HYPERPLANE ARRANGEMENTS: BIGRAPHICAL ARRANGEMENTS

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1. Introduction

In this paper we build the theory behind hyperplane arrangements and their connection to parking functions. We culminate the paper with the Corollary 29, the sliding conjecture which states gives a bijection between the Pak-Stanley labels of a bigraphical arrangement $\Sigma_G(A)$ with the parking functions G°.

2. Basic Background in Hyperplane Arrangements

In this section, we rely on Stanley [6] for the definitions.

Let V be our vector space isomorphic to k^n for some field k.

Definition 1 (Linear Hyperplane)**.** A linear hyperplane is an $(n-1)$-dimensional subspace W of V such that for some some nonzero $w \in V$,

$$W = \{v \in V \mid v \cdot w = 0\}.$$

Definition 2 (Affine Hyperplane)**.** An affine hyperplane, A, is a translate of a linear hyperplane, such that for $a \in \mathsf{k}$

$$A = \{v \in V \mid v \cdot w = a\}.$$

Definition 3 (Finite Hyperplane Arrangements)**.** A finite hyperplane arrangement, $\mathcal{A}$ is a finite set of affine hyperplanes within some n dimensional vector space, $V \cong \mathsf{k}^n$ where k is some field.

Definition 4 (Defining Polynomial of $\mathcal{A}$)**.** Let $f_i(\mathbf{x}) = a_i$ define each hyperplane in $\mathcal{A}$, where $\mathbf{x} = (x_1, \ldots, x_n)$, then the defining polynomial of $\mathcal{A}$ is

$$\mathcal{D}_{\mathcal{A}}(\mathbf{x}) = \prod_{n=1}^{m} (f_n(\mathbf{x}) - a_n)$$

Note, when we state the dimension of $\mathcal{A}$, we refer to the dimension of the ambient vector space, V. On the other hand, the rank of $\mathcal{A}$ is the dimension of the space spanned by the normal vectors to the hyperplanes in $\mathcal{A}$. A hyperplane arrangement is essential if the rank equals the dimension.

Definition 5 (Region of an Arrangement)**.** A region, X, of $\mathcal{A}$ is a connected component of the complement of the hyperplanes,

$$X = \mathsf{k}^n \setminus \left(\bigcup_{H \in \mathcal{A}} H \right).$$

We then define $\mathcal{R}(\mathcal{A})$ to be the set of regions, and

$$r(\mathcal{A}) = |\mathcal{R}(\mathcal{A})|.$$

Date: December 15, 2021.

3. Background on the Shi Arrangement

In this section, we heavily rely on Fishel [2] for definitions, lemmas, and theorems.

Definition 6 (Root System)**.** Let V bs a finite dimensional real vector space with a fixed inner product $\langle \quad , \quad \rangle$, then the root system, denoted by Δ, is a finite set of vectors in V such that $\Delta \cap \mathbb{R}\alpha = \{\alpha, -\alpha\}$ for all $\alpha \in \Delta$ and $s_\alpha \Delta = \Delta$ for all $\alpha \in \Delta$ where s_α is the reflection around the hyperplane H defined where α is the normal vector to H.

Let Δ^+ be the positive roots of Δ, then $\Delta = \Delta^+ \cup -\Delta^+$. Also let Π be the simple roots which form a basis for the $\mathbb{R}-$ span of Δ. Formerly mentioned reflections of $S = \{s_\alpha\}_{\alpha \in \Pi}$ will generate a finite reflection group G. The rank of the system and of G is the dimension of the space spanned by Δ. Coxeter groups generalize the finite reflection groups. If G is a group with the set of generators $S \subseteq G$, then define the order of the element st for $s, t \in S$ as m_{st}, with the special case that if there is no relation then $m_{st} = \infty$.

If G has a presentation such that $m_{ss} = 1$ and for $s, t \in S$, $s \neq t$, $1 < m_{st} \leq \infty$, then G is a Coxeter group. In the case that $m_{st} \in \{2, 3, 4, 6\}$ then the Coxeter group is crystallographic and in fact is finite then it is also a Weyl group. The product of all elements in S, in any order, is a Coxeter element, and all these elements are conjugate and their order is the Coxeter number of G.

We define

$$s_\alpha = v - 2\frac{\langle v|\alpha\rangle}{\langle \alpha|\alpha\rangle}\alpha$$

and the affine reflection

$$s_{\alpha,k} = v - 2\frac{\langle v|\alpha\rangle - k}{\langle \alpha|\alpha\rangle}\alpha$$

for any $v \in V$ and $k \in \mathbb{R}$. The affine Weyl group is the group generated by all the affine reflections for $\alpha \in \Delta$ and $k \in \mathbb{Z}$. Note this is a Coxeter group.

For A_{n-1}, the vector space V is $\{(a_1, \ldots, a_n) \in \mathbb{R}^n \mid a_1 + hdots + a_n = 0\}$. Let the standard basis for $\mathbb{R}^n$ be $\{e_1, \ldots, e_n\}$, where $\langle \quad , \quad \rangle$ is the bilinear form that makes this an orthonormal basis. Then the set of roots is $\Delta\{e_i - e_j \mid i \neq j\}$ and a root is positive if $\alpha \in \Delta^+ = \{e_i - e_j \mid i < j\}$. The simple roots are $\{\alpha_1, \ldots, \alpha_{n-1}\}$ where $\alpha_i = e_i - e_{i+1}$, then Π is a basis of V.

We investigate the affine symmetric group, $\widehat{\mathfrak{S}}_n$. Defined as the set of permutations σ of $\mathbb{Z}$ such that

$$\sigma(i+n) = \sigma(i) + n \text{ for all } i \in \mathbb{Z}$$
$$\sum_{i=1}^{n} \sigma(i) = \binom{n}{2}.$$

This is a Coxeter group, for all i such that $0 \leq i < n$, s_i will correspond to

$$t \mapsto \begin{cases} t & \text{if } t \mod n \neq i \text{ and } t \mod n \neq i+1 \\ t-1 & \text{if } t \mod n = i+1 \\ t+1 & \text{if } t \mod n = i \end{cases}$$

then the set of reflections S is $\{s_1, \ldots, s_{n-1}, s_0\}$ and $\widehat{\mathfrak{S}}_n = \langle s_1, \ldots, s_{n-1}, s_0\rangle$, so $\mathfrak{S}_n \subseteq \widehat{\mathfrak{S}}_n$. Note, s_i is the simple transposition $(i, i+1)$.

Now, we may connect the above theory to hyperplanes. Note, our vector space V is the span of some root system Δ with a fixed inner product $\langle \quad , \quad \rangle$ where G_Δ is invariant. We also denote the closure of a region R as $\overline{R}$ which will be a convex polyhedron. A face of a hyperplane arrangement is a nonempty set of the form $\overline{R} \cap x$ where x is the intersection of hyperplanes in the arrangement. The dimension of a face is the dimension of its affine span. Now, a wall H of a region R is a hyperplane such that $\dim(H \cap R) = \dim(H)$. Finally, $r(\mathcal{A})$ and $b(\mathcal{A})$ will be the number of regions as well as the number of bounded regions respectively of the arrangement.

The roots, plus the integers, will define a system of affine hyperplanes

$$H_{\alpha,k} = \{v \in V \mid \langle v, \alpha\rangle = k\}$$

noting that $H_{-\alpha,-k} = H_{\alpha,k}$. In type A, we will write $x_i - x_j = k$ instead of $H_{\alpha_i+\ldots+\alpha_{j-1},k}$.

Definition 7 (Coxeter Arrangement)**.** The Coxeter arrangement (or the braid arrangement) is

$$\mathrm{Cox}_\Delta = \{H_{\alpha,0} \mid \alpha \in \Delta^+\}.$$

In the Coxeter arrangement, the regions are called chambers and each chamber corresponds to an element of $G = G(\Delta)$. The dominant chamber of V is $\bigcap_{i=1}^{n-1} H^+_{\alpha_i,0}$ where $H^+_{\alpha_i,0}$ is the half space defined as

$$H^+_{\alpha_i,0} = \{v \in V \mid \langle v, \alpha\rangle \geq k\}$$

which will be the identity of G.

Definition 8 (Affine Coxeter Arrangement)**.** The affine Coxeter arrangement is the integer translates of the hyperplanes in Cox_Δ, or $\{H_{\alpha,k}\}_{\alpha\in\Delta^+,k\in\mathbb{Z}}$. In this case, each region is an alcove and the fundamental alcove is

$$A_0 = H^-_{\theta,1} \cap \bigcap_{\alpha\in\Pi} H^+_{\alpha,0}$$

where θ is the highest root. Finally, the dominant alcove is the one contained in the dominant chamber.

We also can consider the m-Catalan arrangement

$$\mathrm{Cat}^m_\Delta = \{H_{\alpha,r} \mid \alpha \in \Delta, 0 \leq r \leq m\}.$$

Suppose W is the affine Weyl group and V the vector space spanned by its roots. W acts on V by affine linear transformation, and acts freely and transitively on the set of alcoves. For affine type A_{n-1}, then s_i will reflect over $H_{\alpha_i,0}$ for $0 < i < n$ and so s_0 will reflect over $H_{\theta,1}$. Each alcove can be identified with the unique $w \in W$ such that $A = A_0 w$. To be more specific for type A_{n-1}, the action on V is such,

$$\begin{aligned} s_i(a_1, \ldots, a_i, a_{i+1}, \ldots, a_n) &= (a_1, \ldots, a_{i+1}, a_i, \ldots, a_n) && \text{for } i \neq 0, \text{ and} \\ s_0(a_1, \ldots, a_n) &= (a_n + 1, a_2, \ldots, a_{n-1}, a_1 - 1). \end{aligned}$$

and we can see $\mathfrak{S}_n$ preserves the inner product while $\widehat{\mathfrak{S}}_n$ does not.

For an alcove A, if we have a positive root α, then there is a unique integer $k = k_\alpha(A)$ such that $k < \langle \alpha, x\rangle < k + 1$ for all $x \in A$, then let $K(A) = \{k_\alpha(A)\}_{\alpha\in\Delta^+}$ denote the set of coordinates for A indexed by positive roots. Shi described the sets of integers which can come from $K(A)$ for some alcove A. If we assume that our root system is an irreducible crystallographic one, then a collection of integers indexed by Δ^+ will correspond to an alcove if and only if $k_\alpha + k_\beta \leq k_{\alpha+\beta} \leq k_\alpha + k_\beta + 1$ for all $\alpha, \beta \in \Delta^+$ such that $\alpha + \beta \in \Delta^+$.

3.1. **Kazhdan-Lisztig Cells.** Most important in this subsection is that Kazhdan and Lustig defined an equivalence relation on the elements of Coxeter group.

Definition 9. Let G be our Coxeter group and S be the set of simple reflections. Then suppose $\mathcal{L}$ is the ring of Laurent polynomials in the indeterminate $q^{1/2}$ with integer coefficients, i.e. $\mathcal{L} = \mathbb{Z}[q^{1/2}, q^{-1/2}]$. Then the Hecke algebra $\mathfrak{h} = \mathfrak{h}(W, S)$ is a free module over $\mathcal{L}$ with a basis T_g where each $g \in G$. The multiplication is defined as

$$T_g T_{g'} = T_{gg'} \quad \text{if } \ell(gg') = \ell(g) + \ell(g')$$

$$(T_s+1)(T_s-q)=0 \quad \text{if } s\in S.$$

where $\ell(g)$ is the length of g.

Involution on $\mathcal{L}$ is defined as $\overline{q^{1/2}}=q^{-1/2}$ and will extend to an involution of the ring $\mathfrak{h}$:

$$\overline{\sum a_g T_g}=\sum \overline{a_g}T_g^{-1}.$$

Below, we will posit the existence of $C_g\in\mathfrak{h}$ for each $g\in G$ and define $P_{y,g}$ where $y\in G$. This order is the Bruhat order on G.

Theorem 10. *For all $g\in G$, there is a unique element $C_g\in\mathfrak{h}$ such that*

$$\overline{C_g}=C_g$$

$$C_g=\sum_{y\leq g}(-1)^{\ell(y)+\ell(g)}q^{\ell(g)/2-\ell(y)}\overline{P_{y,g}}T_y$$

where $P_{y,x}\in\mathcal{L}$ is polynomial in q of at most $\frac{1}{2}(\ell(g)-\ell(y)-1)$ for $y<g$ and $P_{g,g}=1$

From these polynomials we can define a graph and from there cells.

For $g,h\in G$, define $\mu(g,h)$ to be the coefficent of $q^{\frac{1}{2}(\ell(h)-\ell(g)-1)}$ in $P_{g,h}(q)$ if $g<h$ and $\frac{1}{2}(\ell(h)-\ell(g)-1)$ is an integer, otherwise $\mu(g,h)=0$. If $L(g)$ be the set of left descents of g, i.e. $L(g)=\{s\in S\mid sg<g\}$. The directed labeled graph $\tilde{\Gamma}^L_{(G,S)}$ with vertices $x\in G$ and edges $x\xrightarrow{(s,\mu)}y\in E$. Then there are two types of edges in E,

(1) $g,h\in G$, $g\neq h$ either $\mu(g,h)\neq 0$ or $\mu(h,g)\neq 0$ and $s\in L(g)\setminus L(h)$. Then let μ either be $\mu(g,h)$ or $\mu(h,g)$, whichever is not 0.
(2) Or we could have a loop at g. Labeled by $s\in S$ and

$$\mu=\begin{cases}1 & \text{if } s\notin L(g)\\ -1 & \text{if } s\in L(g)\end{cases}$$

The graph $\Gamma^R_{(G,S)}$ is defined symmetrically, the right descents of g: $R(g)=\{s\in S\mid gs<g\}$. The graph $\Gamma^{LR}_{(G,S)}$ is the superposition of both $\Gamma^R_{(G,S)}$ and $\Gamma^L_{(G,S)}$. The cells can then be described in graph theoretic terms. A direct graph is strongly connected if there is a direct path between all pairs of vertices. A strongly connected component of a directed graph is a maximal strongly connected subgraph. Finally, the left cells are the strongly connected components of $\Gamma^L_{(G,S)}$, the right cells are the strongly connected components of $\Gamma^R_{(G,S)}$, and the two-sided cells are the strongly connected components of $\Gamma^{LR}_{(G,S)}$.

Shi studied cells on the affine Weyl groups of type A. Lusztig defined a map σ from $\widehat{\mathfrak{S}}_n$ to the partitions of n. From there he conjectured for any λ partition of n, $\sigma^{-1}(\lambda)$ is a two sided cell. Moreover, he conjectured a formula for the number of left of right cells that make up the two sided cell $\sigma^{-1}(\lambda)$.

We now define σ. Let $w\in\widehat{\mathfrak{S}}_n$ and define $d_k=d_k(w)$ to be the maximum size of a subset of $\mathbb{Z}$ whose elements are non-congruent to each other modulo n and which is a disjoint union of k subsets each of which has its natural order reversed by w. Then, the partition λ is given by $(d_1,d_2-d_1,\ldots,d_n-d_{n-1})$.

Shi actually proved Lusztig's conjectures. He used the correspondence of $\widehat{\mathfrak{S}}_n$ with alcoves to describes the cells of affine type A. He showed that the two sided-cells correspond to connected sets of alcoves, one set of alcoves for each partition λ of n. A two-sided cell is a disjoint union of left-cells. In each of the two-sided cell corresponding to the partition λ, there is one left-cell for each tabloid of shape λ.

The Shi arrangement was not initially defined in terms of hyperplanes. First, he defined rank n sign types as triangular arrays $X = (x_{ij})_{1\leq i<j\leq n}$ with entries $\{+, -, \bigcirc\}$. Admissible sign types correspond to the regions of his arrangement. Shi defined them as the sign types which satisfy the condition, for all $1 \leq i < t < j \leq n$, the triple (x_{ij}, x_{it}, x_{tj}) is a member of the set G_A of admissible sign types of rank 3 (A_2). G_A is the set

$$\{(+++), (++\bigcirc), (\bigcirc++), (++-), (-++), (\bigcirc+\bigcirc), (\bigcirc\bigcirc\bigcirc), (+\bigcirc-)$$
$$(-\bigcirc+), (\bigcirc\bigcirc-), (-\bigcirc\bigcirc), (---), (+--), (--+), (--\bigcirc), (\bigcirc--)\}.$$

Note, if we order the symbols with $- < 0 < +$, then G_A can be seen as the rank 3 sign types where $x_{12} \leq x_{13} \leq x_{23}$ or $x_{23} \leq x_{13} \leq x_{12}$ with $x_{13} = +$, $x_{12} = x_{23} = 0$. We can also see $|G_A| = (n+1)^{n-1}$ for $n = 3$.

Shi then connected the admissible sign types to geometry using the $K(A)$ coordinates and the map ζ. If K is the set of coordinates for an alcove A, then define the sign type $X = \zeta(A)$ as

$$x_{ij} = \begin{cases} + & \text{if } k_{ij} > 0 \\ \bigcirc & \text{if } k_{ij} = 0 \\ - & \text{if } k_{ij} < 0 \end{cases}.$$

Shi then found the hyperplanes so the regions defined by them were made of the alcoves under with the same image under ζ. Shi region and admissible sign type are interchangeable terms.

Shi then showed that left-cells for affine type A are disjoint unions of admissible sign types. Later extending the definition of admissible sign types, and generalizing the Shi arrangement, we arrive at the the desired definition.

Definition 11 (Shi Arrangement)**.** For any irreducible, crystallographic root system Δ, the Shi arrangement

$$\mathrm{Shi}_\Delta = \{H_{\alpha,k} \mid \alpha \in \Delta^+, 0 \leq k \leq 1\}.$$

A dominant region of the Shi arrangement is the connected component of the hyperplane arrangement complement

$$V \setminus \bigcup_{H \in \mathrm{Shi}_\Delta} H$$

that is contained in the dominant chamber.

4. Parking Functions and Partial Orientations

In this section, we rely on Postnikov and Shapiro [5] for definitions, lemmas, and theorems as well as Benson, Chakrabarty, and Tetali [1] and Hopkins and Perkinson [3].

We will now define the Shi arrangement for our purposes.

Definition 12 (Shi Arrangement)**.** The Shi arrangement is defined as

$$\mathrm{Shi}(n) = \{x_i - x_j = 0 \text{ or } x_i - x_j = 1 \mid 1 \leq i < j \leq n\}$$

where the ambient vector space is $\mathbb{R}^n$

Theorem 13. *For all $n \in \mathbb{N}$,*

$$r(\mathit{Shi}(n)) = (n+1)^{n-1}.$$

More interestingly though, is that $\kappa(K_{n+1}) = (n+1)^{n-1}$, so we want a bijection between the regions of $\mathrm{Shi}(n)$ and the spanning trees of K_{n+1}. This was found by doing a procedure called the "Pak-Stanley labeling", which will connect the Shi arrangement to parking functions (where we know of connections to spanning trees). In order to understand this procedure, we must define parking functions.

Definition 14 (Parking Function)**.** A parking function of length n is a sequence, $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{N}^n$ such that its weakly increasing rearrangement, $x_{\alpha_1} \leq \dots \leq x_{\alpha_n}$ satisfies $(x_{\alpha_1}, \dots x_{\alpha_n}) \leq (0, 1, \dots, n-1)$.

Definition 15 (Pak-Stanley Labeling)**.** We begin by labeling the regions, first with the center region to be $(0, 0, \dots, 0)$. For an adjacent region, if we cross a hyperplane of the form $x_i - x_j = 0$, then we will increase the j-th coordinate of the vector by 1. If we cross a hyperplane of the form $x_i - x_j = 1$, then we will increase the i-th coordinate by 1.

Consider now, the sandpile graph, where $G = (V, E)$ and v_0 is the sink vertex, then $\tilde{V} = V \setminus \{v_0\}$. Now we can define a $G-$parking function:

Definition 16 ($G-$parking Function)**.** A $G-$parking function with respect to v_0 is

$$c = \sum_{i=1}^{n} c_i v_i \in \mathbb{Z}\tilde{V}$$

such that for all $U \subseteq \tilde{V}$, there is a vertex $v_i \in U$ such that $0 \leq c_i < d_U(v_i)$ where we define $d_U(v_i)$ as, (for $u \in U$),

$$d_U(u) = |\{\{u, v\} \in E \mid v \in V \setminus U\}|$$

Similar to the Shi arrangement, we define the $G-$Shi Arrangement,

Definition 17 (G-Shi Arrangement)**.** Let $G = (V, E)$ be a simple finite graph with $V = \{v_1, \dots, v_n\}$, then we define the $G-$Shi arrangement

$$\text{Shi}(G) = \{x_i - x_j = 0 \text{ or } x_i - x_j = 1 \mid \{v_i, v_j\} \in E \text{ such that } i < j\}.$$

In light of this definition, Duval, Klivans, and Martin proposed that for $G^\circ = (V \cup \{v_0\}, E \cup \{\{v_0, v_i\} \mid \text{for all } v_i \in E\})$ (where v_0 is the sink), there is a bijection between the Pak-Stanley labels of $\text{Shi}(G)$ and the parking functions of G°.

To show this bijection, we need a few definitions.

Definition 18 (Partial Orientation)**.** On a graph $G = (V, E)$, a partial orientation is when some edges become directed. Let $\mathcal{O} \subseteq V^2$ denote the oriented edge set. We demand that the partial orientations label $\text{Shi}(G)$ in such a way that $\mathcal{O} \to \text{indeg}(\mathcal{O})$ will recover the Pak-Stanley labels.

Note, that a partial orientation is called Shi-admissible if $\mathcal{O}$ is a label of a region of $\text{Shi}(G)$.

One class of partial orientations that is Shi-admissible is acyclic orientations. These acyclic orientations are in bijection with maximal parking functions of G° with respect to the partial order on $\mathbb{Z}V$.

Note, an orientation is Shi-admissible if and only if every potential cycle of the orientation has a positive ν_{Shi} score.

Definition 19 (Potential Cycle)**.** In G a potential cycle, C, is an oriented cycle of G such that if $\{v_i, v_j\} \in C$, then $\{v_j, v_i\} \notin \mathcal{O}$. Now we define the score,

$$\nu_{\text{Shi}}(C) = |\{\{v_i, v_j\} \in C, i < j \mid \{v_i, v_j\} \text{ is blank in } \mathcal{O}\}| - |\{\{v_j, v_i\}, i < j \mid \{v_j, v_i\} \text{ is oriented in } \mathcal{O}\}|.$$

Definition 20 (Bigraphical Arrangement)**.** For each $\{v_i, v_j\} \in E$, choose a_{ij}, $a_{ji} \in \mathbb{R}$ such that there exists some $x \in .\mathbb{R}^n$ with $x_i - x_j < a_{ij}$ and $x_j - x_i < a_{ji}$ for all i, j. Then $A = \{a_{ij}\}$ is called the parameter list and the bigraphical arrangement $\Sigma_G(A)$ is the set of $2|E|$ hyperplanes

$$\Sigma_G(A) = \{x_i - x_j - a_{ij} \mid \{v_i, v_j\} \in E\}$$

Definition 21 (G-Semiorder Arrangement)**.** A $G-$semiorder arrangement is a set of $2|E|$ hyperplanes such that

$$\text{Semi}(G) = \{x_i - x_j = 1 \mid \{v_i, v_j\} \in E\}$$

noting that this is not dependent on the labeling of the vertices.

An orientation is Semi-admissible if it appears as a label of Semi(G).

5. Bigraphical Arrangements

In this section, we rely on Hopkins and Perkinson [4] for definitions, lemmas, and theorems.

Theorem 22. *The regions of $\Sigma_G(A)$ is in bijection with A-admissible partial orientations of G.*

Proof. We follow Theorem 8 of [3] and Theorem 14 of [4] (numbered in outline), and also see Example 15 of [4] for a easy demo.

Given any partial order O, we want to find the correspondence between O and the area R of regions $\Sigma_G(A)$(with regards to graph G and rules of labelling A, so this generalizes beyond Shi-arrangement $\Sigma_G(Shi)$) defined by the rules: for $\mathbf{x} \in R$,

(1) $x_i - x_j < a_{ij}$ and $x_j - x_i < a_{ji}$ if $(v_i, v_j) \notin O$, i.e., a blank edge implies a region being bounded in between the two hyperplanes corresponding to edge (v_i, v_j).
(2) $x_i - x_j < -a_{ji} \Leftrightarrow x_j - x_i > a_{ij}$ if $(v_i, v_j) \in O$.

Notice that by definition $(u, v) \in O$ implies $(v, u) \notin O$, so at least this definition itself is sound. We also want to encode the inequalities above by defining

(1) matrix $A \in M_{k \times n}$ (where $k = |O|$, $n = |V|$) with rows $r_1, ..., r_k$ as follows:
    (a) for every $e_l = (v_i, v_j) \in O$, let r_l be the vector having 1 in the ith entry and -1 in the jth entry.
    (b) for every blank edge $(v_{i'}, v_{j'}) \in O$, define two row in A; one with 1 in the ith entry and -1 in the jth entry, and the other the opposite.
(2) vector $\mathbf{b} \in \mathbb{R}^k$ such that if $e_l \in O$ is (v_i, v_j), then $b_l = -a_{ji}$.

So any point $\mathbf{x} \in R$ by definition satisfies $A\mathbf{x} < \mathbf{b}$.

To show the forward direction, given O it is sufficient to show that R defined this way isn't empty. The trick is to use Farkas' lemma included below and show the equivalence condition: there doesn't exist a row vector $\mathbf{y} \in \mathbb{R}^k$ satisfying (Perkinson's version of the lemma is a little bit different from the wiki version, I included the wiki version but here I'll follow Perkinson)

$$y_i \geq 0 \text{ for every i}, \;\; \mathbf{y} \neq 0, \;\; \mathbf{y}A = 0, \;\; \mathbf{y} \cdot \mathbf{b} \leq 0.$$

Consider any vector $\mathbf{y}$ that satisfy $\mathbf{y}A = 0$. Wlog, say that $y_{l_1} = 1$, and $e_{l_1} = (v_i, v_j) \in O$, then r_{l_1} has 1 in i'th entry and -1 in j'th entry. Because we want

$$\mathbf{y}A = y_1 r_1 + \cdots + y_k r_k = 0,$$

it implies that there must be some $y_{l_2} \neq 0$ such that r_{l_2} have a 1 in the j'th entry. Because O is a partial ordering $e_{l_1} \neq e_{l_2}$, so then we are set out on a journey to find yet another vertex and edge. But the graph is finite, so at some point there's a repeat, and we have acquired a potential cycle.

The interpretation of this procedure is simple, on graph G think of every 1 you marked in the previous procedure as the tail of an edge, and every -1 as the head of an edge. Edges in O is allowed traversed in one way, and blank edges is allowed to be traversed in both ways. A vector $\mathbf{y}$ such that $\mathbf{y}A = 0$ is then an ordered sub-graph of G such that every node has even degree. It follows that $\mathbf{y}$ is a sum of potential cycles. Then, because we assumed our partial order O is admissible, every potential cycle have positive score. Thus,

$$\mathbf{y} \cdot \mathbf{b} = \sum_{\mathbf{c}} \mathbf{c} \cdot \mathbf{b} = \sum_{\mathbf{c}} \sum_{e \in \mathbf{c}} v_A(e, O) \geq 0$$

where the set of $\mathbf{c}$ are potential cycles, and $v_A(e, O)$ is the score of an edge e in O. Contradiction, we've shown that the region R isn't empty.

It remains to show that the order O defined by a region $R \in \Sigma_A(G)$ using rules listed above is A-admissible. For the sake of contradiction, suppose there's a bad ass, just name it $\mathbf{y}$ (so necessarily it satisfy the badassness of $\mathbf{y}$ listed above). Then Farkas' lemma implies there isn't any element in R. Contradiction. □

Theorem 23 (Farkas' lemma). *Let $\mathbf{A} \in \mathbb{R}^{m \times n}$ and $\mathbf{b} \in \mathbb{R}^m$. Then exactly one of the following two assertions is true:*

(1) $\mathbf{A}\mathbf{x} \leq \mathbf{b}$ has a solution with $\mathbf{x} \in \mathbb{R}^n$;
(2) $\mathbf{A}^\mathsf{T}\mathbf{y} = 0$ has a solution with $\mathbf{b}^\mathsf{T}\mathbf{y} < 0$ and $\boldsymbol{y} \geq 0$.

Definition 24. Let A be a hyperplane arrangement in $\mathbb{R}^n$, and let $W \subset \mathbb{R}^n$ be the subspace spanned by the normals of hyperplanes in A. Define a region R to be *relatively bounded* if $R \cap W$ is bounded.

Remark: The normal of a hyperplane $z_i - z_j = 0$ is the vector with 1 on i'th entry and -1 on j'th entry. Because we are assuming that G is connected, the subspace spanned by normal vectors is $\mathrm{Span}(\mathbf{1})^\perp$. The details of the proof is left as an exercise to the reader (or see the notes in tex).

Theorem 25. *The relatively bounded regions of $\Sigma_G(A)$ is in bijection with A-admissible partial orientations of G where every step belongs to some potential cycle. The mapping is the same as defined in Theorem 22.*

Proof. Assume the settings of theorem 22, we first want to prove the following claim:

Claim: region R is relatively bounded if and only if all vectors $\mathbf{z} \lneq \mathbf{0} \in \mathbb{R}^k$, there exists $\mathbf{y} \geq \mathbf{0} \in \mathbb{R}^k$ such that $\mathbf{y}A = 0$ and $\mathbf{y} \cdot \mathbf{z} < \mathbf{0}$.

Proof. By Farkas' lemma, the existence of such $\mathbf{y}$ is equivalent to the insolvability of equation $A\mathbf{x} \leq z$ for every $\mathbf{z} \lneq 0$, so we are going to prove the alternative that: region R is relatively bounded if and only if there exists $\mathbf{z} \lneq 0$, and $\mathbf{x}$ such that $A\mathbf{x} \leq \mathbf{z} \lneq \mathbf{0}$. This would prove the claim.

Recall from Theorem 22 that $\mathbf{x} \in R$ iff $A\mathbf{x} \leq \mathbf{b}$ for some vector $\mathbf{b}$. Then R isn't bounded iff there exists some $\mathbf{x_0} \in \mathbb{R}$ and $\mathbf{x} \notin \mathrm{Span}(\mathbf{1})$ such that

$$A(\mathbf{x_0} + t\mathbf{x}) < \mathbf{b} \Leftrightarrow tA\mathbf{x} \leq \mathbf{b} - A\mathbf{x_0}$$

for all $t \geq 0$. The above condition implies that $A\mathbf{x} \leq \mathbf{0}$, and since $\mathbf{x} \notin \mathrm{Span}(\mathbf{1})$, we have $A\mathbf{x} < \mathbf{0}$. Let $\mathbf{z} = A\mathbf{x}$, the two directions easily follows from here. □

Now assume that $R \in \Sigma_G(A)$ is bounded, and given an edge $e_l \in O_R$, we show that e_l belongs to a potential cycle. Define $\mathbf{z} \lneq \mathbf{0} \in \mathbb{R}^k$ by

$$z_i = \begin{cases} -1 & \text{if } i = l, \\ 0 & \text{otherwise.} \end{cases}$$

then there exists $\mathbf{y}$ satisfying the criteria listed in the claim. Because $\mathbf{y}A = 0$, $\mathbf{y}$ is in the span of potential cycles, implying at least there one potential cycle $\mathbf{c} \cdot \mathbf{b} < 0$ contains the edge e_l.

For the converse direction, begin by assuming O is a partial orientation where each step is a part of a potential cycle, and encode its corresponding area R by the inequalities listed in Theorem 22, i.e., say $\mathbf{x} \in R$ iff $A\mathbf{x} \leq \mathbf{b}$. Let arbitrary $\mathbf{z} \lneq 0$ have entries $\{z_l < 0\}$ corresponding to row l of A and edge e_l. Then, define

$$\mathbf{y} = \sum_{\mathbf{c}_i, e_i \in \mathbf{c}_i} \mathbf{c}_i$$

where $\mathbf{c}$ encodes potential cycles. Necessarily $\mathbf{y}$ satisfy the criterion listed in the claim. Thus $\mathbf{R}$ is bounded.

□

Proposition 26. $\{indeg(\mathcal{O})\} = \{set\ of\ parking\ functions\ of\ G^\circ\}$

Proof. As noticed in previous section, maximal parking functions correspond to acyclic total orientations of G. Consider any parking functions c, $\exists c' s.t. c \leq c'$. Correspondingly, let $O' s.t. \text{indeg}(\mathcal{O}') = c'$, and we can find $\mathcal{O} \subseteq \mathcal{O}'$ so that $\text{indeg}(\mathcal{O}) = c$.

Conversely, let $\mathcal{O}$ be a partial orientation of G. Consider any blank edge $e = \{v_i, v_j\}$ and $e' = \{v_j, v_i\}$. Suppose two graphs $\mathcal{O} \cup e$ and $\mathcal{O} \cup e'$. Suppose both have directed cycles, then there should exist a path in e ande' direction, which is a contradiction. Thus, we can go through all black edges repeatedly and get a acyclic total orientation (every edge is oriented) $\mathcal{O}' \supseteq \mathcal{O}$. So $\text{indeg}(\mathcal{O}) \leq \text{indeg}(\mathcal{O}')$. Where $\text{indeg}(\mathcal{O}')$ is the maximal parking function. So $\text{indeg}(\mathcal{O})$ is a parking function. □

Lemma 27. *Let $\mathcal{O}$ be an A-admissible partial orientation, and let $W \subseteq V$ be a subset of the vertices of G that satisfy: 1. there do not exist $u \in W^c$, $w \in W$ with $(w, u) \in \mathcal{O}$. 2. there is some $u \in W^c$, $w \in W$ such that (u, w) is a blank edge in $\mathcal{O}$. Then there exists $u \in W^c$ and $w \in W$ such that $\mathcal{O} \cup \{(u, w)\}$ is also A-admissible.*

Theorem 28. *$\mathcal{O}$ is an acyclic partial orientation of G, then there exist an A-admissible partial orientation $\mathcal{O}'$ such that indeg($\mathcal{O}$) = indeg($\mathcal{O}'$).*

Proof. By induction. If $\mathcal{O} = \emptyset$ it is satisfied. For induction step, let W_i be subset of V such that $\text{indeg}_{\mathcal{O}_{\rangle - \infty}}(v) < \text{indeg}_{\mathcal{O}_{\rangle}}(v)$. If W_i is non-empty, apply Lemma 21 according to $\mathcal{O}_{\rangle - \infty}$ and W_i and define $\mathcal{O}_i := \mathcal{O}_{i-1} \cup (u, w)$ where $u \in W_i^c$ and $w \in W_i$. Suppose the lemma applies, let $\mathcal{O}' := \mathcal{O}_{|\mathcal{O}|}$, and we get a A-admissible partial orientation as desired.

The rest is to show that the lemma indeed apply. Suppose it does not apply. The there is at least one i $\mathcal{O}_0 \subset \mathcal{O}_{\rangle} \subseteq \mathcal{O}'$ such that corresponding (u, w) is already oriented. (Notice by construction, 1 always holds.) Then we have for each $w \in W_i$, $\text{indeg}(\mathcal{O})(w) > |\{(u, w) \in E : u \in W_i^c\}|$. So there is a w have an arrow in $\mathcal{O}$ coming into it from other $w \in W_i$. Then we have at least one cycle $\in \mathcal{O}$, which is contradiction. □

Corollary 29. *If we denote Pak-Stanley label of region R by $\lambda(R)$, then the set*

$$\{\lambda(R) : \ R \text{ is a region of } \sum_G (A)\}$$

is the set of all parking functions of G°.

Proof. The proof goes by induction. To show the theorem is sufficient to show $\lambda(R) = \text{indego}(\mathcal{O}_R)$. For central region it is true. We prove this by induction which goes by the construction of Pak-Stanley labeling. Suppose $\lambda(R) = \text{indego}(\mathcal{O}_R)$ holds, notice in R if we have $x_j - x_i < a_{ji}$, then passing the line gives us $x_j > x_i + a_{ji}$ in R'. So

$$\lambda(R') = \lambda(R) + v_j = \text{indeg}(\mathcal{O}_R) + v_j = \text{indeg}(\mathcal{O}_{\mathcal{R}'})$$

By Proposition 20 and Theorem 22, for any parking function c, there is some acyclic partial orientation s.t. $\mathcal{O} = c$, and there exist an A-admissible orientation with same indegree sequence and $\lambda(R : \text{region corresponding to } \mathcal{O}') = c$ □

6. Furture Directions

One important direction that could be taken is asking if bigraphical arrangements can be generalized to higher dimension. Specifically using simplicial complexes, can the definitions be extended to find a similar bijection. As of right now, no papers exist that explore this direction.

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