The model of measurement
测量模型

Outline

1. **Some basics**

2. A small history

3. The types
   - Established CFA models
   - One-factor and two-factor models
   - Mixed models

4. Models with link functions

Some basics: important concepts

- ***parameter参数:***

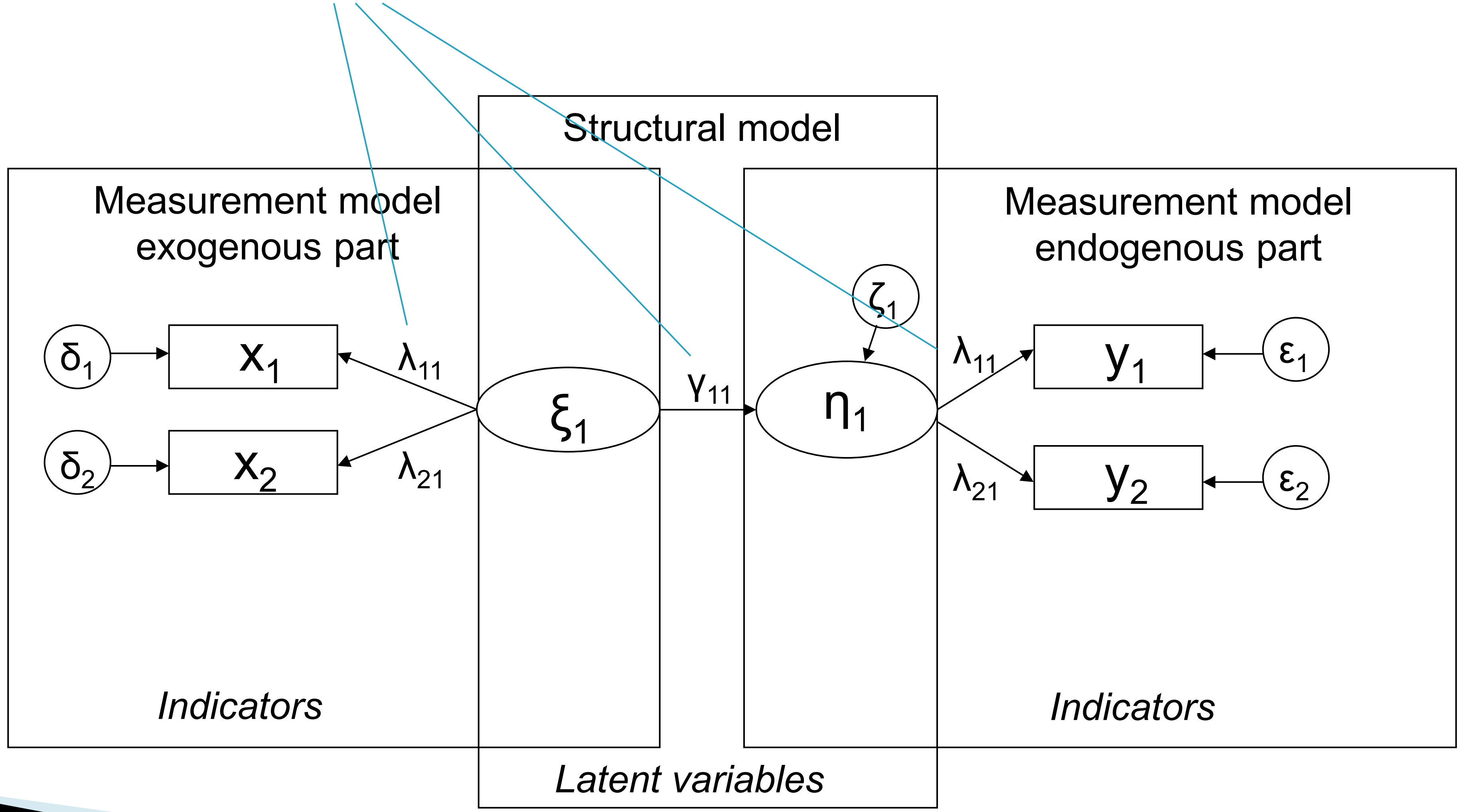
..... is a variable that is part of a model and refers to a property of data (e.g. a regression weight 回归系数 or path coefficient 路径系数)

….. is a quantity

….. is something that needs estimation

….. is something regarding a population

Parameters


Structural model
Measurement model
exogenous part
Measurement model
endogenous part
δ1
δ2
x1
x2
λ11
λ21
ξ1
γ11
η1
ζ1
λ11
λ21
y1
y2
ε1
ε2
Indicators
Indicators
Latent variables

Some basics: important concepts

- ***parameter参数:***

  ..... is a variable that is part of a model and measures a characteristic of data (e.g. a regression weight 回归系数 or path coefficient 路径系数)

  ….. is a quantity

  ….. is something that needs estimation

  ….. is something regarding a population

- ***statistic 变量***:

  …. it is something characterizing a sample

  …. mostly it is an estimated value (estimated mean, estimated variance, etc.)

-

Some basics: a remark

The terms „factor“因子 and „latent variable“潜变量 are often used in this section. Please, be aware

– *they are considered as* ***equivalent***.

„factor“ is the originally introduced term; in the attempt to relate the different latent variable approaches to each other it is replace by „latent variable“.

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4. Models with link functions

A small history

- The idea of the „model of measurement "has grown out of different sources.

- (1) Factor analysis. In an early stage of the development of factor analysis the researchers agreed on assuming a *functional relationship* between a latent source (of responding) and the response.

This assumption suggests $X \sim \xi \quad [X = f(\xi)]$

that is a kind of a model!

A small history

- The idea of the „model of measurement "has grown out of different sources.

- (2) CCT. In early assessment research the idea was established that measurement includes *error*. Even different types of error were distinguished from each other (systematic and random error, etc.) (see Gulliksen, 1950).

  This suggested that measurement was a composite of different components instead of a something treated as a whole.

A small history

- The idea of the „model of measurement " has grown out of different sources.

- (3) IRT. Item response theory (see Baker & Kim, 2017) (IRT项目反应理论) introduced the idea that an observation / a response could be described by several differing sets of parameters.

A small history

- The idea of the „model of measurement " has grown out of different sources.

- Item response theory (IRT项目反应理论) introduced the idea that an observation / a response could be described by several ...

- Furthermore ...

  ... the term *model* is introduced for the first time

  ... This term is used *in the sense of a hypothesis* instead of using it as description of reality.

A small history

- The idea of the „model of measurement “has grown out of different sources.

- Item response theorie (IRT项目反应理论) introduced the idea ...

  In the meantime IRT was especially sucessful in creating many different IRT models showing different properties. These models refer to different types of data and effects.

A small history

What is a model of measurement in CFA and SEM today?

- It is a hypothesis regarding the structure of data
- It describes the manifest variables as composites of latent variables and error variables
- Its appropriateness can be checked by goodness-of-fit testing

A note

A model of measurement always includes
vectors 向量, but **not single variables 单个方程**.

$>$ ~~equation of variables:~~ $a_1 = b_1 + c_1$

$>$ equation of vectors: $\mathbf{a} = \mathbf{b} + \mathbf{c}$, $\begin{bmatrix} a_1 \\ a_2 \\ . \\ a_p \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ . \\ b_p \end{bmatrix} + \begin{bmatrix} c_1 \\ c_2 \\ . \\ c_p \end{bmatrix}$

A note

A model of measurement always includes
vectors 向量, but **not single variables 单个方程**.

Reason:
There must be the possibility to disprove the model of measurement (remember model identification)

A note

A model of measurement always includes

vectors 向量, but **not single variables 单个方程**.

i.e. investigating a model of measurement can yield a *good* or *bad* account of the data.

A bad account means that the model has to be rejected and replaced by another one.

Outline

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    - Mixed models

3. Models with link functions

The types

There is *not* one model of measurement. A number of specific models of measurement were proposed (vgl. Graham, 2006). They differ according to …

- … the *number* of components

  (the standard is one true and one error component; but there may be several true components)

The types

There is *not* one model of measurement. A number of specific model of measurement were proposed (vgl. Graham, 2006). They differ according to …

- … the *number* of components,
- … the *relationships* among the components,
- … the consideration of *intercepts.*
- … the parameters are either *estimated* or *fixed.*

The types

A list of the types:

–model of parallel measurement

–model of τ–equivalent measurement

–model of essentially τ–equivalent measurement

–congeneric model

The types : established CFA models

The **model of parallel measurement** (= model of CTT)

$$X = \tau + \varepsilon$$

X: observations观测值, τ : true components真值, ε : error components误差

- It is assumed that all items of a scale are exactly equivalent in all measurements.

The types : established CFA models

The **model of parallel measurement** (= model of CTT)

$$X = \tau + \varepsilon$$

X: observations, τ : true components, ε : error components

- It is assumed that all items of a scale are exactly equivalent in all measurements.

(i.e. the items of a scale are assumed not to differ from each other

but the persons may differ)

The types : established CFA models

The **model of parallel measurement** (= model of CTT)

$$X = \tau + \varepsilon$$

X: observations, τ : true components, ε : error components

- It is assumed that all items of a scale are exactly equivalent in all measurements.

This is the justification for computing scores by the summation of the responses coded as numbers!

The types : established CFA models

The **model of parallel measurement** (= model of CTT)

$$X = \tau + \varepsilon$$

Example: a simple math problem (e.g. 3+4=?)

- X : … correctness of the response (coded as 1 or 0)
- τ : … true score contribution of e.g. latent fluid intelligence
- ε : … error contribution to response

The types : established CFA models

The **model of parallel measurement** (= model of CTT)

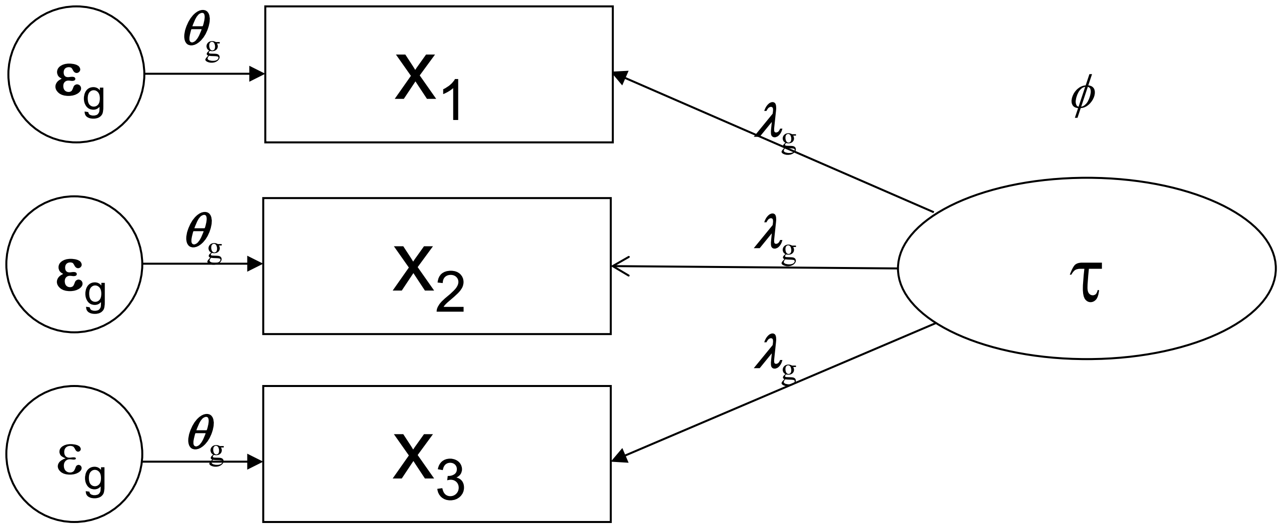
$$X = \tau + \varepsilon$$

Example: a simple math problem

- 3+4=?
- ….. = 7
- ………. -> correct -> 1
- $= f_{cognitive\_processes}(\tau + \varepsilon)$

The types : established CFA models

The **model of parallel measurement** (= model of CTT)



The types : established CFA models

The **model of parallel measurement** (= model of CTT)

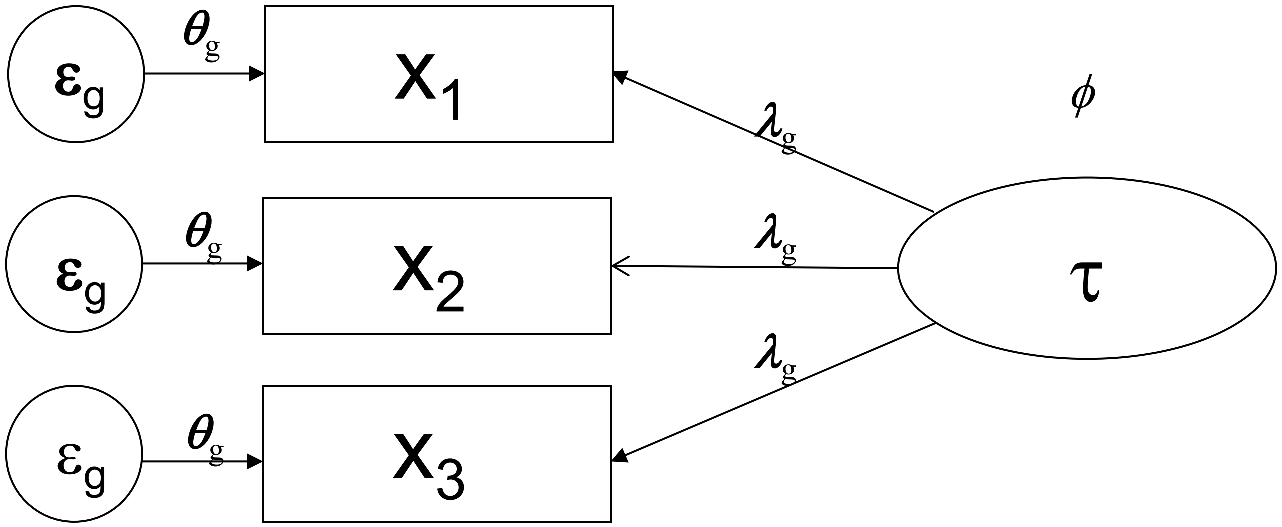
… as formal model of measurement

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ . \\ . \\ x_p \end{bmatrix} = \begin{bmatrix} \tau_g \\ \tau_g \\ \tau_g \\ . \\ . \\ \tau_g \end{bmatrix} + \begin{bmatrix} \varepsilon_g \\ \varepsilon_g \\ \varepsilon_g \\ . \\ . \\ \varepsilon_g \end{bmatrix}$$

$$e \in \Re > 0$$

The types : established CFA models

Practice: *determine the degree of freedom* (df = s−t)

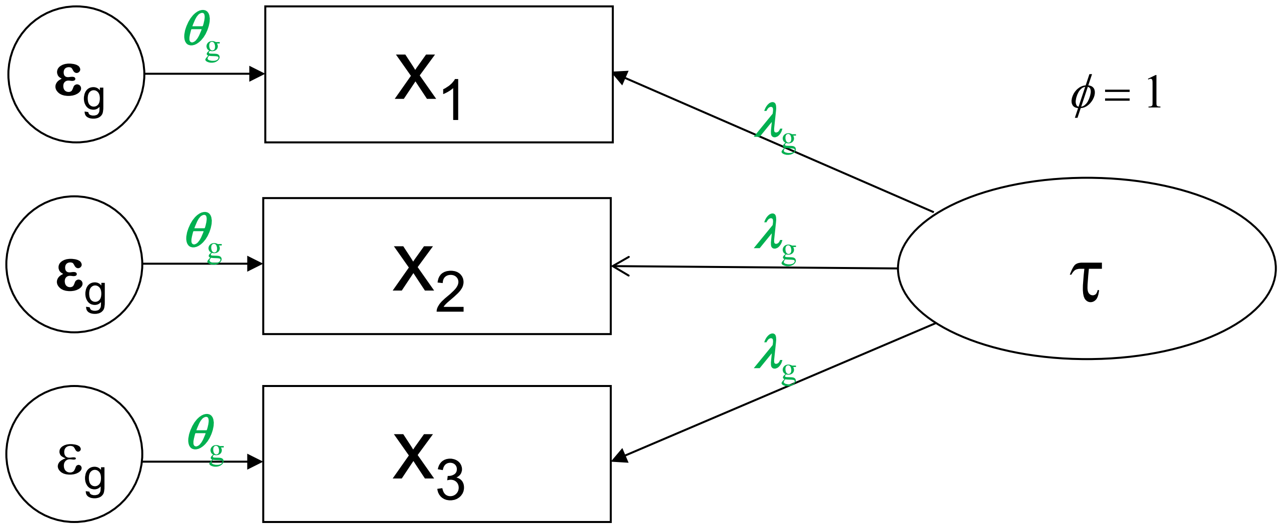


Which one of the following numbers gives the degree of freedom?

0
1
2
3
4
5

The types : established CFA models

Practice: *determine the degree of freedom* (df = s–t)

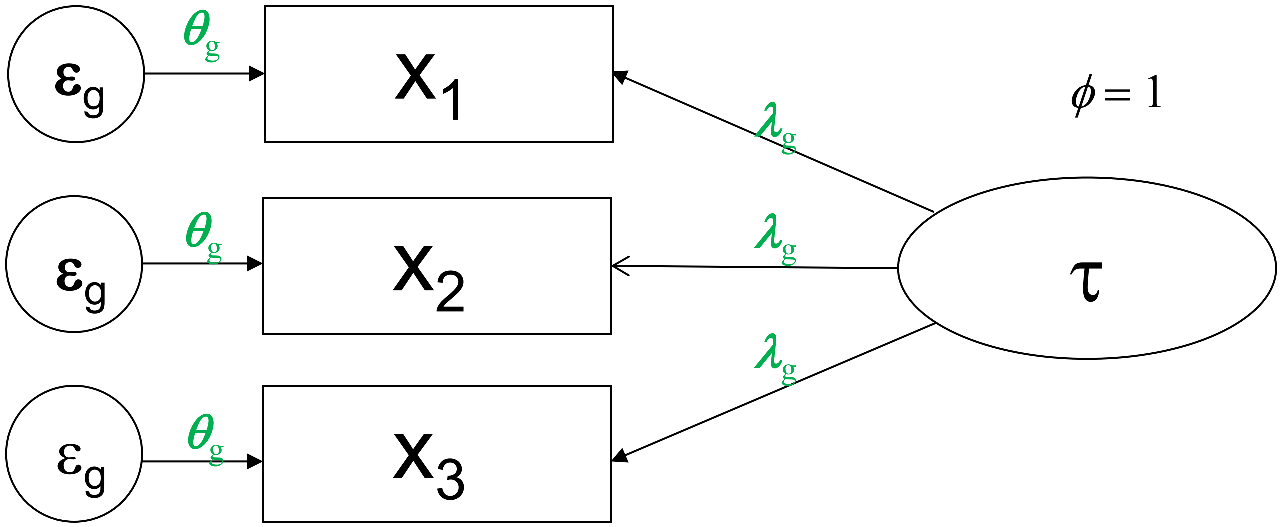


Which one of the following numbers gives the degree of freedom?

df = 6 – 2 = 4

The types : established CFA models

Practice: *determine the degree of freedom* (df = s−t)



A Note

In this case ϕ is set equal to 1 because of the need for scaling.
Scaling will be explained in another course unit.

The types : established CFA models

The **model of τ–equivalent measurement**

$$X = \tau + \varepsilon$$

X: observations, τ : true components, ε : error components

- It is assumed that only the *true components* of all items of a scale are exactly equivalent in measurement.

The types : established CFA models

The **model of τ-equivalent measurement**

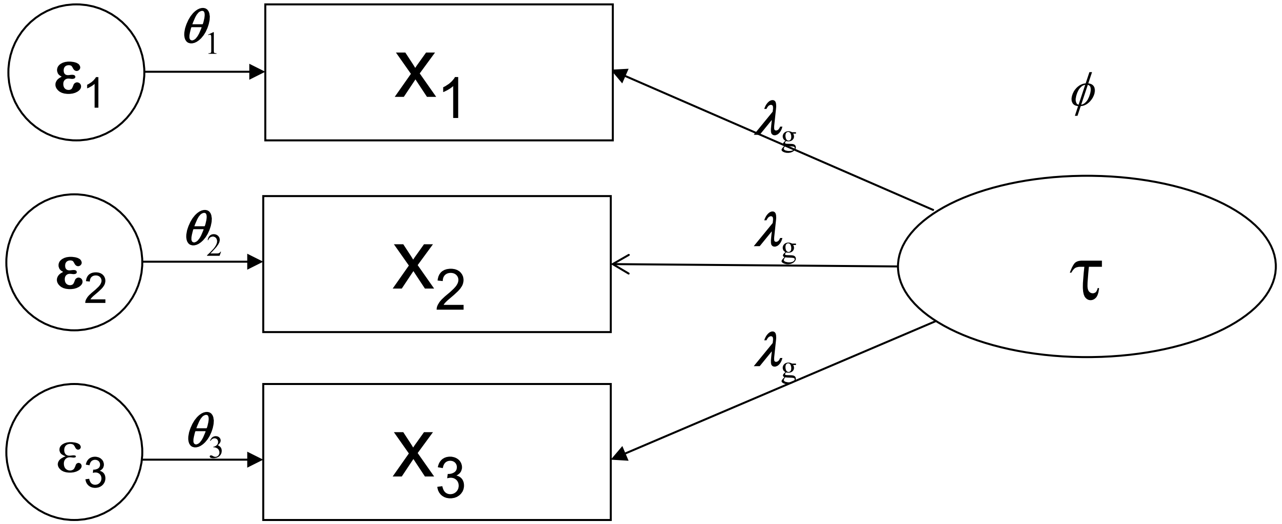
$$X = \tau + \varepsilon$$

X: observations, τ : true component, ε : error component

In contrast, the responses to items may show *different error contributions (i.e. the error components differ across the items)*

The types : established CFA models

The model of τ–equivalent measurement



The types : established CFA models

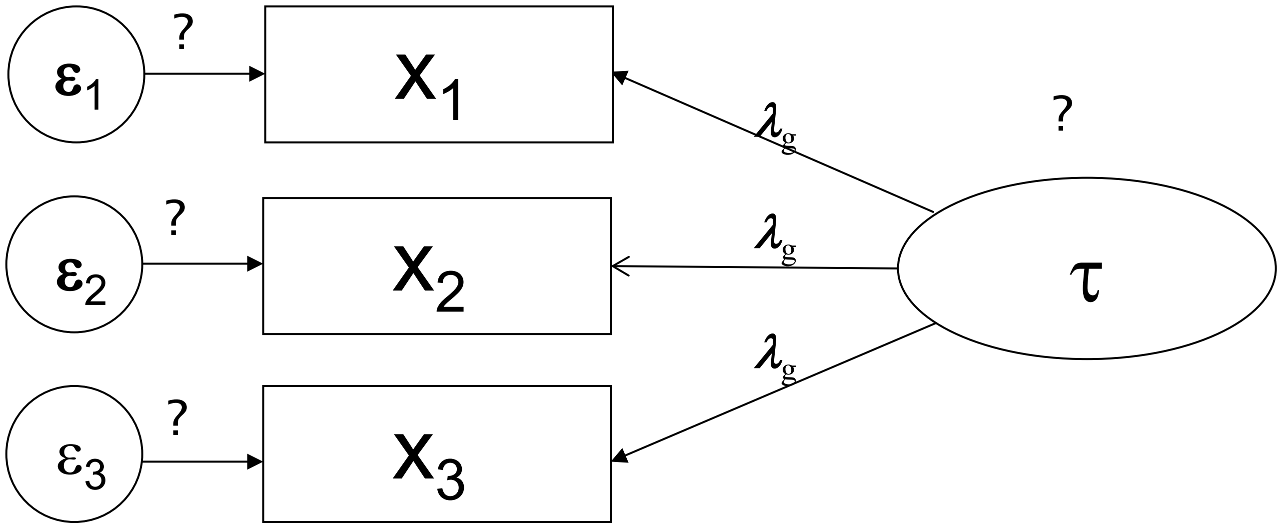
The **model of τ–equivalent measurement**

… as formal model of measurement

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ . \\ . \\ x_p \end{bmatrix} = \begin{bmatrix} \tau_g \\ \tau_g \\ \tau_g \\ . \\ . \\ \tau_g \end{bmatrix} + \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ . \\ . \\ \varepsilon_p \end{bmatrix}$$

The types : established CFA models

Practice: *determine the degree of freedom* (df = s–t)

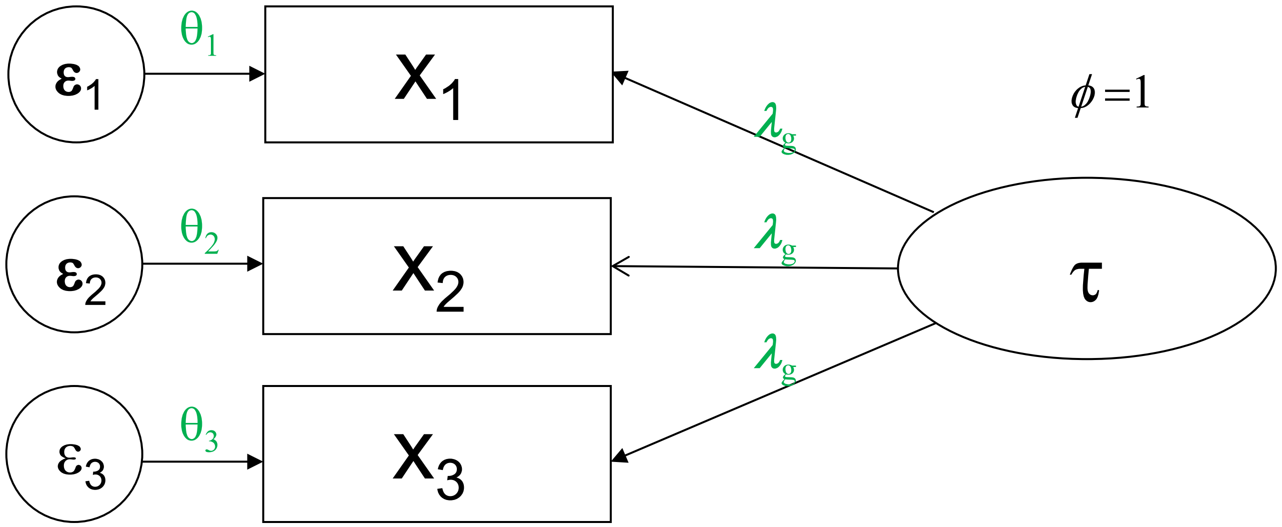


Which one of the following numbers gives the degree of freedom?

0
1
2
3
4

The types : established CFA models

Practice: *determine the degree of freedom* (df = s−t)



Which one of the following numbers gives the degree of freedom?

df = 6 – 4 = 2

The types : established CFA models

The **model of the <u>essentially</u> τ–equivalent measurement**

$$X = (\alpha + \tau) + \varepsilon$$

X: observations, α : item–characteristic constants, τ : true components,

ε: error components,

- It is assumed that the true parts of the items differ by a constant and all the rest is the same as for the τ–equivalent model.

The types : established CFA models

The model of the essentially τ–equivalent measurement

… as formal model of measurement

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ . \\ . \\ x_p \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ . \\ . \\ \alpha_p \end{bmatrix} + \begin{bmatrix} \tau_g \\ \tau_g \\ \tau_g \\ . \\ . \\ \tau_g \end{bmatrix} + \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ . \\ . \\ \varepsilon_p \end{bmatrix}$$

The types : established CFA models

Practice: *determine the degree of freedom* (df = s–t)

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} + \begin{bmatrix} \tau_g \\ \tau_g \\ \tau_g \end{bmatrix} + \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \end{bmatrix}$$

Which one of the following numbers gives the degree of freedom?

0
1
2
3
4

The types : established CFA models

Practice: *determine the degree of freedom* (df = s–t)

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} + \begin{bmatrix} \tau_g \\ \tau_g \\ \tau_g \end{bmatrix} + \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \end{bmatrix}$$

Which one of the following numbers gives the degree of freedom?

df = 6 – 7 = –1

To-be-considered parameters: $\alpha_1, \alpha_2, \alpha_3, \varepsilon_1, \varepsilon_2, \varepsilon_3, \tau_g$

… more items should be considered!

The types : established CFA models

The **congeneric model** (Jöreskog, 1971)

- standard model of CFA 验证性因素分析and SEM结构方程模型):

$$X = (\lambda \times \xi) + \varepsilon \qquad (\lambda \times \xi \text{ takes the role of } \tau)$$

X: observations观测值,

λ: factor loadings因子载荷 (also known as item-scale correlations/ discriminability parameters),

ξ : factor 因子,

ε : error components误差,

λ is specific for the item, ξ can be considered specific for the person (– not specified)

The types : established CFA models

The **congeneric model:** ***a supplement – an innovation***

*… the introduction of this model is accompanied by a change of what was so far considered as **error***

*… the remainder could now be considered as **unique systematic variation***

*Therefore, it is now addressed as <u>**residual**</u>*

The types : established CFA models

The **congeneric model** (Jöreskog, 1971)

- updated version (standard model of CFA 验证性因素分析and SEM结构方程模型):

$$\mathbf{x} = (\lambda \times \xi) + \varepsilon \qquad (\lambda \times \xi \text{ takes the role of } \tau)$$

(*... after change to modern notation used for models of measurement*)

x: vector of manifest variables 观测值,

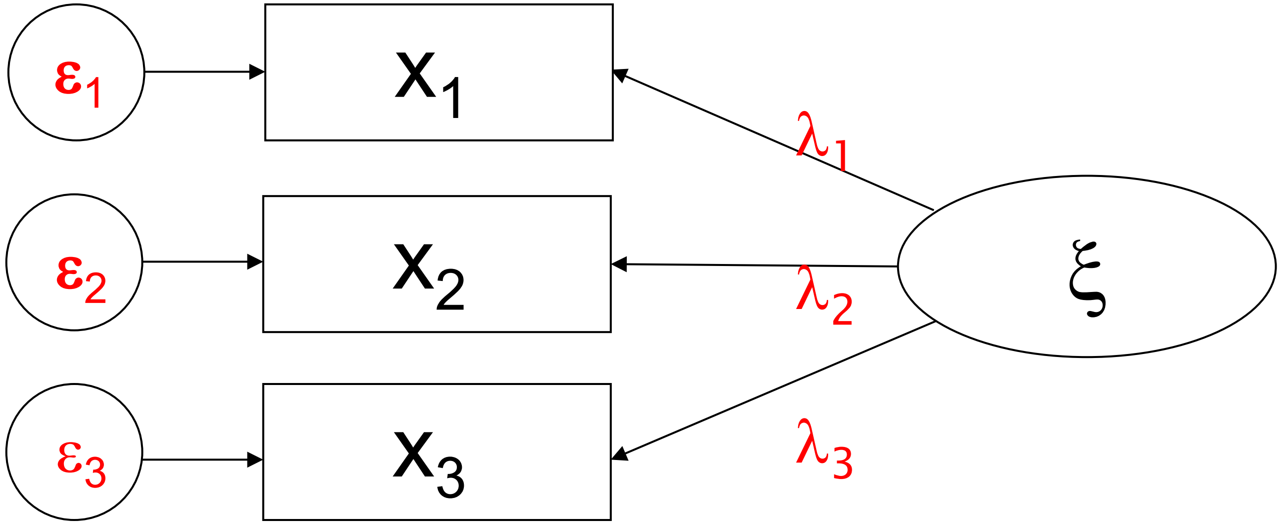
λ: vector of factor loadings因子载荷 (also known as item-scale correlations/ discriminability parameters),

ξ : factor 因子,

ε : vector of residual components误差,

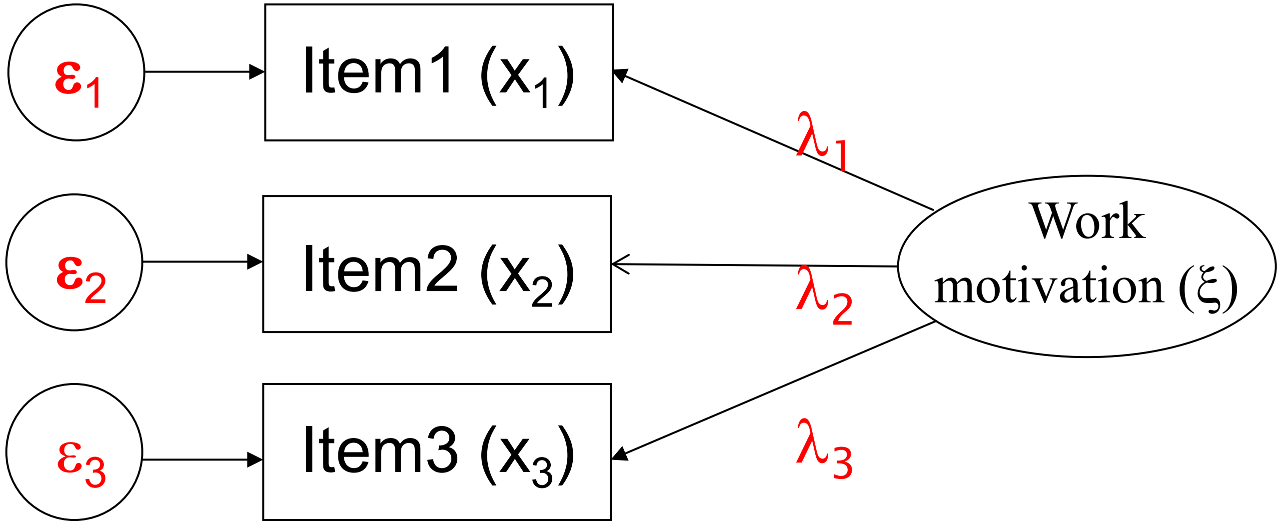
The types : established CFA models

The **congeneric model** (Jöreskog, 1971)



The types : established CFA models

The **congeneric model** (Jöreskog, 1971) – *example*



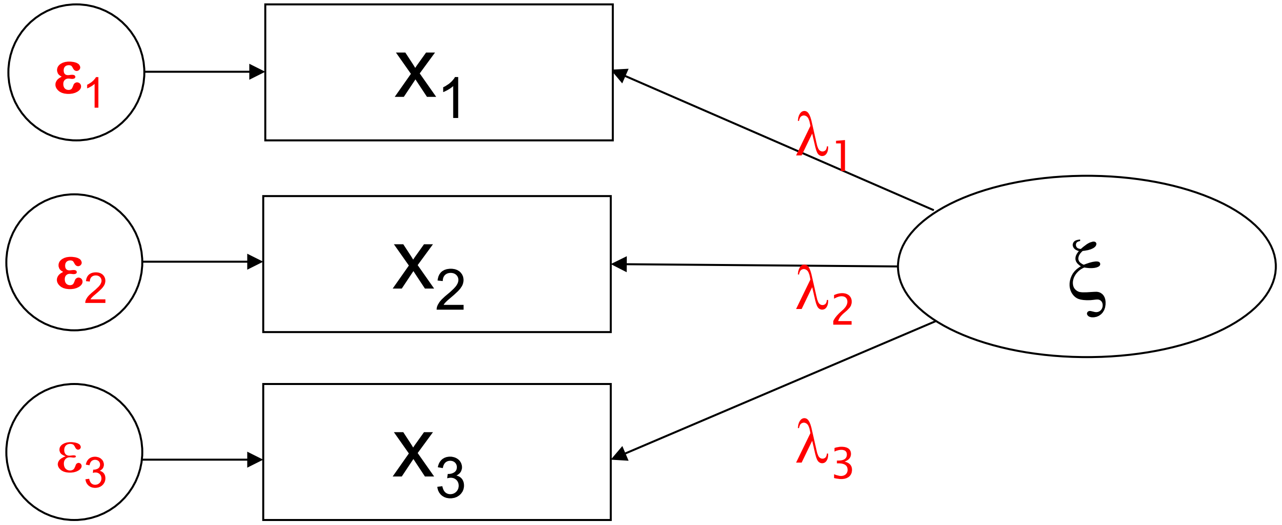
The types : established CFA models

The **congeneric model** (Jöreskog, 1971)

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ . \\ . \\ x_p \end{bmatrix} = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ . \\ . \\ \lambda_p \end{bmatrix} \xi + \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ . \\ . \\ \varepsilon_p \end{bmatrix}$$

The types : established CFA models

Practice: *determine the degrees of freedom* (df = s−t)



Which one of the following numbers gives the degree of freedom?

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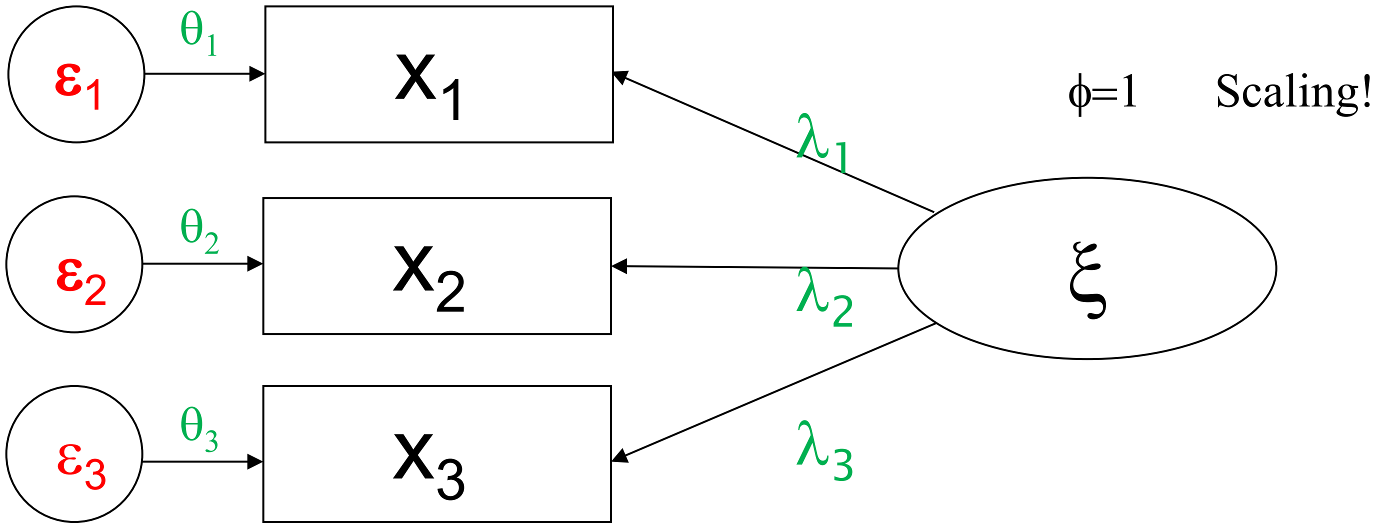
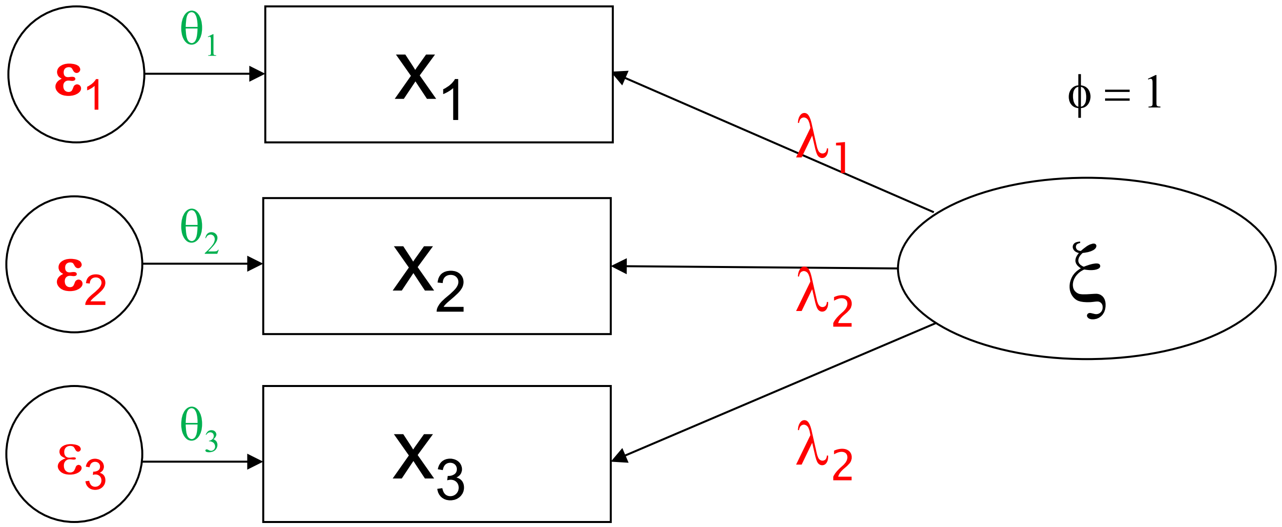
The types : established CFA models

Practice: *determine the degrees of freedom* (df = s−t)



The types : established CFA models

Practice: *determine the degrees of freedom* (df = s−t)



Which one of the following numbers gives the degree of freedom?

df = 6 – 6 = 0

The types : established CFA models

The **congeneric model** (Jöreskog, 1971)

- original version (standard model of CFA and SEM)
- extended version: *linear mixed model*

$$\mathbf{X} = (\boldsymbol{\mu} + \boldsymbol{\lambda}_x \xi) + \boldsymbol{\varepsilon}$$

X: vector of manifest variables, μ : vector of item characteristic constants, ξ : latent variable (= factor), δ : vector of residual variables, λ : vector of factor loadings (item-scale correlations/ discriminability parameters)

- μ and λ are item specific (与特定项目相关)

The types : established CFA models

Ranking of models of measurement regarding degree of freedom

from simple to elaborate

- ✦ *model of parallel measurement*
- ✦ *model of τ-equivalent measurement*
- ✦ *model of essentially t-equivalent measurement*
- ✦ *congeneric model*

Linear mixed congeneric model

The types : established CFA models

How to select ***the model of measurement?***

- *follow the* <u>*simplicity principle*</u>

(theories, hypotheses … should be as simple as possible)

The types : established CFA models

An example of selecting the model of measurement:

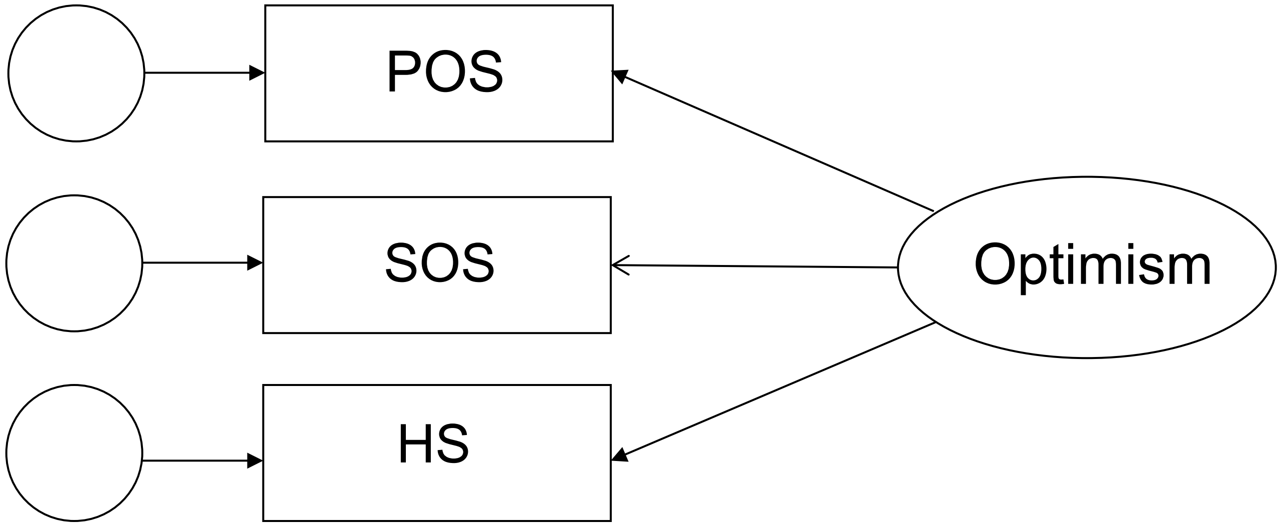
Assume that it is necessary to design a model of measurement for the personality construct „optimism“. In order to establish such a construct, it is necessary to find appropriate indicators in the first step. Since the construct optimism shows several facets (e.g. personal facet, social facet, opt. regarding the future , etc), it is possible to look for operationalizations of facets in the second step. Assume this search for indicators leads to three scales as candidates :

- ***Personal optimism scale (POS)***
- ***Social optimism scale (SOS)***
- ***Hopefulness scale (HS)***

The types : established CFA models

An example of selecting the model of measurement :

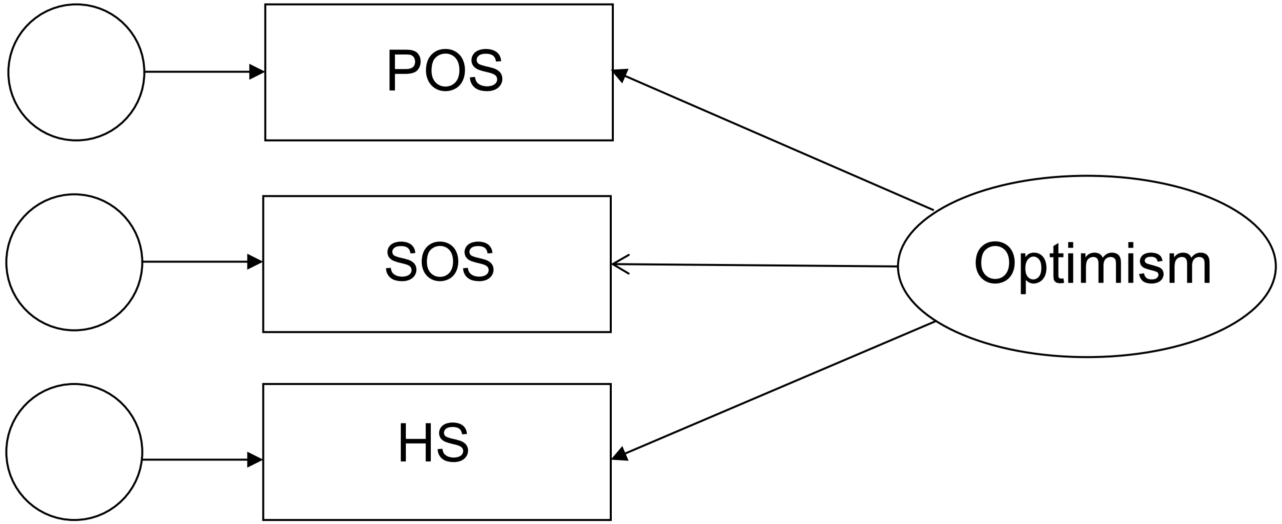
One latent variable and three indicators (manifest variables) lead to the following structure of the model of measurement:



The types : established CFA models

An example of selecting the model of measurement :

Next, the type of the model has to be selected:

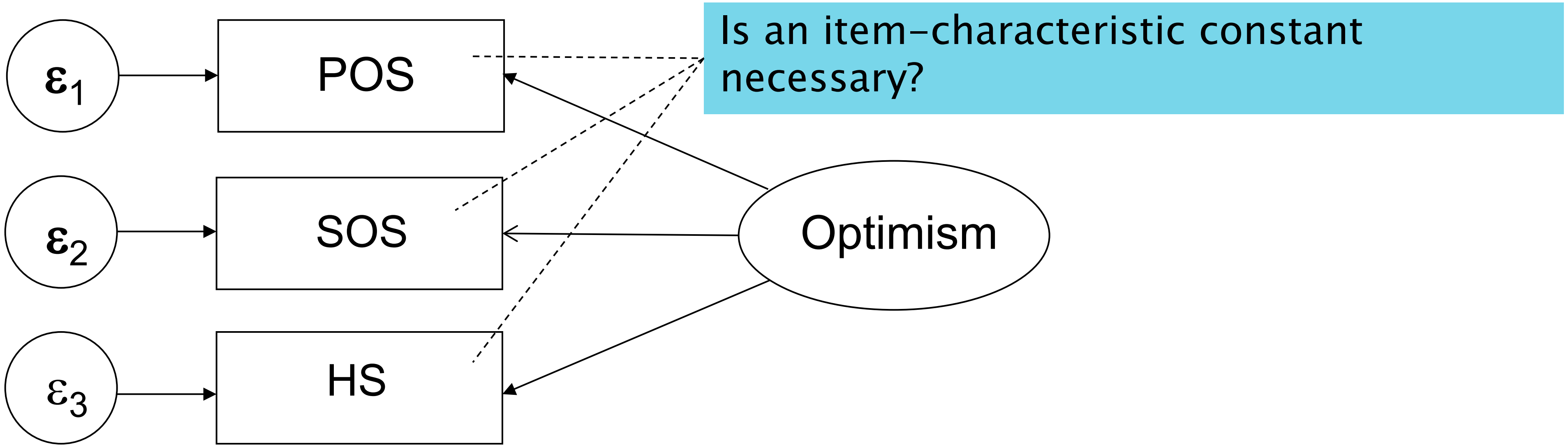


The *principle* suggests to check whether there is reason for selecting a model type with a large degree of freedom!

The types : established CFA models

An example of selecting the model of measurement :

Next, the type of the model has to be selected:



The types : established CFA models

An example of selecting the model of measurement :

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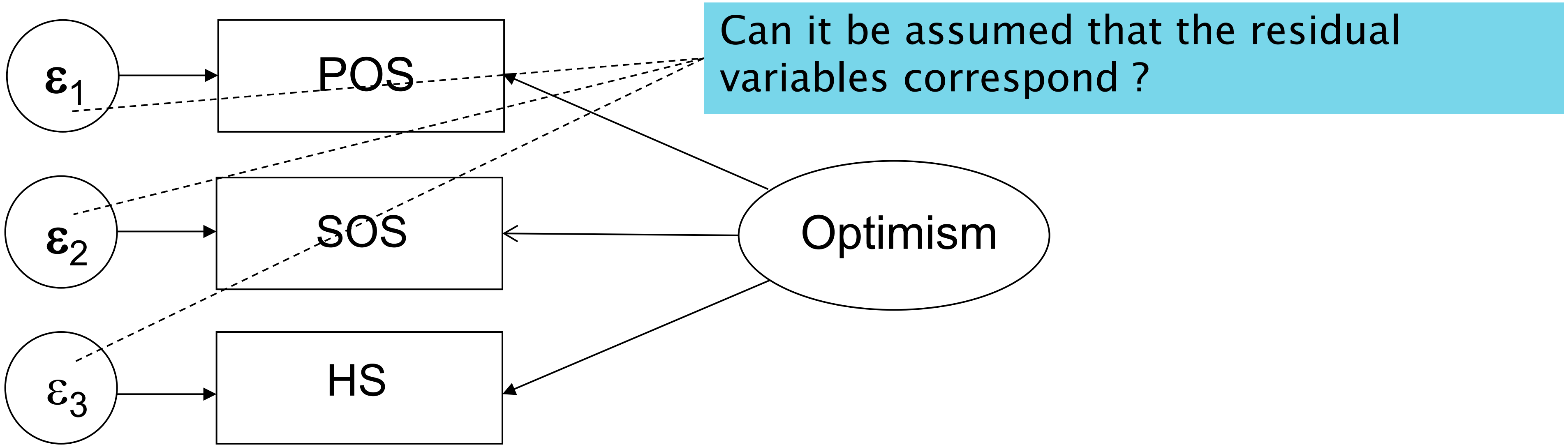
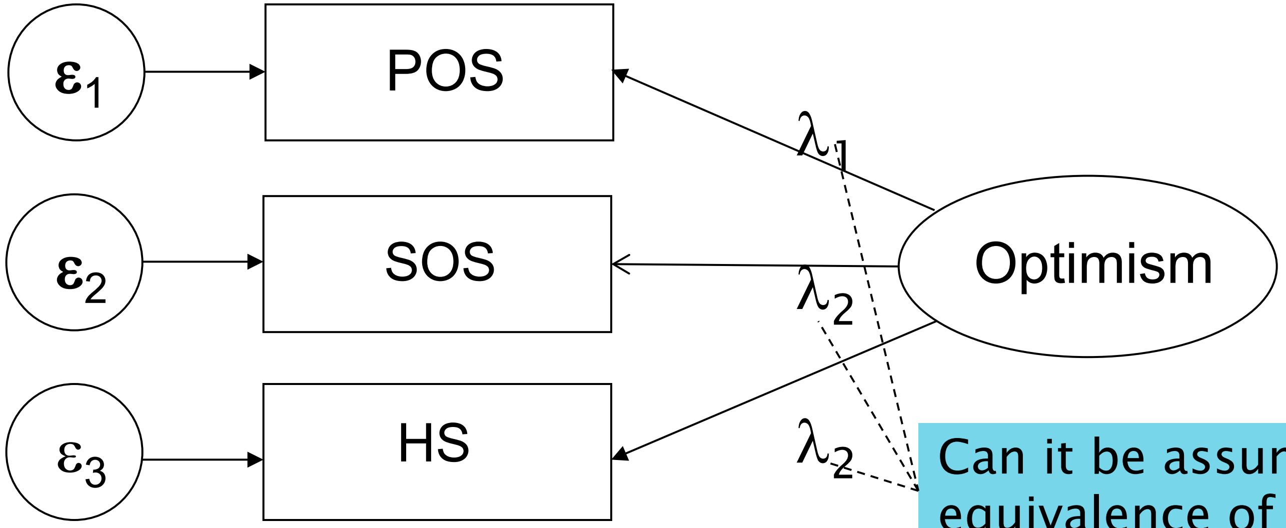


The types : established CFA models

An example of selecting the model of measurement :

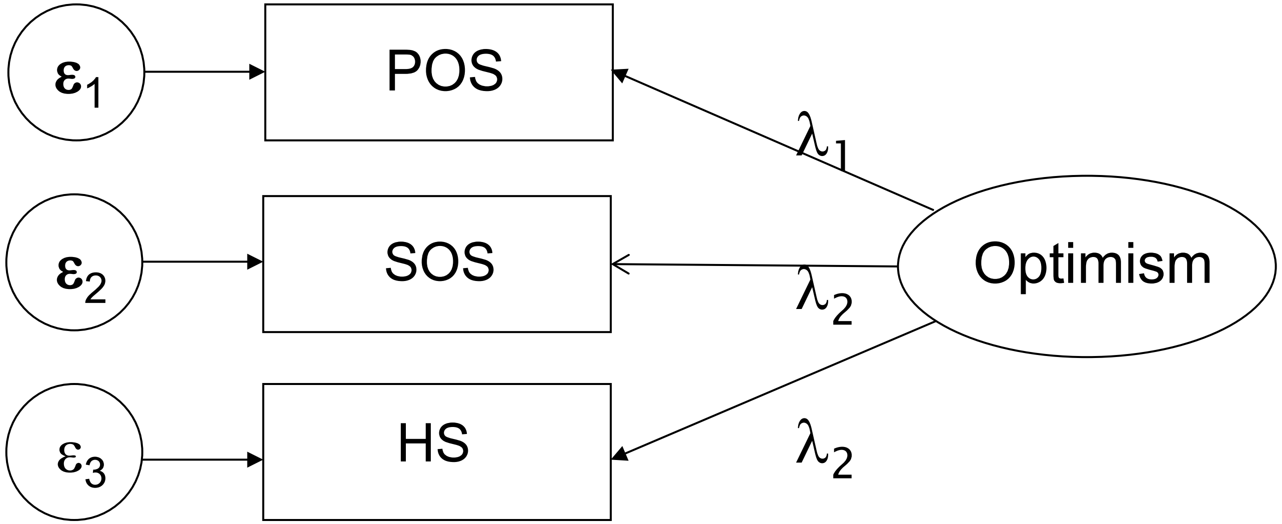
Next, the type of the model has to be selected:



Can it be assumed that there is equivalence of the factor loadings ?

The types : established CFA models

Finally, if there are only „no“s, the search ends up with the congeneric model :



The types : two-factor models

Models can include a second factor so that there are …

- two overlapping factors
- two non-overlapping (but correlated) factors

The types : two-factor models

Models can include a second factor so that there are …

- two overlapping factors
- two non-overlapping (but correlated) factors

The **bifactor model 双因子模型**:

(individual equations of such a model)

$x_i = \lambda_{i1} \times \xi_1 + 0 + \varepsilon_i$ (for (at least) one item)

$x_k = \lambda_{k1} \times \xi_1 + \lambda_{k2} \times \xi_2 + \varepsilon_k$ (for all other items)

The types : two-factor models

Models can include a second factor so that there are …

- two overlapping factors
- two non-overlapping (but correlated) factors

The **<u>bifactor model </u>双因子模型**:

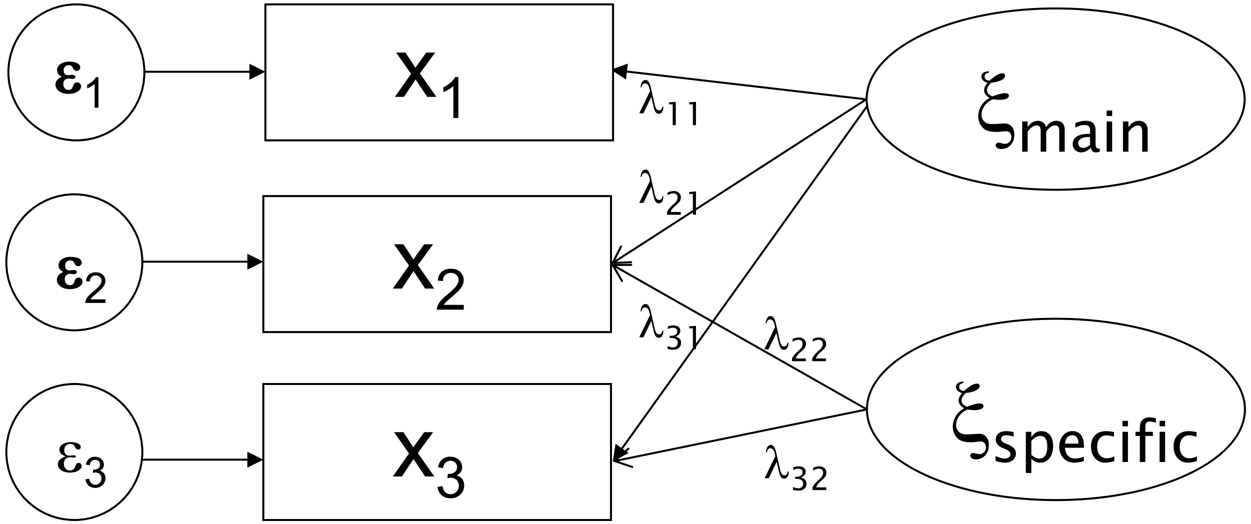
(individual equations of a model)

$x_1 = \lambda_{11} \times \xi_1 + 0 \times \xi_2 + \varepsilon_1$ (for (at least) one item)

$x_i = \lambda_{i1} \times \xi_1 + \lambda_{i2} \times \xi_2 + \varepsilon_i$ (for all other items)

The types : two-factor models

An example of a bifactor model:



The types : two-factor models

Formal description of the bifactor **model**

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ . \\ . \\ x_p \end{bmatrix} = \begin{bmatrix} \lambda_{1\text{main}} \\ \lambda_{2\text{main}} \\ \lambda_{3\text{main}} \\ . \\ . \\ \lambda_{p\text{main}} \end{bmatrix} \xi_{\text{main}} + \begin{bmatrix} 0 \\ \lambda_{2\text{specific}} \\ \lambda_{3\text{specific}} \\ . \\ . \\ \lambda_{p\text{specific}} \end{bmatrix} \xi_{\text{specific}} + \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ . \\ . \\ \varepsilon_{\text{p}} \end{bmatrix}$$

The types : two-factor models

Models can include a second factor so that there are …

- two overlapping factors
- two non-overlapping (but correlated) factors

The types : two-factor models

Models can include a second factor so that there are …

- two overlapping factors
- two non-overlapping (but correlated) factors

(individual equations of a model)

$$x_i = \lambda_{i1} \times \xi_1 + 0 \times \xi_2 + \varepsilon_i$$ (for some items)

$$x_k = 0 \times \xi_1 + \lambda_{k2} \times \xi_2 + \varepsilon_k$$ (for all other items)

The types : two-factor models

Formal description of the two-factor **model**

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ . \\ . \\ x_p \end{bmatrix} = \begin{bmatrix} \lambda_{1\mathrm{A}} \\ \ldots \\ \lambda_{i\_\mathrm{A}} \\ 0 \\ \ldots \\ 0 \end{bmatrix} \xi_{\mathrm{A}} + \begin{bmatrix} 0 \\ \ldots \\ 0 \\ \lambda_{j\_\mathrm{B}} \\ \ldots \\ \lambda_{p\_B} \end{bmatrix} \xi_{\mathrm{B}} + \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ . \\ . \\ \varepsilon_{\mathrm{p}} \end{bmatrix}$$

The types : mixed models

◆Mixed models include *factors* representing *random* effects and *factors* representing *fixed effects*.

◆Distinguishing between fixed and random effects is taken over from analysis of variance 方差分析.

The types : mixed models

◆ ….

◆Distinguishing between fixed and random effects is taken over from analysis of variance.

- **Fixed effects**: there are clearly distinguishable levels

  e.g. the effects due to the different experimental treatments

- **Random effects**: there are no distinguishable levels

  e.g. the dosage of a drug administered to sick persons varies according to their sickness

The types : mixed models

◆Mixed models include *fixed factors* and *random factors* for representing fixed effects and random effects respectively.

◆The formal mixed model of measurement is given by

$$X = \alpha + \beta_{fixed\_effect} \times \xi_{fixed\_effect} + \beta_{random\_effect} \times \xi_{random\_effect} + \varepsilon$$

- β_{fixed_effect} : regression weight for fixed effects
- β_{random_effect} : regression weight for random effects

The types : mixed models

- ….
- The formal mixed model of measurement: example

– extended version of congeneric model: *linear mixed model*

$$X = (\mu + \lambda \times \xi) + \varepsilon$$

The types : mixed models

A Remark:

The mixed models are used in the context of multilevel modeling 多水平模型.

适用于不同组(如，男生/女生)之间的比较

Outline

1. A small history

2. The types
    - Established CFA models
    - One-factor and two-factor models
    - Mixed models

3. Models with link functions

Models with link functions

... (for example) used in combination with probability-based covariances

Models with link functions

Link functions are used as part of generalized linear models for relating variables following different distributions to each other.

e.g. there may be one variable A follwing the normal distribution and another variable B following the binomial distribution. In this case the corresponding link function g( ) is used for establishing equivalence:

$$B \neq A$$

$$B = g(A)$$

Models with link functions

Turning the congeneric model into a generalized linear model gives

$$\mathbf{x} = g(\lambda \times \xi) + \boldsymbol{\varepsilon}$$

$\mathbf{x}$: vector of manifest variables, ξ: latent variable (= factor), λ: vector of factor loadings,
ε: vector of residual variables, g( ): link function

Note: residuals are assumed to always follow the normal distribution

,

Summary and brush up:

1. A small history

… models developed to represent hypotheses

2. The types
   - Established CFA models
   - One-factor and two-factor models
   - Mixed models

… measurement models show a typical linear structure (… a sum of two components)

… remember: the congeneric model of measurement

3. Models with link functions

… there are generalized linear models

Questions regarding course unit 4

- What are the components of the congeneric model of measurement?
- Which parameter is only specific for the item?
- What is the major difference between the congeneric model and the bifactor model?

Literature

Basic literature:

- Baker, F. B., & Kim, S.-H. (2017). The basics of itm response theory using R. Heidelberg: Springer.
- Graham, J. M. (2006). Congeneric and (essentially) tau-equivalent estimates of score reliability. *Educational and Psychological Measurement, 66*, 930-944*.*
- Schweizer, K. (2012). The position effect in reasoning items considered from the CFA perspective. *International Journal of Educational and Psychological Assessment, 11*, 44-58.