

# Linear Systems

## Vectors

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## References

- ▶ S Boyd, Applied Linear Algebra: Chapters 1, 2, 3 and 5.

# Vectors

- **Vectors** are ordered list of numbers (scalars).  $\mathbf{v} = \begin{bmatrix} 1.2 \\ -0.1 \\ \vdots \\ -1.24 \end{bmatrix}$ . **Note:** Small bold letter will represent vectors.

e.g.  $\mathbf{a}, \mathbf{x}, \dots$

- Scalars can be any *field*  $\mathbb{R}, \mathbb{C}, \mathbb{Z}, \mathbb{Q}$ . Scalars will be represented using lower case normal font, e.g.  $x, y, \alpha, \beta, \dots$
- We will typically only encounter  $\mathbb{R}$  in this course.
- Individual elements of a vector  $\mathbf{v}$  are indexed. The  $i^{th}$  element of  $\mathbf{v}$  is referred to as  $v_i$ .
- *Dimension* or *size* of a vector is number of elements in the vector.
- Set of  $n$ -real vectors is denoted by  $\mathbb{R}^n$  (similarly,  $\mathbb{C}^n$ )
- Vectors  $\mathbf{a}$  and  $\mathbf{b}$  are equal, if
- both have the same size; and
  - $a_i = b_i, i \in \{1, 2, 3, \dots, n\}$

# Vectors

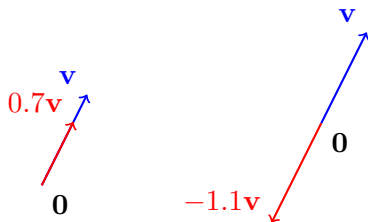
- ▶ **Unit vector**  $e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$  **Zero vector**  $0 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$  **One vector**  $1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$
- ▶ Geometrically, real  $n$ -vectors can be thought of as points in  $\mathbb{R}^n$  space.



# Vectors

- **Vector scaling:** Multiplication of a scalar and a vector.

$$\mathbf{w} = a\mathbf{v} = a \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} av_1 \\ av_2 \\ av_3 \\ \vdots \\ av_n \end{bmatrix} \quad a \in \mathbb{R}; \mathbf{w}, \mathbf{v} \in \mathbb{R}^n$$



## Properties

- Scalar multiplication is *commutative*.

$$\alpha\mathbf{v} = \mathbf{v}\alpha$$

- Scalar multiplication is *associative*.

$$(\alpha\beta)\mathbf{v} = \alpha(\beta\mathbf{v})$$

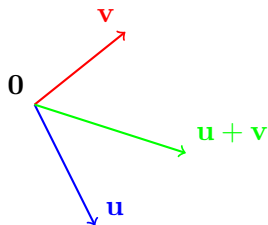
- Scalar multiplication is *distributive*.

$$(\alpha + \beta)\mathbf{v} = \alpha\mathbf{v} + \beta\mathbf{v}$$

# Vectors

- ▶ **Vector addition:** Adding two vectors of the same dimension, element by element.

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ u_3 + v_3 \\ \vdots \\ u_n + v_n \end{bmatrix} \quad \mathbf{u}, \mathbf{v} \in \mathbb{R}^n$$



## Properties

- ▶ Vector addition is *commutative*.

$$\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$$

- ▶ Vector addition is *associative*.

$$(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$$

- ▶ Zero vector has no effect.

$$\mathbf{a} + \mathbf{0} = \mathbf{a}$$

- ▶ Subtraction of vectors.

$$\mathbf{a} + (-1)\mathbf{a} = \mathbf{a} - \mathbf{a} = \mathbf{0}$$

## Vector spaces

- ▶ A set of vectors  $V$  that is closed under **vector addition** and **vector scaling**.

$$\forall \mathbf{x}, \mathbf{y} \in V, \quad \mathbf{x} + \mathbf{y} \in V$$

$$\forall \mathbf{x} \in V, \text{ and } \alpha \in F, \quad \alpha \mathbf{x} \in V$$

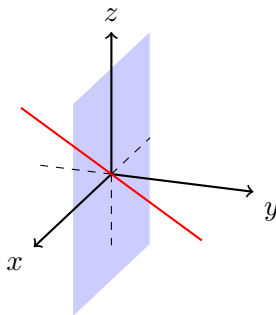
- ▶ For a set to be a vector space, it must satisfy the following properties:  $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$ 
  - ▶ *Commutativity*:  $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$
  - ▶ *Associativity of vector addition*:  $(\mathbf{x} + \mathbf{y}) + \mathbf{z} = \mathbf{x} + (\mathbf{y} + \mathbf{z})$
  - ▶ *Additive identity*:  $\mathbf{x} + \mathbf{0} = \mathbf{0} + \mathbf{x} = \mathbf{x}$  ( $\mathbf{0} \in V$ )
  - ▶ *Additive inverse*:  $\exists -\mathbf{x} \in V, \mathbf{x} + (-\mathbf{x}) = \mathbf{0}$
  - ▶ *Associativity of scalar multiplication*:  $\alpha(\beta \mathbf{x}) = (\alpha\beta) \mathbf{x}$
  - ▶ *Distributivity of scalar sums*:  $(\alpha + \beta) \mathbf{x} = \alpha \mathbf{x} + \beta \mathbf{x}$
  - ▶ *Distributivity of vector sums*:  $\alpha(\mathbf{x} + \mathbf{y}) = \alpha \mathbf{x} + \alpha \mathbf{y}$
  - ▶ *Scalar multiplication identity*:  $1\mathbf{x} = \mathbf{x}$
- ▶ We will mostly deal with  $\mathbb{R}^n$  vectors spaces in this course.

# Subspaces

- ▶ A **subspace**  $S$  of a vector space  $V$  is a subset of  $V$  and is itself a vector space.

$$S \subset V, \quad \forall \mathbf{x}, \mathbf{y} \in S, \alpha \mathbf{x} + \beta \mathbf{y} \in S, \quad \alpha, \beta \in F$$

- ▶ The zero vector is called the **trivial subspace** of a vector space  $V$ .
- ▶ For example in, in  $\mathbb{R}^3$  all planes and lines passing through the origin are subspaces of  $\mathbb{R}^3$ .





## Linear independence

- ▶ A collection of vectors  $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \dots, \mathbf{x}_n\}$ ,  $\mathbf{x}_i \in \mathbb{R}^m$   $i \in \{1, 2, 3, \dots, n\}$  is called *linear dependent* if,

$$\sum_{i=1}^n \alpha_i \mathbf{x}_i = 0, \text{ hold for some } \alpha_1, \alpha_2, \dots, \alpha_n \in \mathbb{R}, \text{ such that } \exists \alpha_i \neq 0$$

- ▶ Another way to state this: A collection of vectors is *linear dependent* if at least one of the vectors in the collection can be expressed as a linear combination of the other vectors in the collection, i.e.

$$\mathbf{x}_i = - \sum_{j=1, j \neq i}^n \left( \frac{\alpha_j}{\alpha_i} \right) \mathbf{x}_j$$

- ▶ A collection of vectors is *linear independent* if it is **not** linearly dependent.

$$\sum_{i=1}^n \alpha_i \mathbf{x}_i = 0 \implies \alpha_1 = \alpha_2 = \alpha_3 \dots = \alpha_n = 0$$

## Span of a set of vectors

- ▶ Consider a set of vectors  $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \dots \mathbf{v}_r\}$  where  $\mathbf{v}_i \in \mathbb{R}^n, 1 \leq i \leq r$ .
- ▶ The **span** of the set  $S$  is defined as the set of all linear combination of the vectors  $\mathbf{v}_i$ ,

$$\text{span}(S) = \{\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \dots + \alpha_r \mathbf{v}_r\}, \alpha_i \in \mathbb{R}$$

- ▶ Is  $\text{span}(S)$  a subspace of  $\mathbb{R}^n$ ?
- ▶ We say that the subspace  $\text{span}(S)$  is spanned by the *spanning set*  $S$ .  $\rightarrow S$  *spans*  $\text{span}(S)$ .
- ▶ **Sum of subspaces**  $X, Y$  is defined as the sum of all possible vectors from  $X$  and  $Y$ .

$$X + Y = \{\mathbf{x} + \mathbf{y} \mid \mathbf{x} \in X, \mathbf{y} \in Y\}$$

- ▶ Sum of two subspace is also a subspace.

# Inner Product

- ▶ **Standard inner product** is defined as the following,

$$\mathbf{x}^T \mathbf{y} = \sum_{i=1}^n x_i y_i, \quad \mathbf{x}, \mathbf{y} \in \mathbb{R}^n$$

For complex vectors:  $\mathbf{x}^* \mathbf{y} = \sum_{i=1}^n \bar{x}_i y_i, \quad \mathbf{x}, \mathbf{y} \in \mathbb{C}^n$

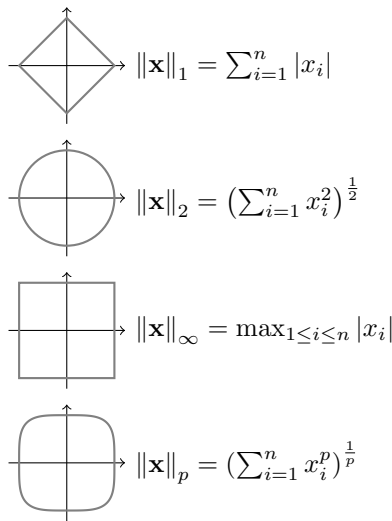
- ▶ **Properties**

- ▶  $\mathbf{x}^T \mathbf{x} > 0, \quad \forall \mathbf{x} \neq 0$  and  $\mathbf{x}^T \mathbf{x} = 0 \Leftrightarrow \mathbf{x} = 0$
- ▶ *Commutative*:  $\mathbf{x}^T \mathbf{y} = \mathbf{y}^T \mathbf{x}$
- ▶ *Associativity with scalar multiplication*:  $(\alpha \mathbf{x})^T \mathbf{y} = \alpha (\mathbf{x}^T \mathbf{y})$
- ▶ *Distributivity with vector addition*:  $(\mathbf{x} + \mathbf{y})^T \mathbf{z} = \mathbf{x}^T \mathbf{z} + \mathbf{y}^T \mathbf{z}$

# Norm

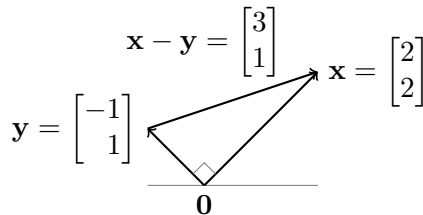
- ▶ Norm is a measure of the size of a vector.
- ▶ *Euclidean norm* of a  $n$ -vector  $\mathbf{x} \in \mathbb{R}^n$  is defined as,  

$$\|\mathbf{x}\|_2 = \sqrt{\mathbf{x}^T \mathbf{x}} = \sqrt{\sum_{i=1}^n x_i^2}.$$
- ▶  $\|\mathbf{x}\|_2$  is a measure of the length of the vector  $\mathbf{x}$ .
- ▶ Any function of the form  $\|\bullet\| : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$  is a valid norm, provided it satisfies the following properties.
- ▶ **Properties**
  - ▶ *Definiteness.*  $\|\mathbf{x}\| = 0 \iff \mathbf{x} = \mathbf{0}$
  - ▶ *Non-negativity.*  $\|\mathbf{x}\| \geq 0$
  - ▶ *Non-negative homogeneity.*  $\|\beta \mathbf{x}\| = |\beta| \|\mathbf{x}\|, \beta \in \mathbb{R}$
  - ▶ *Triangle inequality.*  $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$
- ▶  $p$ -norm:  $\|\mathbf{x}\|_p = (\sum_{i=1}^n |x_i|^p)^{\frac{1}{p}}$
- ▶ Norm of difference between two vectors is a measure of the distance between the vectors.  $d = \|\mathbf{x} - \mathbf{y}\|_2$ .



# Orthogonality

- ▶ Orthogonality is the idea of two vectors being perpendicular,  $\mathbf{x} \perp \mathbf{y}$ .



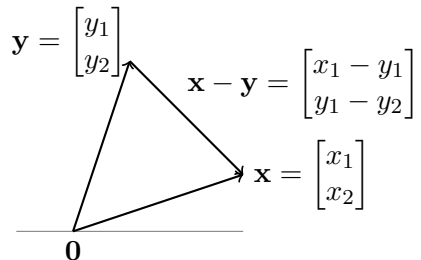
Using the Pythagorean theorem,  $\|\mathbf{x} - \mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2$

$$\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 - 2\mathbf{x}^T \mathbf{y} = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 \implies \mathbf{x}^T \mathbf{y} = 0$$

- ▶ We extend this to the  $n$ -dimensional case and define two vectors  $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$  being orthogonal, if

$$\mathbf{x}^T \mathbf{y} = \sum_{i=1}^n x_i y_i = 0$$

## Angle between vectors



- ▶ Inner products are used for projecting a vector onto another vector or a subspace.
- ▶ It is also a measure of similarity between two vectors,  $\cos(\theta) = \frac{\mathbf{x}^T \mathbf{y}}{\|\mathbf{x}\| \|\mathbf{y}\|}$
- ▶ **Cauchy-Bunyakovski-Schwartz Inequality:**

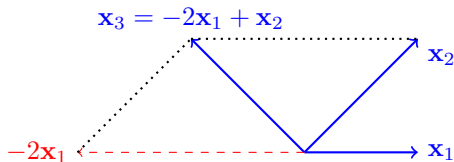
$$|\mathbf{x}^T \mathbf{y}| \leq \|\mathbf{x}\| \|\mathbf{y}\|, \quad \mathbf{x}, \mathbf{y} \in \mathbb{R}^n$$

## Basis

Consider a vector  $\mathbf{y} = \sum_{i=1}^n \alpha_i \mathbf{x}_i$ . What can we say about the coefficients  $\alpha_i$ s when the collection  $\{\mathbf{x}_i\}_{i=1}^n$  is,

- ▶ linearly independent  $\implies \alpha_i$ s are *unique*.
- ▶ linearly dependent  $\implies \alpha_i$ s are not *unique*.

Consider  $\mathbb{R}^2$  vector space.  $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ ,  $\mathbf{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ ,  $\mathbf{x}_3 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ .



**Independence-Dimension inequality:** What is the maximum possible size of a linearly independent collection?

*A linear independent collection of  $n$ -vectors can at most have  $n$  vectors.*

# Basis

- ▶ A linearly independent set of  $n$   $n$ -vector is called a *basis*. In particular, it is a basis of  $\mathbb{R}^n$ .
- ▶ Any  $n$ -vector can be represented as a *unique* linear combination of the elements of the basis.
- ▶ Consider the basis  $\{\mathbf{x}_i\}_{i=1}^n$ . A  $n$ -vector  $\mathbf{y}$  can be represented as a linear combination of  $\mathbf{x}_i$ s,  $\mathbf{y} = \sum_{i=1}^n \alpha_i \mathbf{x}_i$ . This is called the *expansion of  $\mathbf{y}$*  in the  $\{\mathbf{x}_i\}_{i=1}^n$  basis.
- ▶ The numbers  $\alpha_i$  are called the *coefficients* of the expansion of  $\mathbf{y}$  in the  $\{\mathbf{x}_i\}_{i=1}^n$  basis.
- ▶ **Orthogonal vectors:** A set of vectors  $\{\mathbf{x}_i\}_{i=1}^n$  is (*mutually*) *orthogonal* if  $\mathbf{x}_i \perp \mathbf{x}_j$  for all  $i, j \in \{1, 2, 3, \dots, n\}$  and  $i \neq j$ .
- ▶ This set is called **orthonormal** if its elements are all of unit length  $\|\mathbf{x}_i\|_2 = 1$  for all  $i \in \{1, 2, 3, \dots, n\}$ .

$$\mathbf{x}_i^T \mathbf{x}_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$



## Representing a Vector in an Orthonormal Basis

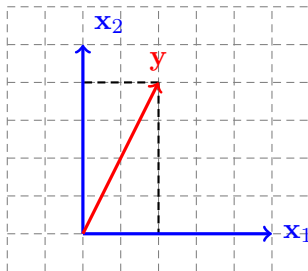
- ▶ An orthonormal collection of vectors is linearly independent.
- ▶ Consider an orthonormal basis  $\{\mathbf{x}_i\}_{i=1}^n$ . The expansion of a vector  $\mathbf{y}$  is given by,

$$\mathbf{y} = \alpha_1 \mathbf{x}_1 + \alpha_2 \mathbf{x}_2 + \alpha_3 \mathbf{x}_3 + \dots + \alpha_n \mathbf{x}_n$$

$$\mathbf{x}_i^T \mathbf{y} = \alpha_1 \mathbf{x}_i^T \mathbf{x}_1 + \alpha_2 \mathbf{x}_i^T \mathbf{x}_2 + \alpha_3 \mathbf{x}_i^T \mathbf{x}_3 + \dots + \alpha_n \mathbf{x}_i^T \mathbf{x}_n = \alpha_i$$

- ▶ Thus, we can rewrite this as,

$$\mathbf{y} = (\mathbf{y}^T \mathbf{x}_1) \mathbf{x}_1 + (\mathbf{y}^T \mathbf{x}_2) \mathbf{x}_2 + (\mathbf{y}^T \mathbf{x}_3) \mathbf{x}_3 + \dots + (\mathbf{y}^T \mathbf{x}_n) \mathbf{x}_n$$



## Dimension of a Vector Space

- ▶ There are an infinite number of bases for a vector space.
- ▶ There is one thing that is common among all these bases – the number of basis vectors.
- ▶ This number is a property of the vector space, and represents the “degrees of freedom” of the space. This is called the **dimension** of the vector space.
- ▶ A subspace of dimension  $m$  can have at most  $m$  independent vectors.
- ▶ Notice that the word “dimension” of a vector space is different from the “dimension” of a vector.
- ▶ E.g. Vectors from  $\mathbb{R}^3$  are three dimensional vectors. But the  $yz$ -plane in  $\mathbb{R}^3$  is a 2 dimensional subspace of  $\mathbb{R}^3$ .

# Linear Functions

- ▶ Let  $f$  be a function which maps real  $n$ -vectors to scalar real numbers. It can be represented as the following,

$$f : \mathbb{R}^n \longrightarrow \mathbb{R}; \quad y = f(\mathbf{x}) = f(x_1, x_2, x_3, \dots, x_n)$$

- ▶ Criteria for  $f$  to be a linear function: **Superposition**:  $f(\alpha \mathbf{x} + \beta \mathbf{y}) = \alpha f(\mathbf{x}) + \beta f(\mathbf{y})$ , where  $\alpha, \beta \in \mathbb{R}$  and  $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ .
- ▶ **Inner product** is a linear function in one of the arguments.

$$f(x) = \mathbf{w}^T \mathbf{x} = w_1 x_1 + w_2 x_2 + w_3 x_3 + \dots + w_n x_n$$

- ▶ Any linear function can be represented in the form  $f(\mathbf{x}) = \mathbf{w}^T \mathbf{x}$  with an appropriately chosen  $\mathbf{w}$ .