MATH 202: VECTOR CALCULUS CAES 9.9, 9.10 HOMEWORK DUE TUESDAY WEEK 11

Problem 1. a) We can check to see that,

$$S^*\omega \wedge S^*\lambda = S^*(\omega \wedge \lambda).$$

We can use the first commutative diagram on pg. 453.

b) We can check to see that

$$S^*(d\omega) = d(S^*\omega).$$

We can use the second commutative diagram on pg. 453.

c) We can check this using the contravariance of the pullback so,

$$(S \circ T)^* \lambda = (T^* \circ S^*) \lambda.$$

We can use the commutative diagram in the middle of pg. 454.

Problem 2. First we can use the polar coordinate mapping,

$$\Phi: \mathbb{R}_{\leq 0} \times \mathbb{R} \to \mathbb{R}^2 \setminus \{(0,0)\}, \qquad \Phi(r,\theta) = (r\cos\theta, r\sin\theta) = (u,v)$$

then the reciprocal mapping becomes,

$$S: \mathbb{R}_{\leq 0} \times \mathbb{R} \to \mathbb{R}_{\leq 0} \times \mathbb{R}, \qquad S(r, \theta) = (1/r, -\theta) = (\tilde{r}, \tilde{\theta})$$

Thus we have the diagram for ω ,

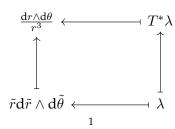
$$d(-\theta) \longleftarrow T^*\omega$$

$$d\tilde{\theta} \longleftarrow \omega$$

thus we see that since $d(-\theta) = -d\theta$ the sought-for pullback $T^*\omega$ must be in the (u,v)-form that pulls back through the polar coordinate to $-d\tilde{\theta}$, so $T^*\omega$ should be the negative of ω but with u and v in place of x and y,

$$T^*\omega = -\frac{u\mathrm{d}v - v\mathrm{d}u}{u^2 + v^2}$$

We can similarly see that for λ we have the diagram, since $\frac{1}{r}d\left(\frac{1}{r}\right) \wedge d(-\theta) = \frac{dr \wedge d\theta}{r^3}$,



thus $T^*\lambda$ must bull back through the polar coordinate mapping to $\frac{\mathrm{d}r\wedge\mathrm{d}\theta}{r^3}$. Since the area-form $\mathrm{d}u\wedge\mathrm{d}v$ pulls back to $r\mathrm{d}r\wedge\mathrm{d}\theta$ the answer is the area form divided by r^4 in (u,v)-coordinates. That is since r in (u,v)-coordinates is $\sqrt{u^2+v^2}$,

$$T^*\lambda = T^*(\mathrm{d}x \wedge \mathrm{d}y) = \frac{\mathrm{d}u \wedge \mathrm{d}v}{(u^2 + v^2)^2}$$

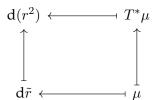
Problem 3. a) We can see, pulling μ back,

$$\Phi^*\mu = \frac{\tilde{r}\cos\tilde{\theta}(\cos\tilde{\theta}\mathrm{d}\tilde{r} - \tilde{r}\sin\tilde{\theta}\mathrm{d}\tilde{\theta}) - \tilde{r}\sin\tilde{\theta}(\sin\tilde{\theta}\mathrm{d}\tilde{r} + \tilde{r}\cos\tilde{\theta}\mathrm{d}\tilde{\theta})}{\tilde{r}} = \mathrm{d}\tilde{r}$$

thus $\int_{\gamma}\mu$ measures change in distance from origin. Or,

$$\int_{\gamma} \mu = |\gamma(1)| - |\gamma(0)|$$

b) Take the diagram,



Since $d(r^2) = 2rdr$. Thus $T^*\mu$ should be $2r = 2\sqrt{u^2 + v^2}$ times μ in (u, v)—coordinates but with u and v in place of x and y,

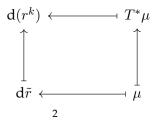
$$T^*\mu = 2\sqrt{u^2 + v^2} \left(\frac{u du + v dv}{\sqrt{u^2 + v^2}} \right) = 2(u du + v dv)$$

c) Take the diagram, noting that $d(\frac{1}{r}) = -\frac{dr}{r^2}$

thus we can see, $T^*\mu$ should be $-\frac{1}{r^2}=-\frac{1}{u^2+v^2}$ multiplied by μ , in (u,v)—coordinates,

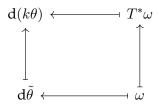
$$T^*\mu = -\frac{1}{u^2 + v^2} \left(\frac{u du + v dv}{\sqrt{u^2 + v^2}} \right) = -\frac{u du + v dv}{(u^2 + v^2)^{3/2}}$$

d) Note First that we will let T be the polar version so $T(r,\theta)=(r^k,k\theta)=(\tilde{r},\tilde{\theta})$ First we will see for μ ,



$$T^*\mu = \left(k\left(\sqrt{u^2 + v^2}\right)^{k-1}\right)\left(\frac{u\mathrm{d}u + v\mathrm{d}v}{\sqrt{u^2 + v^2}}\right)$$

Next we will see for ω ,



Thus,

$$T^*\omega = k\left(\frac{u\mathrm{d}v - v\mathrm{d}u}{u^2 + v^2}\right)$$

Finally we will see for λ ,

First note that $r^k \mathrm{d}(r^{2k-1}) \wedge \mathrm{d}(k\theta) = k^2 r^k \mathrm{d}r \wedge \mathrm{d}\theta$. Thus,

$$T^*\lambda = T^*(\mathrm{d}x \wedge \mathrm{d}y) = k^2 (u^2 + v^2)^{k-1} (\mathrm{d}u \wedge \mathrm{d}v)$$

Problem 4. The theorem used here is change of variable for differential forms.