Homework: Preface

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Math 202: Vector Calculus

Due September 2nd, 2020

Exercise 1

a.

Suppose there exists 2 surfaces in space. With any given point there is a tangent plane to it, and thus a normal line to the plane. For the two closest points between the surfaces, the normal lines must in fact be the same line. Equivalently, the tangent planes must be parallel. Note, this is not only true for the closest (global minimum in a sense), but also for all the points that are local maximum/minimum and the global maximum.

b.

Suppose there exists a surface and a curve in space. For the surface there is a tangent plane and a normal line. On the curve there is a tangent line and thus a normal plane. The two closest points between the surface and the curve is when the normal line of the surface is embedded in the normal plane of the curve. Equivalently, the tangent line to the curve is perpendicular to the normal line of the surface, the tangent plane of the surface is parallel to the tangent line of the curve, or the tangent plane of the surface is perpendicular to the normal plane of the curve.

c.

Suppose there exists two curves in space, and with them each point has a tangent line and a normal plane. Then, the two closest points have shared normal planes. Equivalently, the tangent line to one point is parallel to the other's tangent line, or the tangent line to one is perpendicular to the other's normal plane.

Exercise 2

a.

The general formula for a ball in $n-$dimensional space is

$$\text{vol}(B_n(r)) = \frac{\pi^{\frac{n}{2}}}{\frac{n}{2}!} r^n$$

Thus we know a $1-$dimensional ball that the formula is $2r$, thus

$$2r = \frac{\sqrt{\pi}}{\frac{1}{2}!} r \implies \frac{1}{2}! = \frac{\sqrt{\pi}}{2}.$$

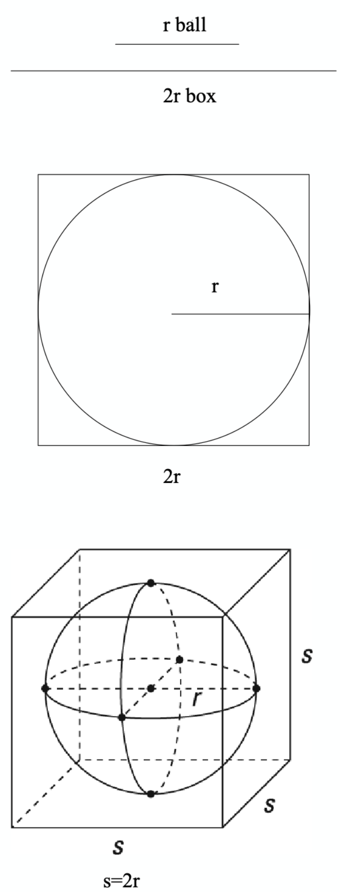
For a 3−dimensional ball,

$$\mathrm{vol}(B_3(r)) = \frac{\pi^{\frac{3}{2}}}{\frac{3}{2}!}r^3 = \frac{\pi^{\frac{3}{2}}}{\frac{3}{2}\frac{\sqrt{\pi}}{2}}r^3 = \pi^{\frac{3}{2}}\frac{4}{3\pi^{\frac{1}{2}}}r^3 = \frac{4}{3}\pi r^3.$$

Thus we verify the formula.
For a 5−dimensional ball,

$$\mathrm{vol}(B_5(r)) = \frac{\pi^{\frac{5}{2}}}{\frac{5}{2}!}r^5 = \frac{\pi^{\frac{5}{2}}}{\frac{5}{2}\frac{3}{2}\frac{\sqrt{\pi}}{2}}r^5 = \pi^{\frac{5}{2}}\frac{8}{15\pi^{\frac{1}{2}}}r^5 = \frac{8}{15}\pi^2 r^5.$$

b.



This limit can be restated as

$$\lim_{n\to\infty}\frac{1}{(2r)^n}\frac{\pi^{\frac{n}{2}}}{\frac{n}{2}!}r^n = \lim_{n\to\infty}\frac{\pi^{\frac{n}{2}}}{2^n(\frac{n}{2})!} = \lim_{n\to\infty}\frac{(\frac{\sqrt{\pi}}{2})^n}{\frac{n}{2}!}$$

Which the limit approaches 0, since there is a term on top that begins to decrease, and the bottom increases.