Homework: 6.7: Part 2

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Math 202: Vector Calculus

Due November 3rd, 2020

6.7.10

a.

First note that we need to show that

$$\text{vol}(S) = \int_S 1 = 2\pi \int_K x$$

by definition of the centroid.
Let us use cylindrical coordinates such that,

$$R = \{(x, \theta, z) : x, z \in K, 0 \le \theta \le 2\pi\}$$
$$\Phi(x, \theta, z) = (x\cos\theta, x\sin\theta, z)$$

then we see,

$$\int_S 1 = \int_{\Phi(R)} 1 = \int_R x = \int_{\theta=0}^{2\pi} 1 \cdot \int_K x = 2\pi \int_K x$$

as desired.

b.

From the previous subsection, by noting $\bar{x} = b$ and $\text{area}(K) = \pi a^2$ by definition, then $\text{vol}(T_{a,b}) = 2\pi b \cdot \pi a^2 = 2\pi^2 a^2 b$

6.7.11

Define our change of variable as $\Phi(x) = rx$, noting that $\Phi'(x) = r$ with $\det(rI_n) = r^n$ then we see that,

$$\text{vol}(rK) = \int_{rK} 1 = \int_{\Phi(K)} 1 = \int_K r^n = r^n \int_K 1 = r^n \text{vol}(K) = r^n v$$

as desired.

6.7.12

a.

Since we have proved the change of scale principle, we know that for any set $K \subset \mathbb{R}^n$(i.e. $\text{vol}B_n(1) = v_n$) that has volume v then the set rK(i.e. $\text{vol}B_n(r)$) has volume $r^n v$, thus we see that $\text{vol}B_n(r) = r^n v_n$

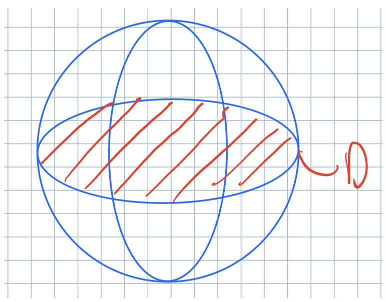
b.

For v_1, this is simply the interval $[-1, 1]$ thus v_1 takes on the value of 2. For v_2 this is from 2-dimensional geometry, or πr^2, thus $v_2 = \pi$

c.

$$\begin{aligned} B_n(1) &= \bigsqcup_{(x_1,x_2)\in D} \{(x_1, x_2)\} \times B_{n-2}\left(\sqrt{1 - x_1^2 - x_2^2}\right) \\ &= \bigsqcup_{(x_1,x_2)\in D} \{(x_1, x_2)\} \times \left\{(x_3, x_4, ..., x_n) : x_3^2 + ... + x_n^2 \leq 1 - x_1^2 - x_2^2\right\} \\ &= \bigsqcup_{(x_1,x_2)\in D} \left\{(x_1, x_2, x_3, x_4, ..., x_n) : x_1^2 + x_2^2 + x_3^2 + ... + x_n^2 \leq 1\right\} \\ &= B_n(1) \end{aligned}$$

Where the last equality holds as we can imagine unioning all of the points that lie within the ball, varying by x_1, x_2 that are within the disk D.



d.

$$\begin{aligned} v_n &= \text{vol}(B_n(1)) \\ &= \int_{B_n(1)} 1 \\ &= \int_{\{\{(x_1,x_2)\}\times B_{n-2}\left(\sqrt{1-x_1^2-x_2^2}\right):(x_1,x_2)\in D\}} 1 \\ &= \int_{(x_1,x_2)\in D} \int_{B_{n-2}\left(\sqrt{1-x_1^2-x_2^2}\right)} 1 && \text{by Fubini's Theorem} \\ &= v_{n-2} \int_{(x_1,x_2)\in D} (1 - x_1^2 - x_2^2)^{\frac{n}{2}-1} && \text{by change of scale principle} \\ &= v_{n-2} \int_{\theta=0}^{2\pi} \int_{r=0}^{1} (1 - r^2)^{\frac{n}{2}-1} \cdot r && \text{by change of variable to polar} \\ &= 2\pi v_{n-2} \left(-\frac{(1 - r^2)^{\frac{n}{2}}}{n}\right)\Bigg|_{r=0}^{1} \\ &= \frac{v_{n-2}\pi}{(n/2)} \end{aligned}$$

as desired.

e.

For the base case we see, as we showed above that $v_2 = \pi$ that our formula gives us,

$$v_2 = \frac{\pi^{2/2}}{(2/2)!} = \pi.$$

For the inductive step, suppose that v_{n-2} works, then we need to see that it works for v_n so note

$$v_{n-2} = \frac{\pi^{(n-2)/2}}{((n-2)/2)!} = \frac{\pi^{n/2-1}}{(n/2-1)!}$$

but we know from before that,

$$v_n = \frac{v_{n-2}\pi}{(n/2)} = \frac{\pi^{n/2-1}}{(n/2-1)!}\frac{\pi}{(n/2)} = \frac{\pi^{n/2}}{(n/2)!}$$

as desired.

6.7.13

a.

$$\begin{aligned}
\int_{S(R)} e^{-x^2-y^2} &= \int_{\{(x,y):0\le x\le R,0\le y\le R\}} e^{-x^2}e^{-y^2} \\
&= \int_{x=0}^{R} e^{-x^2} \cdot \int_{y=0}^{R} e^{-y^2} && \text{by Fubini's Theorem} \\
&= I(R)\cdot I(R) \\
&= I(R)^2
\end{aligned}$$

as desired.

b.

We know that by definition,

$$Q(R) \subset S(R) \subset Q(\sqrt{2}R)$$

and that $e^{-x^2-y^2}$ is positive, thus the inequality follows.

c.

Using polar coordinates we see that,

$$\Phi(Q(R)) = \{(r,\theta) : 0 \le \theta \le \frac{\pi}{2}, 0 \le r \le R\}$$

so then,

$$\int_{Q(R)} = \int_{\theta=0}^{\frac{\pi}{2}} \int_{r=0}^{R} r\cdot e^{-r^2} = \frac{\pi}{2}\left(\frac{1}{2} - \frac{e^{-R^2}}{2}\right)$$

substituting $\sqrt{2}R$ for R above gives,

$$\int_{Q(\sqrt{2}R)} = \frac{\pi}{2}\left(\frac{1}{2} - \frac{e^{-2R^2}}{2}\right)$$

and if we were to take the limit of either we see that as $R \to \infty$,

$$\lim_{R\to\infty} \frac{\pi}{2}\left(\frac{1}{2} - \frac{e^{-R^2}}{2}\right) = \frac{\pi}{4} \qquad \lim_{R\to\infty} \frac{\pi}{2}\left(\frac{1}{2} - \frac{e^{-2R^2}}{2}\right) = \frac{\pi}{4}$$

d.

As seen in **b** we know that the integral of $S(R)$ lies between the two, and we know that their integral is $\frac{\pi}{4}$, thus $S(R)$ is at that same value. Hence,

$$\sqrt{S(R)} = I = \frac{\sqrt{\pi}}{2}$$

6.7.14

a.

$$\Gamma(1) = \int_{x=0}^{\infty} x^{1-1}e^{-x}dx = \int_{x=0}^{\infty} e^{-x}dx = -e^{-x}\Big|_{x=0}^{\infty} = 0 - (-1) = 1$$

$$\Gamma(1/2) = \int_{x=0}^{\infty} x^{(1/2)-1}e^{-x}dx = \int_{x=0}^{\infty} \frac{e^{-x}}{\sqrt{x}}dx$$

let $u = \sqrt{x}$ and so $du = \frac{1}{2}x^{-1/2}dx \implies dx = 2udu$ then,

$$\Gamma(1/2) = 2\int_{u=0}^{\infty} \frac{e^{-u^2}}{u}udu = 2\int_{u=0}^{\infty} e^{-u^2} = \sqrt{\pi}$$

Finally,

$$\Gamma(s+1) = \int_{x=0}^{\infty} x^{s+1-1}e^{-x}dx = \int_{x=0}^{\infty} x^s e^{-x}dx$$

Let $u = x^s$, $v = -e^{-x}$ so then

$$\int_{x=0}^{\infty} x^s e^{-x}dx = -x^s e^{-x}\Big|_{x=0}^{\infty} - \int_{x=0}^{\infty} -se^{-x}x^{s-1}dx = 0 + s\cdot\int_{x=0}^{\infty} e^{-x}x^{s-1}dx = s\Gamma(s)$$

b.

We can see that, $\Gamma(s) = 1 = 0!$, and recursively we can see that,

$$\Gamma(1+1) = \Gamma(2) = 1\Gamma(1) = 1 = 1!$$

$$\Gamma(2+1) = \Gamma(3) = 2\Gamma(2) = 2 = 2!.$$

Thus we see the base case works plus more. Suppose this holds for s, then $\Gamma(s) = (s-1)!$, but as shown above we know,

$$\Gamma(s+1) = s\Gamma(s) = s(s-1)! = s!$$

as desired.

c.

Let us use induction. First note that $v_n = \frac{\pi^{n/2}}{\Gamma((n/2)+1)}$ and for the base case of 1 we see,

$$v_1 = \frac{\pi^{1/2}}{(1/2)\Gamma(1/2)} = \frac{\sqrt{\pi}}{(1/2)\sqrt{\pi}} = 2$$

as expected and needed. For the inductive step, suppose the formula works for $n-2$, then,

$$v_{n-2} = \frac{\pi^{(n/2)-1}}{\Gamma(n/2)}$$

From before we know,

$$v_n = \frac{\pi}{n/2} v_{n-2} = \frac{\pi}{n/2} \frac{\pi^{(n/2)-1}}{\Gamma(n/2)} = \frac{\pi^{n/2}}{\Gamma(n/2+1)}$$

as desired.

d.

First note, in order to be equivalent the factor of 2^n should equal 2^{2s-1}, thus, $n = 2s-1 \implies s = (n+1)/2$, and we see,

$$v_n = \frac{\pi^{n/2}}{\Gamma(n/2+1)} = \frac{\pi^{(2s-1)/2}}{\Gamma((2s-1)/2+1)} = \frac{\pi^{s-(1/2)}}{\Gamma(s+1/2)} = \pi^{s-(1/2)} \frac{2^{2s-1}\pi^{-1/2}\Gamma(s)}{\Gamma(2s)}$$

$$= \frac{2^{2s-1}\pi^{s-1}\Gamma(s)}{\Gamma(2s)} = \frac{2^{2((n+1)/2)-1}\pi^{((n+1)/2)-1}\Gamma((n+1)/2)}{\Gamma(2((n+1)/2))} = \frac{2^n \pi^{(n-1)/2}((n-1)/2)!}{n!}$$

as desired.