Linear Control and Estimation

Matrix Inverses

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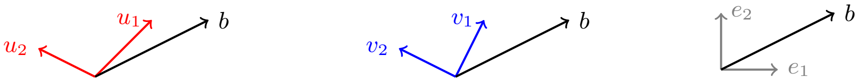
References

Representation of vectors in a basis

- Consider the vector space $\mathbb{R}^n$ with basis $\{v_1, v_2, \ldots v_n\}$. Any vector in $b \in \mathbb{R}^n S$ can be representated as a linear combination of of v_is,

$$b = \sum_{i=1}^{n} v_i \alpha_i = V\alpha; \;\; \alpha \in \mathbb{R}^n, \;\; V = \begin{bmatrix} v_1 & v_2 & \ldots & v_n \end{bmatrix} \in \mathbb{R}^{n \times n}$$



$\{v_1, v_2\}$, $\{u_1, u_2\}$ and $\{e_1, e_2\}$ are valid basis for $\mathbb{R}^2$, and the presentation for b in each one of them is different.

- Finding out α is easiest when we are dealing with an orthonormal basis, in which case α is given by,

$$\alpha = \begin{bmatrix} u_1^T b \\ u_2^T b \\ \vdots \\ u_n^T b \end{bmatrix} = U^T b \implies b_U = \sum_{i=1}^{n} \left(u_i^T b\right) u_i = UU^T b$$

Matrix Inverse

- Consider the equation $Ax = y$, where $A \in \mathbb{R}^{n \times n}$ and $x, y \in \mathbb{R}^n$.
- Let us assume A is non-singular, which means that the columns of A represent a basis for $\mathbb{R}^n$.
- What does x represent? It is the representation of y in the basis consisitng of the columns of A.

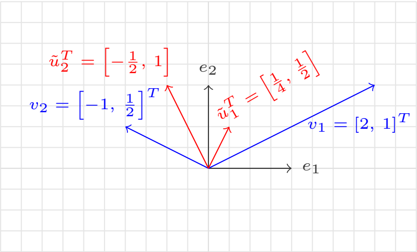
$$y = Ax = \begin{bmatrix} a_1 & a_2 & \ldots & a_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \sum_{i=1}^{n} a_i x_i$$

$$x = A^{-1} y = \begin{bmatrix} \tilde{b}_1^T \\ \tilde{b}_2^T \\ \ldots \\ \tilde{b}_n^T \end{bmatrix} y = \begin{bmatrix} \tilde{b}_1^T y \\ \tilde{b}_2^T y \\ \ldots \\ \tilde{b}_n^T y \end{bmatrix}$$

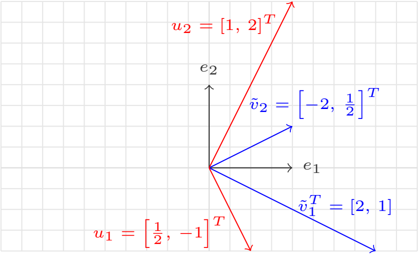
- A^{-1} is a matrix that allows change of basis to the columns of A from the standard basis!

Matrix Inverse

Columns of A and rows of A^{-1}



Rows of A and columns of A^{-1}



$$V = \begin{bmatrix} v_1 & v_2 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & \frac{1}{2} \end{bmatrix}$$

$$V^{-1} = \begin{bmatrix} \tilde{u}_1^T \\ \tilde{u}_2^T \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \frac{1}{2} & 1 \\ -1 & 2 \end{bmatrix}$$

$$v_1^T \tilde{u}_1 = v_2^T \tilde{u}_2 = \tilde{v}_1^T u_1 = \tilde{v}_2^T u_2 = 1$$

$$v_1^T \tilde{u}_2 = v_2^T \tilde{u}_1 = \tilde{v}_1^T u_2 = \tilde{v}_2^T u_1 = 0$$

Left Inverse

- Consider a rectangular matrix $A \in \mathbb{R}^{m \times n}$. There exists no inverse A^{-1} for this matrix.
- But, does there exist two matrices $B \in \mathbb{R}^{n \times m}$ and $C \in \mathbb{R}^{m \times n}$, such that,

$$AB = I_m \quad \text{and} \quad CA = I_n$$

  For a rectangular matrix only on these two can be ture, and thus a rectangular matrix can only have either a left or a right inverse.

  Note: *A non-signular square matrix has both left and right inverses, and both are equal and unique.*

- Any non-zero vector $a \in \mathbb{R}^{n \times 1}$ is left invertible,.

$$ba = 1, \;\; b \in \mathbb{R}^{1 \times n}; \quad b^T = \frac{a}{\|a\|^2} + \alpha a^{\perp}$$

- There can be more than one left inverse! If there is more than one there are an infinite number of them.
- What is condition for the left inverse of A to exist? *The colmuns of A must be independent.* $\longrightarrow rank\,(A) = n \longrightarrow Ax = 0 \implies x = 0.$
- Only fat and wide martices can have left inverses.
- $Ax = b$ can be solved, if and only if $A\,(Bb) = b$, where $BA = I_n$.

Right Inverse

- For a rectangular matrix $A \in \mathbb{R}^{m \times n}$, we can have,

$$AB = I_m \quad \longrightarrow B \text{ is the right inverse.}$$

- Right inverse of A exists only if the rows of A are independent, i.e.

$$rank\,(A) = m \longrightarrow A^T x = 0 \implies x = 0$$

- Only tall, skinny matrices can have right inverses.
- $Ax = b$ can be solved for any b. $x = Bb \implies A\,(Bb) = b$. There will be an infinite number of solutions for x.
- Let $AB = I_m \implies B^T A^T = I_m \implies B^T$ is the left inverse of A^T.

Pseudo Inverse

- Consider a tall, skinny matrix $A \in \mathbb{R}^{m \times n}$ with independent columns. It turns out the Gram matrix $A^T A \in \mathbb{R}^{n \times n}$ is invertible. If that is the case then,

$$\left(A^T A\right)^{-1} A^T A = I_n; \quad B = \left(A^T A\right)^{-1} A^T \text{ is a left inverse.}$$

- $A^\dagger = \left(A^T A\right)^{-1} A^T$ is called the *pseudo inverse* or the *Moore-Penrose inverse*.
- For the case of a fat, wide matrix, we have $A^\dagger = A^T \left(A A^T\right)^{-1}$.
- When A is square and invertible, $A^\dagger = A^{-1}$.

Matrix Inverse and Pseudo Inverse through QR factorization

- Consider an invertible, square matrix $A \in \mathbb{R}^{n \times n}$.

$$A = QR \implies A^{-1} = (QR)^{-1} = R^{-1}Q^{-1} = R^{-1}Q^{T}$$

where, $R, Q \in \mathbb{R}^{n \times n}$. R is upper triangular, and Q is an orthogonal matrix.

- In the case of a left invertible rectangular matrix $A \in \mathbb{R}^{m \times n}$, we can factorize $A = QR$, with $Q \in \mathbb{R}^{m \times n}$ and $R \in \mathbb{R}^{m \times m}$.

$$A^{\dagger} = \left(A^T A\right)^{-1} A^T = \left(R^T Q^T Q R\right)^{-1} R^T Q^T = \left(R^T R\right)^{-1} R^T Q^T = R^{-1} Q^T$$

- For a righ invertible wide, fat matrix, we can find out the pseudo-inverse of A^T, and then take the transpose of the pseudo-inverse.

$$AA^{\dagger} = I \implies \left(A^{\dagger}\right)^T A^T = \left(A^T\right)^{\dagger} A^T = I$$

$$A^T = QR \implies \left(A^T\right)^{\dagger} = \left(A^{\dagger}\right)^T = R^{-1}Q \implies A^{\dagger} = QR^{-T}$$