 \cdot 0.01 + 5 \cdot 2 \cdot 0.2 + 5 \cdot 4 \cdot 0.56 \\ &\quad + 5 \cdot 6 \cdot 0.04 - (0.2 + 5 \cdot 0.8) \cdot (2 \cdot 0.25 + 4 \cdot 0.7 + 6 \cdot 0.05) \\ &= 15.12 - 15.12 = 0\end{aligned}$$

Notice that X_1 and X_2 are stochastically independent!

Calculation of $\mathrm{Cov}(X_1, X_2)$ in R:

```
i <- 1:3 ; ProbTable6_20 <- ProbTable6_20[-3,-4]
Mean_X1 <- weighted.mean(x = X1_6_20,
                         w = prop.table(ProbTable6_20[,2]))
Mean_X2 <- weighted.mean(x = X2_6_20,
                         w = prop.table(ProbTable6_20[2,]))

Cov_X1_X2 <- (sum(X1_6_20[1] * X2_6_20[i] *
                  ProbTable6_20[1,i]) +
              sum(X1_6_20[2] * X2_6_20[i] *
              ProbTable6_20[2,i])) - Mean_X1 * Mean_X2
Cov_X1_X2
```

```
[1] 0
```

Independence and uncorrelatedness

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If $\operatorname{Cov}(X_1, X_2) = 0$, then the random variables X_1 and X_2 are called uncorrelated.
We have:

$$\text{Independence} \quad \begin{array}{c}\Rightarrow \\ \not\Leftarrow\end{array} \quad \text{Uncorrelatedness}$$

Example 6.23:

$X_1 \mid X_2$	-2	0	1	$\sum$
0	0.125	0.000	0.250	0.375
1	0.125	0.250	0.250	0.625
$\sum$	0.250	0.250	0.500	1.000

We have $\operatorname{Cov}(X_1, X_2) = 0$ but $f(0,0) \neq f_{X_1}(0) \cdot f_{X_2}(0)$ as well. Therefore, X_1 and X_2 are uncorrelated but not independent.

Correlation of two random variables

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The correlation coefficient of Bravais-Pearson for two random variables X_1 and X_2 is defined as:

$$\varrho_{X_1,X_2} = \frac{\operatorname{Cov}(X_1, X_2)}{\sqrt{\operatorname{Var} X_1 \cdot \operatorname{Var} X_2}}$$

Example 6.24: see Ex. 6.21

By using $\operatorname{Cov}(X_1, X_2) = -0.18$ as well as $\operatorname{Var} X_1 = 0.630$ and $\operatorname{Var} X_2 = 0.480$, we finally have:

$$\varrho_{X_1,X_2} = \frac{-0.18}{\sqrt{0.630 \cdot 0.480}} = -0.3273\,.$$

Calculation of ϱ_{X_1,X_2} in R:

```
Var_X1 <- sum(f_X1_6_17 * (X1_6_17 - Mean_X1_old)^2)

Var_X2 <- sum(f_X2_6_17 * (X2_6_17 - Mean_X2_old)^2)

Cor_X1_X2 <- Cov_X1_X2_old / (sqrt(Var_X1 * Var_X2))

round(Cor_X1_X2, digits = 4)
```

```
[1] -0.3273
```

Properties of the correlation coefficient

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1. Let Z_1 and Z_2 be the standardised random variables of the random variables X_1 and X_2. We then have:

$$\varrho_{X_1,X_2} = \mathrm{Cov}\left(Z_1, Z_2\right).$$

2. Generally $-1 \leq \varrho_{X_1,X_2} \leq 1$.

3. If $X_2 = a_0 + a_1 \cdot X_1$ and $a_1 \neq 0$, it follows that $|\varrho_{X_1,X_2}| = 1$ (where the reverse holds as well).

4. If

$$\begin{aligned} U_1 &= a_0 + a_1 \cdot X_1 \qquad && (a_1 \neq 0) \\ U_2 &= b_0 + b_1 \cdot X_2 \qquad && (b_1 \neq 0) \end{aligned}$$

are linear transformations of the random variables X_1 and X_2, we have

$$\varrho_{U_1,U_2} = \mathrm{sgn}(a_1 \cdot b_1) \cdot \varrho_{X_1,X_2} \qquad .$$

Example 6.25: (see Example 6.24)

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X_1:	Number of red balls	Y_1:	Share of red balls
X_2:	Number of white balls	Y_2:	Share of white balls

We have $Y_1 = X_1/3$ and $Y_2 = X_2/3$. Furthermore, we already know that $\varrho_{X_1,X_2} = -0.3273$.

Finally, we get

a) $\varrho_{Y_1,Y_2} = -0.3273$,

b) $\varrho_{X_1,Y_1} = 1$,

c) $\varrho_{X_1,Y_2} = -0.3273\,.$

More than two random variables (1)

1. The n random variables $X_1, \ldots, X_n$ are called collectively stochastically independent, if

$$F(x_1, \ldots, x_n) = F(x_1) \cdot \ldots \cdot F(x_n)$$

(analoguously for density and probability functions).

2. The n random variables $X_1, \ldots, X_n$ are called pairwise stochastically independent, if for two arbitrary but different random variables X_i and X_j we have:

$$F(x_i, x_j) = F(x_i) \cdot F(x_j)$$

(analoguously for density and probability functions).

3. We have:
collectively stochastically independent $\Rightarrow$
pairwise stochastically independent $\Rightarrow$ pairwise uncorrelated

More than two random variables (2)

4. Variance-covariance matrix:

$$\Sigma = \begin{pmatrix} \operatorname{Var} X_1 & \operatorname{Cov}(X_1, X_2) & \cdots & \operatorname{Cov}(X_1, X_n) \\ \operatorname{Cov}(X_2, X_1) & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \operatorname{Cov}(X_{n-1}, X_n) \\ \operatorname{Cov}(X_n, X_1) & \cdots & \operatorname{Cov}(X_n, X_{n-1}) & \operatorname{Var} X_n \end{pmatrix}$$

If Σ is a diagonal matrix, then the n random variables are pairwise uncorrelated.

Functions of random variables

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Example 6.26:

a) Two rolls of a dice: We are interested in the overall number of pips $Y = X_1 + X_2$.

b) $N = 101$ balls $(0, \dots, 100)$: n balls are drawn with replacement.

$$Y_1 = \tfrac{1}{2}(X_1 + X_2)$$

$$Y_2 = \tfrac{1}{20}(X_1 + \cdots + X_{20})$$

c) Construction of cylindric components (technical QC): X_1 is the component's diameter and X_2 is its length. Then

$$Y = \tfrac{\pi}{4} \cdot X_1^2 \cdot X_2$$

is its volume.

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Expected value and variance of linearly transformed random variables

If $Y = a_0 + \sum_{i=1}^{n} a_i X_i$ is a general linear transformation of n random variables, then

$$\mathrm{E}\,Y = a_0 + \sum_{i=1}^{n} a_i \mathrm{E}\,X_i$$

is the expected value of the transformed random variable Y. We call E a linear operator! Furthermore

$$\begin{aligned}\mathrm{Var}\,Y &= \sum_{i=1}^{n}\sum_{j=1}^{n} a_i \cdot a_j \cdot \mathrm{Cov}\,(X_i, X_j) \\ &= \sum_{i=1}^{n} a_i^2 \cdot \mathrm{Var}\,X_i + 2 \cdot \sum\sum_{i<j} a_i \cdot a_j \cdot \mathrm{Cov}\,(X_i, X_j)\end{aligned}$$

is the variance of the transformed random variable Y.

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Example 6.27: see Ex. 6.26 b) (1)

An urn contains $N = 100$ balls (1,...,100). $n = 3$ balls are drawn with replacement, where X_i is the number drawn in the i-th draw. Then we have:

$$\mathrm{E}\, X_i = \frac{1}{100}(1 + \cdots + 100) = 50.5$$

$$\mathrm{Var}\, X_i = \frac{1}{100}(1^2 + \cdots + 100^2) - 50.5^2 = 833.25 \qquad .$$

Calculation of E X_i and Var X_i in R:

```
X6_27 <- 1:100
f_x6_27 <- rep(x = 1/100, times = 100)

Mean_X <- sum(f_x6_27 * X6_27)
Var_X <- sum(f_x6_27 * (X6_27 - Mean_X)^2)
```

```
Mean_X
```

```
[1] 50.5
```

```
Var_X
```

```
[1] 833.25
```

Example 6.27: see Ex. 6.26 b) (2)

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Now we are interested in the sample mean $\overline{X} = \frac{1}{3}(X_1 + X_2 + X_3)$. We get:

$$\operatorname{E}\overline{X} = \frac{1}{3}(\operatorname{E}X_1 + \operatorname{E}X_2 + \operatorname{E}X_3) = 50.5$$

$$\operatorname{Var}\overline{X} = \left(\frac{1}{3}\right)^2 (\operatorname{Var}X_1 + \operatorname{Var}X_2 + \operatorname{Var}X_3) = \frac{1}{3} \cdot 833.25 = 277.75 \qquad .$$

Notice that the draws are stochastically independent (with replacement). In the model without replacement we would have $\operatorname{E}\overline{X} = 50.5$ and $\operatorname{Var}\overline{X} = 272.139$.

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Example 6.28: see Ex. 6.21 (1)

We are now interested in $Y = 2X_1 + 4X_2 - 1$. We get:

$$\begin{aligned} \mathrm{E}\, Y &= 2\mathrm{E}\, X_1 + 4\mathrm{E}\, X_2 - 1 \\ &= 2 \cdot 0.9 + 4 \cdot 0.6 - 1 = 3.2 \end{aligned}$$

and

$$\begin{aligned} \mathrm{Var}\, Y &= \sum_{i=1}^{2} \sum_{j=1}^{2} a_i a_j \cdot \mathrm{Cov}\,(X_i, X_j) \\ &= 2^2 \cdot \mathrm{Var}\, X_1 + 2 \cdot 2 \cdot 4 \cdot \mathrm{Cov}\,(X_1, X_2) + 4^2 \cdot \mathrm{Var}\, X_2 \\ &= 4 \cdot 0.63 - 16 \cdot 0.18 + 16 \cdot 0.48 \\ &= 2.52 + 4.8 = 7.32 \qquad . \end{aligned}$$

Example 6.28: see Ex. 6.21 (2)

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Calculation of E Y and Var Y in R

```
a0 <- -1
a1 <- 2
a2 <- 4

# ATTENTION: Means etc. from Ex. 6.21!
Mean_Y <- a0 + a1 * Mean_X1_old + a2 * Mean_X2_old

Var_Y <- a1^2 * Var_X1 + 2 * a1 * a2 * Cov_X1_X2_old +
         a2^2 * Var_X2
```

```
Mean_Y
```

```
[1] 3.2
```

```
Var_Y
```

```
[1] 7.32
```