Linear Control and Estimation

Orthogonality

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References

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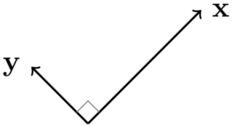
- G Strang, Linear Algebra: Chapters 3.

Orthogonality

- Two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ are orthognal if $\mathbf{x}^T\mathbf{y} = 0$.



- If we have sa set of non-zero vectors $V = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_r\}$, we say this a set of mutually orthogonal vectors, if and only if, $\mathbf{v}_i^T\mathbf{v}_j = 0, \;\; 1 \leq i, j \leq r$ and $i \neq j$. V is also a linearly independent set of vectors.
- When the length of the vectors is 1, it is called an **orthonormal** set of vectors.
- A set of orthonormal vectors V also form an **orthonormal basis** of the subsape $span\,(V)$.

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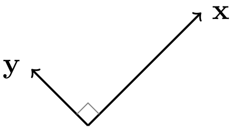

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Is $\left\{ \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \right\}$ an orthonormal set?. If no, how will you make it one?

Orthogonal Subspaces

- Two subspaces V, W are orthogonal if every vector in one subspace is orthogonal to every vector in the other subspace.

$$\mathbf{v}^T \mathbf{w} = 0, \;\; \forall \mathbf{v} \in V \text{ and } \mathbf{w} \in W$$

  Both subspaces V, W are from the same space, e.g. $\mathbb{R}^n$

- Consider two subspaces $V, W \subset \mathbb{R}^n$, such that $V + W = \mathbb{R}^n$. If V and W are orthogonal subspaces, then V and W are **orthogonal complements** of each other.

$$W \perp V \;\; \rightarrow \;\; V^\perp = W \text{ or } W^\perp = V; \quad \left(V^\perp\right)^\perp = V$$

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$V = span\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}^T, \begin{bmatrix}1\\0\\-1\end{bmatrix}^T\right\}$ and $W = span\left\{\begin{bmatrix}1\\-2\\1\end{bmatrix}^T\right\}$. Is $V^\perp = W$? If we add $\begin{bmatrix}1\\-1\\0\end{bmatrix}^T$ to W, is $V^\perp = W$ still true?

Relationship between the Four Fundamental Spaces

- $C(\mathbf{A}), N(\mathbf{A}^T) \subseteq \mathbb{R}^m$ are orthogonal complements.

$$C(\mathbf{A}) \perp N(\mathbf{A}^T)$$

- $C(\mathbf{A}^T), N(\mathbf{A}) \subseteq \mathbb{R}^n$ are orthogonal complements.

$$C(\mathbf{A}^T) \perp N(\mathbf{A})$$

- $\dim C(\mathbf{A}) + \dim N(\mathbf{A}^T) = m \implies C(\mathbf{A}) + N(\mathbf{A}^T) = \mathbb{R}^m$
- $\dim C(\mathbf{A}^T) + \dim N(\mathbf{A}) = n \implies C(\mathbf{A}^T) + N(\mathbf{A}) = \mathbb{R}^n$

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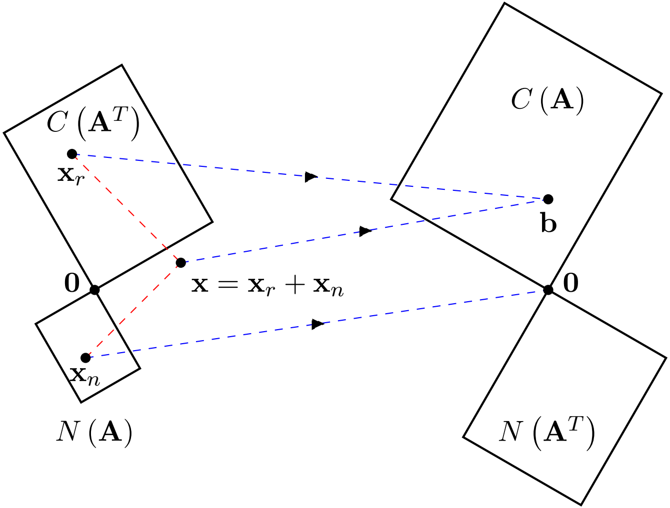
$$\mathbf{A} = \begin{bmatrix} 1 & -2 & 1 & 0 & 1 \\ 2 & -4 & 1 & -1 & -2 \\ -1 & 2 & 1 & 1 & 2 \\ 2 & -4 & -2 & -2 & -4 \end{bmatrix}$$

$$\mathbf{E} = \begin{bmatrix} \mathbf{E}_1 \\ \mathbf{E}_2 \end{bmatrix} = \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ -1 & 2 & 1 & 1 \\ \hline 2 & -4 & -2 & -2 \end{array}\right]$$

$$\mathbf{R} = \begin{bmatrix} \mathbf{R}_1 \\ \mathbf{R}_2 \end{bmatrix} = \left[\begin{array}{ccccc} 1 & -2 & 1 & 0 & 1 \\ 0 & 0 & -1 & -1 & -4 \\ 0 & 0 & 0 & -1 & -5 \\ \hline 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

- Is $C(\mathbf{A}) \perp N(\mathbf{A}^T)$?
- Is $C(\mathbf{A}^T) \perp N(\mathbf{A})$?
- What is $\dim C(\mathbf{A})$, $\dim N(\mathbf{A}^T)$, $\dim C(\mathbf{A}^T)$, $\dim N(\mathbf{A})$?

Relationship between the Four Fundamental Spaces



- $\mathbf{x}_r$ and $\mathbf{x}_n$ are the components of $x \in \mathbb{R}^n$ in the row and nullspaces of $\mathbf{A}$.
- **Nullspace** $N(\mathbf{A})$ is mapped to $\mathbf{0}$.

$$\mathbf{A}\mathbf{x}_n = \mathbf{0}$$

- **Row space** $C(\mathbf{A}^T)$ is mapped to the **column space** $C(\mathbf{A})$.

$$\mathbf{A}\mathbf{x}_r = \mathbf{A}(\mathbf{x}_r + \mathbf{x}_n) = \mathbf{A}\mathbf{x} = \mathbf{b}$$

- The mapping from the **row space** to the **column space** is invertible, i.e. every $\mathbf{x}_r$ is mapped to a unique element in $C(\mathbf{A})$
- What sort of mapping does $\mathbf{A}^T$ do?

Gram-Schmidt Orthogonalization

- Given a linearly independent set of vectors $B = \{\mathbf{x}_1, \mathbf{x}_2, \ldots \mathbf{x}_n\}$, where $\mathbf{x}_i \in \mathbb{R}^m, \ \forall i \in \{1, 2, \ldots n\}$, how can we find a orthonormal basis $\{\mathbf{u}_1, \mathbf{u}_2, \ldots \mathbf{u}_n\}$ for $span\,(B)$? $\longrightarrow$ **Gram-Schmidt Algorithm**
- Its an iterative procedure that can also detect if a given set B is linearly dependent.

Data: $\{\mathbf{x}_i\}_{i=1}^n$
Result: Return an orthonormal basis $\{\mathbf{u}_i\}_{i=1}^n$ if the set B is linearly independent, else return nothing.
for $i = 1, 2, \ldots n$ **do**
1. $\tilde{\mathbf{q}}_i = \mathbf{x}_i - \sum_{j=1}^{i-1} \left(\mathbf{u}_j^T \mathbf{x}_i\right) \mathbf{u}_i$ $\longrightarrow$(**Orthogonalization step**);
2. **If** $\tilde{\mathbf{q}}_i = 0$ **then return**;
3. $\mathbf{u}_i = \tilde{\mathbf{q}}_i / \left\|\tilde{\mathbf{q}}_i\right\|$ $\longrightarrow$(**Normalization step**);

end
return $\{\mathbf{u}_i\}_{i=1}^n$;

Gram-Schmidt Orthogonalization

- The algorithm can also be conveniently represented in a matrix form.

$$B = \{\mathbf{a}_1, \mathbf{a}_2, \ldots \mathbf{a}_n\}$$

$$\text{Let } \mathbf{U}_1 = 0_{m\times 1} \quad \text{and} \quad \mathbf{U}_i = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \ldots & \mathbf{u}_{i-1} \end{bmatrix} \in \mathbb{R}^{m\times(i-1)}$$

$$\mathbf{U}_i^T x_i = \begin{bmatrix} \mathbf{u}_1^T x_i \\ \mathbf{u}_2^T x_i \\ \vdots \\ \mathbf{u}_{i-1}^T x_i \end{bmatrix} \quad \text{and} \quad \mathbf{U}_i \mathbf{U}_i^T x_i = \sum_{j=1}^{i-1} \left(\mathbf{u}_j^T x_i\right) \mathbf{u}_j$$

$$\mathbf{u}_i = \frac{\left(I - \mathbf{U}_i \mathbf{U}_i^T\right) \mathbf{x}_i}{\left\|\left(I - \mathbf{U}_i \mathbf{U}_i^T\right) \mathbf{x}_i\right\|}$$

QR Decomposition

- Gram-Schmidt procedure leads us to another form of matrix decomposition – **QR decompostion**.
- Given a matrix $\mathbf{A} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \ldots & \mathbf{a}_n \end{bmatrix} \in \mathbb{R}^{n \times n}$, whose columns form a linearly independent set. Gramm-Schmidt algorithm produces a orthonormal basis $\{\mathbf{q}_1, \mathbf{q}_2, \ldots \mathbf{q}_n\}$ for $C(\mathbf{A})$.

$$\mathbf{q}_1 = \frac{\mathbf{a}_1}{r_1} \quad \text{and} \quad \mathbf{q}_i = \frac{\mathbf{a}_i - \sum_{j=1}^{i-1} (\mathbf{q}_j^T \mathbf{a}_i) \mathbf{q}_j}{r_k}$$

where, $r_1 = \|\mathbf{a}_1\|$ and $r_k = \left\| \mathbf{a}_i - \sum_{j=1}^{i-1} (\mathbf{q}_j^T \mathbf{a}_i) \mathbf{q}_j \right\|$.

$$\mathbf{a}_1 = r_1 \mathbf{q}_1 \quad \text{and} \quad \mathbf{a}_i = \sum_{j=1}^{i-1} \left(\mathbf{q}_j^T \mathbf{a}_i \right) + r_i \mathbf{q}_i$$

$$\mathbf{A} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 \ldots & \mathbf{a}_n \end{bmatrix} = \begin{bmatrix} \mathbf{q}_1 & \mathbf{q}_2 \ldots & \mathbf{q}_n \end{bmatrix} \begin{bmatrix} r_1 & \mathbf{q}_1^T \mathbf{a}_2 & \mathbf{q}_1^T \mathbf{a}_3 & \ldots & \mathbf{q}_1^T \mathbf{a}_n \\ 0 & r_2 & \mathbf{q}_2^T \mathbf{a}_3 & \ldots & \mathbf{q}_2^T \mathbf{a}_n \\ 0 & 0 & r_2 & \ldots & \mathbf{q}_3^T \mathbf{a}_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & r_n \end{bmatrix} = \mathbf{QR}$$

QR Decomposition

Find the $\mathbf{QR}$ factorization for the following, if possible.

$$\mathbf{A} = \begin{bmatrix} -1 & 2 & 1 \\ 0 & 0 & 2 \\ -4 & 1 & 3 \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} 1 & 2 & -1 & -7 \\ 1 & 2 & 0 & -5 \\ -4 & 1 & 0 & -16 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} 0 & 2 & -1 \\ 1 & 3 & -1 \\ -1 & 1 & 0 \\ 1 & 1 & 1 \\ 2 & 1 & 0 \end{bmatrix}$$

$\mathbf{QR}$ Decomposition

$$\mathbf{A} = \mathbf{QR}; \;\; \mathbf{A}, \mathbf{Q} \in \mathbb{R}^{m \times n}, \;\; \mathbf{R} \in \mathbb{R}^{n \times n}$$

- The columns of $\mathbf{Q}$ form an orthonormal basis for $C\left(\mathbf{A}\right)$, and $\mathbf{R}$ is upper-triangular.
- Similar to $\mathbf{A} = \mathbf{LU}$, $\mathbf{A} = \mathbf{QR}$ can be used for used to solve $\mathbf{Ax} = \mathbf{b}$.

$$\mathbf{Ax} = \mathbf{QRx} = \mathbf{b} \implies \mathbf{Rx} = \mathbf{Q}^{-1}\mathbf{b} = \mathbf{Q}^{T}\mathbf{b}$$

QR Decomposition

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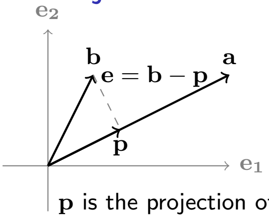
Solve the following through $\mathbf{LU}$ and $\mathbf{QR}$ factorization.

$$\mathbf{Ax} = \begin{bmatrix} -1 & 2 & 1 \\ 0 & 0 & 2 \\ -4 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \\ -2 \end{bmatrix} = \mathbf{b}$$

Orthogonal Projection onto Subspaces



$\mathbf{p}$ is the projection of $\mathbf{b}$ onto $\mathbf{a}$.

$\mathbf{P} = \frac{\mathbf{a}\mathbf{a}^T}{\mathbf{a}^T\mathbf{a}}$ is the projection matrix onto the line $\mathbf{a}$.

$\|\mathbf{e}\|$ is the distance of the point $\mathbf{b}$ from the line along $\mathbf{a}$. This distance is shortest when, $\mathbf{e} \perp \mathbf{a}$.

$$\mathbf{a}^T(\mathbf{b} - \mathbf{p}) = \mathbf{a}^T(\mathbf{b} - \alpha\mathbf{a}) = \mathbf{a}^T\mathbf{b} - \alpha\mathbf{a}^T\mathbf{a} = 0$$

$$\alpha = \frac{\mathbf{a}^T\mathbf{b}}{\mathbf{a}^T\mathbf{a}} \implies \mathbf{p} = \frac{\mathbf{a}^T\mathbf{b}}{\mathbf{a}^T\mathbf{a}}\mathbf{a}$$

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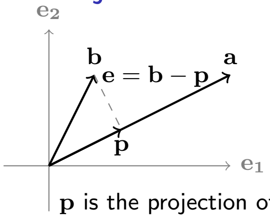

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$\mathbf{P} = \frac{\mathbf{a}\mathbf{a}^T}{\mathbf{a}^T\mathbf{a}}$ is the projection matrix onto the line $\mathbf{a}$.

Find the orthogonal projection matrix associated $\mathbf{a}$, and find the projection of $\mathbf{b}$ on to $span(\{\mathbf{a}\})$.

- $\mathbf{a} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$; $\mathbf{b} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$
- $\mathbf{a} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$; $\mathbf{b} = \begin{bmatrix} 6 \\ 3 \end{bmatrix}$
- $\mathbf{a} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$; $\mathbf{b} = \begin{bmatrix} 3 \\ -2 \\ 0 \end{bmatrix}$

Orthogonal Projection onto Subspaces

- We can also project vectors onto other subspaces, which is the generalization of the projection to a 1 dimensional subspace, i.e. the line.
- Consider a vector $\mathbf{b} \in \mathbb{R}^n$ and a subspace $S \subseteq \mathbb{R}^n$ spanned by the orthonormal basis $\{\mathbf{u}_1, \mathbf{u}_2, \ldots \mathbf{u}_r\}$.
  $\mathbf{b}_S$ – the orthogonal projection of $\mathbf{b}$ onto S is given by the following,
  $$\mathbf{b}_S = \mathbf{U}\mathbf{U}^T\mathbf{b}; \;\; \mathbf{U} = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \ldots & \mathbf{u}_r \end{bmatrix}$$
  $$\text{Projection matrix } \mathbf{P}_S = \mathbf{U}\mathbf{U}^T$$
- A projection matrix is **idempotent**, i.e. $\mathbf{P}^2 = \mathbf{P}$. What does this mean in terms of projecting a vector on to a subspace?

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- A projection matrix is **idempotent**, i.e. $\mathbf{P}^2 = \mathbf{P}$. What does this mean in terms of projecting a vector on to a subspace?

Find the orthogonal projection matrix associated $U = \left\{ \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \right\}$, and find the projection of $\mathbf{b} = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$ on to $span\left(U\right)$.