Homework: 6.2, 6.3

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Math 202: Vector Calculus

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6.2.4

We can take positive subintervals, J_i, in a partition P that all contain at least one rational and one irrational value in J_i [1] and then,

$$m_J(f) = 0 \qquad M_J(f) = 1$$

Then we see that

$$U(f,P) = \sum_J M_J(f) \cdot \text{length}(J) = \sum_J \text{length}(J) = 1$$

$$L(f,P) = \sum_J m_J(f) \cdot \text{length}(J) = \sum_J 0 \cdot \text{length}(J) = 0$$

However,

$$U(f,P) - L(f,P) < \varepsilon$$

for $\varepsilon > 0$, but

$$U(f,P) - L(f,P) = \sum_J \text{length}(J) = 1 \not< \varepsilon$$

for all $\varepsilon > 0$ thus f is not integratable by the criteria.

6.2.5

We can see that f is integrable. First f is bounded as $f(x,y) < 1 + \varepsilon$ (for $\varepsilon > 0$) for all x, y. Secondly, (as the criteria only requires the existence of 1 partition that satisfies the conditions), let the partition be (for $\varepsilon > 0$)

$$P = \{0, (1-\varepsilon)/2, 1/2, 1\}$$

then,

$$L(f,P) = 0 \cdot (1-\varepsilon)/2 + 1 \cdot \varepsilon/2 + 1 \cdot 1/2 = (1+\varepsilon)/2$$

and

$$U(f,P) = 0 \cdot (1-\varepsilon)/2 + 0 \cdot \varepsilon/2 + 1 \cdot 1/2 = (1+\varepsilon)/2 = 1/2$$

Thus we see,

$$U(f,P) - L(f,p) = \varepsilon/2 < \varepsilon$$

[1] For example, one may take $P = \{0, 1/2, 1\}$ in which $\frac{1}{4} \in \mathbb{Q}$ and $\frac{1}{4} \in J_1$, and $\sqrt{\frac{3}{17}} \notin \mathbb{Q}$ and $\sqrt{\frac{3}{17}} \in J_1$, and $\frac{3}{4} \in \mathbb{Q}$ and $\frac{3}{4} \in J_2$, and $\sqrt{\frac{3}{2}} \notin \mathbb{Q}$ and $\sqrt{\frac{3}{2}} \in J_2$

which satisfies the integratability criteria.

Let us take the partition $P = \{0, 1/2, 1\} \times \{0, 1\}$, which can be refined but for our purposed need not be. The lower sum will be,

$$L(f, P) = m_{J_1}(f) \cdot \mathrm{vol}(J_1) + m_{J_2}(f) \cdot \mathrm{vol}(J_2) = 0 \cdot 1/2 + 1 \cdot 1/2 = 1/2$$

Then, $L \int_B f \geq 1/2$, and we know from before that $U \int_B f \leq (1+\varepsilon)/2$, but since ε is an arbitrarily small number, let it be 0. Thus,

$$U \int_B f \leq (1+\varepsilon)/2 = 1/2 \leq L \int_B f$$

where we can then see, since they are equal, that

$$\int_B f = 1/2$$

6.2.6

Note first that a rearrangement of **a** produces,

$$U(f+g, P) - L(f+g, P) \leq (U(f, P) - L(f, P)) + (U(g, P) - L(g, P)).$$

Suppose there exists a partition P' and P'' (which is allowed since both f and g are integrable) such that

$$U(f, P') - L(f, P') < \varepsilon/2 \qquad U(g, P'') - L(g, P'') < \varepsilon/2$$

Let their common refinement be P, then

$$U(f, P) - L(f, P) < \varepsilon/2 \qquad U(g, P) - L(g, P) < \varepsilon/2$$

Then our first equality becomes

$$U(f+g, P) - L(f+g, P) < \varepsilon$$

which satisfies our criterion, in conjunction with the fact that $f+g$ is bounded(as both f and g are bounded). Thus $\int_B (f+g)$ exists.

Now suppose there exists a partition Q and R such that P is their refinement, then we see,

$$L(f, Q) + L(g, R) \leq L(f, P) + L(g, P) \leq L(f+g, P) \leq \int_B (f+g)$$

taking the supremum of both sides we see,

$$\int_B f + \int_B g \leq \int_B (f+g)$$

We can also see,

$$U(f, Q) + U(g, R) \geq U(f, P) + U(g, P) \geq U(f+g, P) \geq \int_B (f+g)$$

taking the infimum of both sides we see,

$$\int_B f + \int_B g \ \geq \int_B (f+g).$$

Observing this, we can then see

$$\int_B (f+g) = \int_B f + \int_B g$$

6.3.2

First note that,

$$|f(\tilde{x}) - f(\tilde{x})| = |\tilde{x}^3 - x^3| = |\tilde{x} - x||\tilde{x}^2 + \tilde{x}x + x^2|$$

We can see that, by letting

$$|\tilde{x} - x| < 1 \implies |\tilde{x}| - |x| < 1 \implies |\tilde{x}| < 1 + |x|$$

then

$$|\tilde{x}^2 + \tilde{x}x + x^2| \leq |\tilde{x}|^2 + |\tilde{x}||x| + |x|^2 \leq |1 + |x||^2 + |1 + |x|||x| + |x|^2 \leq (1 + |x|)^2 + (1 + |x|)|x| + |x|^2 =$$

$$= 1 + 3|x| + 3|x|^2$$

Hence, $\delta = \min\{1, \frac{\varepsilon}{1+3|x|+3|x|^2}\}$ where the 1 comes from the choice that $|\tilde{x} - x| < 1$. Then we see, if $|\tilde{x} - x| < \delta$,

$$\begin{aligned} |f(\tilde{x}) - f(x)| &= |\tilde{x}^3 - x^3| \\ &\leq |\tilde{x} - x|(1 + 3|x| + 3|x|^2) \\ &< \delta(1 + 3|x| + 3|x|^2) \\ &= \frac{\varepsilon}{1 + 3|x| + 3|x|^2}(1 + 3|x| + 3|x|^2) \\ &= \varepsilon \end{aligned}$$

6.3.3

Let $x = \frac{1}{\sqrt{\delta}}$ and $\tilde{x} = \frac{1}{\sqrt{\delta}} + \frac{\delta}{3}$ Then,

$$|\tilde{x} - x| = |\frac{1}{\sqrt{\delta}} + \frac{\delta}{3} - \frac{1}{\sqrt{\delta}}| = \frac{\delta}{3} < \delta$$

Then we see,

$$|f(\tilde{x}) - f(x)| = \left|\left(\frac{\delta}{3} + \frac{1}{\sqrt{\delta}}\right)^3 - \left(\frac{1}{\sqrt{\delta}}\right)^3\right| = \left|\frac{\delta^{3/2}}{3} + \frac{1}{\delta^{3/2}} + \frac{\delta^3}{27} + 1 - \frac{1}{\delta^{3/2}}\right| = \frac{\delta^{3/2}}{3} + \frac{\delta^3}{27} + 1 > \varepsilon$$

Thus we see that this contradicts uniform continuity on $\mathbb{R}$. By theorem 6.3.6 we can see $[0, 500] \subset \mathbb{R}$, f is point-wise continuous on $[0, 500]$ and $[0, 500]$ is compact as it is closed and bounded, thus the conditions are satisfied and f is uniformly continuous on the interval $[0, 500]$.

6.3.4

a.

We will use the Mean Value Theorem directly, note that for all $\tilde{x}, x \in I$ there exists a c in between $\tilde{x}$ and x such that,

$$|f(\tilde{x}) - f(x)| = |f'(c)||(\tilde{x} - x)| \leq R|\tilde{x} - x|$$

We know by Proposition 4.3.4 that since f is diffrentiable, it is point-wise continuous. Let $\delta = \varepsilon/R$ and $\varepsilon > 0$, by point wise continuity we see,

$$|f(\tilde{x}) - f(x)| \leq R|\tilde{x} - x| < R \cdot \varepsilon/R = \varepsilon$$

Thus giving us exactly what we desired, uniform continuity.

b.

We can see that both sine and cosine have derivatives less than a positive constant R, in particular $R = 1$. Of course they are both diffrentiable, thus satisfying the conditions set in **a** and hence both sine and cosine are uniformly continuous on $\mathbb{R}$.