MATH 202: VECTOR CALCULUS
CAES 9.9, 9.10
HOMEWORK DUE TUESDAY WEEK 11

Problem 1. a) We can check to see that,

$$S^*\omega \wedge S^*\lambda = S^*(\omega \wedge \lambda).$$

We can use the first commutative diagram on pg. 453.

b) We can check to see that

$$S^*(\mathrm{d}\omega) = \mathrm{d}(S^*\omega).$$

We can use the second commutative diagram on pg. 453.

c) We can check this using the contravariance of the pullback so,

$$(S \circ T)^*\lambda = (T^* \circ S^*)\lambda.$$

We can use the commutative diagram in the middle of pg. 454.

Problem 2. First we can use the polar coordinate mapping,

$$\Phi : \mathbb{R}_{\leq 0} \times \mathbb{R} \to \mathbb{R}^2 \backslash \{(0,0)\}, \qquad \Phi(r,\theta) = (r\cos\theta, r\sin\theta) = (u,v)$$

then the reciprocal mapping becomes,

$$S : \mathbb{R}_{\leq 0} \times \mathbb{R} \to \mathbb{R}_{\leq 0} \times \mathbb{R}, \qquad S(r,\theta) = (1/r, -\theta) = (\tilde{r}, \tilde{\theta})$$

Thus we have the diagram for ω,

$$\begin{tikzcd} \mathrm{d}(-\theta) & T^*\omega \arrow[l, mapsto] \\ \mathrm{d}\tilde{\theta} \arrow[u, mapsto] & \omega \arrow[l, mapsto] \arrow[u, mapsto] \end{tikzcd}$$

thus we see that since $\mathrm{d}(-\theta) =$ -dθ the sought-for pullback $T^*\omega$ must be in the (u,v)-form that pulls back through the polar coordinate to -d$\tilde{\theta}$, so $T^*\omega$ should be the negative of ω but with u and v in place of x and y,

$$T^*\omega = -\frac{u\mathrm{d}v - v\mathrm{d}u}{u^2+v^2}$$

We can similarly see that for λ we have the diagram, since $\frac{1}{r}\mathrm{d}\left(\frac{1}{r}\right) \wedge d(-\theta) = \frac{\mathrm{d}r \wedge \mathrm{d}\theta}{r^3}$,

$$\begin{tikzcd} \frac{\mathrm{d}r \wedge \mathrm{d}\theta}{r^3} & T^*\lambda \arrow[l, mapsto] \\ \tilde{r}\mathrm{d}\tilde{r} \wedge \mathrm{d}\tilde{\theta} \arrow[u, mapsto] & \lambda \arrow[l, mapsto] \arrow[u, mapsto] \end{tikzcd}$$

thus $T^*\lambda$ must bull back through the polar coordinate mapping to $\frac{\mathrm{d}r\wedge\mathrm{d}\theta}{r^3}$.. Since the area-form $\mathrm{d}u\wedge\mathrm{d}v$ pulls back to $r\mathrm{d}r\wedge\mathrm{d}\theta$ the answer is the area form divided by r^4 in $(u,v)-$coordinates. That is since r in (u,v)-coordinates is $\sqrt{u^2+v^2}$,

$$T^*\lambda = T^*(\mathrm{d}x\wedge\mathrm{d}y) = \frac{\mathrm{d}u\wedge\mathrm{d}v}{(u^2+v^2)^2}$$

Problem 3. a) We can see, pulling μ back,

$$\Phi^*\mu = \frac{\tilde{r}\cos\tilde{\theta}(\cos\tilde{\theta}\mathrm{d}\tilde{r} - \tilde{r}\sin\tilde{\theta}\mathrm{d}\tilde{\theta}) - \tilde{r}\sin\tilde{\theta}(\sin\tilde{\theta}\mathrm{d}\tilde{r} + \tilde{r}\cos\tilde{\theta}\mathrm{d}\tilde{\theta})}{\tilde{r}} = \mathrm{d}\tilde{r}$$

thus $\int_\gamma \mu$ measures change in distance from origin. Or,

$$\int_\gamma \mu = |\gamma(1)| - |\gamma(0)|$$

b) Take the diagram,

$$\begin{tikzcd} \mathrm{d}(r^2) & T^*\mu \arrow[l, mapsto] \\ \mathrm{d}\tilde{r} \arrow[u, mapsto] & \mu \arrow[l, mapsto] \arrow[u, mapsto] \end{tikzcd}$$

Since $\mathrm{d}(r^2) = 2r\mathrm{d}r$. Thus $T^*\mu$ should be $2r = 2\sqrt{u^2+v^2}$ times μ in $(u,v)-$coordinates but with u and v in place of x and y,

$$T^*\mu = 2\sqrt{u^2+v^2}\left(\frac{u\mathrm{d}u + v\mathrm{d}v}{\sqrt{u^2+v^2}}\right) = 2(u\mathrm{d}u + v\mathrm{d}v)$$

c) Take the diagram, noting that $\mathrm{d}\left(\frac{1}{r}\right) = -\frac{\mathrm{d}r}{r^2}$

$$\begin{tikzcd} -\frac{\mathrm{d}r}{r^2} & T^*\mu \arrow[l, mapsto] \\ \mathrm{d}\tilde{r} \arrow[u, mapsto] & \mu \arrow[l, mapsto] \arrow[u, mapsto] \end{tikzcd}$$

thus we can see, $T^*\mu$ should be $-\frac{1}{r^2} = -\frac{1}{u^2+v^2}$ multiplied by μ, in $(u,v)-$coordinates,

$$T^*\mu = -\frac{1}{u^2+v^2}\left(\frac{u\mathrm{d}u + v\mathrm{d}v}{\sqrt{u^2+v^2}}\right) = -\frac{u\mathrm{d}u + v\mathrm{d}v}{(u^2+v^2)^{3/2}}$$

d) Note First that we will let T be the polar version so $T(r,\theta) = (r^k, k\theta) = (\tilde{r}, \tilde{\theta})$
First we will see for μ,

$$\begin{tikzcd} \mathrm{d}(r^k) & T^*\mu \arrow[l, mapsto] \\ \mathrm{d}\tilde{r} \arrow[u, mapsto] & \mu \arrow[l, mapsto] \arrow[u, mapsto] \end{tikzcd}$$

Thus,

$$T^*\mu = \left(k\left(\sqrt{u^2+v^2}\right)^{k-1}\right)\left(\frac{u\mathrm{d}u + v\mathrm{d}v}{\sqrt{u^2+v^2}}\right)$$

Next we will see for ω,

$$\begin{tikzcd}
\mathrm{d}(k\theta) & T^*\omega \arrow[l, mapsto] \\
\mathrm{d}\tilde{\theta} \arrow[u, mapsto] & \omega \arrow[l, mapsto] \arrow[u, mapsto]
\end{tikzcd}$$

Thus,

$$T^*\omega = k\left(\frac{u\mathrm{d}v - v\mathrm{d}u}{u^2+v^2}\right)$$

Finally we will see for λ,

$$\begin{tikzcd}
r^k\mathrm{d}(r^k)\wedge\mathrm{d}(k\theta) & T^*\lambda \arrow[l, mapsto] \\
\tilde{r}\mathrm{d}\tilde{r}\wedge\mathrm{d}\tilde{\theta} \arrow[u, mapsto] & \lambda \arrow[l, mapsto] \arrow[u, mapsto]
\end{tikzcd}$$

First note that $r^k\mathrm{d}(r^{2k-1})\wedge\mathrm{d}(k\theta) = k^2r^k\mathrm{d}r\wedge\mathrm{d}\theta$. Thus,

$$T^*\lambda = T^*(\mathrm{d}x\wedge\mathrm{d}y) = k^2\left(u^2+v^2\right)^{k-1}(\mathrm{d}u\wedge\mathrm{d}v)$$

Problem 4. The theorem used here is change of variable for differential forms.