Assignment 6

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Problem 1

With a utility function $U(x) = x - \frac{\alpha}{2}x^2$ and $x \sim \mathcal{N}(\mu, \sigma^2)$ we calculate as follows:

$$\mathbb{E}[U(x)] = \mathbb{E}[x - \frac{\alpha}{2}x^2] = \mathbb{E}[x] - \frac{\alpha}{2}\mathbb{E}[x^2] = \mu - \frac{\alpha}{2}(\mu^2 + \sigma^2) \tag{1}$$

If $x_{CE} = U^{-1}(\mathbb{E}[U(x)])$ then using standard variance identities,

$$U(x_{CE}) = x_{CE} - \frac{\alpha}{2}x_{CE}^2 = \mu - \frac{\alpha}{2}(\mu^2 + \sigma^2) \tag{2}$$

thus

$$0 = \frac{\alpha}{2}x_{CE}^2 - x_{CE} + \mu - \frac{\alpha}{2}(\mu^2 + \sigma^2) \tag{3}$$

hence

$$x_{CE} = \frac{1}{\alpha}\left(1 \pm \sqrt{1 - \alpha(2\mu - \alpha(\mu^2 + \sigma^2))}\right) \tag{4}$$

thus

$$\pi_A := \mathbb{E}[x] - x_{CE} = \mu - \frac{\alpha}{2}(\mu^2 + \sigma^2) - \frac{1}{\alpha}\left(1 \pm \sqrt{1 - \alpha(2\mu - \alpha(\mu^2 + \sigma^2))}\right). \tag{5}$$

I use an equivalent formulation of the investment scenerio where z is the fraction of wealth invested in the risky asset and units are such that 1 is the total amount to be invested. In this investment scenario, we wish to compute $\arg\max_{z \in [0,1]} \mathbb{E}[U(zx + (1-z)r)]$ where $x \sim \mathcal{N}$ and $r \in \mathbb{R}_{>0}$ is a constant.

Expanding $U(zx + (1-z)r$, we get with a little algebra that the above is equivalent to

$$\underset{z \in [0,1]}{\arg\max} \left(\frac{\alpha}{2}(\mu^2 + \sigma^2) - \alpha r\mu + \mu^2 + \sigma^2\right) z^2 + (\mu - r + \alpha r\mu - 2(\mu^2 + \sigma^2))z + (r + \mu^2 + \sigma^2) \tag{6}$$

which is an elementary constrained optimization problem depending on μ, σ^2, r, α.

For problem instances with $\mu = 0, \sigma^2 = 1, r = 1$, we can compute the optimal investment fraction z as a function of a. We restrict to $\alpha >= 0$ so the utility function is concave. Setting the coefficients as above, the problem reduces to

$$\underset{z \in [0,1]}{\arg\max} \left(1 - \frac{\alpha}{2}\right) z^2 - 3z + 2 \tag{7}$$

The expression inside the maximum attains the value 2 when $z = 0$ and $\frac{\alpha}{2}$ when $z = 1$, and attains a local maximum at $z = \frac{3}{2-\alpha}$. However, the constraint that $z \in [0,1]$ means that the local maximum is never attained inside the feasible region, hence $z = 0$ is the argument maximum. Interpretation: never invest in the risky asset if its returns follow a standard normal!

Problem 3

With notation as in the problem description,

$$W = \begin{cases} W_0(1+\alpha f) & \text{w.p. } p \\ W_0(1-\beta f) & \text{w.p. } 1-p \end{cases}$$

and so

$$U(W) = \begin{cases} \log W_0 + \log(1+\alpha f) & \text{w.p. } p \\ \log W_0 + \log(1-\beta f) & \text{w.p. } 1-p \end{cases}$$

hence

$$\begin{aligned} \mathbb{E}[U(W)] &= p(\log W_0 + \log(1+\alpha f)) + (1-p)(\log W_0 + \log(1-\beta f)) &(8)\\ &= \log W_0 + p\log(1+\alpha f) + (1-p)\log(1-\beta f). &(9) \end{aligned}$$

From this expression we can take derivatives:

$$\frac{d\mathbb{E}U(W)}{df} = \frac{p\alpha}{1+\alpha f} - \frac{(1-p)\beta}{1-\beta f} \tag{10}$$

$$\frac{d^2\mathbb{E}U(W)}{df^2} = \frac{-p\alpha^2}{(1+\alpha f)^2} + \frac{-(1-p)^2\beta^2}{(1-\beta f)^2} < 0. \tag{11}$$

From the second derivative, we see that the utility function of wealth is concave and so we can solve directly for the maximum by setting the first order derivative to 0:

$$0 = \frac{p\alpha}{1+\alpha f^*} - \frac{(1-p)\beta}{1-\beta f^*} \tag{12}$$

which yields, with a little algebra,

$$f^* = \frac{p}{\beta} - \frac{1-p}{\alpha}. \tag{13}$$

This is intuitive: the higher our edge of gain relative to the loss of what we wager, the higher the resultant fraction of wealth should be bet. For $\alpha = \beta = 1$ we have $f^* = 2\left(p - \frac{1}{2}\right)$ as in the simplified Kelly betting problem.

Note that the optimal fraction only makes sense in the context of the problem when $p \geq \frac{\beta}{\alpha+\beta}$. If this condition fails to be satisfied, this means that the "optimal" bet is negative, so the game shouldn't be played at all.