Exam 2: Chapter 4

Monroe Stephenson
Math 202: Vector Calculus

Due October 12th, 2020

I used a derivative calculator on Problem 4. I also used Apple's Grapher and Wolfram Mathematica for visual aids.

Problem 1

a.

We will first find the partial derivatives of f on $\mathbb{R}^2 - \{(0,0)\}$:

$$\begin{aligned} D_1 f(a,b) &= \frac{b^3}{a^2+b^2} - \frac{2b^3a^2}{(a^2+b^2)^2} \\ &= \frac{-b^3(a^2-b^2)}{(a^2+b^2)^2} \\ D_2 f(a,b) &= \frac{3ab^2}{a^2+b^2} - \frac{2ab^4}{(a^2+b^2)^2} \\ &= \frac{ab^2(b^2+3a^2)}{(a^2+b^2)^2} \end{aligned}$$

We can use the sufficiency theorem to show that f is differentiable at each such (a,b). First, as seen above, all of the partial derivatives exist. Second note that, that on $\mathbb{R}^2 - \{(0,0)\}$ f is continuous as the denominator is never 0. Lastly, every point (a,b) away from $(0,0)$ lies in some $\varepsilon-$ball that is also away from $(0,0)$. Thus all (a,b) are interior points. In total, for all points $(a,b) \neq (0,0)$, the partial derivatives exist at and about (a,b) and they are continuous at (a,b). Thus the sufficient conditions from Theorem 4.5.3 are met and f is differentiable at (a,b). Now Theorem 4.5.2 says the derivative matrix at (a,b) is the matrix of partial derivatives,

$$f'(a,b) = [D_1 f(a,b)\ \ D_2 f(a,b)] = \begin{bmatrix} \frac{-b^3(a^2-b^2)}{(a^2+b^2)^2} & \frac{ab^2(b^2+3a^2)}{(a^2+b^2)^2} \end{bmatrix}$$

Furthermore, the derivative of f at every nonzero (a,b) is the linear map,

$$Df_{(a,b)}(h,k) = \frac{-b^3(a^2-b^2)}{(a^2+b^2)^2}h + \frac{ab^2(b^2+3a^2)}{(a^2+b^2)^2}k$$

b.

Since f is zero on both axes, we can see both the partial derivatives at the origin are 0.

$$D_1 f(0,0) = \lim_{t\to 0} \frac{f(t,0) - f(0,0)}{t} = \frac{0 \cdot 0}{t} = 0$$

$$D_2 f(0,0) = \lim_{t\to 0} \frac{f(0,t) - f(0,0)}{t} = \frac{0 \cdot 0}{t} = 0$$

So the only possibility for derivative of f at $(0,0)$ is the zero map.

c.

In order to determine if f is differentiable at $(0,0)$ we determine whether,

$$f(h,k) - f(0,0) - 0 \text{ is } o(h,k)$$

because the denominator $h^2 + k^2$ of f away from the origin is $|(h,k)|^2$,

$$0 \leq |f(h,k)| = \frac{|h||k|^3}{|(h,k)|^2} \leq \frac{|(h,k)|^4}{|(h,k)|^2} = |(h,k)|^2$$

Thus,

$$|f(h,k) - f(0,0) - 0| = |f(h,k)| \leq |(h,k)|^2 \leq c|(h,k)| \implies |(h,k)| \leq c$$

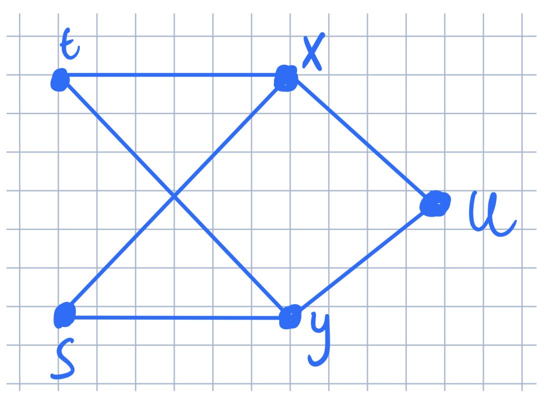
For all c we can have a smaller (h,k) that satisfies the inequality. Thus, $f(h,k) - f(0,0) - 0$ is $o(h,k)$, and hence f is differentiable at $(0,0)$.

Problem 2

We will first take the derivative of u with respect to s,

$$u_s = u_y y_s + u_x x_s \qquad \text{by the chain rule}$$

Where we can see the chain rule visually below:



Now we will differentiate with respect to t,

$$\begin{aligned}
u_{st} &= (u_y)_t y_s + u_y y_{st} + (u_x)_t x_s + u_x x_{st} && \text{by chain rule} \\
&= (u_{yy} y_t + u_{yx} x_t) y_s + u_y y_{st} + (u_{xy} y_t + u_{xx} x_t) x_s + u_x x_{st} && \text{by chain rule} \\
&\text{since } u \text{ is a } \mathcal{C}^2 \text{ function and } x_{st} = x, y_{st} = -y, x_s = x, y_s = y, x_t = x, y_t = -y \\
&= -y^2 u_{yy} + xyu_{xy} - yu_y - xyu_{xy} + x^2 u_{xx} + xu_{xx} \\
&= -(y^2 u_{yy} + yu_y) + (x^2 u_{xx} + xu_{xx}) \\
&= 0 && \text{by assumption}
\end{aligned}$$

Thus we see, $u_{st} = 0$.

Problem 3

We will first take the partial derivatives,

$$D_1 m(x, y) = 6y^2 - 6a^2$$

$$D_2 m(x, y) = 12xy - 12y^3$$

These are both zero when,

$$6y^2 = 6x^2 \implies x = \pm y$$

$$12xy = 12y^3 \implies x = y^2$$

Thus the critical points are $(1, 1), (1, -1), (0, 0)$, as these are the only three points that satisfy both equations. Now we will find the second derivative matrix to determine if they are min/max/saddle points:

$$f''(x, y) = \begin{bmatrix} D_{11} f(x, y) & D_{12} f(x, y) \\ D_{21} f(x, y) & D_{22} f(x, y) \end{bmatrix} = \begin{bmatrix} -12x & 12y \\ 12y & 12x - 36y^2 \end{bmatrix}$$

Thus,

$$f''(1, 1) = \begin{bmatrix} -12 & 12 \\ 12 & -24 \end{bmatrix}$$

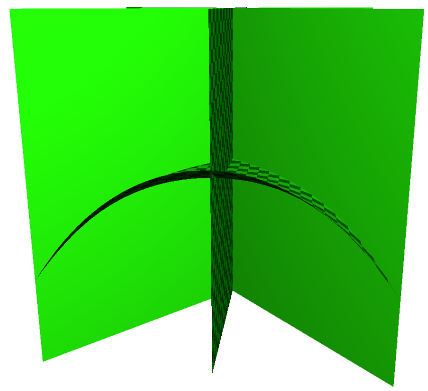
So $(1, 1)$ is a local maximum since $(-12 \cdot -24) - (12 \cdot 12) > 0$ and $-12 < 0$.

$$f''(1, -1) = \begin{bmatrix} -12 & -12 \\ -12 & -24 \end{bmatrix}$$

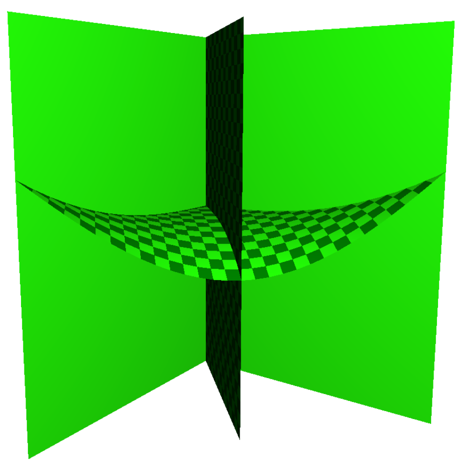
So $(1, -1)$ is a local maximum since $(-12 \cdot -24) - (-12 \cdot -12) > 0$ and $-12 < 0$.
However, for $(0, 0)$ we see,

$$f''(0, 0) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

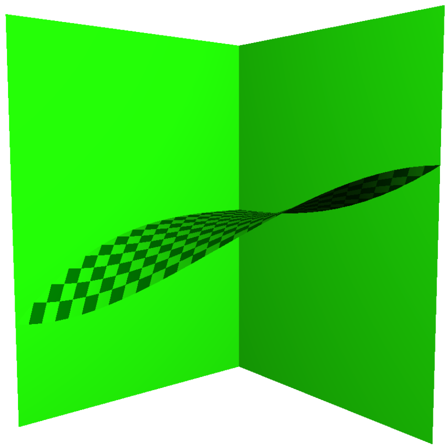
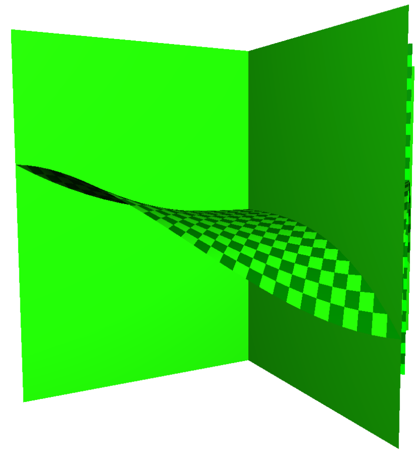
Thus the min/max test is inconclusive. However, we may look at the inflection points to see how the graph changes around a $\varepsilon-$ball centered at the origin. First note that by $D_2 m(x, y)$ that if we pass the $y-$axis that we have a change in signs of $D_2 m(x, y)$, meaning that along the cross section of the $y-$axis there is a curve that is sloped. Secondly, note that from $D_1 m(x, y)$ that if we pass the $x = y$ or $x = -y$ lines then $D_1 m(x, y)$ also changes signs, once again implying that along the cross section of those lines there is a curve that is sloped. If we do calculations of values in $B(\mathbf{0}, \varepsilon)$, we see that if we observe a cross section, holding a positive x constant, of the monkey saddle we see an downward facing parabola, as expected as this represents 2 minimum from the $x = y$ and $x = -y$ lines, and a maximum from the $y-$axis. See picture below.



If we observe a cross section, holding a negative x constant, of the monkey saddle we see an upwards facing parabola, as expected as this represents 2 maximum from the $x = y$ and $x = -y$ lines, and a minimum from the $y-$axis. See picture below.



Finally, if we observe a cross section, holding a positive/negative y constant, of the monkey saddle we see an upwards/downwards sloping curve, as expected as this represents 1 maximum/minimum from the $x = y$ line, and a minimum/maximum from the $x = -y$. See pictures below.

The name is called a monkey's saddle because f has a sort of saddle point at $(0,0)$ but with three up-directions and three down-directions, thus it is a **monkey's saddle** because there is room for both of the monkey's legs and its tail.

Problem 4

First we will note the general form,

$$\gamma'(t) = (\nabla f)(\gamma(t)), \qquad \gamma(0) = \vec{a}$$

Let $\gamma(t) = (x(t), y(t))$, then we see,

$$x'(t) = -\frac{y(t)}{x(t)^2 + y(t)^2} \qquad y(0) = 1$$

$$y'(t) = \frac{x(t)}{x(t)^2 + y(t)^2} \qquad x(0) = 1$$

We can see that if we have $x(t)^2 + y(t)^2 = 1$, then $x'(t) = -y(t)$ and $y'(t) = x(t)$.
If we divide the second equation by the first we see,

$$\frac{x'(t)}{y'(t)} = -\frac{y(t)}{x(t)},$$

thus since we noticed earlier that this can be easily evaluated if $x(t)^2 + y(t)^2 = 1$, we will suppose $x(t) = \cos(t)$ and $y(t) = \sin(t)$ (as that is the only $x(t)$ and $y(t)$ I know that behaves as such), and we have equality on all of the differential equations above as,

$$\frac{x'(t)}{y'(t)} = \frac{-\sin(t)}{\cos(t)} = -\frac{y(t)}{x(t)}.$$

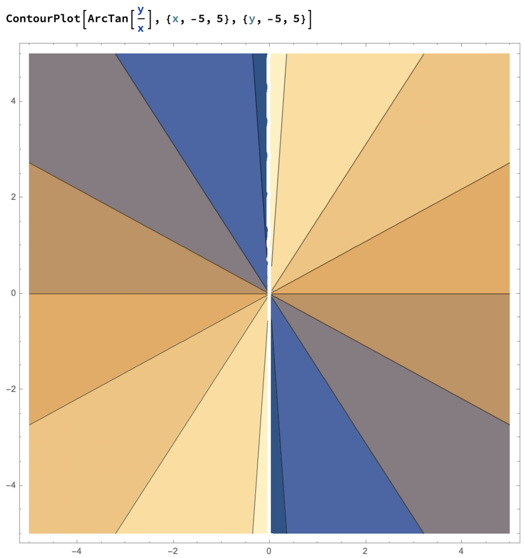
Now $x(0) = 1 = \cos(0)$ and $y(0) = 0 = \sin(0)$, thus we have now also shown that this anzantz satisfies not only the differential equations but also the boundary conditions. Hence the solution is,

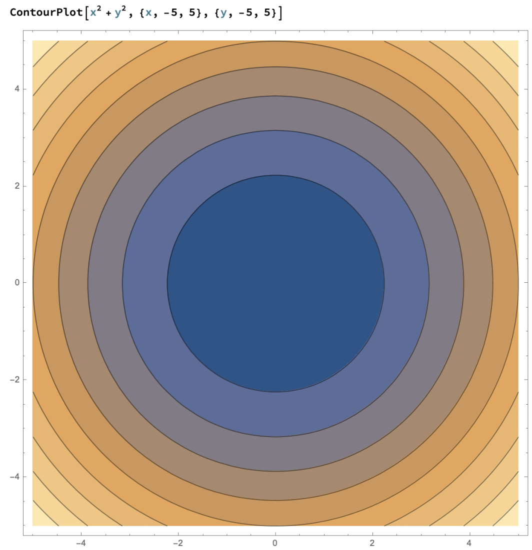
$$(x(t), y(t)) = (\cos(t), \sin(t))$$

and by a little algebra we see,

$$x^2 + y^2 = \cos^2(t) + \sin^2(t) = 1 \implies x^2 + y^2 = 1$$

which follows visually as seen below because if the graphs are overlaid they are orthogonal.





In total the bug will follow the $x^2 + y^2 = 1$ curve starting from $(1, 0)$, and this path is curved. The bug will effectively follow the top right quarter of a circle with radius 1.