Linear Systems

Stability

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Internal stability

- There are two types of stability one can associate with a system $\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t))$ – **Internal stability** and **Input-Output stability**.
- **Internal stability**: Deals with the stability of the zero-input response of the system states, i.e. $\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t))$.
- An *equilibrium point* $\mathbf{x}_e$ of this system is defined as a point in the state space where, $\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}_e) = \mathbf{0}$, i.e. if the system starts in this state, it stays in that state for all time.
- In the case of linear systems, we have $\mathbf{A}\mathbf{x}_e = \mathbf{0}$. The nullspace of $\mathbf{A}$ is the set of all equilibrium points of the linear system.

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- In the case of linear systems, we have $\mathbf{A}\mathbf{x}_e = \mathbf{0}$. The nullspace of $\mathbf{A}$ is the set of all equilibrium points of the linear system.

Find the equilibrium points for the following systems with $\mathbf{f}(\mathbf{x}(t))$: (a) $\begin{bmatrix} x_2 \\ \sin x_1 \end{bmatrix}$; (b) $\begin{bmatrix} x_1 + 2x_2 \\ 2x_1 + 3x_2 \end{bmatrix}$; (c) $\begin{bmatrix} -x_1 + x_2 \\ x_1 - x_2 \end{bmatrix}$; and (d) $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Internal stability

- Definition of stability in the Lyapunov sense for linear systems:
  - The zero-input response of a linear system $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t)$ is *stable or marginally stable* if every finite initial condition $\mathbf{x}(0^-)$ results in a bounded state trajectory $\mathbf{x}(t)\ \forall t \geq 0$.
$$\|\mathbf{x}(t)\| \leq d, \quad \forall t \geq 0$$
  - The zero-input response is *asymtotically stable* if everyf initial condition $\mathbf{x}(0^-)$ results in a bounded state trajectory $\mathbf{x}(t)$ that coverges to 0 as $t \to \infty$.
$$\|\mathbf{x}(t)\| \leq d \text{ and } \lim_{t\to\infty} \|\mathbf{x}(t)\| = 0$$
- The system $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t)$ is marginally stable if and only if all eigenvales of $\mathbf{A}$ have either zero or negative real parts, and the eigenvalues with zero real parts have the same algebraic and geometric multiplicity.
- The system $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t)$ is asymptotically stable if and only if all eigenvales of $\mathbf{A}$ have negative real parts.

Internal stability

- Consider the solution, $\mathbf{x}(t) = e^{t\mathbf{A}}\mathbf{x}(0^-)$, $t \geq 0$, and $\mathbf{A} = \mathbf{VJV}^{-1}$.

$$\|\mathbf{x}(t)\| = \left\|e^{t\mathbf{A}}\mathbf{x}(0^-)\right\| \leq \left\|e^{t\mathbf{J}}\right\| \left\|\mathbf{x}(0^-)\right\|$$

- When $\mathbf{A}$ is diagonalizable (λ_i are the eigenvalues of $\mathbf{A}$),
  - $\|\mathbf{x}(t)\| \leq e^{\sigma t} \|\mathbf{x}(0^-)\|$, where $\sigma = \max_i \Re\{\lambda_i\}$.
  - When $\sigma = 0$, $\|\mathbf{x}(t)\|$ is bounded $\forall t \geq 0$.
  - When $\sigma < 0$, $\lim_{t\to\infty} \|\mathbf{x}(t)\| = 0$.
- When $\mathbf{A}$ is not diagonalizable, then $\mathbf{J}$ is block diagonal.
  - Consider the i^{th} Jordan block, $\mathbf{J}_i = \lambda_i\mathbf{I} + \mathbf{N}$, Thus, $e^{t\mathbf{J}_i} = e^{\lambda_i t\mathbf{I}}e^{t\mathbf{N}} \implies \|\mathbf{x}(t)\| \leq e^{\sigma_i t} \left\|e^{t\mathbf{N}}\right\| \left\|\mathbf{x}(0^-)\right\|$
  - When $\sigma_i = 0$, $\left\|e^{t\mathbf{N}}\right\|$ grows with time, and thus $\mathbf{x}(t)$ is not bounded.
  - When $\sigma_i < 0$, the $e^{\sigma_i t}$ term does not allow $\mathbf{x}(t)$ to grow.

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Comment of the stability: (a) $\begin{bmatrix}-1 & 1\\ 0 & -2\end{bmatrix}$; (b) $\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}$; (c) $\begin{bmatrix}1 & 2\\ 0 & -2\end{bmatrix}$; and (d) $\begin{bmatrix}-1 & -1 & 1\\ 0 & -2 & 1\\ 0 & 0 & 0\end{bmatrix}$

Internal stability – Lyapunov stability criteria

- A general approach to evaluating the the stability of a dynamic system $\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t))$ was proposed by Lyapunov.
- Stability is inferred by looking at the energy associated with a system, and how it changes as the system evolves. i.e, whether the system dissipates, conserves or generates energy with time.
- The idea of the energy associated with the system and its change with time is captured through a *Lyapunov function* $V(\mathbf{x}) : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$,

$$V(\mathbf{0}) = 0 \text{ and } V(\mathbf{x}) > 0 \; \forall \mathbf{x} \neq \mathbf{0}, \quad \text{and } \dot{V}(\mathbf{x}) \leq 0$$

- $\dot{V}(\mathbf{x}) = \left(\frac{\partial}{\partial \mathbf{x}} V(\mathbf{x})\right) \dot{\mathbf{x}} = \left(\frac{\partial}{\partial \mathbf{x}} V(\mathbf{x})\right) \mathbf{f}(\mathbf{x})$ is the time rate of change of energy of the system.
  - Stable (marginally) systems conserve energy, i.e. $\dot{V}(\mathbf{x}) = 0$.
  - Asymptotically stable systems dissipate energy, i.e. $\dot{V}(\mathbf{x}) < 0$.
  - Unstable systems generate energy, i.e. $\dot{V}(\mathbf{x}) > 0$.
- For a given system, if we can find a Lyapunov function, then the system is stable or asymptotically stable if $\dot{V}(\mathbf{x}) < 0$.

Internal stability – Lyapunov stability criteria

- Consider, $\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix} \mathbf{x}(t)$. The energy associated with this system is $V(\mathbf{x}) = \frac{1}{2}kx_1^2 + \frac{1}{2}mx_2^2 \implies \dot{V}(\mathbf{x}) = -bx_2^2$. Is this system stable?
- Consider a general LTI system, $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t)$, with non-singular $\mathbf{A}$.
  A necessary and sufficient condition for this system to be asymptotically stable is for a given symmetric, positive definite matrix $\mathbf{Q}$, there exists a symmetric, positive definite matrix $\mathbf{P}$ such that

$$\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A} = -\mathbf{Q}$$

- We can arbitrarily choose $\mathbf{Q}$ and solve for $\mathbf{P}$. The positive definiteness of $\mathbf{P}$ is a necessary and sufficient condition for the asymptotic stability of the LTI system.

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Is this system asymptotically stable? $\dot{\mathbf{x}}(t) = \begin{bmatrix} 1 & 2 \\ -1 & 2 \end{bmatrix} \mathbf{x}(t)$

Internal stability – Discrete-time LTI systems

- The system $\mathbf{x}[k+1] = \mathbf{A}\mathbf{x}[k]$ is marginally stable if and only if all eigenvales of $\mathbf{A}$ either of magnitude 1 or less than 1, and the eigenvalues with magnitude 1 have the same algebraic and geometric multiplicity.
- The system $\mathbf{x}[k+1] = \mathbf{A}\mathbf{x}[k]$ is asymptotically stable if and only if all eigenvales of $\mathbf{A}$ have magnitude less than 1.
- $\mathbf{x}[k] = \mathbf{A}^k\mathbf{x}[0]\,,\ k > 0$, and $\mathbf{A} = \mathbf{V}\mathbf{J}\mathbf{V}^{-1}$

$$\|\mathbf{x}[k]\| = \left\|\mathbf{A}^k\mathbf{x}\left(0^-\right)\right\| \leq \left\|\mathbf{J}^k\right\| \left\|\mathbf{x}\left(0^-\right)\right\|$$

- When $\mathbf{A}$ is diagonalizable (λ_i are the eigenvalues of $\mathbf{A}$),
  - $\|\mathbf{x}[k]\| \leq |\lambda|^k \|\mathbf{x}[0]\|$, where $\lambda = \max_i |\lambda_i|$.
  - When $|\lambda| = 1$, $\|\mathbf{x}[k]\|$ is bounded $\forall k > 0$.
  - When $|\lambda| < 1$, $\lim_{k\to\infty} \|\mathbf{x}[k]\| = 0$.
- When $\mathbf{A}$ is not diagonalizable, then $\mathbf{J}$ is block diagonal.
  - Consider the i^{th} Jordan block, $\mathbf{J}_i^k = \left(\lambda_i\mathbf{I} + \mathbf{N}\right)^k = \sum_{l=0}^{k} \frac{k!}{(k-l)!l!}\lambda_i^l\mathbf{N}^{k-l}$
  - When $|\lambda_i| = 1$, $\left\|\mathbf{J}_i^k\right\|$ grows with time, and thus $\mathbf{x}[k]$ is not bounded.
  - When $|\lambda_i| < 1$, the λ_i^l term does not allow $\mathbf{x}[k]$ to grow.

Internal stability – Lyapunov stability criteria (discrete-time system)

- For a discrete-time system, $\mathbf{x}[k+1] = \mathbf{A}\mathbf{x}[k]$, we again start with a scalar, positive definite, continuous ("energy" like) function $V(\mathbf{x})$.
- The rate of change of energy is captured by successive differences in the values of $V(\mathbf{x})$ for different values of k, i.e. $\Delta V(\mathbf{x}) = V(\mathbf{x}[k+1]) - V(\mathbf{x}[k])$.
    - Stable (marginally) systems conserve energy, i.e. $\Delta V(\mathbf{x}) = 0$.
    - Asymptotically stable systems dissipate energy, i.e. $\Delta V(\mathbf{x}) < 0$.
    - Unstable systems generate energy, i.e. $\Delta V[\mathbf{x}] > 0$.
- A necessary and sufficient condition for this system $\mathbf{x}[k+!] = \mathbf{A}\mathbf{x}[k]$ to be asymptotically stable is for a given symmetric, positive definite matrix $\mathbf{Q}$, there exists a symmetric, positive definite matrix $\mathbf{P}$ such that

$$\mathbf{A}^T\mathbf{P}\mathbf{A} + \mathbf{P} = -\mathbf{Q}$$

- We can arbitrarily choose $\mathbf{Q}$ and solve for $\mathbf{P}$. The positive definiteness of $\mathbf{P}$ is a necessary and sufficient condition for the asymptotic stability of the LTI system.

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Is this system asymptotically stable? $\mathbf{x}[k+1] = \begin{bmatrix} 1 & 2 \\ -1 & 2 \end{bmatrix} \mathbf{x}[k]$

Input-Output stability

- Input-output stability or external stability deals with the forced response of a system, assuming the system is relaxed.
- Input-output stability is also known as BIBO (bounded input, bounded output) stability, i.e. a bounded input $\mathbf{u}(t)$ applied to the system produces a bounded output $\mathbf{y}(t)$.
- A single input, single output (SISO) LTI system with impulse response $h(t)$ is BIBO stable, if and only if
$$\int_0^{\infty} |h(t)|\, dt < \infty$$
When $h(t)$ is not absolutely integrable, then we are not guaranteed that bounded inputs will produce bounded outputs.
- A SISO system with a rational transfer function $H(s)$ is BIBO stable if and only if all its poles lie in the left half of the s-plane.
$$H(s) = \frac{B(s)}{A(s)} \xrightarrow{\mathcal{L}^{-1}} h(t) \text{ contains } e^{p_i t}, te^{p_i t}, \ldots t^{m-1} e^{p_i t}$$

Input-Output stability

- In the case of a muti-input, multi-output (MIMO) LTI system, the impulse response and transfer function matrices are given by,

$$\mathbf{G}(t) = \mathbf{C}e^{t\mathbf{A}}\mathbf{B} + \mathbf{D}\delta(t) \text{ and } \mathbf{H}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$$

- A MIMO system is BIBO stable, if and only if each element of the impulse response matrix $\mathbf{G}(t)$ is absolutely integrable.

$$\int_0^\infty |g_{ij}(t)|\, dt < \infty, \;\; \forall 1 \leq i, j \leq n$$

- A MIMO LTI system is BIBIO stable, if and only if the poles of each element of the transfer function matrix $H(s)$ lie in the left half of the s-plane.
  Even if we have eigenvalue that have positive real parts, the system migth still be BIBO stable because of pole-zero cancellations in the individual elements of $\mathbf{G}(s)$.

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Is this system externally stable? $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\mathbf{C} = \begin{bmatrix} 1 & -2 \end{bmatrix}$. Is this system internally stable?

Input-Output stability (discrete-time system)

- A SISO discrete-time LTI system with impulse response $h[k]$ is BIBO stable, if and only if

$$\sum_{k=0}^{\infty} |h[k]| < \infty$$

- A SISO system with a rational transfer function $H(z)$ is BIBO stable if and only if all its poles lie within the unit circle $|z| = 1$.

$$H(z) = \frac{B(z)}{A(z)} \xrightarrow{\mathcal{L}^{-1}} h[k] \text{ contains } p_i^k, kp_i^k, \ldots k^{m-1}p_i^k$$

- A MIMO discret-time LTI system is BIBO stable, if and only if each element of the impulse response matrix $\mathbf{G}[k]$ is absolutely summable.

$$\sum_{k=0}^{\infty} |g_{ij}[k]| < \infty, \ \ \forall 1 \leq i, j \leq n$$

- A MIMO discrete-time LTI system is BIBO stable, if and only if the poles of each element of the transfer function matrix $H(z)$ lie in the unit circle.