Vectors and Matrices

1 Key Concepts

Zero matrices

A matrix A is a zero matrix if all its elements are equal to zero, and we write [eq9]

A = $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Square matrices

Example $\begin{bmatrix} 2 & 1 \\ 9 & 4 \end{bmatrix}$

Transpose of a matrix and vector The transpose of a matrix is simply a flipped version of the original matrix. We can transpose a matrix by switching its rows with its columns. We denote the transpose of matrix A by AT A = $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$

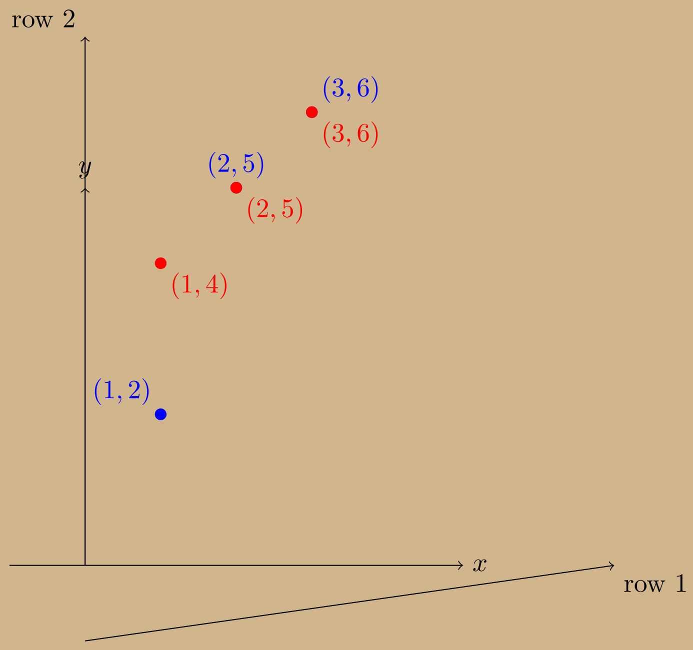
then the transpose of A is $\mathbf{A}^\intercal$

$= \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$

if vector A = $\begin{bmatrix} 1 & 2 & 4 & 5 \end{bmatrix}$

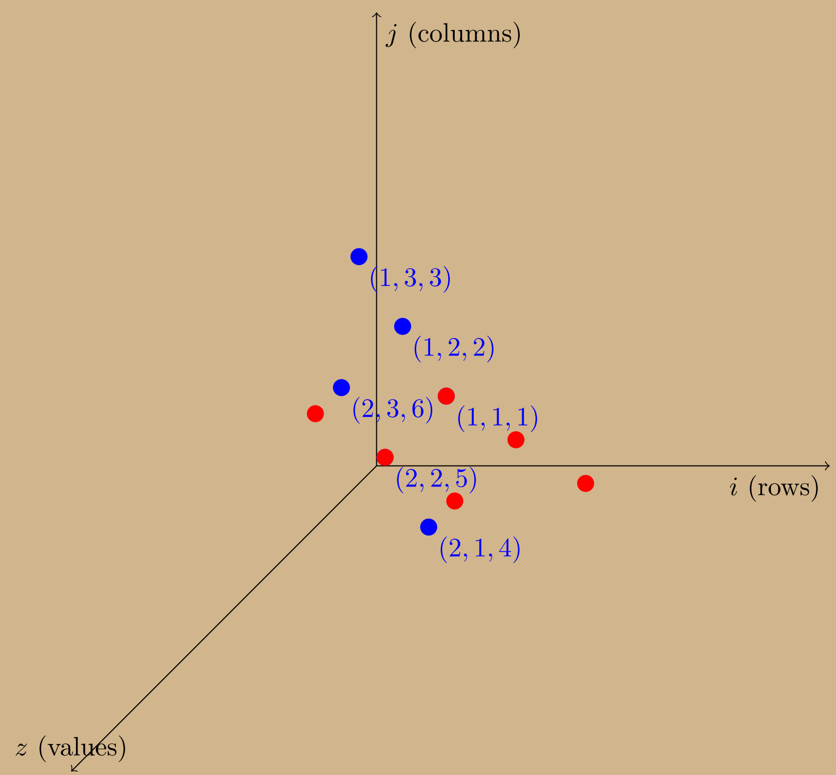
then the transpose version is $\mathbf{A}^\intercal$

$= \begin{bmatrix} 1 \\ 2 \\ 4 \\ 5 \end{bmatrix}$



row 2
$(3, 6)$
$(3, 6)$
y
$(2, 5)$
$(2, 5)$
$(1, 4)$
$(1, 2)$
x
row 1

2 3d vector graph



Symmetric matrices A square matrix is said to be symmetric if it is equal to its transpose.

Example if A is a

$$2 times 2$$

matrix defined by A = $\begin{bmatrix} 4 & 5 \\ 5 & 3 \end{bmatrix}$ its transpose is the following matrix

$\mathbf{A}^\intercal$
$= \begin{bmatrix} 4 & 5 \\ 5 & 3 \end{bmatrix}$

3 Definitions

Scalar - a number. Examples could be temperature, distance, speed, or mass; all quantities that have a magnitude but no direction, other than perhaps positive or negative.

Vector - a list of numbers. There are at least two ways to think of it, one as a point in space. Vectors are denoted by $\vec{A}$. Example of a vector:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \tag{1}$$

Magnitude - of a vector is the distance from the endpoint of the vector to the origin so in short its length. The magnitude of a vector is a scalar value denoted by:

$$\|\mathbf{v}\|$$

Unit vectors - have a magnitude of 1, denoted by a small caret or hat. A unit vector can be used to express the direction of a vector independent of its magnitude.

Calculation of a particular vector to a unit vector: we take the original vector and divide it by its magnitude. In mathematical terms, this process is written as $\hat{v} = \frac{\vec{v}}{\|\vec{v}\|}$.

4 Calculations for Vectors

Multiplying - We have demonstrated how to create a unit vector that has a magnitude of 1 but a direction identical to the vector. Taking together the magnitude and the unit vector, we have all of the information contained in the vector, but neatly separated into its magnitude and direction components. We can use these two components to re-create the vector by multiplying the vector by the scalar like so: $\vec{v} = \|\vec{v}\| \cdot \hat{v}$

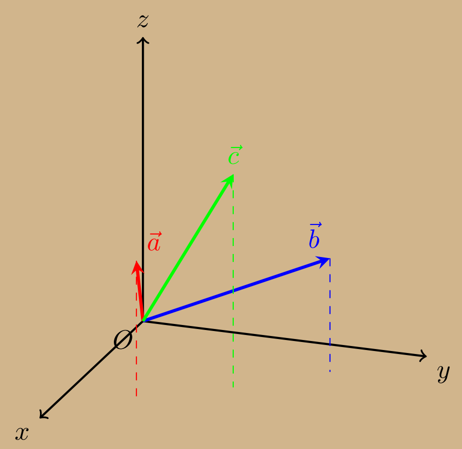
Addition and Subtraction Numerically, we add vectors component-by-component. That is to say, we add the x-components together, and then separately we add the y-components together. For example, if $\vec{a} = (a_x, a_y)$ and $\vec{b} = (b_x, b_y)$, then:

Adding formula: $\vec{c} = \vec{a} + \vec{b} = (a_x + b_x, a_y + b_y)$

Subtracting formula: $\vec{c} = \vec{a} - \vec{b} = (a_x - b_x, a_y - b_y)$

Vector addition has a very simple interpretation in the case of things like displacement. If in the morning a ship sailed 4 miles east and 3 miles north, and then in the afternoon it sailed a further 1 mile east and 2 miles north, what was the total displacement for the whole day? 5 miles east and 5 miles north – vector addition at work.

5 Vector Graph



6 Linear Independence

If two vectors point in different directions, even if they are not very different directions, then the two vectors are said to be linearly independent.

7 Linear Independence Scalar Info

We can multiply a vector by a constant, scalar value and get a vector, and vice versa to get from $\vec{v}$ to $k\vec{v}$. If the two vectors point in different directions, then it is not possible to make one out of the other because multiplying a vector by a scalar will never change the direction of the vector, it will only change the magnitude. This concept generalizes to families of more than two vectors. Three vectors are said to be linearly independent if there is no way to construct one vector by combining scaled versions of the other two. The same definition applies to families of four or more vectors by applying the same rules.

Definition: A family of vectors is linearly independent if no one of the vectors can be created by any linear combination of the other vectors in the family. For example, $\vec{c}$ is linearly independent of $\vec{a}$ and $\vec{b}$ if and only if it is impossible to find scalar values of α and β such that $\vec{c} = \alpha\vec{a} + \beta\vec{b}$.

8 Vector Multiplication and Dot Product

There are two principal ways of multiplying vectors, called dot products (a.k.a. scalar products) and cross products.

The dot product: $d = \vec{a} \cdot \vec{b}$ The dot product generates a scalar value from the product of two vectors.

The cross product: $\vec{d} = \vec{a} \times \vec{b}$ The cross product generates a vector from the product of two vectors.

9 Orthogonality

Orthogonality - as the angle between the two vectors opens up to approach 90°, the dot product of the vectors approaches zero.

10 Matrices

A matrix, like a vector, is also a collection of numbers. The difference is that a matrix is a table of numbers rather than a list. Many of the same rules we just outlined for vectors above apply equally well to matrices. (In fact, you can think of vectors as matrices that happen to only have one column or one row.) First, let's consider matrix addition and subtraction. This part is uncomplicated. You can add and subtract matrices the same way you add vectors – element by element.

$M = \begin{bmatrix} 1 & 2 & 3 \\ a & 2 & 5 \end{bmatrix}$

11 Matrix Addition and Subtraction

Let A and B be matrices of the same size, $m \times n$.

11.1 Matrix Addition

$$A + B = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \cdots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} & \cdots & a_{2n} + b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} + b_{m1} & a_{m2} + b_{m2} & \cdots & a_{mn} + b_{mn} \end{bmatrix}$$

11.2 Matrix Subtraction

$$A - B = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} - \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} - b_{11} & a_{12} - b_{12} & \cdots & a_{1n} - b_{1n} \\ a_{21} - b_{21} & a_{22} - b_{22} & \cdots & a_{2n} - b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} - b_{m1} & a_{m2} - b_{m2} & \cdots & a_{mn} - b_{mn} \end{bmatrix}$$

12 Linear Matrix/Transformations

Linear Transformations are functions (functions that take vectors as inputs), but they're a special subset of functions. Linear transformations are the set of all functions that can be written as a matrix multiplication:

$$\vec{y} = A\vec{x} \tag{2}$$

Where A is a matrix, $\vec{x}$ is the input vector, and $\vec{y}$ is the output vector.

13 Formulas and Equations

14 Equations of Planes

- **Normal Form:** Equation of a plane at a perpendicular distance d from the origin and having a unit normal vector $\hat{n}$ is:
$$\vec{r} \cdot \hat{n} = d$$
- **Perpendicular to a given Line and through a Point:** The equation of a plane perpendicular to a given vector $\vec{N}$, and passing through a point $\vec{a}$ is:
$$(\vec{r} - \vec{a}) \cdot \vec{N} = 0$$
- **Through three Non-Collinear Points:** The equation of a plane passing through three non-collinear points $\vec{a}$, $\vec{b}$, and $\vec{c}$, is:
$$(\vec{r} - \vec{a}) \cdot [(\vec{b} - \vec{a}) \times (\vec{c} - \vec{a})] = 0$$
- **Intersection of Two Planes:** The equation of a plane passing through the intersection of two planes $\vec{r} \cdot \hat{n}_1 = d_1$ and $\vec{r} \cdot \hat{n}_2 = d_2$, is:
$$\vec{r} \cdot (\hat{n}_1 + \lambda \hat{n}_2) = d_1 + \lambda d_2$$

15 Examples

16 Practice Problems

17 Notes / quick refrence

- transpose of a matrix notaion is - $\mathbf{A}^\intercal$

- m x n = m = number of rows(horizontal lines) , n = number of columns(vertical lines) example If you have a 2×3 matrix, it looks like this: V = $\begin{bmatrix} a11 & a12 & a13 \\ a21 & a22 & a23 \end{bmatrix}$

  Without transpose, matrix multiplication wouldn't even make sense.

  Linear Transformations - If you think of a matrix as a function that transforms vectors, the input vector has to match the dimensions correctly. Often, vectors are expected to be columns. So you may have to transpose a row vector into a column vector before applying the transformation. To compute dot products, projections, or anything fancy, you need that transpose.

  Changing Perspective - Sometimes you need to switch how you think about a vector: A row vector describes a linear function. A column vector describes a point or position. In deep math (like dual spaces and forms), the transpose switches between these perspectives.

  When you think of a column vector like: V = $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ you're thinking of it like a point or a direction in space: "Go 3 units right and 4 units up." — it's an object you can plot or move

  But when you think of a row vector: $V^{\mathrm{T}} = \begin{bmatrix} 3 & 4 \end{bmatrix}$ you're thinking of it like a rule that acts on other vectors: It takes a vector and outputs a number — it evaluates something, like a function does. **Why?** Because when you take a row vector and multiply it by a column vector, you get a scalar (one number). That's exactly like what a function does: it eats an input and spits out an output. **Example** $\begin{bmatrix} 3 & 4 \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix} = 3x + 4y$ *Transpose* flips you between: Thinking of vectors as points (columns), and vectors as functions (rows).

Type	Role	Example
Column Vector	Point/Direction	(3,4) T
Row Vector	Function/evaluate	(3, 4)

Table 1: Vector Table