Linear Systems

Course Notes

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Basics of Linear Algebra and Matrix Theory

1.1 Vector space

This chapter mainly consists of a bunch of definitions and concepts about vector spaces that one must be clear about to understand linear algebra. Like most topics in mathematics, or any discipline with a formal structure, the importance of being absolutely clear about the foundational ideas cannot be overstated.

Definition 1.1 Vector Space

A vector space $\mathcal{V}$ over the field $\mathcal{F}$ is a non-empty set of elements that is closed under the operations of scalar multiplication and vector addition.

There are four things involved in a vector space as defined above.

1. A non-empty set of elements, where the are called **vectors**.
2. A field $\mathcal{F}$ of scalar that we will be dealing with when working vector spaces. The numbers associated with the elements of the vectors spaces will be from the field $\mathcal{F}$. We will be mostly dealing with the real $\mathbb{R}$ or complex $\mathbb{C}$ scalar fields in this course.
3. A vector addition operation denoted by $\mathbf{v}_1 + \mathbf{v}_2$, where $\mathbf{v}_1, \mathbf{v}_2 \in \mathcal{V}$.
4. A scalar multiplication operation denoted by $\alpha\mathbf{v}_1$ between an element of $\mathbf{v} \in \mathcal{V}$ and $\alpha \in \mathcal{F}$.

Note: The exact nature of the vector addition and scalar multiplication operations need to be defined depending on the type of vector space we are dealing with.

Vector spaces generalize the idea of Euclidean (flat) spaces that we are very familiar with from our everyday experience and also most our basic math from school.

The most common example of vector spaces that we will encounter in this course are the n-tuple of numbers, which could be thought of as points in an n-dimensional Euclidean space. Consider the all familiar xy-plane shown in Fig. 1.1. The set of all points in this plane forms a vector space. Each point in this plane can be represented by 2-tuple of real numbers,

$$\mathbb{R}^2 = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \middle| x_1, x_2 \in \mathbb{R} \right\} = \{\dots \mathbf{p}_1, \dots \mathbf{p}_2, \dots\}$$

The points $\mathbf{p}_1$, $\mathbf{p}_2$ shown in Fig. 1.1 would be somewhere in the set $\mathbb{R}^2$.

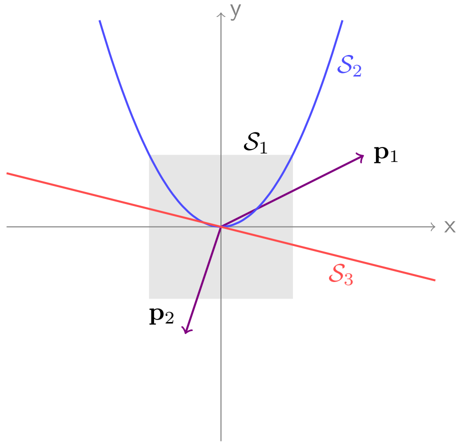

Figure 1.1: xy-plane.

Similarly, the set of all points in n-dimensional Euclidean space form a vector space,

$$\mathbb{R}^n = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \middle| x_1, x_2, \dots x_n \in \mathbb{R} \right\}$$

Vector addition between two n-tuples $\mathbf{v}, \mathbf{w}$ is defined as the sum of the corresponding scalar elements,

$$\mathbf{v} + \mathbf{w} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} \triangleq \begin{bmatrix} v_1 + w_1 \\ v_2 + w_2 \\ \vdots \\ v_n + w_n \end{bmatrix}$$

Scalar multiplication of an n-tuple by a scalar $\alpha \in \mathbb{R}$ is defined as,

$$\alpha\mathbf{v} = \alpha \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \triangleq \begin{bmatrix} \alpha v_1 \\ \alpha v_2 \\ \vdots \\ \alpha v_n \end{bmatrix}$$

The elements of the n-tuple results from vector addition and scalar multiplication are all real number and thus belong to $\mathbb{R}^n$.

The vector addition and scalar multiplication operations associated with a vector space $\mathcal{V}$,

1. $\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$.
2. $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$.
3. There is a zero element in the vector space $\mathcal{V}$, $\mathbf{0} \in \mathcal{V}$, such that $\mathbf{v} + \mathbf{0} = \mathbf{v}$.

4. For every $\mathbf{v} \in \mathcal{V}$, there is an element $-\mathbf{v}$, such that $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$.
5. $(\alpha\beta)\,\mathbf{v} = \alpha\,(\beta\mathbf{v})$, $\forall \alpha, \beta \in \mathcal{F}$ and $\forall \mathbf{v} \in \mathcal{V}$.
6. $\alpha\,(\mathbf{v} + \mathbf{w}) = \alpha\mathbf{v} + \alpha\mathbf{w}$, $\forall \alpha \in \mathcal{F}$ and $\forall \mathbf{v}, \mathbf{w} \in \mathcal{V}$.
7. $(\alpha + \beta)\,\mathbf{v} = \alpha\mathbf{v} + \beta\mathbf{v}$, $\forall \alpha, \beta \in \mathcal{F}$ and $\forall \mathbf{v} \in \mathcal{V}$.
8. $1\mathbf{v} = \mathbf{v}$, $\forall \mathbf{v} \in \mathcal{V}$.

1.2 Subspace

Consider a vector space $\mathcal{V}$[1]. Let us now consider $\mathcal{S}$, a subset of $\mathcal{V}$, i.e. $\mathcal{S} \subseteq \mathcal{V}$. This means that if $x \in \mathcal{S}$ then $x \in \mathcal{V}$. A subset of $\mathcal{V}$ qualifies to a subspace of $\mathcal{V}$ only if $\mathcal{S}$ itself is a vector space.

Definition 1.2 Subspace

Let $\mathcal{S}$ be a non-empty subset of a vector space $\mathcal{V}$ over the field $\mathcal{F}$. $\mathcal{S}$ is a subspace over the same field $\mathcal{F}$ with the same vector addition and scalar multiplication operations as $\mathcal{V}$, if and only if $\mathcal{S}$ is closed under the vector addition and scalar multiplication operations.

$$\begin{aligned} \mathbf{x}, \mathbf{y} \in \mathcal{S} &\implies \mathbf{x} + \mathbf{y} \in \mathcal{S} \\ \mathbf{x} \in \mathcal{S} \text{ and } \alpha \in \mathcal{F} &\implies \alpha\mathbf{x} \in \mathcal{S} \end{aligned}$$

Let us look at some examples to get a good grasp of what subspaces are. $\mathcal{V} = \mathbb{R}^2$ is the parent vector space.

$$\mathcal{V} = \mathbb{R}^2 = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \middle| x_1, x_2 \in \mathbb{R} \right\}$$

The following are some of non-empty subsets of $\mathbb{R}^2$ (shown in Fig. 1.1, but not all of them are subspaces.

1. $\mathcal{S}_1 = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \middle| |x_1| \leq 1, |x_2| \leq 1 \right\}$. This is the set of all points in the gray square shown in Fig. 1.1.

   This is **not** a subspace of $\mathcal{V}$. Because, it is not closed under scalar multiplication or vector addition. For example, $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \in \mathcal{S}_1$, but $\mathbf{v}_1 + \mathbf{v}_2 = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \notin \mathcal{S}_1$.

2. $\mathcal{S}_2 = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \middle| x_1 \in \mathbb{R},\, x_2 = x_1^2 \right\}$. This is the set of all points on the blue colored parabola in Fig. 1.1.

   This is also **not** a subspace of $\mathcal{V}$, as it is not closed under scalar multiplication or vector addition. Let, $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \in \mathcal{S}_2$, but $2\mathbf{v}_1 = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \notin \mathcal{S}_2$.

3. $\mathcal{S}_3 = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \middle| x_1 \in \mathbb{R},\, x_2 = \beta x_1, \beta \in \mathbb{R} \right\}$. This is the set of all points on the red line passing through the origin in Fig. 1.1.

   This is a subspace of $\mathcal{V}$; it is closed under scalar multiplication or vector addition.

   Let $\mathbf{v} = \begin{bmatrix} v_1 \\ \beta v_1 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} w_1 \\ \beta w_1 \end{bmatrix}$. Then, $\alpha\mathbf{v} = \begin{bmatrix} \alpha v_1 \\ \alpha\beta v_1 \end{bmatrix} = \begin{bmatrix} \alpha v_1 \\ \beta\,(\alpha v_1) \end{bmatrix} \in \mathcal{S}_3$.

   And, $\mathbf{v} + \mathbf{w} = \begin{bmatrix} v_1 \\ \beta v_1 \end{bmatrix} + \begin{bmatrix} w_1 \\ \beta w_1 \end{bmatrix} = \begin{bmatrix} v_1 + w_1 \\ \beta v_1 + \beta w_1 \end{bmatrix} = \begin{bmatrix} v_1 + w_1 \\ \beta\,(v_1 + w_1) \end{bmatrix} \in \mathcal{S}_3$.

Problem Subspaces

Which of the following subsets are subspaces of $\mathbb{R}^2$?

1. $\left\{ \alpha\mathbf{p} \mid \mathbf{p} \in \mathbb{R}^2, \alpha = \mathbb{R} \right\}$.
2. $\left\{ \alpha\mathbf{p} + \mathbf{q} \mid \mathbf{p}, \mathbf{q} \in \mathbb{R}^2, \alpha = \mathbb{R} \right\}$.
3. $\left\{ \mathbf{p} = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} \middle| \mathbf{p} \in \mathbb{R}^2,\, p_1^2 + p_2^2 = 1 \right\}$.
4. $\left\{ \mathbf{p} = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} \middle| \mathbf{p} \in \mathbb{R}^2,\, p_1^2 + p_2^2 = 0 \right\}$.

1.3 Span of a set of vectors

Consider a set of vectors $\mathcal{S} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_n\}$, $\mathbf{v}_i \in \mathcal{V}$, $1 \leq i \leq n$, where $\mathcal{V}$ is a vector space over the field $\mathcal{F}$[2]. A linear combination of the elements of $\mathcal{S}$ is a vector $\mathbf{u} \in \mathcal{V}$ defined as the following,

$$\mathbf{u} = \sum_{i=1}^{n} \alpha_i \mathbf{v}_i,\; \alpha_i \in \mathcal{F},\; \mathbf{v}_i \in \mathcal{S}$$

Definition 1.3 Span of set of vectors

The span of a set of vectors $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_n\}$ is defined as the set of all linear combinations of a the elements of $\mathcal{S}$, i.e.

$$span\,(\mathcal{S}) = \left\{ \sum_{i=1}^{n} \alpha_i \mathbf{v}_i \middle| \alpha \in \mathcal{F} \right\}$$

What can we say about $span\,(\mathcal{S})$? Is it a subset of $\mathcal{V}$? Is it a subspace of $\mathcal{V}$? It turns out that the span of a set of vector does form a subspace of $\mathcal{V}$. It is left as an exercise for the reader to prove this.

The subspace $span\,(\mathcal{S})$ is called the subspace spanned by $\mathcal{S}$, or $\mathcal{S}$ is said to span the subspace $span\,(\mathcal{S})$. $\mathcal{S}$ is also called the spanning set of $span\,(\mathcal{S})$.

Problem Span of a set of vectors

1. Consider a set of vectors $\mathcal{S} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_n\}$, $\mathbf{v}_i \in \mathcal{V}, 1 \leq i \leq n$, where $\mathcal{V}$ is a vector space. Prove that a linear combination of the vectors of $\mathcal{S}$ belong to $\mathcal{V}$.

[1] Whenever you hear see the term 'vector space', you must train yourself to immediately think of the the Def. 1.1

[2] From now on, we will assume that a given vector space is over the field $\mathbf{F}$, unless specified otherwise.

2. Prove that the span of a set $\mathcal{S}$ is a subspace of $\mathcal{V}$.

1.4 Linear independence

Like the span, linear independence (LI) is also a concept associated with a set of vectors $\mathcal{S} = \{\mathbf{v}_1, \mathbf{v}_2, \dots \mathbf{v}_n\}$, $\mathbf{v}_i \in \mathcal{V}$, $1 \leq i \leq n$ where $\mathcal{V}$ is a vector space. LI tells us if one or more vectors in $\mathcal{S}$ is in the subspace spanned by other vectors in $\mathcal{S}$ (or if one or more vectors is a linear combination of the other vectors in $\mathcal{S}$).

Definition 1.4 Linear Independene

A set of vectors $\mathcal{S} = \{\mathbf{v}_1, \mathbf{v}_2, \dots \mathbf{v}_n\}$ is a linearly independent set if and only if,

$$\sum_{i=1}^{n} \alpha_i \mathbf{v}_i = 0 \implies \alpha_i = 0, \forall i$$

The set is linearly dependent if at least one of the $\alpha_i \neq 0$.

1.5 Basis and dimension of a vector space

Vector spaces can consist of infinite number of elements. The special structure of vector spaces (closure under scalar multiplication and addition) allows us to generate all elements of a vector space using only a finite number of elements from that vector space[3]. We have seen this earlier: a spanning set $\mathcal{S}$ (consisting of a finite number of elements) can be used to generate all elements in the subspace $span(\mathcal{S})$.

Consider a set $\mathcal{S} = \{\mathbf{v}_1, \mathbf{v}_2, \dots \mathbf{v}_n\}$, $\mathbf{v}_i \in \mathcal{V}$, $1 \leq i \leq n$, a subset of the vector space $\mathcal{V}$, whose elements can be used to generate every element of $\mathcal{V}$ through a linear combination, i.e. $\forall \mathbf{x} \in \mathcal{V}, \exists \alpha_i \in \mathcal{F}, 1 \leq i \leq n$ such that ,

$$\mathbf{x} = \sum_{i=1}^{n} \alpha_i \mathbf{v}_i$$

A basis of a vector space $\mathcal{V}$ is a set with the least number of elements, which can be used to generate all elements of $\mathcal{V}$. Thus, a basis is a set with the sufficient of elements from $\mathcal{V}$ that are necessary to generate all elements of $\mathcal{V}$.

Definition 1.5 Basis of a vector space

A linearly independent spanning set of a vector space $\mathcal{V}$ is called a basis of $\mathcal{V}$.

There can be an infinite number of basis for a vector space $\mathcal{V}$. Apart from the all of them sharing a property of being linearly independent spanning sets of $\mathcal{V}$, there share another common property. They all have the same number of elements. This number is called the dimension of the vector space $\mathcal{V}$.

[3] This is only valid for finite dimensional vector space, and in this course we will only talk about finite dimensional vector space.

Definition 1.6 Dimension of a vector space

The dimension of a vector space $\dim\mathcal{V}$ is the number of elements of a basis the basis of $\mathcal{V}$.

One of the advantages of a basis is that there is a unique representation, in terms of the elements of the basis, for every element from the span of the basis. Consider a basis $\mathcal{S} = \{\mathbf{v}_i\}_{i=1}^{n}$ of the vetor space $\mathcal{V}$, there $\forall \mathbf{x} \in \mathcal{V}$, we can write

$$\mathbf{x} = \sum_{i=1}^{n} \alpha_i \mathbf{v}_i$$

where the α_i are unique. There is only one way to write $\mathbf{x}$ as a linear combnation of the elements $\mathbf{v}_i$ of the basis $\mathcal{S}$.

Problem Basis and representation of vectors

Consider linear dependent set of vector $\mathcal{W} = \{\mathbf{w}_i\}_{i=1}^{n}$. Prove that there there is not unqiue way to represent a vector $\mathbf{x} \in span(\mathcal{W})$ in terms of the elements of $\mathcal{W}$.

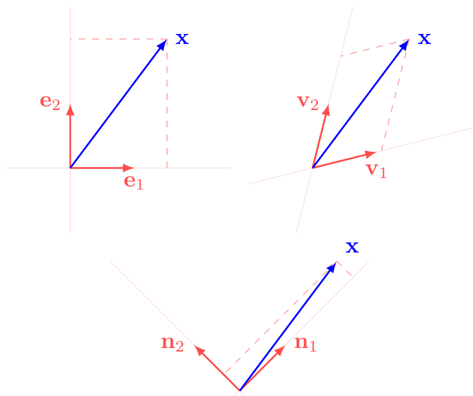

Figure 1.2: $\mathbb{R}^2$ vector space

1.6 Orthonormal basis

Although there can be infinitely many basis for a vector space, some basis are natural or more easy to work with. For example, consider the unit vectors,

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} \dots \mathbf{e}_n = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

The set of unit vectors $\mathcal{E} = \{\mathbf{e}_i\}_{i=1}^{n}$ form a basis for $\mathbb{R}^n$. It is easy to see how any vector $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$ can be represented in the basis $\mathcal{E}$.

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \sum_{i-1}^{n} x_i \mathbf{e}_i$$

With the basis $\mathcal{E}$ it is easy to find out the scalars that one needs to use to represent a vector $\mathbf{x}$ as a linear combination of the elements of $\mathcal{E}$. For other abitrary basis it is not easy to find out the exact scalars that go in the linear combination (e.g. the second plot shown in Fig. 1.2). However, when a basis is orthonormal, then finding the scalars in the linear comination are relatively easy. An orthonormal basis $\mathcal{N} = \{\mathbf{n}_i\}_{i=1}^n$ is a basis with the following properties,

$$\mathbf{n}_i^T \mathbf{n}_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

All vectors have unit length (-2-norm), and mutually orthogonal. Suppose that the representation of $\mathbf{x}$ in the basis $\mathcal{N}$ is given by the following,

$$\mathbf{x} = \sum_{i=1}^{n} \alpha_i \mathbf{n}_i, \quad \alpha_i = \mathbf{n}_i^T \mathbf{x}$$

Problem

Prove the for an orthonormal basis $\mathcal{N} = \{\mathbf{n}_i\}_{i=1}^n$ of $\mathbb{R}^n$, any vector $\mathbf{x} \in \mathbb{R}^n$ can be written as,

$$\mathbf{x} = \sum_{i=1}^{n} \left(\mathbf{n}_i^T \mathbf{x}\right) \mathbf{n}_i$$

1.7 Matrix Multiplication

Consider two matrices $\mathbf{A} \in \mathbb{R}^{m \times p}$ and $\mathbf{B} \in \mathbb{R}^{p \times n}$, the product of these two matrices is $\mathbf{C} \in \mathbb{R}^{m \times n}$,

$$\mathbf{C} = \mathbf{AB}$$

where, the ij^{th} element of $\mathbf{C}$ is given by,

$$c_{ij} = \sum_{k=1}^{p} a_{ik} b_{kj}$$

There are four different ways to make sense of the above equation,

1. **Column view**. The columns of $\mathbf{C}$ are equal to the linear combination of the columns of $\mathbf{A}$, and the scalars in the linear combination come from the rows of the corresponding column of $\mathbf{B}$.

$$i^{th} \text{ column of } \mathbf{C} = \mathbf{c}_i = \sum_{k=1}^{p} b_{ik} \mathbf{a}_k$$

2. **Row view**. The rows of $\mathbf{C}$ are equal to the linear combination of the rows of $\mathbf{B}$, and the scalars in the linear combination come from the columns of the corresponding row of $\mathbf{A}$.

$$i^{th} \text{ column of } \mathbf{C} = \tilde{\mathbf{c}}_i^T = \sum_{k=1}^{p} a_{ki} \tilde{\mathbf{b}}_k^T$$

3. **Inner product view**. The ij^{th} element of $\mathbf{C}$ is given by the inner product of the i^{th} row of $\mathbf{A}$ and the j^{th} columns of $\mathbf{B}$.

$$c_{ij} = \tilde{\mathbf{a}}_i^T \mathbf{b}_j$$

4. **Outer product view**. The matrix $\mathbf{C}$ can be written as the sum of p rank-1 matrices[4] $\mathbf{a}_k \tilde{\mathbf{b}}_k^T$, $1 \leq k \leq p$.

$$\mathbf{C} = \sum_{k=1}^{p} \mathbf{a}_k \tilde{\mathbf{b}}_k^T$$

1.8 Linear Equations $\mathbf{y} = \mathbf{Ax}$

Linear equations of the following form are encountered in numerous application across disciplines,

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 \ldots + a_{2n}x_n &= b_2 \\ a_{31}x_1 + a_{32}x_2 \ldots + a_{3n}x_n &= b_3 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 \ldots + a_{mn}x_n &= b_m \end{aligned}$$

There are m equations and n unknowns. The above equations can be more compactly represented using the matrix notation,

$$\mathbf{y} = \mathbf{Ax}, \quad \mathbf{x} \in \mathbb{R}^n, \mathbf{y} \in \mathbb{R}^m, \mathbf{A} \in \mathbb{R}^{m \times n}$$

$\mathbf{A}$ and $\mathbf{y}$ are known and $\mathbf{x}$ is to be determined.

Linear equations play an important role in engineering as many engineering problems can be mathematically approximated to have a linear structure.

We can broadly classify a vast majority of applications of linear equations into two categories: **Control problem** and **Estimation problem**.

- **Control Problem**. Control problems deal with linear systems with a set of inputs and outputs (Fig. 1.3), and our goal is to control the output behavior of the system by appropriately choosing the system inputs. The system has n scalar inputs $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$ that we can freely choose, and it has m scalar outputs $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} \in \mathbb{R}^m$. The matrix $\mathbf{A}$ contains information about the physics of the system relating the system inputs to the the system outputs

$$\mathbf{y} = \mathbf{Ax}$$

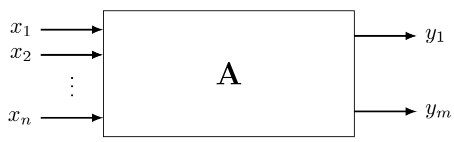

Figure 1.3: Control Problem

The columns of $\mathbf{A}$ show how all the m outputs of the system are affected by the n individual inputs, i.e. the

[4] Matrix whose rank is 1.

i^{th} column is the amount by which the output vector $\mathbf{y}$ changes due to a unit change in x_i.

$$\mathbf{y} = \ldots + x_i \mathbf{a}_i + \ldots$$

The rows of $\mathbf{A}$ tell us how the all the n input of the system affect a particular output, i.e. the i^{th} row of $\mathbf{A}$ tell us the complete story about how the different inputs affect the i^{th} output.

$$y_i = \tilde{\mathbf{a}}_i^T \mathbf{x}$$

Example 1.8.1 Applying a force on an object through a multi-joint system

Assume that one a person is trying to apply a force on a wall using his/her arm consisting of three joints (shoulder, elbow and wrist), as shown in Fig. 1.3.

The force $\mathbf{f} = \begin{bmatrix} f_x \\ f_y \end{bmatrix}$ applied on the wall can be controlled through the torque applied at the three joints $\boldsymbol{\tau} = \begin{bmatrix} \tau_S \\ \tau_E \\ \tau_W \end{bmatrix}$. The relationship between the torque and the endpoint force are linearly related for a fixed shoulder, elbow and wrist angle.

$$\mathbf{f} = \begin{bmatrix} f_x \\ f_y \end{bmatrix} = \begin{bmatrix} j_{11} & j_{12} & j_{13} \\ j_{21} & j_{22} & j_{23} \end{bmatrix} \begin{bmatrix} \tau_S \\ \tau_E \\ \tau_W \end{bmatrix} = \mathbf{J}\boldsymbol{\tau}$$

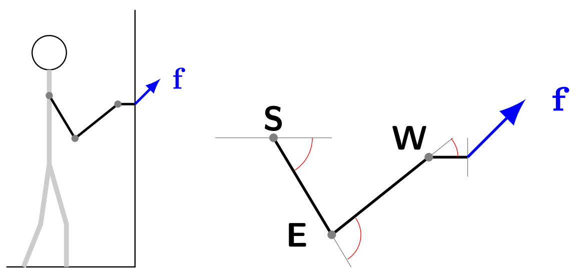

Figure 1.4: Applying a force with the arm.

- **Estimation problem**. Estimation problems deal with determining unknown parameters or inputs associates with a system using measurements from the system $\mathbf{y}$ and knowledge of how the system parameters/input affect these measurements $\mathbf{A}$. Assuming a linear relationship between the measurements and the system inputs/parameters, we have

$$\mathbf{y} = \mathbf{Ax}$$

2

Problems

"I hear and I forget. I see and I remember. I do and I understand."
– Confucius,

2.1 Linear Time-Invariant (LTI) Systems

1. Prove that for a memoryless LTI system the output is a scaled version of the input. Is this also true for: (a) a linear time-variant system; and (b) a non-linear system? Explain your answer.