Problem Set 7

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Due October 30th, 2020

Problem 1

a.

$$\text{tr}(\mathbb{A}\mathbb{B}) = \sum_{i=1}^{n}(ab)_{ii} = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}b_{ji} = \sum_{i=1}^{n}\sum_{j=1}^{n} b_{ji}a_{ij} = \sum_{j=1}^{n}(ba)_{jj} = \text{tr}(\mathbb{B}\mathbb{A})$$

b.

Let $\mathbb{W} = \mathbb{V}\mathbb{L}\mathbb{V}^T$, decomposing $\mathbb{W}$ into the orthogonal matrix $\mathbb{V}$ with the eigenvectors of $\mathbb{W}$ as its columns, and a diagonal $\mathbb{L}$ with the eigenvalues of $\mathbb{W}$ down its diagonal, then we see,

$$\text{tr}(\mathbb{W}) = \text{tr}(\mathbb{V}\mathbb{L}\mathbb{V}^T) = \text{tr}(\mathbb{V}\mathbb{V}^T\mathbb{L}) = \text{tr}(\mathbb{I}\mathbb{L}) = \text{tr}(\mathbb{L}) = \sum_{i=0}^{n}\lambda_i$$

Problem 2

Note, $\det(\mathbb{A}\mathbb{B}) = \det(\mathbb{B}\mathbb{A})$ so,

$$\det(\mathbb{M}) = \det(\mathbb{V}\mathbb{L}\mathbb{V}^T) = \det(\mathbb{V}\mathbb{V}^T\mathbb{L}) = \det(\mathbb{I}\mathbb{L}) = \det(\mathbb{L}) = \prod_{j=1}^{n}(\mathbb{L})_{jj}$$

thus the determinant of $\mathbb{M}$ is equal to the product of its eigenvalues.

Problem 3

Let $\mathbb{A}\mathbf{v} = \lambda\mathbf{v}$ and $\mathbb{A}\mathbf{w} = \lambda\mathbf{w}$, so, let us take,

$$a(\mathbb{A}\mathbf{v} = \lambda\mathbf{v})$$

$$+b(\mathbb{A}\mathbf{w} = \lambda\mathbf{w})$$

$$= \mathbb{A}(a\mathbf{v} + b\mathbf{w}) = \lambda(a\mathbf{v} + b\mathbf{w}) \implies \mathbb{A}\mathbf{z} = \lambda\mathbf{z}$$

thus, the linear combination of 2 eigenvectors with a shared eigenvalue, is another eigenvector with eigenvalue λ.

Now suppose, $\mathbb{A}\mathbf{v} = \lambda_1\mathbf{v}$ and $\mathbb{A}\mathbf{w} = \lambda_2\mathbf{w}$, then,

$$a(\mathbb{A}\mathbf{v} = \lambda_1\mathbf{v})$$

$$+b(\mathbb{A}\mathbf{w} = \lambda_2 \mathbf{w})$$

$$= \mathbb{A}(a\mathbf{v} + b\mathbf{w}) = \lambda_1 a\mathbf{v} + \lambda_2 b\mathbf{w} \implies \mathbb{A}\mathbf{z} = \lambda_1 a\mathbf{v} + \lambda_2 b\mathbf{w}$$

Thus, $\mathbf{z}$ is not an eigenvector.

Problem 4

Suppose there exists an eigenvector $\mathbf{u}$=$\mathbf{v}$ such that the eigenvalue is λ and $\mathbf{u}\cdot\mathbf{u} = 1$. Then suppose that $\mathbf{x} = \mathbf{v} + \alpha\mathbf{w}$ which is also an eigenvector with eigenvalue λ (by Problem 3), so

$$\mathbf{u}\cdot\mathbf{x} = 0 = 1 + \alpha\mathbf{v}\cdot\mathbf{w} \implies \alpha = \frac{-1}{\mathbf{v}\cdot\mathbf{w}}$$

thus,

$$\mathbf{x} = \mathbf{v} - \frac{\mathbf{w}}{\mathbf{v}\cdot\mathbf{w}}$$

and let $\mathbf{g} = \frac{\mathbf{x}}{\sqrt{\mathbf{x}\cdot\mathbf{x}}}$, hence $\mathbf{g}\cdot\mathbf{g} = 1$, so $\mathbf{u}$ and $\mathbf{g}$ are eigenvectors with eigenvalue λ that are orthogonal.

Problem 5

We know in general we have the solution

$$f(x) = A\cos(\omega x) + B\sin(\omega x)$$

then using the boundary conditions we see,

$$A = 0 \qquad B\sin(\omega L) = 0$$

but for $B = 0$ would be problematic, so suppose $\sin(\omega L) = 0$ then $\omega L = n\pi$ so, we see the eigenfunction and eigenvalues are,

$$f(x) = B\sin\left(\frac{n\pi}{L}x\right) \qquad \lambda = -\left(\frac{n\pi}{L}\right)^2$$

Problem 6

Suppose $\mathbf{v}^j$ is an eigenvector of $\mathbb{M}$, or $\mathbb{M}\mathbf{v}^j = \lambda_j\mathbf{v}^j$, then we see that if we dot $(\mathbf{v}^j)^\dagger$,

$$(\mathbf{v}^j)^\dagger\mathbb{M}\mathbf{v}^j = \lambda_j(\mathbf{v}^j)^\dagger\mathbf{v}^j$$

if we also transpose and take the conjugate of the whole equation we see,

$$(\mathbf{v}^j)^\dagger\mathbb{M}^T = \lambda_j^*(\mathbf{v}^j)^\dagger$$

then multiply by $\mathbf{v}^j$ on the right side we see,

$$(\mathbf{v}^j)^\dagger\mathbb{M}^T\mathbf{v}^j = \lambda_j^*(\mathbf{v}^j)^\dagger\mathbf{v}^j$$

if we add the last and the first equations we see,

$$(\mathbf{v}^j)^\dagger(\mathbb{M} + \mathbb{M}^T)\mathbf{v}^j = (\lambda_j + \lambda_j^*)(\mathbf{v}^j)^\dagger\mathbf{v}^j$$

but we know that $\mathbb{M} = -\mathbb{M}^T$, thus the left hand side is 0. Assuming $\mathbf{v}^j$ is a nonzero vector, the only way the right hand side can be zero is if $\lambda_j = -\lambda_j^*$ is a complex number or 0, thus with $\mathbb{M}$ its eigenvalues have no nonzero real parts.

If we do the same as above, except no complex conjugation, we see,

$$(\mathbf{v}^j)^T(\mathbb{M} + \mathbb{M}^T)\mathbf{v}^j = (\lambda_j + \lambda_j)(\mathbf{v}^j)^T\mathbf{v}^j$$

where once again, the left hand side is 0. In order to be 0 on the left hand side, either $(\mathbf{v}^j)^T\mathbf{v}^j = 0$ or $\lambda_j = 0$, which is clearly not always the case.

Problem 7

Our given equations are,

$$m_1\ddot{x}_1(t) = -k_1(x_1(t) - a) + k_2(x_2(t) - x_1(t) - a)$$

$$m_2\ddot{x}_2(t) = -k_2(x_2(t) - x_1(t) - a)$$

So when we set $m_1 = 0$, we see,

$$k_1(x_1(t) - a) = k_2(x_2(t) - x_1(t) - a) \implies (x_1(t) - a) = \frac{k_2}{k_1}(x_2(t) - x_1(t) - a)$$

Imagining the mass 1 as zero we can imagine the equation of motion to be, and by using what we have just shown we see,

$$m_2\ddot{x}_2(t) = -K(x_2(t) - x_1(t) - a + x_1(t) - a) \implies -k_2 = K\left(1 + \frac{k_2}{k_1}\right) \implies \frac{1}{K} = \frac{1}{k_1} + \frac{1}{k_2}$$

Thus $K = \left(\frac{1}{k_1} + \frac{1}{k_2}\right)^{-1}$

Problem 8

First note that,

$$\tilde{p}(f) = \int_{-\infty}^{\infty} p(t)e^{-i2\pi ft}dt$$

thus if $f = 0$, then

$$\tilde{p}(0) = \int_{-\infty}^{\infty} p(t)dt$$

which is the mean of the signal.