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Numerical Optimization - Sheet 12

If you are a student in mathematics please solve the exercises with no tag and the ones with the tag **Mathematics**. If you are a data science student please solve the problems with no tag and those with the tag **Data Science**. Submissions with tags other than your subject count as bonus points. The tag **Programming** marks programming exercises.

Ex 1 Programming (5 Points)

Consider the problem

$$\begin{aligned} &\min x_1^2 + 4x_1x_2 + 5x_2^2 - 10x_1 - 20x_2 \\ &\text{s.t. } 2 - x_1 - x_2 = 0. \end{aligned}$$

(i) Implement the quadratic penalty algorithm as described in Section 6.1.1 in the script. Use `scipy.optimize.minimize` to solve the unconstrained sub-problems.

(ii) Test it on the above problem with `tol`$= 10^{-7}$, $\mu = (10, 10^3, 10^5, 10^7)$.

- Plot the error of the solution against the number of steps (i.e. the index k of μ_k) in the penalty method.
- Plot the condition of the inverse hessian in the solution against μ_k. You can use `np.linalg.cond`, and you might want to use a log-log-scaling.

Hint: The result of `scipy.optimize.minimize` contains the solution and the hessian inverse (or its approximation) which you can use to obtain a condition number. You find the details in the documentation.

Ex 2 (3 Points)

Let $z \in \mathbb{R}^2$, and $\lambda \in \mathbb{R}$. Consider the problem

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix} \begin{pmatrix} z_1 \\ z_2 \\ \lambda \end{pmatrix} = \begin{pmatrix} 1 - x_1 \\ 1 \\ 0 \end{pmatrix}.$$

Recover the quadratic problem which is solved by the linear system.

<u>Solution 2:</u>

$$\begin{aligned} &\min_{z \in \mathbb{R}^2} \frac{1}{2} z^\top \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} z - \begin{pmatrix} 1 - x_1 \\ 1 \end{pmatrix}^\top z \\ &\text{s.t.} \quad \begin{bmatrix} 1, & -1 \end{bmatrix} z = 0 \end{aligned}$$

Ex 3 (2 Points)

Assume $f, g\colon \mathbb{R}^n \to \mathbb{R}$ are sufficiently smooth functions, and $A, B \in \mathbb{R}^{m\times n}$. Construct the KKT-system of the quadratic subproblem of the SQP-method corresponding to the following nonlinear problems.

(i)

$$\begin{aligned}&\min_{\theta,z} \frac{1}{2}z^2\\ &\text{s.t.}\\ &z = g(\theta)\end{aligned}$$

(ii)

$$\begin{aligned}&\min_{\alpha,\beta} f(\alpha) + g(\beta)\\ &\text{s.t.}\\ &A\alpha + B\beta = 0\end{aligned}$$

Hint: The relevant material is presented in lecture 20 and 21.

Solution 3:
The KKT-System is given by

$$\begin{bmatrix}\text{Hess}f(x^k) & C(x^k)^\top\\ C(x^k) & 0\end{bmatrix}\begin{pmatrix}\Delta x^k\\ \lambda^{k+1}\end{pmatrix} = \begin{pmatrix}-\nabla f(x^k)\\ -c(x^k)\end{pmatrix}$$

(i)

$$\mathcal{L}(z,\theta,\lambda) = \frac{1}{2}z^2 - \lambda(z - g(\theta))$$

$$\begin{bmatrix}1 & 0 & 1\\ 0 & \lambda\ \text{Hess}g(\theta^k) & -\nabla(g(\theta^k))\\ 1 & -\nabla(g(\theta^k))^\top & 0\end{bmatrix}\begin{pmatrix}\Delta z^k\\ \Delta\theta^k\\ \lambda^{k+1}\end{pmatrix} = \begin{pmatrix}-z^k\\ 0\\ -z^k + g(\theta^k)\end{pmatrix}$$

(ii)

$$\mathcal{L}(z,\theta,\lambda) = f(\alpha) + g(\beta) - \lambda^\top(A\alpha + B\beta)$$

$$\begin{bmatrix}\text{Hess}f(\alpha^k) & 0 & A\\ 0 & \text{Hess}g(\beta^k) & B\\ A & B & 0\end{bmatrix}\begin{pmatrix}\Delta\alpha^k\\ \Delta\beta^k\\ \lambda^{k+1}\end{pmatrix} = \begin{pmatrix}-\nabla f(\alpha^k)\\ -\nabla g(\beta^k)\\ -A\alpha^k - B\beta^k\end{pmatrix}$$