Linear Systems

Controllability

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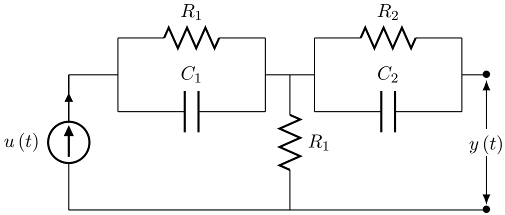
Controllability and observability

- This lecture and the next deal with two important aspects of linear system theory – controllability and observability.
- These two concepts deal with how the input and output interact with the system states.
- Consider the following system:



How are the capacitor voltages affected by input voltage $u(t)$? What does $y(t)$ measure?

Controllability

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)$$

- Controllability tells us if a desired state can be achieved in finite time through an appropriate to choice of inputs. For this, we only have to deal with the state equation.
- **Definition**: The system or the pair $(\mathbf{A}, \mathbf{B})$ is **controllable** if for any initial state $\mathbf{x}_i$ and any final state $\mathbf{x}_f$, there exists an input $\mathbf{u}(t)$ that transfers the initial state to the final state in finite time. Otherwise, the system is uncontrollable.
- It should be noted that,
  - The trajectory from $\mathbf{x}_i$ to $\mathbf{x}_f$ does not matter.
  - $\mathbf{u}(t)$ can be anything, including impulses and derivatives of impulses.

Controllability

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$

Assuming the system starts at $t = 0$, the output at $t = t_f$ is given by,

$$\mathbf{x}(t_f) = e^{t_f\mathbf{A}}\mathbf{x}(0) + \int_0^{t_f} e^{(t_f-\tau)\mathbf{A}}\mathbf{B}\mathbf{u}(\tau)\,d\tau \implies e^{-t_f\mathbf{A}}\mathbf{x}(t_f) - \mathbf{x}(0) = \int_0^{t_f} e^{-\tau\mathbf{A}}\mathbf{B}\mathbf{u}(\tau)\,d\tau$$

From the Cayley-Hamilton theorem we have,

$$e^{-\tau\mathbf{A}} = \sum_{k=0}^{n-1}\alpha_k(\tau)\,\mathbf{A}^k \implies e^{-t_f\mathbf{A}}\mathbf{x}(t_f) - \mathbf{x}(0) = \sum_{k=0}^{n-1}\mathbf{A}^k\mathbf{B}\int_0^{t_f}\alpha_k(\tau)\,\mathbf{u}(\tau)\,d\tau$$

$$\implies e^{-t_f\mathbf{A}}\mathbf{x}(t_f) - \mathbf{x}(0) = \begin{bmatrix}\mathbf{B} & \mathbf{A}\mathbf{B} & \mathbf{A}^2\mathbf{B} & \ldots & \mathbf{A}^{n-1}\mathbf{B}\end{bmatrix}\begin{bmatrix}\int_0^{t_f}\alpha_0(\tau)\,\mathbf{u}(\tau)\,d\tau \\ \vdots \\ \int_0^{t_f}\alpha_{n-1}(\tau)\,\mathbf{u}(\tau)\,d\tau\end{bmatrix}$$

A necessary condition for achieving any arbitrary LHS is that the *controllability matrix* $\mathcal{C} = \begin{bmatrix}\mathbf{B} & \mathbf{A}\mathbf{B} & \ldots & \mathbf{A}^{n-1}\mathbf{B}\end{bmatrix}$ is full rank. This is also a sufficient condition, which we do not show here.

Controllability

- A system or the pair $(\mathbf{A}, \mathbf{B})$ is controllable, if and only if the controllability matrix $\mathcal{C} = \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \dots & \mathbf{A}^{n-1}\mathbf{B} \end{bmatrix}$ has full rank, i.e. $rank\,(\mathcal{C}) = n$.
- Controllability is a system property and is not affected by the choice of coordinate system used for representing the state. Changing the basis of the state to the columns of the matrix $\mathbf{T}$ does not affect the rank of $\mathcal{C}$.
- Let $\tilde{\mathbf{x}} = \mathbf{T}^{-1}\mathbf{x}$, then we have $\tilde{\mathbf{A}} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T}$ and $\tilde{\mathbf{B}} = \mathbf{T}^{-1}\mathbf{B}$. This implies that $\tilde{\mathcal{C}} = \mathbf{T}^{-1}\mathcal{C}$, and $rank\left(\tilde{\mathcal{C}}\right) = rank\,(\mathcal{C})$.

Controllability

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- Let $\tilde{\mathbf{x}} = \mathbf{T}^{-1}\mathbf{x}$, then we have $\tilde{\mathbf{A}} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T}$ and $\tilde{\mathbf{B}} = \mathbf{T}^{-1}\mathbf{B}$. This implies that $\tilde{\mathcal{C}} = \mathbf{T}^{-1}\mathcal{C}$, and $rank\left(\tilde{\mathcal{C}}\right) = rank\left(\mathcal{C}\right)$.
- Are the following systems $(\mathbf{A}, \mathbf{B})$ controllable. (a) $\left(\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$; (b) $\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$; and (c) $\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}\right)$.

Controllability

When $\mathbf{A}$ is diagonalizable $\left(\mathbf{A} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^{-1}\right)$, we have,

$$\dot{\tilde{\mathbf{x}}}(t) = \mathbf{\Lambda}\tilde{\mathbf{x}}(t) + \tilde{\mathbf{B}}\mathbf{u}(t), \quad \text{where } \tilde{\mathbf{x}} = \mathbf{V}^{-1}\mathbf{x} \; \text{ and } \tilde{\mathbf{B}} = \mathbf{V}^{-1}\mathbf{B}$$

$$\tilde{\mathbf{x}}(t_f) = e^{t_f\mathbf{\Lambda}}\tilde{\mathbf{x}}(0) + \int_0^{t_f} e^{(t_f-\tau)\mathbf{\Lambda}}\tilde{\mathbf{B}}\mathbf{u}(\tau)\,d\tau \implies e^{-t_f\mathbf{\Lambda}}\tilde{\mathbf{x}}(t_f) - \tilde{\mathbf{x}}(0) = \int_0^{t_f} e^{-\tau\mathbf{\Lambda}}\tilde{\mathbf{B}}\mathbf{u}(\tau)\,d\tau$$

When a row of $\tilde{\mathbf{B}}$ is zero, then the system is not controllable, i.e. that particular first order block does not receive any input.

When the two or more eigenvalue of the system are repeated, and we only have a single input to the system, then the system is not controllable (problem (b) in the previous slide).
For example, let $\mathbf{\Lambda} = \lambda\mathbf{I}$, then we have, $e^{-\lambda t_f}\tilde{\mathbf{x}}(t_f) - \tilde{\mathbf{x}}(0) = \left(\int_0^{t_f} e^{-\lambda\tau}u(\tau)\,d\tau\right)\tilde{\mathbf{B}}$.

Thus, the RHS can only be along the column space of $\tilde{\mathbf{B}}$.

Controllability

When $\mathbf{A}$ is not diagonalizable $\left(\mathbf{A} = \mathbf{V}\mathbf{J}\mathbf{V}^{-1}\right)$, where,

$$\mathbf{J} = \left[\begin{array}{cc|ccc|ccc} \lambda_1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & \lambda_2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_2 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & \lambda_3 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & \lambda_k \end{array}\right]$$

$$\dot{\tilde{\mathbf{x}}}(t) = \mathbf{J}\tilde{\mathbf{x}}(t) + \tilde{\mathbf{B}}\mathbf{u}(t), \text{ where } \tilde{\mathbf{x}} = \mathbf{V}^{-1}\mathbf{x} \text{ and } \tilde{\mathbf{B}} = \mathbf{V}^{-1}\mathbf{B}$$

When the different Jordan blocks have distinct eigenvalues, and any of the rows of $\tilde{\mathbf{B}}$ corresponding the to the last row of a Jordan block is zero, then the system is not controllable, i.e. that particular Jordan order block does not receive any input.

When the two or more Jordan blocks have the same eigenvalue, and we only have a single input to the system, then the system is not controllable (problem (b) in the previous slide).

Controllability

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$$\dot{\tilde{\mathbf{x}}}(t) = \mathbf{J}\tilde{\mathbf{x}}(t) + \tilde{\mathbf{B}}\mathbf{u}(t), \text{ where } \tilde{\mathbf{x}} = \mathbf{V}^{-1}\mathbf{x} \text{ and } \tilde{\mathbf{B}} = \mathbf{V}^{-1}\mathbf{B}$$

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Is a mass M in free space acted upon by a force $\mathbf{f} = \begin{bmatrix} f_1 & f_2 & f_3 \end{bmatrix}^T$ controllable?

Controllability – Discrete-time system

$$\mathbf{x}\left[k+1\right] = \mathbf{A}\mathbf{x}\left[k\right] + \mathbf{B}\mathbf{u}\left[k\right]$$

Assuming the system starts at $k = 0$, the output at k is given by,

$$\mathbf{x}\left[k\right] = \mathbf{A}^k\mathbf{x}\left[0\right] + \sum_{l=0}^{k-1}\mathbf{A}^{k-l-1}\mathbf{B}\mathbf{u}\left[l\right]$$

$$\mathbf{x}\left[k\right] - \mathbf{A}^k\mathbf{x}\left[0\right] = \begin{bmatrix}\mathbf{B} & \mathbf{A}\mathbf{B} & \mathbf{A}^2\mathbf{B} & \dots & \mathbf{A}^{k-1}\mathbf{B}\end{bmatrix}\begin{bmatrix}\mathbf{u}\left[k-1\right] \\ \mathbf{u}\left[k-2\right] \\ \dots \\ \mathbf{u}\left[0\right]\end{bmatrix} = \mathcal{C}_k\tilde{\mathbf{u}}_{0:k-1}$$

Assuming $\mathbf{x}\left[0\right] = \mathbf{0}$, what all values can $\mathbf{x}\left[k\right]$ take?

Controllability – Discrete-time system

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$$\mathbf{x}[k] - \mathbf{A}^k\mathbf{x}[0] = \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B} & \dots & \mathbf{A}^{k-1}\mathbf{B} \end{bmatrix} \begin{bmatrix} \mathbf{u}[k-1] \\ \mathbf{u}[k-2] \\ \dots \\ \mathbf{u}[0] \end{bmatrix} = \mathcal{C}_k \tilde{\mathbf{u}}_{0:k-1}$$

Assuming $\mathbf{x}[0] = \mathbf{0}$, what all values can $\mathbf{x}[k]$ take? $\longrightarrow \mathbf{x}[k] \in C(\mathcal{C}_k)$. $\mathbf{x}[k]$ can only be in the subspace $C(\mathcal{C}_k)$.

Starting from $k=0$, the possible values $\mathbf{x}[k]$ can take grows until, $C(\mathcal{C}_{k-1}) = C(\mathcal{C}_k)$, i.e.

$$C(\mathcal{C}_0) \subseteq C(\mathcal{C}_1) \subseteq C(\mathcal{C}_2 \cdots) \subseteq C(\mathcal{C}_{n-1}) \subseteq C(\mathcal{C}_n) \subseteq \mathbb{R}^n$$

Controllability – Discrete-time system

- The subspace of reachable states can at most grow for n time steps, as we add more columns from $\mathbf{A}^k\mathbf{B}$.
- $\mathcal{C}_n$ is the subspace that can be reached by the system and nothing more. Because, the columns of $\mathbf{A}^n\mathbf{B}$ are a linear combination of the columns of $\mathbf{B}, \mathbf{AB}, \mathbf{A}^2\mathbf{B}, \ldots \mathbf{A}^{n-1}\mathbf{B}$.
- Thus, the system is controllable if and only if $\mathcal{C}_n = \mathbb{R}^n$, or the $rank\left(\mathcal{C}_n\right) = n$.
- In the discrete case, the controllability matrix $\mathcal{C}_n$ can also be used to determine the input sequence $\tilde{\mathbf{u}}_{0:n-1}$ that takes you from state $\mathbf{x}\left[0\right]$ to $\mathbf{x}\left[n\right]$.

$$\mathbf{x}\left[k\right] - \mathbf{A}^k\mathbf{x}\left[0\right] = \mathcal{C}_n\tilde{\mathbf{u}}_{0:n-1} \implies \tilde{\mathbf{u}}_{0:n-1} = \mathcal{C}_n^{\dagger}\left(\mathbf{x}\left[k\right] - \mathbf{A}^k\mathbf{x}\left[0\right]\right)$$

Where, $\mathcal{C}_n^{\dagger} = \mathcal{C}_n^{-1}$ for a single input system, and it is a right inverse when there are more than one inputs to the system.

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Where, $\mathcal{C}_n^{\dagger} = \mathcal{C}_n^{-1}$ for a single input system, and it is a right inverse when there are more than one inputs to the system.

Which of the following systems are controllable? (a) $\left(\begin{bmatrix}1 & 0\\0 & 2\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}\right)$; (b) $\left(\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}\right)$; and (c) $\left(\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}\right)$.

Controllability – Discrete-time system

When $\mathbf{A}$ is diagonalizable $\left(\mathbf{A} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^{-1}\right)$, we have,

$$\tilde{\mathbf{x}}[k+1] = \mathbf{\Lambda}\tilde{\mathbf{x}}[k] + \tilde{\mathbf{B}}\mathbf{u}[k], \text{ where } \tilde{\mathbf{x}} = \mathbf{V}^{-1}\mathbf{x} \text{ and } \tilde{\mathbf{B}} = \mathbf{V}^{-1}\mathbf{B}$$

$$\tilde{\mathbf{x}}[k] = \mathbf{\Lambda}^k\tilde{\mathbf{x}}[0] + \sum_{l=0}^{k-1}\mathbf{\Lambda}^{k-l-1}\tilde{\mathbf{B}}\mathbf{u}[l] \implies \tilde{\mathbf{x}}[k] - \mathbf{\Lambda}^k\tilde{\mathbf{x}}[0] = \sum_{l=0}^{k-1}\mathbf{\Lambda}^{k-l-1}\tilde{\mathbf{B}}\mathbf{u}(\tau)\,d\tau$$

When a row of $\tilde{\mathbf{B}}$ is zero, then the system is not controllable, i.e. that particular first order block does not receive any input.

When the two or more eigenvalue of the system are repeated, and we only have a single input to the system, then the system is not controllable (problem (b) in the previous slide).
For example, let $\mathbf{\Lambda} = \lambda\mathbf{I}$, then we have, $\tilde{\mathbf{x}}[k] - \lambda^k\tilde{\mathbf{x}}[0] = \left(\sum_{l=0}^{k-1}\lambda^{k-l-1}u[l]\,d\right)\tilde{\mathbf{B}}$.

Thus, the RHS can only be along the column space of $\tilde{\mathbf{B}}$.

Controllability – Discrete-time system

When $\mathbf{A}$ is not diagonalizable $\left(\mathbf{A} = \mathbf{V}\mathbf{J}\mathbf{V}^{-1}\right)$, where,

$$\mathbf{J} = \left[\begin{array}{cc|ccc|ccc} \lambda_1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & \lambda_2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_2 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & \lambda_3 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & \lambda_k \end{array}\right]$$

$$\tilde{\mathbf{x}}\left[k+1\right] = \mathbf{J}\tilde{\mathbf{x}}\left[k\right] + \tilde{\mathbf{B}}\mathbf{u}\left[k\right], \text{ where } \tilde{\mathbf{x}} = \mathbf{V}^{-1}\mathbf{x} \text{ and } \tilde{\mathbf{B}} = \mathbf{V}^{-1}\mathbf{B}$$

When the different Jordan blocks have distinct eigenvalues, and any of the rows of $\tilde{\mathbf{B}}$ corresponding the to the last row of a Jordan block is zero, then the system is not controllable, i.e. that particular Jordan order block does not receive any input.

When the two or more Jordan blocks have the same eigenvalue, and we only have a single input to the system, then the system is not controllable (problem (b) in the previous slide).

Observability

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)$$

- Observability tells us if we can determine all the states of a system using our knowledge of the inputs and the outputs to the system over a finite duration of time. For this, we only have to deal with the measurement equation.
- **Definition**: The system or the pair $(\mathbf{A}, \mathbf{C})$ is **observable** if every state $\mathbf{x}(0)$ can be determined from the observation of the output $\mathbf{y}(t)$ over a finite period of time, $0 \leq t \leq t_1$. Otherwise, the system is unobservable.
- Observability is important to be able to estimate states that cannot be directly measured. Knowledge of the states is essential for implementing state feedback controllers.

Observability

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)$$

Assuming the system starts at $t = 0$, the output at time t is given by,

$$\mathbf{y}(t) = \mathbf{C}e^{t\mathbf{A}}\mathbf{x}(0) + \int_0^t \mathbf{C}e^{(t-\tau)\mathbf{A}}\mathbf{B}\mathbf{u}(\tau)\,d\tau + \mathbf{D}\mathbf{u}(t)$$

Let $\hat{\mathbf{y}}(t) = \mathbf{y}(t) - \int_0^t \mathbf{C}e^{(t-\tau)\mathbf{A}}\mathbf{B}\mathbf{u}(\tau)\,d\tau - \mathbf{D}\mathbf{u}(t)$. Then we have,

$$\hat{\mathbf{y}}(t) = \mathbf{C}e^{t\mathbf{A}}\mathbf{x}(0) \implies \hat{\mathbf{y}}(t) = \sum_{k=0}^{n-1} \alpha_k(t)\,\mathbf{C}\mathbf{A}^k\mathbf{x}(0) = \begin{bmatrix}\alpha_0(t) & \dots & \alpha_{n-1}(t)\end{bmatrix} \begin{bmatrix} \mathbf{C} \\ \mathbf{C}\mathbf{A} \\ \mathbf{C}\mathbf{A}^2 \\ \vdots \\ \mathbf{C}\mathbf{A}^{n-1} \end{bmatrix} \mathbf{x}(0)$$

A necessary condition to uniquely determine $\mathbf{x}(0)$ from $\mathbf{y}(t)$ is that the *observability matrix* $\mathcal{O} = \begin{bmatrix}\mathbf{C} & \mathbf{C}\mathbf{A} & \dots & \mathbf{C}\mathbf{A}^{n-1}\end{bmatrix}^T$ is full rank. This is also a sufficient condition, which we do not show here.

Observability

- A system or the pair $(\mathbf{A}, \mathbf{C})$ is controllable, if and only if the controllability matrix $\mathcal{O} = \begin{bmatrix}\mathbf{C} & \mathbf{CA} & \dots & \mathbf{CA}^{n-1}\end{bmatrix}^T$ has full rank, i.e. $rank\left(\mathcal{O}\right) = n$.
- Observability is a system property and is not affected by the choice of coordinate system used for representing the state. Changing the basis of the state to the columns of the matrix $\mathbf{T}$ does not affect the rank of $\mathcal{O}$.
- Let $\tilde{\mathbf{x}} = \mathbf{T}^{-1}\mathbf{x}$, then we have $\tilde{\mathbf{A}} = \mathbf{T}^{-1}\mathbf{AT}$ and $\tilde{\mathbf{C}} = \mathbf{BT}$. This implies that $\tilde{\mathcal{O}} = \mathcal{O}\mathbf{T}$, and $rank\left(\tilde{\mathcal{O}}\right) = rank\left(\mathcal{O}\right)$.
- When $\mathbf{A}$ is diagonalizable $\left(\mathbf{A} = \mathbf{V\Lambda V}^{-1}\right)$, we have, $\hat{\mathbf{y}}\left(t\right) = \tilde{\mathbf{C}}e^{t\mathbf{\Lambda}}\tilde{\mathbf{x}}\left(0\right)$
  - When a column of $\tilde{\mathbf{C}}$ is zero, then the system is not observable, i.e. that particular first order block does not contribute anything to the output.
  - When the two or more eigenvalue of the system are repeated, and the system has only one output, then the system is not observable.

Observability – Discrete-time system

$$\mathbf{y}\left[k\right] = \mathbf{C}\mathbf{x}\left[k\right] + \mathbf{D}\mathbf{u}\left[k\right]$$

Assuming the system starts at $k = 0$, the output at k is given by,

$$\mathbf{y}\left[k\right] = \mathbf{C}\mathbf{A}^{k}\mathbf{x}\left[0\right] + \sum_{l=0}^{k-1} \mathbf{C}\mathbf{A}^{k-l-1}\mathbf{B}\mathbf{u}\left[l\right] + \mathbf{D}\mathbf{u}\left[k\right]$$

Let $\hat{\mathbf{y}}\left[k\right] = \mathbf{y}\left[k\right] - \sum_{l=0}^{k-1} \mathbf{C}\mathbf{A}^{k-l-1}\mathbf{B}\mathbf{u}\left[l\right] - \mathbf{D}\mathbf{u}\left[k\right]$. Then we have,

$$\hat{\mathbf{y}}\left[k\right] = \mathbf{C}\mathbf{A}^{k}\mathbf{x}\left[0\right] \implies \begin{bmatrix} \hat{\mathbf{y}}\left[0\right] \\ \hat{\mathbf{y}}\left[1\right] \\ \vdots \\ \hat{\mathbf{y}}\left[k\right] \end{bmatrix} = \begin{bmatrix} \mathbf{C} \\ \mathbf{C}\mathbf{A} \\ \vdots \\ \mathbf{C}\mathbf{A}^{k} \end{bmatrix} \mathbf{x}\left(0\right) = \mathcal{O}_k \mathbf{x}\left[0\right]$$

When $rank\left(\mathcal{O}_k\right) = n$, then we can uniquely determine any $\mathbf{x}\left(0\right)$ from $\mathbf{y}\left[k\right]$. Starting with $k = 0$, the $rank\left(\mathcal{O}_k\right)$ can grow till $k = n - 1$.

Observability – Discrete-time system

- In the discrete case, the observability matrix $\mathcal{O}_{n-1}$ can also be used to determine the the state $\mathbf{x}(0)$.

$$\hat{\mathbf{y}}[n-1] = \mathcal{O}_{n-1}\mathbf{x}[0] \implies \mathbf{x}[0] = \mathcal{O}_{n-1}^{\dagger}\hat{\mathbf{y}}[n-1]$$

  Where, $\mathcal{O}_{n-1}^{\dagger} = \mathcal{O}_{n-1}^{-1}$ for a single output system, and it is a left inverse when there is more than one output from the system.
- When $\mathbf{A}$ is diagonalizable $\left(\mathbf{A} = \mathbf{V\Lambda V}^{-1}\right)$, we have, $\hat{\mathbf{y}}[k] = \tilde{\mathbf{C}}\mathbf{\Lambda}^k\tilde{\mathbf{x}}[0]$
  - When a column of $\tilde{\mathbf{C}}$ is zero, then the system is not observable, i.e. that particular first order block does not contribute anything to the output.
  - When the two or more eigenvalue of the system are repeated, and the system has only one output, then the system is not observable.