Homework: 4.8

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Math 202: Vector Calculus

Due October 9th, 2020

Note, small changes were made after Friday's class

4.8.2

First we will find the unit vector d.

$$\frac{(3,1,5)-(1,2,3)}{|(3,1,5)-(1,2,3)|} = \frac{1}{3}(2,-1,2)$$

Now we will find $D_d g(a,b,c)$,

$$D_d g(a,b,c) = bcd_x + acd_y + abd_z$$

Evaluated at at $(1,2,3)$, we see,

$$D_d g(1,2,3) = 2\cdot 3\cdot (\frac{2}{3},0,0) + 1\cdot 3\cdot (0,-\frac{1}{3},0) + 1\cdot 2\cdot (0,0,\frac{2}{3}) = (4,-1,\frac{4}{3})$$

Finding the magnitude we see,

$$\sqrt{4^2+(-1)^2+\left(\frac{4}{3}\right)^2} = \frac{13}{3}$$

4.8.6

First note that the set does contain $\left(1,2,\frac{1}{3}\right)$. We will find the gradient at the point, then since it is a normal line to the surface, we can find the tangent plane.

$$Df_{(a,b,c)}(h,k,j) = (2a+3c)h + (4b)k + (3a)j$$

Thus the gradient here would be

$$\nabla f\left(1,2,\frac{1}{3}\right) = (3,8,3)$$

Then using the tangent plane equation we get,

$$3(x-1)+8(y-2)+3\left(z-\frac{1}{3}\right) = 3x+8y+3z-20 \implies 3x+8y+3z=20$$

4.8.9

a.

Note that the given equation can become,

$$z'(t)e^{-\alpha z(t)} = A\alpha.$$

Integrating both sides we see,

$$\begin{aligned}
\int_{\tau=0}^{t} z'(\tau)e^{-\alpha z(\tau)}d\tau &= \int_{\tau=0}^{t} A\alpha d\tau \\
-\frac{e^{-(t)}}{\alpha} + \frac{e^{-\alpha z(0)}}{\alpha} &= A\alpha t \\
\frac{1}{\alpha}(1 - e^{-\alpha z(t)}) &= A\alpha t \\
z(t) &= -\frac{\ln(1 - A\alpha^2 t)}{\alpha}
\end{aligned}$$

b.

We will first take the gradient,

$$\nabla f(x,y) = \begin{bmatrix} 2e^{2x} & 4e^{y} \end{bmatrix}.$$

We can see that our initial conditions are,

$$\begin{aligned}
x'(t) &= 2e^{2x(t)} \qquad & x(0) = 0 \\
y'(t) &= 4e^{y(t)} \qquad & y(0) = 0
\end{aligned}$$

Thus using part **a** we can solve for these differential equations. For $x(t)$ let $A = 1, \alpha = 2$, and for $y(t)$ let $A = 4, \alpha = 1$, thus we see

$$(x(t), y(t)) = \left(-\frac{\ln(1-4t)}{2}, -\ln(1-4t)\right)$$

Note that

$$(x(t), y(t)) = -\ln(1-4t)\left(\frac{1}{2}, 1\right)$$

thus the bug follows the $y = 2x$ path for $0 \le t < \frac{1}{4}$, and the bug goes to infinity during that time.

4.8.9

I will use Mathematica for the plots, as my drawings are rather miserable. (but the green is me for the integral curves so it's not great)

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ContourPlot[x^2 + y, {x, -5, 5}, {y, -5, 5}, PlotLegends → Automatic]
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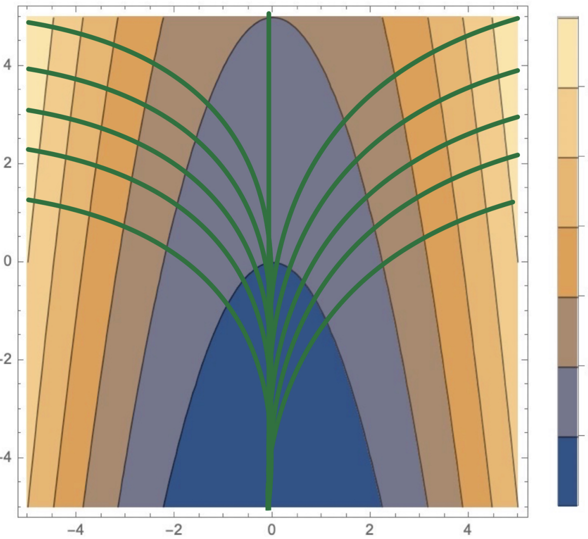
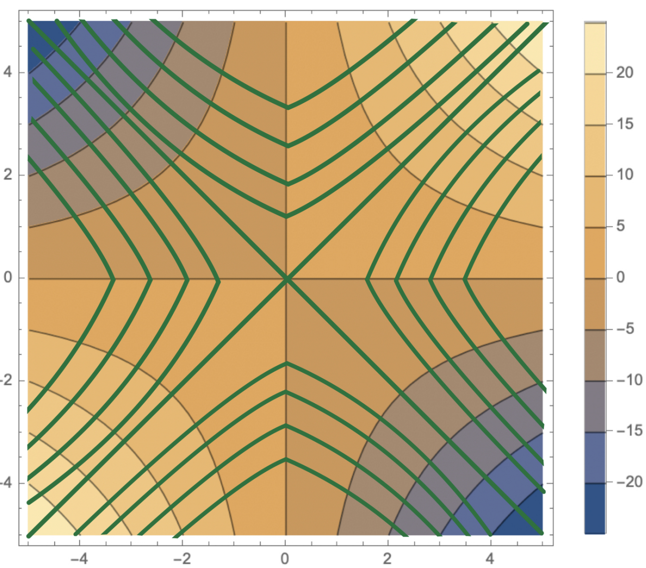



```
ContourPlot[x y, {x, -5, 5}, {y, -5, 5}, PlotLegends → Automatic]
```



For the first, we see the gradient to be,

$$\nabla f = [2x \;\; 1]$$

Thus our differential equations are,

$$x'(t) = 2x(t) \qquad x(0) = a$$

$$y'(t) = 1 \qquad y(0) = b$$

Solving we can see that,

$$x(t) = ae^{2t} \qquad y(t) = t + b.$$

Thus,

$$x = ae^{2(y-b)}$$

Note the general form with initial conditions are:

$$\gamma'(t) = (\nabla f)(\gamma(t)) \qquad \gamma(0) = \vec{a}$$

Let $f(x, y) = xy$ and $\gamma(t) = (x(t), y(t))$. We can see the gradient to be

$$\nabla f = [y \;\; x]$$

Thus our differential equations become,

$$x'(t) = y(t) \qquad x(0) = a$$

$$y'(t) = x(t) \qquad y(0) = b$$

By adding the two and subtracting the two we see,

$$(x + y)'(t) = (x + y)(t) \qquad (x + y)(0) = a + b$$

$$(x - y)'(t) = -(x - y)(t) \qquad (x - y)(0) = a - b$$

Solving these two equations we see,

$$(x + y)(t) = (a + b)e^t$$

$$(x - y)(t) = (a - b)e^{-t}$$

Since these are linear we can add an subtract again, and we see,

$$(x(t), y(t)) = \left(\frac{1}{2}\left((a + b)e^t + (a - b)e^{-t}\right), \frac{1}{2}\left((a + b)e^t - (a - b)e^{-t}\right)\right)$$

Thus, through a little algebra,

$$x^2 - y^2 = a^2 - b^2$$