Mock exam

Remarks:

- Place your bag out of reach and switch off your mobile phone.
- You are allowed to carry three hand written sheets (DIN A4) with notes.
- The following media are prohibited: calculator, lecture notes, other literature, mobile phone.
- Write your name and matriculation number on this exam sheet, and on every sheet you use.
- If you need more sheets please get in touch with the supervising person.
- The exam contains 6 tasks. Please check whether you received all tasks when you start.
- You might answer in German or in English.
- The total number of points of all exercises is 61.

Task 1. (10+10 points) Find a point which fulfills the first order necessary conditions of optimality in the following problems.

a)

$$
\begin{aligned}
&\min_{a,b\in\mathbb{R}} a+b\\
&\text{s.t. } a^2-b=0
\end{aligned}
$$

b)

$$
\begin{aligned}
&\min_{x\in\mathbb{R}^2} \frac{1}{2}x^\top \begin{bmatrix}12 & 6\\ 6 & 3\end{bmatrix} x - x^\top \begin{pmatrix}19\\ 8\end{pmatrix}\\
&\text{s.t. } \begin{bmatrix}1, -1\end{bmatrix} x = 0.
\end{aligned}
$$

Task 2. (4+2 points) The following Python-Code solves a minimization problem for some matrices $B \in \mathbb{R}^{n\times n}$, $C \in \mathbb{R}^{m\times n}$, vectors $b \in \mathbb{R}^n$, $c \in \mathbb{R}^m$, starting values $x \in \mathbb{R}^n$, $\lambda \in \mathbb{R}^m$, and a step-size $\tau > 0$.

```
1  from numpy.linalg import solve, norm
2
3  def func(B, C, b, c, x, lmbda, tau):
4      while True:
5          res = norm(B.dot(x) + lmbda.dot(C) +b) + norm(C.dot(x) + c)
6          if res < 1e-8:
7              break
8          x = - solve(B, lmbda.dot(C) + b)
9          lmbda = lmbda + tau * (C.dot(x) + c)
10     return x
```

a) Which minimization problem does the above code solve and what is the name of the algorithm?

b) Which requirement must the matrix B fulfill such that the algorithm works? Which line gives an error if it is violated?

Task 3. (5+5 points)

Write, for each of the two matrices given below, a pseudocode which performs the matrix-vector multiplication $Ax = y$ without storing the matrix.

a)

$$
A := \begin{bmatrix}\mathrm{Id}_n & -\mathrm{Id}_n\\ \mathrm{Id}_n & \mathrm{Id}_n\end{bmatrix},
$$

where Id_n is the identity matrix in $\mathbb{R}^{n\times n}$.

b) Let the matrix $A \in \mathbb{R}^{n\times n}$ be of rank 1 with a given singular value decomposition $A = u\,\sigma\,v^\top$ for some $u, v \in \mathbb{R}^n$, and $\sigma \in \mathbb{R}$.

Task 4. ((6+2)+(6+1) points) State the KKT-system corresponding to an SQP step for the following problems.

Note, that as the KKT-system is a part of a sequence of KKT-systems some entries might depend on an iteration count $k \in \mathbb{N}$.

a) i) Let $r : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be twice continuously differentiable.

$$\begin{aligned} \min_{x,y\in\mathbb{R}} \ & \frac{1}{2}(x^2 + y^2) \\ \text{s.t. } & r(x,y) = 0 \end{aligned} \tag{1}$$

ii) Assume you omit the term involving the Hessian $\mathrm{Hess}_{x,y} r(x^k, y^k)$ in the KKT system of the k-th SQP step in (1). Which algorithm for solving a QP could be used, and why would it be particularly cheap?

b) i) Let $f, g : \mathbb{R}^n \to \mathbb{R}$ be twice continuously differentiable and $A, B \in \mathbb{R}^{m\times n}$.

$$\begin{aligned} \min_{x,y\in\mathbb{R}^n} \ & f(x) + g(y) \\ \text{s.t. } & Ax + By = 0 \end{aligned} \tag{2}$$

ii) Assume that f and g are convex. Propose an alternative algorithm to solve (2).

Task 5. (2 points) Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be convex, and $h\colon \mathbb{R} \to \mathbb{R}$ be convex and nondecreasing. Show that the function composition $h \circ f$ is convex.
Remark: $(h \circ f)(x) = h(f(x))$

Task 6. (1+2+3+2 points) Please answer the following questions.

a) Consider the optimization problem

$$\min_{x\in\mathbb{R}^d} x^\top Q x + x^\top b.$$

for some positive definite, symmetric $Q \in \mathbb{R}^{d\times d}$ and $b \in \mathbb{R}^d$. How many iterations does the CG-Algorithm need at most?

b) State the quadratic penalty function, which can be used to solve equality constrained optimization problems. Name a disadvantage of the quadratic penalty method and a possible alternative.

c) Let $a \in \mathbb{R}^n$ and $\lambda > 0$ be fixed. Formulate the problem

$$\min_{x\in\mathbb{R}^n} \frac{1}{2}\|x - a\|_2^2 + \lambda \sum_{i=1}^{n-1} |x_{i+1} - x_i|,$$

where $x = (x_1, \ldots, x_n)$, in a way which is treatable by the ADMM.

d) You are given an optimization problem of the form

$$\begin{aligned} \min_{x\in\mathbb{R}^n} \ & x^\top B x - \mu^\top x \\ \text{s.t. } & Cx = c, \end{aligned}$$

for some $m, n \in \mathbb{N}, m < n$, $B \in \mathbb{R}^n$ symmetric, positive definite, $C \in \mathbb{R}^{m\times n}$ with full rank, $c \in \mathbb{R}^m$, and $\mu \in \mathbb{R}^n$. Which optimization algorithm could be used to solve the problem?