Canonical Coin Systems for Change-Making Problems

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***Abstract*—The Change-Making Problem is to represent a given value with the fewest coins under a given coin system. As a variation of the knapsack problem, it is known to be NP-hard. Nevertheless, in most real money systems, the greedy algorithm yields optimal solutions. In this paper, we study what type of coin systems that guarantee the optimality of the greedy algorithm. We provide new proofs for a sufficient and necessary condition for the so-called *canonical* coin systems with four or five types of coins, and a sufficient condition for non-canonical coin systems, respectively. Moreover, we present an $O(m^2)$ algorithm that decides whether a tight coin system is canonical.**

I. Introduction

The Change-Making Problem comes from the following scenario: in a shopping mall, the cashier needs to make change for many values of money based on some *coin system*[2] $\$ = \langle c_1, c_2, \cdots, c_m \rangle$ with $1 = c_1 < c_2 < \cdots < c_m$, where c_i denotes the value of the i-th type of coin in $\$$. For example, the cent, nickel, dime and quarter are four types of US coins, and the corresponding coin system is $\$ = \langle 1, 5, 10, 25 \rangle$. Since the approach, can produce the optimal solutions for many practical instances, especially canonical coin systems.

Definition 1: A coin system $\$$ is canonical if $|\mathrm{GRD}_\$(x)| = |\mathrm{OPT}_\$(x)|$ for all x.

For example, the coin system $\$ = \langle 1, 5, 10, 25 \rangle$ is canonical. Accordingly, the cashier can easily create the optimal solution by repeatedly taking the largest coin whose value is no larger than the remaining amount.

Definition 2: A coin system $\$$ is non-canonical if there is an x with $|\mathrm{GRD}_\$(x)| > |\mathrm{OPT}_\$(x)|$, and such x is called a counterexample of $\$$.

Definition 3: A coin system $\$ = \langle 1, c_2, \cdots, c_m \rangle$ is tight if it has no counterexample smaller than c_m.

For example, both $\$_1 = \langle 1, 7, 10, 11 \rangle$ and $\$_2 = \langle 1, 7, 10, 50 \rangle$ are non-canonical, and $\$_1$ is tight but $\$_2$ is not. It is easy to verify that 14 is the counterexample for them, i.e., $\mathrm{GRD}_{\$_1}(14) = \mathrm{GRD}_{\$_2}(14) = (3, 0, 0, 1)$ and $\mathrm{OPT}_{\$_1}(14) = \mathrm{OPT}_{\$_2}(14) = (0, 2, 0, 0)$.