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Excellent!

STAT 314: APPLIED TIME SERIES ANALYSIS
Mid-Term Exam (Take Home)

Due Date & Time: 4:00 pm, Thursday October 13, 2016.
Number of Questions: 3
Number of pages: 9

Show all steps to receive full credit. If you are using a theorem in text, please state theorem number and page number where you can find the theorem. If theorem is from notes, please indicate what the theorem is. **Please use these sheets to answer. Do not add more pages. You may use the reverse side of each page if you need additional space.**

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1. (20 points). Consider the covariance stationary time series $\{X_t : t \in \mathbb{Z}\}$ that satisfy the stochastic difference equation

$$(1 - 0.5B)(1 - 0.5B)(1 + 0.2B)X_t = Z_t \quad t \in \mathbb{Z}, \text{ where } \{Z_t\} \sim WN(0,1).$$

(a) Derive the Yule-Walker equations

$$\left(1 - B + \frac{B^2}{4}\right)(1 + 0.2B)x_t = Z_t$$

$$(1 - 0.8B + 0.05B^2 + 0.05B^3)x_t = Z_t$$

$$x_t - 0.8x_{t-1} + 0.05x_{t-2} + 0.05x_{t-3} = Z_t$$

$$x_t = (0.8)x_{t-1} + (-0.05)x_{t-2} + (-0.05)x_{t-3} + Z_t$$

$\{x_t \mid x_t = 0.8x_{t-1} - 0.05x_{t-2} - 0.05x_{t-3} + Z_t ; t \in \mathbb{Z}\}$ is AR(p=3) process. $\phi_1 = 0.8$; $\phi_2 = -0.05$ & $\phi_3 = -0.05$

here $\phi(z) = \left(1 - \frac{z}{2}\right)^2\left(1 + \frac{z}{5}\right) = 0 \Rightarrow |z_1| = 2 > 1; |z_2| = 2 > 1 \;\&\; |z_3| = 5 > 1$

$\forall |z| > 1 \Rightarrow X_t$ is causal

$X_t = \sum_{i=0}^{\infty} \psi_i Z_{t-i}$ &

$E(X_t) = 0 \Rightarrow cov(X_t \cdot X_{t+h}) = E(X_t \cdot X_{t+h})$

$$E[X_t \cdot X_{t-k}] = \phi_1 E[X_{t-1} \cdot X_{t-k}] + \phi_2 E[X_{t-2} \cdot X_{t-k}] + \phi_3 E[X_{t-3} \cdot X_{t-k}] + E\left[Z_t \cdot \sum_{i=0}^{\infty} \psi_i Z_{t-k-i}\right] \quad ; k = 0, 1, \ldots$$

$$\gamma(k) = \phi_1 \gamma(k-1) + \phi_2 \gamma(k-2) + \phi_3 \gamma(k-3) + \sum_{i=0}^{\infty} E[Z_t \cdot Z_{t-k-i}]$$

$\underline{k=0}$.

$$\gamma(0) = \phi_1 \gamma(1) + \phi_2 \gamma(2) + \phi_3 \gamma(3) + 1$$

$\underline{k=1}$.

$$\gamma(1) = \phi_1 \gamma(0) + \phi_2 \gamma(1) + \phi_3 \gamma(2)$$

$\underline{k=2}$.

$$\gamma(2) = \phi_1 \gamma(1) + \phi_2 \gamma(0) + \phi_3 \gamma(1)$$

$\underline{k=3}$.

$$\gamma(3) = \phi_1 \gamma(2) + \phi_2 \gamma(1) + \phi_3 \gamma(0)$$

In general $\Rightarrow$

$$\gamma(0) = 0.8\, \gamma(1) + (-0.05)\, \gamma(2) + (-0.05)\, \gamma(3) + 1$$

$$\gamma(h) = 0.8\, \gamma(h-1) + (-0.05)\, \gamma(h-2) + (-0.05)\, \gamma(h-3)$$

$$\forall\ h \geq 1$$

$$\gamma(0) - 0.8\gamma(1) + 0.05\gamma(2) + 0.05\gamma(3) + 0.\gamma(4) = 1 \quad -①$$

$$-0.8\gamma(0) + 1.05\gamma(1) + 0.05\gamma(2) + 0.\gamma(3) + 0.\gamma(4) = 0 \quad -②$$

$$.05\gamma(0) - 0.75\gamma(1) + 1.0\gamma(2) + 0.\gamma(3) + 0.\gamma(4) = 0 \quad -③$$

$$0.05\gamma(0) + 0.05\gamma(1) - 0.8\gamma(2) + 1.\gamma(3) + 0.\gamma(4) = 0 \quad -④$$

$$0.\gamma(0) + 0.05\gamma(1) + 0.05\gamma(2) - 0.8\gamma(3) + 1.\gamma(4) = 0 \quad -⑤$$

(b) Determine the autocovariance function at lags 0 through 4 using the Yule-Walker Equations.

$$\underbrace{\begin{bmatrix} 1 & -0.8 & 0.05 & 0.05 & 0 \\ -0.8 & 1.05 & 0.05 & 0 & 0 \\ 0.05 & -0.75 & 1 & 0 & 0 \\ 0.05 & 0.05 & -0.8 & 1 & 0 \\ 0 & 0.05 & 0.05 & -0.8 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} \gamma(0) \\ \gamma(1) \\ \gamma(2) \\ \gamma(3) \\ \gamma(4) \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{=b}$$

$X = \text{inv}(a) * b$ // by using matlab.

$$x = \begin{bmatrix} 2.2192 \\ 1.6376 \\ 1.1172 \\ 0.7009 \\ 0.4230 \end{bmatrix}$$

$\therefore \gamma(0) = 2.2192$

$\gamma(1) = 1.6376$

$\gamma(2) = 1.1172$

$\gamma(3) = 0.7009$

$\gamma(4) = 0.4230$

Part C

Theorem : →

Let $\{X_t ; t \in \mathbb{Z}\}$ be a covariance stationary time series satisfying $[\phi(B)] X_t = [\Theta(B)] Z_t$; $t \in \mathbb{Z}$ where $\{Z_t\} \sim WN(0; \sigma^2)$. Suppose $\Phi(z) = 0 \Rightarrow |z| > 1$.

Let ξ_i ; $i = 1, 2, \ldots, k$ be the distinct roots of $\phi(z) = 0$ &

let γ_i be the multiplicity of ξ_i ; $i = 1, 2, \ldots, k$.

Then we can write $X_t \overset{a.s}{=} \sum_{j=0}^{\infty} \psi_j Z_{t-j}$; $\forall t \in \mathbb{Z}$, where

$$\psi_j = \sum_{i=1}^{k} \sum_{l=0}^{\gamma_i - 1} \alpha_{il} (j)^l \xi_i^{-j} \quad \text{for } j$$

Note : The α_{il}'s are constant determined by the boundary conditions

$$\psi_j = \sum_{0 < l \leq j} \phi_l \psi_{j-l} + \theta_j \quad ; \quad 0 < j \leq \max(p; q+1).$$

$\theta_1 = 0;\ \phi_1 = 0.8;\ \phi_2 = -0.05;\ \phi_3 = -0.05$

$p = 3$

(c) Express the time series as an infinite moving average process and derive a closed form expression for the coefficients of this moving average representation in terms of the roots of the autoregressive polynomial.

$$\phi(z) = \left(1 - \frac{z}{2}\right)^2\left(1 + \frac{z}{5}\right) = 0 \Rightarrow \forall |z| > 1 \left\{ \begin{array}{l} |z_1| = |z_2| = 2 > 1 \\ |z_3| = 5 > 1 \end{array} \right\}$$

$\therefore \{x_t : t \in \mathbb{Z}\}$ is a causal process. Here $x_t \sim ARMA(p=3; q=0)$
$\sim AR(3)$.

$$\psi_j - \sum_{k=1}^{p} \phi_k \psi_{j-k} = \theta_j \quad ; j = 0, 1, \ldots$$

where $\theta_0 = 1$; $\theta_j = 0$ for $j > q$ &
$\psi_j = 0$ $j < 0$

$j=0$ • $\psi_0 = 1$

$j=1$ • $\psi_1 - \phi_1 \psi_0 = 0 \Rightarrow \psi_1 = 0.8$

$j=2$ • $\psi_2 - [\phi_1 \psi_1 + \phi_2 \psi_0] = 0 \Rightarrow \psi_2 = \phi_1 \psi_1 + \phi_2 \psi_0 = 0.59$

$j=3$ • $\psi_3 - [\phi_1 \psi_2 + \phi_2 \psi_1 + \phi_3 \psi_0] = 0 \Rightarrow \psi_3 = \phi_1 \psi_2 + \phi_2 \psi_1 + \phi_3 \psi_0 = 0.382$

$$\psi_j = \sum_{i=1}^{k} \sum_{l=0}^{r_i - 1} \alpha_{il}\, j^l \xi_i^{-j}$$

• Here ξ_i ; $i = 1, 2, \ldots, k$ be the distinct roots of $\phi(z) = 0$ &

r_i be the multiplicity of ξ_i ; $i = 1, 2, \ldots, k$

- $\xi_1 = 2$; $r_1 = 2$
- $\xi_2 = -5$; $r_2 = 1$
- $k = 2$

$$\psi_1 = \sum_{i=1}^{2} \sum_{l=0}^{r_i - 1} \alpha_{il} (1)^l \xi_i^{-1} = \sum_{l=0}^{1} \alpha_{1l} \xi_1^{-1} + \sum_{l=0}^{0} \alpha_{2l} \xi_2^{-1}$$

$$\psi_1 = \alpha_{10} \xi_1^{-1} + \alpha_{11} \xi_1^{-1} + \alpha_{20} \xi_2^{-1} \quad \text{—} (1)$$

$$\psi_2 = \sum_{i=1}^{2} \sum_{l=0}^{r_i - 1} \alpha_{il} (2)^l \xi_i^{-2} = \sum_{l=0}^{1} \alpha_{1l} (2)^l \xi_1^{-2} + \sum_{l=0}^{0} \alpha_{2l} (2)^l \xi_2^{-2}$$

$$\psi_2 = \alpha_{10} \xi_1^{-2} + 2\alpha_{11} \xi_1^{-2} + \alpha_{20} \xi_2^{-2} \quad \text{—} (2)$$

$$\psi_3 = \sum_{i=1}^{2} \sum_{l=0}^{r_i - 1} \alpha_{il} (3)^l \xi_i^{-3} = \sum_{l=0}^{1} \alpha_{1l} (3)^l \xi_1^{-3} + \sum_{l=0}^{0} \alpha_{2l} (3)^l \xi_2^{-3}$$

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$$\psi_3 = \alpha_{10} \xi_1^{-3} + 3\alpha_{11} \xi_1^{-3} + \alpha_{22} \xi_2^{-3} \quad \text{—} (3)$$

$$\begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}_{3\times 1} = \begin{pmatrix} \xi_1^{-1} & \xi_1^{-1} & \xi_2^{-1} \\ \xi_1^{-2} & 2\xi_1^{-2} & \xi_2^{-2} \\ \xi_1^{-3} & 3\xi_1^{-3} & \xi_2^{-3} \end{pmatrix}_{3\times 3} \begin{pmatrix} \alpha_{10} \\ \alpha_{11} \\ \alpha_{20} \end{pmatrix}_{3\times 1}$$

$$\begin{pmatrix} 0.8 \\ 0.59 \\ 0.382 \end{pmatrix}_{3\times 1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{5} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{25} \\ \frac{1}{8} & \frac{3}{8} & -\frac{1}{125} \end{pmatrix}_{3\times 3} \begin{pmatrix} \alpha_{10} \\ \alpha_{11} \\ \alpha_{20} \end{pmatrix}_{3\times 1}$$

• $\alpha_{10} = \left(\frac{45}{49}\right)$ // ; • $\alpha_{11} = \left(\frac{35}{49}\right)$ // ; • $\alpha_{20} = \left(\frac{4}{49}\right)$ //

$$\psi_j = \sum_{i=1}^{2} \sum_{\ell=0}^{\gamma_i - 1} \alpha_{i\ell} (j)^{\ell} \xi_i^{-j} = \sum_{\ell=0}^{\gamma_1 - 1} \alpha_{1\ell} (j)^{\ell} \xi_1^{-j} + \sum_{\ell=0}^{\gamma_2 - 1} \alpha_{2\ell} (j)^{\ell} \xi_2^{-j}$$

$$\psi_j = \alpha_{10} (j)^0 \xi_1^{-j} + \alpha_{11} (j)^1 \xi_1^{-j} + \alpha_{20} \xi_2^{-j}$$

$$\psi_j = \left[\alpha_{10} \xi_1^{-j} + j\alpha_{11} \xi_1^{-j} + \alpha_{20} \xi_2^{-j}\right]$$

$$\psi_j = \left(\frac{45}{49}\right)(2)^{-j} + j\left(\frac{35}{49}\right)(2)^{-j} + \left(\frac{4}{49}\right)(-5)^{-j}$$

$$\psi_j = \frac{[45 + 35j](2)^{-j} + (4)(-5)^{-j}}{49}$$

//

2. (15 points). Consider the covariance stationary time series $\{X_t : t \in \mathbb{Z}\}$ that satisfies

$$X_t - 1.0X_{t-1} + 0.24X_{t-2} = Z_t - 0.5Z_{t-1}, \text{ for } t \in \mathbb{Z}, \text{ where } \{Z_t\} \sim WN(0,1).$$

(a) Show that $\{X_t : t \in \mathbb{Z}\}$ is both causal and invertible.

$$\underbrace{[1 - 1.0z + 0.24z^2]x_t}_{\phi(z)\cdot x_t} = \underbrace{[1 - 0.5z]Z_t}_{\Theta(z)\cdot Z_t}$$

Theorem:
Text book: page 85
Def^n: Causality

$$\phi(z) = (1 - z + 0.24z^2) = 0 \Rightarrow |z_1| = |5/2| > 1 \; ; \; [\,|z_2| = |5/3| > 1$$

$\therefore$ X_t is causal process.

$$\Theta(z) = (1 - 0.5z) = 0 \Rightarrow |z| = 2 > 1$$

$\therefore$ X_t is invertible process.

Theorem:
Text book: page 86
Def^n: Invertibility

$$\{X_t : t \in \mathbb{Z}\} \sim ARMA(p=2; q=1)$$

(b) Given that $X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j}$ for all $t \in \mathbb{Z}$, find ψ_j for $j = 1, 2, 3$, and 4.

[Text book : Page 85]

$$X_t = \frac{[1 - 0.5z]}{[1 - z + 0.24z^2]} \cdot Z_t$$

$\phi_1 = 1 \; ; \; \phi_2 = (-0.24)$

$\theta_1 = (-0.5) \; ; \; p = 2$

where

$$\psi_j - \sum_{k=1}^{p} \phi_k \psi_{j-k} = \theta_j \; ; \; j = 0, 1, 2, 3, 4, \ldots$$

$\left[\theta_0 = 1 \; ; \; \theta_j = 0 \text{ for } j > q \;\&\; \psi_j = 0 \text{ for } j < 0\right]$

$j=0$ • $\psi_0 = 1$

$j=1$ • $\psi_1 - [\phi_1 \psi_0] = \theta_1 \Rightarrow \psi_1 = -0.5 + 1 = 1/2 \Rightarrow \psi_1 = 1/2$

$j=2$ • $\psi_2 - [\phi_1 \psi_1 + \phi_2 \psi_0] = \theta_2 \Rightarrow \psi_2 = 1 \cdot 1/2 + (-0.24) = 0.26$

• $\psi_2 = 0.26$

$j=3$ • $\psi_3 - [\phi_1 \psi_2 + \phi_2 \psi_1] = \theta_3 \Rightarrow \psi_3 = 0.26 + (-0.24)(1/2) = 0.14$

$\psi_3 = 0.14$

$j=4$

- $\psi_4 - [\phi_1 \psi_3 + \phi_2 \psi_2] = \theta_4 \Rightarrow \psi_4 = 1 \times 0.14 + -0.24 \times 0.26 = 0.0776$

$\underline{\underline{\psi_4 = 0.0776}}$

$$\boxed{\begin{array}{l} \bullet\ \psi_0 = 1.0000 \\ \bullet\ \psi_1 = 0.5000 \\ \bullet\ \psi_2 = 0.2600 \\ \bullet\ \psi_3 = 0.1400 \\ \bullet\ \psi_4 = 0.0776 \end{array}}$$

[Text book : page 86]

(c) Given that $Z_t = \sum_{j=0}^{\infty} \pi_j X_{t-j}$ for all $t \in \mathbb{Z}$, find π_j for $j = 1. 2, 3,$ and 4.

$$\pi_j + \sum_{k=1}^{q} \theta_k \pi_{j-k} = -\phi_j$$

where • $\left[\phi_0 = -1 \; ; \; \phi_j := 0 \text{ for } j > p \;\; \& \;\; \pi_j = 0 \text{ for } j < 0\right]$

$\theta_1 = -1/2$ //

$\phi_1 = 1$ //

$\phi_2 = -0.24$ //

$q = 1$

j=0

- $\pi_0 = 1$

j=1

- $\pi_1 + \theta_1 \pi_0 = -\phi_1$

  $\pi_1 = 1/2 - 1 = -1/2$

  $\pi_1 = -1/2$ //

j=2

- $\pi_2 + \theta_1 \pi_1 = -\phi_2 \Rightarrow \pi_2 = \frac{1}{2}(-\frac{1}{2}) + 0.24 = \underline{\underline{-0.01}}$

  $\pi_2 = \underline{\underline{-0.01}}$

j=3

- $\pi_3 + \theta_1 \pi_2 = -\phi_3 \Rightarrow \pi_3 = \frac{1}{2}(-0.01) = \underline{\underline{-0.005}}$

  $\pi_3 = \underline{\underline{-0.005}}$

j=4

- $\pi_4 + \theta_1 \pi_3 = -\phi_4 \Longrightarrow \pi_4 = 1/2(-0.005) = \underline{\underline{-0.0025}}$

  $\pi_4 = \underline{\underline{-0.0025}}$

- $\pi_0 = 1$
- $\pi_1 = -0.5$
- $\pi_2 = -0.01$
- $\pi_3 = -0.005$
- $\pi_4 = -0.0025$

- For the Q3 : Text book Page 15
  1.4. Stationary models & the Autocorrelation Function
  Def^n 1.4.1 , Def^n 1.4.2 , Def^n 1.4.3

3. (15 points). Determine which of the following time series are covariance stationary. In all cases assume that $\{Z_t : t \in \mathbb{Z}\}$ is a collection of i.i.d random variables with zero mean and variance σ^2. Also let $\{V_t : t \in \mathbb{Z}\}$ and $\{W_t : t \in \mathbb{Z}\}$ be two covariance stationary time series with zero mean and autocovarince functions γ_V and γ_W respectively. You may assume that $\{V_t : t \in \mathbb{Z}\}$ and $\{W_t : t \in \mathbb{Z}\}$ are independent of each other as well as of $\{Z_t : t \in \mathbb{Z}\}$.

- $Z_t \overset{iid}{\sim} WN(0, \sigma^2)$; $E(V_t) = 0$; $E(W_t) = 0$
- $E[V_t \cdot Z_s] = 0$; $E[W_t \cdot Z_s] = 0$; $E(V_t \cdot W_s) = 0 \ \forall t, s$.

(a) $\{X_t : t \in \mathbb{Z}\}$, where $X_t = \alpha V_t + Z_t$ for all $t \in \mathbb{Z}$, where α is a real non-zero constant.

- $\mu_X = E[X_t] = \alpha E[V_t] + E[Z_t] = \alpha \cdot 0 + 0 = 0$ // [doesn't depend on t]

$X_t = \alpha V_t + Z_t$

$X_{t+h} = \alpha V_{t+h} + Z_{t+h}$

$$\text{cov}(X_t, X_{t+h}) = E[X_t \cdot X_{t+h}] = \alpha^2 E[V_t \cdot V_{t+h}] + \alpha E[V_t \cdot Z_{t+h}] + \alpha E[V_{t+h} \cdot Z_t] + E[Z_t \cdot Z_{t+h}]$$

$$\text{cov}(X_t ; X_{t+h}) = \alpha^2 \gamma_V(h) + E[Z_t \cdot Z_{t+h}]$$

$\underline{h=0}$.

$$\text{cov}(X_t ; X_t) = \alpha^2 \gamma_V(0) + E[Z_t \cdot Z_t] = \alpha^2 \gamma_V(0) + \sigma^2$$

$|h| \geq 1$

$$\text{cov}(X_t ; X_{t+h}) = \alpha^2 \gamma_V(h)$$

$\therefore$ • $$\text{cov}(X_t ; X_{t+h}) = \gamma_X(h) = \begin{cases} \alpha^2 \gamma_V(0) + \sigma^2 ; & h = 0 \\ \alpha^2 \gamma_V(h) ; & |h| \geq 1 \end{cases}$$

$\gamma_X(h)$ doesn't depend on t. [Only a function of h].

$\therefore$ $\{X_t ; t \in \mathbb{Z}\}$ is a covariance stationary process.

$E(V_t) = 0$

(b) $\{X_t : t \in \mathbb{Z}\}$, where $X_t = V_t - .5V_{t-1}$ for all $t \in \mathbb{Z}$.

- $E[X_t] = E[V_t] - 0.5E[V_{t-1}] = 0 - 0.5 \cdot 0 = \underline{\underline{0}} = \mu_X$ [doesn't depend on t]

$X_t = V_t - 0.5V_{t-1}$

$X_{t+h} = V_{t+h} - 0.5V_{t+h-1}$

- $$cov(X_t; X_{t+h}) = E[X_t \cdot X_{t+h}] = E[V_t \cdot V_{t+h}] - \frac{1}{2}E(V_t \cdot V_{t+h-1}] - \frac{1}{2}E[V_{t+h} \cdot V_{t-1}] + \frac{1}{4}E[V_{t-1} \cdot V_{t+h-1}]$$

$$= \gamma_v(h) - \frac{1}{2}\gamma_v(h-1) - \frac{1}{2}\gamma_v(h+1) + \frac{1}{4}\gamma_v(h)$$

$$cov(X_t; X_{t+h}) = \frac{5}{4}\gamma_v(h) - \frac{1}{2}[\gamma_v(h-1) + \gamma_v(h+1)]$$ $\Rightarrow$ this is a function of h only.

$cov(X_t; X_{t+h}) \rightarrow$ It doesn't depend on $\underline{t}$.

$$\gamma_x(h) = \begin{cases} \frac{5}{4}\gamma_v(0) - \frac{1}{\cancel{2}} \times \cancel{2}\gamma_v(1) = \frac{5}{4}\gamma_v(0) - \gamma_v(1); & h = 0 \\ \frac{5}{4}\gamma_v(h) - \frac{1}{2}[\gamma_v(h-1) + \gamma_v(h+1)] & ; |h| \geq 1 \end{cases}$$

$\therefore$ $\gamma_x(h) = \frac{5}{4}\gamma_v(|h|) - \frac{1}{2}[\gamma_v(|h-1|) + \gamma_v(|h+1|)]$; $\forall h \in \mathbb{Z}$

- $E(X_t) = \mu_X$ & $cov(X_t; X_{t+h}) = \gamma_x(h)$ doesn't depend on t

$\therefore$ $\{X_t \mid X_t = V_t - \frac{1}{2}V_{t-1} ; t \in \mathbb{Z}\}$ is a covariance stationary process.

$$0 = \mathrm{cov}(V_t; W_s) = E(V_t)\cdot E(W_s) - E(V_t \cdot W_s) \Rightarrow E(V_t \cdot W_s) = E(V_t)\cdot E(W_s)$$

$$0 = \mathrm{cov}(V_t; Z_s) = E(V_t \cdot Z_s) - E(V_t)E(Z_s) \Rightarrow E(V_t \cdot Z_s) = E(V_t)\cdot E(Z_s)$$

$$\gamma_V(h) = \mathrm{cov}(V_t; V_{t+h}) = E[V_t \cdot V_{t+h}]$$

(c) $\{X_t : t \in \mathbb{Z}\}$, where $X_t = V_t W_t + Z_t$ for all $t \in \mathbb{Z}$.

Recall

$$\mathrm{cov}(V_t; W_t) = E(V_t \cdot W_t) - \overbrace{E(V_t)E(W_t)}^{0} = 0$$

$\therefore$ $E[V_t \cdot W_t] = 0$

- $E(X_t) = \mu_X = E[V_t \cdot W_t] + E[Z_t] = 0$

  $E(X_t) = \mu_X = 0$ [This doesn't depend on t]

- $\mathrm{cov}(X_t; X_{t+h}) = E[X_t \cdot X_{t+h}] = E[(V_t \cdot W_t + Z_t)(V_{t+h} \cdot W_{t+h} + Z_{t+h})]$

$$= E[V_t \cdot V_{t+h} \cdot W_t \cdot W_{t+h}] + E[V_t \cdot W_t \cdot Z_{t+h}] + E[Z_t \cdot V_{t+h} \cdot W_{t+h}] + E[Z_t \cdot Z_{t+h}]$$

$$\mathrm{cov}(X_t; X_{t+h}) = E[V_t \cdot V_{t+h}] \cdot E[W_t \cdot W_{t+h}] + \overbrace{E[V_t]\cdot E[W_t]\cdot E[Z_{t+h}]}^{0} + \underbrace{E[Z_t]\cdot E[V_{t+h}]\cdot E[W_{t+h}]}_{0} + E[Z_t\ Z_{t+h}]$$

$$v(X_t; X_{t+h}) = E(V_t \cdot V_{t+h}) \cdot E(W_t \cdot W_{t+h}) + E[Z_t\ Z_{t+h}]$$

$\underline{h=0}$.

$$\mathrm{cov}(X_t; X_t) = E[V_t^2]\,E[W_t^2] + E[Z_t^2]. = \gamma_V(0).\gamma_W(0) + \sigma^2$$

$|h| \geq 1$

$$\mathrm{cov}(X_t; X_{t+h}) = \gamma_V(h) \cdot \gamma_W(h)$$

$$\therefore\ \gamma_X(h) = \begin{cases} \gamma_V(0).\gamma_W(0) + \sigma^2 & ; h = 0 \\ \gamma_V(h).\gamma_W(h) & ; |h| \geq 1 \end{cases}$$

$\gamma_X(h)$ doesn't depend on t [function of h only]

- $\{X_t \mid X_t = V_t \cdot W_t + Z_t ;\ t \in \mathbb{Z}\}$ is a covariance stationary ~~p~~process.

$X_t = V_t + (-1)^t W_t$

$X_{t+h} = V_{t+h} + (-1)^{t+h} W_{t+h}$

(d) $\{X_t : t \in \mathbb{Z}\}$, where $X_t = V_t + (-1)^t \times W_t,\ t \in \mathbb{Z},$.

- $E(X_t) = \mu_x = \underbrace{E[V_t]}_{0} + (-1)^t \underbrace{E(W_t)}_{0} = 0$ // [doesn't depend on t]

- $\operatorname{cov}(X_t ; X_{t+h}) = E[X_t \cdot X_{t+h}] = E[V_t \cdot V_{t+h}] + (-1)^{t+h} \underbrace{E[V_t \cdot W_{t+h}]}_{\Rightarrow 0} + (-1)^t \underbrace{E[W_t \cdot V_{t+h}]}_{\Rightarrow 0} + (-1)^{2t+h} E[W_t \cdot W_{t+h}]$

$\operatorname{cov}(X_t ; X_{t+h}) = \gamma_V(h) + (-1)^h \gamma_W(h)$. [This is a function of h only. It doesn't depend on t]

$$\gamma_X(h) = \begin{cases} \gamma_V(|h|) + \gamma_W(|h|) ; & h = 2k ; k \in \mathbb{Z} \\ \gamma_V(|h|) - \gamma_W(|h|) ; & h = 2k+1 ; k \in \mathbb{Z} \end{cases}$$

- μ_X & $\gamma_X(h)$ don't depend on t

$\therefore \{X_t \mid X_t = V_t + (-1)^t \cdot W_t ; t \in \mathbb{Z}\}$ is a covariance stationary process.