Homework: 4.6

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Math 202: Vector Calculus

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4.6.2

Note that, $x_s = x_t = y_s = -y_t = 1$ and $y_{st} = x_{st} = 0$. From the chain rule we see,

$$u_s = u_x + u_y$$

and applying the chain rule again, since the functions are $\mathcal{C}^2$, and the fact that $u_{xx} - u_{yy} = 0$ we see,

$$u_{st} = u_{xx} - u_{xy} + u_{yx} - u_{yy} \implies u_{st} = 0$$

Thus we have found the relationship.

4.6.3

a.

Let w be defined in terms of p, q, then by the chain rule we can see,

$$w_x = w_p p_x + w_q q_x$$

and furthermore, noting $p_{xx} = 0$, then

$$w_{xx} = p_x^2 w_{pp} + 2p_x q_x w_{qp} + w_{qq} q_x^2$$

we can define w_{tt} identically.
$(\implies)$ Let $c^2 w_{xx} = w_{tt}$ so our equality becomes,

$$c^2(p_x^2 w_{pp} + 2p_x q_x w_{qp} + w_{qq} q_x^2) = p_t^2 w_{pp} + 2p_t q_t w_{qp} + w_{qq} q_t^2$$

and let us note that $p_t = -q_t = c$ and $p_x = q_x = 1$, thus our equality becomes,

$$c^2(w_{pp} + 2w_{qp} + w_{qq}) = c^2 w_{pp} - c^2 w_{qp} + c^2 w_{qq} \implies w_{qp} = -w_{qp} \implies w_{qp} = 0$$

$(\impliedby)$ Let $w_{pq} = 0$, then

$$c^2(w_{pp} + w_{qq}) = c^2(w_{pp} + w_{qq}) \implies c^2(w_{pp} + w_{qp} + w_{qq}) = c^2 w_{pp} - c^2 w_{qp} + c^2 w_{qq} \implies c^2 w_{xx} = w_{tt}$$

b.

As above we defined w in terms of p, q. We can see that since $w = F(x+ct) + G(x-ct)$ and $p = x+ct$, $q = x-ct$, then

$$w = F(p) + G(q).$$

We then see that $w_p = F'(p)$ and $w_{pq} = 0$, thus w satisfies the wave equation, as this is equivalent as shown in **a**.

c.

We begin,

$$\begin{aligned}
y + cu &= \gamma\left((x - vt) + c\left(t - \frac{v}{c}x\right)\right) \\
&= \gamma\left(x - \frac{v}{c}x - vt + ct\right) \\
&= \gamma\left(x\left(1 - \frac{v}{c}\right) + ct\left(1 - \frac{v}{c}\right)\right) \\
&= \gamma\left(1 - \frac{v}{c}\right)(x + ct)
\end{aligned}$$

and similarly,

$$\begin{aligned}
y - cu &= \gamma\left((x - vt) - c\left(t - \frac{v}{c}x\right)\right) \\
&= \gamma\left(x + \frac{v}{c}x - vt - ct\right) \\
&= \gamma\left(x\left(1 + \frac{v}{c}\right) - ct\left(1 + \frac{v}{c}\right)\right) \\
&= \gamma\left(1 + \frac{v}{c}\right)(x - ct)
\end{aligned}$$

d.

Note that w is a function of r, s and so using the chain rule we see,

$$w_p = w_r r_p$$

and note that $r_{pq} = 0$, $s_q = \gamma(1 + v/c)$, and $r_p = \gamma(1 - v/c)$ so

$$w_{pq} = w_{rs} s_q r_p + w_r r_{pq} = w_{rs}\gamma^2(1 - v/c)(1 + v/c) = w_{rs} \implies w_{rs} = 0 \iff c^2 w_{yy} = w_{uu}$$

Therefore we see that if w satisfies the wave equation in the original time ans space variables, then it also satisfies the wave equation in the new time and space variables.

4.6.4

a.

Note that if u is in terms of x, y then $u_s = u_x x_s$ and $u_{ss} = (u_x)_s x_s + u_x x_{ss} = u_{xx} x_s^2 + u_x x_{ss}$ and similarly for u_{tt}. Letting $x = e^s$ and $y = e^t$ (and noting $x = x_s = x_{ss}$ and $y = y_t = y_{tt}$) we see,

$$e^{2s} u_{xx} + e^{2t} u_{yy} + e^s u_x + e^t u_y = (u_{xx} x_s^2 + u_x x_{ss}) + (u_{yy} y_t^2 + u_y y_{tt}) = u_{ss} + u_{tt} = 0$$

and this satisfies the wave equation as desired.

4.6.5

a.

Let us look at u_s,

$$u_s = u_x x_s + u_y y_s$$

and

$$u_{ss} = (u_x)_s x_s + u_x x_{ss} + (u_y)_s y_s + u_y y_{ss}$$

$$= (x_s)(u_{xx}x_s + u_{xy}y_s) + (y_s)(u_{xy}x_s + u_{yy}y_s) + u_x x_{ss} + u_y y_{ss}$$

and similarly for u_{tt} so we can see (noting, $x_s = 2s$, $x_{ss} = 2$, $y_s = 2s$, $y_{ss} = 0$, $x_t = -2t$, $x_{tt} = -2$, $y_t = 2t$, $y_{tt} = 0$),

$$u_{ss} = 4u_{xx}s^2 + 8u_{xy}st + 4u_{yy}t^2 + 2u_x$$

$$u_{tt} = 4u_{xx}t^2 - 8u_{xy}st + 4u_{yy}s^2 - 2u_x$$

so we can see the sum of the right is,

$$4(s^2+t^2)(u_{xx}+u_{yy}) = 0 \implies u_{ss} + u_{tt} = 0$$

where $u_{ss} + u_{tt}$ is from the left hand of the sum.

b.

We can see, $u_\rho = u_r r_\rho$ and $u_{\rho\rho} = u_{rr}r_\rho^2 + u_r r_{\rho\rho}$ and similar for $u_{\phi\phi}$. We know, $r_\rho = k\rho^{k-1}, r_{\rho\rho} = k(k-1)\rho^{k-2}, \theta_\phi = k, \theta_{\phi\phi} = 0$ so we have,

$$u_\rho = k\rho^{k-1}u_r$$

$$u_{\rho\rho} = k^2\rho^{2k-2}u_{rr} + k(k-1)\rho^{k-2}u_r$$

$$u_{\phi\phi} = k^2 u_{\theta\theta}$$

We can see that,

$$\rho^2 u_{\rho\rho} + \rho u_\rho + u_{\phi\phi} = k^2\rho^{2k}u_{rr} + k(k-1)p^k u_r + kp^k u_r + k^2 u_{\theta\theta} = k^2(r^2 u_{rr} + r u_r + u_{\theta\theta}) = 0$$