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Homework 5

Physics 201: Oscillations and Waves

Due October 2nd, 2020

Exercise 1. We can take an ansatz, $\bar{x}(t) = X_0 e^{i\alpha t}$ and then take the real part, so we can find the cosine that is desired. So,

$$- \alpha^2 X_0 e^{i\alpha t} = -2bi\alpha X_0 e^{i\alpha t} + f_0 e^{i\sigma t}$$

and for the equality to hold $\alpha = \sigma$, so,

$$-\sigma^2 X_0 = -2b\sigma i X_0 + f_0 \implies X_0 = \frac{f_0}{2b\sigma i - \sigma^2} \implies \bar{x}(t) = \frac{f_0(-2b\sigma i - \sigma^2)}{\sigma^4 + 4b^2\sigma^2} e^{i\sigma t}$$

So the real part

$$\Re\left(\bar{x}(t)\right) = \frac{-f_0\sigma^2}{\sigma^4 + 4b^2\sigma^2}\cos(\sigma t) + \frac{2bf_0}{4b^2\sigma + \sigma^3}\sin(\sigma t) = \frac{-f_0}{\sigma^2 + 4b^2}\cos(\sigma t) + \frac{2bf_0}{4b^2\sigma + \sigma^3}\sin(\sigma t)$$

For the homogenous solution we see, after cancelling m and letting $x(t) = Ge^{ut}$,

$$\ddot{x}(t) = -2b\dot{x}(t) \implies u^2 Ge^{ut} = -2buGe^{ut} \implies u(u + 2b) = 0$$

Thus we see, taking the same form,

$$h(t) = c_1 e^{-2bt} + c_2 e^0$$

Adding the homogenous solution we see,

$$x(t) = \frac{-f_0}{\sigma^2 + 4b^2}\cos(\sigma t) + \frac{2bf_0}{4b^2\sigma + \sigma^3}\sin(\sigma t) + c_1 e^{-2bt} + c_2$$

Now let $x(0) = x_0$ and $\dot{x}(0) = v_0$ and we get,

$$x(0) = \frac{-f_0}{\sigma^2 + 4b^2} + c_1 + c_2 = x_0$$

and

$$\dot{x}(0) = \frac{2bf_0}{4b^2 + \sigma^2} - 2bc_1 = v_0$$

thus we see,

$$c_1 = -\frac{v_0}{2b} + \frac{f_0}{4b^2 + \sigma^2}$$

and,

$$c_2 = x_0 + \frac{v_0}{2b}$$

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So in total our final solution is,

$$x(t) = \frac{-f_0}{\sigma^2 + 4b^2}\cos(\sigma t) + \frac{2b f_0}{4b^2\sigma + \sigma^3}\sin(\sigma t) + \left(-\frac{v_0}{2b} + \frac{f_0}{4b^2 + \sigma^2}\right)e^{-2bt} + \left(x_0 + \frac{v_0}{2b}\right)$$

Exercise 2. We begin with the fact that by Kirchhoff's law,

$$V_0 - L\dot{I}(t) - RI(t) = 0 \implies -L\dot{I}(t) - R\left(I(t) - \frac{V_0}{R}\right) = 0$$

Then one solution may be,

$$I(t) = \frac{V_0}{R}.$$

For our homogeneous solution we have,

$$-L\dot{I}(t) = RI(t)$$

and suppose $I(t) = Ge^{ut}$ then,

$$-uLGe^{ut} = RGe^{ut} \implies -ul = R \implies u = -\frac{R}{L}$$

so our homogeneous solution becomes,

$$I(t) = Ge^{-\frac{R}{L}t}.$$

Finally, combine our solutions and set our initial values to find G,

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$$I(0) = \frac{V_0}{R} - Ge^{\frac{R}{l}0} \implies \frac{V_0}{R} - G = 0 \implies G = \frac{V_0}{R}$$

so our final solution is,

$$I(t) = \frac{V_0 - V_0e^{-\frac{R}{L}t}}{R}$$

Exercise 3. For a resistor it is clear that if $V(t) = Z(\omega)I(t) = RI(t) \implies Z(\omega) = R$.

For a capacitor, we know that $I(t) = C\frac{dV(t)}{dt}$, and from there

$$V_0e^{i\omega t} = CZ(\omega)\frac{d}{dt}\left(V_0e^{i\omega t}\right) \implies V_0e^{i\omega t} = CZ(\omega)i\omega V_0e^{i\omega t} \implies Z(\omega) = \frac{1}{i\omega C}$$

For a inductor, we know that $V(t) = L\frac{dI(t)}{dt}$, and from there

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$$L\frac{dI(t)}{dt} = V_0e^{i\omega t}$$

$$\text{integrate both sides} \implies I(t) = \frac{V_0e^{i\omega t}}{i\omega L}$$

$$\text{divide by } I(t) \implies \frac{I(t)}{I(t)} = \frac{Z(\omega)}{i\omega L} \implies Z(\omega) = \frac{1}{i\omega L}$$

should be $i\omega L$

$\implies 1 = \frac{Z(\omega)}{i\omega L} \implies Z(\omega) = i\omega L$

Exercise 4. If we use the ansatz, we see,

$$u^2tGe^{ut} + 2\alpha uGe^{ut} = 0 \implies ut + 2\alpha = 0 \implies u = -\frac{2\alpha}{t}.$$

But we stated that u is a time-independent constant, but as seen with the ansatz, u would not be time-independent.

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Exercise 5. Using the ansatz we see,

$$t(Gt^u)'' = -2\alpha(Gt^u)'$$

$$tGu(u-1)t^{u-2} = -2\alpha uGt^{u-1}$$

$$u(u-1)t^{u-1} + 2\alpha ut^{u-1} = 0$$

$$t^{u-1}(u(u-1) + 2u\alpha) = 0$$

Thus, $u = 0, u = 1 - 2\alpha$, and so we have two solutions, in the form,

$$At^0 = A$$

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$$Bt^{1-2\alpha}$$

and by adding them together we get the full solution of,

$$x(t) = A + Bt^{1-2\alpha}$$

Exercise 6.

$$\begin{aligned}
a_j &= \frac{1}{T}\left(\int_0^{T/2} p_0 t e^{-i2\pi jt/T} dt - \int_{T/2}^{T} p_0(t-T)e^{-i2/T} dt\right) \\
&= \frac{1}{T}\left(\left(\frac{ip_0T(2i\pi jt + T)e^{-i2\pi jt/T}}{4\pi^2 j^2}\bigg|_0^{T/2}\right) + \left(\frac{-ip_0T(2\pi jt + (-2\pi j - i)T)e^{-2i\pi jt/T})}{4\pi^2 j^2}\bigg|_{T/2}^{T}\right)\right) \\
&= \frac{1}{T}\left(\frac{((-1)^j(i\pi j + 1) - 1)p_0T^2}{4\pi^2 j^2} + \frac{(-1 + (1 - i\pi j)(-1)^j)p_0T^2}{4\pi^2 j^2}\right) \\
&= \frac{p_0T(-1)^j(i\pi j + 1) + (-1)^j(-i\pi j + 1) - 1 - 1}{4\pi^2 j^2} \\
&= p_0T\frac{(-1)^j(2) - 2}{4\pi^2 j^2} \\
&= p_0T\frac{(-1)^j - 1}{2\pi^2 j^2}
\end{aligned}$$

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Thus, we can see, noting that $a_0 = \frac{p_0T}{4}$ since,

$$a_0 = \frac{p_0}{T}\left(\int_0^{T/2} tdt - \int_{T/2}^{T}(t-T)dt\right) = \frac{p_0}{T}\left(\left(\frac{1}{8}T^2\right) - \left(\frac{1}{2}T^2 - T^2 - \frac{1}{8}T^2 + \frac{1}{2}T^2\right)\right) = \frac{p_0T}{4},$$

if $j =$odd,

$$a_j = -\frac{p_0T}{\pi^2 j^2}$$

if $j =$even,

$$a_j = 0$$

Exercise 7. We can see the units for α would be $\frac{kg\ m}{s^3}$ in order to have F to have the units of Newtons.

We can first find the homogeneous solution which would be, letting $\omega^2 \equiv \frac{k}{m}$,

$$h(t) = A\sin(\omega t) + B\cos(\omega t).$$

Now in order to find our ansatz, we will consider only the driving force with no spring. We can see,

$$\ddot{x}(t) = \frac{\alpha}{m}t$$

then integrating twice,

$$x(t) = \frac{\alpha}{6m}t^3$$

so our ansatz should be of the form At^B, and we can see by setting $A = \frac{\alpha}{k}$ and $B = 1$ that this solves our differential equation as,

$$m\ddot{x} = -k\left(x(t) - \frac{\alpha}{k}t\right)$$

with $x(t) = -\frac{\alpha}{k}t$ as that would allow equality. Thus we have

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$$x(t) = \frac{\alpha}{k}t + A\sin(\omega t) + B\cos(\omega t)$$

with the initial conditions we see, $B = x_0$ and $0 = \frac{\alpha}{k} + A\omega \implies A = -\frac{\alpha}{k\omega}$ so the full solution becomes,

$$x(t) = \frac{\alpha}{k}t - \frac{\alpha}{k\omega}\sin(\omega t) + x_0\cos(\omega t)$$