MATH 311: COMPLEX ANALYSIS
PROBLEM SET 9
DUE WEDNESDAY WEEK 13
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1. (a). $T_{-b} \circ T_a$ maps $a \mapsto 0 \mapsto b$.

(b). Following the hint, $T \in G$ takes $a_1 \mapsto b_1$ and $a_2 \mapsto b_2$ exactly when $T_{b_2} \circ T \circ T_{a_2}^{-1}$ takes $0 \mapsto 0$ and $T_{a_2}(a_1) \mapsto T_{b_2}(b_1)$. $T_{b_2} \circ T \circ T_{a_2}^{-1}$ takes $0 \mapsto 0$ when $T = T_{-b_2} \circ T_{a_2}$, and in this case $T_{b_2} \circ T \circ T_{a_2}^{-1}$ is a rotation. Suppose it takes $T_{a_2}(a_1) \mapsto T_{b_2}(b_1)$, then for sure

$$|T_{a_2}(a_1)| = |T_{b_2}(b_1)| \Leftrightarrow \left|\frac{a_1 - a_2}{1 - a_1\bar{a}_2}\right| = \left|\frac{b_1 - b_2}{1 - b_1\bar{b}_2}\right|.$$

For the converse direction, suppose

$$\left|\frac{a_1 - a_2}{1 - a_1\bar{a}_2}\right| = \left|\frac{b_1 - b_2}{1 - b_1\bar{b}_2}\right|.$$

Then there's a unique rotation r that maps $T_{a_2}(a_1) \mapsto T_{b_2}(b_1)$, let $T = T_{-b2} \circ r \circ T_{a_2}$, then by hint T is the unique automorphism desired, because $T_{b_2} \circ T \circ T_{a_2}^{-1} = r$ is a rotation that takes $T_{a_2}(a_1) \mapsto T_{b_2}(b_1)$.

(c). Because diameters of D is orthogonal to the boundary ∂D, so if γ is some circle orthogonal to ∂D at the some point P, the tangent line of γ there is the diameter that pass through 0 and P. That means the diameter of γ is orthogonal to the diameter of D with the two intersecting at P, and in particular the center of γ is on the line orthogonal to the diameter of D through P. Then by convexity, γ do not pass through 0 unless the center of γ is ∞, in which case γ is exactly the diameter of D through P. That means the only circle through 0 and perpendicular to ∂D at some points are the diameters.

One should check that G maps ∂D to itself. For sure rotation does, so it is sufficient to check the case of T_a for $a \in D$. By brute force, see that for $z \in \partial D$,

$$\left\|\frac{z-a}{1-\bar{a}z}\right\| = \frac{z-a}{1-\bar{a}z} \cdot \frac{\bar{z}-\bar{a}}{1-a\bar{z}} = \frac{\|z\| + \|a\| - a\bar{z} - z\bar{a}}{1 + \|z\|\|a\| - a\bar{z} - z\bar{a}} = 1$$

following $\|z\| = 1$. The rest is easy, for any two points $a, b \in D$, consider $T_a \in G$ such that it maps $a \mapsto 0$, $b \mapsto T_a(b)$. Then there is a unique circle that pass through $0, T_a(b)$ and intersects ∂D orthogonally, namely the diameter through $T_a(b)$. Because the automorphisms of the unit disk are bijective conformal, i.e., preserving circles and angles, this implies that there is a unique circle passing through a, b that is orthogonal to ∂D. To specify, existence follows by taking T_{-a} of each point on diameter, and uniqueness follows that if there's another distinct line passing through a, b that is orthogonal to ∂D, then the image of this line under T_a pass through $0, T_a(b)$ and intersects ∂D orthogonally, but is not the diameter, thus contradiction to the uniqueness shown in part (b).

□

2. (a). (1). It follows 1(c) that two unique points decides a unique circle passing through them and meeting ∂D orthogonally. It is exactly to say that two distinct points in D determines a unique nE line.

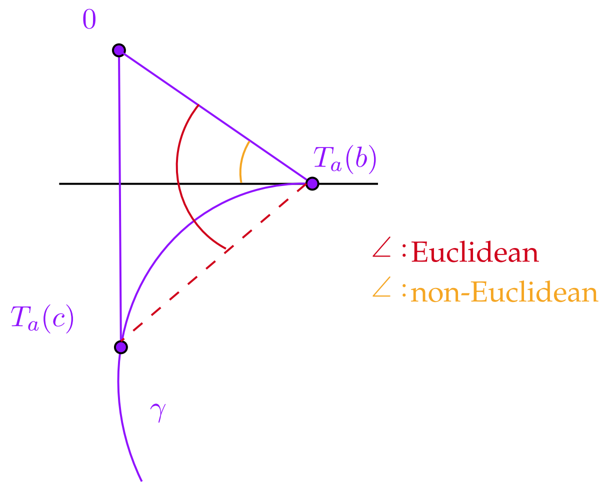

FIGURE 1.

(2). Suppose two nE lines cross at two (or more than two) points, then, there are two nE lines passing through the two intersection points. This contradicts with the previous statement that two points decide a unique nE line. So it must be true that any two nE lines either meet in a single point or in no points.

(3). nE motions preserve the angles of intersections of nE lines because nE motions are a subgroup of flr which are conformal, and nE lines are circles in the complex plane (sphere).

(b). Consider the circle that is orthogonal to ∂D at 1 and i (which specific construction can be found with easy geometric arguments). The tangent lines of the circle at its intersection with ∂D are the two diameters passing through 1 and i. In particular, this implies that the circle is always located within the first quadrant of the coordinate system (with complex sphere viewed as one point compactification $\mathbb{R}^2$ with unit vector 1 and i). Then, there are infinitely many diameter lines passing through 0 and not passing through the interior of the first quadrant, and is not the real or imaginary axis thus none of them would intersect with the circle described. This provides the desired example.

(c). First prove the following lemma

Lemma 0.1. *For non-diameter nE line c, suppose that one can divide the area of $D\backslash c$ as inside or outside based on the convexity of c in the natural Euclidean sense, then 0 is never "inside".*

Proof. Because c is not a diameter, its two intersection with ∂D could be contained in the same side of some diameter d. Now, suppose 0 is in the "inside" of area of $D\backslash c$ partitioned by c, then, c must have crossed d at least two times. Contradiction. □

Let the three vertices of the nE triangle be a, b, c, and consider the $T_a \in G$ which maps $a \mapsto 0$ and preserves the interior angles. Observe that the interior angles of the triangle $\triangle 0T_a(b)T_a(c)$ are less than or equal to the interior angles of the Euclidean triangle $0, T_a(b), T_a(c)$.

(i). Because all nE lines through origin are the diameters of D, the nE angle $\angle T_a(b)0T_a(c)$ is exactly the Euclidean angle $\angle T_a(b)0T_a(c)$.

(ii). Denote the nE line passing through $T_a(b), T_a(c)$ to be γ, then the nE angle of $\angle 0T_a(b)T_a(c)$ is the Euclidean angle between the diameter through $T_a(b)$ and the tangent line of γ at $T_a(b)$ (with correct orientation). Furthermore, by the lemma just proved, 0 is in the "outside" of area of $D\backslash c$ partitioned by c, so by convexity of c, the nE angle $\angle 0T_a(b)T_a(c)$ must be strictly smaller than the Euclidean angle $\angle 0T_a(b)T_a(c)$.

(iii). The case is the same for nE angle $\angle 0T_a(c)T_a(b)$. Therefore the sum of interior angles of $\triangle 0T_a(b)T_a(c)$ is strictly smaller than the sum of interior angles of the Euclidean one, which implies that the sum of interior angles are less than π.

0 is the infimum of the sum of interior angles. One can imagine this to be the sum of interior angle of the triangles which have all three vertices infinitely close to the boundary of the disk. When a vertex is infinitely close to the boundary, the interior angle it contributes goes to 0, because the two adjacent side of the nE triangle - the two nE lines passing through that vertex, are almost tangent to each other.

(d). The statement is proved in 1(b). To describe the circles we will first pick a point p in D and a radius r. We then can take the circle C in D whose hyperbolic distance from p is r for all points on C. When creating a circle as such, we will have a Euclidean circle in fact. However, the difference between a Euclidean circle and C is that the p tends towards the edge of D as compared to the center Euclidean circle. Specifically, because distances are greater the farther the points are away from center of the disk. Hence, the explicit difference between Euclidean and nE circles is that that they do not share the same center. □

3. The linear transformation is the cross-ratio $(z, 1, -i, i)$. First, call this cross ratio T. Now we will show that D maps to $\mathcal{H}$ with T. Note that $\mathfrak{Im}(w) = \frac{w-\overline{w}}{2i}$ so when $r = |z|$ we find

$$\mathfrak{Im}(T(z)) = \frac{1}{2}\left(\frac{1-z}{1+z} + \frac{1-\overline{z}}{1+\overline{z}}\right) = \frac{1-r^2}{|1+z|^2} > 0.$$

If $\mathfrak{Im}(T(z)) > 0$, this forces $r < 1$. Moreover, noting $w = u + iv$, we can see that

$$|T^{-1}(w)|^2 = \left|\frac{i-w}{i+w}\right|^2 = \frac{u^2 + (1-v)^2}{u^2 + (1+v)^2} < 1$$

which tells us that $v > 0$. Hence, we find that T maps D to $\mathcal{H}$ since $\mathfrak{Im}(T(z)) > 0$ if and only if $|T^{-1}(w)| < 1$. Because lfr is conformal, and the nE lines in unit disk model are "circles" perpendicular to the boundary of the unit disk, therefore after the transformation the nE lines are still circles perpendicular to the boundary, although this time the boundary is the real axis. These are the Euclidean circles and lines perpendicular to the real axis. □