CSE 431/531: Algorithm Analysis and Design	Fall 2022
Homework 6	
Instructor: Shi Li	**Deadline: 12/11/2022**

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Problems	1	2	3	Total
Max. Score	24	24	32	80
Your Score				

Problem 1 For each of the following 4problems, state (i) whether the problem is known to be in NP, and (ii) whether the problem is known to be in Co-NP. For problems (1b) and (1d), if your answer for (i) (or (ii)) is yes, you need to give the certificate and the certifier that establishes that the problem is in NP (or Co-NP).

(1a) Given a graph $G = (V, E)$ and $s \leq |V|$, the problem asks whether G contains an independent set of size s.

(1b) Given two circuits C_1 and C_2, each with m input variables $z_1, z_2, \cdots, z_m$, decide if the two circuits compute the same function. That is, whether C_1 and C_2 give the same output for every boolean assignment of z-variables.

(1c) Given a graph $G = (V, E)$, decide if G is 3-colorable.

(1d) Given a graph $G = (V, E)$, decide if G is 2-colorable.

Problem 2 Let NPC be the set of NP-Complete problems. Prove the following statements:

- If P $\neq$ NP, then P $\cap$ NPC $= \emptyset$.
- If P $=$ NP then P $=$ Co-NP.

Problem 3 In the class, you learned that we can solve shortest path problem on graphs with negative weights, with one caveat: If the graph contains a negative cycle, we can not force the algorithm to find the *shortest simple path* from s to t (a simple path is a path that does not visit a vertex twice).

In this problem, you need to prove that HP $\leq_P$ Shortest-Simple-Path, where Shortest-Simple-Path is the shortest simple path problem, and HP is the Hamiltonian path problem. (Though we did not prove this in class, it is known that HP is NP-complete. So this suggests Shortest-Simple-Path is also NP-complete.)

You can assume the Shortest-Simple-Path problem is a decision problem: we are given a graph $G = (V, E)$ with (possibly negative) edge weights $w : E \to \mathbb{R}$, two different vertices $s, t \in V$ and a threshold $L \in \mathbb{R}$. We need to decide if the shortest simple path from s to t in G has weight at most L or not.

Also, recall that in the HP problem, we are given a graph $G = (V, E)$ and two different vertices s and t, and we need to output if there is a Hamiltonian path between s and t in G or not.

Problem 1: a) (i) NP : True (ii) Co-NP: False

b) (i) NP: False (ii) Co-NP: True

Certificate for Co-NP: one instance of the boolean assignment of z-variables that will make G_1 and G_2 give different output.

Certifier for Co-NP: given the above boolean assignment of z-variables, the certifier checks whether two circuits give the same output.

c) (i) NP : True (ii) Co-NP : False

d) (i) NP: True (ii) Co-NP = True

This problem can be solved in polynormial time since it's identical to the graph bipartite problem. Thus it's both in NP and Co-NP and do not need a certificate. The certifier just take the graph and check if it is 2-colorable or not in polynormial time.

Problem 2: 1) Assume if $P \cap NPC \neq \phi$, then there exists a problem both in P and NPC. In the NPC definition, every problem in NP can be polynomial-time reduced to that problem and thus can be solved in polynomial time. In this case $P = NP$

So if $P \neq NP$, $P \cap NPC = \phi$

2) Let X be any problem in Co-NP. In this case $\bar{X} \in NP$. Because of $P = NP$, $\bar{X} \in P$. Therefore $\bar{X}$ can be solved with a polynormial time algorithm, and with this algorithm, we can also solve X in polynormial time, thus $X \in P$

Since at the beginning we have $X \in$ Co-NP, we can have the conclusion that $P =$ Co-NP.

Problem 3 For a given graph $G=(V,E)$, we need to decide whether there exist a path that passes through every vertex in G exactly once. (HP)

We construct a graph G' from G with the following properties:

- G' has the same set of vertices as G
- For each undirected edge in G, make two directed edges of weight -1

Suppose there is a simple path of weight $-(n-1)$ from s to t in G'. Since every edges in G' have weight -1, this path must contain $n-1$ edges. Thus this corresponds to a Hamiltonian path in the graph G.

Conversely, if G has a Hamiltonian path, then we can simply take this path and use it as the shortest path from s to t in G'. Since the path has $n-1$ edges and the weight of each edge is $-n$, the sum of the weights of the edges on the path is minimized, and thus the path is the shortest path from s to t in G'

Therefore, we have shown that a Hamiltonian path exists in G if and only if there exists a shortest simple path from s to t in G' with weight at most $-(n-1)$.
This means that $HP \leq_P$ Shortest-Simple-Path.