

Linear Systems

Matrix Inverses

Sivakumar Balasubramanian

Department of Bioengineering
Christian Medical College, Bagayam
Vellore 632002

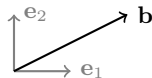
References

- ▶ S Boyd, Applied Linear Algebra: Chapters 11.
- ▶ G Strang, Linear Algebra: Chapters 1.

Representation of vectors in a basis

- Consider the vector space \mathbb{R}^n with basis $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$. Any vector in $\mathbf{b} \in \mathbb{R}^n$ can be represented as a linear combination of \mathbf{v}_i s,

$$\mathbf{b} = \sum_{i=1}^n \mathbf{v}_i \mathbf{a}_i = \mathbf{V} \mathbf{a}; \quad \mathbf{a} \in \mathbb{R}^n, \quad \mathbf{V} = [\mathbf{v}_1 \quad \mathbf{v}_2 \quad \dots \quad \mathbf{v}_n] \in \mathbb{R}^{n \times n}$$



$\{\mathbf{v}_1, \mathbf{v}_2\}$, $\{\mathbf{u}_1, \mathbf{u}_2\}$ and $\{\mathbf{e}_1, \mathbf{e}_2\}$ are valid basis for \mathbb{R}^2 , and the presentation for \mathbf{b} in each one of them is different.

- Finding out \mathbf{a} is easiest when we are dealing with an orthonormal basis, in which case \mathbf{a} is given by,

$$\mathbf{a} = \begin{bmatrix} \mathbf{u}_1^T \mathbf{b} \\ \mathbf{u}_2^T \mathbf{b} \\ \vdots \\ \mathbf{u}_n^T \mathbf{b} \end{bmatrix} = \mathbf{U}^T \mathbf{b} \implies \mathbf{b}_U = \sum_{i=1}^n (\mathbf{u}_i^T \mathbf{b}) \mathbf{u}_i = \mathbf{U} \mathbf{U}^T \mathbf{b}$$

Representation of vectors in a basis

Consider a vector \mathbf{b} whose representation in the standard basis is $\mathbf{b} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

- Consider a basis $V = \left\{ \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}, \begin{bmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix} \right\}$. Find out \mathbf{b}_V .

Representation of vectors in a basis

Consider a vector \mathbf{b} whose representation in the standard basis is $\mathbf{b} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

- Consider a basis $V = \left\{ \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}, \begin{bmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix} \right\}$. Find out \mathbf{b}_V .
- $U = \left\{ \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} \right\}$. Find out \mathbf{b}_U .

Representation of vectors in a basis

Consider a vector \mathbf{b} whose representation in the standard basis is $\mathbf{b} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

- Consider a basis $V = \left\{ \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}, \begin{bmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix} \right\}$. Find out \mathbf{b}_V .
- $U = \left\{ \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} \right\}$. Find out \mathbf{b}_U .
- $W = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ \frac{1}{2} \end{bmatrix} \right\}$. Find out \mathbf{b}_W .

Matrix Inverse

- ▶ Consider the equation $\mathbf{Ax} = \mathbf{y}$, where $\mathbf{A} \in \mathbb{R}^{n \times n}$ and $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.
- ▶ Let us assume \mathbf{A} is non-singular \implies columns of \mathbf{A} represent a basis for \mathbb{R}^n .
- ▶ What does x represent? It is the representation of \mathbf{y} in the basis consisting of the columns of \mathbf{A} .

$$\mathbf{y} = \mathbf{Ax} = [\mathbf{a}_1 \quad \mathbf{a}_2 \quad \dots \quad \mathbf{a}_n] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \sum_{i=1}^n \mathbf{a}_i x_i \implies \mathbf{x} = \mathbf{A}^{-1} \mathbf{y} = \begin{bmatrix} \tilde{\mathbf{b}}_1^T \\ \tilde{\mathbf{b}}_2^T \\ \vdots \\ \tilde{\mathbf{b}}_n^T \end{bmatrix} \mathbf{y} = \begin{bmatrix} \tilde{\mathbf{b}}_1^T \mathbf{y} \\ \tilde{\mathbf{b}}_2^T \mathbf{y} \\ \vdots \\ \tilde{\mathbf{b}}_n^T \mathbf{y} \end{bmatrix}$$

- ▶ \mathbf{A}^{-1} is a matrix that allows change of basis to the columns of \mathbf{A} from the standard basis!

Matrix Inverse

- ▶ Consider the equation $\mathbf{Ax} = \mathbf{y}$, where $\mathbf{A} \in \mathbb{R}^{n \times n}$ and $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.
- ▶ Let us assume \mathbf{A} is non-singular \implies columns of \mathbf{A} represent a basis for \mathbb{R}^n .
- ▶ What does x represent? It is the representation of \mathbf{y} in the basis consisting of the columns of \mathbf{A} .

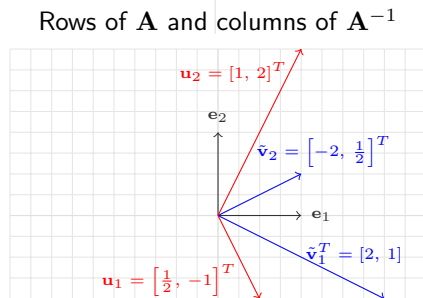
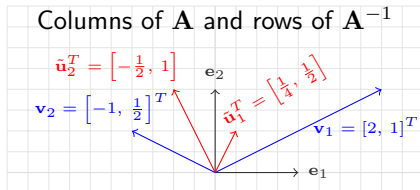
$$\mathbf{y} = \mathbf{Ax} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \dots & \mathbf{a}_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \sum_{i=1}^n \mathbf{a}_i x_i \implies \mathbf{x} = \mathbf{A}^{-1} \mathbf{y} = \begin{bmatrix} \tilde{\mathbf{b}}_1^T \\ \tilde{\mathbf{b}}_2^T \\ \vdots \\ \tilde{\mathbf{b}}_n^T \end{bmatrix} \mathbf{y} = \begin{bmatrix} \tilde{\mathbf{b}}_1^T \mathbf{y} \\ \tilde{\mathbf{b}}_2^T \mathbf{y} \\ \vdots \\ \tilde{\mathbf{b}}_n^T \mathbf{y} \end{bmatrix}$$

- ▶ \mathbf{A}^{-1} is a matrix that allows change of basis to the columns of \mathbf{A} from the standard basis!

• $W = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ \frac{1}{2} \end{bmatrix} \right\}$. Find \mathbf{b}_W by calculating the inverse of the matrix $\mathbf{W} = \begin{bmatrix} 1 & -1 \\ 1 & \frac{1}{2} \end{bmatrix}$. Does your answer match that of the previous approach?

• What about $V = \left\{ \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}, \begin{bmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix} \right\}$. What is \mathbf{b}_V ?

Matrix Inverse



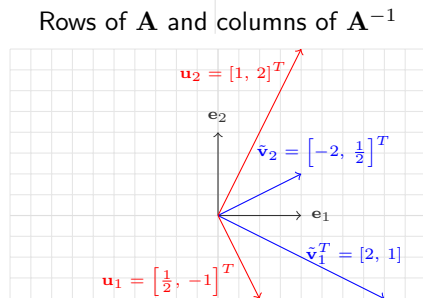
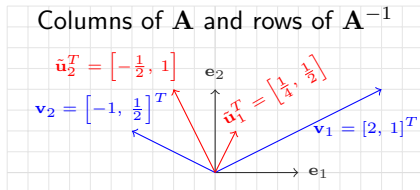
$$\mathbf{V} = [\mathbf{v}_1 \quad \mathbf{v}_2] = \begin{bmatrix} 2 & -1 \\ 1 & \frac{1}{2} \end{bmatrix}$$

$$\mathbf{V}^{-1} = \begin{bmatrix} \tilde{\mathbf{u}}_1^T \\ \tilde{\mathbf{u}}_2^T \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \frac{1}{2} & 1 \\ -1 & 2 \end{bmatrix}$$

$$\mathbf{v}_1^T \tilde{\mathbf{u}}_1 = \mathbf{v}_2^T \tilde{\mathbf{u}}_2 = \tilde{\mathbf{v}}_1^T \mathbf{u}_1 = \tilde{\mathbf{v}}_2^T \mathbf{u}_2 = 1$$

$$\mathbf{v}_1^T \tilde{\mathbf{u}}_2 = \mathbf{v}_2^T \tilde{\mathbf{u}}_1 = \tilde{\mathbf{v}}_1^T \mathbf{u}_2 = \tilde{\mathbf{v}}_2^T \mathbf{u}_1 = 0$$

Matrix Inverse



$$\mathbf{V} = [\mathbf{v}_1 \quad \mathbf{v}_2] = \begin{bmatrix} 2 & -1 \\ 1 & \frac{1}{2} \end{bmatrix}$$

$$\mathbf{V}^{-1} = \begin{bmatrix} \tilde{\mathbf{u}}_1^T \\ \tilde{\mathbf{u}}_2^T \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \frac{1}{2} & 1 \\ -1 & 2 \end{bmatrix}$$

$$\mathbf{v}_1^T \tilde{\mathbf{u}}_1 = \mathbf{v}_2^T \tilde{\mathbf{u}}_2 = \tilde{\mathbf{v}}_1^T \mathbf{u}_1 = \tilde{\mathbf{v}}_2^T \mathbf{u}_2 = 1$$

$$\mathbf{v}_1^T \tilde{\mathbf{u}}_2 = \mathbf{v}_2^T \tilde{\mathbf{u}}_1 = \tilde{\mathbf{v}}_1^T \mathbf{u}_2 = \tilde{\mathbf{v}}_2^T \mathbf{u}_1 = 0$$

Verify these for $\mathbf{W} = \begin{bmatrix} 1 & -1 \\ 1 & \frac{1}{2} \end{bmatrix}$ and

$$\mathbf{V} = \begin{bmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}.$$

Left Inverse

- ▶ Consider a rectangular matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$. There exists no inverse \mathbf{A}^{-1} for this matrix.
- ▶ But, does there exist two matrices $\mathbf{B}, \mathbf{C} \in \mathbb{R}^{n \times m}$, such that,

$$\mathbf{CA} = \mathbf{I}_n \quad \text{and} \quad \mathbf{AB} = \mathbf{I}_m$$

- ▶ Both cannot be true for a rectangular matrix, only one can be true when the matrix is full rank.
- ▶ A rectangular matrix can only have either a left or a right inverse.

Left Inverse

- ▶ Consider a rectangular matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$. There exists no inverse \mathbf{A}^{-1} for this matrix.
- ▶ But, does there exist two matrices $\mathbf{B}, \mathbf{C} \in \mathbb{R}^{n \times m}$, such that,

$$\mathbf{CA} = \mathbf{I}_n \quad \text{and} \quad \mathbf{AB} = \mathbf{I}_m$$

- ▶ Both cannot be true for a rectangular matrix, only one can be true when the matrix is full rank.
- ▶ A rectangular matrix can only have either a left or a right inverse.

Consider a matrix $\mathbf{A} = \begin{bmatrix} 1 & -2 \\ 2 & 1 \\ 1 & 1 \end{bmatrix}$. Let $\mathbf{B}, \mathbf{C} \in \mathbb{R}^{2 \times 3}$. Can you explain why only $\mathbf{CA} = \mathbf{I}_2$ can be true and not $\mathbf{AB} = \mathbf{I}_3$? Can you also explain why \mathbf{C} is not unique?

Left Inverse

- ▶ Any non-zero $\mathbf{a} \in \mathbb{R}^{n \times 1}$ is left invertible: $\mathbf{b}\mathbf{a} = 1$, $\mathbf{b} \in \mathbb{R}^{1 \times n}$; $\mathbf{b}^T = \frac{\mathbf{a}}{\|\mathbf{a}\|^2} + \alpha \mathbf{a}^\perp$
- ▶ This can be generalized to $\mathbf{A} \in \mathbb{R}^{m \times n}$, $m > n$.

$$(\mathbf{C} + \hat{\mathbf{C}}) \mathbf{A} = \mathbf{I}_m \text{ where } \mathbf{C}, \hat{\mathbf{C}} \in \mathbb{R}^{n \times m}, \hat{\mathbf{C}}\mathbf{A} = \mathbf{0}$$

- ▶ Condition for left inverse of \mathbf{A} to exist: *Columns of \mathbf{A} must be independent.*
 $\longrightarrow \text{rank}(\mathbf{A}) = n \longrightarrow \mathbf{A}\mathbf{x} = \mathbf{0} \implies \mathbf{x} = \mathbf{0}.$
- ▶ $\mathbf{A}\mathbf{x} = \mathbf{b}$ can be solved, if and only if $\mathbf{A}(\mathbf{C}\mathbf{b}) = \mathbf{b}$, where $\mathbf{C}\mathbf{A} = \mathbf{I}_n$.

Left Inverse

- ▶ Any non-zero $\mathbf{a} \in \mathbb{R}^{n \times 1}$ is left invertible: $\mathbf{b}\mathbf{a} = 1$, $\mathbf{b} \in \mathbb{R}^{1 \times n}$; $\mathbf{b}^T = \frac{\mathbf{a}}{\|\mathbf{a}\|^2} + \alpha \mathbf{a}^\perp$
- ▶ This can be generalized to $\mathbf{A} \in \mathbb{R}^{m \times n}$, $m > n$.

$$\left(\mathbf{C} + \hat{\mathbf{C}}\right) \mathbf{A} = \mathbf{I}_m \text{ where } \mathbf{C}, \hat{\mathbf{C}} \in \mathbb{R}^{n \times m}, \hat{\mathbf{C}}\mathbf{A} = \mathbf{0}$$

- ▶ Condition for left inverse of \mathbf{A} to exist: *Columns of \mathbf{A} must be independent.*
 $\longrightarrow \text{rank}(\mathbf{A}) = n \longrightarrow \mathbf{A}\mathbf{x} = \mathbf{0} \implies \mathbf{x} = \mathbf{0}.$
- ▶ $\mathbf{A}\mathbf{x} = \mathbf{b}$ can be solved, if and only if $\mathbf{A}(\mathbf{C}\mathbf{b}) = \mathbf{b}$, where $\mathbf{C}\mathbf{A} = \mathbf{I}_n$.

- Let $\mathbf{A} = \begin{bmatrix} 1 & -2 \\ 2 & 1 \\ 1 & 1 \end{bmatrix}$. Find a complete solution for the left inverse of \mathbf{A} such that $\left(\mathbf{C} + \hat{\mathbf{C}}\right) = \mathbf{I}_n$.

Left Inverse

- ▶ Any non-zero $\mathbf{a} \in \mathbb{R}^{n \times 1}$ is left invertible: $\mathbf{b}\mathbf{a} = 1$, $\mathbf{b} \in \mathbb{R}^{1 \times n}$; $\mathbf{b}^T = \frac{\mathbf{a}}{\|\mathbf{a}\|^2} + \alpha \mathbf{a}^\perp$
- ▶ This can be generalized to $\mathbf{A} \in \mathbb{R}^{m \times n}$, $m > n$.

$$(\mathbf{C} + \hat{\mathbf{C}}) \mathbf{A} = \mathbf{I}_m \text{ where } \mathbf{C}, \hat{\mathbf{C}} \in \mathbb{R}^{n \times m}, \hat{\mathbf{C}}\mathbf{A} = \mathbf{0}$$

- ▶ Condition for left inverse of \mathbf{A} to exist: *Columns of \mathbf{A} must be independent.*
 $\longrightarrow \text{rank}(\mathbf{A}) = n \longrightarrow \mathbf{A}\mathbf{x} = \mathbf{0} \implies \mathbf{x} = \mathbf{0}$.
- ▶ $\mathbf{A}\mathbf{x} = \mathbf{b}$ can be solved, if and only if $\mathbf{A}(\mathbf{C}\mathbf{b}) = \mathbf{b}$, where $\mathbf{C}\mathbf{A} = \mathbf{I}_n$.

• Let $\mathbf{A} = \begin{bmatrix} 1 & -2 \\ 2 & 1 \\ 1 & 1 \end{bmatrix}$. Find a complete solution for the left inverse of \mathbf{A} such that $(\mathbf{C} + \hat{\mathbf{C}}) = \mathbf{I}_n$.

• Consider the system, $\mathbf{A}\mathbf{x} = \mathbf{b}$. $\mathbf{A} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\mathbf{x} = [x]$, and $\mathbf{b} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$. Find \mathbf{x} .

Left Inverse

- ▶ Any non-zero $\mathbf{a} \in \mathbb{R}^{n \times 1}$ is left invertible: $\mathbf{b}\mathbf{a} = 1$, $\mathbf{b} \in \mathbb{R}^{1 \times n}$; $\mathbf{b}^T = \frac{\mathbf{a}}{\|\mathbf{a}\|^2} + \alpha \mathbf{a}^\perp$
- ▶ This can be generalized to $\mathbf{A} \in \mathbb{R}^{m \times n}$, $m > n$.

$$(\mathbf{C} + \hat{\mathbf{C}}) \mathbf{A} = \mathbf{I}_m \text{ where } \mathbf{C}, \hat{\mathbf{C}} \in \mathbb{R}^{n \times m}, \hat{\mathbf{C}}\mathbf{A} = \mathbf{0}$$

- ▶ Condition for left inverse of \mathbf{A} to exist: *Columns of \mathbf{A} must be independent.*
 $\longrightarrow \text{rank}(\mathbf{A}) = n \longrightarrow \mathbf{A}\mathbf{x} = \mathbf{0} \implies \mathbf{x} = \mathbf{0}$.
- ▶ $\mathbf{A}\mathbf{x} = \mathbf{b}$ can be solved, if and only if $\mathbf{A}(\mathbf{C}\mathbf{b}) = \mathbf{b}$, where $\mathbf{C}\mathbf{A} = \mathbf{I}_n$.

• Let $\mathbf{A} = \begin{bmatrix} 1 & -2 \\ 2 & 1 \\ 1 & 1 \end{bmatrix}$. Find a complete solution for the left inverse of \mathbf{A} such that $(\mathbf{C} + \hat{\mathbf{C}}) = \mathbf{I}_n$.

• Consider the system, $\mathbf{A}\mathbf{x} = \mathbf{b}$. $\mathbf{A} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\mathbf{x} = [x]$, and $\mathbf{b} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$. Find \mathbf{x} .

• What happens when $\mathbf{b} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$. What is \mathbf{x} ?

Right Inverse

- ▶ For $\mathbf{A} \in \mathbb{R}^{m \times n}$, $n > m$ with full rank, $\mathbf{AB} = \mathbf{I}_m \longrightarrow \mathbf{B}$ is the right inverse.
- ▶ Right inverse of \mathbf{A} exists only if the rows of \mathbf{A} are independent, i.e. $\text{rank}(\mathbf{A}) = m \longrightarrow \mathbf{A}^T \mathbf{x} = \mathbf{0} \implies \mathbf{x} = \mathbf{0}$
- ▶ $\mathbf{Ax} = \mathbf{b}$ can be solved for any \mathbf{b} . $\mathbf{x} = \mathbf{Bb} \implies \mathbf{A}(\mathbf{Bb}) = \mathbf{b}$.
- ▶ There are an infinite number of $\mathbf{Bs} \implies$ an infinite number of solutions \mathbf{x} .

Right Inverse

- ▶ For $\mathbf{A} \in \mathbb{R}^{m \times n}$, $n > m$ with full rank, $\mathbf{AB} = \mathbf{I}_m \longrightarrow \mathbf{B}$ is the right inverse.
- ▶ Right inverse of \mathbf{A} exists only if the rows of \mathbf{A} are independent, i.e. $\text{rank}(\mathbf{A}) = m \longrightarrow \mathbf{A}^T \mathbf{x} = \mathbf{0} \implies \mathbf{x} = \mathbf{0}$
- ▶ $\mathbf{Ax} = \mathbf{b}$ can be solved for any \mathbf{b} . $\mathbf{x} = \mathbf{Bb} \implies \mathbf{A}(\mathbf{Bb}) = \mathbf{b}$.
- ▶ There are an infinite number of $\mathbf{Bs} \implies$ an infinite number of solutions \mathbf{x} .

- Let $\mathbf{A} = \begin{bmatrix} 1 & -2 & 1 \\ 2 & 1 & -1 \end{bmatrix}$. Find a complete solution for the right inverse of \mathbf{A} .

Right Inverse

- ▶ For $\mathbf{A} \in \mathbb{R}^{m \times n}$, $n > m$ with full rank, $\mathbf{AB} = \mathbf{I}_m \longrightarrow \mathbf{B}$ is the right inverse.
- ▶ Right inverse of \mathbf{A} exists only if the rows of \mathbf{A} are independent, i.e. $\text{rank}(\mathbf{A}) = m \longrightarrow \mathbf{A}^T \mathbf{x} = \mathbf{0} \implies \mathbf{x} = \mathbf{0}$
- ▶ $\mathbf{Ax} = \mathbf{b}$ can be solved for any \mathbf{b} . $\mathbf{x} = \mathbf{Bb} \implies \mathbf{A}(\mathbf{Bb}) = \mathbf{b}$.
- ▶ There are an infinite number of $\mathbf{Bs} \implies$ an infinite number of solutions \mathbf{x} .

• Let $\mathbf{A} = \begin{bmatrix} 1 & -2 & 1 \\ 2 & 1 & -1 \end{bmatrix}$. Find a complete solution for the right inverse of \mathbf{A} .

• Solve $\mathbf{Ax} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Compare the solutions from Gauss-Jordan method and the ones obtained using right-inverses.

Right Inverse

- ▶ For $\mathbf{A} \in \mathbb{R}^{m \times n}$, $n > m$ with full rank, $\mathbf{AB} = \mathbf{I}_m \longrightarrow \mathbf{B}$ is the right inverse.
- ▶ Right inverse of \mathbf{A} exists only if the rows of \mathbf{A} are independent, i.e. $\text{rank}(\mathbf{A}) = m \longrightarrow \mathbf{A}^T \mathbf{x} = \mathbf{0} \implies \mathbf{x} = \mathbf{0}$
- ▶ $\mathbf{Ax} = \mathbf{b}$ can be solved for any \mathbf{b} . $\mathbf{x} = \mathbf{Bb} \implies \mathbf{A}(\mathbf{Bb}) = \mathbf{b}$.
- ▶ There are an infinite number of $\mathbf{Bs} \implies$ an infinite number of solutions \mathbf{x} .

- Let $\mathbf{A} = \begin{bmatrix} 1 & -2 & 1 \\ 2 & 1 & -1 \end{bmatrix}$. Find a complete solution for the right inverse of \mathbf{A} .
- Solve $\mathbf{Ax} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Compare the solutions from Gauss-Jordan method and the ones obtained using right-inverses.
- Let $\mathbf{AB} = \mathbf{I}_m$. What about the relationship between \mathbf{A}^T and \mathbf{B}^T ?

Fundamental subspaces of left and right inverses

Left Inverse

- ▶ $\mathbf{A} \in \mathbb{R}^{m \times n}$, $\text{rank}(\mathbf{A}) = n$
- ▶ Subspaces of \mathbf{A} :
$$\begin{aligned} C(\mathbf{A}) &\in \mathbb{R}^m & N(\mathbf{A}^T) &\in \mathbb{R}^m \\ C(\mathbf{A}^T) &= \mathbb{R}^n & N(\mathbf{A}) &= \{\mathbf{0}\} \end{aligned}$$
- ▶ Let $\mathbf{C} \in \mathbb{R}^{n \times m}$ be the left inverse of \mathbf{A} , such that $\mathbf{CA} = \mathbf{I}_n$. What is $\text{rank}(\mathbf{C})$?
- ▶ What about the subspaces of the left inverse?
 - ▶ $C(\mathbf{C})$
 - ▶ $N(\mathbf{C}^T)$
 - ▶ $C(\mathbf{C}^T)$
 - ▶ $N(\mathbf{C})$

Fundamental subspaces of left and right inverses

Left Inverse

- ▶ $\mathbf{A} \in \mathbb{R}^{m \times n}$, $\text{rank}(\mathbf{A}) = n$
- ▶ Subspaces of \mathbf{A} :
 $C(\mathbf{A}) \in \mathbb{R}^m$ $N(\mathbf{A}^T) \in \mathbb{R}^m$
 $C(\mathbf{A}^T) = \mathbb{R}^n$ $N(\mathbf{A}) = \{\mathbf{0}\}$
- ▶ Let $\mathbf{C} \in \mathbb{R}^{n \times m}$ be the left inverse of \mathbf{A} , such that $\mathbf{CA} = \mathbf{I}_n$. What is $\text{rank}(\mathbf{C})$?
- ▶ What about the subspaces of the left inverse?
 - ▶ $C(\mathbf{C}) = C(\mathbf{A}^T) = \mathbb{R}^n$
 - ▶ $N(\mathbf{C}^T)$
 - ▶ $C(\mathbf{C}^T)$
 - ▶ $N(\mathbf{C})$

Fundamental subspaces of left and right inverses

Left Inverse

- ▶ $\mathbf{A} \in \mathbb{R}^{m \times n}$, $\text{rank}(\mathbf{A}) = n$
- ▶ Subspaces of \mathbf{A} :
 $C(\mathbf{A}) \in \mathbb{R}^m$ $N(\mathbf{A}^T) \in \mathbb{R}^m$
 $C(\mathbf{A}^T) = \mathbb{R}^n$ $N(\mathbf{A}) = \{\mathbf{0}\}$
- ▶ Let $\mathbf{C} \in \mathbb{R}^{n \times m}$ be the left inverse of \mathbf{A} , such that $\mathbf{CA} = \mathbf{I}_n$. What is $\text{rank}(\mathbf{C})$?
- ▶ What about the subspaces of the left inverse?
 - ▶ $C(\mathbf{C}) = C(\mathbf{A}^T) = \mathbb{R}^n$
 - ▶ $N(\mathbf{C}^T) = N(\mathbf{A}) = \{\mathbf{0}\}$
 - ▶ $C(\mathbf{C}^T)$
 - ▶ $N(\mathbf{C})$

Fundamental subspaces of left and right inverses

Left Inverse

- ▶ $\mathbf{A} \in \mathbb{R}^{m \times n}$, $\text{rank}(\mathbf{A}) = n$
- ▶ Subspaces of \mathbf{A} :
 $C(\mathbf{A}) \in \mathbb{R}^m$ $N(\mathbf{A}^T) \in \mathbb{R}^m$
 $C(\mathbf{A}^T) = \mathbb{R}^n$ $N(\mathbf{A}) = \{\mathbf{0}\}$
- ▶ Let $\mathbf{C} \in \mathbb{R}^{n \times m}$ be the left inverse of \mathbf{A} , such that $\mathbf{CA} = \mathbf{I}_n$. What is $\text{rank}(\mathbf{C})$?
- ▶ What about the subspaces of the left inverse?
 - ▶ $C(\mathbf{C}) = C(\mathbf{A}^T) = \mathbb{R}^n$
 - ▶ $N(\mathbf{C}^T) = N(\mathbf{A}) = \{\mathbf{0}\}$
 - ▶ $C(\mathbf{C}^T) = C(\mathbf{A}) \in \mathbb{R}^m$
 - ▶ $N(\mathbf{C})$

Fundamental subspaces of left and right inverses

Left Inverse

- ▶ $\mathbf{A} \in \mathbb{R}^{m \times n}$, $\text{rank}(\mathbf{A}) = n$
- ▶ Subspaces of \mathbf{A} :
 $C(\mathbf{A}) \in \mathbb{R}^m$ $N(\mathbf{A}^T) \in \mathbb{R}^m$
 $C(\mathbf{A}^T) = \mathbb{R}^n$ $N(\mathbf{A}) = \{\mathbf{0}\}$
- ▶ Let $\mathbf{C} \in \mathbb{R}^{n \times m}$ be the left inverse of \mathbf{A} , such that $\mathbf{CA} = \mathbf{I}_n$. What is $\text{rank}(\mathbf{C})$?
- ▶ What about the subspaces of the left inverse?
 - ▶ $C(\mathbf{C}) = C(\mathbf{A}^T) = \mathbb{R}^n$
 - ▶ $N(\mathbf{C}^T) = N(\mathbf{A}) = \{\mathbf{0}\}$
 - ▶ $C(\mathbf{C}^T) = C(\mathbf{A}) \in \mathbb{R}^m$
 - ▶ $N(\mathbf{C}) = N(\mathbf{A}^T) \in \mathbb{R}^m$

Fundamental subspaces of left and right inverses

Left Inverse

- ▶ $\mathbf{A} \in \mathbb{R}^{m \times n}$, $\text{rank}(\mathbf{A}) = n$
- ▶ Subspaces of \mathbf{A} :

$$C(\mathbf{A}) \in \mathbb{R}^m \quad N(\mathbf{A}^T) \in \mathbb{R}^m$$

$$C(\mathbf{A}^T) = \mathbb{R}^n \quad N(\mathbf{A}) = \{\mathbf{0}\}$$
- ▶ Let $\mathbf{C} \in \mathbb{R}^{n \times m}$ be the left inverse of \mathbf{A} , such that $\mathbf{CA} = \mathbf{I}_n$. What is $\text{rank}(\mathbf{C})$?
- ▶ What about the subspaces of the left inverse?
 - ▶ $C(\mathbf{C}) = C(\mathbf{A}^T) = \mathbb{R}^n$
 - ▶ $N(\mathbf{C}^T) = N(\mathbf{A}) = \{\mathbf{0}\}$
 - ▶ $C(\mathbf{C}^T) = C(\mathbf{A}) \in \mathbb{R}^m$
 - ▶ $N(\mathbf{C}) = N(\mathbf{A}^T) \in \mathbb{R}^m$

Right Inverse

- ▶ $\mathbf{A} \in \mathbb{R}^{m \times n}$, $\text{rank}(\mathbf{A}) = m$
- ▶ Subspaces of \mathbf{A} :

$$C(\mathbf{A}) = \mathbb{R}^m \quad N(\mathbf{A}^T) = \{\mathbf{0}\}$$

$$C(\mathbf{A}^T) \in \mathbb{R}^n \quad N(\mathbf{A}) \in \mathbb{R}^n$$
- ▶ Let $\mathbf{B} \in \mathbb{R}^{n \times m}$ be the right inverse of \mathbf{A} , such that $\mathbf{AB} = \mathbf{I}_m$. What is $\text{rank}(\mathbf{B})$?
- ▶ What about the subspaces of the right inverse?
 - ▶ $C(\mathbf{B})$
 - ▶ $N(\mathbf{B}^T)$
 - ▶ $C(\mathbf{B}^T)$
 - ▶ $N(\mathbf{B})$

Fundamental subspaces of left and right inverses

Left Inverse

- ▶ $\mathbf{A} \in \mathbb{R}^{m \times n}$, $\text{rank}(\mathbf{A}) = n$
- ▶ Subspaces of \mathbf{A} :

$$C(\mathbf{A}) \in \mathbb{R}^m \quad N(\mathbf{A}^T) \in \mathbb{R}^m$$

$$C(\mathbf{A}^T) = \mathbb{R}^n \quad N(\mathbf{A}) = \{\mathbf{0}\}$$
- ▶ Let $\mathbf{C} \in \mathbb{R}^{n \times m}$ be the left inverse of \mathbf{A} , such that $\mathbf{CA} = \mathbf{I}_n$. What is $\text{rank}(\mathbf{C})$?
- ▶ What about the subspaces of the left inverse?
 - ▶ $C(\mathbf{C}) = C(\mathbf{A}^T) = \mathbb{R}^n$
 - ▶ $N(\mathbf{C}^T) = N(\mathbf{A}) = \{\mathbf{0}\}$
 - ▶ $C(\mathbf{C}^T) = C(\mathbf{A}) \in \mathbb{R}^m$
 - ▶ $N(\mathbf{C}) = N(\mathbf{A}^T) \in \mathbb{R}^m$

Right Inverse

- ▶ $\mathbf{A} \in \mathbb{R}^{m \times n}$, $\text{rank}(\mathbf{A}) = m$
- ▶ Subspaces of \mathbf{A} :

$$C(\mathbf{A}) = \mathbb{R}^m \quad N(\mathbf{A}^T) = \{\mathbf{0}\}$$

$$C(\mathbf{A}^T) \in \mathbb{R}^n \quad N(\mathbf{A}) \in \mathbb{R}^n$$
- ▶ Let $\mathbf{B} \in \mathbb{R}^{n \times m}$ be the right inverse of \mathbf{A} , such that $\mathbf{AB} = \mathbf{I}_m$. What is $\text{rank}(\mathbf{B})$?
- ▶ What about the subspaces of the right inverse?
 - ▶ $C(\mathbf{B}) = C(\mathbf{A}^T) \in \mathbb{R}^n$
 - ▶ $N(\mathbf{B}^T)$
 - ▶ $C(\mathbf{B}^T)$
 - ▶ $N(\mathbf{B})$

Fundamental subspaces of left and right inverses

Left Inverse

- ▶ $\mathbf{A} \in \mathbb{R}^{m \times n}$, $\text{rank}(\mathbf{A}) = n$
- ▶ Subspaces of \mathbf{A} :

$$C(\mathbf{A}) \in \mathbb{R}^m \quad N(\mathbf{A}^T) \in \mathbb{R}^m$$

$$C(\mathbf{A}^T) = \mathbb{R}^n \quad N(\mathbf{A}) = \{\mathbf{0}\}$$
- ▶ Let $\mathbf{C} \in \mathbb{R}^{n \times m}$ be the left inverse of \mathbf{A} , such that $\mathbf{CA} = \mathbf{I}_n$. What is $\text{rank}(\mathbf{C})$?
- ▶ What about the subspaces of the left inverse?
 - ▶ $C(\mathbf{C}) = C(\mathbf{A}^T) = \mathbb{R}^n$
 - ▶ $N(\mathbf{C}^T) = N(\mathbf{A}) = \{\mathbf{0}\}$
 - ▶ $C(\mathbf{C}^T) = C(\mathbf{A}) \in \mathbb{R}^m$
 - ▶ $N(\mathbf{C}) = N(\mathbf{A}^T) \in \mathbb{R}^m$

Right Inverse

- ▶ $\mathbf{A} \in \mathbb{R}^{m \times n}$, $\text{rank}(\mathbf{A}) = m$
- ▶ Subspaces of \mathbf{A} :

$$C(\mathbf{A}) = \mathbb{R}^m \quad N(\mathbf{A}^T) = \{\mathbf{0}\}$$

$$C(\mathbf{A}^T) \in \mathbb{R}^n \quad N(\mathbf{A}) \in \mathbb{R}^n$$
- ▶ Let $\mathbf{B} \in \mathbb{R}^{n \times m}$ be the right inverse of \mathbf{A} , such that $\mathbf{AB} = \mathbf{I}_m$. What is $\text{rank}(\mathbf{B})$?
- ▶ What about the subspaces of the right inverse?
 - ▶ $C(\mathbf{B}) = C(\mathbf{A}^T) \in \mathbb{R}^n$
 - ▶ $N(\mathbf{B}^T) = N(\mathbf{A}) \in \mathbb{R}^n$
 - ▶ $C(\mathbf{B}^T)$
 - ▶ $N(\mathbf{B})$

Fundamental subspaces of left and right inverses

Left Inverse

- ▶ $\mathbf{A} \in \mathbb{R}^{m \times n}$, $\text{rank}(\mathbf{A}) = n$
- ▶ Subspaces of \mathbf{A} :

$$C(\mathbf{A}) \in \mathbb{R}^m \quad N(\mathbf{A}^T) \in \mathbb{R}^m$$

$$C(\mathbf{A}^T) = \mathbb{R}^n \quad N(\mathbf{A}) = \{\mathbf{0}\}$$
- ▶ Let $\mathbf{C} \in \mathbb{R}^{n \times m}$ be the left inverse of \mathbf{A} , such that $\mathbf{CA} = \mathbf{I}_n$. What is $\text{rank}(\mathbf{C})$?
- ▶ What about the subspaces of the left inverse?
 - ▶ $C(\mathbf{C}) = C(\mathbf{A}^T) = \mathbb{R}^n$
 - ▶ $N(\mathbf{C}^T) = N(\mathbf{A}) = \{\mathbf{0}\}$
 - ▶ $C(\mathbf{C}^T) = C(\mathbf{A}) \in \mathbb{R}^m$
 - ▶ $N(\mathbf{C}) = N(\mathbf{A}^T) \in \mathbb{R}^m$

Right Inverse

- ▶ $\mathbf{A} \in \mathbb{R}^{m \times n}$, $\text{rank}(\mathbf{A}) = m$
- ▶ Subspaces of \mathbf{A} :

$$C(\mathbf{A}) = \mathbb{R}^m \quad N(\mathbf{A}^T) = \{\mathbf{0}\}$$

$$C(\mathbf{A}^T) \in \mathbb{R}^n \quad N(\mathbf{A}) \in \mathbb{R}^n$$
- ▶ Let $\mathbf{B} \in \mathbb{R}^{n \times m}$ be the left inverse of \mathbf{A} , such that $\mathbf{AB} = \mathbf{I}_m$. What is $\text{rank}(\mathbf{B})$?
- ▶ What about the subspaces of the left inverse?
 - ▶ $C(\mathbf{B}) = C(\mathbf{A}^T) \in \mathbb{R}^n$
 - ▶ $N(\mathbf{B}^T) = N(\mathbf{A}) \in \mathbb{R}^n$
 - ▶ $C(\mathbf{B}^T) = C(\mathbf{A}) = \mathbb{R}^m$
 - ▶ $N(\mathbf{B})$

Fundamental subspaces of left and right inverses

Left Inverse

- ▶ $\mathbf{A} \in \mathbb{R}^{m \times n}$, $\text{rank}(\mathbf{A}) = n$
- ▶ Subspaces of \mathbf{A} :

$$C(\mathbf{A}) \in \mathbb{R}^m \quad N(\mathbf{A}^T) \in \mathbb{R}^m$$

$$C(\mathbf{A}^T) = \mathbb{R}^n \quad N(\mathbf{A}) = \{\mathbf{0}\}$$
- ▶ Let $\mathbf{C} \in \mathbb{R}^{n \times m}$ be the left inverse of \mathbf{A} , such that $\mathbf{CA} = \mathbf{I}_n$. What is $\text{rank}(\mathbf{C})$?
- ▶ What about the subspaces of the left inverse?
 - ▶ $C(\mathbf{C}) = C(\mathbf{A}^T) = \mathbb{R}^n$
 - ▶ $N(\mathbf{C}^T) = N(\mathbf{A}) = \{\mathbf{0}\}$
 - ▶ $C(\mathbf{C}^T) = C(\mathbf{A}) \in \mathbb{R}^m$
 - ▶ $N(\mathbf{C}) = N(\mathbf{A}^T) \in \mathbb{R}^m$

Right Inverse

- ▶ $\mathbf{A} \in \mathbb{R}^{m \times n}$, $\text{rank}(\mathbf{A}) = m$
- ▶ Subspaces of \mathbf{A} :

$$C(\mathbf{A}) = \mathbb{R}^m \quad N(\mathbf{A}^T) = \{\mathbf{0}\}$$

$$C(\mathbf{A}^T) \in \mathbb{R}^n \quad N(\mathbf{A}) \in \mathbb{R}^n$$
- ▶ Let $\mathbf{B} \in \mathbb{R}^{n \times m}$ be the left inverse of \mathbf{A} , such that $\mathbf{AB} = \mathbf{I}_m$. What is $\text{rank}(\mathbf{B})$?
- ▶ What about the subspaces of the left inverse?
 - ▶ $C(\mathbf{B}) = C(\mathbf{A}^T) \in \mathbb{R}^n$
 - ▶ $N(\mathbf{B}^T) = N(\mathbf{A}) \in \mathbb{R}^n$
 - ▶ $C(\mathbf{B}^T) = C(\mathbf{A}) = \mathbb{R}^m$
 - ▶ $N(\mathbf{B}) = N(\mathbf{A}^T) = \{\mathbf{0}\}$

Pseudo Inverse

- ▶ Consider a tall, skinny matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$ with independent columns. It turns out the Gram matrix $\mathbf{A}^T \mathbf{A} \in \mathbb{R}^{n \times n}$ is invertible. If that is the case then,

$$(\mathbf{A}^T \mathbf{A})^{-1} \mathbf{A}^T \mathbf{A} = \mathbf{I}_n; \quad (\mathbf{A}^T \mathbf{A})^{-1} \mathbf{A}^T \text{ is a left inverse.}$$

- ▶ $\mathbf{A}^\dagger = (\mathbf{A}^T \mathbf{A})^{-1} \mathbf{A}^T$ is called the *pseudo inverse* or the *Moore-Penrose inverse*.
- ▶ For the case of a fat, wide matrix, we have $\mathbf{A}^\dagger = \mathbf{A}^T (\mathbf{A} \mathbf{A}^T)^{-1}$.
- ▶ When \mathbf{A} is square and invertible, $\mathbf{A}^\dagger = \mathbf{A}^{-1}$.

Pseudo Inverse

- ▶ Consider a tall, skinny matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$ with independent columns. It turns out the Gram matrix $\mathbf{A}^T \mathbf{A} \in \mathbb{R}^{n \times n}$ is invertible. If that is the case then,

$$(\mathbf{A}^T \mathbf{A})^{-1} \mathbf{A}^T \mathbf{A} = \mathbf{I}_n; \quad (\mathbf{A}^T \mathbf{A})^{-1} \mathbf{A}^T \text{ is a left inverse.}$$

- ▶ $\mathbf{A}^\dagger = (\mathbf{A}^T \mathbf{A})^{-1} \mathbf{A}^T$ is called the *pseudo inverse* or the *Moore-Penrose inverse*.
- ▶ For the case of a fat, wide matrix, we have $\mathbf{A}^\dagger = \mathbf{A}^T (\mathbf{A} \mathbf{A}^T)^{-1}$.
- ▶ When \mathbf{A} is square and invertible, $\mathbf{A}^\dagger = \mathbf{A}^{-1}$.

- Solve $\mathbf{A} \mathbf{x} = \mathbf{b}$ using the \mathbf{A}^\dagger . $\mathbf{A} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$. Find \mathbf{x} .

Pseudo Inverse

- ▶ Consider a tall, skinny matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$ with independent columns. It turns out the Gram matrix $\mathbf{A}^T \mathbf{A} \in \mathbb{R}^{n \times n}$ is invertible. If that is the case then,

$$(\mathbf{A}^T \mathbf{A})^{-1} \mathbf{A}^T \mathbf{A} = \mathbf{I}_n; \quad (\mathbf{A}^T \mathbf{A})^{-1} \mathbf{A}^T \text{ is a left inverse.}$$

- ▶ $\mathbf{A}^\dagger = (\mathbf{A}^T \mathbf{A})^{-1} \mathbf{A}^T$ is called the *pseudo inverse* or the *Moore-Penrose inverse*.
- ▶ For the case of a fat, wide matrix, we have $\mathbf{A}^\dagger = \mathbf{A}^T (\mathbf{A} \mathbf{A}^T)^{-1}$.
- ▶ When \mathbf{A} is square and invertible, $\mathbf{A}^\dagger = \mathbf{A}^{-1}$.

- Solve $\mathbf{A} \mathbf{x} = \mathbf{b}$ using the \mathbf{A}^\dagger . $\mathbf{A} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$. Find \mathbf{x} .

- Compare \mathbf{A}^\dagger with that of the general left inverse \mathbf{C} . Calculate $\|\mathbf{C}\|^2$ and find out the $\min \|\mathbf{C}\|^2$. What is $\|\mathbf{A}^\dagger\|^2$?

Matrix Inverse and Pseudo Inverse through QR factorization

- ▶ Consider an invertible, square matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$.

$$\mathbf{A} = \mathbf{QR} \implies \mathbf{A}^{-1} = (\mathbf{QR})^{-1} = \mathbf{R}^{-1}\mathbf{Q}^{-1} = \mathbf{R}^{-1}\mathbf{Q}^T$$

where, $\mathbf{R}, \mathbf{Q} \in \mathbb{R}^{n \times n}$. \mathbf{R} is upper triangular, and \mathbf{Q} is an orthogonal matrix.

- ▶ In the case of a left invertible rectangular matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$, we can factorize $\mathbf{A} = \mathbf{QR}$, with $\mathbf{Q} \in \mathbb{R}^{m \times n}$ and $\mathbf{R} \in \mathbb{R}^{m \times m}$.

$$\mathbf{A}^\dagger = (\mathbf{A}^T \mathbf{A})^{-1} \mathbf{A}^T = (\mathbf{R}^T \mathbf{Q}^T \mathbf{Q} \mathbf{R})^{-1} \mathbf{R}^T \mathbf{Q}^T = (\mathbf{R}^T \mathbf{R})^{-1} \mathbf{R}^T \mathbf{Q}^T = \mathbf{R}^{-1} \mathbf{Q}^T$$

- ▶ For a right invertible wide, fat matrix, we can find out the pseudo-inverse of \mathbf{A}^T , and then take the transpose of the pseudo-inverse.

$$\mathbf{A} \mathbf{A}^\dagger = \mathbf{I} \implies \left(\mathbf{A}^\dagger\right)^T \mathbf{A}^T = \left(\mathbf{A}^T\right)^\dagger \mathbf{A}^T = \mathbf{I}$$

$$\mathbf{A}^T = \mathbf{QR} \implies \left(\mathbf{A}^T\right)^\dagger = \mathbf{R}^{-1} \mathbf{Q}^T = \left(\mathbf{A}^\dagger\right)^T \implies \mathbf{A}^\dagger = \mathbf{QR}^{-T}$$