MBMT Tiebreaker Round — Weierstrass

March 9, 2025

DO NOT BEGIN UNTIL YOU ARE INSTRUCTED TO DO SO.

This round consists of **3** questions. You will have **15** minutes to complete the round.

For each of the 3 problems, you may submit an answer up to **3 times, whenever you want,** as long as it is during the round. **Only the last submission to each problem will be counted.**

In order to submit an answer, write your answer on the paper slip with the corresponding problem number. Then, fold the slip in half and place it on the grader's desk behind you.

Your placement will be determined first by the number of problems you solve and then by time of last correct counted submission to break any remaining ties. Please write your answers in a reasonably simplified form.

1. The numbers a, b, c, d, e, f, g, h, and i are the numbers 1, 2, 3, 4, 5, 6, 7, 8, and 9, not necessarily in that order. Given that $a+b=16$, $\frac{c}{d}=4$, and $e+f+g=8$, find hi.

Proposed by Ivy Guo.

Answer: $\boxed{30}$

Solution: Since $a+b=16$, a and b must be 7 and 9 in some order. Since $\frac{c}{d}=4$, either $c=4$ and $d=1$, or $c=8$ and $d=2$.

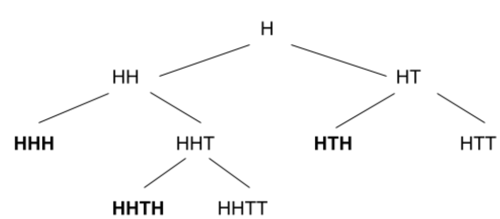
If $c=4$ and $d=1$, then the smallest $e+f+g$ could be is $2+3+5>8$. Therefore, we must have $c=8$ and $d=2$. Then, e, f, and g must be 1, 3, and 4 in some order, leaving h and i to be 5 and 6, so $hi=30$.

2. A fair coin is flipped repeatedly. What is the probability the sequence HTH appears before the sequence HHH?

Proposed by Ivy Guo.

Answer: $\boxed{\frac{3}{5}}$

Solution: If we ever get two T's in a row, it's equivalent to starting over. Consider the following diagram:



The bolded strings are end-states. The probability of reaching any given state on the fourth row is half the probability of reaching any given state on the third row, so the probability of ending with HTH is $\frac{1.5}{2.5}=\frac{3}{5}$.

3. Evaluate

$$\sum_{i=1}^{256} \gcd(256, i).$$

Proposed by Lewis Lau.

Answer: $\boxed{1280}$

Solution: There are 128 odd integers between 1 and 256; these integers have a gcd of 1 with 256. Of the even integers, exactly half of them are also multiples of 4, so there are 64 integers that have a gcd of 2 with 256. We can continue this pattern to see that there are 32 integers with a gcd of 4 with 256, and so on until 1 integer with a gcd of 128 with 256.

Lastly, $\gcd(256, 256) = 256$, so the final sum is

$$128 \cdot 1 + 64 \cdot 2 + \cdots + 1 \cdot 128 + 256 = 128 \cdot 8 + 256 = 1280.$$