Chapter 2 Exam Redo

Monroe Stephenson
Math 202: Vector Calculus
Time: 12:00PM to 2:45PM

Due September 14th, 2020

1.

Let $A = (x, y), B = (z, w) \in \mathbb{R}^2$ and define the inner product as such

$$\langle\langle , \rangle\rangle : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$$

$$\langle\langle A, B\rangle\rangle = xz - xw - yz + 2yw.$$

In order for this definition to stand we need to verify the three inner product properties.

First, positive definiteness.

$$\begin{aligned}
\langle\langle A, A\rangle\rangle &= x^2 - 2xy - 2x^2 && \text{by definition of inner product} \\
&= (x-y)^2 + y^2 && \text{by factoring} \\
&\geq 0 && \text{by the sum of squares over a field}
\end{aligned}$$

On the last part, this implies that $\langle\langle A, A\rangle\rangle \geq 0$ with equality if and only if both $x - y$ and x equal 0 or $A = \mathbf{0}$. Thus we have shown our inner product's positive definiteness.

For symmetry,

$$\begin{aligned}
\langle\langle A, B\rangle\rangle &= xz - xw - yz + 2yw && \text{by definition of inner product} \\
&= zx - wx - zy + 2wy && \text{by commutativity of multiplication field axiom} \\
&= zx + (-wx) + (-zy) + 2wy && \text{by additive inverse field axiom} \\
&= zx + (-zy) + (-wx) + 2wy && \text{by commutativity of addition field axiom} \\
&= zx - zy - wx + 2wy && \text{by additive inverse field axiom} \\
&= \langle\langle B, A\rangle\rangle && \text{by definition of inner product}
\end{aligned}$$

Thus we have shown symmetry of inner product.

Finally, for bilinearity of inner product of which will prove one of the conditions and take the other three for granted. Let $A' = (x', y')$

$$\begin{aligned}
\langle\langle A + A', B\rangle\rangle &= (x + x')z - (x + x')w - (y + y')z + 2(y + y')w && \text{by definition of inner product}\\
&= xz + x'z - xw - x'w - yz - y'z + 2yw + 2y'w && \text{by distributive field axiom}\\
&= xz + x'z + (-xw) + (-x'w) + (-yz) + (-y'z) + 2yw + 2y'w && \text{by additive inverse field axiom}\\
&= (xz + (-xw) + (-yz) + 2yw) && \text{by commutativity of}\\
&+ (x'z + (-x'w) + (-y'z) + 2y'w) && \text{addition field axiom}\\
&= (xz - xw - yz + 2yw) && \text{by additive}\\
&+ (x'z - x'w - y'z + 2y'w) && \text{inverse field axiom}\\
&= \langle\langle A, B\rangle\rangle + \langle\langle A', B\rangle\rangle && \text{by definition of inner product}
\end{aligned}$$

Thus we have shown the third and final property of the inner product, bilinearity.

We will define modulus as

$$|| \quad || : \mathbb{R}^2 \to \mathbb{R}$$

For $A \in \mathbb{R}^2$

$$||A|| = \sqrt{\langle\langle A, A\rangle\rangle}.$$

Note all the properties of modulus follow from the inner product properties.

Now we will prove the inequality. Note that still, for brevity, $A = (x, y), B = (z, w)$.
First note that the left hand side

$$|xz - xw - yz + 2yw| = |\langle\langle A, B\rangle\rangle|$$

and the right hand side,

$$\sqrt{x^2 - 2xy - 2x^2}\sqrt{z^2 - 2zw - 2w^2} = \sqrt{\langle\langle A, A\rangle\rangle}\sqrt{\langle\langle B, B\rangle\rangle} = ||A||\,||B||.$$

Thus our inequality becomes

$$|\langle\langle A, B\rangle\rangle| \leq ||A||\,||B||,$$

which is the Cauchy-Schwarz Inequality which holds with this inner product since the inequality is reliant only on the inner product properties. Thus the inequality is proven. □

2.

a.

The composition of the two functions makes sense. Given an $x \in A \subset \mathbb{R}^p$ (where A is the domain of f), then $f(x) \in f(A) = B \subset \mathbb{R}^n$ (where $f(A)$ is the range of f, B is the domain of g, and $\mathbb{R}^n$ is the codomain of f). Since the range of f and domain of g are equal then given $f(x) \in B$, $g(f(x))$ maps to $\mathbb{R}^m$, as the codomain of g is $\mathbb{R}^m$ and the range of g is $g(B)$. Thus the chain follows. One could see as such,

$$g \circ f : A \to \mathbb{R}^p \supset f(A) = B \to \mathbb{R}^m \supset g(B)$$

$$g \circ f : A \to \mathbb{R}^m \supset g(B)$$

Thus the dimension of the final Euclidean space is m, and the inputs to $g \circ f$ are elements of A

b.

Suppose there exists a sequence in A, $\{x_\nu\}$ that converges to $x \in A$. By the continuity of f, this implies, that if $\{x_\nu\}$ converges to x then $\{f(x_\nu)\} \subset f(A) = B$ converges to $f(x) \in B$. Since $\{f(x_\nu)\} \in B$ and g is continuous, it follows by the definition of continuity that if $\{f(x_\nu)\} \in B$ converges to $f(x) \in B$, then the sequence $\{g(f(x_\nu))\}$ converges to $g(f(x))$. Thus we have now shown that if f and g are continuous then so if their composition, $g \circ f$. $\square$

3.

First we will use the straight line test to test for discontinuity. Take a sequence $\{(x_\nu, y_\nu)\}$ approaching $\mathbf{0}$ along $y = mx$. Then

$$f(x_\nu, y_\nu) = f(x_\nu, mx_\nu) = \frac{x_\nu^2(mx_\nu)^3}{x_\nu^4 + (mx_\nu)^6} = \frac{m^3 x_\nu^5}{x_\nu^4(1 + m^6 x_\nu^2)} = \frac{m^3 x_\nu}{(1 + m^6 x_\nu^2)},$$

which as $x_\nu \to 0$, then the sequence goes to 0.

Now we will use the curved line test to test for discontinuity. Take a sequence $\{(x_\nu, y_\nu)\}$ approaching $\mathbf{0}$ along $y^{3/2} = x$. Then

$$f(x_\nu, y_\nu) = f(y_\nu^{3/2}, y_\nu) = \frac{(y_\nu^{3/2})^2 y_\nu^3}{(y_\nu^{3/2})^4 + (y_\nu)^6} = \frac{y_\nu^6}{y_\nu^6 + y_\nu^6} = \frac{y_\nu^6}{2y_\nu^6} = \frac{1}{2},$$

which as $y_\nu \to 0$, then the sequence goes to $\frac{1}{2}$.

Thus a contradiction. There cannot be a single value that makes f continuous at $\mathbf{0}$.

For g we will use the size bounds, which show us,

$$0 \le |g(x,y)| = \frac{|xy - y^2|}{\sqrt{x^2+y^2}} \le \frac{|xy|}{\sqrt{x^2+y^2}} + \frac{|y^2|}{\sqrt{x^2+y^2}} = \frac{|x||y|}{\sqrt{x^2+y^2}} + \frac{|y|^2}{\sqrt{x^2+y^2}}$$

$$\le \frac{|(x,y)|^2}{\sqrt{|(x,y)|^2}} + \frac{|(x,y)|^2}{\sqrt{|(x,y)|^2}} = 2\frac{|(x,y)|^2}{|(x,y)|} = 2|(x,y)|.$$

Thus as a sequence $\{(x_\nu, y_\nu)\}$ of nonzero input vectors converge to $\mathbf{0}$, the corresponding sequence of outputs $\{g(x_\nu, y_\nu)\}$ is squeezed to 0 in absolute value and hence converges to 0. Thus g is continuous at $\mathbf{0}$ when $b = 0$.$\square$

4.

a.

Take $A = B(\mathbf{0}, R)$ such that $R > 0$, and $p = \mathbf{0}$. Then, since the ball is centered at p, the same ball can be taken and seen that it does not have a point from A^c, thus p is not a boundary point. However, it is a limit point as any given ball would contain a point $x \in A$ such that $x \neq p$.

b.

Take $A = \{\mathbf{0}\}$ and $p = \mathbf{0}$. Then any ball around p contains points from A and A^c thus it is a boundary point. It is not a limit point as there is no ball that can contain a point other than p itself.

c.

Let $p \in A^c$.
$(\implies)$ If p is a limit point of A, that means for all $\varepsilon > 0$ there exists a ball $B(p, \varepsilon)$ and a point $x \in A$ such that $B(p, \varepsilon) \cap x \neq \emptyset$ and $x \neq p$. This ball must always contain a point in A by the definition of a limit point and a point in A^c by assumption, thus this ball satisfies the conditions for a ball $B(p, \varepsilon)$ which implies p as a boundary point. Thus if $p \in A^c$ and p is a limit point of A, p must be a boundary point of A.

$(\impliedby)$ If p is a boundary point of A that means that for all $\varepsilon > 0$, there exists a ball $B(p, \varepsilon)$ that contains a point in A and a point in A^c. We know that $B(p, \varepsilon)$ contains a point in A^c by assumption. $B(p, \varepsilon)$ must also contain a point $x \in A$ by definition of boundary point. That point cannot be p by our assumption $p \in A^c$. Thus for all $\varepsilon > 0$ there exists a ball $B(p, \varepsilon)$ that contains $x \in A$ such that $x \neq p$, which is the definition of limit point. Thus if p is a boundary point of A and $p \in A^c$ then p is a limit point of A

Hence we have show that if $p \in A^c$, then p is a limit point of A if and only if p is a boundary point of A.$\square$

d.

One of the key assumption in proving **c** is that $p \in A^c$, which is not true in either a or b. Had $p \notin A^c$, then for $\implies$ we would not need to have a point in A^c, thus the implication to the right fails. For $\impliedby$, p could be in A, thus not requiring any points to be in A^c.

e.

$(\implies)$ Let A be closed, thus containing all of its limit points. Suppose A does not contain all of its boundary points. Take a point $p \in A^c$ that is a boundary point. By **c**, p is a limit point. Hence a contradiction, as A contains all of its limit points but p is not in A. Thus if A is closed, A must contain all of its boundary points.

$(\impliedby)$ Let A contains all of its boundary points. Since all limit points of A are in A or in A^c, suppose there are 2 cases. Trivially, the limit points that are in A are in A. Non-trivially, the limit points that are in A^c are boundary points by **c**, but A contains all the boundary points. Hence a contradiction for the second case. Thus if A contains all of its boundary points, then A contains all of its limit points and hence is closed.

Thus we have shown that A is closed of and only if A contains all of its boundary points.$\square$