Linear Systems

Solution of LDS

Sivakumar Balasubramanian

Department of Bioengineering
Christian Medical College, Bagayam
Vellore 632002

State space representation of LDS

- A state space representation of a LTI system takes the following form,

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)$$

- Obtaining the solution to the above equations can be posed as the following prolbem,

$$\left.\begin{array}{l}\text{Determine: } \mathbf{x}(t), \mathbf{y}(t) \quad \forall t \geq 0 \\ \text{Given: } \mathbf{u}(t), \forall t \geq 0 \text{ and } \mathbf{x}(0^{-})\end{array}\right\}$$

- We first solve the state equation to obtain $\mathbf{x}(t)$, which is then used to obtain $\mathbf{y}(t)$.
- Because the system is linear, we can separate the solution into zero-input and zero-state solutions.

Zero-input solution for $\mathbf{x}(t)$

- **Zero-Input Solution**: We will start by assuming $\mathbf{u}(t) = \mathbf{0}$.

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t)$$

- For the scalar case, $\dot{x}(t) = ax(t)$, we know the solution to be the following, $x(t) = e^{at}x(0^-)$.
- A similar approach works for the vector case. Let us assume to the solution to zero-input state equation to be of the form, $\mathbf{x}(t) = e^{t\mathbf{A}}\mathbf{x}(0^-)$.
- What is $e^{t\mathbf{A}}$? Functions of matrices are often defined to have properties consistent with that of their scalar coutnerparts.

$$e^{t\mathbf{A}} = \mathbf{I} + t\mathbf{A} + \frac{1}{2!}t^2\mathbf{A}^2 + \ldots = \sum_{k=0}^{\infty} \frac{1}{k!}t^k\mathbf{A}^k \implies \frac{d}{dt}e^{t\mathbf{A}} = \mathbf{A}e^{t\mathbf{A}}$$

- Thus, $\mathbf{x}(t) = e^{t\mathbf{A}}\mathbf{x}(0^-)$ is the zero-input solution.

$f(\mathbf{A})$: functions of square matrices

- How do we evaluate $e^{t\mathbf{A}}$? We do not need to evaluate the infinite series.

Cayley-Hamilton Theorem

> Every square matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$ satisfies it own characteristic equation $p(\lambda) = 0$, i.e. $p(\mathbf{A}) = \mathbf{0}$.
>
> $$p(\mathbf{A}) = \mathbf{A}^n + a_1 \mathbf{A}^{n-1} + a_2 \mathbf{A}^{n-2} + \ldots + a_n \mathbf{I} = \mathbf{0}$$

- Consider a analytic function, $f(x) = \sum_{k=0}^{\infty} a_k x^k$, and a matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$ with characteristic polynomial $p(x)$. Then,

$$f(x) = q(x)\, p(x) + r(x)$$

where, $q(x)$ and $r(x)$ are the quotient and reminder polynomials, respectively, and $r(x) = c_0 + c_1 x + c_2 x^2 + \ldots + c_{n-1} x^{n-1}$.
Since $p(\mathbf{A}) = \mathbf{0}$, we have $f(\mathbf{A}) = r(\mathbf{A}) = \sum_{k=0}^{n-1} c_k \mathbf{A}^k$. Determining c_ks will allow us to calulate $f(\mathbf{A})$.

$f(\mathbf{A})$: functions of square matrices

- If $\mathbf{A}$ has n distinct eigenvalues, c_ks solved through the n equations, $f(\lambda_i) = q(\lambda_i)\, p(\lambda_i) + r(\lambda_i) = r(\lambda_i)$.
- For repeated repeated eigenvalues, we will need the following,

$$\left.\frac{d^{m-1}}{dx^{m-1}} f(x)\right|_{x=\lambda_i} = \left.\frac{d^{m-1}}{dx^{m-1}} r(x)\right|_{x=\lambda_i}$$

- For a diagonalizable matrix, $e^{t\mathbf{A}} = \mathbf{X} e^{t\mathbf{\Lambda}} \mathbf{X}^{-1}$, with $e^{t\mathbf{\Lambda}} = \text{diag}\left(e^{\lambda_1 t} \ldots e^{\lambda_n t}\right)$.
- For non-diagonalizable matrix we have, $e^{t\mathbf{A}} = \mathbf{X} \begin{bmatrix} e^{t\mathbf{J}_1} & & & \\ & e^{t\mathbf{J}_2} & & \\ & & \ddots & \\ & & & e^{t\mathbf{J}_k} \end{bmatrix} \mathbf{X}^{-1}$

$f\left(\mathbf{A}\right)$: functions of square matrices

- If $\mathbf{A}$ has n distinct eigenvalues, c_ks solved through the n equations, $f\left(\lambda_i\right) = q\left(\lambda_i\right)p\left(\lambda_i\right) + r\left(\lambda_i\right) = r\left(\lambda_i\right)$.
- For repeated repeated eigenvalues, we will need the following,

$$\frac{d^{m-1}}{dx^{m-1}} f\left(x\right)\Big|_{x=\lambda_i} = \frac{d^{m-1}}{dx^{m-1}} r\left(x\right)\Big|_{x=\lambda_i}$$

- For a diagonalizable matrix, $e^{t\mathbf{A}} = \mathbf{X}e^{t\mathbf{\Lambda}}\mathbf{X}^{-1}$, with $e^{t\mathbf{\Lambda}} = \text{diag}\left(e^{\lambda_1 t} \ldots e^{\lambda_n t}\right)$.
- For non-diagonalizable matrix we have, $e^{t\mathbf{A}} = \mathbf{X}\begin{bmatrix} e^{t\mathbf{J}_1} & & & \\ & e^{t\mathbf{J}_2} & & \\ & & \ddots & \\ & & & e^{t\mathbf{J}_k} \end{bmatrix}\mathbf{X}^{-1}$

Evaluate $e^{t\mathbf{A}}$ for the following $\mathbf{A}$: (a) $\begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$; (b) $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$; (c) $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

$f(\mathbf{A})$: functions of square matrices

- If $\mathbf{A}$ has n distinct eigenvalues, c_ks solved through the n equations, $f(\lambda_i) = q(\lambda_i)\, p(\lambda_i) + r(\lambda_i) = r(\lambda_i)$.
- For repeated repeated eigenvalues, we will need the following,

$$\left.\frac{d^{m-1}}{dx^{m-1}} f(x)\right|_{x=\lambda_i} = \left.\frac{d^{m-1}}{dx^{m-1}} r(x)\right|_{x=\lambda_i}$$

- For a diagonalizable matrix, $e^{t\mathbf{A}} = \mathbf{X} e^{t\boldsymbol{\Lambda}} \mathbf{X}^{-1}$, with $e^{t\boldsymbol{\Lambda}} = \text{diag}\left(e^{\lambda_1 t} \ldots e^{\lambda_n t}\right)$.
- For non-diagonalizable matrix we have, $e^{t\mathbf{A}} = \mathbf{X} \begin{bmatrix} e^{t\mathbf{J}_1} & & & \\ & e^{t\mathbf{J}_2} & & \\ & & \ddots & \\ & & & e^{t\mathbf{J}_k} \end{bmatrix} \mathbf{X}^{-1}$

Evaluate $e^{t\mathbf{A}}$ for the following $\mathbf{A}$: (a) $\begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$; (b) $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$; (c) $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

Evaluate $e^{t\mathbf{J}}$ for $\mathbf{J}$ with $GM = 1$: (a) $AM = 2$; (b) $AM = 3$; (c) $AM = n$

Laplace transform approach to zero-input response $\mathbf{x}(t)$

- Taking the Unilateral Laplace transform of the zero-input state equation,

$$s\mathbf{x}_{\mathcal{L}}(s) - \mathbf{x}\left(0^-\right) = \mathbf{A}\mathbf{x}_{\mathcal{L}}(s)$$

where, $\mathbf{x}_{\mathcal{L}}(s) = \begin{bmatrix} X_1(s) & X_2(s) \ldots X_n(s) \end{bmatrix}^T$, where $x_i(t) \overset{\mathcal{L}}{\longleftrightarrow} X_i(s)$.

$$(s\mathbf{I} - \mathbf{A})\,\mathbf{x}_{\mathcal{L}}(s) = \mathbf{x}\left(0^-\right) \implies \mathbf{x}_{\mathcal{L}}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{x}\left(0^-\right)$$

$$\implies \mathbf{x}(t) = \mathcal{L}^{-1}\left\{(s\mathbf{I} - \mathbf{A})^{-1}\right\}\mathbf{x}\left(0^-\right)$$

- Using the analogy from the scalar case, we could guess that $\mathbf{x}(t) = e^{t\mathbf{A}}\mathbf{x}\left(0^-\right)$. One can obtain the same solution by first finding the $(s\mathbf{I} - \mathbf{A})^{-1}$ and taking the inverse Laplace of each entry of this matrix.

Laplace transform approach to zero-input response $\mathbf{x}(t)$

- Taking the Unilateral Laplace transform of the zero-input state equation,

$$s\mathbf{x}_{\mathcal{L}}(s) - \mathbf{x}\left(0^-\right) = \mathbf{A}\mathbf{x}_{\mathcal{L}}(s)$$

where, $\mathbf{x}_{\mathcal{L}}(s) = \begin{bmatrix} X_1(s) & X_2(s) \dots X_n(s) \end{bmatrix}^T$, where $x_i(t) \overset{\mathcal{L}}{\longleftrightarrow} X_i(s)$.

$$(s\mathbf{I} - \mathbf{A})\,\mathbf{x}_{\mathcal{L}}(s) = \mathbf{x}\left(0^-\right) \implies \mathbf{x}_{\mathcal{L}}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{x}\left(0^-\right)$$

$$\implies \mathbf{x}(t) = \mathcal{L}^{-1}\left\{(s\mathbf{I} - \mathbf{A})^{-1}\right\}\mathbf{x}\left(0^-\right)$$

- Using the analogy from the scalar case, we could guess that $\mathbf{x}(t) = e^{t\mathbf{A}}\mathbf{x}\left(0^-\right)$. One can obtain the same solution by first finding the $(s\mathbf{I} - \mathbf{A})^{-1}$ and taking the inverse Laplace of each entry of this matrix.

Find $\mathbf{x}(t)$ for $t \geq 0$: $\dot{\mathbf{x}}(t) = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}\mathbf{x}(t)$.

Properties of $e^{t\mathbf{A}}$

- The columns of $e^{t\mathbf{A}} = \begin{bmatrix} \mathbf{a}_1(t) & \mathbf{a}_2(t) & \ldots \mathbf{a}_n(t) \end{bmatrix}$ represent the solutions to different initial conditions, i.e. $\mathbf{x}(t) = \mathbf{a}_i(t) = e^{t\mathbf{A}}\mathbf{e}_i$.

$$\mathbf{x}(t) = e^{t\mathbf{A}}\mathbf{x}\left(0^-\right) = e^{t\mathbf{A}}\sum_{i=1}^{n} x_i\left(0^-\right)\mathbf{e}_i = \sum_{i=1}^{n} x_i\left(0^-\right)\mathbf{a}_i$$

- If we know the response of a system to a set of n linearly independent initial conditions. Let $\mathbf{X}(t)$ represent the matrix whose columns are the solutions to the different initial conditions, then for any given intial condition $\mathbf{x}\left(0^-\right)$, we have the solution,

$$\mathbf{x}(t) = \mathbf{X}(t)\left(\mathbf{X}\left(0^-\right)\right)^{-1}\mathbf{x}\left(0^-\right)$$

- For any arbitrary initial time τ, instead of 0, we can still use the exponential formula to find out the solution,

$$\mathbf{x}(t) = e^{(t-\tau)\mathbf{A}}\mathbf{x}(\tau)$$

- $e^{t\mathbf{A}}$ is called the *state transition matrix*, which takes the state at any given time to its value t seconds forward in time.

Diagonalization of a linear system

- Consider the case where, $\mathbf{A}$ is diagonalizable. Let $\{\lambda_i, \mathbf{v}_i\}_{i=1}^n$ be the eigenpairs of $\mathbf{A}$. Then, $\mathbf{A} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^{-1}$, and we could write the zero-input state equation as,

$$\dot{\mathbf{x}}(t) = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^{-1}\mathbf{x}(t) \xrightarrow{\tilde{\mathbf{x}}(t)=\mathbf{V}^{-1}\mathbf{x}(t)} \dot{\tilde{\mathbf{x}}}(t) = \mathbf{\Lambda}\tilde{\mathbf{x}}(t)$$

  The set of coupled first order differential equations are decoupled by this transforamtion.

- The individual states of $\tilde{\mathbf{x}}(t)$ evolve independently of each other.

$$\tilde{\mathbf{x}}(t) = e^{t\mathbf{\Lambda}}\tilde{\mathbf{x}}\left(0^-\right) = \begin{bmatrix} e^{\lambda_1 t} & & & \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ & & & e^{\lambda_n t} \end{bmatrix} \tilde{\mathbf{x}}\left(0^-\right)$$

- An arbitrary initial state $\mathbf{x}\left(0^-\right) = \sum_{i=1}^n \alpha_i \mathbf{v}_i$ evolves as follows,

$$\mathbf{x}(t) = \alpha_1 e^{\lambda_1 t}\mathbf{v}_1 + \alpha_2 e^{\lambda_2 t}\mathbf{v}_2 + \ldots + \alpha_n e^{\lambda_n t}\mathbf{v}_n$$

Diagonalization of a linear system

When $\mathbf{A}$ is not diagonalizable:
$\mathbf{A} = \mathbf{V}\mathbf{J}\mathbf{V}^{-1} \implies \dot{\tilde{\mathbf{x}}}(t) = \mathbf{J}\tilde{\mathbf{x}}(t)$.

$$\tilde{\mathbf{x}}(t) = e^{t\mathbf{J}}\tilde{\mathbf{x}}\left(0^-\right) = \begin{bmatrix} e^{t\mathbf{J}_1} & & \\ & \ddots & \\ & & e^{t\mathbf{J}_k} \end{bmatrix} \tilde{\mathbf{x}}\left(0^-\right)$$

Consider
$$\mathbf{A} = \mathbf{V}\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}\mathbf{V}^{-1} \implies \dot{\tilde{\mathbf{x}}}(t) = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}\tilde{\mathbf{x}}(t)$$

$$\dot{\tilde{x}}_1(t) = \lambda\tilde{x}_1(t) + \tilde{x}_2(t)$$
$$\dot{\tilde{x}}_2(t) = \lambda\tilde{x}_2(t)$$

We do not have complete decoupling as in the case where $\mathbf{A}$ was diagonalizable.

The exponential of a Jordan block has terms $e^{\lambda t}$, $te^{\lambda t}$, $te^{\lambda t}$, ...

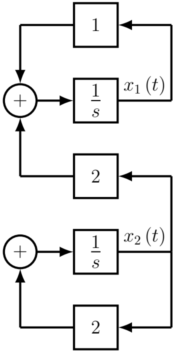
$$e^{t\mathbf{J}_1} = e^{\lambda_1 t}\begin{bmatrix} 1 & t & \frac{t^2}{2!} & \dots & \frac{t^n}{n!} \\ 0 & 1 & t & \dots & \frac{t^{n-1}}{(n-1)!} \\ 0 & 0 & 1 & \dots & \frac{t^{n-2}}{(n-2)!} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}$$

Thus,

$$\tilde{x}_1(t) = \tilde{x}_1\left(0^-\right)e^{\lambda t} + \tilde{x}_2\left(0^-\right)te^{\lambda t}$$
$$\tilde{x}_2(t) = \tilde{x}_2\left(0^-\right)e^{\lambda t}$$

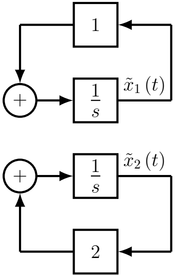
Diagonalization of a linear system

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix} \mathbf{x}(t)$$



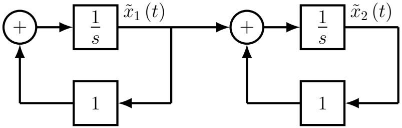
When $\mathbf{A}$ is diagonalizable.

$$\dot{\tilde{\mathbf{x}}}(t) = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \tilde{\mathbf{x}}(t)$$



When $\mathbf{A}$ is not-diagonalizable.

$$\dot{\tilde{\mathbf{x}}}(t) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \tilde{\mathbf{x}}(t)$$



A Jordan block results in series of simple (scalar) first order blocks, where the output of a block acts as the input to another.

Modes of a system

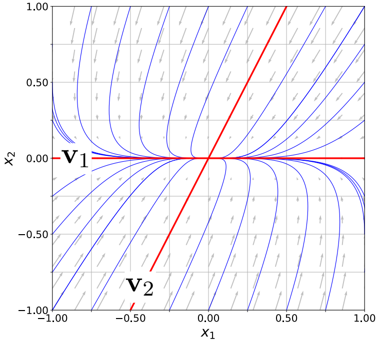
$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) \implies \mathbf{x}(t) = \sum_{i=1}^{n} \alpha_i e^{\lambda_i t} \mathbf{v}_i$$

- The eigvenvalues $\{\lambda_i\}_{i=1}^{n}$ of the system matrix $\mathbf{A}$ characterize the "natural" behavior of the system. These are called the *modes of the system*.
- The modes are exclusively expressed when the system starts in some specific set of states. When the system starts in an arbitrary state, the response contains a linear mixtute of these modes.
- **Dominant mode**: Determines the long-term behavior of the system. In the case of continuous-time systems, this would be the eigenvalue with the largest real part.
- If λ_i is a dominant mode $\implies \left|\alpha_i e^{\lambda_i t}\right| \gg \left|\alpha_j e^{\lambda_j t}\right|, \forall j \neq i$ and $t > T$. This implies that after some time, the response almost only has that particular mode,
$$\mathbf{x}(t) \approx \alpha_i e^{\lambda_i t} \mathbf{v}_i, \ \forall t > T$$
- **Subdominant mode**: These are the other modes of the system, and these essentially determine how fast the system moves to the dominant mode.

Modes of a system

Consider the system, $\dot{\mathbf{x}}(t) = \begin{bmatrix} -1 & 2 \\ 0 & -5 \end{bmatrix} \mathbf{x}(t)$.

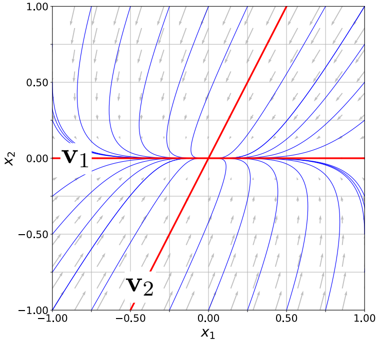
Modes: $\begin{cases} \lambda_1 = -1, & \mathbf{v}_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}^T \\ \lambda_2 = -5, & \mathbf{v}_2 = \begin{bmatrix} 1 & 2 \end{bmatrix}^T \end{cases}$



Modes of a system

Consider the system, $\dot{\mathbf{x}}(t) = \begin{bmatrix} -1 & 2 \\ 0 & -5 \end{bmatrix} \mathbf{x}(t)$.

Modes: $\begin{cases} \lambda_1 = -1, & \mathbf{v}_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}^T \\ \lambda_2 = -5, & \mathbf{v}_2 = \begin{bmatrix} 1 & 2 \end{bmatrix}^T \end{cases}$

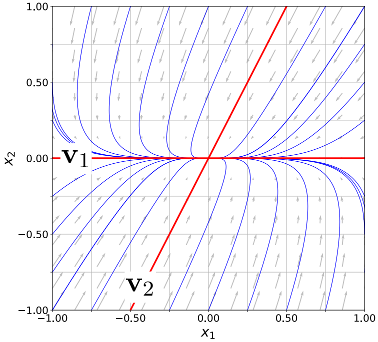


1. Consider a system with modes: $(-1, \mathbf{v}_1)$, $(-1, \mathbf{v}_2)$, $(-3, \mathbf{v}_3)$, and $(-10, \mathbf{v}_4)$. What are the dominant modes? How does any arbitrary state evolve?

Modes of a system

Consider the system, $\dot{\mathbf{x}}(t) = \begin{bmatrix} -1 & 2 \\ 0 & -5 \end{bmatrix} \mathbf{x}(t)$.

Modes: $\begin{cases} \lambda_1 = -1, & \mathbf{v}_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}^T \\ \lambda_2 = -5, & \mathbf{v}_2 = \begin{bmatrix} 1 & 2 \end{bmatrix}^T \end{cases}$



1. Consider a system with modes: $(-1, \mathbf{v}_1)$, $(-1, \mathbf{v}_2)$, $(-3, \mathbf{v}_3)$, and $(-10, \mathbf{v}_4)$. What are the dominant modes? How does any arbitrary state evolve?
2. Describe the state equation of a mass M in free space. What are its modes?

Zero-solution for $\mathbf{x}(t)$

- Let us now assume that the LTI system is relaxed when the input the applied to the system, i.e. $\mathbf{x}(0^-) = \mathbf{0}$. The effect of the input $\mathbf{u}(t)$ can be obtained by working in the Laplace domain,

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \implies \mathbf{x}_{\mathcal{L}}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{u}_{\mathcal{L}}(s)$$

Taking the inverse Laplace transform, we get,

$$\mathbf{x}(t) = \int_0^{\infty} e^{(t-\tau)\mathbf{A}}\mathbf{B}\mathbf{u}(\tau)\,d\tau$$

Zero-solution for $\mathbf{x}(t)$

- Let us now assume that the LTI system is relaxed when the input the applied to the system, i.e. $\mathbf{x}(0^-) = \mathbf{0}$. The effect of the input $\mathbf{u}(t)$ can be obtained by working in the Laplace domain,

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \implies \mathbf{x}_{\mathcal{L}}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{u}_{\mathcal{L}}(s)$$

Taking the inverse Laplace transform, we get,

$$\mathbf{x}(t) = \int_0^{\infty} e^{(t-\tau)\mathbf{A}}\mathbf{B}\mathbf{u}(\tau)\,d\tau$$

What do the columns of $e^{t\mathbf{A}}\mathbf{B}$ represent? What about the row of $e^{t\mathbf{A}}\mathbf{B}$? What about the ij^{th} element of $e^{t\mathbf{A}}\mathbf{B}$?

Complete solution for $\mathbf{x}(t)$ and $\mathbf{y}(t)$

- The complete solution for the state equations is given by the following,

$$\mathbf{x}(t) = e^{t\mathbf{A}}\mathbf{x}(0^-) + \int_0^\infty e^{(t-\tau)\mathbf{A}}\mathbf{B}\mathbf{u}(\tau)\,d\tau$$

- The output of the system is given by,

$$\mathbf{y}(t) = \mathbf{C}e^{t\mathbf{A}}\mathbf{x}(0^-) + \int_0^\infty \mathbf{C}e^{(t-\tau)\mathbf{A}}\mathbf{B}\mathbf{u}(\tau)\,d\tau + \mathbf{D}\mathbf{u}(t) = \mathbf{C}e^{t\mathbf{A}}\mathbf{x}(0^-) + \int_0^\infty \mathbf{G}(t-\tau)\,u(\tau)\,d\tau$$

  where, $\mathbf{G}(t) = \mathbf{C}e^{t\mathbf{A}}\mathbf{B} + \mathbf{D}\delta(t)$ is the *impulse response matrix* of the system.
- The transfer function of the system is given by: $\mathbf{H}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$.

Complete solution for $\mathbf{x}(t)$ and $\mathbf{y}(t)$

- The complete solution for the state equations is given by the following,

$$\mathbf{x}(t) = e^{t\mathbf{A}}\mathbf{x}(0^-) + \int_0^\infty e^{(t-\tau)\mathbf{A}}\mathbf{B}\mathbf{u}(\tau)\,d\tau$$

- The output of the system is given by,

$$\mathbf{y}(t) = \mathbf{C}e^{t\mathbf{A}}\mathbf{x}(0^-) + \int_0^\infty \mathbf{C}e^{(t-\tau)\mathbf{A}}\mathbf{B}\mathbf{u}(\tau)\,d\tau + \mathbf{D}\mathbf{u}(t) = \mathbf{C}e^{t\mathbf{A}}\mathbf{x}(0^-) + \int_0^\infty \mathbf{G}(t-\tau)\,u(\tau)\,d\tau$$

  where, $\mathbf{G}(t) = \mathbf{C}e^{t\mathbf{A}}\mathbf{B} + \mathbf{D}\delta(t)$ is the *impulse response matrix* of the system.
- The transfer function of the system is given by: $\mathbf{H}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$.

Find the impulse response matrix for $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 1 & -0.5 \\ 1 & 1 \end{bmatrix}$, $\mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}$, and $\mathbf{D} = 0$.

Complete solution for $\mathbf{x}(t)$ and $\mathbf{y}(t)$

- The complete solution for the state equations is given by the following,

$$\mathbf{x}(t) = e^{t\mathbf{A}}\mathbf{x}(0^-) + \int_0^\infty e^{(t-\tau)\mathbf{A}}\mathbf{B}\mathbf{u}(\tau)\,d\tau$$

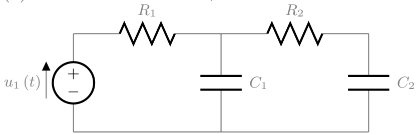
- The output of the system is given by,

$$\mathbf{y}(t) = \mathbf{C}e^{t\mathbf{A}}\mathbf{x}(0^-) + \int_0^\infty \mathbf{C}e^{(t-\tau)\mathbf{A}}\mathbf{B}\mathbf{u}(\tau)\,d\tau + \mathbf{D}\mathbf{u}(t) = \mathbf{C}e^{t\mathbf{A}}\mathbf{x}(0^-) + \int_0^\infty \mathbf{G}(t-\tau)\,u(\tau)\,d\tau$$

  where, $\mathbf{G}(t) = \mathbf{C}e^{t\mathbf{A}}\mathbf{B} + \mathbf{D}\delta(t)$ is the *impulse response matrix* of the system.
- The transfer function of the system is given by: $\mathbf{H}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$.

Find the impulse response matrix for $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 1 & -0.5 \\ 1 & 1 \end{bmatrix}$, $\mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}$, and $\mathbf{D} = 0$.

Find the expression for $\mathbf{y}(t) = \begin{bmatrix} v_{C_1}(t) & v_{R_2}(t) \end{bmatrix}^T$ for the following system, such that $v_{C_1}(0^-) = 1V$, $v_{C_2}(0^-) = -0.5V$, $u_1(t) = 1(t)\,V$, and $R = 1k\Omega, C = 1mF$.



Solution for discrete-time LTI system

- System equations:
$$\mathbf{x}[k+1] = \mathbf{A}\mathbf{x}[k] + \mathbf{B}\mathbf{u}[k]$$
$$\mathbf{y}[k] = \mathbf{C}\mathbf{x}[k] + \mathbf{D}\mathbf{u}[k]$$
- **Zero-input solution**:
$$\mathbf{x}[k] = \mathbf{A}^k\mathbf{x}[0]$$
- **Zero-state solution**:
$$\mathbf{x}[k] = \sum_{l=0}^{k-1} \mathbf{A}^{k-l-1}\mathbf{B}\mathbf{u}[l]$$
- **Complete solution**:
$$\mathbf{x}[k] = \mathbf{A}^k\mathbf{x}[0] + \sum_{l=0}^{k-1} \mathbf{A}^{k-l-1}\mathbf{B}\mathbf{u}[l]$$
- $\mathbf{A}^k$ is the *state transition matrix* and $\mathbf{G}[k] = \mathbf{A}^{k-1}\mathbf{B}$ is the *impulse response matrix*.

Solution for discrete-time LTI system

- We can approach this problem through the z-transform. Taking the unilateral z-transform of the state equation,

$$z\mathbf{X}_{\mathcal{Z}}(z) - z\mathbf{x}(0) = \mathbf{A}\mathbf{X}_{\mathcal{Z}}(z) + \mathbf{B}\mathbf{U}_{\mathcal{Z}}(z)$$

$$\mathbf{X}_{\mathcal{Z}}(z) = z\left(z\mathbf{I} - \mathbf{A}\right)^{-1}\mathbf{x}[0] + \left(z\mathbf{I} - \mathbf{A}\right)^{-1}\mathbf{B}\mathbf{U}_{\mathcal{Z}}(z)$$

  The inverse z-tansform of this leads us to,

$$\mathbf{x}[k] = \mathbf{A}^k\mathbf{x}[0] + \sum_{l=0}^{k-1}\mathbf{A}^{k-l-1}\mathbf{B}\mathbf{u}[l]$$

- Output:

$$\mathbf{y}[k] = \mathbf{C}\mathbf{A}^k\mathbf{x}[0] + \sum_{l=0}^{k-1}\mathbf{C}\mathbf{A}^{k-l-1}\mathbf{B}\mathbf{u}[l] + \mathbf{D}\mathbf{u}[k]$$

- The transfer function of the system is, $\mathbf{H}(z) = \mathbf{C}\left(z\mathbf{I} - \mathbf{A}\right)^{-1}\mathbf{B} + \mathbf{D}$

Diagonalization of a linear system

When $\mathbf{A}$ is diagonalizable, then we have

$$\mathbf{x}[k+1] = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^{-1}\mathbf{x}[k] \implies \tilde{\mathbf{x}}[k+1] = \mathbf{\Lambda}\tilde{\mathbf{x}}[k]$$

where, $\tilde{\mathbf{x}}[k] = \mathbf{V}^{-1}\mathbf{x}[k]$.

$$\tilde{\mathbf{x}}[k] = \mathbf{\Lambda}^k\tilde{\mathbf{x}}[0] = \begin{bmatrix} \lambda_1^k & & & \\ & \lambda_2^k & & \\ & & \ddots & \\ & & & \lambda_n^k \end{bmatrix} \tilde{\mathbf{x}}[0]$$

An arbitrary initial state $\mathbf{x}[0] = \sum_{i=1}^n \alpha_i \mathbf{v}_i$ evolves as follows,

$$\mathbf{x}[k] = \alpha_1\lambda_1^k\mathbf{v}_1 + \alpha_2\lambda_2^k\mathbf{v}_2 + \ldots + \alpha_n\lambda_n^k\mathbf{v}_n$$

When $\mathbf{A}$ is not diagonalizable, $\mathbf{A} = \mathbf{V}\mathbf{J}\mathbf{V}^{-1}$

$$\tilde{\mathbf{x}}[k+1] = \mathbf{J}\tilde{\mathbf{x}}[k+1]$$

$$\tilde{\mathbf{x}}[k] = \mathbf{J}^k\tilde{\mathbf{x}}[0] = \begin{bmatrix} \mathbf{J}_1^k & & \\ & \ddots & \\ & & \mathbf{J}_l^k \end{bmatrix} \tilde{\mathbf{x}}[0]$$

Consider

$$\mathbf{A} = \mathbf{V}\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}\mathbf{V}^{-1} \implies \tilde{\mathbf{x}}[k+1] = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}\tilde{\mathbf{x}}[k]$$

$$\tilde{x}_1[k+1] = \lambda\tilde{x}_1[k] + \tilde{x}_2[k]$$
$$\tilde{x}_2[k+1] = \lambda\tilde{x}_2[k]$$

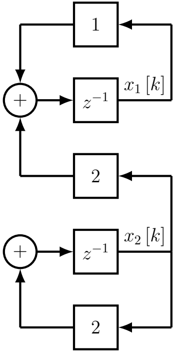
$$\mathbf{J}^k = \lambda^k \begin{bmatrix} 1 & \frac{k!\lambda^{-1}}{(k-1)!1!} & \frac{k!\lambda^{-2}}{(k-2)!2!} & \ldots & \frac{k!\lambda^{-(n-1)}}{(k-n+1)!(n-1)!} \\ 0 & 1 & \frac{k!\lambda^{-1}}{(k-1)!1!} & \ldots & \frac{k!\lambda^{-(n-2)}}{(k-n+2)!(n-2)!} \\ 0 & 0 & 1 & \ldots & \frac{k!\lambda^{-(n-3)}}{(k-n+3)!(n-3)!} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \end{bmatrix}$$

Thus,

$$\tilde{x}_1[k] = \tilde{x}_1[0]\lambda^k + \tilde{x}_2[0]k\lambda^k$$
$$\tilde{x}_2[k] = \tilde{x}_2[0]\lambda^k$$

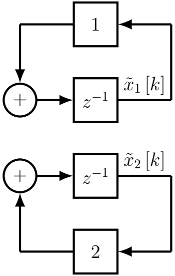
Diagonalization of a linear system

$$\mathbf{x}[k+1] = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix} \mathbf{x}[k]$$



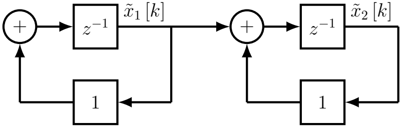
When $\mathbf{A}$ is diagonalizable.

$$\tilde{\mathbf{x}}[k+1] = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \tilde{\mathbf{x}}[k]$$



When $\mathbf{A}$ is not-diagonalizable.

$$\tilde{\mathbf{x}}[k+1] = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \tilde{\mathbf{x}}[k]$$



A Jordan block results in series of simple (scalar) first order blocks, where the output of a block acts as the input to another.

Modes of a discrete-time system

$$\mathbf{x}[k] = \alpha_1 \lambda_1^k \mathbf{v}_1 + \alpha_2 \lambda_2^k \mathbf{v}_2 + \ldots + \alpha_n \lambda_n^k \mathbf{v}_n$$

- The eigvenvalues $\{\lambda_i\}_{i=1}^n$ of the system matrix $\mathbf{A}$ characterize the "natural" behavior of the system. These are called the *modes of the system*.
- **Dominant mode**: Determines the long-term behavior of the system. In the case of discrete-time systems, this would be the eigenvalue with the largest magnitude.
- If λ_i is a dominant mode $\implies \left|\alpha_i \lambda_i^k\right| \gg \left|\alpha_j \lambda_j^k\right|, \forall j \neq i$ and $k > N$.
  This implies that after some time, the response almost only has that particular mode,
  $$\mathbf{x}[k] \approx \alpha_i \lambda_i^k \mathbf{v}_i, \;\; \forall t > T$$
- **Subdominant mode**: These are the other modes of the system, and these essentially determine how fast the system moves to the dominant mode.