GEODESICS ON HOMOGENEOUS COMPACT LORENTZIAN MANIFOLDS

Abstract. The aim of this work is the study of geodesics on homogeneous spaces of the form $M = G/\Lambda$, where G is a Lie group endowed with a bi-invariant Lorentzian metric and $\Lambda < G$ is a cocompact lattice. For the case $G = \mathrm{SL}(2, \mathbb{R})$ it is proved that no lightlike geodesic is closed, situation which does not hold for oscillator groups. Several situations to determine conditions for a (ligh, time or spacelike) geodesic to be closed are given. Also some trivial extensions of the previous cases are studied.

1. Introduction

Known results: stephan Schur, Galloway, medina, Ovando, flat 3-manifolds.

2. Lie groups with Lorentzian bi-invariant metrics

In this section one can find an introduction to general results about Lie groups with bi-invariant Lorentzian metrics. That is, basic notions, features, geodesics and the construction of the induced homogeneous spaces, by taking a lattice in the Lie group.

Let G denote a (real) Lie group with Lie algebra $\mathfrak{g}$. A *bi-invariant* metric on G is a pseudo-Riemannian metric $\langle\, , \rangle$ for which the translation on the left L_g and on the right R_g by elements of the group $g \in G$, are isometries. That is, the conjugation map $I_g : G \to G$, $I_g(x) = g^{-1}xg$ is an isometry. Thus the diferentail of the Adjoint map is a linear isometry on $\mathfrak{g}$, $d(I_g)_e = Ad(g)$. One has the following equivalences (see Chapter 11 in [8]):

(1) $\langle\, , \rangle$ bi-invariant;
(2) $\langle\, , \rangle$ $\mathrm{Ad}(G)$-invariant;
(3) $\langle [X, Y], Z\rangle + \langle Y, [X, Z]\rangle = 0$ for all $X, Y, Z \in \mathfrak{g}$;
(4) the geodesics of G starting at the identity element e are the one-parameter subgroups of G, that is:

$$\alpha(t) = \exp(tX), \qquad \text{for } X \in \mathfrak{g}, \tag{1}$$

and the geodesic through $g \in G$ with initial left-invariant vector X is given by the translation of the curve above, that is $g \exp(tX)$.

If the bi-invariant metric on a Lie group G of dimension n has signature $(1, n-1)$, the metric is called a *Lorentzian metric*. Given a vector field X, it is called

- *spacelike* whenever $\langle X, X\rangle > 0$;
- *timelike* whenever $\langle X, X\rangle < 0$;

- *lightlike* or *null* if $\langle X, X\rangle = 0$.

More generally this extended to geodesics: a geodesic on G with initial condition X, namely $\gamma_X(t)$, is called *spacelike, timelike or lightlike* if X is in the respective class above.

The special linear Lie group $\mathrm{SL}(2,\mathbb{R})$ consisting of 2×2 real matrices with determinant one, admits a bi-invariant Lorentzian metric. The construction of this metric starts on the Lie algebra of the group, $\mathfrak{sl}(2,\mathbb{R})$: identified with the real traceless 2×2-matrices. Here one can consider the Killing form B:

$$\text{Let} \quad X = \begin{pmatrix} x_1 & x_2 \\ x_3 & -x_1 \end{pmatrix}, Y = \begin{pmatrix} y_1 & y_2 \\ y_3 & -y_1 \end{pmatrix} \in \mathfrak{sl}(2,\mathbb{R})$$

then

$$\mathrm{B}(X,Y) = 4Tr(XY) = 8x_1y_1 + 4x_2y_3 + 4x_3y_2. \tag{2}$$

By extending this bilinear form to the group using translations on the left, one obtains a left-invariant metric on the Lie group which is also right-invariant.

Example 2.1. **Geodesics on** $\mathrm{SL}(2,\mathbb{R})$. Let $X \in \mathfrak{sl}(2,\mathbb{R})$. An easy computation shows that X is lightlike by satisfying

$$\mathrm{B}(X,X) = 0 \quad \textit{if and only if} \quad X^2 = 0 \quad \textit{if and only if} \quad det(X) = 0.$$

In fact for any $X \in \mathfrak{sl}(2,\mathbb{R})$ of the form

$$X = \begin{pmatrix} x_1 & x_2 \\ x_3 & -x_1 \end{pmatrix}, \qquad \text{one has} \qquad X^2 = \begin{pmatrix} x_1^2 + x_2x_3 & 0 \\ 0 & x_1^2 + x_2x_3 \end{pmatrix}.$$

More generally, for any $X \in \mathfrak{sl}(2,\mathbb{R})$ one has $B(X,X) = 4Tr(X^2) = 8(-\det(X))$. Thus the geodesics of $\mathrm{SL}(2,\mathbb{R})$ starting at the identity and with initial condition $X \in \mathfrak{sl}(2,\mathbb{R})$ are the one-parameter subgroups (I denotes the identity matrix):

$$\begin{aligned} \cosh(-det(tX))^{1/2} I + \frac{\sinh(-det(tX))^{1/2}}{(-det(tX))^{1/2}} X & \qquad \text{if } det(X) < 0 \\ \cos(det(tX))^{1/2} I + \frac{\sin(det(tX))^{1/2}}{(det(tX))^{1/2}} X & \qquad \text{if } det(X) > 0 \\ I + tX & \qquad \text{if } det(X) = 0, \end{aligned}$$

which correspond to spacelike, timelike and lightlike geodesics respectively, (see Exercises and further results Ch. II in [11]). In this work an alternative presentation of geodesics will be used, presentation that is written by using the conjugation by an element C of $\mathrm{SL}(2,\mathbb{R})$

Proposition 2.2. *Let* $\mathrm{SL}(2,\mathbb{R})$ *denote the Lie group equipped with the Lorentzian metric induced by the Killing form* B. *The geodesics starting at the identity are* *curves defined for every $t \in \mathbb{R}$ of one of the following families:*

$$\textbf{\textit{spacelike}}: \quad C^{-1}\begin{pmatrix} \cosh(kt) & \sinh(kt) \\ \sinh(kt) & \cosh(kt) \end{pmatrix} C, \qquad 0 \neq k \in \mathbb{R},$$

$$\textbf{\textit{timelike}}: \quad C^{-1}\begin{pmatrix} \cos(kt) & -\sin(kt) \\ \sin(kt) & \cos(kt) \end{pmatrix} C, \qquad 0 \neq k \in \mathbb{R},$$

$$\textbf{\textit{lightlike:}} \quad \begin{pmatrix} 1+x_1 t & x_2 t \\ x_3 t & 1-x_1 t \end{pmatrix}, \qquad x_1^2 + x_2 x_3 = 0,$$

for any $C \in \mathrm{SL}(2,\mathbb{R})$.

Proof. Since geodesics at the identity are one-parameter subgroups, a geodesic starting at the identity $\gamma_X(t)$ is given by the usual exponential of matrices.

Take as initial condition of the geodesic, a general element $X \in \mathfrak{sl}(2,\mathbb{R})$ of the form $X = \begin{pmatrix} x_1 & x_2 \\ x_3 & -x_1 \end{pmatrix} \in \mathfrak{sl}(2,\mathbb{R})$.

- Assume X is a lightlike vector. As seen in Example 2.1, one has $X^2 = 0$ and therefore the geodesic $\gamma_X(t)$ is given by $\gamma_X(t) = Id + tX$, which gives the respective expression in Equation (17).
- Consider now the timelike vector $M_k = \begin{pmatrix} 0 & -k \\ k & 0 \end{pmatrix}$, where k is a non-zero real constant. Usual computations give

$$e^{tM_k} = \begin{pmatrix} cos(kt) & -sen(kt) \\ sen(kt) & cos(kt) \end{pmatrix},$$

which are timelike geodesics for any $k \in \mathbb{R}-\{0\}$. Moreover, since translations on the left and on the right are isometries, for any $C \in \mathrm{SL}(2,\mathbb{R})$, a curve of the form

$$C^{-1}\begin{pmatrix} cos(kt) & -sen(kt) \\ sen(kt) & cos(kt) \end{pmatrix} C$$

is also a timelike geodesic through the identity. The claim is that every timelike geodesic starting at the identity is of this form.

In fact, let $X \in \mathfrak{sl}(2,\mathbb{R})$ be any timelike vector field. Clearly $\det(X) > 0$. Note that $e^{tX} = C^{-1}e^{tM_k}C$ comes as a consequence of taking the exponential whenever $X = C^{-1}M_k C$. Thus, one needs to verify that there exist $k \in \mathbb{R} - \{0\}$ and $C \in \mathrm{SL}(2,\mathbb{R})$ such that

$$X = C^{-1}M_k C \qquad \text{equivalently} \qquad CX = M_k C. \tag{3}$$

Notice that $k^2 = det(X)$. In order to solve $CX = M_k C$ for k and C, let $M_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and noticed that $M_k = kM_1$ where k is chosen by satisfying $k^2 = det(X)$. Now the equation $CX - kM_1 C = 0$ is a linear system in the entries of C: (c_1, c_2, c_3, c_4), where $C = \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R})$.

Thus finding solutions of $CX = kM_1C$ is equivalent to find solutions of the following system:

$$\begin{pmatrix} x_1 & x_3 & k & 0 \\ x_2 & -x_1 & 0 & k \\ -k & 0 & x_1 & x_3 \\ 0 & -k & x_2 & -x_1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{4}$$

Since $k \neq 0$ satisfies $k^2 = det(X)$, the resulting solution space to Equation 16 is given by C with entries

$$C = \begin{pmatrix} u\frac{x_1}{k} + v\frac{x_3}{k} & u\frac{x_2}{k} - v\frac{x_1}{k} \\ u & v \end{pmatrix} : u, v \in \mathbb{R}. \tag{5}$$

The solutions have determinant equals 1 whenever

$$2uvx_1 + v^2x_3 - u^2x_2 - k = 0. \tag{6}$$

Choose k such that $k^2 = det(X)$ and $x_2k < 0$. Thus, the equation above has real solutions u, v (here start by thinking the equation above by fixing v and solving the quadratic equation for u).

- For the spacelike case, the reasoning is analogous to that one given in the timelike case. Firstly consider the spacelike vectors $P_k = k\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $k \neq 0$, which produce the next family of spacelike geodesics starting at the idenity
$$e^{tP_k} = \begin{pmatrix} cosh(kt) & senh(kt) \\ senh(kt) & cosh(kt) \end{pmatrix},$$
and one also has the family of geodesics
$$C^{-1}\begin{pmatrix} cosh(kt) & senh(kt) \\ senh(kt) & cosh(kt) \end{pmatrix} C,$$
for any $C \in \mathrm{SL}(2, \mathbb{R})$.

To prove that every spacetime geodesic starting at the identity can be written as above, one needs to prove that there exist $0 \neq k \in \mathbb{R}$ and $C \in \mathrm{SL}(2, \mathbb{R})$ such that $X = C^{-1}P_kC$ where X is a spacelike vector (therefore $\det(X) < 0$). Clearly it holds $det(X) = -k^2$. Thus the equation $CX - P_kC = 0$ has a solution if and only if the following linear system has a non-trivial solution $C = (c_1, c_2, c_3, c_4)$:

$$\begin{pmatrix} x_1 & x_3 & -k & 0 \\ x_2 & -x_1 & 0 & -k \\ -k & 0 & x_1 & x_3 \\ 0 & -k & x_2 & -x_1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

and with $C \in \mathrm{SL}(2, \mathbb{R})$. Usual computations show that the solution space is given by the set of matrices C, whose entries satisfy Equation (5). Thus, C

belongs to $\mathrm{SL}(2,\mathbb{R})$ if and only if Equation (6) holds. In this case, choose k such that $det(X) = -k^2$, and one gets solutions if $v^2 \geq x_2/k$.

□

Note that $\mathrm{SL}(2,\mathbb{R})$ is *complete* in the sense that geodesics are defined on $\mathbb{R}$.

Another examples of Lie groups with bi-invariant Lorentzian metrics arise from the so called *oscillator groups*. Denoted by $\mathrm{Osc_n}(\lambda_1,\ldots,\lambda_\mathrm{n})$, an oscillator Lie group is the simply connected Lie group with real Lie algebra of dimension $2n+2$ $\mathfrak{osc}_n(\lambda_1,...,\lambda_n)$, with $\lambda_i \in \mathbb{R}_{>0}$, spanned by the basis Z,$\{X_i,Y_i\}_{i=1}^n$, T and satisfying the non-trivial Lie bracket relations

$$[X_i,Y_i]=Z,\quad [T,X_i]=\lambda_i Y_i,\quad [T,Y_i]=-\lambda_i X_i$$

where the ad-invariant metric on $\mathfrak{osc}_n(\lambda_1,...,\lambda_n)$ is given by the non-zero relations

$$\lambda_i\langle X_i,X_i\rangle=\lambda_i\langle Y_i,Y_i\rangle=\langle Z,T\rangle=1.$$

The oscillator Lie groups have the differential structure of $\mathbb{R}\times\mathbb{R}^{2n}\times\mathbb{R}$ with the following group product

$$(z_1,v_1,t_1).(z_2,v_2,t_2)=(z_1+z_2+\frac{1}{2}v_1^TJe^{t_1N_\lambda}v_2, v_1+e^{t_1N_\lambda}v_2, t_1+t_2),$$

$$\text{where}\quad N_\lambda=\begin{pmatrix} J_{\lambda_1} & & \mathbf{0}\\ & \ddots & \\ \mathbf{0} & & J_{\lambda_n}\end{pmatrix},\qquad J_{\lambda_i}=\begin{pmatrix}0 & -\lambda_i\\ \lambda_i & 0\end{pmatrix},\qquad J=N_{(-1,\ldots,-1)}.$$

for $v_1,v_2\in\mathbb{R}^{2n}$. Take the corresponding left-invariant metric on the Lie group, which for usual coordinates $(z,x_1,y_1,\ldots,x_n,y_n,t)$ in $\mathbb{R}^{2n+2}$ can be written as

$$g=dt(dz+\frac{1}{2}(\sum_{j=1}^n(y_jdx_j-x_jdy_j))+\sum_{j=1}^n\frac{1}{\lambda_j}(dx_j^2+dy_j^2). \tag{7}$$

To find the differential equations for the geodesics one can compute the Christoffel symbols corresponding to the metric above to get

$$\Gamma^1_{2n+2\ 2i}=-\frac{x_i\lambda_i}{4}\quad \Gamma^1_{2n+2\ 2i+1}=-\frac{y_i\lambda_i}{4},\quad i=1,...,n$$

$$\Gamma^{2i}_{2n+2\ 2i}=\frac{\lambda_i}{2}\quad \Gamma^{2i+1}_{2n+2\ 2i}=-\frac{\lambda_i}{2},\quad i=1,...,n$$

being the others trivial and following symmetry relations.

The resulting equations for the geodesics can be written in the usual coordinates of $\mathbb{R}^{2n}$ as:

$$\begin{array}{rcl} z''(s) &=& \frac{t'(s)}{2}\sum_{k=1}^n\lambda_k\left(x_k'(s)x_k(s)+y_k'(s)y_k(s)\right)\\ x_i''(s) &=& -\lambda_i y_i'(s)t'(s),\\ y_i''(s) &=& \lambda_i x_i'(s)t'(s),\\ t''(s) &=& 0,\end{array} \tag{8}$$

which follows from the general geodesic equation, $\frac{d^2\gamma^k}{dt^2}+\sum_{i,j}\Gamma^k_{ij}(\gamma)\frac{d\gamma^i}{dt}\frac{d\gamma^j}{dt}=0$ (see [8], page 67 FALTA PARAGRAFo!!).

In particular, those geodesic starting at the identity element with initial condition $X = d\ Z+\sum_j(b_jX_j+c_jY_j)+aT$ are:

- for $a\neq 0$:

$$z(s) = \left(d+\frac{1}{2a}\sum_{k=1}^{n}\frac{b_k^2+c_k^2}{\lambda_k}\right)s-\frac{1}{2a^2}\left(\sum_{k=1}^{n}\frac{b_j^2+c_j^2}{\lambda_k^2}\sin(\lambda_k as)\right), \tag{9}$$

$$x_j(s) = \frac{1}{a\lambda_j}\left(b_j sin(\lambda_j as)+c_j cos(\lambda_j as)-c_j\right), \tag{10}$$

$$y_j(s) = \frac{1}{a\lambda_j}\left(-b_j cos(\lambda_j as)+c_j sin(\lambda_j as)+b_j\right), \tag{11}$$

$$t(s) = as, \tag{12}$$

- while for $a=0$:

$$(z,(x_j,y_j),t)(s)=(ds,(b_js,c_js),0). \tag{13}$$

It is not hard to check that for the initial velocity $X\in\mathfrak{osc}_n(\lambda_1,...,\lambda_n)$ as above, the corresponding geodesic is:

- lightlike if $2ad+\sum_{k=1}^n\frac{b_k^2+c_k^2}{\lambda_k}=0$,
- timelike if $2ad+\sum_{k=1}^n\frac{b_k^2+c_k^2}{\lambda_k}<0$,
- or spacelike if $2ad+\sum_{k=1}^n\frac{b_k^2+c_k^2}{\lambda_k}>0$.

Note that the oscillator Lie groups are also complete spaces.

Remark 1. Medina and Revoy in [4, 5] proved that the Lie algebras $\mathfrak{osc}_n(\lambda_1,...,\lambda_n)$ ($\lambda_i>0$) and $\mathfrak{sl}(2,\mathbb{R})$ are the only indecomposable ones admitting a Lorentzian ad-invariant metric. Recall that a Lie algebra provided with a metric is called **indecomposable** if the restriction of the metric to any proper ideal is degenerate.

2.1. **Quotient spaces.** Let G denote a Lie group and let $\Gamma\subset G$ be a discrete cocompact subgroup. The quotient space $M=G/\Gamma$ consists of elements of the form $g\Gamma$ with $g\in G$. Since Γ is closed, there exists a unique manifold structure on M for which the canonical projection $g\mapsto g\Gamma$ is a smooth submersion (see [10]). Finally, the geometry of M is provided by requiring the projection, named π, to be a local isometry. Whenever the Lie group G is provided with a Lorentzian metric, (G,π) is called a *Lorentzian covering.*

Assume G is equipped with a bi-invariant metric. It follows that the geodesics of M starting at $o:=\pi(e)$ are of the form $\hat\alpha=\pi(\alpha(t))$, where α is a one parameter subgroup of G (see [8]). In addition to this, G acts on M by the "translations on the left" which are isometries:

$$\tau_g : M\to M \qquad \text{given by} \quad \tau_g(h\Gamma):=gh\Gamma,$$

by showing that $M = G/\Gamma$ is a homogeneous space.

One can notice that:

(1) A geodesic of G/Γ starting at $g\Gamma$ is the translation via τ_g of some geodesic starting at o.
(2) Every geodesic in G/Γ is the projection via π of some geodesic in G.
(3) Lighlike, timelike and spacelike geodesics of G project to lightlike, timelike and spacelike geodesics of M respectively.

Note that $\pi \circ L_g = \tau_g \pi$ and one gets $\tau(g)\pi \circ \alpha = \pi \circ L_g \circ \alpha$ for a curve $\alpha : (a, b) \to G$ starting at the identity element $e \in G$.

A curve $\beta : (a, b) \to G$ (or to M) is said *closed* when it passes through a same point more than once, that is, there exist $t_2 \neq t_1$ such that $\beta(t_1) = \beta(t_2)$.

(4) A geodesic $\alpha : (-\varepsilon, \varepsilon) \to G$, with $\varepsilon > 0$ and $\alpha(0) = e$ giving rise the the curve $\pi \circ \alpha$ in M is closed in M, if and only if $\alpha(t) \in \Gamma$ for some $t > 0$.

In particular the projection of a closed geodesic in G is always a closed curve in M.

A final result for closed geodesics comes from the following lemma, which, when combined with item (4) states that every closed geodesic in the quotient manifold is actually a periodic curve.

Lemma 2.3. [2] *Let G be a Lie group, let $K < G$ be any closed Lie subgroup of G such that $\pi : G \to G/K$ denotes the usual projection. Let $\alpha : \mathbb{R} \to G$ denote a one-parameter subgroup of G. If $\pi \circ \alpha$ is closed in G/K then it is periodic.*

3. Geodesics on compact manifolds from $\mathrm{SL}(2, \mathbb{R})$

The goal in this section is the study of the geodesics of Lorentzian manifolds of the form $M = G/\Gamma$, where Γ is a closed cocompact discrete subgroup (a lattice) of the Lie group $G = \mathrm{SL}(2, \mathbb{R})$, which is provided with a bi-invariant Lorentzian metric.

The easier case corresponds to timelike geodesics, since from Proposition (2.2) one knows that timelike geodesics of $\mathrm{SL}(2, \mathbb{R})$ are periodic, then their projection to the quotient $\mathrm{SL}(2, \mathbb{R})/\Gamma$ are periodic timelike geodesics.

The latter result is independent of weather Γ is cocompact or not, however cocompactnes is important when studying the lightlike geodesics, where the next lemma (3.1) is used. Before presenting the result, recall the usual classification of elements of $\mathrm{SL}(2, \mathbb{R})$.

For $A \in \mathrm{SL}(2, \mathbb{R})$ one names:

- A is *elliptic*, if $|tr(A)| < 2$.
- A is *parabolic*, if $|tr(A)| = 2$ and $A \neq Id$.
- A is *hyperbolic* , if $|tr(A)| > 2$ or $A = Id$.

Lemma 3.1. [6] *Let Γ be a lattice of $\mathrm{SL}(2, \mathbb{R})$ and let $u \in \Gamma$ be a parabolic element, then u is the identity element.*

In Proposition (2.2) it was proved that lightlike geodesics of $\mathrm{SL}(2, \mathbb{R})$ starting at the identity have the form $\alpha(t) = \begin{pmatrix} 1 + x_1 t & x_2 t \\ x_3 t & 1 - x_1 t \end{pmatrix}$. These elements of these curves have trace 2 (parabolic) and therefore, by the previous lemma, they never intersect any lattice for $t > 0$.

Now the goal is the study of spacelike geodesics in the quotients. If M has a closed spacelike geodesic, there exists a spacelike geodesic α on $\mathrm{SL}(2,\mathbb{R})$ such that for some $t > 0$, $\alpha(t)$ belongs to Γ. Denote by $\gamma := \alpha(t)$ with

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{14}$$

Notice that for any $s \in \mathbb{R}$ any element $\alpha(s)$ of a spacelike geodesic is hyperbolic. In particular γ itself must be a hyperbolic element.

By making use of Proposition 2.2 one gets four equations for the entries a, b, c, d above. Take $C \in \mathrm{SL}(2,\mathbb{R})$ with $C = \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix}$. Compute

$$\gamma = C^{-1} \begin{pmatrix} \cosh kt & \sin kt \\ \sinh kt & \cosh kt \end{pmatrix} C$$

to obtain the next linear system

$$\begin{array}{rcl} a & = & -c_2c_3\cosh(kt) - c_1c_2\sinh(kt) + c_4c_1\cosh(kt) + c_3c_4\sinh(kt) \\ b & = & (c_4^2 - c_2^2)\sinh(kt) \\ c & = & (c_1^2 - c_3^2)\sinh(kt) \\ d & = & -c_2c_3\cosh(kt) + c_1c_4\cosh(kt) + c_1c_2\sinh(kt) - c_3c_4\sinh(kt). \end{array}$$

This system is equivalent to the following one

$$\left\{ \begin{array}{l} a + d = 2\cosh(kt) \\ a - d = 2\sinh(kt)(c_3c_4 - c_1c_2) \\ b = (c_4^2 - c_2^2)\sinh(kt) \\ c = (c_1^2 - c_3^2)\sinh(kt), \end{array} \right. \tag{15}$$

where it is required that $C \in \mathrm{SL}(2,\mathbb{R})$, that is $c_1c_4 - c_2c_3 = 1$ and $a + c > 0$ that is γ hyperbolic. To study if the geodesic is closed one can assume $k = 1$, since k only affects the speed of the geodesic.

The space of solutions of this system of equations is described in the next proposition.

Proposition 3.2. *Any non-diagonal hyperbolic element of* $\mathrm{SL}(2,\mathbb{R})$ *with positive trace can be written as follows*

$$C^{-1} \begin{pmatrix} \cosh(\mu) & \sinh(\mu) \\ \sinh(\mu) & \cosh(\mu) \end{pmatrix} C$$

for some $\mu \in \mathbb{R}^+$ *and* $C \in \mathrm{SL}(2,\mathbb{R})$.

Proof. Any hyperbolic element with positive trace has the form of γ in (14), which may satisfy the system (15) with $a + d > 0$ and b, c not both zero.

Setting $k=1$ in (15) and looking for solutions with $t>0$ it follows from $a+d=2\cosh(t)$ that $a+d\geq 2$.

Let μ such that $\cosh(\mu)=\frac{tr(\gamma)}{2}$. Define now $w:=(a+d)^2-4$, then since $\cosh^2 t-\sinh^2 t=1$ it can be seen that $2\sinh t=\sqrt{w}$.

Set $\chi=\begin{cases}1 & \text{if } a-d\geq 0,\\ -1 & \text{if } a-d<0.\end{cases}$. To find the entries c_i, the following situations should be considered:

(1) If $\mathbf{c}<0$, one can define C as

$$\pm\begin{pmatrix}0 & -\sqrt{\frac{-2c}{w}}\\ \sqrt{\frac{-2c}{w}} & \chi\sqrt{\frac{2b}{w}-\frac{w}{2c}}\end{pmatrix}$$

(2) If $\mathbf{c}>0$, one could take C as

$$\pm\begin{pmatrix}\sqrt{\frac{2c}{w}} & -\chi\sqrt{\frac{w}{2c}-\frac{2b}{w}}\\ 0 & \sqrt{\frac{w}{2c}}\end{pmatrix}$$

(3) If $\mathbf{b}<0$, one could define C as

$$\pm\begin{pmatrix}-\chi\sqrt{\frac{2c}{w}-\frac{w}{2b}} & \sqrt{\frac{-2b}{w}}\\ -\sqrt{\frac{-w}{2b}} & 0\end{pmatrix}$$

(4) If $\mathbf{b}>0$, one can take the following C:

$$\pm\begin{pmatrix}\sqrt{\frac{w}{2b}} & 0\\ \chi\sqrt{\frac{w}{2b}-\frac{2c}{w}} & \sqrt{\frac{2b}{w}}\end{pmatrix}.$$

Thus, Equation (15) is satisfied, which finishes the proof.

□

In Paragraph 8.1, Ch. 8 in [1] , Beardon proved that every non-finite discrete subgroup contains some hyperbolic element.

Moreover, every lattice Γ has hyperbolic elements with positive trace. In fact, if $\gamma\in\Gamma$ is hyperbolic, then γ^2 is hyperbolic and it has positive trace. Finally the next lemma shows that a lattice Γ must contain non-diagonal hyperbolic elements.

Lemma 3.3. *The only diagonal element in a lattice* Γ *of* $\mathrm{SL}(2,\mathbb{R})$ *is the identity.*

Proof. Assume $\gamma\in\Gamma$ is a diagonal element of the form $\gamma=\begin{pmatrix}\lambda & 0\\ 0 & 1/\lambda\end{pmatrix}$, with $\lambda\neq 0,1$. Then it generates a discrete subgroup of Γ. Clearly $\gamma^n=\begin{pmatrix}\lambda^n & 0\\ 0 & 1/\lambda^n\end{pmatrix}$, so that the limit $\lim_{n\to\infty}\gamma^n\in\Gamma$ may contain three null entries, and this would imply $\lim_{n\to\infty}\gamma^n$ not in $\mathrm{SL}(2,\mathbb{R})$. □

From Criterion (2.3.1) in Beardon [1] any lattice in $\mathrm{SL}(2,\mathbb{R})$ is countable.

Proposition 3.4. *For any lattice* Γ *of* $\mathrm{SL}(2,\mathbb{R})$ *there is a spacelike geodesic of* $\mathrm{SL}(2,\mathbb{R})$ *that does not go through the lattice.*

Proof. Suppose that every spacelike geodesic of $\mathrm{SL}(2,\mathbb{R})$ contains an element of a lattice Γ. In particular, for the family of spacelike geodesics α_c given by

$$\begin{pmatrix}1 & 0\\ c & 1\end{pmatrix}\begin{pmatrix}\cosh t & \sinh t\\ \sinh t & \cosh t\end{pmatrix}\begin{pmatrix}1 & 0\\ -c & 1\end{pmatrix} = \begin{pmatrix}\ldots & \sinh(t)\\ \ldots & \ldots\end{pmatrix}$$

it must be that for every $c \in \mathbb{R}$ there exists $t_c > 0$ such that $\alpha_c(t_c) \in \Gamma$.

Consider $T := \{t \in \mathbb{R} : \alpha_c(t) \in \Gamma, c \in \mathbb{R}\}$. Since Γ is countable and the function $\sinh t$ is bijective, it must occur that T is countable.

Consider now any function $F : I \to T$ such that $F(c) = t_c \iff \alpha_c(t_c) \in \Gamma$. It can be proved that such function must contain at least one element in its image that is reached infinitely many times, call it t_∞. Let $I' := \{c \in I : F(c) = t_\infty\}$ which is a subset of the compact set I, and since I' is an infinite set it must contain a convergent subsequence, $\{c_n\}_{n\in\mathbb{N}} \to c'$. Finally consider the following sequence of different elements of Γ

$$\begin{aligned}\alpha_{c_n}(t_\infty) &= \begin{pmatrix}1 & 0\\ c_n & 1\end{pmatrix}\begin{pmatrix}\cosh t_\infty & \sinh t_\infty\\ \sinh t_\infty & \cosh t_\infty\end{pmatrix}\begin{pmatrix}1 & 0\\ -c_n & 1\end{pmatrix}\\ &= \begin{pmatrix}c_n \sinh t_\infty + \cosh t_\infty & \sinh(t)\\ \sinh t_\infty(1-c_n^2) & \cosh t_\infty - c_n \sinh t_\infty\end{pmatrix}.\end{aligned}$$

The convergence of this sequence contraditcs the fact that Γ is a discrete subgroup, therefore not every α_c can contain elements of Γ

□

Theorem 3.5. *Let* Γ *be a lattice of* $\mathrm{SL}(2,\mathbb{R})$ *such that the compact space* $M = \mathrm{SL}(2,\mathbb{R})/\Gamma$ *is equipped with the induced metric from the Killing form on* $\mathfrak{sl}(2,\mathbb{R})$. *Then*

- *no lightlike geodesic is closed;*
- *every timelike geodesic is periodic;*
- *there are both open and closed spacelike geodesics.*

Proof. Since lightlike geodesic have initial vector X given by a unipotent element, the statement about lightlike geodesics is a consequence of Lemma 3.1.

Note that every timelike geodesic on $\mathrm{SL}(2,\mathbb{R})$ is a periodic curve, thus the projection on M is also closed.

The proof for the statement on spacelike geodesics follows from Proposition 3.2, Lemma 3.3 and Proposition 3.4. □

Example 3.6. Two rich families of examples of lattices of $\mathrm{SL}(2,\mathbb{R})$ arise from the study of arithmetic lattices, which in a broad sense are those commensurable [1] to some subgroup of $SL(2,\mathbb{Z})$, see [6].

[1]Two subgroups H,F of an abstract group are *commensurable* if $[H : H\cap F] < \infty$ and $[F : H\cap F] < \infty$.

First family: [6] Let $a, b \in \mathbb{Z}$ such that the only integer solution of $w^2 - ax^2 - by^2 + abz^2 = 0$ is $(0,0,0,0)$, then

$$\Lambda_{a,b} = \left\{ \begin{pmatrix} p + q\sqrt{a} & r + s\sqrt{a} \\ rb - sb\sqrt{a} & p - q\sqrt{a} \end{pmatrix} : p, q, r, s \in \mathbb{Z} \right\} \cap \mathrm{SL}(2, \mathbb{R})$$

is a arithmetic lattice of $\mathrm{SL}(2, \mathbb{R})$.

Second family: [6] Let χ be a real root of a polynomial of integer coefficients and let $\mathcal{O}$ be the ring of integers of the field $K := \mathbb{Q}[\chi]$. If $a, b > 0 \in \mathcal{O}$ are such that for every field embedding $\sigma : K \to \mathbb{C}$, one has $\sigma(a)$ and $\sigma(b)$ are negative, then

$$\Lambda_{a,b}^{\mathcal{O}} = \left\{ \begin{pmatrix} p + q\sqrt{a} & r + s\sqrt{a} \\ rb - sb\sqrt{a} & p - q\sqrt{a} \end{pmatrix} : p, q, r, s \in \mathcal{O} \right\} \cap \mathrm{SL}(2, \mathbb{R})$$

is a cocompact arithmetic lattice of $\mathrm{SL}(2, \mathbb{R})$.

These are not empty, particularly for the first family, when taking $a = b = 3$ the required condition holds.

Lemma 3.7. *If (p, q, r, s) correspond to an element of a lattice of the First family above such that $p^2 = 1$ then $q = r = s = 0$.*

Proof. Note that the determinant of a member of $\Lambda_{a,b}$ is $p^2 - aq^2 - br^2 + abs^2 = 1$. Let (p, q, r, s) as in the lemma statement, then $1 - aq^2 - br^2 + abs^2 = 1 \implies -aq^2 - br^2 + abs^2 = 0$, then $(0, q, r, s)$ is an integer solution of $w^2 - ax^2 - by^2 + abz^2 = 0$, therefore $q = r = s = 0$. □

Closed spacelike geodesics of $M = \mathrm{SL}(2, \mathbb{R}) / \Lambda_{a,b}$ will occur if for some spacelike geodeic α of $\mathrm{SL}(2, \mathbb{R})$, $\alpha(t) \in \Lambda_{a,b}$ for some $t > 0$. Using the expression of the proposition 2.2, and imposing $\alpha(t) = \gamma \in \lambda_{a,b}$ one can work with the system to get the following equivalent four equations

$$c_4^2 - c_2^2 = d_4 \tag{16}$$

$$c_1^2 - c_3^2 = d_1 \tag{17}$$

$$c_3 c_4 - c_1 c_2 = \frac{q\sqrt{a}}{w} \tag{18}$$

$$c_1 c_4 - c_2 c_3 = 1 \tag{19}$$

where $w = \sqrt{p^2 - 1}$, $d_1 := \frac{br - bs\sqrt{a}}{w}$ and $d_4 := \frac{r + s\sqrt{a}}{w}$. If this system holds for some C and p, q, r, s corresponding to some γ then the associated spacelike curve intersects $\Lambda_{a,b}$ for some $t > 0$, notice that equation 18 implies that $p^2 \neq 0$ so by lemma 3.7 $\gamma \neq Id$.

Since the system has 4 equations and 4 variables, one can suspect that the space of solutions might have dimension 0, however it was possible to find a one-dimensional subset of solutions.

Theorem 3.8. *Take $\gamma \in \Lambda_{a,b}$ such that $q=0$, then the set of spacelike geodesics corresponding to $c_2 \in \mathbb{R}$ such that $c_2^2 + d_4 > 0$, $c_3 = \frac{c_2}{d_4}$, c_1, c_4 such that $c_1^2 = d_1 + c_3^{faltaalgo}$, $c_4^2 = d_4 + c_2^2$ and $\operatorname{sign}(c_1 c_4) = \operatorname{sign}(d_4)$ project to closed spacelike geodesics in $\mathrm{SL}(2,\mathbb{R})\,/\Lambda_{a,b}$.*

Proof. The non-closed spacelike geodesics correspond to the projection of those presented in proposition 2.2 such that $C = Id$. To see this, let α be one of those geodesics, note that:

(1) the trace of the geodesic is $2\cosh(kt)$ while the trace of an element of the lattice is $2p$, then $\cosh(kt) = p$,
(2) multiplying the $(1,2)$ elements by b on both matrices and adding that to the $(2,1)$ elements gives $(b+1)\sinh(kt) = 2br$

Since $\cosh^2(x) - \sinh^2(x) = 1$, then combining the two results above, one gets that $p^2 - 1 = (\frac{2br}{b+1})^2$, in particular $\frac{2br}{b+1}$ must be an integer, call it d, then $p^2 - 1 = d^2 \iff p^2 - d^2 = 1$ whose only integer solution is $p = \pm 1$ and $d = 0$. Finally, since $p^2 = 1$, then $q = r = s = 0$ and it is clear that $\alpha(t) \neq \pm Id$ for $t > 0$.

$$\begin{pmatrix} \frac{\sqrt{w}}{r+s\sqrt{a}} & 0 \\ \frac{q\sqrt{a}}{\sqrt{w(r+s\sqrt{a})}} & \sqrt{\frac{r+s\sqrt{a}}{w}} \end{pmatrix}$$ □

Corollary 3.9. *Let Γ be a lattice such that it contains an element γ with its two diagonal elements equal, then there is a one dimensional set of closed spacelike geodesics in $\mathrm{SL}(2,\mathbb{R})\,/\Gamma$.*

3.1. **The case of $\widetilde{\mathrm{SL}}(2,\mathbb{R})$.** The universal covering group of the special linear group $\mathrm{SL}(2,\mathbb{R})$ is denoted by $\widetilde{\mathrm{SL}}(2,\mathbb{R})$. These Lie groups share the same Lie algebra and therefore the bilinear form of Equation (2) can be extended to the whole group $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ producing a bi-invariant Lorentzian metric.

Since $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ is the simply connected Lie group with Lie algebra $\mathfrak{sl}(2,\mathbb{R})$, the group $\mathrm{SL}(2,\mathbb{R})$ is a quotient group obtained as $\widetilde{\mathrm{SL}}(2,\mathbb{R})\,/N$, where N is a normal discrete subgroup of $\widetilde{\mathrm{SL}}(2,\mathbb{R})$. This means that the geodesics of $\mathrm{SL}(2,\mathbb{R})$ are the projection of geodesics of $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ via $\widetilde{g} \mapsto \widetilde{g}N$, which is a Lorentzian covering and a group homomorphism whose kernel is precisely N.

Remark. Let $\widetilde{\Gamma}$ be a discrete subgroup in $\widetilde{\mathrm{SL}}(2,\mathbb{R})$. If the quotient space $\widetilde{\mathrm{SL}}(2,\mathbb{R})\,/\,\widetilde{\Gamma}$ is compact, then N cannot be contained in $\widetilde{\Gamma}$. In fact, if $N < \widetilde{\Gamma}$ then one could construct a continuous map from $\widetilde{\mathrm{SL}}(2,\mathbb{R})\,/\,\widetilde{\Gamma}$ to $\mathrm{SL}(2,\mathbb{R})$, which is suryective, giving a contradiction.

On the other hand for any cocompact lattice Γ in $\mathrm{SL}(2,\mathbb{R})$ there exists a cocompact lattice $\widetilde{\Gamma} < \widetilde{\mathrm{SL}}(2,\mathbb{R})$. Indeed $N < \widetilde{\Gamma}$.

Theorem 3.10. *Let $\widetilde{\Gamma}$ be a lattice of $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ and let $\widetilde{M} = \widetilde{\mathrm{SL}}(2,\mathbb{R})\,/\,\widetilde{\Gamma}$. Then*

- *timelike geodesics of $\widetilde{M}$ are either all closed or all open;*

Proof. Consider the timelike geodesics of $\mathrm{SL}(2,\mathbb{R})$ with $k=1$, of course the graph of the geodesics is independent of k. By equation 2.2 these have period 2π and can be written as $\alpha_C(t) = C^{-1}R(t)C$, for $C \in \mathrm{SL}(2,\mathbb{R})$. Define $\widetilde{\alpha}_C$ the geodesic of $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ such that $p \circ \widetilde{\alpha}_C = \alpha_C$, then it must be $\widetilde{\alpha}_C(2\pi m) \in N$. Additionally $\widetilde{\alpha}_C$ is given by $\widetilde{\alpha}_C(t) = \widetilde{exp}\left(tC^{-1}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} C\right)$ which is a continuous function of C for a fixed t, then since $\widetilde{\alpha}_C(2\pi m) \in N$ it must be constant as a function of C, therefore if one $\widetilde{\alpha}_C$ is closed they must all be closed. $\square$

A special case happens when the lattice $\widetilde{\Gamma}$ of $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ contains the normal subgroup N, so that $N < \widetilde{\Gamma} < \widetilde{\mathrm{SL}}(2,\mathbb{R})$. In this case $\widetilde{\mathrm{SL}}(2,\mathbb{R})\,/\,\widetilde{\Gamma}$ has the same geometric structure as the quotient $\mathrm{SL}(2,\mathbb{R})\,/p(\widetilde{\Gamma})$ according to next proposition.

Proposition 3.11. *Let $\widetilde{\Gamma}$ and $\widetilde{\alpha}$ be respectively a lattice and a geodesic of $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ such that $N < \widetilde{\Gamma}$, and consider the following diagrams*

$$\begin{tikzcd}
 & \widetilde{SL}(2,\mathbb{R}) \arrow[ddl, "\widetilde{\pi}"'] \arrow[d, "p"] \\
 & \mathrm{SL}(2,\mathbb{R}) \arrow[d, "\pi"] \\
\widetilde{SL}(2,\mathbb{R})/\widetilde{\Gamma} \arrow[r, "R"] & \mathrm{SL}(2,\mathbb{R})\,/p(\widetilde{\Gamma})
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \widetilde{\alpha} \arrow[ddl, "\widetilde{\pi}"'] \arrow[d, "p"] \\
 & p \circ \widetilde{\alpha} \arrow[d, "\pi"] \\
\widetilde{\pi} \circ \widetilde{\alpha} \arrow[r, "R"] & \pi \circ p \circ \widetilde{\alpha}
\end{tikzcd}$$

where $R(\widetilde{g}\,\widetilde{\Gamma}) = p(\widetilde{g})p(\widetilde{\Gamma})$ and $p, \pi, \widetilde{\pi}$ are the covering projections. Then:

1. *The diagram is commutative,*
2. *$p(\widetilde{\Gamma})$ is a lattice of $\mathrm{SL}(2,\mathbb{R})$.*
3. *If $\pi \circ p \circ \widetilde{\alpha}$ is an open geodesic of $\mathrm{SL}(2,\mathbb{R})\,/p(\widetilde{\Gamma})$ then $\widetilde{\pi} \circ \widetilde{\alpha}$ is an open geodesic of $\widetilde{\mathrm{SL}}(2,\mathbb{R})\,/\,\widetilde{\Gamma}$*
4. *R has inverse $R^{-1}(p(\widetilde{g})p(\widetilde{\Gamma})) = \widetilde{g}\,\widetilde{\Gamma}$ and*
5. *R is an isometry*

Proof. For item (1), see first that R is well defined. For this let $\widetilde{g}, \widetilde{h} \in \widetilde{\mathrm{SL}}(2,\mathbb{R})$ such that $\widetilde{g}\,\widetilde{\Gamma} = \widetilde{h}\,\widetilde{\Gamma}$, which implies $\widetilde{h}^{-1}\widetilde{g} \in \widetilde{\Gamma}$. Now since p is a homomorphism one has that

$$R(\widetilde{g}\,\widetilde{\Gamma}) = R(\widetilde{h}\,\widetilde{\Gamma}) \iff p(\widetilde{g})p(\widetilde{\Gamma}) = p(\widetilde{h})p(\widetilde{\Gamma}) \iff p(\widetilde{h}^{-1}\widetilde{g}) \in p(\widetilde{\Gamma}).$$

To prove that $R \circ \widetilde{\pi} = \pi \circ p$ take $\widetilde{g} \in \widetilde{\mathrm{SL}}(2,\mathbb{R})$ and see that by definition $R \circ \widetilde{\pi}(\widetilde{g}) = \pi \circ p(\widetilde{g}) = \widetilde{g}N.p(\widetilde{\Gamma})$. In particular, since p and π are covering maps, R must be surjective. Moreover, locally R coincides with $\pi \circ p \circ \widetilde{\pi}^{-1}$ and therefore it is continuous. Since $\widetilde{\mathrm{SL}}(2,\mathbb{R}) \,/\, \widetilde{\Gamma}$ is compact it follows that $SL(2,\mathbb{R})/p(\widetilde{\Gamma})$ is compact and $p(\widetilde{\Gamma})$ is a lattice in $\mathrm{SL}(2,\mathbb{R})$.

Finally, since all the maps involved are Lorentzian coverings, the geodesic $\widetilde{\alpha} \in \widetilde{\mathrm{SL}}(2,\mathbb{R})$ projects to a geodesic on each of the three spaces below. In particular, if the projection $\pi \circ p(\widetilde{\alpha})$ on $\mathrm{SL}(2,\mathbb{R}) \,/ p(\widetilde{\Gamma})$ is open, it must hold that $p \circ \widetilde{\alpha}$ is also open in $\mathrm{SL}(2,\mathbb{R})$ and again $\widetilde{\alpha}$ is open in $\widetilde{\mathrm{SL}}(2,\mathbb{R})$. Now, since the diagram is commutative it holds that $\widetilde{\pi} \circ \widetilde{\alpha}$ is open in $\widetilde{\mathrm{SL}}(2,\mathbb{R}) \,/\, \widetilde{\Gamma}$.

□

In the conditions of last proposition there is an equivalent theorem for $\widetilde{\mathrm{SL}}(2,\mathbb{R}) \,/\, \widetilde{\Gamma}$ as theorem (3.10) for $\mathrm{SL}(2,\mathbb{R}) \,/\Gamma$.

Corollary 3.12. *Let $\widetilde{\Gamma}$ be a lattice of $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ with $N < \widetilde{\Gamma}$, then the geodesics of the quotient $M = \widetilde{\mathrm{SL}}(2,\mathbb{R}) \,/\, \widetilde{\Gamma}$ have the next properties:*

- *no lightlike geodesic is closed;*
- *every timelike geodesic is closed;*
- *there are both open and closed spacelike geodesics.*

Let $\widetilde{\Gamma}$ be a lattice of $\widetilde{\mathrm{SL}}(2,\mathbb{R})$, then no lightlike geodesic on the compact space $\widetilde{M} = \widetilde{\mathrm{SL}}(2,\mathbb{R}) \,/\, \widetilde{\Gamma}$ is closed.

Proof. By Proposition 3.11 $p(\widetilde{\Gamma})$ is a lattice of $\mathrm{SL}(2,\mathbb{R})$. Since every lighlike geodesic of $\mathrm{SL}(2,\mathbb{R})$ is open and has the form $\pi \circ p \circ \widetilde{\alpha}$, with $\widetilde{\alpha}$ a lightlike geodesic of $\widetilde{\mathrm{SL}}(2,\mathbb{R})$, then $\widetilde{\pi} \circ \widetilde{\alpha}$ is open and lightlike in $\widetilde{\mathrm{SL}}(2,\mathbb{R}) \,/\, \widetilde{\Gamma}$. □

This could be different on $\widetilde{\mathrm{SL}}(2,\mathbb{R}) \,/\, \widetilde{\Gamma}$. There, a geodesic $\widetilde{\pi} \circ \widetilde{\alpha}$ starting at $o = \pi(\widetilde{e})$ is closed if and only if $\alpha(t) \in \widetilde{\Gamma}$ for some $t > 0$ and $p \circ \widetilde{\alpha}$ being closed in $\mathrm{SL}(2,\mathbb{R})$ means that $\widetilde{\alpha}(t') \in N$ for some $t' > 0$, so in particular, $\pi \circ \widetilde{\alpha}$ would be closed if $N < \widetilde{\Gamma}$; it is unknown to the authors of this paper if there are lattices of $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ with that property. ????????????????

4. The case of the Oscillator groups

In this section we study geodesics on Lorentian compact spaces $M = \mathrm{Osc}_{\mathrm{n}}(\lambda_1, \ldots, \lambda_{\mathrm{n}}) \,/\Gamma$, where Γ is a cocompact lattice in $\mathrm{Osc}_{\mathrm{n}}(\lambda_1, \ldots, \lambda_{\mathrm{n}})$. The following results shows a condition to construct such lattices.

Lemma 4.1. [5] *An oscillator group $\mathrm{Osc}_{\mathrm{n}}(\lambda_1, \ldots, \lambda_{\mathrm{n}})$ admits a lattice if and only if the numbers λ_j generate an additive discrete subgroup of $\mathbb{R}$.*

In the demonstration of the lemma it is also shown that for a lattice Γ, the set $T(\Gamma) := \{t \in \mathbb{R} : (z,u,t) \in \Gamma\}$ is an additive discrete subgroup of $\mathbb{R}$, also shown in [3], page 93.

To find the smallest positive element of $T(\Gamma)$ one can do the following computation, found useful for many purposes in [5] and [3], where $(z,u,t),(w,b,0) \in \Gamma$

$$(z,u,t)^n.(w,b,0).(z,u,t)^{-n} = (w, e^{ntN_\lambda}b, 0).$$

For any n, these elements have a fixed norm and form a sequence in the lattice contained in a compact set, therefore it is a finite set. By computing the exponential e^{ntN_λ} it follows that, taking $t = t_0 := min_{t>0}\{t \in T(\Gamma)\}$, it follows that

$$t_0 = \frac{2\pi k_i}{N\lambda_i}, \tag{20}$$

for some positive integers N, k_i with $i = 1, ..., n$.

In [3], the author presents a very detailed description of the lattices of the oscillator groups, he does this by presenting a family of groups named $Osc_n(w,B)$ and isomorphisms between them, all this groups have base set $\mathbb{R}^{2n+2}$. The oscilator groups of this work can be recover from such family as $\mathrm{Osc_n}(\lambda_1,\ldots,\lambda_\mathrm{n}) = Osc_n(w_{(1,...,1)}, N_\lambda)$ where $w_{(1,...,1)}(v_1,v_2) = v_1^T J v_2$, see page 6 of Fisher's paper.

Theorem 5 states that for a given lattice L of $\mathrm{Osc_n}(\lambda_1,\ldots,\lambda_\mathrm{n})$, there exists an isomorphism between $\mathrm{Osc_n}(\lambda_1,\ldots,\lambda_\mathrm{n})$ and some $Osc_n(w,B)$ such that $\Phi(L) = L(\xi_0)$, where $L(\xi_0)$ is notation for a lattice in $Osc_n(w,B)$ generated by

$$\{(1,0,0),(0,e_i,0),(0,\xi_0,1)\}$$

, with ξ_0 a fixed element of R^{2n}.

In particular $(1,0,0) \in L(\xi_0)$ is in the image of Φ, and it is easy to check, since the explicit form of Φ is provided in the paper that $\Phi^{-1}(1,0,0) = (w,0,0)$ for some non-zero w.

For lattices where $N = 1$, it happens that $e^{t_0 N_\lambda} = Id$, and by using Φ again, one can construct another tipe of elements, as shown in the next lemma.

Lemma 4.2. [3] *Let* Γ *be any lattice of* $\mathrm{Osc_n}(\lambda_1,\ldots,\lambda_\mathrm{n})$*, then there always exists* $w \neq 0 \in \mathbb{R}$ *such that*

$$(w,0,0) \in \Gamma. \tag{21}$$

Also if Γ *is such that* $t_0 = \frac{2\pi k_i}{\lambda_i}$ *for positive integers* k_i *then there exists an element of the form*

$$\gamma = (z,0,t) \tag{22}$$

with z *and* t *not zero.*

Proof. The first part of the lemma was already proved.

From equation (4), page 9 of Fischer's paper one can see that in this scenario where $e^B = Id$ that $\xi_0^j \in \{-1,0,1\}$, $j = 1, ..., 2n$ and so: starting with the element

$(0, \xi_0, 1)$ and multiplying it, at most $2n$-times by either $(0, e_i, 0)$ or $(0, -e_i, 0)$ one obtains and element of the desired form. Finally since $e^B = Id$ the isomorphism Φ^{-1} takes this element in $Osc_n(w, B)$ to $\mathrm{Osc_n}(\lambda_1, \ldots, \lambda_n)$ preserving the nullity of the central component. □

Example 4.3. The first part of the lemma above has immediate consequences for the lightlike geodesics. In fact according to the Equations (13), (for the case $a = 0$), the condition of being lightlike imposes $b_j = c_j = 0$, and so the geodesics take the form

$$\alpha_d(s) = (ds, 0, 0).$$

which must neccesarily intersect any lattice and the associated lightlike geodesics in the quotient manifold will be closed.

Example 4.4. In the case where Γ contains an element of the form $(0, 0, t)$, with $t \neq 0$ it follows that any lightlike geodesic, say α, with $a \neq 0$ intersect the element $(0, 0, Nt) \in \Gamma$ since $\alpha(Nt) = (0, 0, Nt)$.

Theorem 4.5. *Let Γ be a cocompact lattice of $\mathrm{Osc_n}(\lambda_1, \ldots, \lambda_n)$, and consider the compact Lorentzian manifold $M = \mathrm{Osc_n}(\lambda_1, \ldots, \lambda_n)/\Gamma$, then only one of the following situations occurs*

- *either every lightlike geodesic of M is closed, or*
- *at every point in M there is exactly one direction for which all lightlike geodesics of M are closed and the rest are not-closed.*

Proof. Recall that it suffices to study the geodesics starting at $o := \pi(e)$ and that every geodesic $\hat{\alpha}$ is the projection of some geodesic, α, on $\mathrm{Osc_n}(\lambda_1, \ldots, \lambda_n)$: $\hat{\alpha} = \pi(\alpha)$ with $\alpha(0) = e$. Also, $\hat{\alpha}$ is closed in M if $\alpha(s) \in \Gamma$ for some $s > 0$.

As discussed after lemma **??**, all lightlike geodesics of the form $\pi((ds, 0, 0))$ are closed in M, and so geodesics pointing on this direction will always be closed. Therefore, to prove the theorem it must be that all the other lightlike geodesics are either closed or non is closed.

Let α be a lightlike geodesic from the remaining family, and suppose it it closed, this means that there exists some $\gamma = (z, u, t) \in \Gamma$ for which $\alpha(s) = \gamma$ for some $s > 0$. Since the curve α is a one-parameter subgroup of $\mathrm{Osc_n}(\lambda_1, \ldots, \lambda_n)$, then for any integer m: $\alpha(ms) = \gamma^m$, which is an element of Γ. Recall also that since $t \in T(\Gamma)$ it is of the form rt_0 for some integer r and $t_0 = \frac{2\pi k_i}{N\lambda_i}$. Finally, since $s = \frac{t}{a}$, one can compute that $\gamma^N = \alpha(N\frac{t}{a}) = (0, (0, 0), Nt) = (0, 0, Nrt_0)$, and therefore, since Nr is an integer, this element is in the lattice and every lightlike geodesic of M is closed.

In conclusion, when an element of the form $(0, 0, kt_0)$ is in the lattice every lightlike geodesic of M is closed, otherwise only $\hat{\alpha}_d(s) = (ds, 0, 0)\Gamma$ are.

□

Corollary 4.6. *Let Γ be a cocompact lattice of* $\mathrm{Osc_n}(\lambda_1,\dots,\lambda_n)$*, then the lightlike geodesics of* $M = \mathrm{Osc_n}(\lambda_1,\dots,\lambda_n)/\Gamma$ *are all closed if and only if* Γ *contains an element of the form* $(0,0,t)$*.*

Example 4.7. Both situations stated in the above theorem are possible. Take for example the three families of cocompact lattices constructed in [2] for $Osc(1)$, all the dimenstion four oscillator groups are isomorphic to $Osc_1(1)$,

$$\begin{aligned}\Lambda_{n,0} &= \frac{1}{2n}\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}\times 2\pi\mathbb{Z},\\ \Lambda_{n,\pi} &= \frac{1}{2n}\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}\times \pi\mathbb{Z},\\ \Lambda_{n,\frac{\pi}{2}} &= \frac{1}{2n}\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}\times \frac{\pi}{2}\mathbb{Z},\end{aligned}$$

where $n\in\mathbb{N}$, for which the authors proved that all lightlike geodesics of $M_{n,0} = Osc_1(1)/\Lambda_{n,0}, M_{n,\pi} = Osc_1(1)/\Lambda_{n,\pi}$ and $M_{n,\pi/2} = Osc_1(1)/\Lambda_{n,\pi/2}$ are closed. However other lattices can be obtained by noticing that

$$\begin{aligned}\phi_m &: Osc_1(1)\to Osc_1(1)\\ \phi_m(z,x,y,t) &= (z+mt,x,y,t),\ m\in\mathbb{R}\end{aligned}$$

are automorphisms of $Osc_1(1)$. So, the lattices $\phi(\Lambda_{n,\bullet})$ most likely do not contain an element of the form $(0,0,t)$. For example, given an integer $p\neq 0$, the lattice $\phi_p(\Lambda_n,0)$ does not contain such element since $\frac{a}{2n} + p\,2\pi b = 0$ has no solution for integers a,b.

Lemma 4.8. *Let $\eta_1\in\Gamma$ intersected by a spacelike geodesic with initial speed defined by (d,b_j,c_j,a), then there exists $x\in\mathbb{Z}$ such that $(z+xw,b_j,c_j,a)$ defines a timelike geodesic intersecting an element η_2 of Γ.*

For any lattice Γ of $\mathrm{Osc_n}(\lambda_1,\dots,\lambda_n)$ *there are closed timelike and spacelike geodesics of* $\mathrm{Osc_n}(\lambda_1,\dots,\lambda_n)/\Gamma$*.*

Proof. Let α be a geodesic with initial velocity defined by parameters (d_0,b_j,c_j,a), with $a\neq 0$ as in (9), and let $\hat\gamma = (\hat z,\hat u,\hat t)$ an element of the lattice Γ and define $\eta_1 = \gamma^N$ and $\eta_2 = \gamma^{-N}$‘. The condition that $\alpha(\hat s) = \hat\gamma$ for some $\hat s > 0$ leads to $\hat s = \frac{\hat t}{a}$ and the following two expresions:

$$\begin{pmatrix}\sin\lambda_j\hat t & \cos\lambda_j\hat t - 1\\ 1-\cos\lambda_j\hat t & \sin\lambda_j\hat t\end{pmatrix}\begin{pmatrix}b_j\\ c_j\end{pmatrix} = \begin{pmatrix}\hat{b_j}\\ \hat{c_j}\end{pmatrix}, \tag{23}$$

$$\hat z = \left(d + \frac{1}{2a}\sum_{k=1}^{n}\frac{b_k^2+c_k^2}{\lambda_k}\right)\frac{\hat t}{a} - \frac{1}{2a^2}\left(\sum_{k=1}^{n}\frac{b_j^2+c_j^2}{\lambda_k^2}\sin(\lambda_k\hat t)\right). \tag{24}$$

Aiming to construct a closed geodesic, one can find solutions for (b_j,c_j) in (23) when $\hat t\neq\frac{2\pi m}{\lambda_j}$. In lattices where $N>1$ (with N as in 20), elements of this kind exists, however if $N=1$ the only other posibility to solve (23) is that $(\hat{b_j},\hat{c_j}) = (0,0)$, equivalently that elements of the form $(\hat{z}, 0, \hat{t})$ exists; and this is exactly the case of the second part of lemma 4.2.

Finally since d is yet to set, equation (24) can define d as a function of a, for any $a \neq 0$.

This system admits solutions for $\hat{t} \neq \frac{2\pi m}{\lambda_j}$. If this holds, one can adjunst the final component $\hat{z}$ of $\hat{\gamma}$ as follows

and since both a and d are still unassinged they can be used to construct a geodesic that holds the desired condition.

□

Lemma 4.9. *ver que se puede decir mas fuerte de este lemma. existen geodeicas inducidas por alsa cerradas que no se cierran.*

For any lattice Γ *of* $\mathrm{Osc_n}(\lambda_1, \ldots, \lambda_n)$ *there are open timelike and spacelike geodesics of* $\mathrm{Osc_n}(\lambda_1, \ldots, \lambda_n)/\Gamma$.

Proof. Let Γ be a lattice and $\hat{\gamma} = (\hat{z}, \hat{u}, \hat{t}) \in \Gamma$, where $\hat{t} = Nt_0 = \frac{2\pi k_i}{\lambda_i}$ with k_i integers as seen in the discussion after Lemma 4.1, this can be achieved by taking any element of the lattice and taking its Nth power. Let $\hat{s}$ such that $\alpha(\hat{s}) = (z(\hat{s}), u(\hat{s}), t(\hat{s})) = \hat{\gamma}$, where α is a geodesic of $\mathrm{Osc_n}(\lambda_1, \ldots, \lambda_n)$ (with $a \neq 0$). Then it must be $t(\hat{s}) = a\hat{s} = Nt_0$, which implies $u(\hat{s}) = 0$ and $z(\hat{s}) = (d + \frac{1}{2a}\sum_{k=1}^{n} \frac{b_k^2 + c_k^2}{\lambda_k})\frac{Nt_0}{a}$, with a, b_k, c_k, d define the initial velocity of α.

For any $\epsilon > 0$, consider $I_d := [d, d+\epsilon]$, if for any d' in I_d the geodesic of initial velocity given by a, b_k, c_k, d' were to intercept the lattice at s', $\alpha_{d'}(s') \in \Gamma$ then it must be $t'(s') = r' t_0$ for some integer r'. This way one can define a function $F : I_d \to \mathbb{Z}$, and it is easy to see that there is an infinitely repeating element in the image of F, call it r_∞. Then $A_d := \{d' \in I_d : F(d') = r_\infty\}$ is bounded and contains a convergent sequence, call it d'_n.

Finally

$$\alpha_{d'_n}(\frac{r_\infty t_0}{a}) = \left(\left(d'_n + \frac{1}{2a}\sum_{k=1}^{n}\frac{b_k^2 + c_k^2}{\lambda_k}\right)\frac{\hat{t}}{a} - \frac{1}{2a^2}\left(\sum_{k=1}^{n}\frac{b_j^2 + c_j^2}{\lambda_k^2}\sin(\lambda_k \hat{t})\right), constant, r_\infty t_0\right)$$

is a convergent sequence of elements of the lattice, which can not be since Γ is discrete.

□

5. Final Remarks

Consider the group $G = \widetilde{\mathrm{SL}}(2, \mathbb{R}) \times \mathbb{R}$, this group is simply connected, has a bi-invariant Lorentzian metric and admits cocompact lattices, however its Lie algebra is not indecomposable. This last fact affects the geometry of $M = G/\Gamma$, for Γ a cocompact lattice. In fact when studying the lightlike geodesics of M, since a geodesic of $\mathbb{R}$ is $\gamma(t) = at$, and let $\hat{\alpha}$ be a geodesic of $\widetilde{\mathrm{SL}}(2, \mathbb{R})$ then

$$0 = < \hat{\alpha}'(0), \hat{\alpha}'(0) > + a^2 \tag{25}$$

must hold, where $<,>$ is the killing form of $\mathrm{SL}(2,\mathbb{R})$. It follows that $\hat{\alpha}$ must be a timelike geodesic of $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ and so $\alpha := p(\hat{\alpha})$ is periodic because of proposition (2.2), additionally, there is $t > 0$ such that $\hat{\alpha(t)} \in N$.

To construct a glosed lightlike geodesic in the compact Lorentzian quotient manifold $M = \widetilde{\mathrm{SL}}(2,\mathbb{R}) \times \mathbb{R} \ / \ \widetilde{\Gamma} \times A$ one can continue as follows:

(1) Choose $\widetilde{\Gamma}$ such that $N \subset \widetilde{\Gamma}$
(2) For a timelike geodeisc $\hat{\alpha}$ of $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ choose $a \in R$ such that (25) holds.
(3) Finally, choose $m = at$ where t is such that $\hat{\alpha(t)} \in N$.

The former procedure ensures that the lightlike geodesic $\hat{\alpha}(t), at$ of the product group is closed when projected to the quotient manifold.

VER: aca volvio a surjir lo de $N < \widetilde{\Gamma}$ aunque no sea necesario, creo que deberiamos verlo porque no creo podamos constuirlas de otra forma.

5.1. **Remarks.** Consider the group $G = \mathrm{Osc_n}(\lambda_1, \ldots, \lambda_n) \times \mathbb{R}$, this group is simply connected, has a bi-invariant Lorentzian metric and admits cocompact lattices, however its Lie algebra is not indecomposable. This last fact affects the geometry of $M = G/\Gamma$, for Γ a cocompact lattice. Take for example the lightlike geodesics of M, since a geodesic of $\mathbb{R}$ is $\gamma(t) = rt$, and let α be a geodesic of $\mathrm{Osc_n}(\lambda_1, \ldots, \lambda_n)$ then $c = (\alpha, \gamma)$ is lightlike in M if

$$0 = < \alpha'(0), \alpha'(0) > + r^2 \tag{26}$$

holds, where $<,>$ is the metric of the oscilator (7) at the identity. It follows that α must be a timelike geodesic of $\mathrm{Osc_n}(\lambda_1, \ldots, \lambda_n)$. Choose $\Lambda_{n,0}$, a lattice of $Osc_1(1)$, and $w\mathbb{Z}$ a lattice of $\mathbb{R}$, which is the case for any real $w \neq 0$, then $\Lambda_{n,0} \times w\mathbb{Z}$ is a lattice of $G = Osc_1(1) \times \mathbb{R}$.

The lightlike condition in this case, the equation (26), is

$$0 = 2ad + \frac{b+c}{2a} + r^2$$

Should c be a closed lightlike geodesic of G then, $\alpha(s) \in \Lambda_{n,0}$ and $r(s) \in w\mathbb{Z}$ for some $s > 0$. From the equations of α it follows that $s = \frac{2\pi k}{a}$ for some $k \in \mathbb{Z}$ and $z(\frac{2\pi k}{a}) = (d + \frac{b+c}{2a})(\frac{2\pi k}{a}) = \frac{m}{2n}$ for some $m \in \mathbb{Z}$. This can be reduced to

$$r^2 = -\frac{a^2 m}{2\pi k},$$

also it must be that $\gamma(\frac{2\pi k}{a}) = r\frac{2\pi k}{a} = wz \to r^2 = \frac{a^2 z^2 w^2}{(2\pi k)^2}$, then, since $a \neq 0$

$$w^2 = -\frac{2\pi km}{z^2}.$$

In conclusion, since it is possible to choose w that never hold for any $k, z \in \mathbb{Z}$, take $w = e$ for example, for such lattices lightlike geodesics of $Osc_1(1) \times \mathbb{R}$ are never closed.

Lo siguiente es para el otro paper

6. Isometries of homogeneous compact Lorentzian manifolds

In this section we study the isometries of the compact quotients considered before.

Let G denote a Lie group endowed with a left-invariant metric. Its isometry group $Iso(G)$ can be written as a product $Iso(G) = L(G)F(G)$ where $L(G)$ denote the subgroup of translations on the left and $F(G)$ is a closed subgroup consisting of those isometries which fix the identity element.

Example 6.1. Let G be a Lie group equipped with a bi-invariant metric. Then inner automorphisms $I_g : G \to G$, as $I_g(x) = gxg^{-1}$ belongs to $F(G)$ so as the inverse map $\iota : G \to G$, given by $\iota : x \to x^{-1}$.

To describe completely the subgroup $F(G)$ for bi-invariant metrics, the following result from Müller [7] will come in handy.

Theorem 6.2. *Let G be a Lie group with Lie algebra $\mathfrak{g}$ such that $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$ and $\mathfrak{g}/\mathfrak{r}$ is simple. Let Q be a pseudo-Riemannian bi-invariant metric on G. Then an isometry Φ belongs to $F(G)$ if and only if it is an automorphism or an anti-automorphism of G such that $d\Phi_e$ leaves the form Q_e on $\mathfrak{g} \times \mathfrak{g}$ invariant.*

Recall that an *anti-automorphism* of a Lie group G is a map $f : G \to G$ such that $f(gh) = f(h)f(g)$ for all $g, h \in G$.

Notice that the inversion map $\iota : G \to G$, $x \mapsto x^{-1}$ is an anti-automorphism of G since $(gh)^{-1} = h^{-1}g^{-1}$.

Furthermore, take an automorphism $\phi : G \to G$, then $\iota \circ \phi$ is an anti-automorphism, since

$\iota \circ \phi(gh) = \iota(\phi(g)\phi(h)) = \iota(\phi(h))\iota(\phi(g))$ for all $g, h \in G$. Similarly one proves that for any anti-automorphism $\psi : G \to G$, $\iota \circ \psi$ is an automorphism of G. This proves the next result.

Lemma 6.3. *Let G be a Lie group with inversion map $\iota : G \to G$. Then ι defines a bijection between the set of automorphisms and the set of anti-automorphisms of G.*

Notice that the set of automorphisms builds a group. Let $Inn(G)$ be the group subgroup of $F(G)$ corresponding to the inner automorphisms of G and let $F_0(G)$ denote the connected component of the identity in $F(G)$.

Example 6.4. Consider the Lie group $\mathrm{SL}(2, \mathbb{R})$. The map $\chi : \mathrm{SL}(2, \mathbb{R}) \to \mathrm{SL}(2, \mathbb{R})$ given by

$$\chi \begin{pmatrix} g_1 & g_2 \\ g_3 & g_4 \end{pmatrix} := \begin{pmatrix} g_1 & -g_2 \\ -g_3 & g_4 \end{pmatrix}$$

is an automorphism of $\mathrm{SL}(2, \mathbb{R})$.

Automorphisms of $\mathrm{SL}(2, \mathbb{R})$. Let Φ be an isometric automorphism of $\mathrm{SL}(2, \mathbb{R})$. Its differential at the identity, namely $A := d\Phi_e$ satisfies

(1) $A[X,Y]=[AX,AY]$,
(2) and the isometry condition: $< AX, AY >=< X, Y >$.

Recall that $\mathfrak{sl}(2,\mathbb{R})$ has a basis $\mathfrak{B}=\{H,E,F\}$ of the form:

$$H := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad E := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad F := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Making use of this basis, one gets the following matrix for $A_{\mathfrak{B}}$

$$A = \begin{pmatrix} u & a & p \\ v & b & q \\ w & c & r \end{pmatrix}.$$

In this basis, through computation, the isometry condition reduces to the following equation

$$u^2 + vw = 1,$$

however, this same equation arises as a consequence of the automorphism condition $[AX, AY] = A[X,Y]$ together with requiring A to be a vector field isomorphism, therefore every automorphism is isometric in the case of $\mathrm{SL}(2,\mathbb{R})$.

By computing the possible matices for $A_{\mathfrak{B}}$ one can parametrise the results as following:

For $x, y \in \mathbb{R} - \{0\}$, $z \in \mathbb{R} - \{-1, 1\}$:

$$\begin{pmatrix} z & x & \frac{1-z^2}{4x} \\ y & \frac{xy}{z-1} & \frac{y(1-z)}{4x} \\ \frac{1-z^2}{y} & \frac{x(1-z)}{y} & \frac{(1+z)^2(z-1)}{4xy} \end{pmatrix} \tag{27}$$

For $x \in \mathbb{R}$, $y \in \mathbb{R} - \{0\}$:

$$\begin{pmatrix} 1 & 0 & -\frac{xy}{2} \\ x & \frac{1}{y} & -\frac{x^2y}{4} \\ 0 & 0 & y \end{pmatrix}, \begin{pmatrix} 1 & -\frac{xy}{2} & 0 \\ 0 & y & 0 \\ x & -\frac{xy^2}{4} & \frac{1}{y} \end{pmatrix}, \begin{pmatrix} -1 & \frac{xy}{2} & 0 \\ x & -\frac{x^2y}{4} & \frac{1}{y} \\ 0 & y & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 & \frac{xy}{2} \\ 0 & 0 & y \\ x & \frac{1}{y} & -\frac{x^2y}{4} \end{pmatrix}$$

The next goal is to see to which automorphisms correspond the former matrices in order to classify them and to make sure that they actually correspond to existing isometries of $\mathrm{SL}(2,\mathbb{R})$.

Let $\mathrm{Ad}(g)$ denote the automorphism of $\mathfrak{sl}(2,\mathbb{R})$ that is the differential at the identity of the inner automorphism of $\mathrm{SL}(2,\mathbb{R})$, $I_g : \mathrm{SL}(2,\mathbb{R}) \to \mathrm{SL}(2,\mathbb{R})$ given by $I_g : h \to ghg^{-1}$. Let $g \in \mathrm{SL}(2,\mathbb{R})$ be:

$$\begin{pmatrix} g_1 & g_2 \\ g_3 & g_4 \end{pmatrix} \qquad \text{with } g_1g_4 - g_2g_3 = 1$$

By computing the matrix of $\mathrm{Ad}(g)$ in the basis $\mathfrak{B}$ it follows that

$$Ad(g)_{\mathfrak{B}} = \begin{pmatrix} g_1g_4 + g_2g_3 & -g_1g_3 & g_2g_4 \\ -2g_1g_2 & g_1^2 & -g_2^2 \\ 2g_3g_4 & -g_3^2 & g_4^2 \end{pmatrix}$$

It can be shown that every automorphism is either an inner automorphism or the composition of χ, from example 6.4, with an automorphism. The procedure to see this is as follows:

- The matrices in (27) with $xy(z-1) > 0$ and correspond to inner automorphisms, to see this, take:

  For $y > 0$, take the following value for g

$$g = \begin{pmatrix} g_1 & g_2 \\ g_3 & g_4 \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{va}{u-1}} & -\sqrt{\frac{v(u-1)}{4a}} \\ \frac{u-1}{2g_2} & \frac{u+1}{2g_1} \end{pmatrix}$$

  For $y < 0$, take the following value for g

$$g = \begin{pmatrix} g_1 & g_2 \\ g_3 & g_4 \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{va}{u-1}} & \sqrt{\frac{v(u-1)}{4a}} \\ \frac{u-1}{2g_2} & \frac{u+1}{2g_1} \end{pmatrix}$$

- The matrices in (27) with $xy(z-1) < 0$ and correspond to automorphisms of the form $\chi \circ I_g$, to see this note first that $d\chi_e = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ and so

$$d\chi_e \circ I_g = \begin{pmatrix} g_1g_4 + g_2g_3 & -g_1g_3 & g_2g_4 \\ +2g_1g_2 & -g_1^2 & +g_2^2 \\ -2g_3g_4 & g_3^2 & -g_4^2 \end{pmatrix}$$

  Now, for $y > 0$, take the following value for g

$$g = \begin{pmatrix} g_1 & g_2 \\ g_3 & g_4 \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{va}{u-1}} & \sqrt{\frac{v(u-1)}{4a}} \\ \frac{u-1}{2g_2} & \frac{u+1}{2g_1} \end{pmatrix}$$

  And for $y < 0$, take the following value for g

$$g = \begin{pmatrix} g_1 & g_2 \\ g_3 & g_4 \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{va}{u-1}} & -\sqrt{\frac{v(u-1)}{4a}} \\ \frac{u-1}{2g_2} & \frac{u+1}{2g_1} \end{pmatrix}$$

  The four matrices parametrised by x, y correspond too to either inner automorphism or composition these with χ.

Proposition 6.5. *Let G denote a Lie group endowed with a bi-invariant metric. Then the following holds*

- $Inn(\mathrm{SL}(2,\mathbb{R})) = F_0(\mathrm{SL}(2,\mathbb{R}))$*;*
- *for any $\Phi \in F(\mathrm{SL}(2,\mathbb{R}))$, either*
  (1) $\Phi \in F_0(\mathrm{SL}(2,\mathbb{R}))$
  (2) $\Phi \in F_1(\mathrm{SL}(2,\mathbb{R})) := \{\chi \circ I_g : g \in \mathrm{SL}(2,\mathbb{R})\}$
  (3) $\Phi \in F_2(\mathrm{SL}(2,\mathbb{R})) := \{\iota \circ I_g : g \in \mathrm{SL}(2,\mathbb{R})\}$
  (4) $\Phi \in F_3(\mathrm{SL}(2,\mathbb{R})) := \{\iota \circ \chi \circ I_g : g \in \mathrm{SL}(2,\mathbb{R})\}$

These are four connected components of $F(G)$ where the first two correspond to the automorphisms and the last two to the anti-automorphisms of G.

The group $F(SL(2,\mathbb{R})) = (\mathbb{Z}_2 \bigoplus \mathbb{Z}_2) \times PSL(2,\mathbb{R})$

Proof. From theorem 6.2 the isometries that fix e are either automorphisms or anti-automorphisms, and by lemma 6.3 every anti-automorphism can be found by componing an automorphsims with the inverse function, then it suffices to find the automorphisms. And as shown before, the automorphisms of $\mathrm{SL}(2,\mathbb{R})$ are either the inner automorphisms, I_g or $\chi \circ I_g$. Finally it is easy to note that $I_g = I_{-g}$, consequantly $Inn(\mathrm{SL}(2,\mathbb{R})) \approx PSL(2,\mathbb{R})$.

The following proposition shows an easy way to recognise whether the differential of an isometry conrresponds to an automorphism or and anti-automorphism, and to which family of proposition 6.5 it belongs.

□

Proposition 6.6. *Let A be the matrix of the differential of some isometry, Φ, of $SL(2,\mathbb{R})$ at e in the basis $\mathfrak{B} = \{h, e, f\}$*

$$A = \begin{pmatrix} u & a & p \\ v & b & q \\ w & c & r \end{pmatrix}$$

then

(1) If $det(A) = 1$ $b > 0$ or $q < 0$, then $\Phi \in F_0(\mathrm{SL}(2,\mathbb{R}))$
(2) If $det(A) = 1$ $b < 0$ or $q > 0$, then $\Phi \in F_1(\mathrm{SL}(2,\mathbb{R}))$
(3) If $det(A) = -1$ $b < 0$ or $q > 0$, then $\Phi \in F_2(\mathrm{SL}(2,\mathbb{R}))$
(4) If $det(A) = -1$ $b > 0$ or $q < 0$ then $\Phi \in F_3(\mathrm{SL}(2,\mathbb{R}))$

7. Fuchsian Groups

Characherization of arithmeticity:

Theorem 7.1 (TAKEUCHI). *Let Γ be a cofinite Fuchsian group. Then Γ is arithmetic if and only if Γ satisfies the following two conditions:*

label=() $K := \mathbb{Q}(Tr(\Gamma))$ is an algebraic number field of finite degree and $Tr(\Gamma)$ is contained in the ring of integers $\mathcal{O}_K$ of K.

lbbel=() *Let K_2 be the field $\mathbb{Q}(tr(g)^2 : g \in \Gamma)$. For any embedding ϕ of K into $\mathbb{C}$, which is not the identity if restricted to K_2, the set $\phi(Tr(\Gamma))$ is bounded in $\mathbb{C}$.*

8. Isometries of oscillator groups

9. Isometries in the quotient

Theorem 9.1 (KATOK-WEIL-Ti)**.** *Every arithmetic is commesurable to a Fuchsian group derived from quaterion algebras over totally real number fields. In addition, only when the field is $\mathbb{Q}$ and the quaterion algebra is isomorphic to $M(2,\mathbb{Q})$ results that the arithmetic lattice is not cocompact.*

Theorem 9.2 (PETE)**.** *The normalizer of a Fuchsian group is a Fuchsian group*

Theorem 9.3. *$SL(2,\mathbb{R})$ does not have normal non trivial normal subgroups. Proof: use the fact that its centre is $\{Id, -Id\}$.*

Theorem 9.4 (Ragu 158)**.** *G conexo, semisimple sin compact factors, Γ un lattice entonces Γ contenido en finitos latices.*

Theorem 9.5 (Ragu 154)**.** *p de un lattice es discreto*

Therefore the inverse function of $\mathrm{SL}(2,\mathbb{R})$ is never fiber-preserving.

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