Homework: 6.1, 6.2

Monroe Stephenson
Math 202: Vector Calculus

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6.1.1

a.

The subintervals of P are $J_1 = [0, 1/2]$ and $J_2 = [1/2, 1]$. The lengths of J_i are $\text{length}(J_1) = 1/2 - 0 = 1/2$ and $\text{length}(J_2) = 1 - 1/2 = 1/2$.

The sub-intervals of P' are $J_1 = [0, 3/8]$, $J_2 = [3/8, 5/8], J_3 = [5/8, 1]$ The lengths of J_i are $\text{length}(J_1) = 3/8 - 0 = 3/8$, $\text{length}(J_2) = 5/8 - 3/8 = 1/4$, and $\text{length}(J_3) = 1 - 5/8 = 3/8$.

Note, in this particular case, $P'' = P \cup P' = \{0, 3/8, 1/2, 5/8, 1\}$. Thus the subintervals are, $J_1 = [0, 3/8], J_2 = [3/8, 1/2], J_3 = [1/2, 5/8], J_4 = [5/8, 1]$. Respectively their lengths are $\text{length}(J_1) = 3/8 - 0 = 3/8$,$\text{length}(J_2) = 1/2 - 3/8 = 1/8$,$\text{length}(J_3) = 5/8 - 1/2 = 1/8$, and $\text{length}(J_4) = 1 - 5/8 = 3/8$

b.

First we know by definition of Cartesian product that the subboxes of Q must be,

$$[0, 1/2] \times [0, 1/2], [0, 1/2] \times [1/2, 1], [1/2, 1] \times [0, 1/2], [1/2, 1] \times [1/2, 1]$$

Note that since the subintervals in **a** were all 1/2 and the subintervals of $\{0, 1/2, 1\}$ are of length 1/2 we have area for all of them to be 1/4.

We know that for Q', by the Cartesian product, that the subboxes must be,

$$[0, 3/8] \times [0, 1/2], [0, 3/8] \times [1/2, 1]$$

$$[3/8, 5/8] \times [0, 1/2], [3/8, 5/8] \times [1/2, 1]$$

$$[5/8, 1] \times [0, 1/2], [5/8, 1] \times [1/2, 1]$$

The first row's area is, $3/8 \cdot 1/2 = 3/16$, the second row's area is, $1/4 \cdot 1/2 = 1/8$, and the third row's area is, $3/8 \cdot 1/2 = 3/16$.

Finally we know for Q'', by the Cartesian product, that the subboxes must be,

$$[0, 3/8] \times [0, 1/2], [0, 3/8] \times [1/2, 1]$$

$$[3/8, 1/2] \times [0, 1/2], [3/8, 1/2] \times [1/2, 1]$$

$$[1/2, 5/8] \times [0, 1/2], [1/2, 5/8] \times [1/2, 1]$$

$$[5/8, 1] \times [0, 1/2], [5/8, 1] \times [1/2, 1]$$

The first row's area is, $3/8 \cdot 1/2 = 3/16$, the second row's area is, $1/8 \cdot 1/2 = 1/16$, the third row's area is, $1/8 \cdot 1/2 = 1/16$, and the fourth row's area is $3/8 \cdot 1/2 = 3/16$.

6.1.3

a.

When $x = 0$, $f(x) = 0$. When $x = 0.5$, $f(x) = 0.25$. We can see that since $0 \leq x \leq 1$ that our value can never be negative. Thus, $m_J(f) = 0$. We also can see that $f(x) = x - x^2$, $f'(x) = 1 - 2x$, and $f''(x) = -2$ thus at $x = 0.5$ is a maximum and $M_J(f) = 0.25$.

b.

We can find an irrational number, say $\sqrt{2}$, then $f(x) = 1$ which is the largest number possible in our range (as $m \in \mathbb{Z}_+$ and thus, $1/m \leq 1$), thus $M_J(f) = 1$. Now we can imagine an infinitesimal rational number in the form $\varepsilon = 1/m$, thus,

$$f(\varepsilon) = \frac{1}{m} = \varepsilon$$

thus we can see $m_J(f) = 0$ since our values get arbitrarily close to 0. Note that we could not have $f(x)$ that is negative since $m > 0$.

c.

Note that $f(x)$ is trapped in between $1-x$ and $-1+x$ and that as it approaches 0 it takes on values arbitrarily close to -1 and 1 but never actually -1 and 1. Hence, $m_J(f) = -1$ and $M_J(f) = 1$

6.1.4

a.

$$L(f, P) = 0^2 \cdot 1/2 + (1/2)^2 \cdot 1/2 = 1/8$$

$$L(f, P') = 0^2 \cdot 3/8 + (3/8)^2 \cdot 1/4 + (5/8)^2 \cdot 3/8 = 93/512$$

$$L(f, P'') = 0^2 \cdot 3/8 + (3/8)^2 \cdot 1/8 + (1/2)^2 \cdot 1/8 + (5/8)^2 \cdot 3/8 = 25/128$$

$$U(f, P) = (1/2)^2 \cdot 1/2 + (1)^2 \cdot 1/2 = 5/8$$

$$U(f, P') = (3/8)^2 \cdot 3/8 + (5/8)^2 \cdot 1/4 + (1)^2 \cdot 3/8 = 269/512$$

$$U(f, P'') = (3/8)^2 \cdot 3/8 + (1/2)^2 \cdot 1/8 + (5/8)^2 \cdot 1/8 + (1)^2 \cdot 3/8 = 65/128$$

We can then see that in fact,

$$L(f, P) = 1/8 \leq 5/8 = U(f, P)$$

$$L(f,P') = 93/512 \leq 269/512 = U(f,P')$$
$$L(f,P'') = 25/128 \leq 65/128 = U(f,P'')$$
$$U(f,P'') = 65/128 \leq 269/512 = U(f,P')$$
$$U(f,P') = 269/512 \leq 5/8 = U(f,P)$$
$$L(f,P'') = 25/128 \leq 93/512 = L(f,P')$$
$$L(f,P') = 93/512 \leq 1/8 = L(f,P)$$
$$L(f,P) = 1/8 \leq 269/512 = U(f,P')$$
$$L(f,P') = 93/512 \leq 65/128 = U(f,P'')$$

or more succinctly,

$$\max\{L(f,P), L(f,P')\} \leq L(f,P'') \leq U(f,P'') \leq \min\{U(f,P), U(f,P')\}$$

b.

$$L(f,Q) = 0 \cdot 1/4 + 0 \cdot 1/4 + 1 \cdot 1/4 + 1 \cdot 1/4 = 1/2$$
$$L(f,Q') = 0 \cdot 3/16 + 0 \cdot 3/16 + 0 \cdot 1/8 + 0 \cdot 1/8 + 1 \cdot 3/16 + 1 \cdot 3/16 = 3/8$$
$$L(f,Q'') = 0 \cdot 3/16 + 0 \cdot 3/16 + 0 \cdot 1/16 + 0 \cdot 1/16 + 1 \cdot 1/16 + 1 \cdot 1/16 + 1 \cdot 3/16 + 1 \cdot 3/16 = 1/2$$
$$U(f,Q) = 1 \cdot 1/4 + 1 \cdot 1/4 + 1 \cdot 1/4 + 1 \cdot 1/4 = 1$$
$$U(f,Q') = 0 \cdot 3/16 + 0 \cdot 3/16 + 1 \cdot 1/8 + 1 \cdot 1/8 + 1 \cdot 3/16 + 1 \cdot 3/16 = 5/8$$
$$U(f,Q'') = (3/8)^2 \cdot 3/8 + (1/2)^2 \cdot 1/8 + (5/8)^2 \cdot 1/8 + (1)^2 \cdot 3/8 = 5/8$$

and note, similarly to above,

$$\max\{L(f,Q), L(f,Q')\} \leq L(f,Q'') \leq U(f,Q'') \leq \min\{U(f,Q), U(f,Q')\}$$

6.2.1

For brevity, let $\mathcal{L} = \{L(f,P) : P \text{ is a partition of } B\}$ and $\mathcal{U} = \{U(f,P) : P \text{ is a partition of } B\}$. By Lemma 6.2.2 we know that $\sup\{\mathcal{L}\} \leq \inf\{\mathcal{U}\}$, since by Proposition 6.1.10 we know that $L(f,P) \leq U(f,P)$ for all P, thus satisfying the conditions for Lemma 6.2.2. Note though that,

$$L\int_B f = \sup\{\mathcal{L}\} \leq \inf\{\mathcal{U}\} = U\int_B f$$

and the desired result is seen,

$$L\int_B f \leq U\int_B f.$$

6.2.3

First note that f is bounded, as $k+\varepsilon$ for $\varepsilon > 0$ is always greater than $f(x)$ for all x. Now, note,

$$m_J(f) = \sup\{f(x) : x \in J\} = \sup\{k : x \in J\} = k$$

$$M_J(f) = \inf\{f(x) : x \in J\} = \inf\{k : x \in J\} = k$$

since $f(x) = k$ for all x. Then,

$$L(f, P) = \sum_J k \cdot \mathrm{vol}(J)$$

$$U(f, P) = \sum_J k \cdot \mathrm{vol}(J)$$

Thus we can see that

$$U(f, P) - L(f, P) = \sum_J k \cdot \mathrm{vol}(J) - \sum_J k \cdot \mathrm{vol}(J) = 0$$

and the conditions for integratability is met as for all $\varepsilon > 0$, $U(f, P) - L(f, P) = 0 < \varepsilon$.

Note that for all P,

$$L\int_B f = \sup\{L(f, P) : P \text{ is a partition of } B\} = L(f, P) =$$

$$= U(f, P) = \inf\{U(f, P) : P \text{ is a partition of } B\} = U\int_B f$$

Thus we see that we can just set,

$$\int_B f = \sum_J k \cdot \mathrm{vol}(J)$$

and we can just let the partition be the box B, thus,

$$\int_B f = k \cdot \mathrm{vol}(B)$$