PHYSICS 201: OSCILLATIONS AND WAVES
HOMEWORK DUE FRIDAY WEEK 10

Problem 1.

$$\frac{\partial f(x,y)}{\partial x} = A\left(2\sin\left(\frac{y}{L}\right)x - y\right)$$
$$\frac{\partial^2 f(x,y)}{\partial y^2} = -\frac{1}{L^2}Ax^2\sin\left(\frac{y}{L}\right)$$
$$\frac{\partial^3 f(x,y)}{\partial x^3} = 0$$
$$\frac{\partial^2 f(x,y)}{\partial x \partial y} = A\left(\frac{2x}{L}\cos\left(\frac{y}{L}\right) - 1\right)$$

Problem 2.

$$\mathbb{M} = \begin{pmatrix} \frac{\partial x}{\partial x} & \frac{\partial y}{\partial x} & \frac{\partial z}{\partial x} \\ \frac{\partial y}{\partial x} & \frac{\partial y}{\partial y} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} & \frac{\partial z}{\partial z} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Problem 3. For $\phi(x,0) = 0$ means that the system is at its equilibrium location at $t = 0$. For $\phi(x,0) = \alpha$ means that the system has been shifted from equilibrium to the right by a factor of α or $x + \alpha$. Finally, for a system $\phi(x,0) = x/2$ means that every mass is shifted to the right $x/2$ beyond equilibrium or $3x/2$, thus the mass at equilibrium at $1/4m$ is at $3/8m$ from the origin.

Problem 4. We know that

$$\frac{\partial^2 \phi_1(x,t)}{\partial t^2} = v^2\frac{\partial^2 \phi_1(x,t)}{\partial t^2}$$
$$\frac{\partial^2 \phi_2(x,t)}{\partial t^2} = v^2\frac{\partial^2 \phi_2(x,t)}{\partial t^2}$$

adding these 2 linear equations we see,

$$\frac{\partial^2 \phi_1(x,t)}{\partial t^2} + \frac{\partial^2 \phi_2(x,t)}{\partial t^2} = v^2\frac{\partial^2 \phi_1(x,t)}{\partial t^2} + v^2\frac{\partial^2 \phi_2(x,t)}{\partial t^2}$$
$$\frac{\partial^2 (\phi_1(x,t) + \phi_2(x,t))}{\partial t^2} = v^2\frac{\partial^2 (\phi_1(x,t) + \phi_2(x,t))}{\partial t^2}$$

Thus we see that $\phi_1(x,t) + \phi_2(x,t)$ satisfies the wave equation.

Problem 5. Let $e_j = ja$ and $\phi(e_j, t) = x_j(t) - e_j$, then we see,

$$E = \frac{1}{2}m\sum_{j=1}^{n}\left(\frac{\partial \phi(e_j,t)}{\partial t}\right)^2 + \frac{1}{2}k\sum_{j=1}^{n-1}(\phi(e_{j+1},t) + (j+1)a - \phi(e_j,t) - ja - a)^2$$

expressing this in terms of fixed M and K

$$\frac{1}{2}\frac{M}{n}\sum_{j=1}^{n}\left(\frac{\partial \phi(e_j,t)}{\partial t}\right)^2 + \frac{1}{2}(K(n-1))\sum_{j=1}^{n-1}(\phi(e_j + a,t) - \phi(e_j,t))^2$$

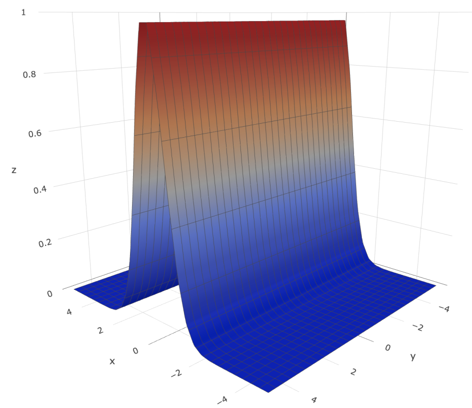

FIGURE 1. For $t = 0$

then using Taylor expansion and noting that as a shrinks that $x \approx e_j$ as well as $L = (n-1)a$,

$$\frac{1}{2}\frac{M}{n}\sum_{j=1}^{n}\left(\frac{\partial\phi(x,t)}{\partial t}\right)^2 + \frac{1}{2}(K\left(\frac{L}{a}\right))\sum_{j=1}^{n-1}\left(\phi(x,t) + a\frac{\partial\phi(x,t)}{\partial x} - \phi(x,t)\right)^2$$

$$\frac{1}{2}\frac{M}{n}\sum_{j=1}^{n}\left(\frac{\partial\phi(x,t)}{\partial t}\right)^2 + \frac{1}{2}(K\left(\frac{L}{a}\right))a^2\sum_{j=1}^{n-1}\left(\frac{\partial\phi(x,t)}{\partial x}\right)^2$$

$$\frac{1}{2}\frac{M}{n}\sum_{j=1}^{n}\left(\frac{\partial\phi(x,t)}{\partial t}\right)^2 + \frac{1}{2}\frac{KL^2}{n-1}\sum_{j=1}^{n-1}\left(\frac{\partial\phi(x,t)}{\partial x}\right)^2$$

since $L \approx na$ and $M \approx (n-1)m$

$$\frac{1}{2}\frac{Ma}{L}\sum_{j=1}^{n}\left(\frac{\partial\phi(x,t)}{\partial t}\right)^2 + \frac{1}{2}\frac{mKL^2}{M}\sum_{j=1}^{n-1}\left(\frac{\partial\phi(x,t)}{\partial x}\right)^2$$

using similar approximations again as well as $\frac{KL^2}{M} = v^2$

$$\frac{1}{2}\frac{Ma}{L}\sum_{j=1}^{n}\left(\frac{\partial\phi(x,t)}{\partial t}\right)^2 + \frac{1}{2}\frac{M}{n}v^2\sum_{j=1}^{n-1}\left(\frac{\partial\phi(x,t)}{\partial x}\right)^2$$

$$\frac{1}{2}\frac{M}{L}\sum_{j=1}^{n}\left(\frac{\partial\phi(x,t)}{\partial t}\right)^2 a + \frac{1}{2}\frac{M}{L}v^2\sum_{j=1}^{n-1}\left(\frac{\partial\phi(x,t)}{\partial x}\right)^2 a$$

Finally, letting $n \to \infty$ we see $\sum \_\_a \to \int \_\_dx$

$$\int \frac{1}{2}\frac{M}{L}\left(\left(\frac{\partial\phi(x,t)}{\partial t}\right)^2 + v^2\left(\frac{\partial\phi(x,t)}{\partial x}\right)^2\right)dx$$

where the integrand is itself the energy density (energy per unit length) of the continuous system.

Problem 6. See **Figure 1 and 2**

First note,

$$\frac{\partial^2\phi^2(x,t)}{\partial x^2} = 4f_0^2\left(4x^2 + 8tvx + 4t^2v^2 - 1\right)\mathrm{e}^{-2(x+tv)^2}$$

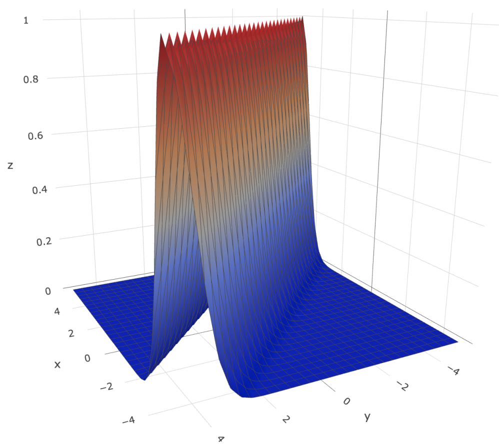

FIGURE 2. For $t' > 0$

and
$$\frac{\partial^2 \phi^2(x,t)}{\partial t^2} = 4f_0^2 v^2 \left(4v^2t^2 + 8vxt + 4x^2 - 1\right) \mathrm{e}^{-2(vt+x)^2}$$
Thus we see,
$$u(x,t) = \frac{4Mf_0^2 v^2 \mathrm{e}^{-2(vt+x)^2}}{L} \left(4v^2t^2 + 8vxt + 4x^2 - 1\right)$$

Problem 7. a) First we can see,
$$\frac{\partial^2 \phi_1(x,t)}{\partial t^2} = -\frac{1}{L^2} 4\pi^2 a v^2 \sin\left(\frac{2\pi x}{L}\right)\cos\left(\frac{2\pi vt}{L}\right)$$
and
$$v^2 \cdot \frac{\partial^2 \phi_1(x,t)}{\partial x^2} = v^2 \cdot \left(-\frac{1}{L^2} 4\pi^2 a \sin\left(\frac{2\pi x}{L}\right)\cos\left(\frac{2\pi vt}{L}\right)\right)$$
Thus we see, $\phi(x,t)$ satisfies the wave equation as the two are equivalent.

b) For $t = 0$, noting for both figures that we we are only looking at one cycle since $t : 0 \to L$ (**Figure 3**) and for an arbitrary later time (**Figure 4**), $t' > 0$

c)
$$f(p) = g(p) = \frac{1}{2} A \sin\left(\frac{2\pi p}{L}\right)$$
Thus,
$$\begin{aligned} f(x+vt) + g(x-vt) &= \frac{1}{2}A\sin\left(\frac{2\pi(x+vt)}{L}\right) + \frac{1}{2}A\sin\left(\frac{2\pi(x-vt)}{L}\right) \\ &= \frac{1}{2}A\left(\sin\left(\frac{2\pi x}{L}\right)\cos\left(\frac{2\pi vt}{L}\right) + \sin\left(\frac{2\pi vt}{L}\right)\cos\left(\frac{2\pi x}{L}\right)\right. \\ &\left. + \sin\left(\frac{2\pi x}{L}\right)\cos\left(\frac{2\pi vt}{L}\right) - \sin\left(\frac{2\pi vt}{L}\right)\cos\left(\frac{2\pi x}{L}\right)\right) \\ &= A\sin\left(\frac{2\pi x}{L}\right)\cos\left(\frac{2\pi vt}{L}\right) \end{aligned}$$

as desired.

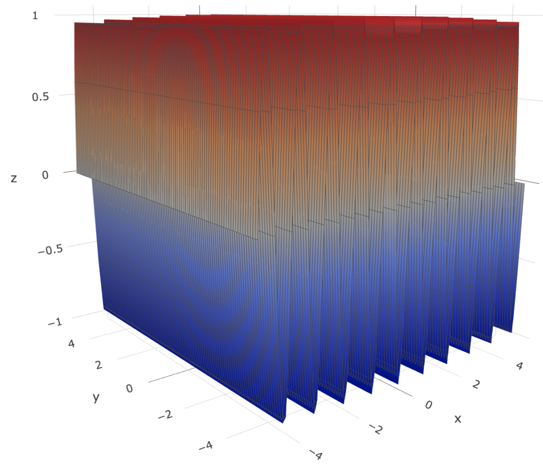

FIGURE 3. For $t = 0$

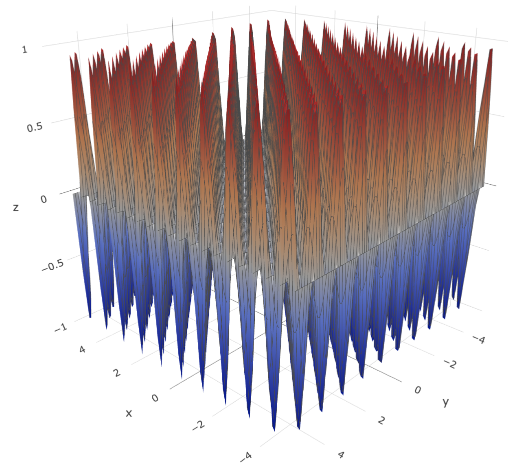

FIGURE 4. For $t' > 0$