Linear Systems

Vectors

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References

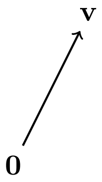
- S Boyd, Applied Linear Algebra: Chapters 1, 2, 3 and 5.

Vectors

- **Vectors** are ordered list of numbers (scalars). $\mathbf{v} = \begin{bmatrix} 1.2 \\ -0.1 \\ \vdots \\ -1.24 \end{bmatrix}$. **Note:** Small bold letter will represent vectors. e.g. $\mathbf{a}, \mathbf{x}, \ldots$
- Scalars can be any *field* $\mathbb{R}, \mathbb{C}, \mathbb{Z}, \mathbb{Q}$. Scalars will be represented using lower case normal font, e.g. $x, y, \alpha, \beta, \ldots$
- We will typically only encouter on $\mathbb{R}$ in this course.
- Individual elements of a vector $\mathbf{v}$ are indexed. The i^{th} element of $\mathbf{v}$ is referred to as v_i.
- *Dimension* or *size* of a vector is number of elements in the vector.
- Set of n-real vectors is denoted by $\mathbb{R}^n$ (similarly, $\mathbb{C}^n$)
- Vectors $\mathbf{a}$ and $\mathbf{b}$ are equal, if
  - both have the same size; and
  - $a_i = b_i, i \in \{1, 2, 3, \ldots n\}$

Vectors

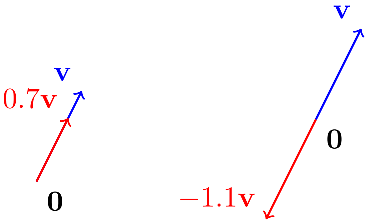
- **Unit vector** $\mathrm{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ **Zero vector** $\mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ **One vector** $\mathbf{1} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$
- Geometrically, real n-vectors can be thought of as points in $\mathbb{R}^n$ space.



Vectors

- **Vector scaling**: Multiplication of a scalar and a vector.

$$\mathbf{w} = a\mathbf{v} = a \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} av_1 \\ av_2 \\ av_3 \\ \vdots \\ av_n \end{bmatrix} \quad a \in \mathbb{R}; \;\; \mathbf{w}, \mathbf{v} \in \mathbb{R}^n$$



Properties

- Scalar multiplication is *commutative*.

$$\alpha \mathbf{v} = \mathbf{v} \alpha$$

- Scalar multiplication is *associative*.

$$(\alpha\beta)\,\mathbf{v} = \alpha\,(\beta\mathbf{v})$$

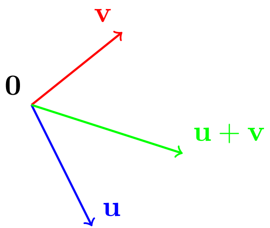
- Scalar multiplication is *distributive*.

$$(\alpha + \beta)\,\mathbf{v} = \alpha\mathbf{v} + \beta\mathbf{v}$$

Vectors

- **Vector addition**: Adding two vectors of the same dimension, element by element.

$$\mathbf{u}+\mathbf{v} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ u_3 + v_3 \\ \vdots \\ u_n + v_n \end{bmatrix} \quad \mathbf{u}, \mathbf{v} \in \mathbb{R}^n$$



Properties

- Vector addition is *commutative*.

$$\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$$

- Vector addition is *associative*.

$$(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$$

- Zero vector has no effect.

$$\mathbf{a} + \mathbf{0} = \mathbf{a}$$

- Subtraction of vectors.

$$\mathbf{a} + (-1)\mathbf{a} = \mathbf{a} - \mathbf{a} = \mathbf{0}$$

Vector spaces

- A set of vectors V that is closed under **vector addition** and **vector scaling**.

$$\forall \mathbf{x}, \mathbf{y} \in V, \quad \mathbf{x} + \mathbf{y} \in V$$

$$\forall \mathbf{x} \in V, \text{ and } \alpha \in F, \quad \alpha \mathbf{x} \in V$$

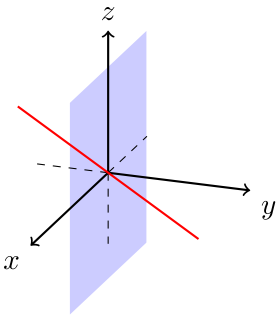
- For a set to be a vector space, it must satisfy the followng properties: $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$
  - *Commutativity*: $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$
  - *Associativity of vector addition*: $(\mathbf{x} + \mathbf{y}) + \mathbf{z} = \mathbf{x} + (\mathbf{y} + \mathbf{z})$
  - *Additive identity*: $\mathbf{x} + \mathbf{0} = \mathbf{0} + \mathbf{x} = \mathbf{x}$ $(0 \in V)$
  - *Additive inverse*: $\exists - \mathbf{x} \in V, \mathbf{x} + (-\mathbf{x}) = \mathbf{0}$
  - *Associativity of scalar multiplication*: $\alpha (\beta \mathbf{x}) = (\alpha \beta \mathbf{x})$
  - *Distributivity of scalar sums*: $(\alpha + \beta) \mathbf{x} = \alpha \mathbf{x} + \beta \mathbf{x}$
  - *Distributivity of vector sums*: $\alpha (\mathbf{x} + \mathbf{y}) = \alpha \mathbf{x} + \alpha \mathbf{y}$
  - *Scalar multiplication identity*: $1 \mathbf{x} = \mathbf{x}$
- We will mostly deal with $\mathbb{R}^n$ vectors spaces in this course.

Subspaces

- A **subspace** S of a vector space V is a subset of V and is itself a vector space.

$$S \subset V, \quad \forall \mathbf{x}, \mathbf{y} \in S, \alpha\mathbf{x} + \beta\mathbf{y} \in S, \quad \alpha, \beta \in F$$

- The zero vector is called the **trivial subspace** of a vector space V.
- For example in, in $\mathbb{R}^3$ all planes and lines passing through the origin are subspaces of $\mathbb{R}^3$.



Linear independence

- A collection of vectors $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots \mathbf{x}_n\}$, $\mathbf{x}_i \in \mathbb{R}^m \;\; i \in \{1, 2, 3, \ldots n\}$ is called *linear dependent* if,

$$\sum_{i=1}^{n} \alpha_i \mathbf{x}_i = 0, \text{ hold for some } \alpha_1, \alpha_2, \ldots \alpha_n \in \mathbb{R}, \text{ such that } \exists \alpha_i \neq 0$$

- Another way to state this: A collection of vectors is *linear dependent* if at least one of the vectors in the collection can be expressed as a linear combination of the other vectors in the collection, i.e.

$$\mathbf{x}_i = - \sum_{j=1, j \neq i}^{n} \left(\frac{\alpha_j}{\alpha_i} \right) \mathbf{x}_j$$

- A collection of vectors is *linear independent* if it is **not** *linearly dependent*.

$$\sum_{i=1}^{n} \alpha_i \mathbf{x}_i = 0 \implies \alpha_1 = \alpha_2 = \alpha_3 \ldots = \alpha_n = 0$$

Span of a set of vectors

- Consider a set of vectors $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \ldots \mathbf{v}_r\}$ where $\mathbf{v}_i \in \mathbb{R}^n, 1 \leq i \leq r$.
- The **span** of the set S is defined as the set of all linear combination of the vectors $\mathbf{v}_i$,

$$span\left(S\right) = \left\{\alpha_1 \mathbf{v}_1 = \alpha_2 \mathbf{v}_2 + \ldots + \alpha_r \mathbf{v}_r\right\}, \ \alpha_i \in \mathbb{R}$$

- Is $span\left(S\right)$ a subspace of $\mathbb{R}^n$?
- We say that the subspace $span\left(S\right)$ is spanned by the *spanning set* S. $\longrightarrow$ S *spans* $span\left(S\right)$.
- **Sum of subspaces** X, Y is defined as the sum of all possible vectors from X and Y.

$$X + Y = \left\{\mathbf{x} + \mathbf{y} \,|\, \mathbf{x} \in X, \mathbf{y} \in Y\right\}$$

- Sum of two subspace is also a subspace.

Inner Product

- **Standard inner product** is defined as the following,

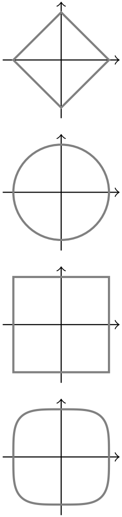
$$\mathbf{x}^T\mathbf{y} = \sum_{i=1}^{n} x_i y_i, \;\; \mathbf{x}, \mathbf{y} \in \mathbb{R}^n$$

  For complex vectors: $\mathbf{x}^*\mathbf{y} = \sum_{i=1}^{n} \overline{x}_i y_i, \;\; \mathbf{x}, \mathbf{y} \in \mathbb{C}^n$

- **Properties**
  - $\mathbf{x}^T\mathbf{x} > 0, \;\; \forall \mathbf{x} \neq 0$ and $\mathbf{x}^T\mathbf{x} = 0 \Leftrightarrow \mathbf{x} = 0$
  - *Commutative*: $\mathbf{x}^T\mathbf{y} = \mathbf{y}^T\mathbf{x}$
  - *Associativity with scalar multiplication*: $(\alpha\mathbf{x})^T\mathbf{y} = \alpha\left(\mathbf{x}^T\mathbf{y}\right)$
  - *Distributivity with vector addition*: $(\mathbf{x}+\mathbf{y})^T\mathbf{z} = \mathbf{x}^T\mathbf{z} + \mathbf{y}^T\mathbf{z}$

Norm

- Norm is a measure of the size of a vector.
- *Euclidean norm* of a n-vector $\mathbf{x} \in \mathbb{R}^n$ is defined as, $\|\mathbf{x}\|_2 = \sqrt{\mathbf{x}^T\mathbf{x}} = \sqrt{\sum_{i=1}^{n} x_i^2}$.
- $\|\mathbf{x}\|_2$ is a measure of the length of the vector $\mathbf{x}$.
- Any function of the form $\|\bullet\| : \mathbb{R}^n \longrightarrow \mathbb{R}_{\geq 0}$ is a valid norm, provided it satisfies the following properties.
- **Properties**
  - *Definiteness.* $\|\mathbf{x}\| = 0 \iff x = 0$
  - *Non-negativity.* $\|\mathbf{x}\| \geq 0$
  - *Non-negative homogeneity.* $\|\beta\mathbf{x}\| = |\beta| \|\mathbf{x}\|, \beta \in \mathbb{R}$
  - *Triangle inequality.* $\|\mathbf{x}+\mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$
- p-norm: $\|\mathbf{x}\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}}$
- Norm of difference between two vectors is a measure of the distance between the vectors. $d = \|\mathbf{x}-\mathbf{y}\|_2$.



$\|\mathbf{x}\|_1 = \sum_{i=1}^{n} |x_i|$

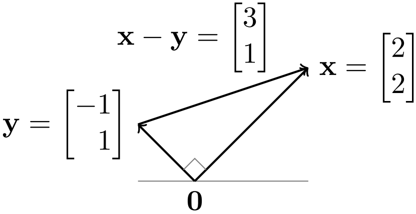
$\|\mathbf{x}\|_2 = \left(\sum_{i=1}^{n} x_i^2\right)^{\frac{1}{2}}$

$\|\mathbf{x}\|_\infty = \max_{1\leq i\leq n} |x_i|$

$\|\mathbf{x}\|_p = \left(\sum_{i=1}^{n} x_i^p\right)^{\frac{1}{p}}$

Orthogonality

- Orthogonality is the idea of two vectors being perperndicular, $\mathbf{x} \perp \mathbf{y}$.



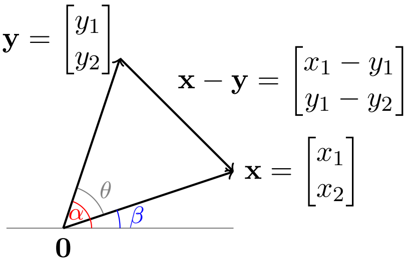
Using the Pythagonean theorem, $\|\mathbf{x} - \mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2$

$$\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 - 2\mathbf{x}^T\mathbf{y} = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 \implies \mathbf{x}^T\mathbf{y} = 0$$

- We extend this to the n-dimensional case and define two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ being orthogonal, if

$$\mathbf{x}^T\mathbf{y} = \sum_{i=1}^{n} x_i y_i = 0$$

Angle between vectors



$$\cos\alpha = \frac{y_1}{\|\mathbf{y}\|}, \quad \cos\beta = \frac{x_1}{\|\mathbf{x}\|}$$

$$\sin\alpha = \frac{y_2}{\|\mathbf{y}\|}, \quad \sin\beta = \frac{x_2}{\|\mathbf{x}\|}$$

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\cos\beta$$

$$\cos(\theta) = \frac{x_1 y_1 + x_2 y_2}{\|\mathbf{x}\|\|\mathbf{y}\|} = \frac{\mathbf{x}^T\mathbf{y}}{\|\mathbf{x}\|\|\mathbf{y}\|}$$

$$\mathbf{x}^T\mathbf{y} = \|\mathbf{x}\|\,\|\mathbf{y}\|\cos(\theta)$$

- Inner products are used for projecting a vector onto another vector or a subspace.
- It is also a measure of similiarity between two vectors, $\cos(\theta) = \frac{\mathbf{x}^T\mathbf{y}}{\|\mathbf{x}\|\|\mathbf{y}\|}$
- **Cauchy-Bunyakovski-Schwartx Inequality**:

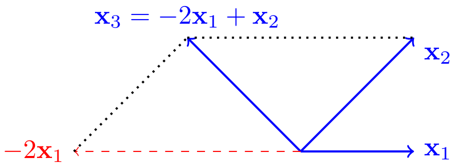
$$\left|\mathbf{x}^T\mathbf{y}\right| \leq \|\mathbf{x}\|\,\|\mathbf{y}\|, \quad \mathbf{x}, \mathbf{y} \in \mathbb{R}^n$$

Basis

Consider a vector $\mathbf{y} = \sum_{i=1}^{n} \alpha_i \mathbf{x}_i$. What can we say about the coefficients α_is when the collection $\{\mathbf{x}_i\}_{i=1}^{n}$ is,

- linearly independent $\implies$ α_is are *unique*.
- linearly dependent $\implies$ α_is are not *unique*.

Consider $\mathbb{R}^2$ vector space. $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\mathbf{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\mathbf{x}_3 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.



Independence-Dimension inequality: What is the maximum possible size of a linearly independent collection?

A linear independent collection of n-vectors can at most have n vectors.

Basis

- A linearly independent set of n n-vector is called a *basis*. In particular, it is a basis of $\mathbb{R}^n$.
- Any n-vector can be represented as a *unique* linear combination of the elements of the basis.
- Consider the basis $\{\mathbf{x}_i\}_{i=1}^n$. A n-vector $\mathbf{y}$ can be represented as a linear combination of $\mathbf{x}_i$s, $\mathbf{y} = \sum_{i=1}^n \alpha_i \mathbf{x}_i$. This is called the *expansion of* $\mathbf{y}$ in the $\{\mathbf{x}_i\}_{i=1}^n$ basis.
- The numbers α_i are called the *coefficients* of the expansion of $\mathbf{y}$ in the $\{\mathbf{x}_i\}_{i=1}^n$ basis.
- **Orthogonal vectors**: A set of vectors $\{\mathbf{x}_i\}_{i=1}^n$ is *(mutually) orthogonal* is $\mathbf{x}_i \perp \mathbf{x}_j$ for all $i, j \in \{1, 2, 3, \ldots n\}$ and $i \neq j$.
- This set is called **orthonormal** if its elements are all of unit length $\|\mathbf{x}_i\|_2 = 1$ for all $i \in \{1, 2, 3, \ldots n\}$.

$$\mathbf{x}_i^T \mathbf{x}_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

Representing a Vector in an Orthonormal Basis

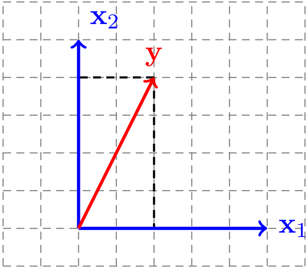
- An orthonormal collection of vectors is linearly independent.
- Consider an orthonormal basis $\{\mathbf{x}_i\}_{i=1}^{n}$. The expansion of a vector $\mathbf{y}$ is given by,

$$\mathbf{y} = \alpha_1\mathbf{x}_1 + \alpha_2\mathbf{x}_2 + \alpha_3\mathbf{x}_3 + \ldots + \alpha_n\mathbf{x}_n$$

$$\mathbf{x}_i^T\mathbf{y} = \alpha_1\mathbf{x}_i^T\mathbf{x}_1 + \alpha_2\mathbf{x}_i^T\mathbf{x}_2 + \alpha_3\mathbf{x}_i^T\mathbf{x}_3 + \ldots + \alpha_n\mathbf{x}_i^T\mathbf{x}_n = \alpha_i$$

- Thus, we can rewrite this as,

$$\mathbf{y} = \left(\mathbf{y}^T\mathbf{x}_1\right)\mathbf{x}_1 + \left(\mathbf{y}^T\mathbf{x}_2\right)\mathbf{x}_2 + \left(\mathbf{y}^T\mathbf{x}_3\right)\mathbf{x}_3 + \ldots + \left(\mathbf{y}^T\mathbf{x}_1\right)\mathbf{x}_n$$



Dimension of a Vector Space

- There an infinite number of bases for a vector space.
- There is one thing that is common among all these bases – the number of bases vectors.
- This number is a property of the vector space, and represents the "degrees of freedom" of the space. This is called the **dimension** of the vector space.
- A subspace of dimension m can have at most m independent vectors.
- Notice that the word "dimension" of a vector space is different from the "dimension" of a vector.
- E.g. Vectors from $\mathbb{R}^3$ are three dimensional vectors. But the yz-plane in $\mathbb{R}^3$ is a 2 dimensional subspace of $\mathbb{R}^3$.

Linear Functions

- Let f be a function which maps real n-vectors to scalar real numbers. It can be represented as the following,
$$f : \mathbb{R}^n \longrightarrow \mathbb{R}; \quad y = f(\mathbf{x}) = f\left(x_1, x_2, x_3, \ldots x_n\right)$$
- Criteria for f to be a linear function: **Superposition**: $f\left(\alpha \mathbf{x} + \beta \mathbf{y}\right) = \alpha f\left(\mathbf{x}\right) + \beta f\left(\mathbf{y}\right)$, where $\alpha, \beta \in \mathbb{R}$ and $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.
- **Inner product** is a linear function in one of the arguments.
$$f\left(x\right) = \mathbf{w}^T \mathbf{x} = w_1 x_1 + w_2 x_2 + w_3 x_3 + \ldots + w_n x_n$$
- Any linear function can be represented in the form $f\left(\mathbf{x}\right) = \mathbf{w}^T \mathbf{x}$ with an appropriately chosen $\mathbf{w}$.