Homework: 4.3, 4.4

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Math 202: Vector Calculus

Due September 25th, 2020

4.3.3

($\Longrightarrow$) Let f be differentiable at a (which needs to be an interior point of A) with derivative T. Then,

$$f(a+h) - f(a) - T_a(h) = o(h).$$

By the componentwise nature of the $o(h)$ condition we know that each component f_i must satisfy,

$$f_i(a+h) - f_i(a) - T_{ia}(h) = o(h)$$

or that each f_i is differentiable at a with derivative T_i. Thus we have shown that if f is differentiable at a with derivative T, then each f_i is differentiable at a with derivative T_i. Hence, we have shown the rightward implication.

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By the componentwise nature of the $o(h)$ condition we know that f must satisfy,

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or that f is differentiable at a with derivative T. Thus we have shown that if each f_i is differentiable at a with derivative T_i, then f is differentiable at a with derivative T. Hence, we have shown the leftward implication.

4.3.4

We will start with,

$$f(a+h, b+k) - f(a,b) - Df_{a,b}(h,k)$$

which implies for the first component, which is allowed by componentwise nature,

$$\begin{aligned}(a+h)^2 - (b+k)^2 - a^2 + b^2 - 2ah + 2bk &= a^2 + 2ah + h^2 - b^2 - 2bk - k^2 - a^2 + b^2 - 2ah \\ &= h^2 - k^2 \\ &= o(h,k) \qquad \text{by the basic family of Landau Functions}\end{aligned}$$

For the second component,

$$\begin{aligned} 2(a+h)(b+k) - 2ab - 2bh - 2ak &= 2ab + 2ak + 2hb + 2hk - 2ab - 2bh - 2ak \\ &= 2hk \\ &= o(h,k) \end{aligned}$$

The last step is justified since, k, f are scalar functions with $k = \mathcal{O}(h) \implies k = o(1)$ and $h = \mathcal{O}(h)$ and by the product property of Landau functions we find, $2hk$ to be $o(h,k)$

Hence, since both components are $o(h,k)$, the derivative is true.

4.3.6

Let $|f(x)| \le |x^2|$. Since the function is underneath the parabola, its derivative at $\mathbf{0}_n$, if it's differentiable, must be $T(h) = \mathbf{0}_m$ for all h. Furthermore, since $|f(x)| \le |x^2|$, we know $|f(\mathbf{0}_n)| \le 0 \implies f(\mathbf{0}_n) = \mathbf{0}_m$. So in total we see,

$$|f(a+h) - f(a) - T_a(h)| = |f(\mathbf{0}_n + h) - f(\mathbf{0}_n) - T_{\mathbf{0}_n}(h)| = |f(h)| \le |h|^2 = o(h)$$

By the basic family of Landau Functions and the dominance principal, we have shown that if $|f(x)| \le |x^2|$ for all $x \in \mathbb{R}^n$, then f is differentiable at $\mathbf{0}_n$.

4.4.2

Since f is differentiable at a with derivative Df_a then we can find,

$$\begin{aligned} (\alpha f)(a+h) - (\alpha f)(a) - (\alpha Df_a)(h) &= \alpha(f)(a+h) - \alpha(f)(a) - \alpha(Df_a)(h) \qquad \text{by linearity of mappings} \\ &= \alpha((f)(a+h) - (f)(a) - (Df_a)(h)) \qquad \text{by distributive laws} \\ &= \alpha(o(h)) \qquad \text{since } f \text{ is differentiable at } a \text{ with derivative } Df_a \\ &= o(h) \qquad \text{by vector space properties of } o(h) \end{aligned}$$

4.4.5

Let $X(x,y,z) = x$,$Y(x,y,z) = y$, and $Z(x,y,z) = z$ Thus we have, by the multivariable product rule,

$$\begin{aligned} Df_{(a,b,c)}(h,k,\ell) &= D((XY)Z)_{(a,b,c)}(h,k,\ell) \\ &= (XY)(a,b,c)D(Z)_{(a,b,c)} + Z(a,b,c)D(XY)_{(a,b,c)}(h,k,\ell) \\ &= (ab)Z(h,k,\ell) + c(Y(a,b,c)DX_{(a,b,c)}(h,k,\ell) + X(a,b,c)DY_{(a,b,c)}(h,k,\ell)) \\ &= (ab)Z(h,k,\ell) + c(bX(h,k,\ell) + aY(h,k,\ell)) \\ &= abl + bch + ack \end{aligned}$$

Hence, we have found the derivative.