Linear Systems

Observability

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Controllability

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)$$

- Observability tells us if we can determine the states of a system using our knowledge of the system's inputs and output.
- **Definition**: The system or the pair $(\mathbf{A}, \mathbf{B})$ is **controllable** if for any initial state $\mathbf{x}_i f$ and any final state $\mathbf{x}_f$ there exists an input $\mathbf{u}(t)$ that transfer the initial state to the final state in finite time. Otherwise, the system is uncontrollable.
- It should be noted that,
  - The trajectory from $\mathbf{x}_i$ to $\mathbf{x}_f$ does not matter.
  - $\mathbf{u}(t)$ can be anything, including impulses and derivatives of impulses.
  - We only need to get to $\mathbf{x}_f$. It is not required that the final state be maintained once we reach there.

Controllability

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$

Assuming that the system starts at $t = t_i$ with initial condition $\mathbf{x}(t_i) = \mathbf{x}_i$, the output at $t = t_f$ is given by,

$$\mathbf{x}(t_f) = e^{(t_f - t_i)\mathbf{A}}\mathbf{x}(t_i) + \int_{t_i}^{t_f} e^{(t_f - \tau)\mathbf{A}}\mathbf{B}\mathbf{u}(\tau)\, d\tau$$

$$e^{-t_f\mathbf{A}}\mathbf{x}(t_f) - e^{-t_i\mathbf{A}}\mathbf{x}(t_i) = \int_{t_i}^{t_f} e^{-\tau\mathbf{A}}\mathbf{B}\mathbf{u}(\tau)\, d\tau$$

From the Cayley-Hamilton theorem we have,

$$e^{-\tau\mathbf{A}} = \sum_{k=0}^{n-1} \alpha_k(\tau)\,\mathbf{A}^k \implies e^{-t_f\mathbf{A}}\mathbf{x}(t_f) - e^{-t_i\mathbf{A}}\mathbf{x}(t_i) = \sum_{k=0}^{n-1} \mathbf{A}^k\mathbf{B} \int_{t_i}^{t_f} \alpha_k(\tau)\,\mathbf{u}(\tau)\, d\tau$$

$$\implies e^{-t_f\mathbf{A}}\mathbf{x}(t_f) - e^{-t_i\mathbf{A}}\mathbf{x}(t_i) = \begin{bmatrix}\mathbf{B} & \mathbf{A}\mathbf{B} & \mathbf{A}^2\mathbf{B} & \dots & \mathbf{A}^{n-1}\mathbf{B}\end{bmatrix} \begin{bmatrix} \int_{t_i}^{t_f} \alpha_0(\tau)\,\mathbf{u}(\tau)\, d\tau \\ \vdots \\ \int_{t_i}^{t_f} \alpha_{n-1}(\tau)\,\mathbf{u}(\tau)\, d\tau \end{bmatrix}$$

A necessary condition for achieving any arbitrary LHS is that the *controllability matrix* $\mathcal{C} = \begin{bmatrix}\mathbf{B} & \mathbf{A}\mathbf{B} & \dots & \mathbf{A}^{n-1}\mathbf{B}\end{bmatrix}$ is full rank. This is also a sufficient condition, which we do not show here.

Controllability

- A system or the pair $(\mathbf{A}, \mathbf{B})$ is controllable, if and only if the controllability matrix $\mathcal{C} = \begin{bmatrix}\mathbf{B} & \mathbf{AB} & \dots & \mathbf{A}^{n-1}\mathbf{B}\end{bmatrix}$ has full rank, i.e. $rank\left(\mathcal{C}\right) = n$.
- Controllability is a system property and is not affected by the choice of coordinate system used for representing the state. Changing the basis of the state to the columns of the matrix $\mathbf{T}$ does not affect the rank of $\mathcal{C}$.
- Let $\tilde{\mathbf{x}} = \mathbf{T}^{-1}\mathbf{x}$, then we have $\tilde{\mathbf{A}} = \mathbf{T}^{-1}\mathbf{AT}$ and $\tilde{\mathbf{B}} = \mathbf{T}^{-1}\mathbf{B}$. This implies that $\tilde{\mathcal{C}} = \mathbf{T}^{-1}\mathcal{C}$, and $rank\left(\tilde{\mathcal{C}}\right) = rank\left(\mathcal{C}\right)$.

Controllability

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- Are the following systems $(\mathbf{A}, \mathbf{B})$ controllable. (a) $\left(\begin{bmatrix}1 & 0\\0 & 2\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}\right)$; (b) $\left(\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}\right)$; and (c) $\left(\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}, \begin{bmatrix}1 & -1\\1 & 2\end{bmatrix}\right)$.
- **Output Controllability**: The system is **output controllable** if for any initial output $\mathbf{y}_i$ and any final output$\mathbf{y}_f$ there exists an input $\mathbf{u}\left(t\right)$ that transfer the initial output to the final output in finite time. This can be done if and only if,

$$rank\left(\begin{bmatrix}\mathbf{D} & \mathbf{CB} & \mathbf{CAB} & \mathbf{CA}^2\mathbf{B} & \dots & \mathbf{CA}^{n-1}\mathbf{B}\end{bmatrix}\right) = m$$

Controllability – Discrete-time system

$$\mathbf{x}[k+1] = \mathbf{A}\mathbf{x}[k] + \mathbf{B}\mathbf{u}[k]$$

Assuming the system starts at $k = 0$, the output at k is given by,

$$\mathbf{x}[k] = \mathbf{A}^k\mathbf{x}[0] + \sum_{l=0}^{k-1} \mathbf{A}^{k-l-1}\mathbf{B}\mathbf{u}[l]$$

$$\mathbf{x}[k] - \mathbf{A}^k\mathbf{x}[0] = \begin{bmatrix} \mathbf{B} & \mathbf{A}\mathbf{B} & \mathbf{A}^2\mathbf{B} & \ldots & \mathbf{A}^{k-1}\mathbf{B} \end{bmatrix} \begin{bmatrix} \mathbf{u}[k-1] \\ \mathbf{u}[k-2] \\ \ldots \\ \mathbf{u}[0] \end{bmatrix} = \mathcal{C}_k \tilde{\mathbf{u}}_{0:k-1}$$

Assuming $\mathbf{x}[0] = \mathbf{0}$, what all values can $\mathbf{x}[k]$ take?

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Assuming $\mathbf{x}[0] = \mathbf{0}$, what all values can $\mathbf{x}[k]$ take? $\longrightarrow \mathbf{x}[k] \in C(\mathcal{C}_k)$. $\mathbf{x}[k]$ can only be in the subspace $C(\mathcal{C}_k)$.

Starting from $k=0$, the possible values $\mathbf{x}[k]$ can take grows until, $C(\mathcal{C}_{k-1}) = C(\mathcal{C}_k)$, i.e.

$$\mathcal{C}_0 \subseteq \mathcal{C}_1 \subseteq \mathcal{C}_2 \cdots \subseteq \mathcal{C}_{n-1} \subseteq \mathcal{C}_n \subseteq \mathbb{R}^n$$

Controllability – Discrete-time system

- The subspace of reachable states can at most grow for n time steps, as we add more columns from $\mathbf{A}^k\mathbf{B}$.
- $\mathcal{C}_n$ is the subspace that can be reached by the system and nothing more. Because, the columns of $\mathbf{A}^n\mathbf{B}$ are a linear combination of the columns of $\mathbf{B}, \mathbf{AB}, \mathbf{A}^2\mathbf{B}, \ldots \mathbf{A}^{n-1}\mathbf{B}$.
- Thus, the system is controllable if and only if $\mathcal{C}_n = \mathbb{R}^n$, or the $rank\left(\mathcal{C}_n\right) = n$.
- In the discrete case, the controllability matrix $\mathcal{C}_n$ can also be used to determine the input sequence $\tilde{\mathbf{u}}_{0:n-1}$ that takes you from state $\mathbf{x}\left[0\right]$ to $\mathbf{x}\left[n\right]$.

$$\mathbf{x}\left[k\right] - \mathbf{A}^k\mathbf{x}\left[0\right] = \mathcal{C}_n\tilde{\mathbf{u}}_{0:n-1} \implies \tilde{\mathbf{u}}_{0:n-1} = \mathcal{C}_n^\dagger\left(\mathbf{x}\left[k\right] - \mathbf{A}^k\mathbf{x}\left[0\right]\right)$$

Where, $\mathcal{C}_n^\dagger = \mathcal{C}_n^{-1}$ for a single input system, and it is a right inverse when there are more than one inputs to the system.

Controllability – Discrete-time system

- Which of the following systems are controllable? (a) $\left(\begin{bmatrix}1 & 0\\0 & 2\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}\right)$; (b) $\left(\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}\right)$; and (c) $\left(\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}\right)$.
- What is the problem with system (b)? The system has repeated eigenvalues, and we have only one input.
- **Output Controllability**:

$$rank\left(\begin{bmatrix}\mathbf{D} & \mathbf{CB} & \mathbf{CAB} & \mathbf{CA}^2\mathbf{B} & \dots & \mathbf{CA}^{n-1}\mathbf{B}\end{bmatrix}\right) = m$$