MATH 202: VECTOR CALCULUS
CAES 9.8
HOMEWORK DUE FRIDAY WEEK 11

Problem 1.

$$\begin{aligned}
\mathrm{d}\omega &= \mathrm{d}(f\mathrm{d}x + g\mathrm{d}y + h\mathrm{d}z) \\
&= (D_1 f\mathrm{d}x + D_2 f\mathrm{d}y + D_3 f\mathrm{d}z) \wedge \mathrm{d}x \\
&+ (D_1 g\mathrm{d}x + D_2 g\mathrm{d}y + D_3 g\mathrm{d}z) \wedge \mathrm{d}y \\
&+ (D_1 h\mathrm{d}x + D_2 h\mathrm{d}y + D_3 h\mathrm{d}z) \wedge \mathrm{d}z \\
&= (D_2 f\mathrm{d}y + D_3 f\mathrm{d}z) \wedge \mathrm{d}x \\
&+ (D_1 g\mathrm{d}x + D_3 g\mathrm{d}z) \wedge \mathrm{d}y \\
&+ (D_1 h\mathrm{d}x + D_2 h\mathrm{d}y) \wedge \mathrm{d}z && \text{since } \mathrm{d}x_i \wedge \mathrm{d}x_i = 0 \\
&= (D_2 h - D_3 g)\mathrm{d}y \wedge \mathrm{d}z \\
&+ (D_3 f - D_1 h)\mathrm{d}z \wedge \mathrm{d}x \\
&+ (D_1 g - D_2 f)\mathrm{d}x \wedge \mathrm{d}y && \text{since } \mathrm{d}x_i = \mathrm{d}x_j \iff (-1)^{kl}\mathrm{d}x_j = \mathrm{d}x_i
\end{aligned}$$

as desired.

Problem 2.

$$\begin{aligned}
\mathrm{d}\omega &= \mathrm{d}(f\mathrm{d}y \wedge \mathrm{d}z + g\mathrm{d}z \wedge \mathrm{d}x + h\mathrm{d}x \wedge \mathrm{d}y) \\
&= (D_1 f\mathrm{d}x + D_2 f\mathrm{d}y + D_3 f\mathrm{d}z) \wedge \mathrm{d}y \wedge \mathrm{d}z \\
&+ (D_1 g\mathrm{d}x + D_2 g\mathrm{d}y + D_3 g\mathrm{d}z) \wedge \mathrm{d}z \wedge \mathrm{d}x \\
&+ (D_1 h\mathrm{d}x + D_2 h\mathrm{d}y + D_3 h\mathrm{d}z) \wedge \mathrm{d}x \wedge \mathrm{d}y \\
&= D_1 f\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z + D_2 g\mathrm{d}y \wedge \mathrm{d}z \wedge \mathrm{d}x + D_3 h\mathrm{d}z \wedge \mathrm{d}x \wedge \mathrm{d}y && \text{since } \mathrm{d}x_i \wedge \mathrm{d}x_i = 0 \\
&= (D_1 f + D_2 g + D_3 h)\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z && \text{since } \mathrm{d}x_i = \mathrm{d}x_j \iff (-1)^{kl}\mathrm{d}x_j = \mathrm{d}x_i
\end{aligned}$$

Problem 3. a) We can see that,

$$f_1 = D_1\phi \qquad f_2 = D_2\phi \qquad f_3 = D_3\phi$$

b) From Problem 1,

$$g_1 = (D_2 f_3 - D_3 f_2) \qquad g_2 = (D_3 f_1 - D_1 f_3) \qquad g_3 = (D_1 f_2 - D_2 f_1)$$

c) From Problem 2,

$$h = (D_1 g_1 + D_2 g_2 + D_3 g_3)$$

Problem 4.

$$\nabla \times F = (D_2 f_3 - D_3 f_2, D_3 f_1 - D_1 f_3, D_1 f_2 - D_2 f_1) = \text{curl}F \qquad \text{by definition of cross product}$$

$$\langle \nabla, G \rangle = D_1 g_1 + D_2 g_2 + D_3 g_3 = \text{div}G \qquad \text{by definition of inner product}$$

Problem 5. First we will show,

$$\begin{tikzcd} \phi \arrow[r, mapsto, "\mathrm{grad}"] \arrow[d, mapsto, "\mathrm{id}"'] & (f_1, f_2, f_3) \arrow[d, mapsto, "\cdot \vec{\mathrm{d}s}"] \\ \phi \arrow[r, mapsto, "\mathrm{d}"'] & f_1 \mathrm{d}x + f_2 \mathrm{d}y + f_3 \mathrm{d}z \end{tikzcd}$$

which we can see as, the grad(ϕ) gives us $(D_1\phi, D_2\phi, D_3\phi) = (f_1, f_2, f_3)$ from **Problem 3a** then, applying $\cdot \vec{\mathrm{d}s}$ gives us

$$(f_1, f_2, f_3) \cdot \vec{\mathrm{d}s} = f_1 \mathrm{d}x + f_2 \mathrm{d}y + f_3 \mathrm{d}z$$

which is equivalent to applying $d(\phi)$ as,

$$d(\phi) = f_1 \mathrm{d}x + f_2 \mathrm{d}y + f_3 \mathrm{d}z$$

by definition. Thus the diagram commutes.
Then we show,

$$\begin{tikzcd} (f_1, f_2, f_3) \arrow[r, mapsto, "\mathrm{curl}"] \arrow[d, mapsto, "\cdot \vec{\mathrm{d}s}"'] & (g_1, g_2, g_3) \arrow[d, mapsto, "\cdot \vec{\mathrm{d}n}"] \\ f_1 \mathrm{d}x + f_2 \mathrm{d}y + f_3 \mathrm{d}z \arrow[r, mapsto, "\mathrm{d}"'] & g_1 \mathrm{d}y \wedge \mathrm{d}z + g_2 \mathrm{d}z \wedge \mathrm{d}x + g_3 \mathrm{d}x \wedge \mathrm{d}y \end{tikzcd}$$

We can see the curl from **Problem 4** is $(D_2 f_3 - D_3 f_2, D_3 f_1 - D_1 f_3, D_1 f_2 - D_2 f_1)$ then applying $\cdot \vec{\mathrm{d}n}$ gives us,

$$g_1 \mathrm{d}y \wedge \mathrm{d}z + g_2 \mathrm{d}z \wedge \mathrm{d}x + g_3 \mathrm{d}x \wedge \mathrm{d}y.$$

From **Problem 3b** and **Problem 1** that

$$\mathrm{d}(f_1 \mathrm{d}x + f_2 \mathrm{d}y + f_3 \mathrm{d}z) = g_1 \mathrm{d}y \wedge \mathrm{d}z + g_2 \mathrm{d}z \wedge \mathrm{d}x + g_3 \mathrm{d}x \wedge \mathrm{d}y,$$

thus the diagram commutes.
and finally,

$$\begin{tikzcd} (g_1, g_2, g_3) \arrow[r, mapsto, "\mathrm{div}"] \arrow[d, mapsto, "\cdot \vec{\mathrm{d}n}"'] & h \arrow[d, mapsto, "\mathrm{d}V"] \\ g_1 \mathrm{d}y \wedge \mathrm{d}z + g_2 \mathrm{d}z \wedge \mathrm{d}x + g_3 \mathrm{d}x \wedge \mathrm{d}y \arrow[r, mapsto, "\mathrm{d}"'] & h \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z \end{tikzcd}$$

We can see that the div from **Problem 4** gives us, $D_1 g_1 + D_2 g_2 + D_3 g_3$, then applying $\mathrm{d}V$ we see,

$$(D_1 g_1 + D_2 g_2 + D_3 g_3)\mathrm{dV} = (D_1 g_1 + D_2 g_2 + D_3 g_3)\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z.$$

From **Problem 3c** and **Problem 2** we can see that

$$\mathrm{d}(g_1 \mathrm{d}y \wedge \mathrm{d}z + g_2 \mathrm{d}z \wedge \mathrm{d}x + g_3 \mathrm{d}x \wedge \mathrm{d}y) = (D_1 g_1 + D_2 g_2 + D_3 g_3)\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$$

as desired, and so the diagram commutes.
Finally, we can glue the diagrams together for the desired result.

$$\begin{tikzcd}
\phi \arrow[r, mapsto, "\mathrm{grad}"] \arrow[d, mapsto, "\mathrm{id}"'] & (f_1, f_2, f_3) \arrow[r, mapsto, "\mathrm{curl}"] \arrow[d, mapsto, "\cdot \vec{\mathrm{d}s}"'] & (g_1, g_2, g_3) \arrow[r, mapsto, "\mathrm{div}"] \arrow[d, mapsto, "\cdot \vec{\mathrm{d}n}"'] & h \arrow[d, mapsto, "\mathrm{d}V"] \\
\phi \arrow[r, mapsto, "\mathrm{d}"] & f_1 dx + f_2 dy + f_3 dz \arrow[r, mapsto, "\mathrm{d}"] & g_1 \mathrm{d}y \wedge \mathrm{d}z + g_2 \mathrm{d}z \wedge \mathrm{d}x + g_3 \mathrm{d}x \wedge \mathrm{d}y \arrow[r, mapsto, "\mathrm{d}"] & h \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z
\end{tikzcd}$$

Problem 6. We can see using the commutative diagram and the nilpotence of d that,

$$\mathrm{d}(\mathrm{d}(\phi)) = (\mathrm{curl} \circ \mathrm{grad})(\phi) \cdot \vec{\mathrm{d}n} \implies (\mathrm{curl} \circ \mathrm{grad})(\phi) = 0$$

$$\mathrm{d}(\mathrm{d}((f_1, f_2, f_3) \cdot \vec{\mathrm{d}s})) = (\mathrm{div} \circ \mathrm{curl})(\phi)\mathrm{d}V \implies (\mathrm{div} \circ \mathrm{curl})(\phi) = 0$$

Finally,

$$\mathrm{grad}(\phi) = (D_1\phi, D_2\phi, D_3\phi) \implies \mathrm{div}(\mathrm{grad}(\phi)) = \mathrm{div}(D_1\phi, D_2\phi, D_3\phi) = D_1\phi + D_2\phi + D_3\phi$$

thus

$$\mathrm{div}(\mathrm{grad}(\phi)) = 0 \implies D_1\phi, +D_2\phi + D_3\phi = 0$$

or the **harmonic equation**, as desired.