Linear Systems

Controllability

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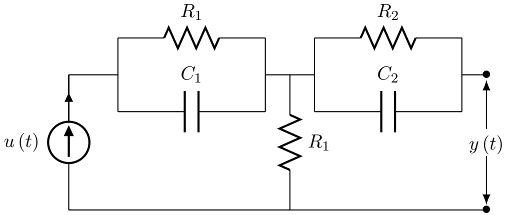
Controllability and observability

- This lecture and the next deal with two important aspects of linear system theory – controllability and observability.
- These two concepts deal with how the input and output interact with the system states.
- Consider the following system:



How are the capacitor voltages affected by input voltage $u(t)$? What does $y(t)$ measure?

Controllability

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)$$

- Controllability tells us if a desired state can be achieved in finite time through an appropriate to choice of inputs. For this, we only have to deal with the state equation.
- **Definition**: The system or the pair $(\mathbf{A}, \mathbf{B})$ is **controllable** if for any initial state $\mathbf{x}_i$ and any final state $\mathbf{x}_f$, there exists an input $\mathbf{u}(t)$ that transfers the initial state to the final state in finite time. Otherwise, the system is uncontrollable.
- It should be noted that,
  - The trajectory from $\mathbf{x}_i$ to $\mathbf{x}_f$ does not matter.
  - $\mathbf{u}(t)$ can be anything, including impulses and derivatives of impulses.

Controllability

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$

Assuming the system starts at $t = 0$, the output at $t = t_f$ is given by,

$$\mathbf{x}(t_f) = e^{t_f \mathbf{A}}\mathbf{x}(0) + \int_0^{t_f} e^{(t_f - \tau)\mathbf{A}}\mathbf{B}\mathbf{u}(\tau)\, d\tau \implies e^{-t_f \mathbf{A}}\mathbf{x}(t_f) - \mathbf{x}(0) = \int_0^{t_f} e^{-\tau \mathbf{A}}\mathbf{B}\mathbf{u}(\tau)\, d\tau$$

From the Cayley-Hamilton theorem we have,

$$e^{-\tau \mathbf{A}} = \sum_{k=0}^{n-1} \alpha_k(\tau)\mathbf{A}^k \implies e^{-t_f \mathbf{A}}\mathbf{x}(t_f) - \mathbf{x}(0) = \sum_{k=0}^{n-1} \mathbf{A}^k\mathbf{B} \int_0^{t_f} \alpha_k(\tau)\mathbf{u}(\tau)\, d\tau$$

$$\implies e^{-t_f \mathbf{A}}\mathbf{x}(t_f) - \mathbf{x}(0) = \begin{bmatrix} \mathbf{B} & \mathbf{A}\mathbf{B} & \mathbf{A}^2\mathbf{B} & \dots & \mathbf{A}^{n-1}\mathbf{B} \end{bmatrix} \begin{bmatrix} \int_0^{t_f} \alpha_0(\tau)\mathbf{u}(\tau)\, d\tau \\ \vdots \\ \int_0^{t_f} \alpha_{n-1}(\tau)\mathbf{u}(\tau)\, d\tau \end{bmatrix}$$

A necessary condition for achieving any arbitrary LHS is that the *controllability matrix* $\mathcal{C} = \begin{bmatrix} \mathbf{B} & \mathbf{A}\mathbf{B} & \dots & \mathbf{A}^{n-1}\mathbf{B} \end{bmatrix}$ is full rank. This is also a sufficient condition, which we do not show here.

Controllability

- A system or the pair $(\mathbf{A}, \mathbf{B})$ is controllable, if and only if the controllability matrix $\mathcal{C} = \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \dots & \mathbf{A}^{n-1}\mathbf{B} \end{bmatrix}$ has full rank, i.e. $rank\left(\mathcal{C}\right) = n$.
- Controllability is a system property and is not affected by the choice of coordinate system used for representing the state. Changing the basis of the state to the columns of the matrix $\mathbf{T}$ does not affect the rank of $\mathcal{C}$.
- Let $\tilde{\mathbf{x}} = \mathbf{T}^{-1}\mathbf{x}$, then we have $\tilde{\mathbf{A}} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T}$ and $\tilde{\mathbf{B}} = \mathbf{T}^{-1}\mathbf{B}$. This implies that $\tilde{\mathcal{C}} = \mathbf{T}^{-1}\mathcal{C}$, and $rank\left(\tilde{\mathcal{C}}\right) = rank\left(\mathcal{C}\right)$.

Controllability

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- Are the following systems $(\mathbf{A}, \mathbf{B})$ controllable. (a) $\left(\begin{bmatrix}1 & 0\\0 & 2\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}\right)$; (b) $\left(\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}\right)$; and (c) $\left(\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}, \begin{bmatrix}1 & -1\\1 & 2\end{bmatrix}\right)$.

Controllability

When $\mathbf{A}$ is diagonalizable $\left(\mathbf{A} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^{-1}\right)$, we have,

$$\dot{\tilde{\mathbf{x}}}\left(t\right) = \mathbf{\Lambda}\tilde{\mathbf{x}}\left(t\right) + \tilde{\mathbf{B}}\mathbf{u}\left(t\right), \quad \text{where } \tilde{\mathbf{x}} = \mathbf{V}^{-1}\mathbf{x} \quad \text{and } \tilde{\mathbf{B}} = \mathbf{V}^{-1}\mathbf{B}$$

$$\tilde{\mathbf{x}}\left(t_f\right) = e^{t\mathbf{\Lambda}}\tilde{\mathbf{x}}\left(0\right) + \int_0^{t_f} e^{\left(t_f - \tau\right)\mathbf{\Lambda}}\tilde{\mathbf{B}}\mathbf{u}\left(\tau\right) d\tau \implies e^{-t_f\mathbf{A}}\tilde{\mathbf{x}}\left(t_f\right) - \tilde{\mathbf{x}}\left(0\right) = \int_0^{t_f} e^{-\tau\mathbf{A}}\tilde{\mathbf{B}}\mathbf{u}\left(\tau\right) d\tau$$

Controllability – Discrete-time system

$$\mathbf{x}\left[k+1\right] = \mathbf{A}\mathbf{x}\left[k\right] + \mathbf{B}\mathbf{u}\left[k\right]$$

Assuming the system starts at $k=0$, the output at k is given by,

$$\mathbf{x}\left[k\right] = \mathbf{A}^k\mathbf{x}\left[0\right] + \sum_{l=0}^{k-1}\mathbf{A}^{k-l-1}\mathbf{B}\mathbf{u}\left[l\right]$$

$$\mathbf{x}\left[k\right] - \mathbf{A}^k\mathbf{x}\left[0\right] = \begin{bmatrix}\mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B} & \dots & \mathbf{A}^{k-1}\mathbf{B}\end{bmatrix}\begin{bmatrix}\mathbf{u}\left[k-1\right] \\ \mathbf{u}\left[k-2\right] \\ \dots \\ \mathbf{u}\left[0\right]\end{bmatrix} = \mathcal{C}_k\tilde{\mathbf{u}}_{0:k-1}$$

Assuming $\mathbf{x}\left[0\right] = \mathbf{0}$, what all values can $\mathbf{x}\left[k\right]$ take?

Controllability – Discrete-time system

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Assuming the system starts at $k = 0$, the output at k is given by,

$$\mathbf{x}[k] = \mathbf{A}^k\mathbf{x}[0] + \sum_{l=0}^{k-1}\mathbf{A}^{k-l-1}\mathbf{B}\mathbf{u}[l]$$

$$\mathbf{x}[k] - \mathbf{A}^k\mathbf{x}[0] = \begin{bmatrix}\mathbf{B} & \mathbf{A}\mathbf{B} & \mathbf{A}^2\mathbf{B} & \dots & \mathbf{A}^{k-1}\mathbf{B}\end{bmatrix}\begin{bmatrix}\mathbf{u}[k-1] \\ \mathbf{u}[k-2] \\ \dots \\ \mathbf{u}[0]\end{bmatrix} = \mathcal{C}_k\tilde{\mathbf{u}}_{0:k-1}$$

Assuming $\mathbf{x}[0] = \mathbf{0}$, what all values can $\mathbf{x}[k]$ take? $\longrightarrow \mathbf{x}[k] \in C(\mathcal{C}_k)$. $\mathbf{x}[k]$ can only be in the subspace $C(\mathcal{C}_k)$.

Starting from $k = 0$, the possible values $\mathbf{x}[k]$ can take grows until, $C(\mathcal{C}_{k-1}) = C(\mathcal{C}_k)$, i.e.

$$\mathcal{C}_0 \subseteq \mathcal{C}_1 \subseteq \mathcal{C}_2 \cdots \subseteq \mathcal{C}_{n-1} \subseteq \mathcal{C}_n \subseteq \mathbb{R}^n$$

Controllability – Discrete-time system

- The subspace of reachable states can at most grow for n time steps, as we add more columns from $\mathbf{A}^k\mathbf{B}$.
- $\mathcal{C}_n$ is the subspace that can be reached by the system and nothing more. Because, the columns of $\mathbf{A}^n\mathbf{B}$ are a linear combination of the columns of $\mathbf{B}, \mathbf{AB}, \mathbf{A}^2\mathbf{B}, \ldots \mathbf{A}^{n-1}\mathbf{B}$.
- Thus, the system is controllable if and only if $\mathcal{C}_n = \mathbb{R}^n$, or the $rank\left(\mathcal{C}_n\right) = n$.
- In the discrete case, the controllability matrix $\mathcal{C}_n$ can also be used to determine the input sequence $\tilde{\mathbf{u}}_{0:n-1}$ that takes you from state $\mathbf{x}[0]$ to $\mathbf{x}[n]$.

$$\mathbf{x}[k] - \mathbf{A}^k\mathbf{x}[0] = \mathcal{C}_n\tilde{\mathbf{u}}_{0:n-1} \implies \tilde{\mathbf{u}}_{0:n-1} = \mathcal{C}_n^{\dagger}\left(\mathbf{x}[k] - \mathbf{A}^k\mathbf{x}[0]\right)$$

Where, $\mathcal{C}_n^{\dagger} = \mathcal{C}_n^{-1}$ for a single input system, and it is a right inverse when there are more than one inputs to the system.

Controllability – Discrete-time system

- Which of the following systems are controllable? (a) $\left(\begin{bmatrix}1 & 0\\0 & 2\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}\right)$; (b) $\left(\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}\right)$; and (c) $\left(\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}\right)$.
- What is the problem with system (b)? The system has repeated eigenvalues, and we have only one input.
- **Output Controllability**:

$$rank\left(\begin{bmatrix}\mathbf{D} & \mathbf{CB} & \mathbf{CAB} & \mathbf{CA}^2\mathbf{B} & \dots & \mathbf{CA}^{n-1}\mathbf{B}\end{bmatrix}\right) = m$$