Homework: 4.2

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Math 202: Vector Calculus

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4.2.2

a.

We can simply evaluate and see that since $\varphi(x) = |x|^e \implies |\varphi_e(h)| = ||h|^e| \leq ||c^{\frac{1}{e}}|^e| = c$ (since $c > 0$). Thus by definition, for all $c > 0$, $|\varphi(h)| \leq c$ for all small enough h, or φ_e is $o(1)$.

b.

$\varphi_1 \implies \varphi(x) = |x|$. Thus we see, $|\varphi(h)| = |h| \leq c|h|$, where $c \geq 1$. Hence, this satisfies the condition for $\mathcal{O}(h)$

c.

φ_{e-1} is $o(1)$, and φ_1 is $\mathcal{O}(h)$ and thus by the product property for Landau functions we see that $\varphi_e = \varphi_{e-1}\varphi_1$ is $o(h)$. $\varphi_{e-1}\varphi_1 = |x|^{e-1}|x| = |x|^e$. Thus we have shown φ_e, such that $e > 1$, is $o(h)$.

d.

This problem has shown that φ_e is $\mathcal{O}(h)$ when $e = 1$ and is $o(h)$ when $e > 1$. We also showed that φ is $o(1)$. Thus, since $o(h) \subset \mathcal{O}(h) \subset o(1)$, we see, φ is $o(1)$ if $e > 0$ (as we showed all e for this are $o(1)$), φ is $\mathcal{O}(h)$ if $e \geq 1$ (as we showed 1 to be $\mathcal{O}(h)$ and $\mathcal{O}(h) \subset o(h)$), and finally φ is $o(h)$ if $e > 1$ (which we showed in part c.)

4.2.4

First note φ is $o(h)$ if and only if $|\varphi|$ is. Let φ have components $\varphi_1, \varphi_2, ..., \varphi_m$. For all h and each $j \in \{1, 2, ..., m\}$, by the size bounds,

$$|\varphi_j(h)| \leq |\varphi(h)| \leq \sum_{i=1}^{m} |\varphi_i(h)|.$$

Using the dominance principal of Landau spaces with the left hand side of size bounds for the implication($|\varphi_j(h)| \leq |\varphi(h)|$), we see that if $|\varphi|$ is $o(h)$ then each component, $|\varphi_i|$ is $o(h)$. For the implied by, using the dominance principal with the vector space properties with the right hand side of the size bounds($|\varphi(h)| \leq \sum_{i=1}^{m} |\varphi_i(h)|$ and $o(h) + o(h) = o(h)$), we see that if each component $|\varphi_j|$ is $o(h)$ then so is $|\varphi|$. In total we get,

$$|\varphi| \text{ is } o(h) \implies \text{each } |\varphi_j| \text{ is } o(h) \implies \sum_{i=1}^{m} \text{ is } o(h) \implies |\varphi| \text{ is } o(h).$$

Thus $|\varphi|$ is $o(h)$ if and only if each $|\varphi_i|$ is. As we noted above, we can drop the absolute values and we have the desired componentwise nature of $o(h)$

4.2.5

We will prove $\mathcal{O}(o(h)) = o(h)$. Suppose $\varphi : B(\mathbf{0}_n, \varepsilon) \to \mathbb{R}^m$ is $o(h)$ and $\psi : B(\mathbf{0}_m, \rho) \to \mathbb{R}^l$ is $\mathcal{O}(k)$. Then,

$$\text{for all } c > 0, |\varphi(h)| \leq c|h| \text{ for all small enough } h$$

Thus if h is small then so is $\varphi(h)$, we see

$$\text{for some } d > 0, |\psi(\varphi(h))| \leq d|\varphi(h)| \leq cd|h| \text{ for all small enough } h$$

Since c can be any positive number, and d can be some positive number, then cd can be any positive number. Thus multiplying the two to creates a new positive number that can be any number, let it be e. Thus, combining we see

$$\text{for all } e > 0, |(\psi \circ \varphi)(h)| \leq e|h| \text{ for all small enough } h$$

Thus $\psi \circ \varphi$ is $o(h)$.