Homework: 2.2, 2.3

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Math 202: Vector Calculus

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2.2.7

First we will show the reverse triangle inequality,

$$\begin{aligned} x &= (x-y)+y \\ &= |(x-y)+y| && \text{by applying modulus to both sides} \\ &\leq |x-y|+|y| && \text{by triangle inequality} \\ &\implies ||x|-|y|| \leq |x-y| \end{aligned}$$

By substituting y for $-y$ in the reverse triangle inequality we find,

$$||x|-|-y|| \leq |x-(-y)| \implies ||x|-|y|| \leq |x+y|$$

Similar can be done by substituting $-y$ in the original triangle inequality,

$$|x+(-y)| \leq |x|+|-y| \implies |x-y| \leq |x|+|y|$$

We can substitute because $|-y| = \sqrt{\langle -y,-y\rangle} = \sqrt{(-1)^2\langle y,y\rangle} = \sqrt{\langle y,y\rangle} = |y|$. Thus we have the full triangle inequality.

2.2.8

First we will prove the left-hand inequality. Since $x_j = \langle x, e_j\rangle$ and $|e_j| = 1$ then

$$|x_j| = |\langle x, e_j\rangle| \leq |x||e_j| = |x|\cdot 1 = |x|$$

where the inequality is by the Cauchy-Schwarz inequality.
For the right hand inequality,

$$\begin{aligned}
|x| &= \left|\sum_{i=1}^{n} x_i e_i\right| && \text{by definition of vectors} \\
&\leq \sum_{i=1}^{n} |x_i e_i| && \text{by generalized triangle inequality} \\
&= \sum_{i=1}^{n} |x_i||e_i| && \text{by definition of modulus and basis vectors} \\
&\leq \sum_{i=1}^{n} |x_i| && \text{by } |e_i| = 1
\end{aligned}$$

Thus we have proved the size bounds.
The first inequality is equality when $x = (0, ..., x_j, ..., 0)$. The second is inequality is equality at the same time.

2.2.10

By the Law of cosines we have, in terms of vectors,

$$|x - y|^2 = |x|^2 + |y|^2 - 2|x||y|\cos\theta$$

which we can parse,

$$\begin{aligned}
\langle x - y, x - y\rangle &= |x|^2 + |y|^2 - 2|x||y|\cos\theta \\
2\langle x, y\rangle + \langle y, y\rangle + \langle x, x\rangle &= |x|^2 + |y|^2 - 2|x||y|\cos\theta \\
\frac{\langle x, y\rangle}{|x||y|} &= \cos\theta
\end{aligned}$$

2.2.12

For the $\implies$ direction

$$\begin{aligned}
|x + y|^2 &= \langle x + y, x + y\rangle && \text{by definition of modulus} \\
&= \langle x, x\rangle + 2\langle x, y\rangle + \langle y, y\rangle && \text{by bilinearity and symmetry of inner product} \\
&= \langle x, x\rangle + \langle y, y\rangle && \text{by orthogonality of } x, y \\
&= |x|^2 + |y|^2 && \text{by definition of modulus}
\end{aligned}$$

For the $\impliedby$ direction

$$\begin{aligned}
&|x + y|^2 = |x|^2 + |y|^2 \\
&\langle x + y, x + y\rangle = \langle x, x\rangle + \langle y, y\rangle && \text{by definition of modulus} \\
&\langle x, x\rangle + 2\langle x, y\rangle + \langle y, y\rangle = \langle x, x\rangle + \langle y, y\rangle && \text{by the bilinearity and symmetry of inner product} \\
&\langle x, y\rangle = 0 && \text{by cancellation}
\end{aligned}$$

Thus the proposition is proved both ways.

2.2.13

Suppose x and y are 2 sides of a parallelogram. In order for the parallelogram to be a rhombus, $|x| = |y|$. Let us show that if the diagonals of the parallelogram are orthogonal, then the parallelogram is a rhombus. The diagonals of our parallelogram are $x - y$ and $x + y$ by the head minus tail rule.

$$\begin{aligned}
\langle x - y, x + y\rangle &= \langle x, x\rangle - \langle y, y\rangle + \langle y, x\rangle - \langle x, y\rangle && \text{by bilinearity of inner product} \\
&= |x|^2 - |y|^2 && \text{by cancellation from symmetry and definition of modulus}
\end{aligned}$$

Since the diagonals are orthogonal, $|x|^2 = |y|^2 \implies |x| = |y|$. Thus the sides are equal and the parallelogram is a rhombus.
If we suppose $|x| = |y| \implies |x|^2 = |y|^2$. Thus,

$$\begin{aligned}
\langle x - y, x + y\rangle &= \langle x, x\rangle - \langle y, y\rangle + \langle y, x\rangle - \langle x, y\rangle && \text{by bilinearity of inner product} \\
&= 0 && \text{by cancellation from symmetry and } |x| = |y|
\end{aligned}$$

Therefore the diagonals are perpendicular.

2.2.15

a.

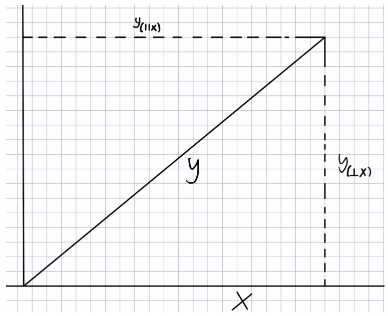
This follows by the definition of $y_{\|x}$ and $y_{\perp x}$.
Since $\langle x, y\rangle$ and $|x||y|$ are both scalars, then by definition of $y_{\perp x}$, $y_{\perp x}$ is a scalar multiple of x.
In order for $y_{\perp x}$ and x to be orthogonal, their inner product must be zero.

$$\begin{aligned}
\langle y_{\perp x}, x\rangle &= \langle y - y_{(\|x)}, x\rangle && \text{by definition of } y_{\perp x} \\
&= \langle y - \frac{\langle x, y\rangle}{|x|^2} x, x\rangle && \text{by definition of } y_{\|x} \\
&= \langle y, x\rangle - \frac{\langle x, y\rangle}{|x|^2}\langle x, x\rangle && \text{by the bilinearity of the inner product} \\
&= 0 && \text{by cancellation, definition of modulus, and symmetry of the inner product}
\end{aligned}$$

Suppose $y = y_{\|x} + y_{\perp x} = y'_{\|x} + y'_{\perp x}$. This implies

$$y_{\|x} - y'_{\|x} = y'_{\perp x} - y_{\perp x}$$

Since $y_{\|x}$ and $y'_{\|x}$ are scalar multiples of x, so $cx = y_{\|x} - y'_{\|x} = y'_{\perp x} - y_{\perp x}$, but the right hand side is perpendicular to x by definition. So the inner product must be zero, $\langle cx, x\rangle = c\langle x, x\rangle = c|x|^2 = 0$ but by the positive definiteness of squaring, $x = 0$ or $c = 0$. Since $x = 0$ is clearly not true, $c = 0$. This entails $0 = y_{\|x} - y'_{\|x} = y'_{\perp x} - y_{\perp x} \implies y'_{\perp x} = y_{\perp x}$ and $y_{\|x} = y'_{\|x}$



b.

$$\begin{aligned}
|y|^2 &= |y_{\|x}|^2 + |y_{\perp x}|^2 \\
\langle y, y\rangle &= \langle y_{\|x}, y_{\|x}\rangle + \langle y_{\perp x}, y_{\perp x}\rangle \\
&= \langle \frac{\langle x, y\rangle}{|x|^2}x, \frac{\langle x, y\rangle}{|x|^2}x\rangle + \langle y - \frac{\langle x, y\rangle}{|x|^2}x, y - \frac{\langle x, y\rangle}{|x|^2}x\rangle \\
&= \frac{\langle x, y\rangle^2}{|x|^4}\langle x, x\rangle + \langle y, y\rangle - \frac{\langle x, y\rangle}{|x|^2}\langle y, x\rangle - \frac{\langle x, y\rangle}{|x|^2}\langle x, y\rangle + \frac{\langle x, y\rangle^2}{|x|^4}\langle x, x\rangle \\
0 &= 2\frac{\langle x, y\rangle^2}{|x|^2} - 2\frac{\langle x, y\rangle^2}{|x|^2} \\
0 &= 0
\end{aligned}$$

Thus equality.

This is parallel (no pun intended) to the Pythagoras's Theorem.

c.

$$\begin{aligned}
|y|^2 &= |y_{\|x}|^2 + |y_{\perp x}|^2 \\
|y|^2 &\geq |y_{\|x}|^2 \\
|y| &\geq |y_{\|x}|
\end{aligned}$$

The inequality follows due to the the subtraction of a term, meaning it at most is equal. For equality,

$$|y| \geq \left|\frac{\langle x, y\rangle}{|x|^2}x\right|$$

Which means that equality occurs when y is a scalar multiple of x.
From here we can see

$$|x||y| \geq \langle x, y\rangle,$$

the Cauchy-Schwarz Inequality.

2.3.4

For the positive definite property,

$$\begin{aligned}\langle f, f\rangle &= \int_0^1 f(t)f(t)dt \\ &= \int_0^1 f(t)^2 dt\end{aligned}$$

For $f(t) \neq 0$ and we know $f(t)^2 > 0$ and suppose $f(t)^2 = g(t)$. Let, $g(t_0) = r$ with $t_0 \in [0,1]$ and $r \in \mathbb{R}_{>0}$ and $g(t_0) \neq \frac{r}{2}$. By the persistence of inequality, $g(t) \neq \frac{r}{2}$. By the intermediate value theorem then, $g(t) > \frac{r}{2} > 0$ for $t \in [0,1]$. Thus

$$\int_0^1 g(t)dt > \frac{r}{2}(1-0) = \frac{r}{2} > 0 \implies \int_0^1 f(t)^2 dt > \frac{r}{2} > 0 \implies \langle f, f\rangle > 0.$$

When $f(t) = 0$,

$$\langle f, f\rangle = \int_0^1 f(t)^2 dt = \int_0^1 0^2 dt = \int_0^1 0 dt = 0\Big|_0^1 = 0,$$

and for the converse, $\langle f, f\rangle = 0$,

$$\int_0^1 f(t)f(t)dt = \int_0^1 f(t)^2 dt = 0$$

and we know $f(t)^2 > 0$, so in order to have the integral evaluate to 0, $f(t)^2$ must be zero, thus $f(t) = 0$. Hence, we have showed the positive definite property.

The section from the heading 2.3.4 to here was revised after in-class discussion and discussion with Bhavana Panchumarthi.

For the symmetric property,

$$\begin{aligned}\langle f, g\rangle &= \int_0^1 f(t)g(t)dt && \text{by definition of inner product} \\ &= \int_0^1 g(t)f(t)dt && \text{by commutativity of mappings} \\ &= \langle g, f\rangle && \text{by definition of inner product}\end{aligned}$$

Next we will show the bilinearity of the inner product.

$$\begin{aligned}\langle f+f', cg\rangle &= \int_0^1 (f+f')(t)\cdot c\cdot g(t)dt\\ &= \int_0^1 (f(t)+f'(t))c\cdot g(t)dt\\ &= \int_0^1 (c\cdot f(t)g(t)+c\cdot f'(t)g(t))dt\\ &= c\int_0^1 f(t)g(t)dt + c\int_0^1 f'(t)g(t))dt\\ &= c\langle f,g\rangle + c\langle f,g\rangle\end{aligned}$$

We can form the Cauchy-Schwarz inequality as

$$|\langle f,g\rangle| \le |f||g| \implies \sqrt{\left(\int_0^1 f(t)g(t)dt\right)^2} \le \sqrt{\int_0^1 f(t)^2dt}\sqrt{\int_0^1 g(t)^2dt} \implies$$

$$\left(\int_0^1 f(t)g(t)dt\right)^2 \le \left(\int_0^1 f(t)^2dt\right)\left(\int_0^1 g(t)^2dt\right)$$

2.3.5

We want to prove component-wise nature of convergence, so we will use the definition of component-wise nullness and convergence. A sequence is null if and only if all of its component sequences is null, and a sequence converges to p if $\{x_\nu - p\}$ is null. Suppose we have a sequence that converges to p, then the sequence $\{x_\nu - p\}$ is null, and when a sequence is null its component sequences must be null, thus $\{(x_{1,\nu} - p_1, ..., x_{n,\nu} - p_n\}$ is null. Thus each component scalar sequence, $\{x_{j,\nu}\}$ converges to p_j.