# Linear Systems

## Linear Dynamical Systems: Transfer function view

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# Overview

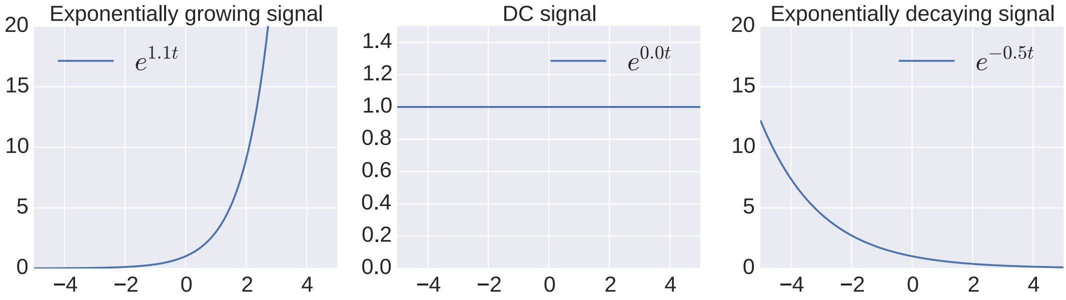
- We look at linear dynamical systems from a tranditional point of view in this lecture, which we choose to call the "transfer function" view.
- We will cover:
  - Definitions of some common signals we will encounter in the rest of the course.
  - Linear-time invariant (LTI) systems
  - Overview of Laplace and z-transforms
  - Impulse response and convolution
  - Transfer function and Frequency response

# Real Exponentials

**Continuous-time** version

$$x(t) = be^{at}$$

where, $a, b, t \in \mathbb{R}$. $b$ is the amplitude and $a$ is the exponential growth or decay rate.

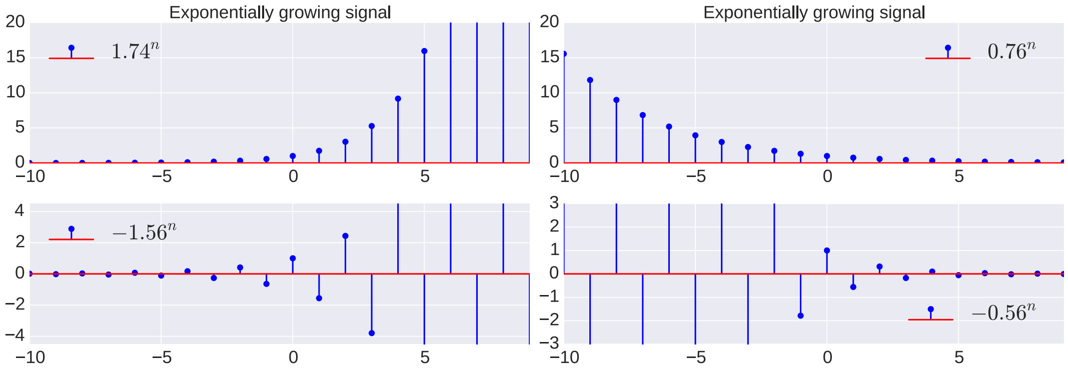


# Real Exponentials

**Discrete-time** version

$$x[n] = b\,(a)^n$$

where, $a, b \in \mathbb{R}$ and $n \in \mathbb{Z}$. $b$ is the amplitude and $a$ is the exponential growth or decay rate.

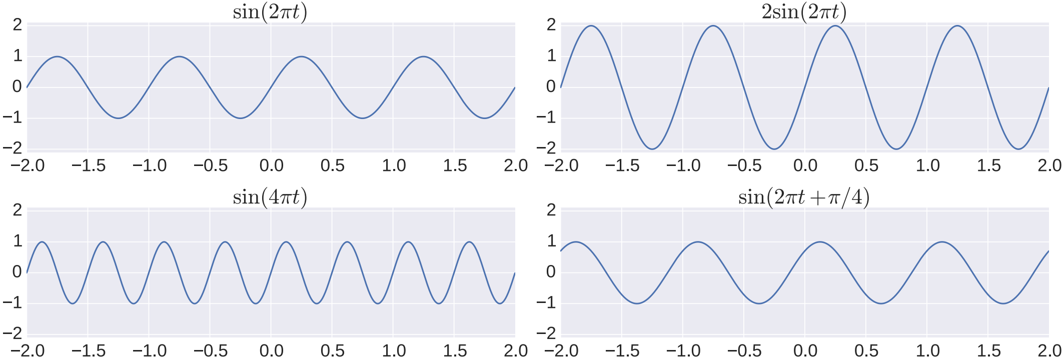


# Sinusoidal signals

**Continuous-time** version

$$x(t) = A \sin\left(\omega t + \phi\right)$$

where, $A$ is the amplitude, $\omega$ is the angular frequency $\left(\mathrm{rad.sec}^{-1}\right)$, and $\phi$ is the phase angle.

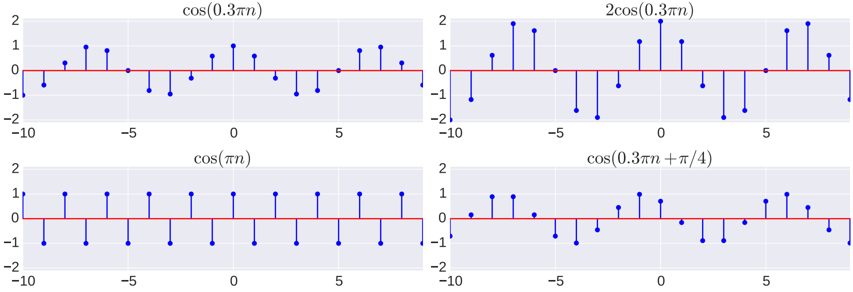


# Sinusoidal signals

**Discrete-time** version

$$x[n] = A \sin\left(\Omega n + \phi\right)$$

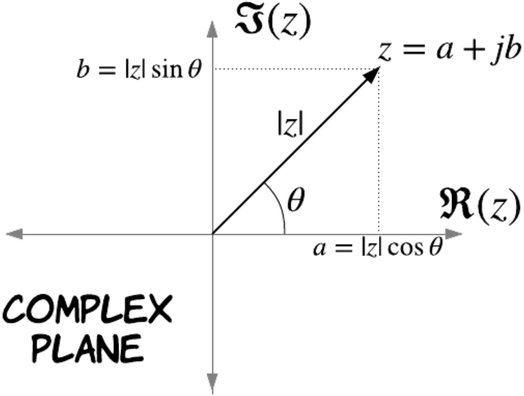
where, $A$ is the amplitude, $\Omega$ is the digital frequency $\left(\text{rad.sample}^{-1}\right)$, and $\phi$ is the phase angle.



# Sinusoidal signals

## Complex exponential representation of sinusoids

$$z = a + jb = |z|\, e^{j\theta} = |z| \cos\theta + j\,|z| \sin\theta$$

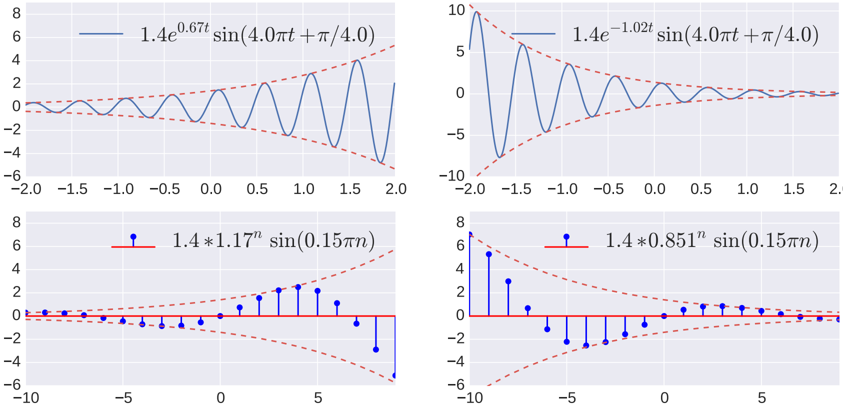


# Exponential sinusoids

**Continuous-time** version

Amplitude modulated sinusoids

$$x(t) = ae^{bt}\sin\left(\omega t + \phi\right), \quad a, b, \omega, \phi \in \mathbb{R}$$



# Impulse function $\delta(t)$, $\delta[n]$

## Dirac delta function $\delta(t)$

- This is **NOT** a conventional function.
- It makes sense only when it is used in an integral.
- It is not characterized by the exact values it takes as a function of the independent variable, but by the following important property.

$$\int_a^b \delta(t) = \begin{cases} 1, & 0 \in [a, b] \\ 0, & 0 \notin [a, b] \end{cases}$$

- It operates like a value selector.

$$\int_{-\infty}^{\infty} f(t)\delta(t) = f(0), \text{, where } f \text{ is continuous at } t = 0.$$

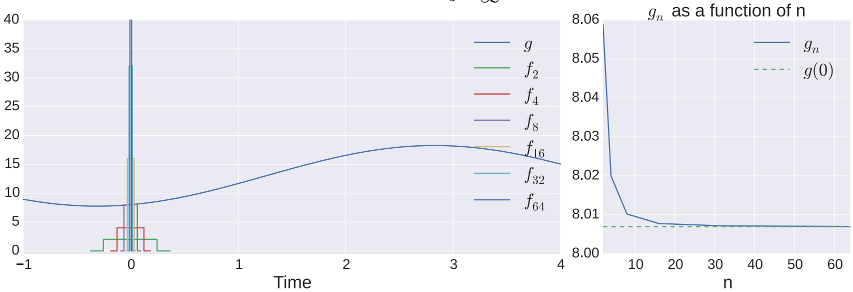
- Impulse function is a very useful theoretical tool for representing: point charges or masses, forces in instantaneous collisions, derivatives of jump discontinuities etc.

# Impulse function $\delta(t)$, $\delta[n]$

$\delta(t)$ can be understood through a limiting operation. Let $f_n(t) = \begin{cases} n, & -\frac{1}{2n} \leq t \leq \frac{1}{2n} \\ 0, & \text{Otherwise} \end{cases}$ and $\int_{-\infty}^{\infty} f_n(t)dt = 1$

$$\int_{-\infty}^{\infty} f_n(t)\, g(t)\, dt = \int_{-\frac{1}{2n}}^{\frac{1}{2n}} ng(t)\, dt = g_n = \lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(t)\, g(t)\, dt$$

$$\lim_{n\to\infty} g_n = g(0) = \int_{-\infty}^{\infty} g(t)\delta(t)dt$$

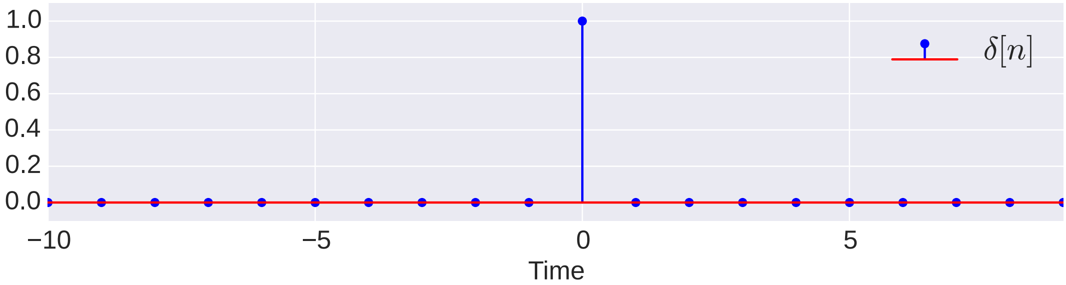


# Impulse function $\delta(t)$, $\delta[n]$

**Kronecker delta function or sequence $\delta[n]$**

- Very easy to understand unlike the continuous-time version.

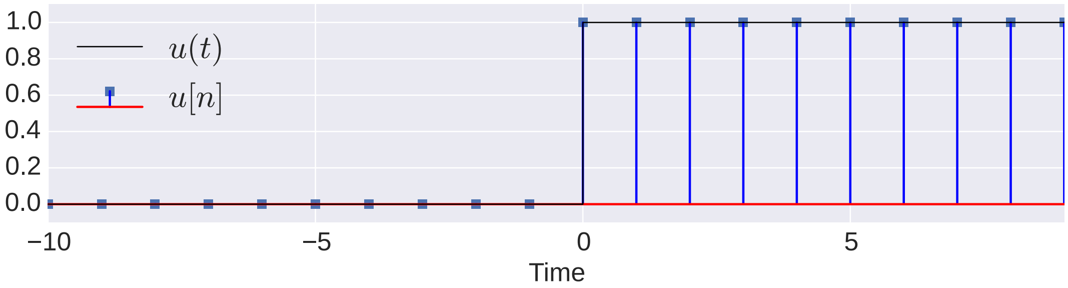
$$\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & \text{Otherwise} \end{cases}$$



# Step function $1(t), 1[n]$

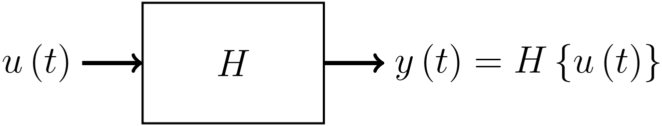
Definition of **continuous-time** unit step function,

$$1(t) = \begin{cases} 1 & t > 0 \\ \frac{1}{2} & t = 0 \\ 0 & t < 0 \end{cases}; \quad 1\left(t\right) = \int_{-\infty}^{t} \delta\left(t\right) dt; \quad \frac{d}{dt} 1\left(t\right) = \delta\left(t\right)$$




What is the corresponding definition of the discrete-time unit step function $1[n]$?

# Linear System

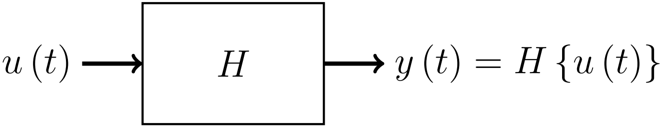



- Behavior of dynamic systems can be described mathematically through differential equations, or difference equations in the case of discrete-time systems.
- A system is **linear** if,

$$y_1(t) = H\{u_1(t)\} \text{ and } y_2(t) = H\{u_2(t)\}$$

$$H\{a_1x_1(t) + a_2u_2(t)\} = a_1H\{u_1(t)\} + a_2H\{u_2(t)\} = a_1y(t) + a_2y_2(t)$$

# Time-Invariant System




- A system is **time-invariant** if,

$$y(t) = H\{u(t)\} \implies H\{u(t-\tau)\} = y(t-\tau)$$

- Characteristics of the system do not change with time. Time-shifted inputs produce correspondingly time-shifted output.

# Linear Time-Invariant System

$$u(t) \longrightarrow \boxed{H} \longrightarrow y(t) = H\{u(t)\}$$

- LTI systems: both **linear** and **time-invariant**. These are described through constant coefficient linear differential (or difference) equations.
- **Continuous-time**:

$$\frac{d^n}{dt^n}y(t) + a_1\frac{d^{n-1}}{dt^{n-1}}y(t) + \ldots + a_n y(t) = u(t) + b_1\frac{d}{dt}u(t) + \ldots + b_m\frac{d^m}{dt^m}y(t)$$

- **Discrete-time**:

$$y[k-n] + a_1 y[k-n] + \ldots + a_n y[k-n] = u[k] + b_1 y[k-1] + \ldots + b_m u[k-m]$$

# Unilateral Laplace Transform

- Unilateral Laplace transform can be used for solving these linear-constatint coefficient differential equations.
- Consider a time-domain signal $x(t)$, such that $x(t) = 0, \forall t < 0$.

$$\mathcal{L}\left\{x(t)\right\} = X(s) = \int_{0^-}^{\infty} x(t)\, e^{-st} dt, \quad s = \sigma + j\omega$$

  wherem $X(s)$ exists only for specific values of $s$, which is called the *region of convergence*.
- A time-domain function is converted to a function in the $s$-domain.
- Unilateral Laplace transform transform $X(s)$ provides a different way to look at the signal $x(t)$.
- The inverse of a unilateral Laplace transform is often not obtained analytically, but by using a table of Laplace transform pairs $(x(t), X(s))$.

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- The inverse of a unilateral Laplace transform is often not obtained analytically, but by using a table of Laplace transform pairs $(x(t), X(s))$.

Evaluate the unilateral Laplace transform of the following: (i) $e^{at} \times 1(t)$; (ii) $e^{(a+jb)t} \times 1(t)$; (iii) $1(t)$; (iv) $\delta(t)$; (v) $\sin \omega t \times 1(t)$

# Unilateral Laplace Transform

- An important property of the unilateral Laplace transform that will be useful in solving differential equations is the Laplace transform on $x'(t)$,

$$\mathcal{L}\left\{x'(t)\right\} = sX(s) - x\left(0^-\right)$$

$$\mathcal{L}\left\{x''(t)\right\} = s^2X(s) - sx\left(0^-\right) - x'\left(0^-\right)$$

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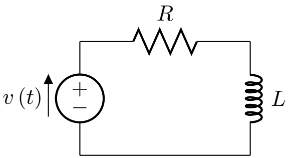
Derive and solve the differential equation representing the voltage-current relationship.




$u(t) = v(t), y(t) = v_L(t)$

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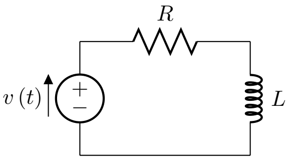
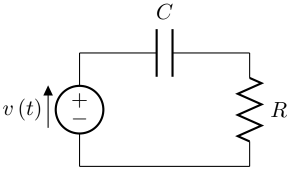



$u(t) = v(t), y(t) = v_L(t)$




$u(t) = v(t), y(t) = v_R(t)$

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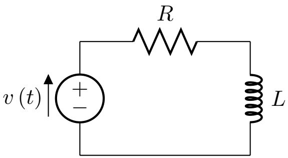
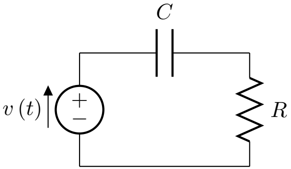



$u(t) = v(t), y(t) = v_L(t)$

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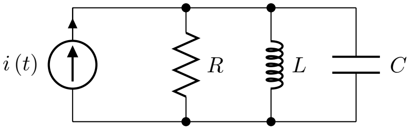


$u(t) = v(t), y(t) = v(t)$

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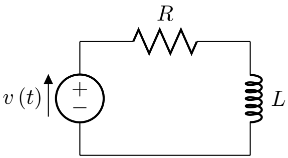


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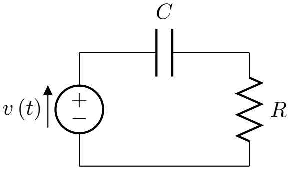


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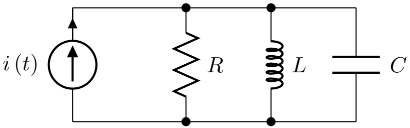



$u(t) = v(t), y(t) = v(t)$

What is the output for each of the three systems when $v(t) = v_0 \times 1(t)$?

# Impulse Response

- In case we do not know the exact composition of the system, and only have access to the input and output ports, i.e. we can manipulate the input and observe (measure) the output. How can we characterize the system in this case?

$$u(t) \longrightarrow \boxed{H} \longrightarrow y(t) = H\left\{u(t)\right\}$$

- When the system $H$ is (or approximately) LTI, there is a nice way to characterize the system.
- If we know the system output $\left\{w_i(t)\right\}_{i=1}^{n}$ for a set of signals, $\left\{v_i(t)\right\}_{i=1}^{n}$, then system output for an arbitrary input $u(t) = \sum_{i=1}^{n} a_i v_i(t - \tau_i)$ is,

$$y(t) = H\left\{u(t)\right\} = H\left\{\sum_{i=1}^{n} a_i v_i(t - \tau_i)\right\} = \sum_{i=1}^{n} a_i w_i(t - \tau_i)$$

# Impulse Response

$$u(t) \rightarrow \boxed{H} \rightarrow y(t) = H\{u(t)\}$$

- Ideally, $V = \{v_i(t)\}_{i=1}^{n}$ is chosen to allows us to represents a wide range of signals using $V$.
- The are two popular choices: **(a)** $\delta(t)$; and **(b)** $e^{st}$.
- A signal $u(t)$ can be represented in the following form,

$$u(t) = \int_{-\infty}^{\infty} u(\tau)\,\delta(\tau - t)\,d\tau$$

The representation arises as the limit of a sequence of functions are are approximated by rectangular pulse,

$$u(t) = \lim_{\Delta \to 0} \sum_{n=-\infty}^{\infty} u(t_n) \left( \frac{1(t - t_n) - 1(t - t_{n+1})}{\Delta} \right) \Delta; \;\; n\Delta \leq t < (n+1)\Delta$$

# Impulse Response

$$u(t) \rightarrow \boxed{H} \rightarrow y(t) = H\{u(t)\}$$

- The output of the system to $u(t)$ is given by,

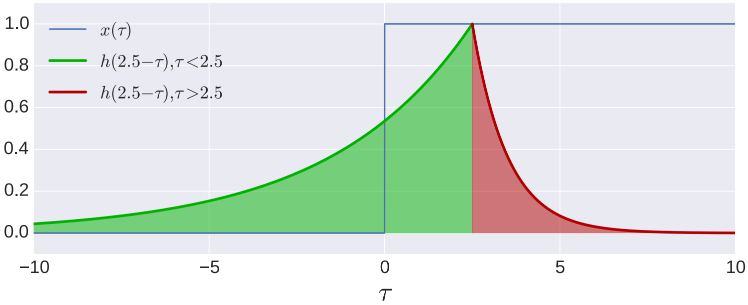
$$y(t) = H\left\{\int_{-\infty}^{\infty} u(\tau)\,\delta(\tau - t)\,d\tau\right\} = \int_{-\infty}^{\infty} u(\tau)\,H\{\delta(\tau - t)\}\,d\tau$$

$$y(t) = \int_{-\infty}^{\infty} u(\tau)\,h(\tau - t)\,d\tau = u(t) * h(t) = h(t) * u(t)$$

$$h(t) = H\{\delta(t)\}$$

- $h(t)$ is the *impulse response* of the LTI system $H$. **Note:** The system must be at rest when the impulse is applied at the input, i.e. the output must be zero $y(t) = 0, \forall t < 0$
- The output of $H$ to an input $u(t)$ is obtained through the *convolution* between $u(t)$ and $h(t)$.

# Convolution Integral




$h(t) = \begin{cases} e^{-t} & t < 0 \\ e^{-0.25t} & t \geq 0 \end{cases}$ and $x(t) = 1(t)$. $h\left(t\right)$ acts as a weighting function. It evlauates the present output by weighting the present, past and future input values.

$$\begin{cases} h(t), \forall t < 0 & \text{Weightage for the future} \\ h(0) & \text{Weightage for the present} \\ h(t), \forall t > 0 & \text{Weightage for the past} \end{cases}$$

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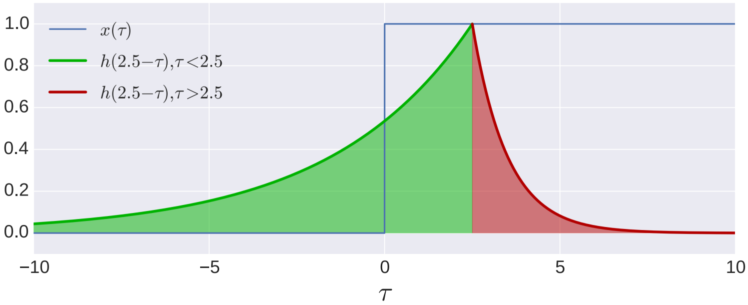


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# Convolution Integral

What is the output $y(t)$ of a system $H$ with the following impulse response $h(t)$ and input $u(t)$?

1. $h(t) = e^{-at} 1(t)$
   - $u(t) = 1(t)$
   - $u(t) = \delta(t-1) + 0.5\delta(t-5)$

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2. $h(t) = 1(t) - 1(t-1)$
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3. $h(t) = e^{-at} 1(t) + e^{bt} 1(-t)$
   - $u(t) = 1(t)$
   - $u(t) = \delta(t - T_2) + \delta(t - T_3)$

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   - $u(t) = 1(t)$
   - $u(t) = \delta(t-T_2) + \delta(t-T_3)$
4. $h(t) = 1(t)$
   - $u(t) = 1(t)$
   - $u(t) = e^{-at}1(t)$

# Transfer function

- **Output of a LTI system to** $e^{st}$:

$$y(t) = h(t) * e^{st} = \int_{-\infty}^{\infty} h(\tau)\, e^{s(t-\tau)} d\tau = \left(\int_{-\infty}^{\infty} h(\tau)\, e^{-s\tau} d\tau\right) e^{st}$$

$$y(t) = H(s)\, e^{st} \implies H(s) = \frac{\text{Response to } e^{st}}{e^{st}}$$

- $e^{st}$ are the *eigenfunctions* of LTI systems, and their corresponding *eigenvalue* is $H(s)$. For these functions, convolution $h(t) * e^{st}$ is simplified to multiplication $H(s) \times e^{st}$.
- $H(s)$ is called the transfer function of the LTI system. This is the *bilateral* Laplace transform of the impulse response $h(t)$. This becomes the unilateral Laplace transform when $h(t) = 0, \forall t < 0$.

# Transfer function

- Laplace transform the convolution integral: $h$

$$y(t) = h(t) * e^{st} = \int_{-\infty}^{\infty} h(\tau)\, e^{s(t-\tau)} d\tau = \left( \int_{-\infty}^{\infty} h(\tau)\, e^{-s\tau} d\tau \right) e^{st}$$

  If we only deal with causal systems,

$$y(t) = \left( \int_{0}^{\infty} h(\tau)\, e^{-s\tau} d\tau \right) e^{st} = H(s)\, e^{st} \implies H(s) = \frac{\text{Response to } e^{st}}{e^{st}}$$

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# Transfer function

Derive the transfer function and impulse response for the following systems.

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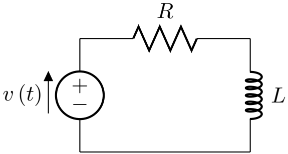


$u(t) = v(t), y(t) = v_L(t)$

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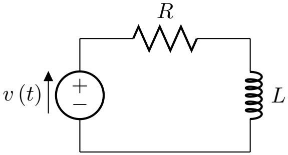
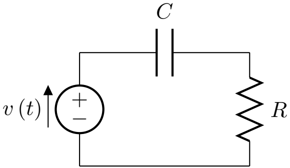


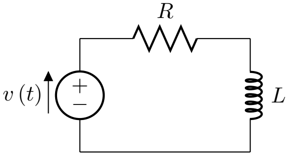

$u(t) = v(t)\,, y(t) = v_L(t)$



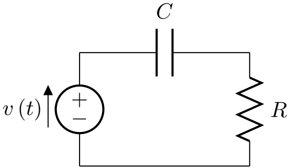

$u(t) = v(t)\,, y(t) = v_R(t)$

# Transfer function

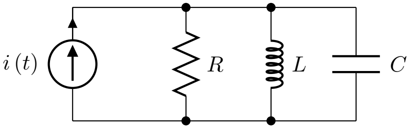
Derive the transfer function and impulse response for the following systems.




$u(t) = v(t), y(t) = v_L(t)$




$u(t) = v(t), y(t) = v_R(t)$




$u(t) = v(t), y(t) = v(t)$

S

# Transfer function

- In general, the Laplace transform of a general linear constant coefficient different equation with zero initial conditions would be of the following form.

$$s^n Y(s) + a_1 s^{n-1} Y(s) + \ldots + a_n Y(s) = U(s) + b_1 s U(s) + \ldots + b_m s^m U(s)$$

$$\frac{Y(s)}{U(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \ldots + b_1 s + 1}{s^n + a_1 s^{n-1} + \ldots + a_{n-1} s + a_n} = \frac{B(s)}{A(s)} = H(s)$$

$$H(s) = b_m \frac{(s - z_1)(s - z_2) \cdots (s - z_m)}{(s - p_1)(s - p_2) \cdots (s - p_n)}$$

- The roots of $B(s)$ and $A(s)$ are the *zeros* and *poles* of the system.
- Poles of the system determine the stability of the system, in the BIBO sense – Bounded Input, Bounded Output.
- $H$ is stable in BIBO sense, if

$$|u(t)| < M_u < \infty \implies |y(t)| = H\{u(t)\} < M_y < \infty$$

# z transform

- The discrete-time equivalent of the Laplace transform is the $z$- transform. The unilateral z-transform is defined as,

$$X(z) = \mathcal{Z}\{x[k]\} = \sum_{k=0}^{\infty} x[k]\, z^{-k}$$

$X(z)$ exists only for specific set of vaues of $z$, which is called the *region of convergence*.

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Evaluate the unilateral z transform of the following: (i) $a^k \times 1\left[k\right]$; (ii) $\cos \Omega k \times 1\left[k\right]$; (iii) $1\left[k\right]$; (iv) $\delta\left[k\right]$

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- Unilateral z-transform can be used for solving linear constant coefficient difference equations.
- If $x[k] \overset{\mathcal{Z}}{\longleftrightarrow} X(z)$, then

$$\mathcal{Z}\{x[k-1]\} = zX(z) + x[-1]$$

# Impulse Response

$$u[k] \longrightarrow \boxed{\;H\;} \longrightarrow y[k] = H\{u[k]\}$$

- Any discrete-time signal can be represented as a linear combination of time-shifted impulse sequences,

$$x[k] = \sum_{l=-\infty}^{\infty} x[l]\,\delta[k-l]$$

- The output of the system $H$ to any input sequence is given by,

$$y[k] = H\{u[k]\} = \sum_{l=-\infty}^{\infty} x[l]\,H\{\delta[k-l]\} = \sum_{l=-\infty}^{\infty} x[l]\,h[k-l]$$

$$y[k] = x[k] * h[k]$$

This is the *convolution sum*, and $h[k]$ is the *impulse response* of the system $H$.