Introduction to Signal Processing

Geometric Signal Theory

Sivakumar Balasubramanian

Department of Bioengineering
Christian Medical College, Bagayam
Vellore 632002

Geometric Signal Theory

An **interesting and important development** that will help understanding signal processing a little easier.

This lecture will present **a geometric view of signal and some important signal processing operation**.

Requires some getting used to new terminology for the generalization of familiar geometric ideas from 2 or 3 dimensions to ∞ dimensions.

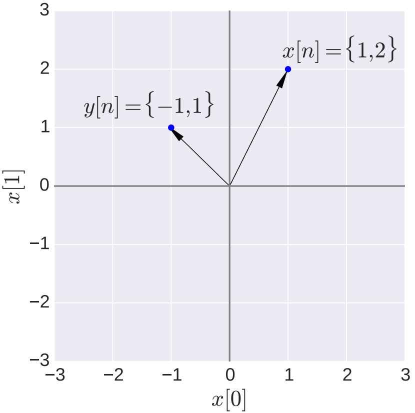
Discrete-time signals as vectors

Consider two finite duration real discrete-time signal

$$x[n] = \{\boxed{x_0}, x_1\} = [x_0, x_1]^T$$

$$y[n] = \{\boxed{y_0}, y_1\} = [y_0, y_1]^T$$

where, $n \in \{0, 1\}$



Some familiar and useful geometric ideas

- **Length** of a vector.

$$\|x\| = \sqrt{x_0^2 + x_1^2}$$

- **Distance** between vectors.

$$\|x - y\| = \sqrt{(x_0 - y_0)^2 + (x_1 - y_1)^2}$$

- **Scalar product** or **Inner product** between vectors.

$$\langle x, y \rangle = x_0 y_0 + x_1 y_1 \implies \|x\| = \langle x, x \rangle$$

$$\langle x, y \rangle = \|x\| \|y\| \cos \theta$$

Extension to N dimensions

Consider the following finite duration signals with N elements,

$$x[n] = \{\boxed{x_0}, x_1, \cdots, x_{N-1}\} = [x_0, x_1, \cdots, x_{N-1}]^T$$

$$y[n] = \{\boxed{y_0}, y_1, \cdots, y_{N-1}\} = [y_0, y_1, \cdots, y_{N-1}]^T$$

where, $x_i, y_i \in \mathbb{C}$

- **Length** of a vector. $\|x\| = \left(\sum_{i=0}^{N-1} |x_i|^2\right)^{\frac{1}{2}}$
- **Distance** between vectors. $\|x - y\| = \left(\sum_{i=0}^{N-1} |x_i - y_i|^2\right)^{\frac{1}{2}}$
- **Inner product** between vectors. $\langle x, y \rangle = \sum_{i=0}^{N-1} x_i y_i^*$

Extension to infinite dimensions!

Can we extend the geometric ideas to infinite dimensional signals?
Yes, but we need to be careful.

$$x[n] = \{\boxed{x_0}, x_1, \cdots\} = [x_0, x_1, \cdots]^T$$

$$y[n] = \{\boxed{y_0}, y_1, \cdots\} = [y_0, y_1, \cdots]^T$$

- **Length** $\|x\| = \left(\sum_{i=0}^{\infty} |x_i|^2\right)^{\frac{1}{2}}$
- **Distance** $\|x - y\| = \left(\sum_{i=0}^{\infty} |x_i - y_i|^2\right)^{\frac{1}{2}}$
- **Inner product** $\langle x, y\rangle = \sum_{i=0}^{\infty} x_i y_i^*$

The above ideas make sense only when the infinite sums are finite converge, i.e. **the sums must converge**.

Extension to infinite dimension!

We will restrict ourselves to the space of **finite energy signals**, i.e.

$$\|x\| = \left(\sum_{i \in \mathbb{Z}} |x_i|^2 \right)^{\frac{1}{2}} < \infty$$

Here we have assumed $x[n]$ to start at $-\infty$ and end at ∞.

$$x[n] = \{\cdots, x_{-1}, \boxed{x_0}, x_1, \cdots\} = [\cdots, x_{-1}, x_0, x_1, \cdots]^T$$

This also leads to meaningful inner products,

$$\|x\|, \|y\| < \infty \implies |\langle x, y \rangle| < \infty$$

This is called the $\ell^2(\mathbb{Z})$ space.

What is the inner product?

- Projection of a signal x onto another signal y, or simply a measure of their relative orientations.

$$\langle x, y \rangle = \sum_{i \in \mathbb{Z}} x_i y_i^* = \|x\| \|y\| \cos \theta$$

  where, θ is the angle between the signals x and y.
- $\langle x, y \rangle$ tells us how much of x is in y and *vice versa*.
- x and y are orthogonal, when $\langle x, y \rangle = 0 \implies x \perp y$
  For example, let $x = [1, 1]^T$ and $y = [1, -1]^T$. What is $\langle x, y \rangle$?

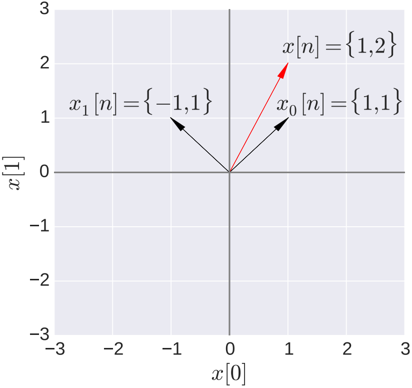
Bases: Representing a signal in terms of other signals

Can $x[n]$ be represented as a linear combination of $x_0[n]$ and $x_1[n]$? **Yes.**

$$x[n] = \alpha_0 x_0[n] + \alpha_1 x_1[n]$$

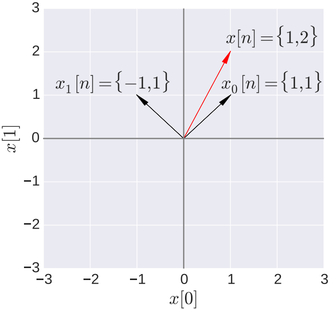
$$\alpha_i = \frac{\langle x_i[n], x[n] \rangle}{\|x_i[n]\|}, i = 0, 1$$

What are the values of α_1 and α_2?



Bases: Representing a signal in terms of other signals

- $\{1, 2\}$ and $\{\alpha_0, \alpha_1\}$ are two presentations of $x[n]$.
- The difference is the basis.
- Basis for $\{\alpha_0, \alpha_1\}$ is $\{x_0[n], x_1[n]\}$.
- What is the basis for $\{1, 2\}$? $\implies$ the *standard basis* $e_0 = \{1, 0\}$ and $e_1[n] = \{0, 1\}$.



Extension of ideas to continuous-time

Consider two continuous time signals, $x(t), y(t) \in \mathbb{C}^{\mathbb{R}}$,

- **Length**.

$$\|x\| = \left(\int_{-\infty}^{\infty} |x(t)|^2 \, dt \right)^{1/2}$$

- **Distance**.

$$\|x - y\| = \left(\int_{-\infty}^{\infty} |x(t) - y(t)|^2 \, dt \right)^{1/2}$$

- **Scalar product** or **Inner product** between vectors.

$$\langle x, y \rangle = \left(\int_{-\infty}^{\infty} x(t) y^*(t) dt \right)^{1/2}$$

Note: The above quantities are meaningful only when the integrals converge.

Extension to infinite dimension!

We will restrict ourselves to the space of **finite energy signals**, i.e.

$$\|x\| = \left(\int_{-\infty}^{\infty} |x(t)|^2 \, dt \right)^{1/2} < \infty$$

This is called the $\mathcal{L}^2(\mathbb{R})$ space.