

# Linear Systems

## Linear Dynamical Systems: State Space View

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## States of a system

- ▶ A characteristic feature of most dynamical systems is their memory, i.e. the system's response (or output) depends on the present and past values of its input; We are only deal with causal systems here.
- ▶ If we get interested in a system at some arbitrary time  $t_0$ , we might not have a complete record of the past input to the system.
- ▶ The idea of a *state* deals with this problem.
- ▶ **Defintion:** *The state  $\mathbf{x}(t_0)$  of a system is the information at time  $t_0$ , which along with the input  $u(t)$ ,  $\forall t \geq t_0$  can be used to uniquely determine the system output  $y(t)$ ,  $\forall t \geq t_0$ .*
- ▶ The state  $\mathbf{x}(t_0)$  summarizes all the information ones needs to know about the system's past in order to predict its future.
- ▶ Examples of states of a system:
  - ▶ Position and velocity of a mass acted up by a force.
  - ▶ Capacitor voltage and inductor current of a electrical network.
  - ▶ Initial conditions of a differential equation describing a system.

## States of a system

In general, the state and the input will determine the system's output.

$$\left. \begin{array}{l} \mathbf{x}(t_0) \\ u(t), \forall t \geq t_0 \end{array} \right\} \rightarrow y(t), \forall t \geq t_0$$

In the case of a linear system, if

$$\left. \begin{array}{l} \mathbf{x}_1(t_0) \\ u_1(t), \forall t \geq t_0 \end{array} \right\} \rightarrow y_1(t), \forall t \geq t_0 \quad \text{and} \quad \left. \begin{array}{l} \mathbf{x}_2(t_0) \\ u_2(t), \forall t \geq t_0 \end{array} \right\} \rightarrow y_2(t), \forall t \geq t_0$$

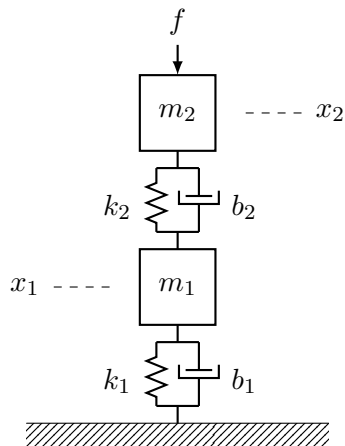
$$\Rightarrow \left. \begin{array}{l} a_1 \mathbf{x}_1(t_0) + a_2 \mathbf{x}_2(t_0) \\ a_1 u_1(t) + a_2 u_2(t), \forall t \geq t_0 \end{array} \right\} \rightarrow a_1 y_1(t) + a_2 y_2(t), \forall t \geq t_0$$

For a linear system, knowing the system output to the states and the input will allow us to know the complete output.

- ▶ **Zero State Response:**  $\mathbf{x}(t_0) = \mathbf{0}; u(t), t \geq t_0 \} \rightarrow y_{zs}(t), \forall t \geq t_0$
- ▶ **Zero Input Response:**  $\mathbf{x}(t_0); u(t) = 0, t \geq t_0 \} \rightarrow y_{zi}(t), \forall t \geq t_0$

$y_{zs}(t) + y_{zi}(t)$  gives the complete response.

## States of a system



- ▶ In the system shown, the input  $u(t)$  is the force  $f(t)$  applied to  $m_2$ , and the output  $y(t)$  is the position of  $m_2$  ( $x_2(t)$ ).
- ▶  $y(t)$  depends not only on  $f(t)$ , but also on:  $\dot{x}_2(t)$ ,  $x_1(t)$  and  $\dot{x}_1(t)$ .
- ▶ For the same input  $u$ , we can obtain different output  $y$  if the starting states are different. Thus, knowledge of the states are essential for correctly predicting the behavior of the system.
- ▶ In general, the dynamics of a system in terms of its states, input(s) and output(s) is mathematically represented as,

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t)) \rightarrow \text{State Equation} \\ \mathbf{y}(t) = \mathbf{g}(\mathbf{x}(t), \mathbf{u}(t)) \rightarrow \text{Measurement Equation} \end{cases}$$

where,  $\mathbf{x} \in \mathbb{R}^n$ ,  $\mathbf{u} \in \mathbb{R}^p$ , and  $\mathbf{y} \in \mathbb{R}^m$ , and  $t \in \mathbb{R}$  represents times.

## State space representation of linear systems

- ▶ In the case of a linear system, the equations representing the dynamics takes a simpler form,

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t) \mathbf{x}(t) + \mathbf{B}(t) \mathbf{u}(t)$$

$$\mathbf{y}(t) = \mathbf{C}(t) \mathbf{x}(t) + \mathbf{D}(t) \mathbf{u}(t)$$

where,

- ▶  $\mathbf{A}(t) \in \mathbb{R}^{n \times n}$  is the *system* matrix.
- ▶  $\mathbf{B}(t) \in \mathbb{R}^{n \times p}$  is the *input* matrix.
- ▶  $\mathbf{C}(t) \in \mathbb{R}^{m \times n}$  is the *output* matrix.
- ▶  $\mathbf{D}(t) \in \mathbb{R}^{m \times p}$  is the *feedforward* matrix.
- ▶ In the case of time-invariant system, the matrices are constant.

$$\dot{\mathbf{x}}(t) = \mathbf{A} \mathbf{x}(t) + \mathbf{B} \mathbf{u}(t)$$

$$\mathbf{y}(t) = \mathbf{C} \mathbf{x}(t) + \mathbf{D} \mathbf{u}(t)$$

- ▶ These two equations represent how the states and the measured outputs of the system are affected by the current states and inputs. The individual terms in these matrices indicate how a particular state/input affects another state/output.

## State space representation of linear systems

Consider a LTI system represented by the following differential equation,

$$\ddot{y}(t) + a_1 \dot{y}(t) + a_2 y(t) = u(t)$$

We can obtain a state space representation of this differential equation by choosing two states,  $x_1(t) = y(t)$  and  $x_2(t) = \dot{y}(t)$ ,

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} \dot{y}(t) \\ \ddot{y}(t) \end{bmatrix} = \begin{bmatrix} x_2(t) \\ -a_2 x_1(t) - a_1 x_2(t) + u(t) \end{bmatrix}$$

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 \\ -a_2 & -a_1 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mathbf{u}(t)$$

$$\mathbf{y}(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}(t)$$

The choice of state for a system is not unique. If for a linear system,  $\mathbf{x}(t)$  is a state, then so is  $\hat{\mathbf{x}}(t) = \mathbf{T}\mathbf{x}(t)$ , where  $\mathbf{T}$  is invertible.

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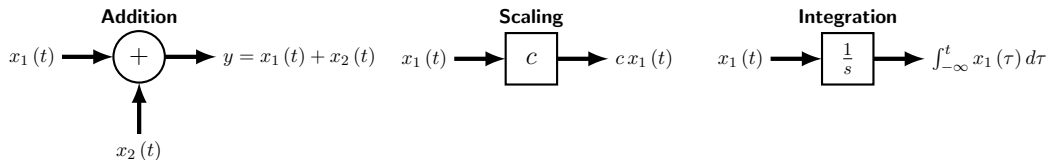
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If  $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$  are the different matrices associated with a LTI system with state  $\mathbf{x}(t)$ . Derive the matrices when  $\mathbf{T}\mathbf{x}(t)$  is chosen as the state.

## Block diagram representation of linear systems

- ▶ Pictorial representation of different components of a system and their inter-connections can provide insights into the behavior of the system.
- ▶ Helps breakdown a complex system into a set of simpler systems connected to each other.
- ▶ Linear systems in general can be built using three basic elements:



Represent the following linear differential equations using the three elementary block,

- ▶  $\dot{y}(t) + 0.1y(t) = u(t)$
- ▶  $\ddot{y}(t) + 2\dot{y}(t) + 5y(t) = u(t) - 2\ddot{u}(t)$



# State space representation of discrete-time linear systems

- ▶ Discrete-time linear system,

$$\mathbf{x}[k+1] = \mathbf{A}[k] \mathbf{x}[k] + \mathbf{B}[k] \mathbf{u}[k]$$

$$\mathbf{y}[k] = \mathbf{C}[k] \mathbf{x}[k] + \mathbf{D}[k] \mathbf{u}[k]$$

where,  $k \in \mathbb{Z}$  correspond to time index.

- ▶  $\mathbf{A}[k] \in \mathbb{R}^{n \times n}$  is the *system* matrix.
  - ▶  $\mathbf{B}[k] \in \mathbb{R}^{n \times p}$  is the *input* matrix.
  - ▶  $\mathbf{C}[k] \in \mathbb{R}^{m \times n}$  is the *output* matrix.
  - ▶  $\mathbf{D}[k] \in \mathbb{R}^{m \times p}$  is the *feedforward* matrix.
- ▶ In the case of time-invariant system, the matrices are constant.

$$\mathbf{x}[k+1] = \mathbf{A}\mathbf{x}[k] + \mathbf{B}\mathbf{u}[k]$$

$$\mathbf{y}[k] = \mathbf{C}\mathbf{x}[k] + \mathbf{D}\mathbf{u}[k]$$

# State space representation of discrete-time linear systems

Consider a LTI system represented by the following differential equation,

$$y[k] + a_1 y[k-1] + a_2 y[k-2] = u[k]$$

We can obtain a state space representation of this difference equation by choosing two states,  $x_1[k] = y[k-1]$  and  $x_2[k] = y[k-2]$ ,

$$\mathbf{x}[k+1] = \begin{bmatrix} x_1[k+1] \\ x_2[k+1] \end{bmatrix} = \begin{bmatrix} y[k] \\ y[k-1] \end{bmatrix} = \begin{bmatrix} -a_1 x_1[k] - a_2 x_2[k] + u[k] \\ x_1[k] \end{bmatrix}$$

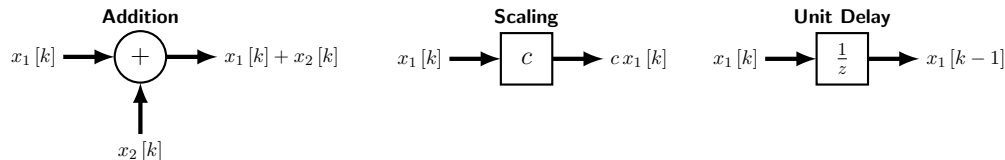
$$\mathbf{x}[k+1] = \begin{bmatrix} -a_1 & -a_2 \\ 1 & 0 \end{bmatrix} \mathbf{x}[k] + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \mathbf{u}[k]$$

$$\mathbf{y}[k] = \begin{bmatrix} -a_1 & -a_2 \end{bmatrix} \mathbf{x}[k] + \begin{bmatrix} 1 \end{bmatrix} \mathbf{u}[k]$$

The choice of state for a system is not unique. If for a linear system,  $\mathbf{x}[k]$  is a state, then so is  $\hat{\mathbf{x}}[k] = \mathbf{T}\mathbf{x}[k]$ , where  $\mathbf{T}$  is invertible.

## Block diagram representation of discrete-time linear systems

- Discrete-time linear systems in general can be built using three basic elements:



Represent the following linear differential equations using the three elementary block,

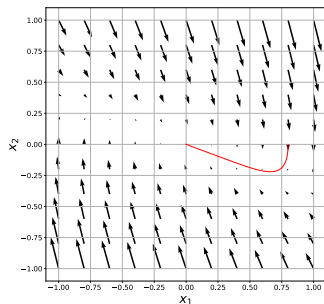
- $y[k] + 0.1y[k] = u[k]$
- $y[k - 2] + 2y[k - 1] + 5y[k] = u[k] - 2u[k - 2]$
- $y[k] = \frac{1}{5} \sum_{l=0}^4 u[k - l]$

# State space visualization

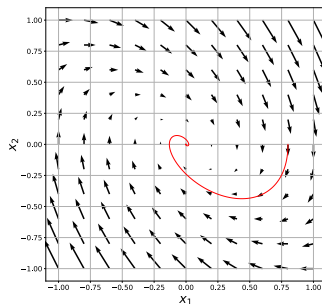
For systems with two states, we can visualize the state space trajectories of the system to gain better understanding of the system dynamics.

The state dynamics of a mass, spring and damper system is given by the following equation,

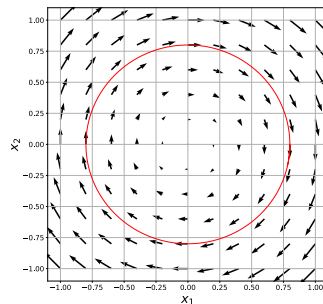
$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} \mathbf{u}(t)$$



$$m = 1, b = 3, k = 1$$



$$m = 1, b = 1, k = 1$$



$$m = 1, b = 0, k = 1$$