Homework 2

Physics 201

Due September 11th, 2020

Exercise 1. We will define $r_0 = R$ where R is the radius of the Earth, and $\Delta r = r$ where r is the distance above Earth.

$$F(r_0 + \Delta r) = GMm \cdot \frac{1}{(r_0 + \Delta r)^2}$$

Thus we can make the approximation

$$F(r_0 + \Delta r) = (F(r_0) + F'(r_0)\Delta r + \frac{1}{2}F''(r_0)\Delta r^2 + ...)$$

So the first non-constant is $F'(r_0)\Delta r = -GMm\frac{2\Delta r}{r_0^3}$.

We will evaluate the constant term $F(r_0) = GMm\frac{1}{r_0^2} = \frac{(6.67\cdot 10^{-11})(5.972\cdot 10^{24})}{(6.3781\cdot 10^6)^2} \approx 9.7918$ which is close to the acceleration of gravity near the Earth's surface.

The first order correction is $F'(r_0)\Delta r = -GMm\frac{2\Delta r}{r_0^3} = -\frac{(6.67\cdot 10^{-11})(5.972\cdot 10^{24})(1)(1)(2)}{(6.3781\cdot 10^6)^3} \approx -0.00000307044736068$ which is absurdly small of course.

Exercise 2. With masses that are d apart they experience the same force,

$$F_1 = \frac{Gm_1m_2}{d^2}\hat{x} \implies a_1 = \frac{F_1}{m_1} = \frac{Gm_2}{d^2}\hat{x}$$

$$F_2 = -\frac{Gm_1m_2}{d^2}\hat{x} \implies a_2 = \frac{F}{m_2} = -\frac{Gm_1}{d^2}\hat{x}.$$

If $m_2 = -m_1$ then,

$$F_1 = -\frac{Gm_1^2}{d^2}\hat{x} \implies a_1 = \frac{F_1}{m_1} = -\frac{Gm_1}{d^2}\hat{x}$$

$$F_2 = \frac{Gm_1^2}{d^2}\hat{x} \implies a_2 = \frac{F}{m_2} = -\frac{Gm_1}{d^2}\hat{x}.$$

We can use Newton's second law to find their motion. First mass 1,

$$\begin{aligned}
m_1\ddot{x}_1(t) &= -\frac{Gm_1^2}{d^2} \\
\ddot{x}_1(t) &= -\frac{Gm_1}{d^2} \\
\dot{x}_1(t) &= -\frac{Gm_1}{d^2}t + A \\
x_1(t) &= -\frac{Gm_1}{2d^2}t^2 + At + B \\
x_1(0) &= B = -\frac{d}{2} \implies \\
x_1(t) &= -\frac{Gm_1}{2d^2}t^2 - \frac{d}{2}
\end{aligned}$$

Now mass 2,

$$\begin{aligned}
m_2\ddot{x}_2(t) &= \frac{Gm_1^2}{d^2} \\
-m_1\ddot{x}_2(t) &= \frac{Gm_1^2}{d^2} \\
\ddot{x}_2(t) &= -\frac{Gm_1}{d^2} \\
\dot{x}_2(t) &= -\frac{Gm_1}{d^2}t + A \\
x_2(t) &= -\frac{Gm_1}{2d^2}t^2 + At + B \\
x_2(0) &= B = \frac{d}{2} \implies \\
x_2(t) &= -\frac{Gm_1}{2d^2}t^2 + \frac{d}{2}
\end{aligned}$$

Exercise 3. We can use the trigonometric identity of $\sin(2x) = \sin(x)\cos(x)$ to translate $U(x) = U_0\frac{\sin(\frac{2x}{l})}{2}$ which implies that the minimum is when $\sin(\frac{2x}{l}) = -1$ which is when $\frac{2x}{l} = \frac{3\pi}{2} \implies$ the minimum is $\frac{x}{l} = \frac{3\pi}{4}$.

To determine the oscillatory motion, we must find the k constant, which comes from $U''(x_e)$, where $x_e = \frac{3\pi}{4}$.

$$U''(x_e) = \frac{d^2}{dt^2}\left(U_0\frac{\sin\left(\frac{2x}{l}\right))}{2}\right)\Bigg|_{x_e} = -U_0\frac{2\sin\left(\frac{2x_e}{l}\right)}{l^2} = \frac{2U_0}{l^2}.$$

$$T = 2\pi\sqrt{\frac{m}{U''(x_e)}} = 2\pi\sqrt{\frac{ml^2}{2U_0}} = \sqrt{2}l\pi\sqrt{\frac{m}{U_0}}$$

Exercise 4. a.

The positions that are allowed are 2,4, and 5 because they are below the total energy line, and nothing can be above due to no negative kinetic energy. 4 is fastest since it is farthest from the

E line meaning the least potential and most kinetic. 4 is slowest since it is closest to the E line meaning the most potential and least kinetic.

b.

2 and 4 since there is an decrease to the local minimum, local fastest speed, and a increase away from the local minimum, a decrease in speed, within the vicinity of the the positions.

c.

4 since there is a slow decent to the local minimum and a slow rise away as opposed to 2 where it is a steep decent and climb.

Exercise 5.

$$\begin{aligned}
f(t) &= f_0 \sum_{j=0}^{\infty} \frac{t^j}{j!} \\
&= f_0 \left(1 + t + \frac{t^2}{2} + \frac{t^3}{6} + ... \right) \\
\frac{df(t)}{dt} &= f_0 \left(\frac{d}{dt}(1) + \frac{d}{dt}(t) + \frac{d}{dt}\left(\frac{t^2}{2}\right) + \frac{d}{dt}\left(\frac{t^3}{6}\right) + ... \right) \\
&= f_0 \left(0 + 1 + t + \frac{t^2}{2} ... \right) \\
&= f_0 \left(1 + t + \frac{t^2}{2} + \frac{t^3}{6} + ... \right) \\
&= f_0 \sum_{j=0}^{\infty} \frac{t^j}{j!} \\
&= f(t)
\end{aligned}$$

Or we may do it more symbolically as opposed to intuitively,

$$\begin{aligned}
f(t) &= f_0 \sum_{j=0}^{\infty} \frac{t^j}{j!} \\
\frac{df(t)}{dt} &= f_0 \frac{d}{dt} \sum_{j=0}^{\infty} \frac{t^j}{j!} \\
&= f_0 \sum_{j=0}^{\infty} \frac{d}{dt} \frac{t^j}{j!} \\
&= f_0 \sum_{j=0}^{\infty} j \frac{t^{j-1}}{j!} \\
\text{Let } l &= j - 1 \\
&= f_0 \sum_{l=-1}^{\infty} (l+1) \frac{t^l}{(l+1)!} = f_0 \sum_{l=0}^{\infty} \frac{t^l}{l!} \\
&= f(t)
\end{aligned}$$

Exercise 6. Following from section 1.4.1,

$$\dot{f}(t) = t^p \sum_{j=0}^{\infty} a_j (j+p) t^{j-1}$$

Since we have the ODE in the form $\dot{f}(t) - \alpha f(t)$ we can rewrite this as,

$$t^p \left[a_0 p t^{-1} + \sum_{j=0}^{\infty} a_{j+1}(j+p+1) t^j \right] - \alpha t^p \sum_{j=0}^{\infty} a_j t^j = 0$$

$$t^p \left[a_0 p t^{-1} + \sum_{j=0}^{\infty} (a_{j+1}(j+p+1) - \alpha a_j) t^j \right] = 0$$

We can find the recursion relation,

$$a_{j+1} = \frac{\alpha a_j}{j+p+1}$$

and once again set $p = 0$, thus

$$a_{j+1} = \frac{\alpha a_j}{j+1}$$

with the first few terms being,

$$a_1 = \alpha a_0$$

$$a_2 = \alpha \frac{a_1}{2} = \alpha^2 \frac{a_0}{2}$$

$$a_3 = \alpha \frac{a_2}{3} = \alpha^3 \frac{a_0}{6}$$

Thus we find,

$$f(t) = a_0 \sum_{j=0}^{\infty} \frac{(\alpha t)^j}{j!}$$

and we can set $a_0 = f_0$, thus

$$f(t) = f_0 \sum_{j=0}^{\infty} \frac{(\alpha t)^j}{j!} = f_0 e^{\alpha x}$$