PHYSICS 201: OSCILLATIONS AND WAVES
FINAL DUE WEEK 12 FRIDAY
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Problem 1. We can see that the following are equivalent, $\kappa = \frac{k}{m}$ to

$$\kappa(-3x_1(t) + x_2(t) + x_3(t)) = \kappa(x_2(t) - x_1(t)) + \kappa(x_3(t) - x_1(t)) - \kappa(x_1(t))$$

or mass one connected to the wall, and to the other 2 masses, i.e. m_2, m_3.

$$\kappa(x_1(t) - 3x_2(t) + x_3(t) + x_4(t)) = \kappa(x_1(t) - x_2(t)) + \kappa(x_3(t) - x_2(t)) + \kappa(x_4(t) - x_2(t))$$

or mass 2 connected to all the other masses.

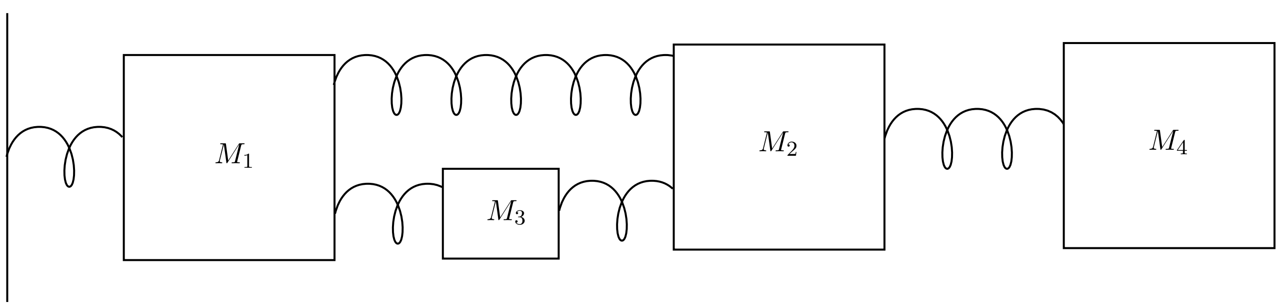
$$\kappa(x_1(t) + x_2(t) - 2x_3(t)) = \kappa(x_1(t) - x_3(t)) + \kappa(x_2(t) - x_3(t))$$

which is mass 3 connected to mass 1 and 2.

$$\kappa(x_2(t) - x_4(t))$$

which means mass 4 is connected to mass 2.

In total we can see the system as such (sizes are irrelevant since masses are equal, so assume they don't have identical densities):



Problem 2. [1]

a) We want to find the eigenvalues, but we know that the eigenvalues for an inverse matrix are $\frac{1}{\lambda_j}$ from the matrix before hand thus (noting $\mathbb{M}^{-1} = (\mathbb{A} - \mu\mathbb{I})$),

$$(\mathbb{A} - \mu\mathbb{I})\mathbf{u}^j = \phi_j\mathbf{u}^j$$

$$(\mathbb{A} - (\mu + \phi_j)\mathbb{I})\mathbf{u}^j = 0$$

and for non-zero $\mathbf{u}^j$,

$$\mathbb{A} - (\mu + \phi_j)\mathbb{I} = 0 \implies \mathbb{A} = (\mu + \phi_j)\mathbb{I}$$

which we can find the eigenvalues by taking the determinant of both sides, noting that $\det(\mathbb{A}) = \det(\mathbb{L})$ so we can take a singular eigenvalue from $\mathbb{A}$, or λ_j (it should be noted

All calculations have been done as scratch work by hand.

[1] In this problem, this page is cited for the properties of eigen-things. Mostly things covered in Math 201, but not Physics 201. `http://linear.ups.edu/html/section-PEE.html`

formally, what is happening is $\prod_{i=1}^{n} \lambda_i = \prod_{i=1}^{n}(\mu + \phi_i)$ and each index must match, so we can simply "pluck" one out), so we see,

$$\lambda_j = \mu + \phi_j \implies \phi_j = \lambda_j - \mu$$

Thus we simply need to invert ϕ_j since these are the eigenvalues for $\mathbb{M}^{-1}$. Thus our eigenvalues for $\mathbb{M}$ are

$$\psi_j = \frac{1}{\phi_j} = \frac{1}{\lambda_j - \mu}$$

For our eigenvectors, our eigenvectors must be the same as those for $\mathbb{M}^{-1}$ so,

$$(\mathbb{A} - \mu\mathbb{I})\mathbf{u}^j = (\lambda_j - \mu)\mathbf{u}^j$$

$$(\mathbb{A} - \mu\mathbb{I} - \lambda_j\mathbb{I} + \mu\mathbb{I})\mathbf{u}^j = 0$$

$$(\mathbb{A} - \lambda_j\mathbb{I})\mathbf{u}^j = 0 \implies \mathbb{A}\mathbf{u}^j = \lambda_j\mathbf{u}^j \implies \mathbf{u}_j = \mathbf{v}_j$$

Since $\mathbb{M}^{-1}$ and $\mathbb{M}$ share eigenvectors, the eigenvectors of $\mathbb{M}$ are simply $\mathbf{v}^j$.

b) First note that $\mathbb{M} = \mathbb{A}^T\mathbb{A} = \mathbb{A}^2$ since $\mathbb{A}$ is symmetric. Note also we know all of the eigenvalues of $\mathbb{A}$ are real. Finally, we know for a matrix $\mathbb{A}$ with eigenvalues λ, that the eigenvalues for $\mathbb{A}^2$ must be λ^2. Thus since we know the eigenvalues for $\mathbb{M}$ are the square of real numbers, which are positive values.

Problem 3. We can see that since Mass 1 and 3 are at equilibrium so, $x_1(0) = x_3(0) = 0$, we can see

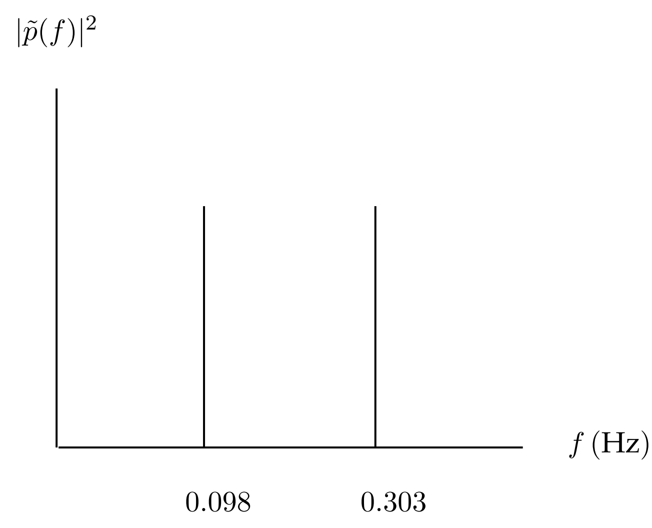
$$\vec{\mathbf{X}}(0) = \mathbf{A} + \alpha(\vec{\mathbf{v}}^1 + \vec{\mathbf{v}}^4)$$

So we know that the 1st and 4th modes are equal. We can calculate the frequencies as such,

$$\sqrt{\lambda_1}\frac{1}{f_1} = 2\pi \implies f_1 = \frac{\sqrt{\frac{1}{2}(5+\sqrt{5})}}{2\pi} \approx 0.303\text{Hz}$$

$$\sqrt{\lambda_4}\frac{1}{f_4} = 2\pi \implies f_4 = \frac{\sqrt{\frac{1}{2}(3-\sqrt{5})}}{2\pi} \approx 0.098\text{Hz}$$

We can see that they are equal heights (as the given positions of $x_2(0)$ and $x_4(0)$ in the graph match the linear combination of $\mathbf{v}^1$ and $\mathbf{v}^4$ of $2(1+\sqrt{5})$ and $2(2)$ respectively when $\alpha = 1$). Thus,



Problem 4. We can see that the general wave equation solution is

$$\phi(x,t) = f(x - vt) + g(x + vt)$$

then let

$$u(x) = \phi_0 e^{-x^2} \qquad w'(x) = 0$$

as given.
Then

$$f(x) = \frac{1}{2}\left(u(x) - \frac{1}{v}w(x)\right) = \frac{1}{2}\left(\phi_0 e^{-x^2} - \frac{C_0}{v}\right)$$
$$g(x) = \frac{1}{2}\left(u(x) + \frac{1}{v}w(x)\right) = \frac{1}{2}\left(\phi_0 e^{-x^2} + \frac{C_0}{v}\right)$$

for some C_0 as $w(x)$ must be a constant.
Thus

$$\phi(x,t) = \frac{1}{2}\left(\phi_0 e^{-(x-vt)^2} - \frac{C_0}{v} + \phi_0 e^{-(x+vt)^2} + \frac{C_0}{v}\right) = \frac{\phi_0}{2}\left(e^{-(x-vt)^2} + e^{-(x+vt)^2}\right)$$

Problem 5. Let $\kappa \equiv \sqrt{\frac{k}{m}}$ then,

$$\mathbb{Q} = -\kappa^2 \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$$

We can see that the eigenvalues are,

$$\lambda_1 = \kappa^2 \qquad \lambda_2 = 3\kappa^2$$

and the eigenvectors are

$$\mathbf{v}^1 = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \end{pmatrix} \qquad \mathbf{v}^2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Thus (from 3.94) we see,

$$\mathbf{X}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = p_1 \cos\left(\sqrt{\lambda_1}t\right)\mathbf{v}^1 + p_2 \cos\left(\sqrt{\lambda_2}t\right)\mathbf{v}^2 + \mathbf{A}$$
$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \frac{p_1}{\sqrt{2}}\cos(\kappa t)\begin{pmatrix} -1 \\ 1 \end{pmatrix} + \frac{p_2}{\sqrt{2}}\cos\left(\kappa t\sqrt{3}\right)\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 3a \end{pmatrix}$$

We can then see that

$$p_1 = -\frac{3a\sqrt{2}}{4} \qquad p_2 = -\frac{a\sqrt{2}}{4}$$

which will give us the desired initial conditions of $x_1(0) = a/2$ and $x_2(0) = 2a$.
Thus the full solution is,

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = -\frac{3a}{4}\cos(\kappa t)\begin{pmatrix} -1 \\ 1 \end{pmatrix} - \frac{a}{4}\cos\left(\kappa t\sqrt{3}\right)\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 3a \end{pmatrix}$$