Linear Control and Estimation: Assignment

Vectors

1. Is set of vectors $\left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ independent? Explain your answer.

2. Prove the following for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$

   (a) **Triangle Inequality**:
   $$\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$$

   (b) **Backward Triangle Inequality**:
   $$\|\mathbf{x} - \mathbf{y}\| \geq |\|\mathbf{x}\| - \|\mathbf{y}\||$$

   (c) **Parallelogram Idenitity**:
   $$\frac{1}{2}\left(\|\mathbf{x} + \mathbf{y}\|^2 + \|\mathbf{x} - \mathbf{y}\|^2\right) = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2$$

3. Consider a set of vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. When is $\|\mathbf{x} - \mathbf{y}\| = \|\mathbf{x} + \mathbf{y}\|$? What can you say about the geometry of the vectors $\mathbf{x}$, $\mathbf{y}$, $\mathbf{x} - \mathbf{y}$ and $\mathbf{x} + \mathbf{y}$?

4. If $S_1, S_2 \subseteq V$ are subspaces of V, the is $S_1 \cap S_2$ a subspace? Demonstrate your answer.

5. Prove that the sum of two subspaces $S_1, S_2 \subseteq V$ is a subspace.

6. Consider a vector $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$. Express the following interms of inner product between a vector $\mathbf{u}$ and $\mathbf{v}$, and in each case specify the vector $\mathbf{u}$.

   (a) $\sum_{i=1}^n v_i$
   (b) $\frac{1}{n}\sum_{i=1}^n v_i$
   (c) $\sum_{i=1}^n v_i x^{(n-i)}$, where $x \in \mathbb{R}$
   (d) $\frac{1}{n-1}\sum_{i=1}^n \left(v_i - \frac{1}{n}\sum_{i=1}^n v_i\right)^2$
   (e) $\frac{1}{5}\sum_{i=3}^5 v_i$
   (f) $\sum_{i=1}^{n-1}(v_{i+1} - v_i)$
   (g) $\sqrt{v_n}$
   (h) $\sum_{i=1}^n w_i v_i^2$

7. Which of the following are linear functions of $\{x_1, x_2, \ldots, x_n\}$?

   (a) $\min_i \{x_i\}_{i=1}^n$
   (b) $\left(\sum_{i=1}^n x_i^2\right)^{1/2}$
   (c) x_6

8. Consider a linear function $f : \mathbb{R}^n \to \mathbb{R}$. Prove that every linear function of this form can be represented in the following form.
   $$y = f(\mathbf{x}) = \mathbf{w}^T\mathbf{x} = \sum_{i=1}^n w_i x_i, \quad \mathbf{x}, \mathbf{w} \in \mathbb{R}^n$$

9. An *affine* function f is defined as the sum of a linear function and a constant. It can in general be represented in the form,
   $$y = f(\mathbf{x}) = \mathbf{w}^T\mathbf{x} + \beta, \quad \mathbf{x}, \mathbf{w} \in \mathbb{R}^n,\ \beta \in \mathbb{R}$$
   Prove that affine functions are not linear. Prove that any affine function can be represented in the form $\mathbf{w}^T\mathbf{x} + \beta$.

10. Consider a function $f : \mathbb{R}^3 \to \mathbb{R}$, such that,
   $$f\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = 2;\ f\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = -3;\ f\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = 1;$$
   Can you determine the following values of $f(\mathbf{x})$, if you are told that f is linear?
   $$f\left(\begin{bmatrix} 2 \\ 2 \\ -2 \end{bmatrix}\right) =?;\ f\left(\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}\right) =?;\ f\left(\begin{bmatrix} 0.5 \\ 0.6 \\ -0.1 \end{bmatrix}\right) =?;$$
   Can you find out these values if you are told that f is affine?

11. For the previous question, (a) assume that f is linear and find out $w \in \mathbb{R}^3$, such that $f(\mathbf{x}) = \mathbf{w}^T\mathbf{x}$; and (b) assume f is affine and find out $\mathbf{w}, \beta$ such that $f() = \mathbf{w}^T\mathbf{x} + \beta$.

12. Consider the weighted norm of vector $\mathbf{v}$, defined as,
   $$\|\mathbf{v}\|_{\mathbf{w}}^2 = \sum_{i=1}^n w_i v_i^2; \quad \mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix}$$
   Is this a valid norm?

13. Prove that the following modified version of the Cauchy-Bunyakovski-Schwartx Inequality is true.
   $$\left|\sum_{i=1}^n u_i v_i w_i\right| \leq \|\mathbf{u}\|_{\mathbf{w}} \|\mathbf{v}\|_{\mathbf{w}}$$

14. Consider a basis $B = \{\mathbf{b}_i\}_{i=1}^n$ of R^n. Let the vector $\mathbf{x}$ with the following representations in the standard and B basis.
   $$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \sum_{i=1}^n x_i e_i \quad \text{and} \quad \mathbf{x}_b = \begin{bmatrix} x_{b1} \\ x_{b2} \\ \vdots \\ x_{bn} \end{bmatrix} = \sum_{i=1}^n x_{bi} b_i$$
   Evaluate the $\|\mathbf{x}\|_2^2$ and $\|\mathbf{x}_b\|_2^2$. Determined what happens to $\|\mathbf{x}_b\|_2^2$ under the following conditions on the basis vectors:

   (a) $\|\mathbf{b}_i\| = 1, \forall i$
   (b) $\left\|\mathbf{b}_i^T\mathbf{b}_j\right\| = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$

15. Consider a set of measurements made from adult male subjects, where their height, weight and BMI (body moass index) were recorded as stored as vectors of length three; the first element is the height in cm, second is the weight in Kg, and the alst the the BMI. Consider the following four subjects,

$$\mathbf{s}_1 = \begin{bmatrix} 167 \\ 102 \\ 36.6 \end{bmatrix}; \ \mathbf{s}_2 = \begin{bmatrix} 180 \\ 87 \\ 26.9 \end{bmatrix}$$

$$\mathbf{s}_3 = \begin{bmatrix} 177 \\ 78 \\ 24.9 \end{bmatrix}; \ \mathbf{s}_4 = \begin{bmatrix} 152 \\ 76 \\ 32.9 \end{bmatrix}$$

You can use the distance between these vectors $\|\mathbf{s}_i - \mathbf{s}_j\|_2$ as a measure of the similiarity between the the four subjects. Generate a 4×4 table comparing the distance of each subject with respect to another subject; the diagonal elements of this table will be zero, and it will be symmetric about the main diagonal.

(a) Based on this table, how do the different subjects compare to each other?

(b) How do the similarities change if the height had been measured in m instead of cm? Can you explain this difference?

(c) Is there a way to fix this problem? Consider the weighted norm presented in one of the earlier problems.

$$\|x\|_{\mathbf{w}} = \left(w_1 x_1^2 + w_2 x_2^2 + \ldots + w_n x_n^2\right)^{\frac{1}{2}}$$

(d) What would be a good choice for $\mathbf{w}$ to address the problems with comparing distance between vectors due to change in units?

(e) Can the angle between two vectors be used as a measure of similarity between vectors? Does this suffer from the problem of $\|\mathbf{x}\|_2$?

Matrices

16. Elements of the matrix $\mathbf{C} \in \mathbb{R}^{m \times n}$ obtained as the product of two matrices $\mathbf{A} \in \mathbb{R}^{m \times p}$ and $\mathbf{B} \in \mathbb{R}^{p \times n}$ is given by,

$$c_{ij} = \sum_{k=1}^{p} a_{ik} b_{kj}$$

We had discussed four different ways to think of matrix multiplication. By algebraically manipulating the previous equation arrive at these four views (inner product view, column view, row view and outer product view)?

17. Given the matrices $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ -1 & 3 & -1 & -1 \\ 2 & -2 & 0 & 2 \\ 1 & 0 & -3 & 0 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} -1 & 1 \\ 0 & 1 \\ -1 & 1 \\ -0 & 1 \end{bmatrix}$ and $\mathbf{C} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Evaluate the following products.

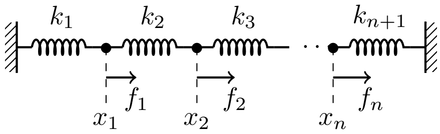
(a) $\mathbf{AB}$ (b) $\mathbf{A}^2\mathbf{B}$ (c) $\mathbf{C}\mathbf{B}^T\mathbf{A}$ (d) $\mathbf{C}^3$ (e) $\mathbf{ABC}$

18. Consider the following matrix,

$$\mathbf{A} = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} 0.1 & 0 \\ 0 & 0.9 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$$

Find out the expression for $\mathbf{A}_n = \mathbf{A}^n$. What is $\mathbf{A}_\infty = \lim_{n \to \infty} \mathbf{A}^n$?

19. Derive force and displacement relationship for a series of $n+1$ springs (with spring constants k_i) connected in a line. There are n nodes, with f_i and x_i representing the force applied and resulting displacement at the i^{th} node.



(a) Represent the relationship in the following form,

$$\mathbf{f} = \mathbf{Kx}; \ \mathbf{f} = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix}; \ \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

(b) What kind of a pattern does $\mathbf{K}$ have?

(c) Consider a specific case where $n = 5$ and $k = 1.5 N.m^{-1}$. What should be forces applied at the four nodes in order to displace the spring $\mathbf{x} = \begin{bmatrix} 0.5 \\ -0.5 \\ 0 \\ 0 \end{bmatrix} m$.

20. Prove that a matrix $\mathbf{M} \in \mathbb{R}^{n \times n}$ can always be written as a sum a symmetric matrix $\mathbf{S}$ and a skew-symmetric matrix $\mathbf{A}$.

$$\mathbf{M} = \mathbf{S} + \mathbf{A}, \ \mathbf{S}^T = \mathbf{S} \text{ and } \mathbf{A}^T = -\mathbf{A}$$

Does this property also hold for a complex matrix $\mathbf{M} \in \mathbb{C}^{n \times n}$?

21. The trace of a matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$ is defined as, $trace\,(\mathbf{A}) = \sum_{i=1}^{n} a_{ii}$. Prove the following,

(a) $trace\,(\mathbf{A})$ is a linear function of $\mathbf{A}$.

(b) $trace\,(\mathbf{AB}) = trace\,(\mathbf{BA})$

(c) $trace\,(\mathbf{A}^T\mathbf{A}) = 0 \implies \mathbf{A} = 0$

22. Prove that the rank of an outer product $\mathbf{xy}^T$ is 1, where $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and $\mathbf{x}, \mathbf{y} \neq \mathbf{0}$.

23. Is there a relationship between the space of solutions to the following two equations?

$$\mathbf{y}^T\mathbf{A} = \mathbf{c}^T \quad \text{and} \quad \mathbf{Ax} = \mathbf{b}$$

If so, how are they related?

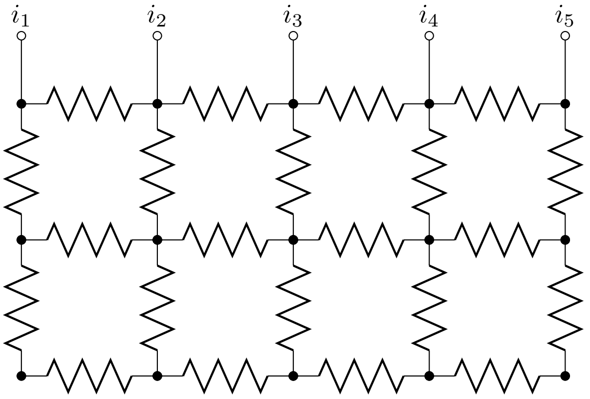
24. Consider an upper triangular and lower triangular matrices $\mathbf{U}$ and $\mathbf{L}$, respectively.

(a) Is the product of two upper triangular matrices $\mathbf{U}_1\mathbf{U}_2$ upper triangular?

(b) Is the product of two lower triangular matrices $\mathbf{L}_1\mathbf{L}_2$ upper triangular?

(c) What is the *trace* $(\mathbf{LU})$?

25. Consider the following electrical circuit with rectangular grid of resistors R. The input to this grid is a set of current injected at the top node as shown in the figure, such that $\sum_{k=1}^{5} i_k = 0$.



Express the relationship between the voltages at the different nodes (represented by • in the figure) and the net current flowing in/out of the node in the following form, $\mathbf{Gv} = \mathbf{i}$. Where, $\mathbf{G}$ is the conductance matrix, $\mathbf{v}$ is the vector of node voltages, and $\mathbf{i}$ is the vector representing the net current flow in/out of the different node.

26. Consider the following system.

$$\begin{bmatrix} 1 & 5 & 1 & 2 \\ 2 & -1 & 0 & 2 \\ 4 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix} \mathbf{x}_i = b_i$$

Solve the above equation using LU factorization for the following $\mathbf{b}_i$s.

$$\mathbf{b}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \ \mathbf{b}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \ \mathbf{b}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \ \mathbf{b}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Construct a matrix $\mathbf{X}$ using the four solutions $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ and $\mathbf{x}_4$ as its columns.

$$\mathbf{X} = \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \mathbf{x}_3 & \mathbf{x}_4 \end{bmatrix}$$

Find out $\mathbf{XA}$ and $\mathbf{AX}$,. Based on this what can you say about $\mathbf{X}$?

27. How many different reduced row echelon forms can a matrix $\mathbf{A} \in \mathbb{R}^{4\times 5}$ have? Hint: *Think in terms of basic and non-basic columns.*

28. Consider the system of equation, $\mathbf{Ax} = \mathbf{b}$, such that a matrix $\mathbf{A} \in \mathbb{R}^{m\times n}$, $\mathbf{x}, \mathbf{b} \in \mathbb{R}^n$. Are the following statements true? Explain your answer.

(a) $rank\,\mathbf{A} \leq \min(m, n)$

(b) The system is consistent if $rank\,\mathbf{A} = m$.

(c) The system has a unique solution if $rank\,\mathbf{A} = n$.

29. If two systems of linear equations are consistent, with augmented matices $[\mathbf{A}|\mathbf{b}]$ and $[\mathbf{A}|\mathbf{c}]$. Is $[\mathbf{A}|\mathbf{b} + \mathbf{c}]$ consistent?

30. If a matrix $\mathbf{A}$ has LU decomposition, such that $\mathbf{A} = \mathbf{LU}$. Demonstrate that it also has a LDU decomposition $\mathbf{A} = \mathbf{LD\hat{U}}$, where $\mathbf{D}$ is a diagonal matrix, and $\mathbf{\hat{U}}$ is upper triangular. What happens to the LU and LDU decompositions when a matrix $\mathbf{A} = \mathbf{A}^T$?

31. Write down a basis for the four fundamental subspaces of the following matrix,

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & -3 & 4 & -1 & 0 \\ 4 & 8 & 12 & -8 & 2 & 1 \\ 2 & 3 & 2 & 1 & -2 & 0 \\ -3 & -1 & 1 & -4 & 0 & -1 \\ 1 & -2 & -1 & 0 & 0 & 0 \end{bmatrix}$$

32. Consider a matrix $\mathbf{A} = \begin{bmatrix} 1 & 4 & 5 \\ 4 & 118 & 26 \\ 3 & 16 & 30 \end{bmatrix}$.

(a) Apply Gaussian elimination to simply this matrix into an upper-triangular matrix $\mathbf{U}$.

(b) What is the corresponding upper-triangular matrix $\tilde{\mathbf{U}}$ obtained by applying Gaussian elimination to $\mathbf{A}^T$?

(c) Could you have arrived at $\tilde{\mathbf{U}}$ without having to repeat the Gaussian elimination process on $\mathbf{A}^T$?

(d) Write down the LDU decompositions of $\mathbf{A}$ and $\mathbf{A}^T$.

33. Derive the inverse of the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

34. Consider the following upper-triangular matrix,

$$U = \begin{bmatrix} u_{11} & u_{12} & u_{13} & \cdots & u_{1n} \\ 0 & u_{22} & u_{23} & \cdots & u_{2n} \\ 0 & 0 & u_{33} & \cdots & u_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & u_{nn} \end{bmatrix}$$

where, $u_{ii} \neq 0, \ 1 \leq i \leq n$. Do the columns of this matrix form a linearly independent set? Explain your answer.

35. Verify that $\mathbf{A}$ and $\mathbf{B}$ are inverses of each other,

(a) $\mathbf{A} = \mathbf{I} - \mathbf{uv}^T$ and $\mathbf{B} = \mathbf{I} + \mathbf{uv}^T / \left(1 - \mathbf{v}^T\mathbf{u}\right)$

(b) $\mathbf{A} = \mathbf{C} - \mathbf{uv}^T$ and $\mathbf{B} = \mathbf{C}^{-1} + \mathbf{C}^{-1}\mathbf{uv}^T\mathbf{C}^{-1} / \left(1 - \mathbf{v}^T\mathbf{C}^{-1}\mathbf{u}\right)$

(c) $\mathbf{A} = \mathbf{I} - \mathbf{UV}$ and $\mathbf{B} = \mathbf{I}_n + \mathbf{U}\left(\mathbf{I}_m - \mathbf{VU}\right)^{-1}\mathbf{V}$

(d) $\mathbf{A} = \mathbf{C} - \mathbf{UD}^{-1}\mathbf{V}$ and $\mathbf{B} = \mathbf{A}^{-1} + \mathbf{A}^{-1}\mathbf{U}\left(\mathbf{D} - \mathbf{VA}^{-1}\mathbf{U}\right)^{-1}\mathbf{VA}^{-1}$

where, $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{n\times n}$, $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, $\mathbf{U} \in \mathbb{R}^{n\times m}$, $\mathbf{V} \in \mathbb{R}^{m\times n}$ and $\mathbf{D} \in \mathbb{R}^{m\times m}$.

References

[1] G. Strang *Introduction to linear algebra.* Wellesley-Cambridge Press Wellesley, MA, USA, 1993