Problem 1: Suppose $Y_t = 5 + 2t + X_t$ where $\{X_t\}$ is a zero-mean stationary time series with autocovariance function γ_k.
(a) Find the mean function for $\{Y_t\}$.
(b) Find the autocovariance function for $\{Y_t\}$.
(c) Is $\{Y_t\}$ stationary?

Problem 2: Suppose that $\{Y_t\}$ is stationary with autocovariance function γ_k
(a) Show that $W_t = \nabla Y_t = Y_t - Y_{t-1}$ is stationary and find its mean and autocovariance.
(b) Show that $U_t = \nabla^2 Y_t = \nabla[Y_t - Y_{t-1}]$ is stationary.

Problem 3: Let $\{X_t\}$ be zero-mean, unit variance stationary process with autocorrelation function ρ_k. Suppose that μ_t is a non-constant function and that σ_t is a positive-valued non-constant function. The observed series is formed as $Y_t = \mu_t + \sigma_t X_t$.
(a) Find the mean and covariance function for $\{Y_t\}$.
(b) Show that the autocorrelation function of $\{Y_t\}$ depends only on the time lag. Is $\{Y_t\}$ stationary?

Problem 4: Let $Y_t = \epsilon_t - \theta(\epsilon_{t-1})^2$. Assume $\{\epsilon_t\}$ is normally distributed.
(a) Find the autocorrelation function of $\{Y_t\}$.
(b) Is $\{Y_t\}$ stationary? (c) Simulate the series $\{Y_t\}$ for $\theta \in \{-1, 0.5, 1\}$ and $\sigma = 1$.

Problem 5: Suppose X is a random variable with zero-mean. Define $Y_t = (-1)^t X$
(a) Find the mean and autocovariance of Y_t.
(b) Is $\{Y_t\}$ stationary?

Problem 6: Let $\{Y_t\}$ be stationary with autocovariance γ_k. Let $\bar{Y} = n^{-1}\sum_{t=1}^{n} Y_t$.
(a) Show that

$$\begin{aligned}\mathrm{Var}(\bar{Y}) &= \frac{\gamma_0}{n} + \frac{2}{n}\sum_{k=1}^{n-1}\left(1 - \frac{k}{n}\right)\gamma_k \\ &= \frac{1}{n}\sum_{k=-n+1}^{n-1}\left(1 - \frac{|k|}{n}\right)\gamma_k\end{aligned}$$

(b) The sample variance is defined by $S^2 = (n-1)^{-1}\sum_{t=1}^{n}(Y_t - \bar{Y})^2$. Use (a) to show that

$$E(S^2) = \frac{n}{n-1}\gamma_0 - \frac{n}{n-1}\mathrm{Var}(\bar{Y}).$$

Problem 7: Random cosine wave.
Suppose that, for $t \in \{0, \pm 1, \pm 2, ...\}$

$$Y_t = R\cos(2\pi(ft + \Phi)),$$

where $0 < f < 0.5$ is a fixed frequency, and R and Φ are uncorrelated random variables with $\Phi \sim \mathrm{U}(0, 1)$.
(a) Show that $E(Y_t) = 0 \ \forall$ t.
(b) Show that the series is stationary with $\gamma_k = \frac{1}{2}E(R^2)\cos(2\pi f k)$.

Problem 8: This problem pertains to the data `gasprices`.
The data lists average price (US dollars per gallon) for a regular gasoline in the US. There are $n = 145$ weekly observations collected from 1/2009 to 11/2011.
(a) Construct a time series plot for these data and describe any observations you see in the plot.
(b) Create a scatter with the observed series Y_t on the vertical axis and Y_{t-1} on the horizontal. This is called a *lag-1 scatterplot*. You can do this with the R command

```
plot(y=gasprices, x=zlag(gasprices, 1), ylab=expression(Y[t]),
xlab=expression(Y[t-1]), type='p')
```

This plot displays the observed data plotted against the lag-1 series, i.e., the scatterplot of the 144 points $\{(Y_1, Y_2), (Y_2, Y_3), ..., (Y_{144}, Y_{145})\}$. What does the plot suggests about the original series Y_t? To aid in interpretation, calculate the sample autocorrelation function using the command

```
cor(gasprices[2:145], zlag(gasprices, 1)[2:145]).
```

(c) What does this plot looks like for larger lags say 2, 3, 5, 15?
(d) When compared to lag-1 series, do the corresponding correlations between Y_t and Y_{t-k}. increase or decrease as $k \to \infty$? Interpret.