Exam 3: Chapter 6a

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Math 202: Vector Calculus

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a.

Let us first note that for all x, that,

$$m_J(f) + m_J(g) \leq f(x) + g(x) = (f+g)(x)$$

from this we see for each f, g that $m_J(f) + m_J(g)$ is a lower bound of $\{(f+g)(x) : x \in J\}$. Since $\{(f+g)(x) : x \in J\}$ is nonempty and has lower bounds, it has a greatest lower bound, or

$$\inf\{(f+g)(x) : x \in J\}.$$

Since $m_J(f) + m_J(g)$ is a lower bound and $\inf\{(f+g)(x) : x \in J\} = m_J(f+g)$ is the greatest lower bound,

$$m_J(f) + m_J(g) \leq m_J(f+g)$$

and this is the desired result.

b.

Let $J = [0,1]$, $f : J \to \mathbb{R}$, $f(x) = c_1$, and $g : J \to \mathbb{R}$, $g(x) = c_2$ for all x where c_1 and c_2 are constants. Then we see,

$$m_J(f) + m_J(g) = c_1 + c_2 = m_J(f+g)$$

c.

Let $J = [0,1]$, $f : J \to \mathbb{R}$, $f(x) = x$, and $g : J \to \mathbb{R}$, $g(x) = -x$ for all x Then we see,

$$m_J(f) = 0$$

$$m_J(g) = -1$$

and since $(f+g)(x) = x - x = 0$ then,

$$m_J(f+g) = 0$$

so,

$$m_J(f) + m_J(g) = -1 < 0 = m_J(f+g)$$

2.

a.

Suppose there does exist $\delta > 0$ that satisfies the definition of uniform continuity for $\varepsilon = \ln(6/5)$. Set $\tilde{x} = 3\delta/2$ and $x = \delta$, so we can see that $|\tilde{x} - x| < \delta$, and we can see,

$$|f(\tilde{x}) - f(x)| = \left|\ln\left(\frac{3}{2}\delta\right) - \ln(\delta)\right| = \left|\ln\left(\frac{3}{2\delta}\delta\right)\right| = \ln(3/2) > \ln(6/5) = \varepsilon$$

Where the last inequality is because $e^{\ln(3/2)} = 3/2$ and $e^{\ln(6/5)} = 6/5 = e^{\varepsilon}$ so $3/2 > 6/5 = e^{\varepsilon}$ and by taking the log of both sides we see $\ln(3/2) > \varepsilon$. Hence, contradicting uniform continuity.

b.

Let $I = [1, \infty)$, $f : I \to \mathbb{R}$, $f(x) = \ln(x)$ and thus, $f'(x) = 1/x$. From Exercise 6.3.4 we know that since f is differentiable on I, and $|f'(x)| \leq 1$ for all $x \in I$, it must be uniformly continuous on I.

3.

a.

Using the Extreme Value Theorem (Theorem 1.2.1) we can show this. Since $I = [a, b]$ is a nonempty, closed, and bounded interval in $\mathbb{R}$ and $\gamma_1' : I \to \mathbb{R}$, and $\gamma_2' : I \to \mathbb{R}$ are continuous functions then γ_1' and γ_2' both take a minimum and a maximum value on I. We can then see that there exists values B_1 and B_2 such that $|\max(\gamma_1')|$, $|\min(\gamma_1')| < B_1$ and $|\max(\gamma_2')|$, $|\min(\gamma_2')| < B_2$ then for all t,

$$|\gamma_1'(t)| < B_1 \qquad |\gamma_2'(t)| < B_2.$$

Let $B = B_1 + B_2$ then,

$$|\gamma_1'(t)| < B_1 \leq B_1 + B_2 = B \qquad |\gamma_2'(t)| < B_2 \leq B_1 + B_2 = B$$

as desired.

b

Let

$$\bar{\varepsilon} = \min\left\{\frac{\varepsilon}{8B(b-a)}, \frac{\sqrt{\varepsilon}}{4}\right\}$$

so that $4B(b-a)\bar{\varepsilon} \leq \varepsilon/2$ and $8\bar{\varepsilon}^2 \leq \varepsilon/2$.

We can use the Mean Value Theorem here(Theorem 1.2.3). By the mean value theorem, since γ_1, γ_2 are continuous and γ_1, γ_2 are both differentiable on (a, b), there exist $c_1, c_2 \in (a, b)$ such that $|\gamma_i(s) - \gamma_i(t)| = |s - t||\gamma_i'(c_i)|$. Thus we see, that by **a** and $|s - t| < \delta = \frac{\bar{\varepsilon}}{B}$,

$$|\gamma_1'(c_1)| < B < \frac{\bar{\varepsilon}}{|s-t|} \qquad |\gamma_2'(c_2)| < B < \frac{\bar{\varepsilon}}{|s-t|}$$

by multiplication becomes,

$$|s - t||\gamma_1'(c_1)| < \bar{\varepsilon} \qquad |s - t||\gamma_2'(c_2)| < \bar{\varepsilon}.$$

By the mean value theorem from above,

$$|\gamma_1(s) - \gamma_1(t)| < \bar{\varepsilon} \qquad |\gamma_2(s) - \gamma_2(t)| < \bar{\varepsilon}$$

as desired.

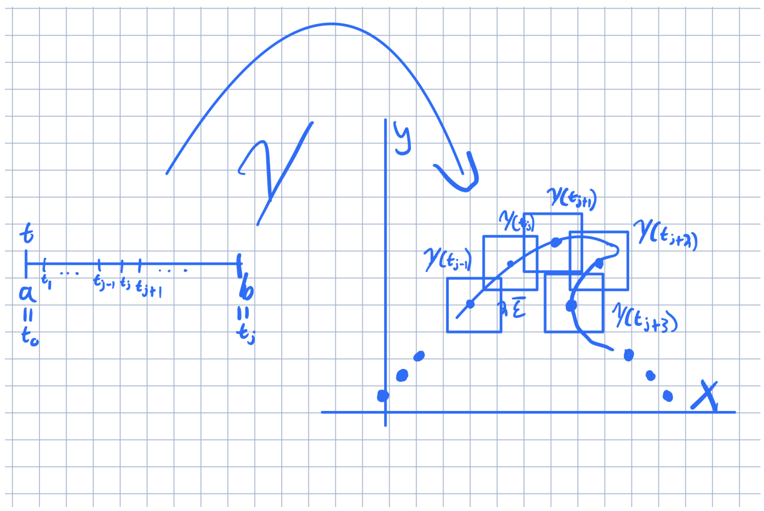
c

Let s sit in between t_j and t_{j+1} (since $s \in [a,b]$ and thus must sit in between some t_j and t_{j+1}) then we can see that

$$t_{j+1} - t_j = a + (j+1)\delta - a - j\delta = \delta$$

so at most s can sit δ away from t_j but then it is at t_{j+1} or t_{j-1}. Thus, s sits within a distance δ of some t_j. Note that for the end point, $t_n - t_{n-1} \leq \delta$ so if s is near t_n, it is still within δ of some t_j.

d.



First we know that by **c** that s sits within δ of t_j ($|s - t_j| < \delta$) and so the conditions of **b** are met, implying

$$|\gamma_1(s) - \gamma_1(t_j)| < \bar{\varepsilon} \qquad |\gamma_2(s) - \gamma_2(t_j)| < \bar{\varepsilon}$$

which translates to mean that for any s there exists a square of $2\bar{\varepsilon}$ (since $-\bar{\varepsilon} < \gamma_i(s) - \gamma_i(t_j) < \bar{\varepsilon}$ and since the image of γ has 2 dimensions) that contains $\gamma(s)$. Thus we can see that the squares cover the image of γ since there exists a corresponding square for all $s \in [a,b]$.

e.

We can calculate the area since we know there are $n+1$ squares of area $(2\bar{\varepsilon})^2$, thus,

$$\begin{aligned}
(n+1)(2\bar{\varepsilon})^2 &< \left(\frac{b-a}{\delta} + 2\right)(4\bar{\varepsilon}^2) && \text{since, } n+1 < \frac{b-a}{\delta} + 2 \\
&= \left(\frac{B(b-a)}{\bar{\varepsilon}} + 2\right)(4\bar{\varepsilon}^2) && \text{since, } \delta = \bar{\varepsilon}/B \\
&= 4B(b-a)\bar{\varepsilon} + 8\bar{\varepsilon}^2 \\
&\leq \varepsilon/2 + \varepsilon/2 && \text{by definition of } \bar{\varepsilon} \\
&= \varepsilon
\end{aligned}$$

as desired.