V. Schulz
M. Klar

SS 2022
Submission: 19.04.21, until 12:15

Numerical Optimization - Sheet 2

If you are a student in mathematics please solve the exercises with no tag and the ones with the tag **Mathematics**. If you are a data science student please solve the problems with no tag and those with the tag **Data Science**. Submissions with tags other than your subject count as bonus points. The tag **Programming** marks programming exercises.

Ex 1 (4 Points)

Let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$.

(i) Show that
$$\mathrm{Ker}(A) = \mathrm{Im}(A^T)^{\perp}.$$

(ii) Show that the following assertions exclude each other:

- There exists an $x \in \mathbb{R}^n$ which fulfills $Ax = b$.
- There is a $y \in \mathbb{R}^m$ which fulfills $b^\top y = 1$ and $A^\top y = 0$.

Ex 2 **Mathematics** (4 Points)

(Schur complement lemma) Let
$$D = \begin{bmatrix} A & b \\ b^\top & c \end{bmatrix},$$
where $A \in \mathbb{R}^{n \times n}$, $b \in \mathbb{R}^n, c \in \mathbb{R}$. Suppose that A is symmetric, positive definite. Prove that D is positive semidefinite if and only if $c - b^\top A^{-1} b \geq 0$.

Ex 3 **Data Science** (4 Points)

(i) Let $f : \mathbb{R}^n \to \mathbb{R}$, $x \mapsto b^\top x + a$ for $a \in \mathbb{R}$ and $b \in \mathbb{R}^n$ be an affine function. Show that f is convex.

(ii) Let $f : \mathbb{R}^n \to \mathbb{R}$ be convex and $c \in \mathbb{R}$. Show that the set
$$M := \{x \in \mathbb{R}^n \mid f(x) \leq c\}$$
is convex.

Ex 4 Programming (5 Points)

(i) Please implement a function `drv(f, x, h)` in Python 3, which returns the central difference for an arbitrary function $f : \mathbb{R} \to \mathbb{R}$ at point $x \in \mathbb{R}$ and step size h. The central difference of a function is given by

$$f'(x) \approx \frac{f(x+h) - f(x-h)}{2h}.$$

(ii) Implement a function `ddrv(f, x, h)` which computes the second derivative. Use the approximation

$$f''(x) \approx \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}.$$

(iii) Implement a function `pdrv(g, x, j, h)` which returns the partial derivative of a function $g : \mathbb{R}^n \to \mathbb{R}$ at point $x \in \mathbb{R}^n$ in direction of its j-th component, where $j = 0, \ldots, n-1$. The function g should take a `numpy`-array of length n as argument.

(iv) Implement a function `grad(g, x, h)` based on `pdrv(g, x, j, h)` which returns the gradient of g with respect to the standard Euclidean norm.

(v) Import the function `test` of the module `numdrv_test`. Execute it for each of your solutions via `test(drv)`, `test(ddrv)` and `test(pdrv)`. Your errors should be in the order of 10^{-7} or smaller.