Assignment 3

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Problem 1

For a deterministic policy $\pi_D : \mathcal{S} \to \mathcal{A}$ we have the following equations which follow from replacing terms of the form $\sum \pi(s,a)g(a)$ with $g(\pi_D(s)$ in the nondeterministic MDP Bellman Policy equations.

$$V^{\pi_D}(s) = Q^{\pi_D}(s, \pi_D(s)) \tag{1}$$
$$Q^{\pi_D}(s,a) = R(s,a) + \gamma \sum_{s' \in \mathcal{S}} P(s,a,s')V^{\pi_D}(s') \tag{2}$$
$$V^{\pi_D} = R(s, \pi_D(s)) + \gamma \sum_{s' \in \mathcal{S}} P(s, \pi_D(s), s')V^{\pi_D}(s') \tag{3}$$

Problem 2

We will exploit the fact that rewards depend only on action, not on the state. Thus $Q^*(s,a) = Q^*(a)$ and so from the Bellman optimality equation, $V^*(s) = \max_a Q^*(s,a) = \max_a Q^*(a)$ and thus $V^*(s) = V^*(t)$ for any $s, t \in \mathcal{S}$. Then computing,

$$\begin{aligned} V^*(s) &= \max_a Q^*(s,a) && (4) \\ &= \max_a \{R(s,a) + \gamma(P(s,a,s)V^*(s) + P(s,a,s+1)V^*(s+1))\} && (5) \\ &= \max_a \{a(1-a) + (1-a)(1+a) + \frac{1}{2}((1-a)V^*(s) + aV^*(s+1))\} && (6) \\ &= \max_a \{1 + a - 2a^2 + \frac{1}{2}V^*(s)\} && (7) \end{aligned}$$

This function of a is maximized at $a = \frac{1}{4}$ by elementary algebra, in which case

$$V^*(s) = 1 + \frac{1}{4} - 2\left(\frac{1}{4}\right)^2 + \frac{V^*(s)}{2}$$

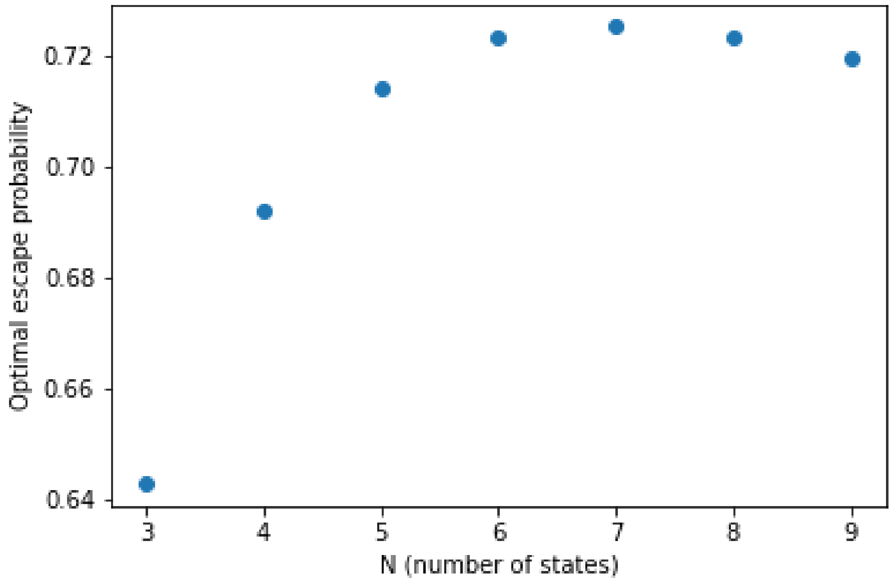
and thus

$$V^*(s) = 2\left(1 + \frac{1}{8}\right) = \frac{9}{4} \quad \forall s \in \mathcal{S}.$$

Incidentally we have also computed an optimal deterministic policy, namely $\pi^D(s) = \frac{1}{4}$ for all $s \in \mathcal{S}$.

Problem 3

For code, see the Jupyter notebook `lilypad.ipynb` in the `assignment3` directory. We observe from the below graph of optimal escape probability against problem size that a maximum occurs at $n = 7$, before which the optimal escape probability is increasing in n and after which the graph is decreasing in n.



Problem 4

When $\gamma = 0$, the problem reduces to the elementary calculation:

$$\min_a \mathbb{E}_{s' \sim \mathcal{N}(s,\sigma)^2}[e^{as'}] = \min_a \int_{-\infty}^{\infty} e^{ax} \cdot \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-s)^2/(2\sigma^2)} dx \tag{8}$$

This can be solved by computing a such that

$$0 = \frac{d}{da} \int_{-\infty}^{\infty} e^{ax} \cdot \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-s)^2/(2\sigma^2)} dx \tag{9}$$

$$= \int_{-\infty}^{\infty} \frac{d}{da} e^{ax} \cdot \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-s)^2/(2\sigma^2)} dx \tag{10}$$

$$= e^{as + a^2\sigma^2/2}(s + a\sigma^2) \tag{11}$$

and thus

$$a = -\frac{s}{\sigma^2} \tag{12}$$

attains the minimum expected myopic cost at for any state s. Substituting this back into the above expectation, we see with a little calculation that the optimal expected cost is $\frac{1}{\sqrt{2}}$ for all s.