$\frac{15}{15}$

Applied Time Series Analysis

STAT 5814 - HW # 1

• Isuru Ratnayake.

(01)

[1.4] $Z_t \overset{ind}{\sim} N(0,\sigma^2) \Rightarrow E(Z_t)=0$

$Var(Z_t) = E(Z_t^2) - \underbrace{E(Z_t)}_{0} = \sigma^2$

a, b & c are constant.

a). $x_t = a + bZ_t + cZ_{t-2}$

$E(x_t) = a + b\underbrace{E(Z_t)}_{\to 0} + c\underbrace{E(Z_{t-2})}_{\to 0} = a = \mu_t$

$E(x_t) = \mu_t = a$ is independent of t.

$cov(x_t, x_{t+h}) = E[x_t - E(x_t)][x_{t+h} - E(x_{t+h})]$

$= E\{\cancel{a} + bZ_t + cZ_{t-2} - \cancel{a}\}[\cancel{a} + bZ_{t+h} + cZ_{t+h-2} - \cancel{a}]$

$= E[b^2 Z_t . Z_{t+h}] + bc\,E[Z_t . Z_{t+h-2}]$

$+ bc\,E[Z_{t-2} . Z_{t+h}] + c^2 E[Z_{t-2} . Z_{t+h-2}]$

<u>If $h=0$</u>

$cov(x_t, x_t) = (b^2+c^2)\sigma^2$

If $|h| = 1$

$cov(x_t, x_{t+|1|}) = b^2 . 0 + bc . 0 + bc . 0 + c^2 . 0 = 0$

If $|h| = 2$

$\left.\begin{array}{l} \cdot\ cov(x_t, x_{t+2}) = bc.E(Z_{t-2} . Z_{t-2}) = bc\sigma^2 \\ \cdot\ cov(x_t, x_{t-2}) = bc.E(Z_t . Z_t) = bc\sigma^2 \end{array}\right\}$ $bc\sigma^2$

$|h| > 2$

$cov(x_t, x_{t+|h|}) = 0$

$\therefore$

$$cov(x_t, x_{t+h}) = \begin{cases} (b^2+c^2)\sigma^2 & h = 0 \\ (bc)\sigma^2 & |h| = 2 \\ 0 & ; \ o/w. \end{cases}$$

$\&\ E(x_t) = \mu_t = a.$

Since μ_t, $cov(x_t, x_{t+h})$ are not function of t. There for $\{X_t \mid X_t = a + bZ_t + cZ_{t-2} \ ; t \in \mathbb{Z}\}$ is (weakly) stationary function.

b]. $X_t = Z_1 \cdot \cos(ct) + Z_2 \cdot \sin(ct)$

$$E(x_t) = \cos(ct) \cdot \underbrace{E(Z_1)}_{0} + \sin(ct) \cdot \underbrace{E(Z_2)}_{0} = 0$$

$$cov(x_t, x_{t+h}) = E[x_t - \overset{\to 0}{\mu_t}] \cdot [x_{t+h} - \overset{\to 0}{\mu_{t+h}}]$$
$$= E(x_t \cdot x_{t+h})$$

$$= E\left\{[Z_1 \cos(ct) + Z_2 \sin(ct)][Z_1 \cos[c(t+h)] + Z_2 \sin[c(t+h)]]\right\}$$

$$= [E(Z_1^2)] \cdot \cos(ct) \cdot \cos[c(t+h)] + \cos(ct) \cdot \sin[c(t+h)] \, E(Z_1 \cdot Z_2) + \cos[c(t+h)] \cdot \sin(ct) \cdot E(Z_1 \cdot Z_2) + \sin[c(t+h)] \cdot \sin(ct) \cdot E(Z_2^2).$$

$\cos(\alpha-\beta) = \cos\alpha \cdot \cos\beta + \sin\alpha \cdot \sin\beta$

$$\operatorname{cov}(x_t, x_{t+h}) = \Big\{\cos(ct) \cdot \cos[c(t+h)] \cdot$$
$$+ \sin(ct) \cdot \sin[c(t+h)]\Big\}\sigma^2$$
$$= \sigma^2 \Big\{\cos[\cancel{ct} + ch - \cancel{ct}]\Big\} = \underline{\underline{\sigma^2 \cdot \cos(ch)}}$$

$\operatorname{cov}(x_t, x_{t+h}) = \sigma^2 \cos(ch)$ is a function of h. (doesn't depentont).

$\therefore \{x_t \mid x_t = z_1 \cos(ct) + z_2 \cdot \sin(ct);\ t \in \mathbb{Z}\}$ is (weakly) stationary function.

(c). $x_t = z_t \cdot \cos(ct) + z_{t-1} \sin(ct)$

$$E(x_t) = \cos(ct) \cdot \underbrace{E(z_t)}_{0} + \sin(ct)\underbrace{E(z_{t-1})}_{0}$$

- $E(x_t) = \mu_t = 0$ ∴ independent of t.

$$\operatorname{cov}(x_t, x_{t+h}) = E(x_t - \overset{\to 0}{E(x_t)})(x_{t+h} - \overset{\to 0}{E(x_{t+h})})$$
$$= E\big[z_t \cdot \cos(ct) + z_{t-1} \cdot \sin(ct)\big] \cdot$$
$$\cdot \big[z_{t+h} \cos[c(t+h)] + z_{t+h-1} \sin[c(t+h)]\big]$$

$$= E\big[z_t \cdot z_{t+h}\big] \cdot \cos(ct) \cdot \cos[c(t+h)] + \cos(ct) \cdot \sin[c(t+h)] \cdot \big[E(z_t \cdot z_{t+h-1})\big]$$
$$+ \sin(ct)\cos[c(t+h)] \cdot E[z_{t-1} \cdot z_{t+h}] + \sin(ct) \cdot \sin[c(t+h)] \cdot \big[E[z_{t-1} \cdot z_{t+h-1}]\big]$$

$= \cos(ct)\cdot\cos[c(t+h)]\cdot E[Z_t \cdot Z_{t+h}] + \cos(ct)\sin[c(t+h)]\cdot E[Z_t\cdot Z_{t+h-1}]$

$+ \sin(ct)\cdot \cos(c(t+h))\cdot E[Z_{t-1}\cdot Z_{t+h}] + \sin(ct)\cdot\sin(c(t+h))\cdot$
$\cdot E[Z_{t-1}\cdot Z_{t+h-1}]$

If $h=0$.

$\sin^2\theta+\cos^2\theta=1$

$[\cos(ct)]^2\sigma^2 + [\sin(ct)]^2\sigma^2 = \sigma^2$

If $h=1$.

$\cos(ct)\cdot\sin[c(t+1)]\cdot\sigma^2$

If $h=-1$

$\sin(ct)\cdot\cos(c(t-1))\,\sigma^2$

If $|h|>1$

$cov(x_t, x_{t+h}) = 0$

$$cov(x_t, x_{t+h}) = \begin{cases} \sigma^2 & ; h=0 \\ \sin(ct)\cdot\cos(c(t-1))\,\sigma^2 & ; h=-1 \\ \cos(ct)\cdot\sin(c(t+1))\,\sigma^2 & ; h=1 \\ 0 & ; o/w. \end{cases}$$

~~Since the autocovariance function is a function of t [if $c \neq \pm k\pi$, $k\in\mathbb{Z}$]; hence the above process is not stationary [non-stationary]~~

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Since autocovariance function is not only a function of t but also it depends on the value of C.

If $C = \pm k\pi \,;\, k \in \mathbb{Z}$ then the $\{X_t : t \in \mathbb{Z}\}$ is Stationary.

For $C \neq \pm k\pi \,;\, k \in \mathbb{Z} \,;\, \{x_t : t \in \mathbb{Z}\}$ is not (weakly) Stationary.

[It depends on t].

(d). $x_t = a + bZ_0$

$E(x_t) = a + bE(Z_0) = a + b[0] = a = \mu_t$.

• $E(x_t) = \mu_t = a \rightarrow$ does not depend on t. [μ_t is independent on t].

$\operatorname{cov}(x_t, x_{t+h}) = E[x_t - \mu_t][x_{t+h} - \mu_{t+h}] = E[b^2 Z_0^2]$

$= b^2 E(Z_0^2)$ Here $E(Z_0^2) - \underset{0}{(E(Z_0))^2} = \sigma^2$

$\therefore \operatorname{cov}(x_t; x_{t+h}) = b^2\sigma^2$ [does not depend on t].

$\therefore$ Both μ_t & $\operatorname{cov}(x_t; x_{t+h})$ are independent on t.

$\therefore \{x_t \mid x_t = a + bZ_0 \,:\, t \in \mathbb{Z}\}$ is (weakly) Stationary process.

(e). $x_t = Z_0 \cdot \cos(ct)$.

• $\mu_t = E(x_t) = \cos(ct)E(Z_0) = \cos(ct) \cdot 0 = 0$

$E(x_t) = \mu_t = 0$ (doesn't depend on t).

• $\operatorname{cov}(x_t; x_{t+h}) = E(x_t - E(x_t))(x_{t+h} - E(x_{t+h})] = E(x_t \cdot x_{t+h})$

$= [\cos(ct)]^2 \cdot E(Z_0^2) = \sigma^2 \cdot \cos^2(ct)$

If $C = \pm k\pi \,;\, k \in \mathbb{Z}$ $\operatorname{cov}(x_t; x_{t+h})$ does not depend on t. Hence $\{X_t ; t \in \mathbb{Z}\}$ process is weakly stationary.

Else $[C \neq k\pi \,;\, k \in \mathbb{Z}]$ x_t is not Stationary.

(f). $x_t = Z_t \cdot Z_{t-1}$.

$E(x_t) = E[Z_t \cdot Z_{t-1}]$

$E(x_t) = E(Z_t) \cdot E(Z_{t-1}) = 0$

$Var(Z_t) = E(Z_t^2) - [E(Z_t)]^2 = \sigma^2$

$E(Z_t) = 0$

$cov(Z_i, Z_j) = 0 \quad \forall i \neq j$

$E(Z_i \cdot Z_j) - E(Z_i)E(Z_j) = 0$

$\therefore E(X_t) = \mu_t = 0$; does not depend on t.

$cov(x_t; x_{t+h}) = E[x_t - \overset{\rightarrow 0}{E(x_t)}][x_{t+h} - \overset{\rightarrow 0}{E(x_{t+h})}]$

$= E(x_t \cdot x_{t+h}) = E(Z_t \cdot Z_{t-1} \cdot Z_{t+h} \cdot Z_{t+h-1})$

$E(x_t \cdot x_{t+h}) = cov(x_t ; x_{t+h}) =$

$E(Z_t \cdot Z_{t-1} \cdot Z_{t+h} \cdot Z_{t+h-1})$.

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If $h=0$.

$$\mathrm{cov}(x_t, x_t) = E[Z_t \cdot Z_{t-1} \cdot Z_t \cdot Z_{t-1}]$$

$$= E[Z_t^2 \cdot Z_{t-1}^2] = \sigma^4$$

$$= E(Z_t^2) \cdot E(Z_{t-1}^2) = \sigma^4$$

If $|h| > 0$

$$\mathrm{cov}(x_t, x_{t+|h|}) = 0 = E[Z_t] \cdot E[Z_{t-1}] \cdot E[Z_{t+h}] \; E[Z_{t+h-1}]$$

$$\therefore \mathrm{cov}(x_t, x_{t+h}) = \begin{cases} \sigma^4 & ; h=0 \\ 0 & ; o/w \end{cases}$$ ACVF doesn't depend on t

$\{x_t \mid x_t = Z_t \cdot Z_{t-1} \; ; t \in \mathbb{Z}\}$

$\therefore$ x_t is stationary function

(e). $X_t = Z_0 \cdot \cos(ct)$

$E(x_t) = \mu_t = \cos[ct] \cdot E(Z_0) = 0$; does not depend on t.

$\mathrm{cov}(x_t, x_{t+h}) = E(x_t - \underbrace{E(x_t)}_{0})(x_{t+h} - \underbrace{E(x_{t+h})}_{0})] = E[x_t \cdot x_{t+h}]$

$= E[Z_0 \cdot \cos(ct)][Z_0 \cos c(t+h)] = \cos[c(t+h)] \cdot \cos(ct)\, \sigma^2$

$\mathrm{cov}(x_t, x_{t+h}) = \sigma^2 \cos(ct) \cdot \cos[c(t+h)]$

The process $\{X_t \mid x_t = Z_0 \cos(ct) \; ; t \in \mathbb{Z}\}$ is stationary if $c = \pm k\pi \; ; k \in \mathbb{Z}$.

The process $\{x_t \mid x_t = Z_0 \cos(ct) \; ; t \in \mathbb{Z}\}$ is not stationary if $c \neq \pm k\pi \; ; k \in \mathbb{Z}$

[1.5] Let $\{x_t\}$ be the moving-average process of order 2 given by

$$x_t = Z_t + \theta \cdot Z_{t-2}$$

$$Z_t \sim WN(0,1).$$

$E(Z_t)=0$

$Var(Z_t) = E(Z_t^2) - [E(Z_t)]^2 = 1$, with $E(Z_t^2) = 1$ and $E(Z_t) = 0$

a).

$$\operatorname{cov}(x_t, x_{t+h}) = E[x_t - E(x_t)][x_{t+h} - E(x_{t+h})]$$

$$E(x_t) = 0 + \theta \cdot 0 = 0$$

$$E(x_{t+h}) = E(Z_t) + \theta \cdot E(Z_{t-2}) = 0$$

$$\operatorname{cov}(x_t, x_{t+h}) = E[Z_t + \theta \cdot Z_{t-2}][Z_{t+h} + \theta \cdot Z_{t+h-2}]$$

$$= E[Z_t \cdot Z_{t+h}] + \theta E[Z_t \cdot Z_{t+h-2}]$$

$$+ \theta E[Z_{t-2} \cdot Z_{t+h}] + \theta^2 E[Z_{t-2} \cdot Z_{t+h-2}]$$

If $h=0$.

$$\operatorname{cov}(x_t, x_t) = E[Z_t^2] + \theta^2 E[Z_{t-2}^2] = (1+\theta^2)$$

If $|h|=1$.

$$\operatorname{cov}(x_t, x_{t+|1|}) = 0$$

If $|h|=2$.

$$\operatorname{cov}(x_t, x_{t+2}) = \theta \cdot E(Z_t^2) = \theta$$

$$\operatorname{cov}(x_t, x_{t-2}) = \theta \cdot E(Z_{t-2}^2) = \theta$$

$|h| > 2$

$$\operatorname{cov}(x_t, x_{t+|h|}) = 0$$

$\therefore$

$$\underbrace{cov(x_t, x_{t+h})}_{\text{auto covariance function.}} = \begin{cases} (1+\theta^2) & ; \quad h=0 \\ \theta & ; \quad |h| = 2 \\ 0 & ; \quad o/w \end{cases}$$

$$\gamma(h) = \begin{cases} 1+(0.8)^2 = 1.64 & ; \quad h=0 \\ 0.8 & ; \quad |h|=2 \\ 0 & ; \quad o/w. \end{cases}$$

$$\rho(h) = \frac{\gamma(h)}{\gamma(0)} = \begin{cases} 1 & ; \quad h=0 \\ \frac{0.8}{1.64} = 0.488 & \quad |h|=2 \\ 0 & ; \quad o/w \end{cases}$$

(b). $x_1 = Z_1 + \theta Z_{-1}$

$x_2 = Z_2 + \theta Z_0$

$x_3 = Z_3 + \theta Z_1$

$x_4 = Z_4 + \theta Z_2$

$$\bar{X} = \frac{x_1 + x_2 + x_3 + x_4}{4} = \frac{\left[(Z_1 + \theta Z_1) + (Z_2 + \theta Z_2) + Z_3 + Z_4 + \theta Z_{-1} + \theta Z_0 \right]}{4}$$

• here $cov(Z_i, Z_j) = 0 \ \forall i \neq j \Rightarrow E(Z_i Z_j) = 0$

$$Var(\bar{X}) = \frac{\left((1+\theta)^2 \, var(Z_1) + (1+\theta)^2 var(Z_2) + var(Z_3) + Var(Z_4) + \theta^2 var(Z_{-1}) + \theta^2 Var(Z_0) \right)}{16}$$

[$\because$ Here $cov(Z_i, Z_j) = 0 \ \forall i \neq j$.]

$$Var(\bar{X}) = \frac{2(1+\theta)^2 + 2 + 2\theta^2}{16}$$

$$Var(\bar{X}) = \frac{\theta^2 + 2\theta + 1 + 1 + \theta^2}{8} = \frac{\theta^2 + \theta + 1}{4}$$

$$= \frac{(0.8)^2 + 1 + 0.8}{4} = \frac{2.44}{4} = 0.61.$$

(c). If $\theta = -0.8$

$$Var(X) = \frac{(-0.8)^2 + 1 - 0.8}{4} = 0.21$$

$\theta = -0.8 < 0$

when Covariance at lag 2 the variance in c). is less than that of b)

(17) • If $\{X_t\}$ & $\{Y_t\}$ are uncorrelated stationary sequences. i.e.

• $\begin{cases} E(x_t) = \mu_x & \text{; does not depend on } t. \\ \text{cov}(x_t, x_{t+h}) = \gamma_x(h). & \text{is a function of } h \text{ only} \end{cases}$

$\therefore \{X_t\}$ is a stationary process

Simillarly

• $\begin{cases} E(y_t) = \mu_y & \text{; does not depend on } t \\ \text{cov}(y_t, y_{t+h}) = \gamma_y(h) & \text{is a function of } h \text{ only.} \end{cases}$

$\therefore \{y_t\}$ is a stationary process.

$\text{cov}(x_r, y_s) = 0$ for $\forall\, r$ & s. where $r \in \mathbb{Z}$ & $s \in \mathbb{Z}$

Now consider $\{x_t + y_t\}$ where $t \in \mathbb{Z}$.

$E\{x_t + y_t\} = E(x_t) + E(y_t) = \mu_x + \mu_y$

• $(\mu_x + \mu_y) = E(x_t) + E(y_t) \Rightarrow$ doesn't depend on t.

$\cancel{\text{cov}\{x_t + y_t\} = E[(x_t + y_t) - (\mu_x + \mu_y)]}[$

$\text{cov}\{(x_t + y_t), (x_{t+h} + y_{t+h})\}$

$= E\{[(x_t + y_t) - (\mu_x + \mu_y)][(x_{t+h} + y_{t+h}) - (\mu_x + \mu_y)]\}$

$$E\{(x_t-\mu_x)+(y_t-\mu_y)\}\{(x_{t+h}-\mu_x)+(y_{t+h}-\mu_y)\}$$

$$= E\{(x_t-\mu_x)(x_{t+h}-\mu_x)\}$$

$$+ \underbrace{E\{(x_t-\mu_x)(y_{t+h}-\mu_y)\}}_{\to 0}$$

$$+ \underbrace{E\{(y_t-\mu_y)(x_{t+h}-\mu_x)\}}_{\to 0}$$

$$+ E\{(y_t-\mu_y)(y_{t+h}-\mu_y)\}$$

$$= \gamma_x(h) + \gamma_y(h) \;//$$

- Hence auto-covariance function equal to sum of the auto-covariance functions of $\{x_t\}$ & $\{y_t\}$ //

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STAT 5814

HW#2.

Isuru Ratnayake.

(3.1). Determine which of the following ARMA processes are causal and which of them one invertible. ; $Z_t \overset{WN}{\sim} (0; \sigma^2)$.

a). $X_t + 0.2X_{t-1} - 0.48\,X_{t-2} = Z_t$
ARMA(2, 0) = AR(2).

$$\underbrace{\left[1 + 0.2Z - 0.48Z^2\right]}\,X_t = \theta_0\,Z_t$$

$$\left[\phi(Z)\right] \cdot x_t = \left[\underbrace{\Theta(Z)}_{=1}\right] \cdot Z_t$$

Now consider
$$\phi(Z) = 1 + 0.2Z - 0.48Z^2 = 0$$

$$Z = -1.25 \implies |Z| = |-1.25| = 1.25 > 1$$
or
$$Z = 1.667 \implies |Z| = |1.667| = 1.667 > 1$$

$\therefore \forall\ |Z| > 1$; which located outside the unit circle
$\therefore$ ARMA(2,0) = AR(2) is also causal process.

$\Theta(Z) = 1 \neq 0$; Hence $\{X_t ; t \in \mathbb{Z}\}$ is invertible.

Or in other words.

$$Z_t = x_t + 0.2\,x_{t-1} - 0.48\,x_{t-2}$$

Here Z_t can be explained by X_s ; $s \leq t$
$\therefore$ ARMA (2,0) = AR(2) is invertible process.

ARMA (2, 2)

(b). $X_t + 1.9 X_{t-1} + 0.88 X_{t-2} = Z_t + 0.2 Z_{t-1} + 0.7 Z_{t-2}$

$$\underbrace{\left(1 + 1.9 z + 0.88 z^2\right)}_{} x_t = \underbrace{\left[1 + 0.2 z + 0.7 z^2\right]}_{} Z_t$$

$$\left[\Phi(z)\right] \cdot x_t = \left[\Theta(z)\right] \cdot Z_t$$

$$\phi(z) = 1 + 1.9 z + 0.88 z^2$$

- If $\phi(z) \neq 0$ for each $|z| \leq 1$ then process is causal $\left[\ \phi(z) = 0 \ \text{ if } \forall |z| > 1\right]$

$$z_1 = -1.25 \quad \text{or} \quad z_2 = -0.9091$$

$$|z_1| = |1.25| > 1 \ ; \ |z_2| = |0.9091| < 1$$

Hence $\{X_t \mid x_t \sim \text{ARMA}(2,2) \ : t \in \mathbb{Z}\}$ is not causal. $(\because \forall |z| > 1)$

Now consider $\Theta(z) = \left(1 + 0.2 z + 0.7 z^2\right)$

$$\Theta(z) = 0 \Rightarrow 1 + 0.2 z + 0.7 z^2 = 0$$

$$z = \frac{-0.2 \pm \sqrt{(0.2)^2 - 4 \times 0.7}}{2 \times 0.7} = \frac{-0.2 \pm \sqrt{-2.76}}{2 \times 0.7}$$

$$= \frac{-0.2 \pm 1.66 i}{2 \times 0.7} =$$

$$z = \frac{-1 \pm \sqrt{69}\, i}{7}$$

$$z_1 = \frac{-(1 - \sqrt{69}\, i)}{7} \quad \text{or} \quad z_2 = \frac{-(1 + \sqrt{69}\, i)}{7}$$

$$|z_1| = |z_2| = \sqrt{70}/7 > 0$$

(03).

$\forall\ |z| > 1$ process $\{x_t : t \in \mathbb{Z}\}$ is invertible.

ARMA(1, 1)

(c). $X_t + 0.6 X_{t-1} = Z_t + 1.2 Z_{t-1}$

$$X_t + 0.6 z X_t = Z_t + 1.2 z \cdot Z_t$$

$$\underbrace{(1 + 0.6z)}\, x_t = \underbrace{(1 + 1.2z)}\, Z_t$$

$$[\phi(z)] \cdot x_t = [\theta(z)] \cdot Z_t$$

Let consider $\Rightarrow \phi(z) = 0$

$$1 + 0.6z = 0$$

$$\frac{-10}{6} = z$$

$$|z| = \left|\frac{-10}{6}\right| = 5/3 > 1$$

Hence $\{x_t : t \in \mathbb{Z}\}$ is Causal process.

Now consider $\theta(z) = 1 + 1.2z = 0$

$$\frac{-1}{1.2} = \frac{-5}{6} = z$$

$$|z| = \left|-5/6\right| < 1$$

$\therefore \{x_t : t \in \mathbb{Z}\}$ is not invertible process.

ARMA(2,0) = AR(2).

(d). $X_t + 1.8 X_{t-1} + 0.81 X_{t-2} = Z_t$

$X_t + 1.8 Z x_t + 0.81 Z^2 x_t = Z_t$

$\left[1 + 1.8Z + 0.81Z^2\right] x_t = \left[\underbrace{\Theta(Z)}\right] Z_t$

$\underbrace{\left(1 + 1.8Z + 0.81Z^2\right)} x_t = 1 \cdot Z_t$

$\phi(Z) \cdot x_t = Z_t$

$\phi(Z) = 1 + 1.8Z + 0.81Z^2 = (0.9Z+1)^2 = 0$

$Z_1 = Z_2 = -10/9$

$|Z| = \left|\frac{-1}{0.9}\right| = \left|\frac{-10}{9}\right| = \frac{10}{9} > 1$

$\therefore \{X_t ; t \in \mathbb{Z}\}$ is a causal process.

$\Theta(Z) = 1 \neq 0 \therefore$

$Z_t = x_t + 1.8 x_{t-1} + 0.81 x_{t-2}$

Z_t can be explained by x_s ; $s \leq t$

$\therefore$ Hence $\{X_t ; t \in \mathbb{Z}\}$ is invertible.

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ARMA(1,2)

(e). $X_t + 1.6X_{t-1} = Z_t - 0.4Z_{t-1} + 0.04Z_{t-2}$

$$\underbrace{(1 + 1.6z)}\,x_t = \underbrace{\left[1 - 0.4z + 0.04z^2\right]}\,Z_t$$

$$[\phi(z)]\cdot x_t = [\Theta(z)]\cdot Z_t$$

$\phi(z) = 0 \implies$

$$1 + 1.6z = 0$$

$$1.6z = -1$$

$$z = \frac{-10}{16} \quad ; \quad |z| = \left|\frac{-10}{16}\right| = \left|\frac{-5}{8}\right| < 1$$

$\therefore$ $\{X_t ; t \in \mathbb{Z}\}$ is not causal process.

$$\Theta(z) = 1 - 0.4z + 0.04z^2 = 0$$

$$z_1 = 5 \quad \& \quad z_1 = 5.$$

$$\therefore \forall\ |z| = |5| > 1$$

$\therefore$ $\{X_t ; t \in \mathbb{Z}\}$ process is invertible.

01

$ARMA(p,q) = ARMA(2,0)$.

(3.3).

a). $X_t + 0.2\, X_{t-1} - 0.48\, X_{t-2} = Z_t$

$$\underbrace{\left[1 + 0.2z - 0.48z^2\right]}\, x_t = Z_t$$

$\phi(z) \cdot x_t = Z_t$; $\phi(z) = 0 \quad \forall |z| > 1$

$\therefore \{x_t : t \in \mathbb{Z}\}$ is causal process.

$\phi(z) = 1 - \phi_1 z - \phi_2 z^2$

here $\phi_1 = -0.2$; $\phi_2 = 0.48$

$$\psi_j - \sum_{k=1}^{p} \phi_k \psi_{j-k} = \theta_j \; ; \; j = 0,1 .$$

- $j = 0 \Rightarrow$

$$\psi_0 - \sum_{k=1}^{2} \phi_k \psi_{0-k} = \theta_0$$

$$\psi_0 = 1 .$$

- $j = 1 \Rightarrow$

$$\psi_1 - \sum_{k=1}^{2} \phi_k \psi_{1-k} = \theta_1 = 0$$

$$\psi_1 = \phi_1 \psi_0 = -0.2 \times 1 = -0.2$$

- $j = 2 \Rightarrow$

$$\psi_2 - \sum_{k=1}^{2} \phi_k \psi_{2-k} = \theta_2 = 0$$

$$\psi_2 - \phi_1 \psi_1 - \phi_2 \psi_0 = 0$$

$$\psi_2 = (-0.2)(-0.2) + (0.48) = \underline{\underline{0.52}}$$

• $j=3$

$$\psi_3 - \sum_{k=1}^{2} \phi_k \psi_{3-k} = \epsilon_3$$

$$\psi_3 - \phi_1 \psi_2 - \phi_2 \psi_1 = 0$$

$$\psi_3 = \phi_1 \psi_2 + \phi_2 \psi_1 = (-0.2)(0.52) + (0.48)(-0.2)$$

$$\psi_3 = \underline{\underline{-0.2}}$$

(c)• $X_t + 0.6 X_{t-1} = Z_t + 1.2 Z_{t-1}$

$$X_t + 0.6 Z X_t = Z_t + 1.2 Z Z_t$$

$$\underbrace{(1 + 0.6 z)}\, x_t = (1 + 1.2 z) Z_t$$

$$[\phi(z)] \cdot x_t = [\Theta(z)] \cdot Z_t$$

$$\phi(z) = 1 + 0.6 z = 0 \; ; \; |z| > 1$$

$\therefore \{X_t \mid t \in Z\}$ is causal process.

$$\phi(z) = 1 + 0.6 z \Rightarrow 1 - \phi_1 z$$

$$\phi_0 = 1 \;\&\; \phi_1 = (-0.6) \;//$$

$$\theta_0 = 1 \;\&\; \theta_1 = 1.2 \;//$$

$$p = 1 \;\&\; q = 1$$

03

$$\psi_j - \sum_{k=1}^{p} \psi_{j-k}\phi_k = \theta_j \quad ; j = 0, 1, \ldots$$

- $j = 0 \Rightarrow \psi_0 = 1$ //

- $j = 1 \Longrightarrow$

$$\psi_1 - \sum_{k=1}^{1} \psi_{1-1}\phi_1 = \theta_1$$

$$\psi_1 - 1(-0.6) = 1.2 \Rightarrow \psi_1 = 0.6 \text{ //}$$

- $j = 2 \Longrightarrow$

$$\psi_2 - \sum_{k=1}^{1} \psi_{2-1}\phi_1 = \theta_2 = 0$$

$$\psi_2 = (0.6)(-0.6) = -0.36 \text{ //}$$

- $j = 3 \Longrightarrow$

$$\psi_3 - \psi_{3-1}\phi_1 = \theta_3 = 0$$

$$\psi_3 = \psi_2\phi_1 = (-0.36)(-0.6) = 0.216 \text{ //}$$

$AR(2) \equiv ARMA(p=2, q=0)$

(d). $X_t + 1.8 X_{t-1} + 0.81 X_{t-2} = Z_t$

$$X_t [1 + 1.8 z + 0.81 z^2] = Z_t$$

$$[\phi(z)] x_t = [\theta(z)] Z_t \qquad \theta_0 = 1, \ \theta_1 = 0$$

$$\phi(z) = 1 - (-1.8) z - (-0.81) z^2$$

$$\phi(z) = 1 - \phi_1 z^1 - \phi_2 z^2$$

$$\therefore \ \phi_0 = 1 \ ; \ \phi_1 = (-1.8) \ \& \ \phi_2 = (-0.81)$$

$$\psi_j - \sum_{k=1}^{p} \psi_{j-k} \phi_k = \theta_j \quad ; j = 0, 1$$

- $j = 0 \Rightarrow \psi_0 = 1$ //
- $j = 1 \Rightarrow$

$$\psi_1 - \sum_{k=1}^{2} \psi_{1-k} \phi_k = 0$$

$$\psi_1 = \psi_0 \phi_1 = 1 \cdot (-1.8) = (-1.8) //$$

- $j = 2 \Rightarrow$

$$\psi_2 - \sum_{k=1}^{2} \psi_{2-k} \phi_k = \theta_2 = 0$$

$$\psi_2 = \psi_1 \phi_1 + \psi_0 \phi_2 = (-1.8)(-1.8) + (1)(-0.81) = \underline{\underline{2.43}}$$

- $j = 3 \Rightarrow \psi_3 - \sum_{k=1}^{2} \psi_{3-k} \phi_k = \theta_3 = 0$

$$\psi_3 = \psi_2 \phi_1 + \psi_1 \phi_2 = (2.43)(-1.8) + (-1.8)(-0.81)$$

$$\psi_3 = -2.916 //$$

(3.6) $X_t = Z_t + \theta \cdot Z_{t-1}$; $\{Z_t\} \sim WN(0; \sigma^2)$

$Y_t = \tilde{Z}_t + \left(\frac{1}{\theta}\right) \hat{Z}_{t-1}$; $\{\hat{Z}_t\} \sim WN(0; \theta^2\sigma^2)$

where $0 < |\theta| < 1$

consider

$X_t = Z_t + \theta \cdot Z_{t-1}$ MA(1) process.

$E(X_t) = 0$; $E(Z_t) = 0$: $var(Z_t) = E(Z_t^2) = \sigma^2$

$X_t \cdot X_{t+h} = Z_t \cdot X_{t+h} + \theta \cdot X_{t+h} \cdot Z_{t-1}$

$E[X_t \cdot X_{t+h}] = E[X_{t+h} \cdot Z_t] + \theta \cdot E[X_{t+h} \cdot Z_{t-1}]$

$\gamma(h) = E[X_{t+h} \cdot Z_t] + \theta \cdot E[X_{t+h} \cdot Z_{t-1}]$

$\underline{h=0}$.

$\gamma(0) = E[X_t \cdot Z_t] + \theta E[X_t \cdot Z_{t-1}]$

$\gamma(0) = E\{(Z_t + \theta Z_{t-1}) Z_t\} + \theta E[(Z_t + \theta Z_{t-1}) Z_{t-1}]$

$= E[Z_t^2] + \theta E[Z_t \cdot Z_{t-1}] + \theta E[Z_t \cdot Z_{t-1}] + \theta^2 E[Z_{t-1} \cdot Z_{t-1}]$

$= \sigma^2 + \theta^2\sigma^2 = (1+\theta^2)\sigma^2$

$\underline{h=1}$.

$\gamma(1) = E[X_{t+1} \cdot Z_t] + \theta \cdot E[X_{t+1} \cdot Z_{t-1}]$

$= E[Z_{t+1} Z_t] + E[\theta \cdot Z_t^2] + \theta E[Z_{t+1} \cdot Z_{t-1}] + \theta[Z_t \cdot Z_{t-1}] = \theta\sigma^2$

$\underline{h = (-1)}$:

$$\gamma(-1) = E[x_{t-1} \cdot z_t] + \theta E[x_{t-1} \cdot z_{t-1}]$$

$$= E[(z_{t-1} + \theta z_{t-2}) z_t] + \theta E\{(z_{t-1} + \theta \cdot z_{t-2}) z_{t-1}\}$$

$$= \theta^2 E[z^2_{t-1}] = \theta \sigma^2 //$$

- $$\gamma_x(h) = \begin{cases} (1+\theta^2)\sigma^2 & ; h = 0 \\ \theta \sigma^2 & ; |h| = 1 \\ 0 & ; |h| > 1 \end{cases}$$

Let consider

$$y_t = \tilde{z}_t + \left(\frac{1}{\theta}\right) \cdot \hat{z}_{t-1} \quad ; \hat{z}_t \overset{WN}{\sim} (0; \theta^2 \sigma^2)$$

$$E(y_t) = 0 \quad ; \operatorname{var}(y_t) = E(z_t^2) - [E(z_t)]^2 = 0$$

$$E(z_t^2) = \theta^2 \sigma^2$$

$$E(y_t \cdot y_{t+h}) = E(\hat{z}_t \cdot y_{t+h}) + \left(\frac{1}{\theta}\right) E(\hat{z}_{t-1} \cdot y_{t+h})$$

$$\gamma(h) = E\left\{\hat{z}_t \left(\hat{z}_{t+h} + \frac{1}{\theta} \cdot \hat{z}_{t+h-1}\right)\right\}$$

$$+ \frac{1}{\theta} E\left\{\hat{z}_{t-1} \cdot \left(\hat{z}_{t+h} + \frac{1}{\theta} \cdot \hat{z}_{t+h-1}\right)\right\}$$

- $h = 0$

$$\gamma(0) = E[z_t^2] + \frac{1}{\theta^2} E[z_{t-1}^2] = (1 + \theta^2)\sigma^2$$

- $h = 1$

$$\gamma(1) = \frac{1}{\theta} E(\hat{z}_t^2) = \frac{1}{\theta} \theta^2 \cdot \sigma^2 = \theta \cdot \sigma^2$$

- $h = (-1)$

$$\gamma(-1) = \frac{1}{\theta} E(\tilde{z}_{t-1}^2) = \frac{1}{\theta} \cdot \theta^2 \cdot \sigma^2 = \theta \sigma^2$$

$$\therefore \quad \gamma_y(h) = \begin{cases} (1+\theta^2)\sigma^2 & ; \; h = 0 \\ \theta \cdot \sigma^2 & ; \; |h| = 1 \\ 0 & ; \; |h| > 1 \end{cases}$$

$$\gamma_x(h) = \gamma_y(h)$$

$\frac{10}{10} + \frac{5}{5}$

STAT 5814 - H/W #3

Isuru Ratnayake.

[2.9]

(a) • $y_t = x_t + w_t \quad ; \{w_t\} \sim WN(0, \sigma_w^2)$ —①

$x_t = \phi x_{t-1} + z_t ; \quad \{z_t\} \sim WN(0, \sigma_z^2)$ —②

$|\phi| < 1$

$E[w_s \cdot z_t] = 0 \quad \forall \; s \,\&\, t.$

$\cancel{y_t = \phi x_{t-1} + z_t + w_t.}$

$\cancel{E(y_t) = \phi E(x_{t-1}) + E(z_t) + E(w_t).}$

$\cancel{E(y_t) = \phi \mu_x}$ ~~this doesn't depend on t.~~

Let consider ② $\Rightarrow$

$x_t - \phi z x_t = z_t$

$[1 - \phi z] x_t = z_t$ here $|\phi| < 1 \therefore |z| > 1$

Causal sol^n exists.

$\therefore x_t = [1 + \phi + \phi^2 + \cdots + \phi^i + \cdots] z_t$ —③

NOW $y_t = \sum_{i=0}^{\infty} \phi^i z_{t-i} + w_t$; here $\phi^0 = 1$

$E(y_t) = 0 + 0 = 0$. doesn't depend on t.

$y_{t+h} = \sum_{j=0}^{\infty} \phi^j \cdot z_{t+h-j} + w_{t+h}$

$$E\left[y_t \cdot y_{t+h}\right] = E\left[\sum_{i=0}^{\infty} \phi^i Z_{t-i}\right]\left[\sum_{j=0}^{\infty} \phi^j Z_{t+h-j} + W_{t+h}\right]$$

$$+E\left[W_t\left[\sum_{j=0}^{\infty} \phi^j Z_{t+h-j} + W_{t+h}\right]\right]$$

$$E(y_t \cdot y_{t+h}) = E\left[\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \phi^i Z_{t-i} \cdot \phi^j Z_{t+h-j}\right] + 0 + 0 + E\left[W_t \cdot W_{t+h}\right]$$

$\left[\because E(W_s \cdot Z_t) = 0\right]$

$-h+j=+i \quad j-h=i$

$j=(i+h)$

$$E\left[y_t \cdot y_{t+h}\right] = E\left\{\sum_{i=0}^{\infty} \phi^i \cdot \phi^{i+|h|} Z_{t-i}^2\right] + E\left[W_t \cdot W_{t+h}\right]$$

$\underline{h=0.}$

$$\gamma(0) = \sigma_Z^2 \sum_{i=0}^{\infty} \phi^{2i} + \sigma_\omega^2. \qquad \because\left[\phi^0 = 1\right]$$

$\underline{|h| \geq 1.}$

$$\gamma(h) = \sigma_Z^2 \sum_{i=0}^{\infty} \phi^{2i+|h|}$$

This doesn't depend on t.

$\therefore \{y_t\}$ is stationary process.

A.C.V.F.

$$\gamma(h) = \begin{cases} \dfrac{\sigma_z^2}{(1-\phi^2)} + \sigma_w^2 & ; \ h=0 \\[2ex] \dfrac{\sigma_z^2 \ \phi^{|h|}}{(1-\phi^2)} & ; \ |h| \geq 1. \end{cases}$$

here $|\phi| < 1$.

(b).

$u_t : y_t - \phi y_{t-1}$

$$y_t = X_t + w_t \quad ; \ \{w_t\} \sim WN(0, \sigma_w^2) \quad -①$$

$$\phi y_{t-1} = \phi X_{t-1} + \phi w_{t-1} \quad -②$$

$$① - ② \Rightarrow \quad u_t := [y_t - \phi y_{t-1}] = [X_t - \phi X_{t-1}] + [w_t - \phi w_{t-1}]$$

$$u_t := (y_t - \phi y_{t-1}) = Z_t + (w_t - \phi w_{t-1})$$

$$u_t := (y_t - \phi z y_t) = z_t + (w_t - \phi z w_t)$$

$$u_t := (1 - \phi z) y_t = z_t + (1 - \phi z) w_t$$

$$u_t := \quad y_t = \sum_{i=0}^{\infty} \phi^i z_{t-i} + w_t.$$

- $E(u_t) = 0$
- $\gamma(h) = \begin{cases} \dfrac{\sigma_z^2}{(1-\phi^2)} + \sigma_w^2 & ; \ h=0 \\[1ex] \dfrac{\sigma_z^2 \ \phi^{|h|}}{(1-\phi^2)} & ; \ h \geq 1 \end{cases}$

$\therefore$ u_t is stationary process.

According to Proposition 2.1.1. $\{u_t\} := y_t - \phi y_{t-1}$ is 1-correlated MA(1) process.

$\because$ $\{u_t\}$ is stationary 1-correlated time series with $E(u_t) = 0$. Hence $\{u_t\}$ can be represented as MA(1) process.

(b).

$$u_t := y_t - \phi y_{t-1} = x_t + w_t - \phi x_{t-1} - \phi w_{t-1}$$

$$u_t := [y_t - \phi y_{t-1}] = [x_t - \phi x_{t-1}] + [w_t - \phi w_{t-1}]$$

$$u_t := z_t + w_t - \phi w_{t-1}$$

$$u_{t+h} := z_{t+h} + w_{t+h} - \phi w_{t+h-1}$$

$E(u_t) = 0$

$$\begin{aligned} cov(u_t; u_{t+h}) = E(u_t \cdot u_{t+h}) = {} & E[z_t \cdot z_{t+h}] \\ & + E[z_t \cdot w_{t+h}] \to 0 \\ & - \phi E[z_t \cdot w_{t+h-1}] \to 0 \\ & + E[w_t \cdot z_{t+h}] \to 0 \\ & + E[w_t \cdot w_{t+h}] \to 0 \\ & + \phi^2 E[w_{t-1} \cdot w_{t+h-1}] \\ & - \phi E[w_t \cdot w_{t+h-1}] \\ & - \phi E[z_{t+h} \cdot w_{t-1}] \to 0 \\ & - \phi E[w_{t-1} \cdot w_{t+h}] \end{aligned}$$

$$\gamma(0) = \sigma_z^2 + \sigma_w^2 + \phi^2 \sigma_w^2 = \sigma_z^2 + [1 + \phi^2]\sigma_w^2$$

$$\gamma(1) = -\phi \sigma_w^2$$

Show more work here

$$\gamma(h) = 0; \quad h \geq 2$$

According to proposition 2.1.1 $\{u_t\} : t \in \mathbb{Z}$ is 1- correlated MA(1) process.

$E(u_t) = 0$ & $\gamma(h) = \underbrace{\begin{cases} \sigma_z^2 + (1+\phi^2)\sigma_w^2 & ; h = 0 \\ -\phi\sigma_w^2 & ; |h| = 1 \\ 0 & ; h \geq 2 \end{cases}}_{\text{doesn't depend on } t}$

$\therefore \{u_t \mid u_t : t \in \mathbb{Z}\}$ is stationary process with $q = 1$ $\therefore \{u_t\} \sim MA(q=1)$

(3.4).

$$X_t = 0.8 X_{t-2} + Z_t \; ; \quad \{Z_t\} \sim WN(0; \sigma^2).$$

$$X_t = \sum_{i=0}^{\infty} \psi_i Z_{t-i} \quad ; E(X_t) = 0 \quad \psi_0 = 1$$

$$X_t \cdot X_{t-h} = 0.8\, X_{t-2} \cdot X_{t-h} + Z_t \cdot X_{t-h}$$

$$E[X_t \cdot X_{t-h}] = 0.8\, E[X_{t-2} \cdot X_{t-2-(h-2)}]$$

$$+ E\left[Z_t \cdot \sum_{i=0}^{\infty} \psi_i Z_{t-h-i}\right]$$

1.0816

$\underline{h=0}$.

$$\gamma(0) = 0.8\,\gamma(2) + \sigma^2 \quad \text{—①}$$

$$\frac{0.64}{(1-0.64)^2}$$

$\underline{h=1}$.

$$\gamma(1) = 0.8\,\gamma(1) \quad \text{—②}$$

$\underline{h=2}$.

$$\gamma(2) = 0.8\,\gamma(0) \quad \text{—③}$$

from ② $\Rightarrow \gamma(1) = 0$

from ① & ③ $\Rightarrow \gamma(0) = (0.8)^2 \gamma(0) + \sigma^2$

$$\gamma(0) = \frac{\sigma^2}{[1-(0.8)^2]} = \left[\frac{\sigma^2}{0.36}\right]$$

$$\gamma(2) = \left(\frac{0.8}{0.36}\right)\sigma^2 = \left(\frac{2}{9}\sigma^2\right)$$

$$\rho(h) = \frac{\gamma(h)}{\gamma(0)} = \begin{cases} 1 & ; \quad h=0 \\ 0 & ; \quad h=1 \\ (0.8)^{h/2} & ; \quad h=2k \, ; \quad k=\pm 1, \pm 2 \\ 0 & ; \quad o/w. \end{cases}$$

$$\therefore * \ \rho(h) = \begin{cases} (0.8)^{h/2} ; & h = 2k ; \ k \in \mathbb{Z} \\ 0 ; & o/w \end{cases}$$

- P.A.C.F of the AR(2). $\Longrightarrow$

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + Z_t$$

$\phi_1 = 0, \ \phi_2 = 0.8$

$$\hat{X}_{h+1} = \phi_{1h} X_h + \phi_{2h} X_{h-1}$$

$$\alpha(p) = \phi_p$$

$\&$ $\alpha(h) = 0$ for $h > p$.

$\alpha(0) = 1$

$\alpha(1) = 0$

$\alpha(2) = 0.8$

$\alpha(k) ; = 0 \quad \forall k \geq 3$

(3.6).

$$X_t = Z_t + \theta \cdot Z_{t-1} \quad ; \quad \{Z_t\} \sim WN(0;\sigma^2)$$
$$Y_t = \tilde{Z}_t + \frac{1}{\theta} \cdot Z_{t-1} \quad ; \quad \{\tilde{Z}_t\} \sim WN(0;\sigma^2\theta^2)$$

$0 < |\theta| < 1$;

$E(X_t) = 0$;

$$E[X_t \cdot X_{t+h}] = E[Z_t \cdot X_{t+h}] + \theta\, E[X_{t+h} \cdot Z_{t-1}]$$

$$\gamma(h) = E[Z_t \cdot X_{t+h}] + \theta\, E[Z_{t-1} \cdot X_{t+h}\}$$

<u>$h=0$</u>.

$$\gamma(0) = E[Z_t \cdot X_t] + \theta\, E[Z_{t-1} \cdot X_t]$$

$$\gamma(0) = \sigma^2 + \theta^2\sigma^2 = \sigma^2[1+\theta^2)$$

<u>$h=1$</u>.

$$\gamma(1) = E[Z_t \cdot X_{t+1}] + \theta \cdot E[Z_{t-1} \cdot X_{t+1}]$$

$$= \theta\sigma^2$$

<u>$h=(-1)$</u>.

$$\gamma(|-1|) = \gamma(1) = E[Z_t \cdot X_{t-1}) \cancel{+} \theta\, E[Z_{t-1} \cdot X_{t-1}]$$
$$\gamma(1) = \theta \cdot \sigma^2$$

$\underline{|h|>1}$.

$$\gamma(h) = 0$$

$$\therefore \gamma_x(h) = \begin{cases} (1+\theta^2)\sigma^2 & ; h=0 \\ \theta\sigma^2 & ; |h|=1 \\ 0 & ; |h|>1 \end{cases}$$

$$Y_t = \hat{Z}_t + \frac{1}{\theta} \cdot \hat{Z}_{t-1} \quad ; \quad \{\hat{Z}_t\} \sim WN(0; \sigma^2\theta^2)$$

$$E(Y_t) = 0 \quad ; \quad V(\hat{Z}_t) = E(\hat{Z}_t^2) - [E(\hat{Z}_t)]^2 = \sigma^2\theta^2 //$$

$$\gamma(h) = E[Y_t \cdot Y_{t+h}] = E[\hat{Z}_t \cdot Y_{t+h}] + \frac{1}{\theta} \cdot E[\hat{Z}_{t-1} \cdot Y_{t+h}]$$

$\underline{h=0}$.

$$\gamma(0) = E[\hat{Z}_t \cdot Y_t] + \frac{1}{\theta} E[\hat{Z}_{t-1} \cdot Y_t]$$

$$\gamma(0) = \theta^2\sigma^2 + \frac{1}{\cancel{\theta^2}} \cdot \cancel{\theta^2}\sigma^2 = \sigma^2[1+\theta^2] //$$

$\underline{h=1}$.

$$\gamma(1) = E[\hat{Z}_t \cdot Y_{t+1}] + \frac{1}{\theta} \cdot E[\hat{Z}_{t-1} \cdot Y_{t+1}] = \frac{1}{\cancel{\theta}} \cdot \cancel{\theta^2}\sigma^2 = \underline{\underline{\theta\sigma^2}}$$

$\underline{h=(-1)}$.

$$\gamma(-1) = E[\hat{Z}_t \cdot Y_{t-1}] + \frac{1}{\theta} \cdot E[\hat{Z}_{t-1} \cdot Y_{t-1}] = \frac{1}{\cancel{\theta}} \cdot \cancel{\theta^2}\sigma^2 = \underline{\underline{\theta\sigma^2}}$$

$$\therefore \gamma_Y(h) = \begin{cases} \sigma^2(1+\theta^2) & h=0 \\ \theta\sigma^2 & |h|=1 \\ 0 & ; |h|>1 \end{cases} \qquad \therefore \underline{\underline{\gamma_X(h) = \gamma_Y(h)}}$$

$\frac{10}{10}$

STAT S814

HW #4

Isuru Ratnayake.

[2.14]

$$X_t = A \cdot \cos(wt) + B \cdot \sin(wt). \quad t = 0; \pm 1, \pm 2 \ldots$$

- A & B are uncorrelated random variables with mean 0 & variance 1 $\Rightarrow \text{cov}(A,B) = \underbrace{E(A \cdot B) - E(A) \cdot E(B)}_{}\xrightarrow{} 0$
  $V(A) = E(A^2) - (E(A))^2 = 0 \therefore E(A^2) = 1 \; ; E(B^2) = 1$
- $\omega \in [0, \pi]$

$$E(X_t) = E(A) \cdot \cos(wt) \cdot E(B) \cdot \sin(wt)$$

$$\therefore E(X_t) = 0 \,//.$$

$$E(X_t \cdot X_{t+h}) = \text{cov}(X_t; X_{t+h})$$

here $X_t = A \cdot \cos(\omega(t)) + B \cdot \sin(wt)$.

$x_{t+h} = A \cdot \cos(\omega(t+h)] + B \cdot \sin[w(t+h)]$

$$E(X_t \cdot X_{t+h}) = E[A^2 \cos(wt) \cdot \cos[w(t+h)]] + E[B^2 \cdot \sin(wt) \cdot \sin[w(t+h)]]$$

$$E[X_t \cdot X_{t+h}] = \cos(wt) \cdot \cos[w(t+h)] \, E(A^2) \rightarrow 1 + \sin(wt) \cdot \sin[w(t+h)] \, E(B^2) \rightarrow 1$$

$$\gamma(h) = \cos[\cancel{wt} + wh - \cancel{wt}] = \cos(wh).$$

when $h = 0, \pm 1, \ldots$

$$P_1 X_2 = \hat{X}_2 = \phi_{11} X_1$$

$$\mathrm{cov}[(X_2 - \hat{X}_2); X_j] = 0 \; ; \; j=1$$

$$\mathrm{cov}[X_2 \cdot X_1) - \mathrm{cov}(\phi_{11} X_1, X_1] = 0$$

$$\gamma(1) - \phi_{11}\gamma(0) = 0$$

$$\phi_{11}\gamma(0) = \gamma(1) \Longrightarrow \phi_{11} = \rho(1) = \frac{\gamma(1)}{\gamma(0)}$$

$$\phi_{11} = \cos(\omega).$$

$$\therefore \quad P_1 X_2 = \hat{X}_2 = [\cos\omega] \cdot X_1$$

$$\cancel{M.S.E = E[X_2 - \hat{X}_2]^2 = \langle (X_2 - \phi_{11} X_1), (X_2 - \phi_{11} X_1) \rangle}$$

$$\cancel{= \gamma(0) - \phi_{11}\gamma(1) - \phi_{11}\gamma(1) + \phi_{11}^2 \gamma(0).}$$

$$\cancel{= \gamma(0)[1 - \phi_{11}^2] - 2\phi_{11}\gamma(1).}$$

- $$M.S.E = \gamma(0) - \phi_{11}\gamma(1)$$

$$= \gamma(0) - \phi_{11}^2 \gamma(0) = [1 - \phi_{11}^2]\gamma(0)$$

$$= \sin^2 \omega$$

(b). $P_2 X_3 = \hat{X}_3 = \phi_{12} X_2 + \phi_{22} X_1$

$$E[(X_3 - \hat{X}_3) ; X_j] = 0 ; j = 1, 2$$

j=1

$$\gamma(2) - \phi_{12}\gamma(1) - \phi_{22}\gamma(0) = 0 \quad -① $$

j=2.

$$\rho(2) - \phi_{12}\rho(1) - \phi_{22} = 0$$

$$\gamma(1) - \phi_{12}\gamma(0) - \phi_{22}\gamma(1) = 0 \quad -②$$

$$\cancel{\gamma(1)[1-\phi_{22}] = \phi_{12}\gamma(0)}$$

$$\begin{pmatrix}\gamma(2)\\ \gamma(1)\end{pmatrix} = \begin{pmatrix}\gamma(1) & \gamma(0)\\ \gamma(0) & \gamma(1)\end{pmatrix}\begin{pmatrix}\phi_{12}\\ \phi_{22}\end{pmatrix}$$

$$\cancel{\rho(1) = \left[\frac{\phi_{12}}{1-\phi_{22}}\right]}$$

$$\cancel{\rho(2) - \left[\frac{\phi_{12}^2}{1-\phi_{22}}\right] = \phi_{22}.}$$

$$\rho(2) - \phi_{12}\rho(1) = \phi_{22}.$$

$$\rho(1) - \phi_{12} = \phi_{22}\rho(1)$$

$$\rho(1) - \phi_{12} = \rho(1)\left[\rho(2) - \phi_{12}\rho(1)\right]$$

$$\rho(1) - \rho(1).\rho(2) = \phi_{12}\left[1 - [\rho(1)]^2\right]$$

$$\frac{\rho(1)\,[1-\rho(2)]}{[1-\rho(1)]^2} = \phi_{12} = \frac{\cos w\,[1 - \cancel{\cos(2w)}]^{2}}{\sin^2 w}$$

$\phi_{12} = 2\cos w$ //

$$1 - 2\frac{\cancel{\cos w}}{\cancel{\cos w}} = \phi_{22} = (-1)$$

$$P_2 X_3 = \hat{X}_3 = (2\cos w) X_2 - 1 \cdot x_1$$

Method I :→

$$\text{M.S.E} = E\left[X_3 - \hat{X}_3\right]^2 =$$

$$= \gamma(0) - (2\cos w)\gamma(1) + \gamma(2)$$

$$= \gamma(0) - \left[(2\cos w) \quad -1\right]\begin{bmatrix}\gamma(1) \\ \gamma(2)\end{bmatrix}$$

$$= 1 - \left[2\cos w \quad -1\right]\begin{bmatrix}\cos w \\ \cos 2w\end{bmatrix}$$

$$= \underbrace{1 - \cancel{2\cos^2 w}}_{\cancel{\cos(2w)}} + \cancel{\cos 2w} = 0$$

(c). $P_n X_{n+1} = \hat{X}_{n+1} = \phi_{1n} X_n + \phi_{2n} X_{n-1} + \cdots + \phi_{nn} X_1$

- Page 77 2.6 section [Simillar example is given]

$\{X_t\}$ is stationary & it follows

$$\hat{X}_{n+1} = (2\cos w) X_n - X_{n-1}$$

$E(X_{n+1} - \hat{X}_{n+1})^2$;

$$MSE = \gamma(0) - \left[2\cos w \quad -1\right]\begin{bmatrix}\gamma(1) \\ \gamma(2)\end{bmatrix} = 0$$

(2.15). Suppose that $\{X_t \mid t \in \mathbb{Z}\}$ is a stationary process & $E(X_t) = 0$

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \cdots + \phi_p X_{t-p} + Z_t.$$

AR(p) ; $p < n$.

$$P_n \cdot X_{n+1} = \hat{X}_{n+1} = a_0 + a_1 X_n + a_2 X_{n-1} + \cdots + a_n X_1$$

$$M.S.E = E\{X_{n+1} - \hat{X}_{n+1}\}^2.$$

we have to find values for a_i's $i = 0, 1, \ldots, n$ which minimize the M.S.E. [find linear combination of X_i's which minimize the MSE]

$$MSE = E\{X_{n+1} - [a_0 + a_1 X_n + a_2 X_{n-1} + \cdots + a_n X_1]\}^2$$

$$= E\{-a_0 + [X_{n+1} - (a_1 X_n + a_2 X_{n-1} + \cdots + a_n X_1)]\}^2$$

$E(X_t) = 0 \ \forall t.$

$$= a_0^2 + 2a_0 \underbrace{E[X_{n+1} - (a_1 X_n + a_2 X_{n-1} + \cdots + a_n X_1)]}_{\to 0}$$

$$+ E[X_{n+1} - (a_1 X_n + a_2 X_{n-1} + \cdots + a_n X_1)]^2$$

$$= \frac{\partial MSE}{\partial a_0} \overset{set}{=} 0$$

OK

$$\therefore 2a_0 = 0$$

$$a_0 = 0$$

Taking the derivative the expectation needs conditions to be satisfied.

$\therefore \quad \hat{X}_{n+1} = a_n X_1 + a_{n-1} X_2 + \cdots + a_1 X_n.$

MMSE

$$E\left[\left(X_{n+1} - \hat{X}_{n+1}\right) X_j\right] = 0 \quad \text{for } \forall j = 1, 2, \ldots, n.$$

NOW $\underbrace{X_{n+1}} = \phi_1 X_n + \phi_2 X_{n-1} + \cdots + \phi_p X_{n+1-p} + Z_{n+1}$

$$\hat{X}_{n+1} = a_1 X_n + a_2 X_{n-1} + \cdots + a_p X_{n-p+1} + \cdots + a_n X_1$$

$$\hookrightarrow \sum_{i=1}^{n} a_i x_{n+1-i}$$

$$E\left\{ \begin{aligned} &(\phi_1 - a_1) X_n + (\phi_2 - a_2) X_{n-1} + \cdots + (\phi_p - a_p) X_{n+1-p} \\ &\quad - a_{p+1} X_{n-p+2} - \cdots - a_n X_1 \\ &\quad + Z_{n+1} \end{aligned} \right\} X_j = 0$$

$$(\phi_1 - a_1)\gamma(n-j) + (\phi_2 - a_2)\gamma(n-1-j) + \cdots + (\phi_p - a_p)\cdots - a_{p+1} E(X_{n-p+2} \cdot X_j) + \cdots - a_n E(X_1 X_j) = 0$$

This is satisfied

$$\text{if} \quad \phi_i = a_i \quad \text{for } 1 \le i \le p$$

$$a_i = 0 \quad \text{for } i > p.$$

$$\therefore \hat{X}_{n+1} = \phi_1 X_n + \phi_2 X_{n-1} + \cdots + \phi_p X_{n+1-p}.$$

$$X_{n+1} = \phi_1 X_n + \phi_2 X_{n-1} + \cdots + \phi_p X_{n+p-1} + Z_{n+1}$$

$$\left(X_{n+1} - \hat{X}_{n+1}\right) = Z_{n+1}$$

$$E\left[X_{n+1} - \hat{X}_{n+1}\right]^2 = E\left[Z^2_{n+1}\right] = \sigma^2$$

$$M.S.E = \sigma^2$$

[4.6]

$$X_t = A \cdot \cos\left(\frac{\pi t}{3}\right) + B \cdot \sin\left(\frac{\pi t}{3}\right) + Y_t$$

$$Y_t = Z_t + 2.5 Z_{t-1} \quad ; \quad \{Z_t\} \sim WN(0, \sigma^2)$$

$$\text{cov}(A, B) = 0 \; ; \; E(A) = E(B) = 0 \quad \&$$
$$V(A) = V(B) = E(A^2) = E(B^2) = \gamma^2$$
$$\text{cov}(Z_t, A) = \text{cov}(Z_t, B) = 0 \;\; \forall t$$

$$\gamma_x(h) = \text{cov}(X_t ; X_{t+h}) = E[X_t \cdot X_{t+h}]$$

$$X_{t+h} = A \cdot \cos\left(\frac{\pi}{3}(t+h)\right) + B \cdot \sin\left(\frac{\pi}{3}(t+h)\right) + Y_{t+h}$$

$$E[X_t \cdot X_{t+h}] = E\left[A \cdot \cos\left(\frac{\pi t}{3}\right) + B \cdot \sin\left(\frac{\pi t}{3}\right) + Y_t\right] \left[A \cdot \cos\left(\frac{\pi}{3}(t+h)\right) + B \sin\left(\frac{\pi}{3}(t+h)\right) + Y_{t+h}\right]$$

$$\gamma_x(h) = E\left[A^2 \cdot \cos\left(\frac{\pi}{3} t\right) \cdot \cos\left(\frac{\pi}{3}(t+h)\right)\right]$$
$$+ E\left[B^2 \cdot \sin\left(\frac{\pi}{3} t\right) \cdot \sin\left(\frac{\pi}{3}(t+h)\right)\right]$$
$$+ E[Y_t \cdot Y_{t+h}].$$

$$\gamma_x(h) = E(A^2) \cdot \cos\left(\frac{\pi}{3}t\right) \cdot \cos\left[\frac{\pi}{3}(t+h)\right]$$

$$+ E(B^2) \cdot \sin\left(\frac{\pi}{3}t\right) \cdot \sin\left(\frac{\pi}{3}(t+h)\right]$$

$$+ E\left\{Z_t + 2.5 Z_{t-1}\right]\left[Z_{t+h} + 2.5 Z_{t+h-1}\right]$$

$$\gamma_x(h) = V^2 \cos\left[\cancel{\frac{\pi}{3}t} + \frac{\pi}{3}h - \cancel{\frac{\pi}{3}t}\right]$$

$$+ E\left[Z_t \cdot Z_{t+h}\right] + E\left[Z_t (2.5) Z_{t+h-1}\right]$$

$$+ (2.5) E\left[Z_{t-1} \cdot Z_{t+h}\right]$$

$$+ (2.5)^2 E\left[Z_{t-1} \cdot Z_{t+h-1}\right]$$

$$\gamma_x(h) = V^2 \cos\left[\frac{\pi}{3}h\right] + E\left[Z_t \cdot Z_{t+h}\right] + (2.5) E\left[Z_t \cdot Z_{t+h-1}\right]$$

$$+ (2.5) E\left[Z_{t-1} \cdot Z_{t+h}\right]$$

$$+ (2.5)^2 E\left[Z_{t-1} \cdot Z_{t+h-1}\right]$$

<u>$h = 0$</u>.

$$\gamma_x(0) = V^2 + \sigma^2 + (2.5)^2 \sigma^2 = V^2 + \sigma^2 (1 + (2.5)^2)$$

<u>$|h| = 1$</u>

$$\gamma_x(1) = \frac{V^2}{2} + \sigma^2 (2.5)$$

$|h| > 1$

$$\gamma_x(h) = V^2 \cos(\pi/3 \, h)$$

let consider

$$\gamma_X(h) = \gamma_S(h) + \gamma_Y(h)$$

$$\therefore F_X(\lambda) = F_S(\lambda) + F_Y(\lambda)$$

$$\gamma_S(h) = v^2 \cos\left(\frac{\pi}{3} h\right)$$

$$\gamma_Y(h) = \begin{cases} \sigma^2 [1+\theta^2] & ; h=0 \\ \sigma^2 \theta & ; |h|=1 \\ 0 & ; o/w \end{cases}$$

$$\gamma_S(h) = \int_{-\pi}^{+\pi} \exp(ih\lambda) \cdot f(\lambda) \cdot d\lambda$$

$$= \int_{(-\pi,\pi]} \exp(ih\lambda)\, dF(\lambda).$$

$$\cancel{v^2 \cos\left(\frac{\pi h}{3}\right)} \cancel{=} \cancel{\int_{-\pi}^{\pi}}$$ According to the proof in page 115

If $X_t = A\cos(\omega t) + B\sin(\omega t)$ where A & B are uncorrelated r.V with $E(A) = E(B) = 0$ & $V(A) = V(B) = 1$, has ACVF $\gamma(h) = \cos(\omega h)$

Then

$$F(\lambda) = \begin{cases} 0 & \text{if } \lambda < -\omega \\ 0.5 = 1/2 & \text{if } -\omega \le \lambda < \omega \\ 1.0 ; & \text{if } \lambda \geq \omega \end{cases}$$

$$\therefore \quad F_S(\lambda) = \begin{cases} 0^2 & \text{if } \lambda < -\pi/3 \\ \frac{V^2}{2} & \text{if } -\pi/3 \le \lambda < \pi/3 \\ V^2 & \text{if } \lambda \geq \pi/3 \end{cases}$$

$$f_y(\lambda) = \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} \exp(-ih\lambda)\, \gamma_y(h)$$

$$= \frac{1}{2\pi} \left[\exp(-i\lambda) \cdot \gamma_y(1) + \exp(i\lambda)\, \gamma_y(1) + \gamma_y(0) \right]$$

$$= \frac{1}{2\pi} \left\{ \sigma^2(1+\theta^2) + \sigma^2\theta \left[\cos(\lambda) - \cancel{i\sin(\lambda)} + \cos(\lambda) + \cancel{i\sin(\lambda)} \right] \right\}$$

$$= \frac{\sigma^2}{2\pi} \left[1 + \theta^2 + 2\theta\cos(\lambda) \right] = \frac{\sigma^2}{2\pi} \left(7.25 + 5\cos\lambda \right)$$

$$F_y(\lambda) = \int_{-\pi}^{\lambda} f_y(\lambda)\, dy = \frac{\sigma^2}{2\pi} \left[7.25(\lambda + \pi) + 5\sin(\lambda) \right]$$

$$\boxed{F_x(\lambda) = F_S(\lambda) + F_y(\lambda)}$$

What is F_X ?

[4.9]

$$\gamma(h) = \int_{-\pi}^{+\pi} \exp(ih\lambda)\cdot f(\lambda)\, d\lambda.$$

here

$$f(\lambda) = \begin{cases} 100 \; ; \text{ if } \frac{\pi}{6} - 0.01 < \lambda < \pi/6 + 0.01 \\ 0 \; ; \text{ o/w} \end{cases}$$

and on $[-\pi, 0]$

(a). Evaluate the ACVF of $\{X_t\}$ at lag 0 & 1.

<u>for h = 0</u>

$$\gamma_x(h) = \gamma_x(0) = 2\int_{\pi/6 - 0.01}^{\pi/6 + 0.01} \exp(i\cdot 0\cdot\lambda)\cdot 100\cdot d\lambda$$

$$\gamma_x(0) = 2\int_{\pi/6-0.01}^{\pi/6+0.01} 100\, d\lambda = 2\times 100\left[\cancel{\frac{\pi}{6}} + 0.01 \; \cancel{-\pi/6} + 0.01\right]$$

$$= 2 \times 100 \times 0.02 = 4 \,//$$

<u>for h = 1</u>

$$\gamma_x(h) = \gamma_x(1) = \int_{-\pi}^{\pi} \exp(i\lambda)\cdot f(\lambda)\cdot d\lambda$$

$$\gamma_x(1) = 100\int_{-\pi/6-0.01}^{-\pi/6+0.01} \exp(i\lambda)\, d\lambda + 100\int_{\pi/6-0.01}^{\pi/6+0.01} \exp(i\lambda)\cdot d\lambda$$

$$\gamma_x(1) = 100\int_{-\pi/6-0.01}^{-\pi/6+0.01} [\cos(\lambda) + \underbrace{i\sin(\lambda)}_{0}]\, d\lambda$$

$$+ 100\int_{\pi/6-0.01}^{\pi/6+0.01} [\cos(\lambda) + \underbrace{i\sin(\lambda)}_{0}]\, d\lambda$$

$$= 100\int_{-\pi/6-0.01}^{-\pi/6+0.01} \cos(\lambda)\cdot d\lambda + 100\int_{\pi/6-0.01}^{\pi/6+0.01} \cos(\lambda)\, d\lambda$$

$$100 \left[\sin(\lambda)\right]_{-\pi/6-0.01}^{-\pi/6+0.01} + 100\left[\sin(\lambda)\right]_{\pi/6-0.01}^{\pi/6+0.01}$$

$$100\Big\{ \sin\left(-\frac{\pi}{6}+0.01\right) - \sin\left(-\pi/6-0.01\right) + \sin\left(\pi/6+0.01\right) - \sin\left(\pi/6-0.01\right) \Big\}$$

$$100\times 2\Big\{ \sin\left(\pi/6+0.01\right) + \sin\left(-\pi/6+0.01\right)\Big\}$$

$$200\sqrt{3}\sin(0.01)$$

$= \{$

(b). Find the spectral density of the process $\{y_t\}$ defined by

$$Y_t = \nabla_{12} X_t = X_t - X_{t-12} = \sum_{k=-\infty}^{\infty} \psi_k X_{t-k}$$

with $\psi_0 = 1$ & $\psi_{12} = -1$ & $\psi_j = 0$ o/w.

$$f_Y(\lambda) = \left|\psi(\exp[-i\lambda])\right|^2 f_X(\lambda) \quad \text{where}$$

$$\psi(\exp[-i\lambda]) = \sum_{j=-\infty}^{\infty} \psi_j \exp[-ij\lambda] \quad j=0 \;\&\; j=12$$

$$\psi[\exp(-i\lambda)] = 1 - 1\exp[-i12\lambda]$$

$$f_Y(\lambda) = \left|1 - \exp(-12i\lambda)\right|^2 f_X(\lambda)$$

$$f_y(\lambda) = \left|1 - \exp(-12i\lambda)\right|^2 f_X(\lambda)$$

$$= \left[1 - \exp(-12i\lambda)\right]\left[1 - \exp(12i\lambda)\right] f_X(\lambda)$$

$$= f_X(\lambda)\left\{1 - \left[\cos(12\lambda) - i\sin(12\lambda)\right]\right\} \cdot \left\{1 - \left[\cos(12\lambda) + i\sin(12\lambda)\right]\right\}$$

$$= f_X(\lambda)\left\{(1 - \cos 12\lambda)^2 - i^2 \sin^2(12\lambda)\right\}$$

$$= f_X(\lambda)\left\{1 + \underbrace{\cos^2(12\lambda) + \sin^2(12\lambda)}_{1} - 2\cos(12)\lambda\right\}$$

$$= 2(1 - \cos(12)\lambda)\, f_X(\lambda) = 200[1 - \cos 12\lambda]$$

(c). What is the variance of Y_t?

$$\gamma_Y(0) = \int_{-\pi}^{\pi} f_Y(\lambda) \cdot d\lambda$$

$$= 200 \int_{-\frac{\pi}{6}-0.01}^{-\pi/6+0.01} [1 - \cos 12\lambda] \, d\lambda$$

$$+ 200 \int_{+\frac{\pi}{6}-0.01}^{\frac{\pi}{6}+0.01} [1 - \cos 12\lambda] \, d\lambda$$

$$= 200 \left\{ \left[\lambda - \frac{\sin(12\lambda)}{12} \right]_{-\frac{\pi}{6}-0.01}^{-\frac{\pi}{6}+0.01} + \left[\lambda - \frac{\sin 12\lambda}{12} \right]_{\pi/6-0.01}^{\pi/6+0.01} \right\}$$

$$= 200 \left\{ 0.02 - \frac{1}{12} \sin\left(+12\left(-\frac{\pi}{6}+0.01\right)\right) + \frac{1}{12} \sin\left(12\left(-\frac{\pi}{6} - 0.01\right)\right) \right.$$

$$\left. + 0.02 - \frac{1}{12} \sin\left(12\left(\pi/6 + 0.01\right)\right) + \frac{1}{12} \sin\left(12\left(\pi/6 - 0.01\right)\right) \right\}$$

$$= 200 \left\{ 0.04 + \frac{\sin(2\pi - 0.12) - \sin(2\pi + 0.12)}{6} \right\}$$

$$= 200 \left[0.04 - \frac{\sin(0.12)}{3} \right] = \underline{\underline{0.0192}}$$