Linear Algebra & Random Processes

Experiments, Sample Spaces and Probability

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Topics Covered & References

Topics

- Random Experiments, Sample Space, and Events
- Review of Set Operations
- Axioms of Probability
- Proability Assignments - Discrete Sample Spaces
- Methods of Counting
- Conditional probability
- Independent Events
- Baye's Rule
- Sequence of Events

References

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Random Experiments and Sample Space

- **Random experiment** – A experiment whose outcome is not predictable.
  - Tossing of a coin.
  - Voltage across a real resistor (R) for a known current.
  - Height and weight of 40 year old person randomly chosen from a population.
- A random experiment is specified by a set of experimental procdures, and a set of measurements/observations.
- The **outcome** of a random experiment is any observable variable of interest.
- **Sample space** of the experiment S is the universe of possible values we can observe for a random experiment's outcome.
- Individual elements of S are called the **sample points**

Random Experiments and Sample Space

- Some simple examples of random experiments:
  - Tossing of a coin: $S = \{\text{Head}, \text{Tail}\}$.
  - Age (in years) of a person randomly chosen from a population. $S = ?$
  - Voltage across a real resistor (R) for a known current. $S = ?$
  - Height and weight of 40 year old person randomly chosen from a population. $S = ?$
- Sample spaces can be discrete (finite or countably infinite) or continuous.
- Give some examples of:
  1. Finite sample spaces.
  2. Countably inifinite sample spaces.
  3. Uncountable sample spaces.

Events

- Any subset of the sample space S is called and **event** of the experiment.
- Consider the experiment tossing a dice, and we observe the count of the dots that turn on the top face of the dice.
  - Observed outcome is an even number.
  - Observed outcome is a positive number.
  - Observed outcome is 0.
- For discrete sample spaces and **elementary event** is an event with just single sample point.
- We can cobmine events to produce other events that might be of interest to us. Set operations can be used to perform algebra on events.

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  - Observed outcome is 0. $A = \{\} \implies$ **Impossible event**
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Operating on Events: Review of Set Operations

- A set is a collection of objects, and an universal set (U) is the collecton of all possible objects of interest.
- **Subset**: $B \subseteq A$; **Proper subset**: $B \subset A$; **Empty set**: $\emptyset$.
- **Equality of sets**: $A = B \iff A \subset B$ and $B \subset A$.
- **Union of sets**: $A \cup B = \{x \,|\, x \in A \,\text{or}\, x \in B\}$.
- **Intersection of sets**: $A \cap B = \{x \,|\, x \in A \,\text{and}\, x \in B\}$.
- **Complement of a set**: $A^c = \{x \,|\, x \notin A \,\text{and}\, x \in U\}$.
- **Relative Complement of a set**: $A - B = \{x \,|\, x \in A \,\text{and}\, x \notin U\}$.
- Union and intersection operations are commutative, associative, and distributive.
- These operations can be extended to an arbitrary nunber of sets.

$$\bigcup_{i=1}^{\infty} A_i = A_1 \cup A_2 \cup \cdots \quad \text{and} \quad \bigcap_{i=1}^{\infty} A_i = A_1 \cap A_2 \cap \cdots$$

Axioms of Probability Theory

Probability is a number assigned to events of a random experiment (and not necessarily the individual sample points).

The assignment of probabilities must satisfy the prorperties. Axioms of probability:

Axiom I: For any event A, $P(A) \geq 0$

Axiom II: $P(S) = 1$; S is the sample space.

Axiom III: For two events A, B,
$A \cap B = \emptyset \implies P(A \cup B) = P(A) + P(B)$

Axiom IIIa: For a countable sequencce of events $A_1, A_2, A_3, \ldots$, with
$A_i \cap A_j = \emptyset$, $i \neq j$

The other rules for proability calculation for events of an experiment can be derived from these three axioms.

- $P(A^c) = 1 - P(A)$
- $P(A) \leq 1$
- $P(\emptyset) = 0$
- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
- $A \subset B \implies P(A) \leq P(B)$

Discrete Sample Spaces

- For any experiment, we need to make some inital probability assingments to events, and then we can calculate the probabilities of other events through the different rules.
- For **discrete sample spaces**, this can be done by assigning probabilities to the elementary events. Consider the following example:
  - Random experiment: Randomly select a person within the CMC hospital campus between 9AM to 5PM during a weekday.
  - Outcome: Age of the person chosen in years.
  - Sample space: We are interested different age ranges. $S = \{$I, II, III, IV, V, $\}$. I: $0-1$, II: $2-16$, III: $17-30$, IV: $31-60$, and V: Above 61.
  - Assign probabilities to the different elementary outcomes: $P(I) =?$, $P(II) =?$, $P(III) =?$, $P(IV) =?$, and $P(V) =?$.

Discrete Sample Spaces

- What are the probabilities of the following events? (a) Age greater than 16; (b) Age between 31 and 60; (c) Either very young or old.
- We consider the sample experiment, but a different outcome:
  - Outcome: The category of the person: Staff (Sta), Student (Stu), and Other (Oth).
  - Sample space: $S = \{\text{Sta, Stu, Oth}\}$?

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  - What are the probabilities of the following events? (a) Peron is a not a staff; (b) Staff or a student; (c) Staff or a patient.
- Once we have the probabilities assigned to all possible events (complements, countable unions and intersections), then we have a complete probability model for the random experiment.

Probability Assignment through Counting

- For **finite workspaces**, the assignment of probabilities are often based on the assumption that all the outcomes of equiprobable.
- Let, $S = \{o_1, o_2, \ldots o_N\}$. The probability of each elementary event $E_i = \{o_i\}$ is, (assuming all outcomes are equiprobable)

$$P(E_i) = \frac{1}{N}$$

- Examples: fair coin, fair dice, the previous example about samping people from the CMC hospial campus etc.
- The probability of any other event,

$$P(A) = \frac{\text{No. of favorable outcomes}}{\text{Total no. of outcomes}} = \frac{n_A}{N}$$

  where, n_A is the number of elements of the set representing the event A.
- In problems where the equi-probable assumption can be made, determing the probability of a event can be simply thought of a problem of counting things.

Probability Assignment through Counting - Useful counting formulae

- Consider a set of n unique item to choose from. We will assume that each of the n elements if equally likely to be chosen in the sampling processes. We are interest in selecting k elements from this set.

	Replacement	**No Replacement**
Order		
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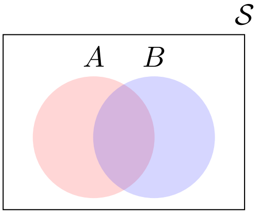
No Order	$\binom{n+k-1}{k}$	$\frac{n^k}{k!(n-k)!} \triangleq \binom{n}{k}$

Conditional probability

Often in random experiments, the knowledge of occurance of an event A has implications on our uncertanity about the occurance of another event B.

Occurance of an event A, changes the sample space that needs to be considered for the calculation of probability of B.



The probability of event B occuring, given that A has already occured is the *conditional probability of B given A.*

$$P(B|A) = \frac{P(A \cap B)}{P(A)}, \; P(A) > 0$$

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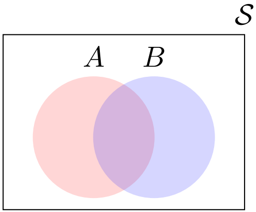

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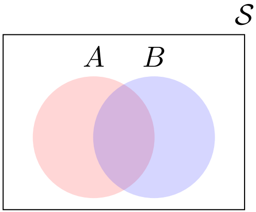

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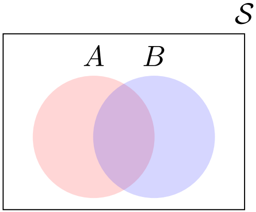

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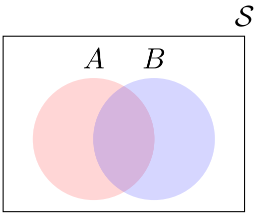

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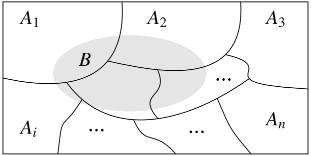
Baye's Rule

Conditional probabilities can be used for calculating probabilities of intersection of events,
$P(A \cap B) = P(B|A)\,P(A) = P(A|B)\,P(B)$
This gives us the *Baye's Rule*,

$$P(A|B) = P(B|A)\,\frac{P(A)}{P(B)}$$

Consider the partition $\{A_1, A_2, \ldots A_n\}$ of the sample space $\mathcal{S}$, such that $A_i \cap A_j = \emptyset$, $i \neq j$ and $\bigcup_{i=1}^{n} A_i = \mathcal{S}$



Theorem of total probability:

$$P(B) = \sum_{i=1}^{n} P(B \cap A_i) = \sum_{i=1}^{n} P(B|A_i)\,P(A_i)$$

General form of Baye's rule:

$$P(A_i|B) = \frac{P(A_i|B)\,P(A_i)}{\sum_{j=1}^{n} P(B|A_j)\,P(A_j)}$$

Independent Events

Two events are said to be *independent*, if the occurance of one does not affect the probability of occurance of the other, i.e.

$$P\left(A|B\right) = P\left(A\right) \quad \iff \quad A \,\&\, B \text{ are independent events.}$$

This, implies that $P\left(A \cap B\right) = P\left(A\right) P\left(B\right)$.

Consider two mutually exclusive events E and F. Are these events independent?

A collection of events $A_1, A_2, \ldots A_n$ are *independent*, if and only if,

$$P\left(\bigcap_{i=1}^{n} A_i\right) = \prod_{i=1}^{n} P\left(A_i\right) \implies P\left(A_k \middle| \bigcap_{j \in \{i_1 \ldots i_l\}} A_j\right) = P\left(A_k\right)$$

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Consider two mutually exclusive events E and F. Are these events independent? No. The occurance of one rules out the occurance of the other.

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Sequence of Events

- Often we are interested in random expriments, which are composed of a series of sub-experiments, which in-turn might or might not be independent.
- **Independent experiments** are ones where the events of the two experiments are indepdent.
- Consider a random experiment consisting of a series of N coin tosses. If the outcome of the current coin toss is not affected by the outcomes of the past tosses, and will not affect the outcomes of the future tosses, then we individual coin tosses (sub-experiments) are independent experiments.

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  If the probability of obtaining H in any sub-experiment is p. What is probability of any specific outcome of this experiment?

  How would our approach for evaluating the probability change, if the coin tosses were not independent?