Linear Algebra & Random Processes

Multiple Random Variables

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Topics Covered & References

Topics

References

Multiple Random Variables

- In any experiment, it is not uncommon for use to be interested in multiple variables related to the experiment. For example, if we were collecting data on obesity in a population, we would at lesat measure the weight and the height of each individual included in the study.
- Thus, often our sample space of interest S is a Cartesian product of "smaller" sample spaces $S_1, S_2, \ldots S_n$: $S = S_1 \times S_2 \times \cdots \times S_n$. Our probability model will not have n random variables associated for each S_i, and our pmf/pdfs will be multi-variate functions. We could now define a n-dimensional random vector as an element of $\mathbb{R}^n$ which can be compacted written as the following,

$$\mathbf{x} = \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_n \end{bmatrix}^T \in \mathbb{R}^n$$

Multiple Random Variables - Discrete Case

- Consider the bivariate case, $\mathbf{z} = \begin{bmatrix} \mathbf{x} & \mathbf{y} \end{bmatrix}^T$. We can define the joint probability mass function $\mathbf{x}$ as the following,

$$f_{\mathbf{x},\mathbf{y}}(x, y) = P(\mathbf{x} = x, \mathbf{y} = y) = P(\mathbf{x} = x \cap \mathbf{y} = y)$$

- Probabilities of events defined on $[\mathbf{x}, \mathbf{y}]^T$ can be determined through the following,

$$P\left([\mathbf{x},\ \mathbf{y}]^T \in A \subset \mathbb{R}^2\right) = \sum_{[x,y]^T \in A} f_{\mathbf{x},\mathbf{y}}(x, y)$$

- Joint probability distribution function of multivariate r.v.,

$$F_{\mathbf{x},\mathbf{y}}(x, y) = P(\mathbf{x} \leq x, \mathbf{x} \leq y) = \sum_{u \leq x} \sum_{v \leq y} f_{\mathbf{x},\mathbf{y}}(u, v)$$

Multiple Random Variables - Discrete Case

Marginal Probability Mass Functions – the pmf of the the individual r.v.s.

$$f_{\mathbf{x}_1}(x_1) = \sum_{x_2} f_{\mathbf{x}_1,\mathbf{x}_2}(x_1,x_2)$$

$$f_{\mathbf{x}_2}(x_2) = \sum_{x_1} f_{\mathbf{x}_1,\mathbf{x}_2}(x_1,x_2)$$

$f_{\mathbf{x}_1,\mathbf{x}_1}(x_1,x_2)$	$\mathbf{x}_1$						$f_{\mathbf{x}_2}(x_2)$
$\mathbf{x}_2$	1	2	3	4	5	6	
1	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{6}$
2	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{6}$
3	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{6}$
4	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{6}$
5	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{6}$
6	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{6}$
$f_{\mathbf{x}_1}(x_1)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	1

Multiple Random Variables - Discrete Case

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$f_{\mathbf{x}_1,\mathbf{x}_1}(x_1, x_2)$	$\mathbf{x}_1$: 1	2	3	4	5	6	$f_{\mathbf{x}_2}(x_2)$
$\mathbf{x}_2$: 1	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{6}$
2	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{6}$
3	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{6}$
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5	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{6}$
6	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{6}$
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Consider a pair of dice that are thrown, and let $\mathbf{x}_1$ and $\mathbf{x}_2$ be random variables representing the number that turns up on each of the two dice.s What will be the PMFs of the following bivariate r.vs $\mathbf{z} =$ (a) $[\mathbf{x}_1 + \mathbf{x}_2, \mathbf{x}_1 - \mathbf{x}_2]^T$; (c) $[\mathbf{x}_1 \cdot \mathbf{x}_2, \mathbf{x}_1/\mathbf{x}_2]^T$.

Multiple Random Variables - Continuous Case

- The joint pdf for the bivariate case gives us the probability of the r.v.s assuming a value in a small interval around the point of interest.

$$P\left(x - \frac{\delta x}{2} < \mathbf{x} \leq x + \frac{\delta x}{2}, y - \frac{\delta y}{2} < \mathbf{y} \leq y + \frac{\delta y}{2}\right) = f_{\mathbf{x},\mathbf{y}}\left(x, y\right) \delta x \delta y$$

- Probabilities of events defined on $[\mathbf{x}, \mathbf{y}]^T$ can be determined through the following,

$$P\left([\mathbf{x},\, \mathbf{y}]^T \in A \subset \mathbb{R}^2\right) = \iint_A f_{\mathbf{x},\mathbf{y}}\left(x, y\right) dx\, dy$$

- Joint probability distribution function of multivariate r.v.,

$$F_{\mathbf{x},\mathbf{y}}\left(x, y\right) = P\left(\mathbf{x} \leq x, \mathbf{x} \leq y\right) = \int_{-\infty}^{y} \int_{-\infty}^{x} f_{\mathbf{x},\mathbf{y}}\left(u, v\right) du\, dv$$

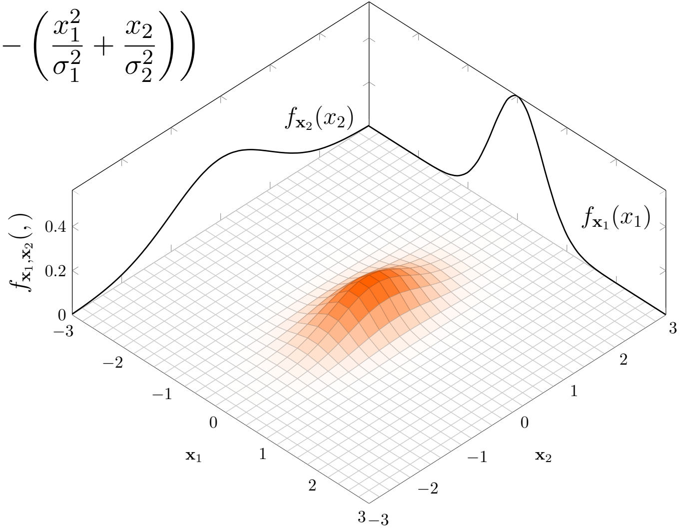
Multiple Random Variables - Continuous Case

$$f_{\mathbf{x}_1,\mathbf{x}_2}(x_1, x_2) = \frac{1}{2\pi\sigma_1\sigma_2} \exp\left(-\left(\frac{x_1^2}{\sigma_1^2} + \frac{x_2}{\sigma_2^2}\right)\right)$$

$$f_{\mathbf{x}_1} = \frac{1}{\sqrt{2\pi\sigma_1^2}} \exp\left(-\frac{x_1^2}{\sigma_1^2}\right)$$

$$f_{\mathbf{x}_2} = \frac{1}{\sqrt{2\pi\sigma_2^2}} \exp\left(-\frac{x_2^2}{\sigma_2^2}\right)$$

This is an example of a multivariate Guassian distribution. In this case the marginal distributions are also Gaussian.



Conditional Distributions and Independence

- Often, the knowledge of one r.v. in a set of r.v.s gives us some information about the other r.v.s.
- Consider a discrete bivariate r.v. $[\mathbf{x},\ \mathbf{y}]^T$, with pmf $f_{\mathbf{x},\mathbf{y}}(x, y)$ and marginal pmfs $f_{\mathbf{x}}(x)$ and $f_{\mathbf{y}}(y)$. The *conditional pmf* of $\mathbf{y}$ given that $\mathbf{x} = x$ is given by,

$$f_{\mathbf{y}|\mathbf{x}}(y|x) = P(y|x) = \frac{P(\mathbf{x} = x, \mathbf{y} = y)}{P(\mathbf{x} = 0)} = \frac{f_{\mathbf{x},\mathbf{y}}(x, y)}{f_{\mathbf{x}}(x)}$$

- We can similarly define the condition pdf of a continuous bivariate r.v. as,

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Consider a pair of dice that are thrown, and let $\mathbf{x}_1$ and $\mathbf{x}_2$ be r.v.s representing the number on each dice. Let $\mathbf{z} = [\mathbf{z}_1,\ \mathbf{z}_2]^T = [\mathbf{x}_1 + \mathbf{x}_2, \mathbf{x}_1 - \mathbf{x}_2]^T$. What is $f_{\mathbf{z}_2|\mathbf{z}_1}(z_2|z_1 = 4)$?

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Consider a joint pdf $f_{\mathbf{x},\mathbf{y}}(x,y)=e^{-y},\ 0<x<y<\infty$. What is $f_{\mathbf{y}|\mathbf{x}}(y|x)$? $f_{\mathbf{x}|\mathbf{y}}(x|y)$

Transformation of multiple random variables

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- The joint pdf in general cannot be determined from the knowledge of the marginal pdfs. However, when we know that the pdfs are independent, the the joint pdf is completely determined by the marginal pdfs.

Joint momemts

- Just like we did in the case of single r.v.s, we can calculate expectations of multiple r.v.s.
- The joint pdf in general cannot be determined from the knowledge of the marginal pdfs. However, when we know that the pdfs are independent, the the joint pdf is completely determined by the marginal pdfs.