# Homework: 3W

Monroe Stephenson
Math 113: Discrete Structures

Due September 16th, 2020

## Problem 1

Let $y$ be an arbitrary element of $A$. Then we can create the function,

$$f(E) = \begin{cases} E - \{y\} & \text{if } y \in E \\ E \cup \{y\} & \text{if } y \notin E \end{cases}$$

This takes E sets to O, as $|f(E)| = |E| - 1 = 2e - 1$ and $|f(E)| = |E| + 1 = 2e + 1$ where $e = \frac{|E|}{2}$. In fact, it also takes E sets to O. $|f(O)| = |O| - 1 = (2o + 1) - 1 = 2o$ and $|f(E)| = |O| + 1 = (2o + 1) + 1 = 2(o + 1)$ where $o = \frac{|O|-1}{2}$. We can see that

$$f \circ f(E) = f(f(E)) = f(O) = \begin{cases} O - \{y\} & \text{if } y \in O \\ O \cup \{y\} & \text{if } y \notin O \end{cases} = E$$

Thus we have showna bijection, $f : E \to O, f : O \to E$. Since they are in bijection, $|O| = |E|$. Since the cardinalities of the set are either even or odd,

$$|E| + |O| = 2|E| = 2|O| = 2^{|A|} \implies |E| = |O| = 2^{|A|-1}$$

## Problem 2

( $\implies$ ) Suppose there does not exist $g : B \to A$ such that, $f \circ g = \mathrm{id}_B$, but $f$ is surjective. If $f$ is surjective then $f(A) = B$. $g(B) \neq A$ since otherwise $f \circ g = \mathrm{id}_B$. However, we can construct a function $g$ that is surjective, i.e. $g(B) = A$, thus a contradiction. If there exists a $g : B \to A$ such that $f \circ g = \mathrm{id}_B$ then $f$ is surjective.

( $\impliedby$ )Suppose $f$ is not surjective. Take the range of $f$ to be $f(A) \subsetneq B$. Let $f \circ g : B \to f(A) \subsetneq B = \mathrm{id}_B$. However, this is clearly false as by assumption, $\mathrm{id}_B$ should have equal domain and range or $\mathrm{id}_B : B \to B$ and $\mathrm{id}_B(B) = B$, $|B| = |f(A)|$ and as by assumption, $|f(A)| < |B|$. Thus a contradiction. Thus $f(A) = B$ or surjectivity. Hence, if $f$ is surjective then there exists $g : B \to A$ such that $f \circ g = \mathrm{id}_B$.

Thus we have proved both ways, and $f$ is surjective if and only if there exists $g : B \to A$ such that $f \circ g = \mathrm{id}_B$.