MATH 202: VECTOR CALCULUS
CAES 9.5
HOMEWORK DUE WEDNESDAY WEEK 11

Problem 1. We can see the shape is a half chalice as in Figure 1,

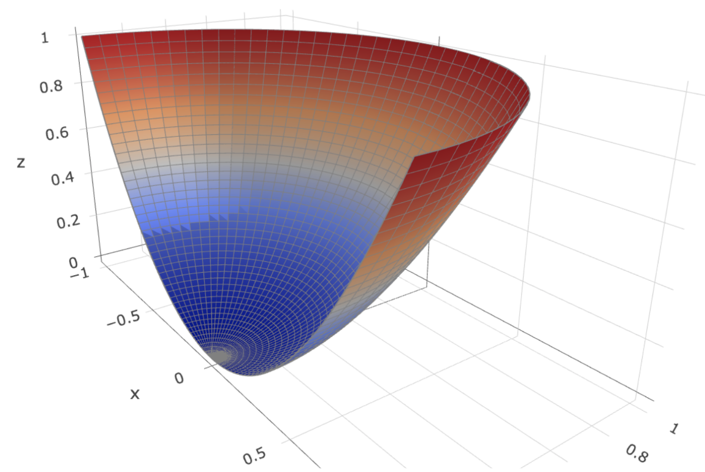

FIGURE 1.

For $dx \wedge dy$ we can see that this is a semicircle and thus is $\frac{\pi a^2}{2}$.
For $dy \wedge dz$ we can see that the projections cancel out and thus the value is 0.
For $dz \wedge dx$ we can see that we want to find the area in the parabola, so we can imagine the parabola in a box with sides a^2 and $2a$, and thus,

$$\pm 2a \cdot a^2 \mp 2 \cdot \int_{x=0}^{a} x^2 = \pm 2a^3 \mp \frac{2a^3}{3} = \pm \frac{4a^3}{3}$$

and we expect the value to be negative because the plane has opposite orientation to the image.
Finally, we can imagine that the $z \wedge dy$ fills the area below the chalice like object, and the $-ydz \wedge dx$ fills the area above the chalice like object, thus we see a half-cylinder, so the value must be, with a radius a and a height a^2 $\frac{\pi a^4}{2}$
We will first find

$$\Phi' = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \\ 2r & 0 \end{pmatrix}$$

then,

$$\int_{\Phi} dx \wedge dy = \int_{r=0}^{a} \int_{\theta=0}^{\pi} \det \Phi'_{1,2} = \int_{r=0}^{a} \int_{\theta=0}^{\pi} r = \frac{\pi a^2}{2}$$
$$\int_{\Phi} dy \wedge dz = \int_{r=0}^{a} \int_{\theta=0}^{\pi} \det \Phi'_{2,3} = \int_{r=0}^{a} \int_{\theta=0}^{\pi} -2r^2 \cos\theta = 0$$

$$\int_{\Phi} dz \wedge dx = \int_{r=0}^{a} \int_{\theta=0}^{\pi} \det \Phi'_{3,1} = \int_{r=0}^{a} -2r^2 \cdot \int_{\theta=0}^{\pi} \sin\theta = -\frac{4a^3}{3}$$

$$\int_{\Phi} zdx \wedge dy - ydz \wedge dx = \int_{r=0}^{a} \int_{\theta=0}^{\pi} r^2 \cdot \det \Phi'_{1,2} - r\sin\theta \cdot \det \Phi'_{3,1}$$

$$= \int_{r=0}^{a} \int_{\theta=0}^{\pi} r^2 \cdot r - r\sin\theta \cdot -2r^2 \sin\theta = \int_{r=0}^{a} \int_{\theta=0}^{\pi} r^3(1 + 2\sin\theta) = \frac{\pi a^4}{2}$$

as desired.

Problem 2. We can see the surface below,

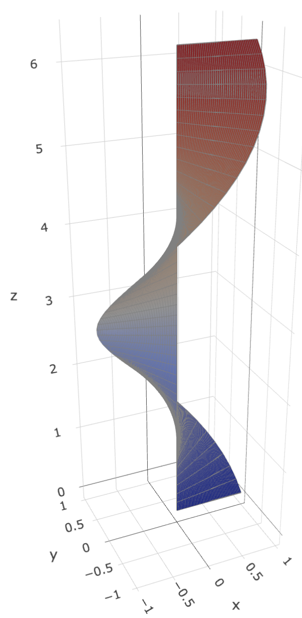

FIGURE 2.

First we will find Φ'

$$\Phi' = \begin{pmatrix} -v\sin u & \cos u \\ v\cos u & \sin u \\ 1 & 0 \end{pmatrix}$$

Thus our integral becomes,

$$
\begin{aligned}
\int_\Phi \omega &= \int_{u=0}^{2\pi}\int_{v=0}^{1} v\cos u \cdot \det \Phi'_{2,3} + v \sin u \cdot \det \Phi'_{1,2} \\
&= \int_{u=0}^{2\pi}\int_{v=0}^{1} -v\cos u \cdot \sin u + v\sin u \cdot (-v \sin^2 u - v\cos^2 u) \\
&= -\int_{u=0}^{2\pi}\int_{v=0}^{1} v\cos u \cdot \sin u + v^2 \sin u \\
&= -\int_{u=0}^{2\pi} \left(\frac{1}{2}v^2 \cos u \cdot \sin u + \frac{1}{3} v^2 \sin u\right)\Bigg|_{v=0}^{1} \\
&= -\int_{u=0}^{2\pi} \frac{1}{4}\sin(2u) + \frac{1}{3}\sin u \\
&= -\left(-\frac{1}{8}\cos(2u) - \frac{1}{3}\cos u\right)\Bigg|_{u=0}^{2\pi} \\
&= 0
\end{aligned}
$$

Problem 3.

$$
\begin{aligned}
\int_\Delta \omega &= \int_D (f\circ\Delta)\cdot \det \Delta' && \text{by 9.14} \\
&= \int_{\Delta(D)} f && \text{by change of variable theorem} \\
&= \int_D f && \text{since } \Delta(D) = D
\end{aligned}
$$

Problem 4. First we will draw a commuting diagram,

$$
\begin{tikzcd}
D \arrow[rr, bend left, "T"] \arrow[dr, bend right, "\Phi"'] & & \tilde{D} \arrow[ll, bend left, "S"] \arrow[dl, bend left, "\Psi"] \\
& A &
\end{tikzcd}
$$

a) If we differentiate both sides we see,

$$(S\circ T)'(u) = \text{id}'(u)$$

$$S'(T(u)))\cdot T'(u) = I$$

then taking the determinants we see,

$$\det(S'(T(u)))\cdot T'(u)) = \det(S'(T(u)))) \cdot \det(T'(u)) = \det(I) = 1$$

which we can then see,

$$\det(T'(u)) = \frac{1}{\det(S'(T(u)))}$$

which implies that $\det T'(u) \neq 0$ for all $u \in D$ as desired.

b) Let

$$M = \begin{pmatrix} M_1 \\ M_2 \\ \vdots \\ M_n \end{pmatrix}$$

where each M_i is a row of k entries. Then,

$$(MN)_I = \left(\begin{pmatrix} M_1 \\ M_2 \\ \vdots \\ M_n \end{pmatrix} N\right)_I = \begin{pmatrix} M_1N \\ M_2N \\ \vdots \\ M_nN \end{pmatrix}_I = \begin{pmatrix} M_{i_1}N \\ M_{i_2}N \\ \vdots \\ M_{i_k}N \end{pmatrix} = \begin{pmatrix} M_{i_1} \\ M_{i_2} \\ \vdots \\ M_{i_k} \end{pmatrix} N = M_I N$$

c) Using the chain rule we see,

$$(\Psi \circ T)' = (\Phi)'$$
$$\Psi'(T) \cdot T' = \Phi'.$$

Using part **b**,

$$(\Psi'(T) \cdot T')_I = \Psi'_I(T) \cdot T' = \Phi'_I$$

then finally taking the determinant of both sides,

$$\det(\Psi'_I(T) \cdot T') = \det(\Psi'_I(T)) \cdot \det(T') = \det(\Phi'_I)$$

as desired.

d) We can see,

$$\int_\Psi \omega = \int_{\tilde{D}} (f \circ \Psi) \cdot \det \Psi'_I = \int_{T(D)} (f \circ \Psi) \cdot \det \Psi'_I$$

from here we can use the change of variable theorem, and

$$\int_{T(D)} (f \circ \Psi) \cdot \det \Psi'_I = \int_D ((f \circ \Psi \cdot \det \Psi'_I) \circ T) \cdot \det T'$$

from here

$$\int_D ((f(\Psi) \cdot \det \Psi'_I)(T) \cdot \det T' = \int_D f(\Psi(T)) \cdot \det \Psi'_I(T) \cdot \det T'$$

from **c** we know

$$\int_D f(\Psi(T)) \cdot \det \Psi'_I(T) \cdot \det T' = \int_D f(\Psi(T)) \cdot \det \Phi'_I$$

we also know from the commuting diagram above that $\Psi \circ T = \Phi$ thus we see,

$$\int_D (f \circ \Phi) \cdot \det \Phi'_I$$

and once again from change of variable,

$$\int_D (f \circ \Phi) \cdot \det \Phi'_I = \int_\Phi \omega \implies \int_\Phi \omega = \int_\Psi \omega$$

as desired.

For an orientation reversing, we know $|\det T'| = -\det T'$ thus,

$$\int_\Phi \omega = -\int_\Psi \omega$$

e) Yes it is still valid, we only would end up pushing sums through if we were to show **d** for the more general form.