V. Schulz
M. Klar

SS 2022
Submission: 26.04.21, until 12:15

Numerical Optimization - Sheet 3

If you are a student in mathematics please solve the exercises with no tag and the ones with the tag **Mathematics**. If you are a data science student please solve the problems with no tag and those with the tag **Data Science**. Submissions with tags other than your subject count as bonus points. The tag **Programming** marks programming exercises.

Ex 1 (6 Points)

The Rosenbrock function is defined by

$$f : \mathbb{R}^2 \to \mathbb{R}, \quad f(x) = (1 - x_1)^2 + 100(x_2 - x_1^2)^2.$$

Show that the point $x = (1, 1)$ is a local minimum of f and that the Hessian of f is positive definite at that point.

Ex 2 (4 Points)

Let $H \in \mathbb{R}^{n \times n}$ be symmetric, $C \in \mathbb{R}^{m \times n}$ surjective, and $m \leq n$. In addition, there is an $\alpha > 0$ such that

$$(v, Hv)_{\mathbb{R}^n} \geq \alpha \|v\|^2, \quad \forall v \in \operatorname{Kern} C$$

Show that

$$\begin{bmatrix} H & C^T \\ C & 0 \end{bmatrix} : \mathbb{R}^{n+m} \to \mathbb{R}^{n+m}$$

is invertible.
Hint: Note the surjectivity of C implies that $\operatorname{Kern}(C^T) = \{0\}$.

Ex 3 **Data Science, Programming** (6 Points)

(i) Implement the Rosenbrock function (see Ex. 1), its exact gradient and its exact Hessian w.r.t to the standard Euclidean scalar product.

(ii) Solve the optimization problem $\min f(x)$ using the function `minimize` of the module `scipy.optimize` with starting values $x_0 = (0, 0)$ and $x_0 = (0.99, 0.99)$.

- without any other parameters except from f and x_0.
- using of the parameter `jac`.
- using of the parameters `jac` and `hess` and `method="Newton-CG"`.

Print the solution of `minimize` into your iPython-Notebook.

Ex 4 **Mathematics** (4 Points)

A mapping $f \in C^1(S, \mathbb{R})$, where $S \subset \mathbb{R}^n$ is convex, is called strongly convex, if

$$(\nabla f(x) - \nabla f(y),\, x - y) \geq m\|x - y\|^2 \tag{1}$$

for some $m > 0$ and all $x, y \in S$. Show that f is strongly convex if, and only if

$$f(y) \geq f(x) + (\nabla f(x),\, y - x) + \frac{m}{2}\|x - y\|^2 \tag{2}$$

for $m > 0$ and all $x, y \in S$.