PHYSICS 201: OSCILLATIONS AND WAVES
HOMEWORK DUE FRIDAY WEEK 10

Problem 1. We will take the general solution, in terms of our eigenvalues and eigenvectors,

$$\mathbf{X}(t) = \sum_{j=1}^{n} \left[F_j \cos(\sqrt{\lambda_j} t) + G_j \sin(\sqrt{\lambda_j} t) \right] \mathbf{v}^j + \mathbf{A}$$

Taking **X** at 0,

$$\mathbf{X}(0) = \sum_{j=1}^{n} [F_j] \mathbf{v}^j + \mathbf{A} = \mathbf{A} \implies F_j = 0$$

Thus we can see

$$\dot{\mathbf{X}}(t) = \sum_{j=1}^{n} [\sqrt{\lambda_j} G_j \cos(\lambda_j)] \mathbf{v}^j \implies \dot{x}_k(0) = \sqrt{\lambda_k} G_k \mathbf{v}^k$$

Problem 2. First note that

$$\mathbb{A} \mathbf{v}^j = \lambda_j \mathbf{v}^j$$

so

$$a \mathbf{v}^j_{k-1} + b \mathbf{v}^j_k + a \mathbf{v}^j_{k+1} = \lambda_j \mathbf{v}^j_k.$$

Suppose

$$\mathbf{v}^j_k = \sin\left(\frac{\pi j k}{n+1}\right)$$

then we see,

$$a \sin\left(\frac{\pi j (k-1)}{n+1}\right) + b \left(\sin \frac{\pi j k}{n+1}\right) + a \left(\sin \frac{\pi j (k+1)}{n+1}\right) = \lambda_j \sin\left(\frac{\pi j k}{n+1}\right).$$

By trigonometric identities we see,

$$\sin\left(\frac{\pi j (k \pm 1)}{n+1}\right) = \sin\left(\frac{\pi j k}{n+1}\right) \cos\left(\frac{\pi j}{n+1}\right) \pm \sin\left(\frac{\pi j}{n+1}\right) \cos\left(\frac{\pi j k}{n+1}\right)$$

thus our previous equality becomes

$$a \left(\sin\left(\frac{\pi j k}{n+1}\right) \cos\left(\frac{\pi j}{n+1}\right) - \sin\left(\frac{\pi j}{n+1}\right) \cos\left(\frac{\pi j k}{n+1}\right) \right) + b \left(\sin \frac{\pi j k}{n+1}\right)$$
$$+a \left(\sin\left(\frac{\pi j k}{n+1}\right) \cos\left(\frac{\pi j}{n+1}\right) + \sin\left(\frac{\pi j}{n+1}\right) \cos\left(\frac{\pi j k}{n+1}\right) \right) = \lambda_j \sin\left(\frac{\pi j k}{n+1}\right).$$

which reducing we see,

$$2a \left(\sin\left(\frac{\pi j k}{n+1}\right) \cos\left(\frac{\pi j}{n+1}\right) \right) + b \left(\sin \frac{\pi j k}{n+1}\right) = \lambda_j \sin\left(\frac{\pi j k}{n+1}\right)$$

$$2a \cos\left(\frac{\pi j}{n+1}\right) + b = \lambda_j$$

as desired.

Problem 3. First, we can translate this into,

$$\begin{aligned}
\ddot{\mathbf{X}}(t) &= \mathbb{Q}(\mathbf{X}(t) - \mathbf{A}) - 2b\dot{\mathbf{X}}(t) \\
&= \mathbb{V}\mathbb{L}\mathbb{V}^T(\mathbf{X}(t) - \mathbf{A}) - 2b\dot{\mathbf{X}}(t) \\
\mathbb{V}^T(\ddot{\mathbf{X}}(t)) &= \mathbb{L}(\mathbb{V}^T(\mathbf{X}(t) - \mathbf{A})) - 2b(\mathbb{V}^T\dot{\mathbf{X}}(t)) \\
\mathbb{V}^T(\ddot{\mathbf{Y}}(t)) &= \mathbb{L}(\mathbb{V}^T(\mathbf{Y}(t)) - 2b(\mathbb{V}^T\dot{\mathbf{Y}}(t)) && \text{letting } \mathbf{Y}(t) = \mathbf{X}(t) - \mathbf{A} \\
\frac{d^2}{dt^2}\mathbf{Z}(t) &= \mathbb{L}\mathbf{Z}(t) - 2b\frac{d}{dt}(\mathbf{Z}(t)) && \text{letting } \mathbf{Z}(t) = \mathbb{V}^T\mathbf{Y}(t) \\
\implies \ddot{z}_k(t) &= \lambda_j z_k(t) - 2b\dot{z}(t)
\end{aligned}$$

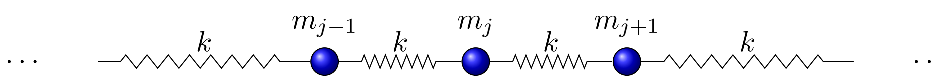
as desired.

Problem 4. If we imagine that each mass is a middle mass in the chain of masses, however taking into account the counting issue for 1 and N, we see,

$$\ddot{x}_1(t) = \kappa^2(x_N(t) - 2x_1(t) + x_2(t))$$

$$\ddot{x}_j(t) = \kappa^2(x_{j-1}(t) - 2x_j(t) + x_{j+1}(t)) \quad \text{for } j = 2 \to n-1$$

$$\ddot{x}_N(t) = \kappa^2(x_{N-1}(t) - 2x_N(t) + x_1(t))$$



Problem 5. We know that the potential must match, so

$$U = \frac{1}{2}K\Delta x^2 = \frac{1}{2}k\left(\frac{\Delta x}{2}\right)^2 + \frac{1}{2}k\left(\frac{\Delta x}{2}\right)^2$$

$$K = \frac{k}{4} + \frac{k}{4} \implies k = 2K$$

which follows the springs in series law, as

$$K = \left(\frac{1}{k} + \frac{1}{k}\right)^{-1} = \left(\frac{2}{k}\right)^{-1} = \frac{k}{2} \implies k = 2K$$

Problem 6. We can simply calculate the initial position and velocity as such,

$$x(0) = \frac{mc^2}{F_0}\sqrt{1 + 0^2} = \frac{mc^2}{F_0}$$

$$x'(t) = \frac{mc^2}{F_0}\frac{\frac{F_0^2}{m^2c^2}t}{\sqrt{1 + \left(\frac{F_0 t}{mc}\right)^2}} = \frac{F_0 t}{m\sqrt{1 + \left(\frac{F_0 t}{mc}\right)^2}} \implies x'(0) = 0$$

To determine the particle's position and velocity, we can take the limit,

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} \frac{mc^2}{F_0}\sqrt{1 + \left(\frac{F_0 t}{mc}\right)^2} = \infty$$

$$\lim_{t\to\infty} x'(t) = \lim_{t\to\infty} \frac{F_0 t}{m\sqrt{1 + \left(\frac{F_0 t}{mc}\right)^2}} = \lim_{t\to\infty} \frac{F_0}{m\sqrt{\frac{1}{t^2} + \frac{F_0^2}{m^2c^2}}} = \frac{\lim_{t\to\infty} F_0}{\lim_{t\to\infty} m\sqrt{\frac{1}{x^2} + \frac{F_0^2}{m^2c^2}}} = \frac{F_0}{m\frac{F_0}{mc}} = c.$$

By Newton's second law we see,

$$F(t) = m\ddot{x}(t) = m\frac{F_0}{m\left(1 + \left(\frac{F_0 t}{cm}\right)^2\right)^{\frac{3}{2}}} = \frac{F_0}{\left(1 + \left(\frac{F_0 t}{cm}\right)^2\right)^{\frac{3}{2}}}$$