with

$$\begin{aligned}[L^2, L_z] &= [L_x^2 + L_y^2 + L_z^2, L_z] \\ &= L_x[L_x, L_z] + [L_x, L_z]L_x \\ &\qquad + L_y[L_y, L_z] + [L_y, L_z]L_y + (L_z^2, L_z) \\ &= L_x(-i\hbar)L_y + (-i\hbar)L_yL_x + L_y(i\hbar)L_x + (i\hbar)L_xL_y + 0 \\ &= 0\end{aligned}$$

Thus operators L^2 and L_z commute and, continuing in the same way, it can be shown that

$$[L^2, L_x] = [L^2, L_y] = [L^2, L_x] = 0 \tag{19.29}$$

Eigenvalues of the angular momentum operators

We will now use the commutation relations for L^2 and its components to find the eigenvalues of L^2 and L_z, without reference to any specific wavefunction. In other words, the eigenvalues of the operators follow from the structure of their commutators. There is nothing particular about L_z,and L_x or L_y could equally well have been chosen, though, in general, it is not possible to find states that are simultaneously eigenstates of two or more of L_x, L_y and L_z.

To help with the calculation, it is convenient to define the two operators

$$U \equiv L_x + iL_y \quad \text{and} \quad D \equiv L_x iL_y.$$

These operators are not Hermitian; they are in fact Hermitian conjugates, in that $\mathrm{U}^\dagger$=D and $\mathrm{D}^\dagger = U$, but they do not represent measurable physical quantities. We first note their multiplication and commutation properties:

$$\begin{aligned}UD &= (L_x + iL_y)(L_x - iL_y) = L_x^2 + L_y^2 + i[L_y, L_x] \\ &= L^2 L_z^2 + hL_z,\end{aligned} \tag{19.30}$$

$$\begin{aligned}DU &= (L_x - iL_y)(L_x + iL_y) = L_x^2 + L_y^2 - i[L_y, L_x] \\ &= L^2 - L_z^2 - \hbar L_z,\end{aligned} \tag{19.31}$$

$$[L_z, U] = [L_z, L_x] + i[L_z, L_y] = i\hbar L_y + \hbar L_x = \hbar U, \tag{19.32}$$

$$[L_z, D] = [L_z, L_x]i[L_z, L_y] = i\hbar L_y - \hbar L_x = -\hbar D \tag{19.33}$$

In the same way as was shown for matrices, it can be demonstrated that if two operators commute they have a common set of eigenstates. Since $L^2$and L_z commute they possess such a set; let one of the set be $|\psi\rangle$with

$$L^2|\psi\rangle = a|\psi\rangle \quad \text{and} \quad L_z|\psi\rangle = b|\psi\rangle.$$

Now consider the state $|\psi'\rangle = \mathrm{U}|\psi\rangle$and the actions of L^2 andL_z upon it.