MATH 202: VECTOR CALCULUS
CAES 9.13
HOMEWORK DUE FRIDAY WEEK 12

Problem 1. We can parametrize the unit disk as

$$\Phi\colon [0,1]^2 \to \mathbb{R}^2, \qquad \Phi(r,\theta) = (r\cos 2\pi\theta, r\sin 2\pi\theta) = (x,y)$$

thus we are looking at

$$\partial H = H \circ \Phi \circ \partial\Delta^2$$

So

$$\partial H = -H \circ \Phi \circ \Delta^2_{1,0} + H \circ \Phi \circ \Delta^2_{1,1} + H \circ \Phi \circ \Delta^2_{2,0} - H \circ \Phi \circ \Delta^2_{2,1}$$

From here we will speak only of the geometry of the boundary. We will simply take how H moves the boundary of the unit disk. We can see that H leaves the boundary as is, and raises the middle of the disk to create a hemispherical shell (with a point at the top that is neglected), with the boundary going counterclockwise from above. Note, there are also transversals going up and down the shell that cancel each other out.

Problem 2. We can use spherical coordinates and see that

$$\Phi(\rho,\theta,\varphi) = (\rho\cos 2\pi\theta \sin \pi\varphi/2, \rho\sin 2\pi\theta\sin\pi\varphi/2, \rho\cos\pi\varphi/2) = (x,y,z)$$

Thus,

$$\partial\Phi = \Phi \circ \partial\Delta^3 = \Phi \circ (-\Delta^3_{1,0} + \Delta^3_{1,1} + \Delta^3_{2,0} - \Delta^3_{2,1} - \Delta^3_{3,0} + \Delta^3_{3,1})$$

Which we can see from the terms builds the dome shape of the hemispherical shell with and inward normal vector, as well as a disk along the $z = 0$ plane that sits underneath the dome shape. It also has a normal vector pointing inwards. There is also a point in the middle of the disk that is neglected. There is also 2 walls inside that have opposite signs that cancel. There is also a rod like object that is neglected in integration.

Problem 3. We can parametrize the unit disk as

$$\Psi\colon [0,1]^2 \to \mathbb{R}^2, \qquad \Psi(r,\theta) = (r\cos 2\pi\theta, r\sin 2\pi\theta) = (x,y)$$

thus we are looking at

$$\partial\Phi = \Phi \circ \Psi \circ \partial\Delta^2$$

So

$$\partial\Phi = -\Phi \circ \Psi \circ \Delta^2_{1,0} + \Phi \circ \Psi \circ \Delta^2_{1,1} + \Phi \circ \Psi \circ \Delta^2_{2,0} - \Phi \circ \Psi \circ \Delta^2_{2,1}$$

From here we will speak only of the geometry of the boundary. We will simply take how Φ moves the boundary of the unit disk. We can see that Φ raises the edge of the disk to create a hemispherical shell, with the boundary going counterclockwise from above. Note, there are also transversals going up and down the shell that cancel each other out.

Problem 4. We can see that this takes us to a spherical shell, but we may reparameterize it as such (so that we go from the standard 2-cube to our ambient space),

$$\Phi(\theta,\varphi) = (\cos 2\pi\theta\sin\pi\varphi, \sin 2\pi\theta\sin\pi\varphi, \cos\pi\varphi/2) = (x,y,z)$$

In which case we can see that

$$\partial\Phi = \Phi \circ \partial\Delta^2 = \Phi \circ (-\Delta^2_{1,0} + \Delta^2_{1,1} + \Delta^2_{2,0} - \Delta^2_{2,1})$$

which we can see, since our object is a spherical shell, that when we compute the boundary, all the points cancel out and we are left with a boundary of 0. We have two lines transversing the side of the sphere that cancel out, and one point at the top and one point at the bottom that are negligible.

Problem 5. We can see that the surface is effectively the spherical coordinates so our surface is a sphere with radius one. Note that θ takes us around the latitude of the sphere, and the φ takes us around the co-latitude of the sphere. One wall will map to the center of the sphere. One wall will map to the outside of the sphere. Two wall maps to be walls on the interior and cancel out. Two walls map to become rods from a pole to the center and are negligible. Thus, our surface is the spherical shell with the normal vectors point inwards. As in Exercise 9.13.5 we are looking at a similar surface, the difference is how many parameters we are considering and the dimension of our image space. Thus, we can truly imagine this exercise to be the surface that we are taking the boundary of in Exercise 9.13.5. Thus, it follows that the boundary in Exercise 9.13.5 is 0, by the nilpotence of ∂, $\partial^2 = 0$.

Problem 6. We can see that the surface is torus and we can find the boundary as such,

$$\partial\Phi = \Phi \circ \partial\Delta^3 = \Phi \circ (-\Delta^3_{1,0} + \Delta^3_{1,1} + \Delta^3_{2,0} - \Delta^3_{2,1} - \Delta^3_{3,0} + \Delta^3_{3,1})$$

Two walls become disks on the x-axis that cancel out because of reversed orientation. One wall becomes an anulus like object from b to $b+a$, and another wall becomes the same thing that cancels out because of reversed orientation. One wall maps to be a circle around the z-axis, or the spinal-cord of the torus, and is negligible. Finally, the last wall becomes the shell of the torus, and so our boundary is the torus simply with the normal vector pointing out.