Linear Control and Estimation

Vectors

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References

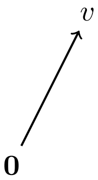
- S Boyd, Applied Linear Algebra: Chapters 1, 2, 3 and 5.

Vectors

- **Vectors** are ordered list of numbers (scalars). $v = \begin{bmatrix} 1.2 \\ -0.1 \\ 2.14 \\ 9.0 \\ -1.24 \end{bmatrix}$
- Scalars can be any *field* $\mathbb{R}, \mathbb{C}, \mathbb{Z}, \mathbb{Q}$.
- We will typically only encouter on $\mathbb{R}$ in this course.
- Individual elements of a vector v are indexed. The i^{th} element of v is referred to as v_i.
- *Dimension* or *size* of a vector is number of elements in the vector.
- Set of n-real vectors is denoted by $\mathbb{R}^n$ (similarly, $\mathbb{C}^n$)
- Vectors a and b are equal, if
  - both have the same size; and
  - $a_i = b_i, i \in \{1, 2, 3, \ldots n\}$

Vectors

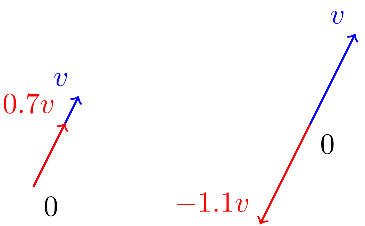
- **Unit vector** $e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ **Zero vector** $\mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ **One vector** $\mathbf{1} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$
- Geometrically, real n-vectors can be thought of as points in $\mathbb{R}^n$ space.



Vectors

- **Vector scaling**: Multiplication of a scalar and a vector.

$$w = av = a \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} av_1 \\ av_2 \\ av_3 \\ \vdots \\ av_n \end{bmatrix} \quad a \in \mathbb{R}; \;\; w, v \in \mathbb{R}^n$$



Properties

- Scalar multiplication is *commutative*.

$$\alpha v = v\alpha$$

- Scalar multiplication is *associative*.

$$(\alpha\beta)\,v = \alpha\,(\beta v)$$

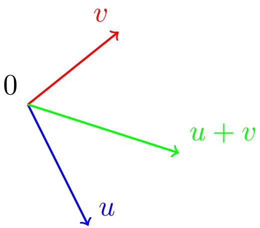
- Scalar multiplication is *distributive*.

$$(\alpha + \beta)\,v = \alpha v + \beta v$$

Vectors

- **Vector addition**: Adding two vectors of the same dimension, element by element.

$$u + v = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ u_3 + v_3 \\ \vdots \\ u_n + v_n \end{bmatrix} \quad u, v \in \mathbb{R}^n$$



Properties

- Vector addition is *commutative*.

$$a + b = b + a$$

- Vector addition is *associative*.

$$(a + b) + c = a + (b + c)$$

- Zero vector has no effect.

$$a + 0 = a$$

- Subtraction of vectors.

$$a + (-1)a = a - a = 0$$

Vector spaces

- A set of vectors V that is closed under **vector addition** and **vector scaling**.

$$\forall x, y \in V, \quad x + y \in V$$

$$\forall x \in V, \text{ and } \alpha \in F, \quad \alpha x \in V$$

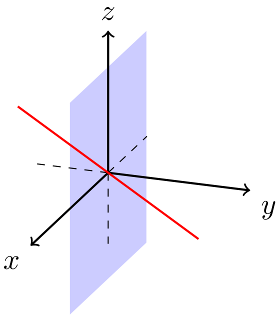
- For a set to be a vector space, it must satisfy the followng properties: $x, y, x \in V$
  - *Commutativity*: $x + y = y + x$
  - *Associativity of vector addition*: $(x + y) + z = x + (y + z)$
  - *Additive identity*: $x + 0 = 0 + x = 0 \; (0 \in V)$
  - *Additive inverse*: $\exists - x \in V, x + (-x) = 0$
  - *Associativity of scalar multiplication*: $\alpha\,(\beta x) = (\alpha \beta x)$
  - *Distributivity of scalar sums*: $(\alpha + \beta)\,x = \alpha x + \beta x$
  - *Distributivity of vector sums*: $\alpha\,(x + y) = \alpha x + \alpha y$
  - *Scalar multiplication identity*: $1x = x$
- We will mostly deal with $\mathbb{R}^n$ vectors spaces in this course.

Subspaces

- A **subspace** S of a vector space V is a subset of V and is itself a vector space.

$$S \subset V, \quad \forall x, y \in S, \alpha x + \beta y \in S, \quad \alpha, \beta \in F$$

- The zero vector is called the **trivial subspace** of a vector space V.
- For example in, in $\mathbb{R}^3$ all planes and lines passing through the origin are subspaces of $\mathbb{R}^3$.



Linear independence

- A collection of vectors $\{x_1, x_2, x_3, \ldots x_n\}, \;\; x_i \in \mathbb{R}^m \;\; i \in \{1, 2, 3, \ldots n\}$ is called *linear dependent* if,

$$\sum_{i=1}^{n} \alpha_i x_i = 0, \text{ hold for some } \alpha_1, \alpha_2, \ldots \alpha_n \in \mathbb{R}, \text{ such that } \exists \alpha_i \neq 0$$

- Another way to state this: A collection of vectors is *linear dependent* if at least one of the vectors in the collection can be expressed as a linear combination of the other vectors in the collection, i.e.

$$x_i = - \sum_{j=1, j \neq i}^{n} \left(\frac{\alpha_j}{\alpha_i} \right) x_j$$

- A collection of vectors is *linear independent* if it is **not** *linearly dependent*.

$$\sum_{i=1}^{n} \alpha_i x_i = 0 \implies \alpha_1 = \alpha_2 = \alpha_3 \ldots = \alpha_n = 0$$

Span of a set of vectors

- Consider a set of vectors $S = \{v_1, v_2, v_3 \ldots v_r\}$ where $v_i \in \mathbb{R}^n, 1 \leq i \leq r$.
- The **span** of the set S is defined as the set of all linear combination of the vectors v_i,

$$span\,(S) = \{\alpha_1 v_1 = \alpha_2 v_2 + \ldots + \alpha_r v_r\}\,,\;\; \alpha_i \in \mathbb{R}$$

- Is $span\,(S)$ a subspace of $\mathbb{R}^n$?
- We say that the subspace $span\,(S)$ is spanned by the *spanning set* S. $\longrightarrow$ S *spans* $span\,(S)$.
- **Sum of subspaces** X, Y is defined as the sum of all possible vectors from X and Y.

$$X + Y = \{x + y \,|\, x \in X, y \in Y\}$$

- Sum of two subspace is also a subspace.

Inner Product

- **Standard inner product** is defined as the following,

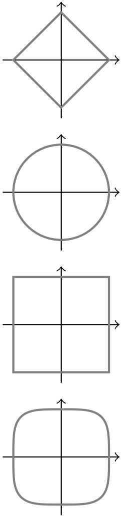
$$x^T y = \sum_{i=1}^{n} x_i y_i, \;\; x, y \in \mathbb{R}^n$$

  For complex vectors: $x^* y = \sum_{i=1}^{n} \overline{x}_i y_i, \;\; x, y \in \mathbb{C}^n$

- **Properties**
  - $x^T x > 0, \;\; \forall x \neq 0$ and $x^T x = 0 \Leftrightarrow x = 0$
  - *Commutative*: $x^T y = y^T x$
  - *Associativity with scalar multiplication*: $(\alpha x)^T y = \alpha \left(x^T y\right)$
  - *Distributivity with vector addition*: $(x + y)^T z = x^T z + y^T z$

Norm

- Norm is a measure of the size of a vector.
- *Euclidean norm* of a n-vector $x \in \mathbb{R}^n$ is defined as, $\|x\|_2 = \sqrt{x^T x} = \sqrt{\sum_{i=1}^n x_i^2}$.
- $\|x\|_2$ is a measure of the length of the vector x.
- Any function of the form $\|\bullet\| : \mathbb{R}^n \longrightarrow \mathbb{R}_{\geq 0}$ is a valid norm, provided it satisfies the following properties.
- **Properties**
  - *Definiteness.* $\|x\| = 0 \iff x = 0$
  - *Non-negativity.* $\|x\| \geq 0$
  - *Non-negative homogeneity.* $\|\beta x\| = |\beta| \|x\|, \beta \in \mathbb{R}$
  - *Triangle inequality.* $\|x + y\| \leq \|x\| + \|y\|$
- p-norm: $\|x\|_p = \left(\sum_{i=1}^n |x_i|^p\right)^{\frac{1}{p}}$
- Norm of difference between two vectors is a measure of the distance between the vectors. $d = \|x - y\|_2$.



$\|x\|_1 = \sum_{i=1}^n |x_i|$

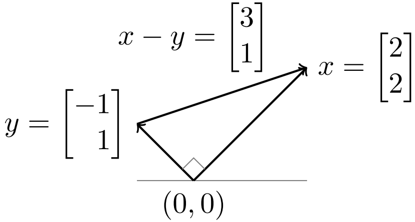
$\|x\|_2 = \left(\sum_{i=1}^n x_i^2\right)^{\frac{1}{2}}$

$\|x\|_\infty = \max_{1 \leq i \leq n} |x_i|$

$\|x\|_p = \left(\sum_{i=1}^n x_i^p\right)^{\frac{1}{p}}$

Orthogonality

- Orthogonality is the idea of two vectors being perperndicular, $x \perp y$.



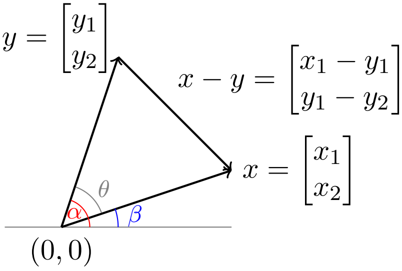
Using the Pythagonean theorem, $\|x - y\|^2 = \|x\|^2 + \|y\|^2$

$$\|x\|^2 + \|y\|^2 - 2x^T y = \|x\|^2 + \|y\|^2 \implies x^T y = 0$$

- We extend this to the n-dimensional case and define two vectors $x, y \in \mathbb{R}^n$ being orthogonal, if

$$x^T y = \sum_{i=1}^{n} x_i y_i = 0$$

Angle between vectors



$$\cos\alpha = \frac{y_1}{\|y\|}, \quad \cos\beta = \frac{x_1}{\|x\|}$$

$$\sin\alpha = \frac{y_2}{\|y\|}, \quad \sin\beta = \frac{x_2}{\|x\|}$$

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\cos\beta$$

$$\cos(\theta) = \frac{x_1 y_1 + x_2 y_2}{\|x\|\|y\|} = \frac{x^T y}{\|x\|\|y\|}$$

$$x^T y = \|x\| \, \|y\| \cos(\theta)$$

- Inner products are used for projecting a vector onto another vector or a subspace.
- It is also a measure of similiarity between two vectors, $\cos(\theta) = \frac{x^T y}{\|x\|\|y\|}$
- **Cauchy-Bunyakovski-Schwartx Inequality**:

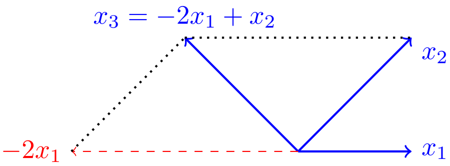
$$\left|x^T y\right| \leq \|x\| \, \|y\|, \quad x, y \in \mathbb{R}^n$$

Basis

Consider a vector $y = \sum_{i=1}^{n} \alpha_i x_i$. What can we say about the coefficients α_is when the collection $\{x_i\}_{i=1}^{n}$ is,

- linearly independent $\implies$ α_is are *unique*.
- linearly dependent $\implies$ α_is are not *unique*.

Consider $\mathbb{R}^2$ vector space. $x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $x_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $x_3 = \begin{bmatrix} -1, 1 \end{bmatrix}$.



Independence-Dimension inequality: What is the maximum possible size of a linearly independent collection?

A linear independent collection of n-vectors can at most have n vectors.

Basis

- A linearly independent set of n n-vector is called a *basis*. In particular, it is a basis of $\mathbb{R}^n$.
- Any n-vector can be represented as a *unique* linear combination of the elements of the basis.
- Consider the basis $\{x_i\}_{i=1}^n$. A n-vector y can be represented as a linear combination of x_is, $y = \sum_{i=1}^n \alpha_i x_i$. This is called the *expansion of* y in the $\{x_i\}_{i=1}^n$ basis.
- The numbers α_i are called the *coefficients* of the expansion of y in the $\{x_i\}_{i=1}^n$ basis.
- **Orthogonal vectors**: A set of vectors $\{x_i\}_{i=1}^n$ is *(mutually) orthogonal* is $x_i \perp x_j$ for all $i, j \in \{1, 2, 3, \ldots n\}$ and $i \neq j$.
- This set is called **orthonormal** if its elements are all of unit length $\|x_i\|_2 = 1$ for all $i \in \{1, 2, 3, \ldots n\}$.

$$x_i^T x_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

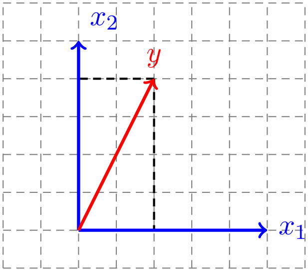
Representing a Vector in an Orthonormal Basis

- An orthonormal collection of vectors is linearly independent.
- Consider an orthonormal basis $\{x_i\}_{i=1}^n$. The expansion of a vector y is given by,

$$y = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \ldots + \alpha_n x_n$$
$$x_i^T y = \alpha_1 x_i^T x_1 + \alpha_2 x_i^T x_2 + \alpha_3 x_i^T x_3 + \ldots + \alpha_n x_i^T x_n = \alpha_i$$

- Thus, we can rewrite this as,

$$y = \left(y^T x_1\right) x_1 + \left(y^T x_2\right) x_2 + \left(y^T x_3\right) x_3 + \ldots + \left(y^T x_1\right) x_n$$



Dimension of a Vector Space

- There an infinite number of bases for a vector space.
- There is one thing that is common among all these bases – the number of bases vectors.
- This number is a property of the vector space, and represents the "degrees of freedom" of the space. This is called the **dimension** of the vector space.
- A subspace of dimension m can have at most m independent vectors.
- Notice that the word "dimension" of a vector space is different from the "dimension" of a vector.
- E.g. Vectors from $\mathbb{R}^3$ are three dimensional vectors. But the yz-plane in $\mathbb{R}^3$ is a 2 dimensional subspace of $\mathbb{R}^3$.

Linear Functions

- Let f be a function which maps real n-vectors to scalar real numbers. It can be represented as the following,

$$f : \mathbb{R}^n \longrightarrow \mathbb{R}; \quad y = f(x) = f\left(x_1, x_2, x_3, \ldots x_n\right)$$

- Criteria for f to be a linear function: **Superposition**: $f\left(\alpha x + \beta y\right) = \alpha f\left(x\right) + \beta f\left(y\right)$, where $\alpha, \beta \in \mathbb{R}$ and $x, y \in \mathbb{R}^n$.
- **Inner product** is a linear function in one of the arguments.

$$f\left(x\right) = w^T x = w_1 x_1 + w_2 x_2 + w_3 x_3 + \ldots + w_n x_n$$

- Any linear function can be represented in the form $f\left(x\right) = w^T x$ with an appropriately chosen w.