# Linear Control and Estimation

## Matrix Inverses

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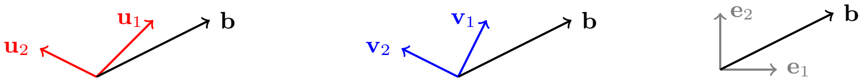
# References

# Representation of vectors in a basis

- Consider the vector space $\mathbb{R}^n$ with basis $\{\mathbf{v}_1, \mathbf{v}_2, \dots \mathbf{v}_n\}$. Any vector in $\mathbf{b} \in \mathbb{R}^n S$ can be representated as a linear combination of of $\mathbf{v}_i$s,

$$\mathbf{b} = \sum_{i=1}^{n} \mathbf{v}_i \mathbf{a}_i = \mathbf{V}\mathbf{a}; \;\; \mathbf{a} \in \mathbb{R}^n, \;\; \mathbf{V} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_n \end{bmatrix} \in \mathbb{R}^{n \times n}$$




$\{\mathbf{v}_1, \mathbf{v}_2\}$, $\{\mathbf{u}_1, \mathbf{u}_2\}$ and $\{\mathbf{e}_1, \mathbf{e}_2\}$ are valid basis for $\mathbb{R}^2$, and the presentation for $\mathbf{b}$ in each one of them is different.

- Finding out $\mathbf{a}$ is easiest when we are dealing with an orthonormal basis, in which case $\mathbf{a}$ is given by,

$$\mathbf{a} = \begin{bmatrix} \mathbf{u}_1^T b \\ \mathbf{u}_2^T b \\ \vdots \\ \mathbf{u}_n^T b \end{bmatrix} = \mathbf{U}^T\mathbf{b} \implies \mathbf{b}_U = \sum_{i=1}^{n} \left(\mathbf{u}_i^T b\right) \mathbf{u}_i = \mathbf{U}\mathbf{U}^T\mathbf{b}$$

# Matrix Inverse

- Consider the equation $\mathbf{A}\mathbf{x} = \mathbf{y}$, where $\mathbf{A} \in \mathbb{R}^{n \times n}$ and $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.
- Let us assume $\mathbf{A}$ is non-singular, which means that the columns of $\mathbf{A}$ represent a basis for $\mathbb{R}^n$.
- What does $x$ represent? It is the representation of $\mathbf{y}$ in the basis consisitng of the columns of $\mathbf{A}$.

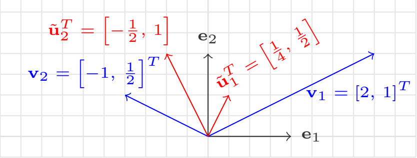
$$\mathbf{y} = \mathbf{A}\mathbf{x} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \ldots & \mathbf{a}_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \sum_{i=1}^{n} \mathbf{a}_i x_i$$

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{y} = \begin{bmatrix} \tilde{\mathbf{b}}_1^T \\ \tilde{\mathbf{b}}_2^T \\ \ldots \\ \tilde{\mathbf{b}}_n^T \end{bmatrix} \mathbf{y} = \begin{bmatrix} \tilde{\mathbf{b}}_1^T \mathbf{y} \\ \tilde{\mathbf{b}}_2^T \mathbf{y} \\ \ldots \\ \tilde{\mathbf{b}}_n^T \mathbf{y} \end{bmatrix}$$

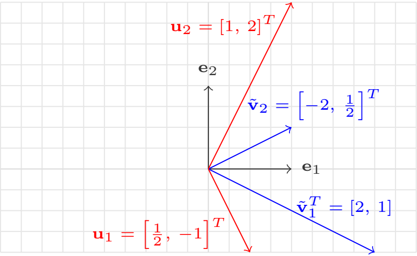
- $\mathbf{A}^{-1}$ is a matrix that allows change of basis to the columns of $\mathbf{A}$ from the standard basis!

# Matrix Inverse

Columns of $\mathbf{A}$ and rows of $\mathbf{A}^{-1}$




Rows of $\mathbf{A}$ and columns of $\mathbf{A}^{-1}$




$$\mathbf{V} = \begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2\end{bmatrix} = \begin{bmatrix}2 & -1\\ 1 & \frac{1}{2}\end{bmatrix}$$

$$\mathbf{V}^{-1} = \begin{bmatrix}\tilde{\mathbf{u}}_1^T\\ \tilde{\mathbf{u}}_2^T\end{bmatrix} = \frac{1}{2}\begin{bmatrix}\frac{1}{2} & 1\\ -1 & 2\end{bmatrix}$$

$$\mathbf{v}_1^T\tilde{\mathbf{u}}_1 = \mathbf{v}_2^T\tilde{\mathbf{u}}_2 = \tilde{\mathbf{v}}_1^T\mathbf{u}_1 = \tilde{\mathbf{v}}_2^T\mathbf{u}_2 = 1$$

$$\mathbf{v}_1^T\tilde{\mathbf{u}}_2 = \mathbf{v}_2^T\tilde{\mathbf{u}}_1 = \tilde{\mathbf{v}}_1^T\mathbf{u}_2 = \tilde{\mathbf{v}}_2^T\mathbf{u}_1 = 0$$

# Left Inverse

- Consider a rectangular matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$. There exists no inverse $\mathbf{A}^{-1}$ for this matrix.
- But, does there exist two matrices $\mathbf{B}, \mathbf{C} \in \mathbb{R}^{n \times m}$, such that, $\mathbf{AB} = I_m$ and $\mathbf{CA} = I_n$

  For a rectangular matrix only on these two can be ture, and thus a rectangular matrix can only have either a left or a right inverse.

  **Note:** *A non-singular square matrix has both left and right inverses, and both are equal and unique.*
- Any non-zero vector $\mathbf{a} \in \mathbb{R}^{n \times 1}$ is left invertible,.

$$\mathbf{ba} = 1, \;\; \mathbf{b} \in \mathbb{R}^{1 \times n}; \quad \mathbf{b}^T = \frac{\mathbf{a}}{\|\mathbf{a}\|^2} + \alpha \mathbf{a}^{\perp}$$

- There can be more than one left inverse! If there is more than one there are an infinite number of them.
- What is condition for the left inverse of $\mathbf{A}$ to exist? *The colmuns of $\mathbf{A}$ must be independent.* $\longrightarrow rank\left(\mathbf{A}\right) = n \longrightarrow \mathbf{Ax} = 0 \implies \mathbf{x} = 0.$
- Only tall and skinny martices can have left inverses.
- $\mathbf{Ax} = \mathbf{b}$ can be solved, if and only if $\mathbf{BAx} = \mathbf{Bb}$, where $\mathbf{BA} = \mathbf{I}_n$.

# Right Inverse

- For a rectangular matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$, we can have,

$$\mathbf{A}\mathbf{B} = \mathbf{I}_m \quad \longrightarrow \mathbf{B} \text{ is the right inverse.}$$

- Right inverse of $\mathbf{A}$ exists only if the rows of $\mathbf{A}$ are independent, i.e.

$$rank\left(\mathbf{A}\right) = m \longrightarrow \mathbf{A}^T\mathbf{x} = \mathbf{0} \implies \mathbf{x} = \mathbf{0}$$

- Only fat and wide matrices can have right inverses.
- $\mathbf{A}\mathbf{x} = \mathbf{b}$ can be solved for any b. $\mathbf{x} = \mathbf{B}\mathbf{b} \implies \mathbf{A}\left(\mathbf{B}\mathbf{b}\right) = \mathbf{b}$. There will be an infinite number of solutions for $\mathbf{x}$.
- Let $\mathbf{A}\mathbf{B} = \mathbf{I}_m \implies \mathbf{B}^T\mathbf{A}^T = \mathbf{I}_m \implies \mathbf{B}^T$ is the left inverse of $\mathbf{A}^T$.

# Pseudo Inverse

- Consider a tall, skinny matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$ with independent columns. It turns out the Gram matrix $\mathbf{A}^T\mathbf{A} \in \mathbb{R}^{n \times n}$ is invertible. If that is the case then,

$$\left(\mathbf{A}^T\mathbf{A}\right)^{-1}\mathbf{A}^T\mathbf{A} = \mathbf{I}_n; \quad \mathbf{B} = \left(\mathbf{A}^T\mathbf{A}\right)^{-1}\mathbf{A}^T \text{ is a left inverse.}$$

- $\mathbf{A}^\dagger = \left(\mathbf{A}^T\mathbf{A}\right)^{-1}\mathbf{A}^T$ is called the *pseudo inverse* or the *Moore-Penrose inverse*.
- For the case of a fat, wide matrix, we have $\mathbf{A}^\dagger = \mathbf{A}^T\left(\mathbf{A}\mathbf{A}^T\right)^{-1}$.
- When $A$ is square and invertible, $\mathbf{A}^\dagger = \mathbf{A}^{-1}$.

# Matrix Inverse and Pseudo Inverse through $\mathbf{QR}$ factorization

- Consider an invertible, square matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$.

$$\mathbf{A} = \mathbf{QR} \implies \mathbf{A}^{-1} = (\mathbf{QR})^{-1} = \mathbf{R}^{-1}\mathbf{Q}^{-1} = \mathbf{R}^{-1}\mathbf{Q}^T$$

  where, $\mathbf{R}, \mathbf{Q} \in \mathbb{R}^{n \times n}$. $\mathbf{R}$ is upper triangular, and $\mathbf{Q}$ is an orthogonal matrix.
- In the case of a left invertible rectangular matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$, we can factorize $\mathbf{A} = \mathbf{QR}$, with $\mathbf{Q} \in \mathbb{R}^{m \times n}$ and $\mathbf{R} \in \mathbb{R}^{m \times m}$.

$$\mathbf{A}^\dagger = \left(\mathbf{A}^T\mathbf{A}\right)^{-1}\mathbf{A}^T = \left(\mathbf{R}^T\mathbf{Q}^T\mathbf{QR}\right)^{-1}\mathbf{R}^T\mathbf{Q}^T = \left(\mathbf{R}^T\mathbf{R}\right)^{-1}\mathbf{R}^T\mathbf{Q}^T = \mathbf{R}^{-1}\mathbf{Q}^T$$

- For a righ invertible wide, fat matrix, we can find out the pseudo-inverse of $\mathbf{A}^T$, and then take the transpose of the pseudo-inverse.

$$\mathbf{A}\mathbf{A}^\dagger = \mathbf{I} \implies \left(\mathbf{A}^\dagger\right)^T\mathbf{A}^T = \left(\mathbf{A}^T\right)^\dagger\mathbf{A}^T = \mathbf{I}$$

$$\mathbf{A}^T = \mathbf{QR} \implies \left(\mathbf{A}^T\right)^\dagger = \left(\mathbf{A}^\dagger\right)^T = \mathbf{R}^{-1}\mathbf{Q} \implies \mathbf{A}^\dagger = \mathbf{Q}\mathbf{R}^{-T}$$