Homework: 2.3, 2.4

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Math 202: Vector Calculus

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Note, minor changes, such as $\mathbf{0}$ to 0, was made after the lecture.

2.3.9

First we will use the straight line test to test for discontinuity. Take a sequence $\{(x_\nu, y_\nu)\}$ approaching $\mathbf{0}$ along $y = mx$. Then

$$f(x_\nu, y_\nu) = f(x_\nu, mx_\nu) = \frac{x_\nu^4 - (mx_\nu)^4}{(x_\nu^2 + (mx_\nu)^2)^2} = \frac{x_\nu^4(1-m^4)}{(x_\nu^2(1+m^2))^2} = \frac{x_\nu^4(1-m^2)(1+m^2)}{x_\nu^4(1+m^2)^2} = \frac{1-m^2}{1+m^2}.$$

Note as the sequence $\{(x_\nu, y_\nu)\}$ converges to $\mathbf{0}$, that $\{f(x_\nu, y_\nu)\}$ is constant. With infinite slopes possible, this cannot take one single value. Thus $f(x, y)$ is discontinuous at $\mathbf{0}$.

For $h(x, y)$ we will use the size bounds to see if the limit exists,

$$\begin{aligned}
0 \le |h(x,y)| &= \frac{|x^3 - y^3|}{x^2 + y^2} && \text{by definition of } h(x,y) \text{ and modulus} \\
&\le \frac{|x^3| + |y^3|}{|(x,y)|} && \text{by the triangle inequality and definition of modulus} \\
&= \frac{2|(x,y)|^3}{|(x,y)|} && \text{by definition of modulus and assumption } |x| \le |(x,y)| \text{ and } |y| \le |(x,y)| \\
&= 2|(x,y)| && \text{by cancellation}
\end{aligned}$$

Thus as a sequence $\{(x_\nu, y_\nu)\}$ of nonzero input vectors converge to $\mathbf{0}$, the corresponding sequence of outputs $\{h(x_\nu, y_\nu)\}$ is squeezed to 0 in absolute value and hence converges to 0. Thus h is continuous at $\mathbf{0}$ when $b = 0$.

For $g(x, y)$ we will use the straight line test. Take a sequence $\{(x_\nu, y_\nu)\}$ approaching $\mathbf{0}$ along $y = mx$. Then

$$g(x_\nu, y_\nu) = g(x_\nu, mx_\nu) = \frac{x_\nu^2 - m^3 x_\nu^3}{x_\nu^2 + m^2 x_\nu^2} = \frac{x_\nu^2(1 - m^3 x_\nu)}{x_\nu^2(1+m^2)} = \frac{(1 - m^3 x_\nu)}{1+m^2}$$

which as $\{x_\nu, y_\nu\}$ tends towards $\mathbf{0}$, $\{g(x_\nu, y_\nu)\}$ tends towards $\frac{1}{1+m^2}$ itself. Since there are infinitely many slopes, b cannot take one value. Thus g is discontinuous at $\mathbf{0}$.

For $k(x,y)$ we will use the straight line test. Take a sequence $\{(x_\nu, y_\nu)\}$ approaching $\mathbf{0}$ along $y = mx$. Then

$$k(x_\nu, y_\nu) = k(x_\nu, mx_\nu) = \frac{x_\nu^2 m^2 x_\nu^2}{x_\nu^2 + m^6 x_\nu^6} = \frac{x_\nu^4 m^2}{x_\nu^2(1 + m^6 x_\nu^4)} = \frac{x_\nu^2 m^2}{1 + m^6 x_\nu^4}$$

which as $\{x_\nu, y_\nu\}$ tends towards $\mathbf{0}$, $\{k(x_\nu, y_\nu)\}$ tends towards 0 itself.

Let us now take a sequence $\{(x_\nu, y_\nu)\}$ approaching $\mathbf{0}$ along $y = x^2$. Then

$$k(x_\nu, y_\nu) = k(x_\nu, x_\nu^2) = \frac{x_\nu^3}{x_\nu^2 + x_\nu^{12}} = \frac{1}{x_\nu^{11}}$$

which as $\{x_\nu, y_\nu\}$ tends towards $\mathbf{0}$, $\{k(x_\nu, y_\nu)\}$ tends towards ∞ itself. Thus a contradiction as there is no singular value that satisfies both of the line tests. Hence, k is discontinuous at $\mathbf{0}$

2.4.1

a.

Since the ball, by definition, does not have a boundary it is not closed. However, it is bounded because we could have the subset contained in $B(\mathbf{0}, 2)$, or even the ball itself as it needs not proper containment. Therefore it is not compact.

b.

This is a closed set since all sequences in A should converge in A, however it is not bounded because the function cannot be contained in a ball since any radius of the ball say R, there is a point in the set at R^2. Thus it is not compact.

c.

The set given is a spherical shell of radius 1 centered at the origin. It is closed and bounded since a ball of radius 2 can encircle the set and it is closed because the sphere contains all limit points. Thus it is compact as well.

d.

The set is closed. If there is a sequence $\{x_\nu\}$ in the set that converges to a point p, and by definition of the set, $f(p) = \mathbf{0}_m$, thus the limit point is contained in the set. However, we cannot know if the set is bounded without more information on $f(x)$ because it could be bounded like c or not bounded like b. Hence its compactness is not determined.

e.

The set is not closed because it is not complete, thus it does not contain all of its limit points like irrational numbers. It is also not bounded because for any R radius of the ball, there always exists in the set say $2R$. This is by the Archimedean property for natural numbers which extends to the rationals. Thus it is not compact.

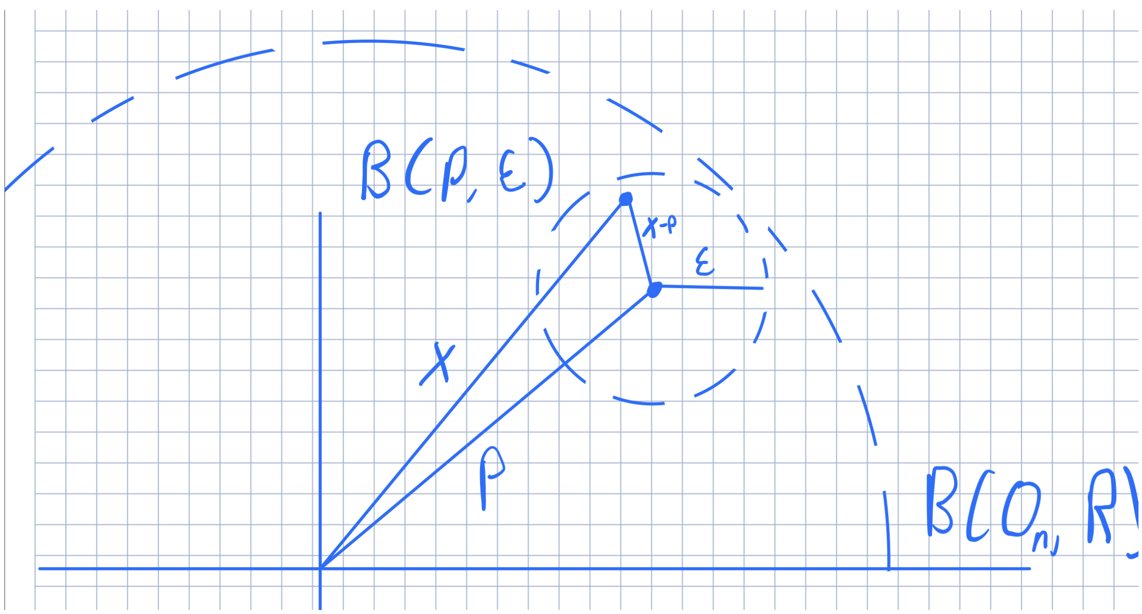
f.

The set is not closed because it does not contain $\mathbf{0}$, which is a limit point of the set as x_i tends towards 0. It also is not bounded because there in the set there is always a larger number, say you have a ball of radius $x_1 + ... + x_n = R$, then consider the element of the set such that $x_i = 2x_i$ and now this exceeds the radius of the ball. Thus this set is not compact.

2.4.2

For the first part we can take the ball $B(\mathbf{0}_2, 1)$ with the limit point on the boundary at $(0, 1)$. For second set we can imagine any set, call it C and a point a that has no intersection with C and does not lie on the boundary of C. Then we have the set $A = C \cup \{a\}$, thus $a \in A$, but it is not a limit point of A because $B(a, \varepsilon) \cap C = \emptyset$.

2.4.5



We can see two things from the picture. First, that the ball creating the boundary should be $R = |p| + \varepsilon$, and second that we can use the triangle inequality from the two sides $x - p$ and x. From here,

$$
\begin{aligned}
|x - p + x| &\leq |x - p| + |p| && \text{by triangle inequality} \\
&\leq |p| + \varepsilon && \text{since } x \text{ is a point in the ball thus} |x - p| \leq \varepsilon \\
&= R && \text{by assumption}
\end{aligned}
$$

Thus we have shown that every ball is bounded in $\mathbb{R}^n$

2.4.7

For the first one we can take the x_1 axis of $\mathbb{R}^n$ which is closed. We can then take the sequence $\{x_{1,\nu}\} = \{1, 2, 3, ...\}$. Of course there is no subsequence of $\{x_\nu\}$ that converges, thus our set is not bounded. Thus we have a closed set that does not satisfy the sequential characterization of bounded sets.

Now for the second piece we can imagine a ball, $B(\mathbf{0}, R)$ that is punctured at the origin, call it D. By Exercise 2.4.5 we know that the ball is bounded. Take the sequence, $\{x_\nu\} = \{1, 1, \frac{1}{2}, \frac{1}{6}, ...\}$, which converges to $\mathbf{0}$. However, this contradicts the sequential characterization of closed sets since $\mathbf{0} \notin D$. Thus we have a bounded set that does not satisfy the sequential characterization of closed sets.

2.4.8

For a closed set whose continuous image need not be closed, take $\mathbb{R}$, which is closed, and the function $f : \mathbb{R} \to \mathbb{R}\backslash\{0\}$, $x \to \frac{1}{x}$. Here the function is continuous, but does not contain the limit point 0, thus the continuous image is not a closed set.

For a closed set whose continuous image need not be bounded, simply take $f : \mathbb{R} \to \mathbb{R}$, $x \to x$, and this is not bounded.

For the continuous image of a bounded set that need not be closed, take the function $f : (0, 1) \to (0, 1), x \to x$. The continuous image is bounded since we can have a ball say, $B(\mathbf{0}, 2)$, however, since the image does not contain 1, a limit point, the image is not closed.

Finally, for the continuous image of a bounded set that need not be bounded, take the function $f : (0, 1] \to \mathbb{R}, x \to \frac{1}{x}$, which the continuous image is not bounded since the function tends to infinity as x approaches 0.