Linear Control and Estimation

Eigenvalues and Eigenvectors

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References

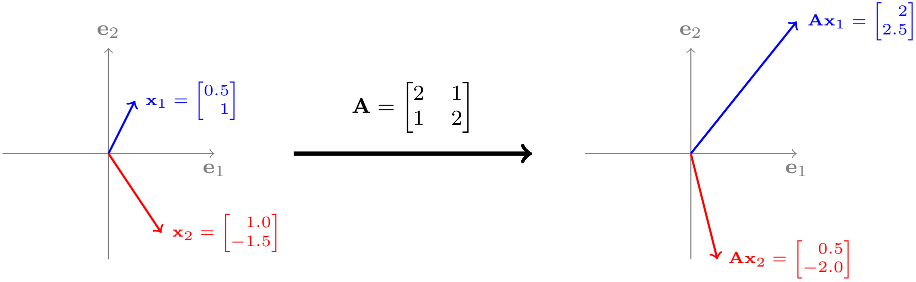
- G Strang, Linear Algebra: Chapters 5.

Linear transformation

- Matrices represent linear transformations, $\mathbf{A} \in \mathbb{R}^{m \times n}$ represents a transformation $T : \mathbb{R}^n \to \mathbb{R}^m$.

$$\mathbf{y} = T(\mathbf{x}) = \mathbf{A}\mathbf{x}, \quad \mathbf{x} \in \mathbb{R}^n \text{ and } \mathbf{y} \in \mathbb{R}^m$$

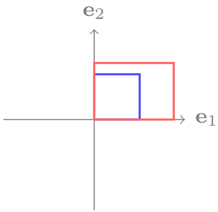
- Consider a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$.
  In general, T scales and rotates the vector $\mathbf{x}$ to produce $\mathbf{y}$.



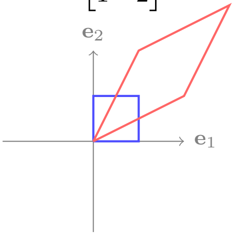
Linear transformation

An easier way is to look at what happens to the standard basis $\{\mathbf{e}_i\}_{i=1}^{n}$.

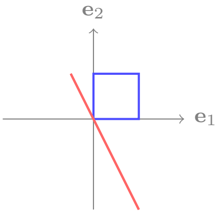
$$\mathbf{A} = \begin{bmatrix} 1.75 & 0 \\ 0 & 1.25 \end{bmatrix}$$



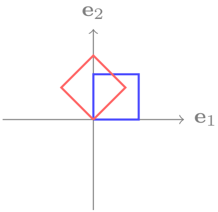
$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$



$$\mathbf{A} = \begin{bmatrix} -0.5 & 1 \\ 1 & -2 \end{bmatrix}$$



$$\mathbf{A} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$



Linear transformation in different basis

- Consider a basis $V = \{\mathbf{v}_i\}_{i=1}^n$ for $\mathbb{R}^n$. Let $\mathbf{x} = \begin{bmatrix} x_1 & x_2 & \dots & x_n \end{bmatrix}^T \in \mathbb{R}^n$ be the representation of $\mathbf{x}$ in the standard basis. Any vector $\mathbf{x} \in \mathbb{R}^n$ can be represented as,

$$\mathbf{x}_V = \sum_{i=1}^n x_{vi}\mathbf{v}_i, \;\; \mathbf{x}_V = \begin{bmatrix} x_{v1} & x_{v2} & \dots & x_{vn} \end{bmatrix}^T$$

- We can go back and forth between these two representations in the following way,

$$\mathbf{x} = \mathbf{V}\mathbf{x}_V \quad \text{and} \quad \mathbf{x}_V = \mathbf{V}^{-1}\mathbf{x}; \;\; \text{where, } \mathbf{V} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_n \end{bmatrix}$$

- When V is an orthonormal basis, then the algebra gets simpler,

$$\mathbf{x} = \mathbf{V}\mathbf{x}_V \quad \text{and} \quad \mathbf{x}_V = \mathbf{V}^T\mathbf{x}$$

Linear transformation in different basis

Consider a linear transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ represented by the matrix $\mathbf{A} = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}$.

Consider a vector $\mathbf{x} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$. What is $\mathbf{y} = \mathbf{A}\mathbf{x}$?

$$\mathbf{y} = \mathbf{A}\mathbf{x} = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ -4 \end{bmatrix}$$

Now, consider a basis $V = \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix} \right\}$ for $\mathbb{R}^2$. The representation of $\mathbf{x}, \mathbf{y}$ in V is,

$$\mathbf{x}_V = \mathbf{V}^{-1}\mathbf{x} = \frac{1}{6} \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 14 \\ 2 \end{bmatrix}, \quad \mathbf{y}_V = \frac{1}{6} \begin{bmatrix} 28 \\ 1 \end{bmatrix}$$

Now, if we apply the linear transformation T on $\mathbf{x}_V$ will we get $\mathbf{y}_V$?

$$\mathbf{A}\mathbf{x}_V = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} \frac{1}{6} \begin{bmatrix} 14 \\ 2 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 16 \\ -12 \end{bmatrix} \neq \mathbf{y}_V$$

It turns out representation of a linear transformation T is basis dependent!

Similarity transformation

- Linear transformations represented in one basis represent a different transformation in another basis. This issue can be addressed by keeping track of the basis one is working in.
- Let $\mathbf{x}, \mathbf{y}$ be representations in the standard basis. Changing basis to V, gives us $\mathbf{x}_V, \mathbf{y}_V$.
$$\mathbf{y}_V = \mathbf{V}^{-1}\mathbf{y} = \mathbf{V}^{-1}\mathbf{A}\mathbf{x} = \mathbf{V}^{-1}\mathbf{A}\mathbf{V}\mathbf{x}_V = \mathbf{A}_V\mathbf{x}_V$$
- Consider a linear transformation $\hat{T}$ whose representation in V is given by the matrix $\hat{\mathbf{A}}_V$. What is the representation of $\hat{T}$ in the standard basis?
- Two matrices $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{n \times n}$ are called *similar* matrices, if there exists a non-singular matrix $\mathbf{Q}$, such that,
$$\mathbf{B} = \mathbf{Q}^{-1}\mathbf{A}\mathbf{Q}$$
- The transformation represented by $\mathbf{Q}^{-1}\mathbf{A}\mathbf{Q}$ is called the *similarity transformation*.
- *Similar* matrices represent the same linear transformation in different basis.

Complex Vectors and Matrices

- Similar to $\mathbb{R}^n$, we can have $\mathbb{C}^n$. $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_{r1} + jx_{i1} \\ x_{r2} + jx_{i2} \\ \vdots \\ x_{rn} + jx_{in} \end{bmatrix}$
- Vector addition and scalar mulitplication are the same. The scalar is a complex number.
- Additive identity, and scalar multiplication identity are the same. So is the **standard basis** $\{\mathrm{e}_i\}_{i=1}^n$
- **Linear independence**: The set $\{\mathbf{v}_i\}_{i=1}^n$ with $\mathbf{v}_i \in \mathbb{C}^n$ is linearly independent, if $\sum_{i=1}^n c_i \mathbf{v}_i = 0, \implies c_i = 0, \ \forall 1 \leq i \leq n, \ \ c_i \in \mathbb{C}^n$
- **Inner product**: $\mathbf{x}^H \mathbf{y} = \begin{bmatrix} \overline{x}_1 & \overline{x}_2 & \dots & \overline{x}_n \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \sum_{i=1}^n \overline{x}_i y_i$

Complex Vectors and Matrices

- **Length**: $\|\mathbf{x}\|_2^2 = \mathbf{x}^H\mathbf{x} = \sum_{i=1}^n \overline{x}_i x_i = \sum_{i=1}^n |x_i|^2$
- **Orthogonality**: $\mathbf{x}^H\mathbf{y} = 0$
- Complex matrices have complex entries. $\mathbf{A} \in \mathbb{C}^{m\times n}$ such that $a_{ij} \in \mathbb{C},\ \forall 1 \leq i \leq m,\ 1 \leq j \leq n$
- The transpose operation is generalized to conjugate transpose known as the *Hermitian*. $\mathbf{A}^H = \overline{\mathbf{A}}^T$.
- The idea of symmertric matrices $\mathbb{R}^{n\times n}$ are now generalized to $\mathbb{C}^{n\times n}$ as $\mathbf{A} = \mathbf{A}^H$. Such matrices are called **Hermitian** matrices.
- Orthogonal matrices in the complex case are called **Unitary** matrices, $\mathbf{U}^H\mathbf{U} = \mathbf{I} \implies \mathbf{U}^{-1} = \mathbf{U}^H$.

Eignenvectors and Eigenvalues

- It turns out for any linear transformation represented by $\mathbf{A} \in \mathbb{C}^{n \times n}$, the are vectors that satisfy with the following property,

$$\mathbf{A}\mathbf{x} = \lambda \mathbf{x}, \quad \mathbf{x} \in \mathbb{C}^n, \lambda \in \mathbb{C}, \;\; \mathbf{x} \neq \mathbf{0}$$

  where, λ and $\mathbf{x}$ are called the eigenvalue and the associated eigenvector of $\mathbf{A}$.
- Any such pair $(\lambda, \mathbf{x})$ is called the eigenpair of $\mathbf{A}$.
- These are important for undertanding and solving linear differential and difference equations:

$$\frac{d\mathbf{x}\left(t\right)}{dt} = \mathbf{A}\mathbf{x}\left(t\right) \quad \text{and} \quad \mathbf{x}\left[n+1\right] = \mathbf{A}\mathbf{x}\left[n\right]$$

Eignenvectors and Eigenvalues

Consider the differential equation, $\frac{d\mathbf{x}(t)}{dt} = \mathbf{A}\mathbf{x}(t) = \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix} \mathbf{x}(t)$. Let us assume that the solution is of the form, $\mathbf{x} = e^{\lambda t}\hat{\mathbf{x}}$. Then we have,

$$\frac{d\mathbf{x}(t)}{dt} = e^{\lambda t}\mathbf{A}\hat{\mathbf{x}} = e^{\lambda t}\lambda\hat{\mathbf{x}} \implies \mathbf{A}\hat{\mathbf{x}} = \lambda\hat{\mathbf{x}}$$

$$\begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} = \begin{bmatrix} \lambda\hat{x}_1 \\ \lambda\hat{x}_2 \end{bmatrix} \implies \begin{bmatrix} 2-\lambda & 1 \\ 1 & 4-\lambda \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

where, $\hat{\mathbf{x}} \in N(\mathbf{A} - \lambda\mathbf{I})$.

This problem can be solved by, $\det(\mathbf{A} - \lambda\mathbf{I}) = |\mathbf{A} - \lambda\mathbf{I}| = 0$

$$(2-\lambda)(4-\lambda) - 1 = \lambda^2 - 6\lambda + 7 = 0 \implies \lambda = 3 \pm \sqrt{2}$$

$$\mathbf{A}\hat{\mathbf{x}} = \left(3+\sqrt{2}\right)\hat{\mathbf{x}} \implies \hat{\mathbf{x}} = \begin{bmatrix} -1+\sqrt{2} \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{A}\hat{\mathbf{x}} = \left(3-\sqrt{2}\right)\hat{\mathbf{x}} \implies \hat{\mathbf{x}} = \begin{bmatrix} -1-\sqrt{2} \\ 1 \end{bmatrix}$$

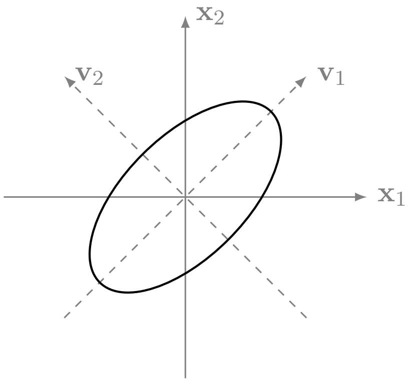
$\left(3+\sqrt{2}, \begin{bmatrix} -1+\sqrt{2} \\ 1 \end{bmatrix}\right)$ and $\left(3-\sqrt{2}, \begin{bmatrix} -1-\sqrt{2} \\ 1 \end{bmatrix}\right)$ are the eigenpairs of $\mathbf{A}$.

Eigenvalues and Eigenvectors

- We can find the eigenpairs using the same approach for $\mathbf{A} \in \mathbb{C}^{n \times n}$, $\det\left(\mathbf{A} - \lambda\mathbf{I}\right) = 0 = p\left(\lambda\right)$.
- $p\left(\lambda\right)$ is the characteristic polynomial of $\mathbf{A}$, and $p\left(\lambda\right) = 0$ is the characteristic equation.
- The eigenvalues are the roots of the polynomial $p\left(\lambda\right)$, and the $\mathbf{x}$ in $\left(\mathbf{A} - \lambda\mathbf{I}\right)\mathbf{x} = 0$ for the different λs are the corresponding eigenvectors.
- The subspace spanned for $\mathbf{x}$ for a particular λ is called the eigenspace.
- $\mathbf{A}$ has n eigenvalues, some of which can be comples, and some might be repeated.
- For a real matrix, all complex eigenvalues occur in conjugate pairs.

Diagonalization of a matrix

Often the right choice of basis can simply an equation or the analysis of a problem. For example,



The equation of the ellipse in standard basis is:

$$\frac{5}{8}x_1^2 + \frac{5}{8}x_2^2 - \frac{3}{4}x_1x_2 = 1$$

This has a much simpled representation in the dashed coordinate frame.

$$\frac{1}{4}v_1^2 + v_2^2 = 1$$

The use of similarity transformation to simplify a matrix is at the heart of diagonalization.

Diagonalization of a matrix

- Consider a matrix $\mathbf{A}$ with n eigenpairs $\{(\lambda_i, \mathbf{x}_i)\}_{i=1}^{n}$.

$$\mathbf{A}\begin{bmatrix}\mathbf{x}_1 & \mathbf{x}_2 & \ldots & \mathbf{x}_n\end{bmatrix} = \begin{bmatrix}\lambda_1\mathbf{x}_1 & \lambda_2\mathbf{x}_2 & \ldots & \lambda_n\mathbf{x}_n\end{bmatrix}$$

$$\mathbf{AX} = \mathbf{X}\begin{bmatrix}\lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda_n\end{bmatrix} = \mathbf{X\Lambda}$$

- If the eignevectors are linearly independent, then we have $\mathbf{X}^{-1}\mathbf{AX} = \mathbf{\Lambda}$
- A change of basis to $\mathbf{X}$ simplifies $\mathbf{A}$ to a diagonal matrix, the simplest possible form.
- If a matrix $\mathbf{A}$ has n distinct eigenvalues, then $\mathbf{A}$ can always be diagonalized.
- When there are repeated eigenvalues, we might not always be able to diagonlize a matrix. This happens when there aren't enough eigenvectors.

Eigenvalues and Eigenvectors of special matrices

- A square matrices with a complete set of eigenvectors, i.e. a linearly independent set of n eigenvectors, can be decomposed into the following,

$$\mathbf{A} = \mathbf{X}\mathbf{\Lambda}\mathbf{X}^{-1}$$

- When $\mathbf{A} \in \mathbb{R}^{n \times n}$ is symmetric, i.e. $\mathbf{A} = \mathbf{A}^T$,
  - All eigenvalues are real.
  - The matrix poses a complete set of eigenvectors, i.e. they form a linearly independent set.
  - The eigenvector can be chosen to be orthogonal to each other. When the eigenvalues are distinct, the eigenvectors are orthogonal. But when the eigenvalues are not distinct, we can choose them to be orthogonal.

This gives us, $\mathbf{A} = \mathbf{A}^T = \mathbf{X}\mathbf{\Lambda}\mathbf{X}^T$.

Jordan Form

- $\mathbf{A}$ cannot be diagonalized. What is the next best thing? This is the *Jordan form*
- Let $\mathbf{A}$ have eigenvalues $(\lambda_1, \lambda_2, \ldots \lambda_k)$. We can find a similarity trasnformation, such that,

$$\mathbf{A} = \mathbf{P}\mathbf{J}\mathbf{P}^{-1}, \ \ \mathbf{J} = \begin{bmatrix} \mathbf{J}\left(\lambda_1\right) & \mathbf{0} & \ldots & \mathbf{0} \\ \mathbf{0} & \mathbf{J}\left(\lambda_2\right) & \ldots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \ldots & \mathbf{J}\left(\lambda_k\right) \end{bmatrix}$$

Each $\mathbf{J}\left(\bullet\right)$ is associated with an eigenvalue and an eigenvector, and is called a Jordan block, and has the form

$$\mathbf{J}\left(\lambda_l\right) = \begin{bmatrix} \lambda_l & 1 & 0 & \ldots & 0 \\ 0 & \lambda_l & 1 & \ldots & 0 \\ 0 & 0 & \lambda_l & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & 1 \\ 0 & 0 & 0 & \ldots & \lambda_l \end{bmatrix}$$

- $\mathbf{J} \in \mathbb{C}^{r \times r}$. $r =$ the albegraic multiplicity of the eigenvalue λ_l.
- $r - 1 =$ the geomtric multiplicity of the eigenvalue $\lambda_l = \dim N\left(\mathbf{A} - \lambda_l \mathbf{I}\right)$.
- A 1-by-1 Jordan block is simply $[\lambda_l]$, corresponding to a eigenvalue with an associated eigenvector.