Midterm

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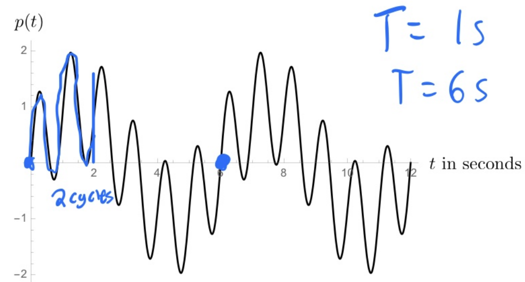
Due October 16th, 2020

Problem 1

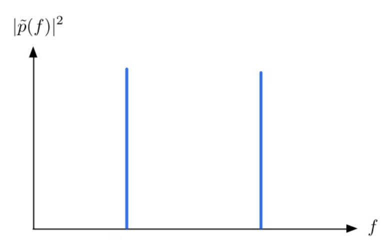
Problem 1

a. For the signal $p(t)$ shown below, sketch the power spectrum ($|\tilde{p}(f)|^2$ vs. f — use appropriate units and values for the frequency axis, and get the correct relative heights, but don't worry about the height values themselves).



T= 1s 2 oscillations

T=6s

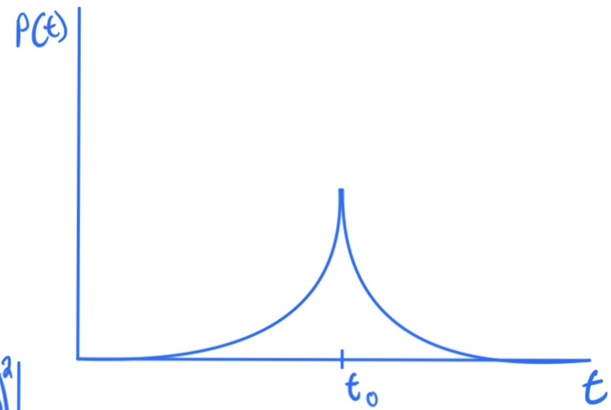


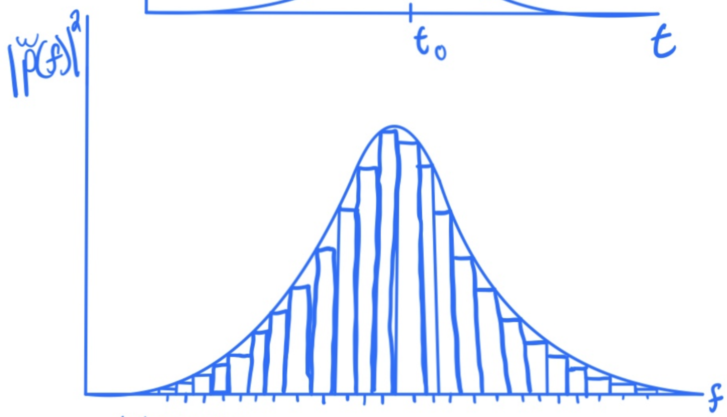
1/6 HZ 1 HZ

they are of equal heights because it looks like an even mix of both.

b. A person claps their hands once at time t_0 generating a distinct audio signal. Sketch the signal $p(t)$ (where the height of $p(t)$ represents the volume of the signal at the source), and sketch the associated power spectrum.

This would likely look like a double exponential that is displaced, similar to #1 PS6





where each bar is a frequency

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Problem 2

a.

Note that we can let $v(m) = \dot{x}(m)$, then our equation becomes,

$$\ddot{x}(m) = -\frac{u}{m}$$

integrate both sides once

$$\int_{M=m}^{m_0} \ddot{x}(M) dM = -u \int_{M=m}^{m_0} \frac{1}{m} dm$$
$$\dot{x}(m_0) - \dot{x}(m) = -u \ln(m_0) + u \ln(m)$$

but since we start at rest($\dot{x}(m_0) = 0$) we see,

$$v(m) = u\ln(m_0) - u\ln(m) = u\left(\ln\left(\frac{m_0}{m}\right)\right)^{1}$$

Also note that when we are at initial mass, $\ln(1) = 0$ satisfying our initial conditions.

When $m = m_0/2$,

$$v(m) = u\left(\ln\left(\frac{m_0}{m_0/2}\right)\right) = u\ln(2)$$

b.

Note that

$$2^i = e^{\ln(2^i)} = e^{i\ln(2)} = \cos(\ln(2)) + i\cos(\ln(2))$$

thus we see,

$$\Re(2^i) = \cos(\ln(2)) \qquad \Im(2^i) = \sin(\ln(2))$$

Problem 3.

$$F = -\frac{dU}{dx} = -U_0\left(\frac{5}{4}x^3 - \frac{21}{2}x^2 + 28x - 24\right)$$

We can then factor to see,

$$F = -U_0(x-4)(x-2)(5x-12)$$

thus the roots are,

$$x = 4 \quad x = 2 \quad x = 2.4$$

Taking the second derivative to determine whether they are minima or maxima,

$$U''(x) = U_0\left(\frac{15}{4}x^2 - 21x + 28\right)$$

We see,

$$U''(2) = U_0 \quad U''(2.4) = -0.8U_0 \quad U''(4) = 4U_0$$

thus we see the minima are $x = 2$, $x = 4$ by the min/max test. To determine their time periods we simply need to evaluate,

$$T = 2\pi\sqrt{\frac{m}{U''(x_e)}}$$

We see, for $x_e = 2$,

$$T = 2\pi\sqrt{\frac{m}{U''(2)}} = 2\pi\sqrt{\frac{m}{U_0}}$$

and for $x_e = 4$,

$$T = 2\pi\sqrt{\frac{m}{U''(4)}} = 2\pi\sqrt{\frac{m}{4U_0}} = \pi\sqrt{\frac{m}{U_0}}$$

[1] It does not matter if m_0/m or m/m_0 as long as the u is the correct sign

Problem 4.

Note for this problem that we will want to take $\Re(h(x)+\tilde{x}(t))$ for the final solution. Let $\tilde{x}(t) = Ge^{iut}$, then Newton's second law becomes, letting $\omega = \sqrt{\frac{k}{m}}$, $2b = \frac{\gamma}{m}$, and $f_0 = \frac{F_0}{m}$,

$$\ddot{\tilde{x}}(t) = -\omega^2 \tilde{x}(t) - 2b\dot{\tilde{x}}(t) + f_0 e^{i\sigma t},$$

then

$$-Gu^2 e^{iut} = -\omega^2 G e^{iut} - 2biGue^{iut} + f_0 e^{i\sigma t}$$

we can see that this equality holds only if $u = \sigma$. Cancelling terms we see,

$$-G\sigma^2 = -\omega^2 G - 2biG\sigma + f_0$$

We then see,

$$G = \frac{f_0}{\omega^2 + 2bi\sigma - \sigma^2} = -\frac{if_0}{2b\sigma + i(\sigma^2 + \omega^2)}$$

Thus,

$$\tilde{x}(t) = -\frac{if_0}{2b\sigma + i(\sigma^2 + \omega^2)} e^{i\sigma t}$$

taking the real part we see,

$$\Re\left\{-\frac{if_0}{2b\sigma + i(\sigma^2 + \omega^2)} e^{i\sigma t}\right\} = \frac{2bf_0\sigma \sin(\sigma t) - f_0\sigma^2 \cos(\sigma t) + f\omega^2 \cos(\sigma t)}{4b^2\sigma^2 + (\sigma^2 - \omega^2)^2}$$

We know that our homogenous solution is,

$$h(t) = e^{-bt}\left(Ae^{\sqrt{b^2-\omega^2}t} + Be^{-\sqrt{b^2-\omega^2}t}\right).$$

Thus our full solution is

$$x(t) = \Re(\tilde{x}(t)+h(t)) = e^{-bt}\left(Ae^{\sqrt{b^2-\omega^2}t} + Be^{-\sqrt{b^2-\omega^2}t}\right) + \frac{2bf_0\sigma \sin(\sigma t) - f_0\sigma^2 \cos(\sigma t) + f\omega^2 \cos(\sigma t)}{4b^2\sigma^2 + (\sigma^2 - \omega^2)^2}$$

Our initial conditions are

$$x(0) = \frac{3}{4} \qquad \dot{x}(0) = 0$$

Thus we see,

$$\frac{3}{4} = A + B + \frac{f_0(\omega^2 - \sigma^2)}{4b^2\sigma^2 + (\sigma^2 - \omega^2)^2}$$

and

$$\dot{x}(0) = (\sqrt{b^2 - \omega^2} - b)A + (-\sqrt{b^2 - \omega^2} - b)B + \frac{2bf_0\sigma^2}{4b^2\sigma^2 + (\sigma^2 - \omega^2)^2} = 0$$

However, note that since $b^2 = ((4/4)/2)^2 = 1/4$ and $\omega^2 = 1/4$ so then

$$x(t) = e^{-bt}(A + B) + \frac{2bf_0\sigma \sin(\sigma t) - f_0\sigma^2 \cos(\sigma t) + f\omega^2 \cos(\sigma t)}{4b^2\sigma^2 + (\sigma^2 - \omega^2)^2}$$

but $A + B$ might as well just be C, so

$$x(t) = Ce^{-bt} + \frac{2bf_0\sigma \sin(\sigma t) - f_0\sigma^2 \cos(\sigma t) + f_0\omega^2 \cos(\sigma t)}{4b^2\sigma^2 + (\sigma^2 - \omega^2)^2}$$

so we can see that our full solution is,

$$x(t) = \left(\frac{3}{4} - \frac{f_0(\omega^2 - \sigma^2)}{4b^2\sigma^2 + (\sigma^2 - \omega^2)^2}\right)e^{-bt} + \frac{2bf_0\sigma\sin(\sigma t) - f_0\sigma^2\cos(\sigma t) + f_0\omega^2\cos(\sigma t)}{4b^2\sigma^2 + (\sigma^2 - \omega^2)^2}$$

let us use all of the given values and we see,

$$x(0) = C + \frac{-2(36) + 2(1/4)}{4(1/4)36 + (36 - 1/4)^2} = \frac{3}{4} \implies C = 0.8044$$

Finally, we can solve for where the mass is at 0.5s,

$$x(0.5) = 0.8044e^{-(1/2)(1/2)} + \frac{2(1/2)2(6)\sin(6 \cdot 0.5) - 2(36)\cos(6 \cdot 0.5) + 2(36)\cos(6(0.5))}{4(1/4)(36) + (36 - 1/4)^2} = 0.62776\text{m}$$