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Numerical Optimization - Sheet 11

If you are a student in mathematics please solve the exercises with no tag and the ones with the tag **Mathematics**. If you are a data science student please solve the problems with no tag and those with the tag **Data Science**. Submissions with tags other than your subject count as bonus points. The tag **Programming** marks programming exercises.

Ex 1 (6 Points)

Apply the range space method to solve the problem

$$\min \ \frac{1}{2}x^\top \begin{bmatrix} 3 & 2 & 1 \\ 2 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} x$$

$$\text{unter} \quad \begin{aligned} x_1 - 2x_2 + x_3 &= -3 \\ -x_1 + x_2 - x_3 &= 2 \end{aligned}$$

in the following steps.

(i) Derive C, B, b, c, and A corresponding to the problem as given in the lecture.

(ii) Solve $A\lambda = \alpha$.

(iii) Solve $Bx = -(C^\top \lambda + b)$.

<u>Solution 1:</u>
We define (as in the lecture)

$$B = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}, \quad b = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad C = \begin{bmatrix} 1 & -2 & 1 \\ -1 & 1 & -1 \end{bmatrix}, \quad c = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$$

1) The range space method requires the following systems:

$$B^{-1} = \begin{bmatrix} 2 & -2 & -1 \\ -2 & 2.5 & 1 \\ -1 & 1 & 1 \end{bmatrix}$$

$$A = CB^{-1}C^T = \begin{bmatrix} 1 & -2 & 1 \\ -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 2 & -2 & -1 \\ -2 & 2.5 & 1 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -2 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 15 & -9 \\ -9 & 5.5 \end{bmatrix}$$

Note that in practice inverses like B^{-1} usually do not exist. It is much more efficient to just solve the corresponding linear system. Moreover, in many cases it is completely impossible to store the inverse of a matrix. Now we compute the right hand side and finally the solution of the system.

$$\alpha = c - CB^{-1}b \overset{b=0}{\Rightarrow} \alpha = c$$

$$\Rightarrow A\lambda = \alpha \Rightarrow \lambda = \begin{pmatrix} -1 \\ -2 \end{pmatrix} \Rightarrow x = -B^{-1}(C^T\lambda + b) \overset{b=0}{=} -B^{-1}(C^T\lambda) \qquad \Rightarrow x = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}.$$

Ex 2 (10 Points)

Consider the equality constrained QP

$$\min_{x\in\mathbb{R}^3} \frac{1}{6}x^\top \begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix} x + \begin{bmatrix} 0 & 0 & \frac{1}{3} \end{bmatrix} x$$

$$\text{s. t.} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} x = 0$$

and solve it using the null space method in the following steps.

(i) Define $C, B, \dots$ and $x = (y, z)$ as given in the lecture.

(ii) Show that the kernel of C is given by $\ker(C) = \{\alpha\,(1,1,1)^\top \mid \alpha \in \mathbb{R}\}$.

(iii) Show that the null-space method is applicable, i.e. that B is positive definite on $\ker(C)$.

(iv) Determine the reduced Hessian and the corresponding right hand side as given in the lecture (see *S Schur complement = reduced Hessian*).

(v) Successively determine z and y.

Solution 2:

(i) $\text{kern}(C) = \alpha \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$B = \frac{1}{3} \begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$

$\alpha \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \frac{1}{3} \begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix} \alpha \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 3\alpha^2 > 0 \Rightarrow B$ pos. def. auf $\text{kern}(C)$.

(ii) $\text{rank} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = 2 \Rightarrow C_1 = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}, C_2 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

$y := (x_1, x_2)^\top, z := x_3, b_1, c_1 = (0,0)^\top, b_2 = \frac{1}{3}, c_2 = 0,$

$B = \frac{1}{3} \left[\begin{array}{cc|c} -1 & 2 & 2 \\ 2 & -1 & 2 \\ \hline 2 & 2 & -1 \end{array}\right] =: \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$

$C_1^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

$S = B_{22} - C_2^\top C_1^{-\top} B_{12} - B_{21} C_1^{-1} C_2 + C_2^\top C_1^{-\top} B_{11} C_1^{-1} C_2 =$

$= -\frac{1}{3} - \frac{1}{3} \begin{bmatrix} 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} - \frac{1}{3} \begin{bmatrix} 2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \end{bmatrix} + \frac{1}{3} \begin{bmatrix} 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \end{bmatrix} =$

$= -\frac{1}{3} + \frac{4}{3} + \frac{4}{3} + \frac{2}{3} = 3$

$rs = -b_2 = -\frac{1}{3} \Rightarrow Sz = rs \Leftrightarrow z = -\frac{1}{9}$

$y = -C_1^{-1} C_2 z = \frac{1}{9} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{9} \\ -\frac{1}{9} \end{bmatrix}$

$\Rightarrow x = \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} -\frac{1}{9} \\ -\frac{1}{9} \\ -\frac{1}{9} \end{bmatrix}$

Ex 3 (3 Points)

Consider the optimization problem

$$\min_{x\in\mathbb{R}^2} \frac{1}{2}x^\top x - \begin{pmatrix}1\\1\end{pmatrix}^\top x \qquad (1)$$
$$\text{s.t. } x_2 = 0.$$

(i) Determine the solution $x^* \in \mathbb{R}^2$ and the corresponding adjoint variable $\lambda^* \in \mathbb{R}$ for problem (1).

(ii) Is x^* also the solution for the constraint $x_2 \leq 0$ instead of $x_2 = 0$?

(iii) Is x^* also the solution for the constraint $x_2 \geq 0$ instead of $x_2 = 0$?

Hint: Have a look at the adjoint variable λ^*.

Solution 3:

(i) The solution is obtained, as usual in this case, by solving the necessary first order conditions. We get $x^* = (1, 0)$ and $\lambda^* = 1$.

(ii) If we assume that $x_2 \leq 0$ is an active constraint, we obtain exactly the same pair of solutions as above, i.e. $x^* = (1, 0)$ and $\lambda^* = 1$. As $\lambda^* \geq 0$ this also solves the problem with constraints $x_2 \leq 0$.

(iii) Here, we do not obtain the same sign in the definition of the constraints as in (ii). So that the intermediate solution, if we assume that $x_2 \leq 0$ is an active constraint, will be $\tilde{x}^* = (1, 0)$ and $\tilde{\lambda}^* = -1$. Hence, $\tilde{x}^* = (1, 0)$ is not a solution to the inequality constrained problem. Dropping the assumption that $x_2 \leq 0$ is an active constraint yields the solution of (iii), namely $x^* = (1, 1)$ and $\lambda^* = 0$.

Ex 4 **Programming** (4 Points)

(i) Implement the Uzawa-iteration as described in the script.

(ii) Test your iteration with parameters $\tau = 0.05$, tol$= 0.001$ on the problem

$$\min_x \frac{1}{2}x^T Bx + x^T b$$
$$\text{s.t. } Cx = c$$

$$B = \begin{bmatrix}3 & 2 & 1\\2 & 2 & 0\\1 & 0 & 2\end{bmatrix}, \quad C = \begin{bmatrix}1 & -2 & 1\\-1 & 1 & -1\end{bmatrix}, \quad b = \begin{pmatrix}0\\0\\0\end{pmatrix}, \quad c = \begin{pmatrix}-3\\2\end{pmatrix}.$$