Linear Systems

Positive Definiteness and Matrix Norm

Sivakumar Balasubramanian

Department of Bioengineering
Christian Medical College, Bagayam
Vellore 632002

References

- G Strang, Linear Algebra: Chapters 6 and 7.

Positive definite matrices

- We know that $\mathbf{x}^T\mathbf{x} > 0$ for all $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{x} \neq \mathbf{0}$.
- What can we say about $\mathbf{x}^T\mathbf{A}\mathbf{x}$, where $\mathbf{A} \in \mathbb{R}^{n \times n}$?
  - Is it positive for all $\mathbf{x} \neq \mathbf{0}$?
  - Can it be zero for some $\mathbf{x} \neq \mathbf{0}$?
  - Can it be negative some $\mathbf{x} \neq \mathbf{0}$?
- Any matrix $\mathbf{A}$ for which $\mathbf{x}^T\mathbf{A}\mathbf{x} > 0$, for all $\mathbf{x} \neq \mathbf{0}$ is called a *positive definite matrix*.
- Positive definite matrices are very useful and are commonly encountered in practice: optmization, mechanics (mass matrix, stiffness matrix), stability analysis, co-variance matrices etc.

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Are the following matrices positive definite: $\begin{bmatrix} 2 & -6 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 5 & -1 \\ 14 & 11 \end{bmatrix}$

Positive Definite Matrix

- The idea of positive definiteness is intimately related to the problem of minimization of a function.
- Consider the following function of a single variable $f(x)$. This function reaches a minimum at $x = 0$, when $\frac{df(x)}{dx}\Big|_{x=0} = 0$ and $\frac{d^2 f(x)}{dx^2}\Big|_{x=0} > 0$. E.g.,

$$f(x) = 3x^2 \rightarrow \frac{df(x)}{dx}\Big|_{x=0} = 0, \ \frac{d^2 f(x)}{dx^2}\Big|_{x=0} = 3 > 0$$

- What about $f(x_1, x_2) = ax_1^2 + 2bx_1x_2 + cx_2^2$? We can extend the previous idea using partial derivatives.

$$\frac{\partial f(x_1, x_2)}{\partial x_1} = 0, \ \frac{\partial f(x_1, x_2)}{\partial x_2} = 0, \ \frac{\partial^2 f(x_1, x_2)}{\partial x_1^2} > 0, \ \frac{\partial^2 f(x_1, x_2)}{\partial x_2^2} > 0$$

Is this enough? $\frac{\partial^2 f(x_1, x_2)}{\partial x_1 \partial x_2}$ must also be taken into account.

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Positive Definite Matrix

- We can rearrange $ax_1^2 + 2bx_1x_2 + cx_2^2$ in the following manner,

$$f\left(x_1, x_2\right) = ax_1^2 + 2bx_1x_2 + cx_2^2 = a\left(x_1 + \frac{b}{a}x_2\right)^2 + \left(c - \frac{b^2}{a}\right)x_2^2$$

$f\left(\bullet\right) > 0, \forall x_1, x_2 \neq 0$ when,

$$a > 0 \quad \text{and} \quad c - \frac{b^2}{a} > 0 \implies ac > b^2$$

$$\frac{\partial^2 f\left(x_1, x_2\right)}{\partial x_1^2} > 0 \quad \text{and} \quad \frac{\partial^2 f\left(x_1, x_2\right)}{\partial x_1^2}\frac{\partial^2 f\left(x_1, x_2\right)}{\partial x_2^2} > \left(\frac{\partial^2 f\left(x_1, x_2\right)}{\partial x_1 \partial x_2}\right)^2$$

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Verfy this on the following functions: 1) $x_1^2 + x_1x_2 + x_2^2$, 2) $x_1^2 + 2x_1x_2 + x_2^2$, 3) $x_1^2 + 3x_1x_2 + x_2^2$

Positive Definite Matrix

$f(\bullet)$ can be expressed as $\mathbf{x}^T\mathbf{A}\mathbf{x}$, where $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, and $\mathbf{A} = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$. $f(\mathbf{x}) > 0$ for all $\mathbf{x} \neq \mathbf{0}$, when $\mathbf{A}$ is positive definite.

$\mathbf{x}^T\mathbf{A}\mathbf{x}$ is called a *quadratic form*. For a symmetric matrix $\mathbf{A}$,

$$\mathbf{x}^T\mathbf{A}\mathbf{x} = \begin{bmatrix} x_1 & x_2 & \dots & x_n \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{12} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix} = \sum_{i=1}^{n} a_{ii} x_i^2 + 2 \sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} a_{ij} x_i x_j$$

In general, $\mathbf{A} \in \mathbb{R}^{n\times n}$ is positive definite, if:

- The eigenvalues of $\mathbf{A}$ are all positive.
- The pivots (without row exchange) are all positive.

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Show that any $\mathbf{A}$ is positive definite if the symmetric matrix $\mathbf{A} + \mathbf{A}^T$ is positive definite. Note: *This should explain why we have only been talking about symmetric matrices.*

Matrix Norm

- Since matrices also form vector spaces, we can talk about norms of matrices, which extent the idea of sizes and distances to spaces of matrices.
- If we think of matrices a set of mn scalars, then we can use the same approach as vectors,

$$\|\mathbf{A}\|_F = \left(\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2 \right)^{\frac{1}{2}}$$

This is called the *Frobinius norm*.

Matrix Norm

- There are other norms defined for matrices that are very useful from the point of the view of linear transformation.
- These are called *induced matrix norms*, that looks at how matrices map vectors from the range to domain spaces.
- Let $\mathbf{A} \in \mathbb{R}^{m \times n} : \mathbb{R}^n \to \mathbb{R}^m$, and $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{y} \in \mathbb{R}^m$

  $$\|\mathbf{y}\| = \|\mathbf{A}\mathbf{x}\| \leq C \|\mathbf{x}\|, \ \ \forall \mathbf{x} \in \mathbb{R}^n$$

  C is the maximum factor by which $\mathbf{A}$ amplifies the vector $\mathbf{x}$.
- The induced norm of a matrix is defined as,

  $$\|\mathbf{A}_p\| = \max_{\mathbf{x}} \frac{\|\mathbf{A}\mathbf{x}\|_p}{\|\mathbf{x}\|_p} = \max_{\|\mathbf{x}\|_p = 1} \|\mathbf{A}\mathbf{x}\|_p$$

Matrix Norm

Consider a matrix $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 \end{bmatrix} = \begin{bmatrix} \tilde{\mathbf{a}}_1^T \\ \tilde{\mathbf{a}}_2^T \end{bmatrix}$.

$\|\mathbf{A}\|_1 = \max_i \|\mathbf{a}_i\|_1$ $\quad$ $\|\mathbf{A}\|_2 = \sqrt{\lambda_{\max}}$ $\quad$ $\|\mathbf{A}\|_\infty = \max_i \|\tilde{\mathbf{a}}_i\|_1$