MATH 202: VECTOR CALCULUS
CHAPTER 6B EXAM
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Problem 1. First we will switch the bounds of integration,

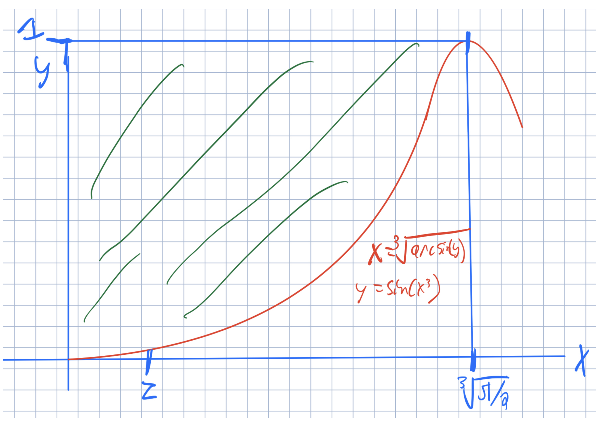

FIGURE 1.

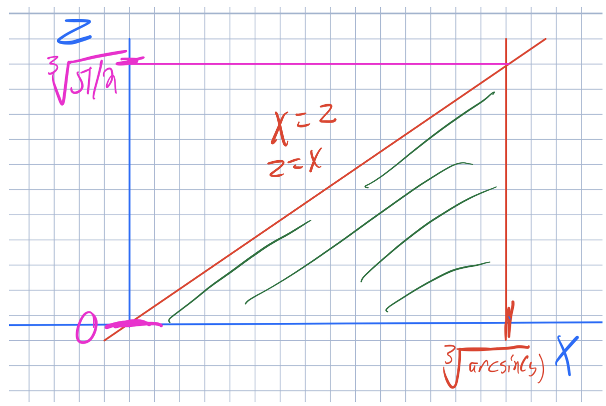

FIGURE 2.

$$I = \frac{24240}{e-1}\int_{z=0}^{\sqrt[3]{\pi/2}}\int_{x=z}^{\sqrt[3]{\pi/2}}\int_{y=\sin(x^3)}^{1}\cos(x^3)e^{(y^2)}z$$

$$= \frac{24240}{e-1}\int_{z=0}^{\sqrt[3]{\pi/2}}\int_{y=0}^{1}\int_{x=z}^{\sqrt[3]{\arcsin(y)}}\cos(x^3)e^{(y^2)}z \qquad \text{see Figure 1}$$

$$= \frac{24240}{e-1}\int_{y=0}^{1}\int_{z=0}^{\sqrt[3]{\pi/2}}\int_{x=z}^{\sqrt[3]{\arcsin(y)}}\cos(x^3)e^{(y^2)}z \qquad \text{since the outer integrals integrate over a box}$$

$$= \frac{24240}{e-1}\int_{y=0}^{1}\int_{x=0}^{\sqrt[3]{\arcsin(y)}}\int_{z=0}^{x}\cos(x^3)e^{(y^2)}z \qquad \text{see Figure 2}$$

Now we may evaluate to see,

$$
\begin{aligned}
I &= \frac{24240}{e-1}\int_{y=0}^{1}\int_{x=0}^{\sqrt[3]{\arcsin(y)}}\int_{z=0}^{x} \cos(x^3)e^{(y^2)}z \\
&= \frac{24240}{e-1}\int_{y=0}^{1}\int_{x=0}^{\sqrt[3]{\arcsin(y)}} \left(\frac{\cos(x^3)e^{(y^2)}z^2}{2}\right)\Bigg|_{z=0}^{x} \\
&= \frac{24240}{e-1}\int_{y=0}^{1}\int_{x=0}^{\sqrt[3]{\arcsin(y)}} \frac{\cos(x^3)e^{(y^2)}x^2}{2} \\
&= \frac{24240}{e-1}\int_{y=0}^{1} \left(\frac{e^{(y^2)}\sin(x^3)}{6}\right)\Bigg|_{x=0}^{\sqrt[3]{\arcsin(y)}} \\
&= \frac{24240}{e-1}\int_{y=0}^{1} \frac{ye^{(y^2)}}{6} \\
&= \frac{24240}{e-1}\left(\frac{e^{(y^2)}}{12}\right)\Bigg|_{y=0}^{1} \\
&= \frac{24240}{e-1}\left(\frac{e-1}{12}\right) \\
&= 2020
\end{aligned}
$$

Problem 2. (1) We can walk through the calculations and find a, b, c, d, e, f, To begin,

$$\phi(0,0) = (1,1) = (a(0)+b(0)+c, d(0)+e(0)+f) \implies c = 1 \quad f = 1$$

$$\phi(1,0) = (3,2) = (a(1)+b(0)+1, d(1)+e(0)+1) \implies a = 2 \quad d = 1$$

$$\phi(0,1) = (2,4) = (2(0)+b(1)+1, 1(0)+e(1)+1) \implies b = 1 \quad e = 3$$

Thus

$$\phi(u,v) = (2u+v+1, u+3v+1).$$

Let us check that $\phi(1,1) = (4,5)$,

$$\phi(1,1) = (2(1)+1+1, 1+3(1)+1) = (4,5).$$

(2) We can see that from **1**,

$$\phi' = \begin{pmatrix} D_1\phi_1 & D_2\phi_1 \\ D_1\phi_2 & D_2\phi_2 \end{pmatrix} = \begin{pmatrix} a & b \\ d & e \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} \implies |\det \phi'| = 5$$

thus for all $(u,v) \in K$, $|\det \phi'(u,v)| = 5$

(3) We can see that, by the change of variable theorem,

$$
\begin{aligned}
I = \int_P f = \int_{\phi(K)} f = \int_K (f \circ \phi) \cdot |\det \phi'| \\
&= \int_{u=0}^{1} \int_{v=0}^{1} f(\phi(u,v)) \cdot 5 \\
&= 5 \cdot \int_{u=0}^{1} \int_{v=0}^{1} e^{(2u+v+1)-(u+3v+1)} \\
&= 5 \cdot \int_{u=0}^{1} \int_{v=0}^{1} e^{u-2v} \\
&= 5 \cdot \int_{u=0}^{1} \left(\frac{-e^{u-2v}}{2} \right) \Big|_{v=0}^{1} \\
&= 5 \cdot \int_{u=0}^{1} \frac{e^{u-2}(e^2-1)}{2} \\
&= 5 \cdot \left(\frac{e^{u-2}(e^2-1)}{2} \right) \Big|_{u=0}^{1} \\
&= \frac{5(e-1)(e^2-1)}{2e^2}
\end{aligned}
$$

Problem 3. We may use 6.7.10b and Exercise 6.7.5, where $(x-1/2)^2 + z^2 = (1/2)^2$ is the cross-sectional area of $\Phi(K)$ along the (x,z) plane and hence $\Phi(K)$ is a torus $T_{a,b}$ with $a = b = \frac{1}{2}$. Now we will calculate and see,

$$
\operatorname{vol}(\Phi(K)) = 2\pi^2 a^2 b = 2\pi^2 \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right) = \frac{\pi^2}{4}.
$$