Claudio Canuto • Anita Tabacco

Mathematical Analysis I

Second Edition

Springer

UNITEXT

Released notes: as of 2018, Canuto and Tabacco's two-volume textbook is one of the best calculus/analysis books published in English, combining the rigor of (what in the US is unfortunately taught separately under the title) “analysis” with the material concerning concrete functions associated (again, sadly in the US) with “calculus”.

The English edition, however, suffers from two drawbacks: (a) the ludicrously bad translation, which was obviously not done by a native English speaker. This provides for some passing amusement, so it might not be a drawback after all. (b) the annoying decision to print the proofs in a light blue font, and the horribly ugly boxes around theorems and definitions. Both of these abominations were removed with Infix Pro and Adobe Acrobat for this release.

If you enjoy these PDFs, please support the authors by purchasing a hardcopy or by sending them some cash, or perhaps one of those embarrassing “math clocks”.

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Preface

This textbook is meant to help students acquire the basics of Calculus in curricula where mathematical tools play a crucial part (so Engineering, Physics, Computer Science and the like). The fundamental concepts and methods of Differential and Integral Calculus for functions of one real variable are presented with the primary purpose of letting students assimilate their effective employment, but with critical awareness. The general philosophy inspiring our approach has been to simplify the system of notions available prior to the university reform; at the same time we wished to maintain the rigorous exposition and avoid the trap of compiling a mere formulary of ready-to-use prescriptions.

In view of the current Programme Specifications, the organization of a first course in Mathematics often requires to make appropriate choices about lecture content, the comprehension level required from the recipients, and which kind of language to use. From this point of view, the treatise is 'stratified' in three layers, each corresponding to increasingly deeper engagement by the user. The intermediate level corresponds to the contents of the eleven chapters of the text. Notions are first presented in a naïve manner, and only later defined precisely. Their features are discussed, and computational techniques related to them are exhaustively explained. Besides this, the fundamental theorems and properties are followed by proofs, which are easily recognisable by the font's colour.

At the elementary level the proofs and the various remarks should be skipped. For the reader's sake, essential formulas, and also those judged important, have been highlighted in blue, and gray, respectively. Some tables, placed both throughout and at the end of the book, collect the most useful formulas. It was not our desire to create a hierachy-of-sorts for theorems, instead to leave the instructor free to make up his or her own mind in this respect.

The deepest-reaching level relates to the contents of the five appendices and enables the strongly motivated reader to explore further into the subject. We believe that the general objectives of the Programme Specifications are in line with the fact that willing and able pupils will build a solid knowledge, in the tradition of the best academic education. The eleven chapters contain several links to the different appendices where the reader will find complements to, and insight in various topics. In this fashion every result that is stated possesses a corresponding proof.

To make the approach to the subject less harsh, and all the more gratifying, we have chosen an informal presentation in the first two chapters, where relevant definitions and properties are typically part of the text. From the third chapter onwards they are highlighted by the layout more discernibly. Some definitions and theorems are intentionally not stated in the most general form, so to privilege a brisk understanding. For this reason a wealth of examples are routinely added along the way right after statements, and the same is true for computational techniques. Several remarks enhance the presentation by underlining, in particular, special cases and exceptions. Each chapter ends with a large number of exercises that allow one to test on the spot how solid one's knowledge is. Exercises are grouped according to the chapter's major themes and presented in increasing order of difficulty. All problems are solved, and at least half of them chaperone the reader to the solution.

We have adopted the following graphical conventions for the constituent building blocks: definitions appear on a gray background, theorems' statements on blue, a vertical coloured line marks examples, and boxed exercises, like $\boxed{12.}$, indicate that the complete solution is provided.

We wish to dedicate this volume to Professor Guido Weiss of Washington University in St. Louis, a master in the art of teaching. Generations of students worldwide have benefited from Guido's own work as a mathematician; we hope that his own clarity is at least partly reflected in this textbook.

This second English edition reflects the latest version of the Italian book, that is in use since over a decade, and has been extensively and successfully tested at the Politecnico in Turin and in other Italian Universities. We are grateful to the many colleagues and students whose advice, suggestions and observations have allowed us to reach this result. Special thanks are due to Dr. Simon Chiossi, for the careful and effective work of translation.

Finally, we wish to thank Francesca Bonadei – Executive Editor, Mathematics and Statistics, Springer Italia – for her encouragement and support in the preparation of this textbook.

Torino, August 2014 *Claudio Canuto, Anita Tabacco*

Contents

1

Basic notions

In this introductory chapter some mathematical notions are presented rapidly, which lie at the heart of the study of Mathematical Analysis. Most should already be known to the reader, perhaps in a more thorough form than in the following presentation. Other concepts may be completely new, instead. The treatise aims at fixing much of the notation and mathematical symbols frequently used in the sequel.

1.1 Sets

We shall denote **sets** mainly by upper case letters $X, Y, \ldots$, while for the members or elements of a set lower case letters $x, y, \ldots$ will be used. When an element x is in the set X one writes $x \in X$ ('x is an element of X', or 'the element x belongs to the set X'), otherwise the symbol $x \notin X$ is used.

The majority of sets we shall consider are built starting from sets of numbers. Due to their importance, the main sets of numbers deserve special symbols, namely:

$$\begin{aligned}
\mathbb{N} &= \text{set of natural numbers}\\
\mathbb{Z} &= \text{set of integer numbers}\\
\mathbb{Q} &= \text{set of rational numbers}\\
\mathbb{R} &= \text{set of real numbers}\\
\mathbb{C} &= \text{set of complex numbers.}
\end{aligned}$$

The definition and main properties of these sets, apart from the last one, will be briefly recalled in Sect. 1.3. Complex numbers will be dealt with separately in Sect. 8.3.

Let us fix a non-empty set X, considered as *ambient set*. A **subset** A of X is a set all of whose elements belong to X; one writes $A \subseteq X$ ('A is contained, or included, in X') if the subset A is allowed to possibly coincide with X, and $A \subset X$ ('A is properly contained in X') in case A is a *proper* subset of X, that

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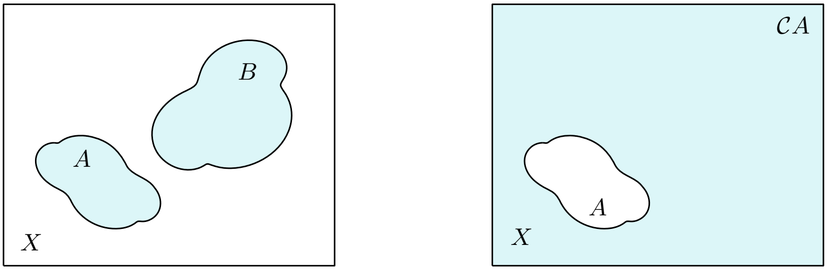


Figure 1.1. Venn diagrams (left) and complement (right)

is, if it does not exhaust the whole X. From the intuitive point of view it may be useful to represent subsets as bounded regions in the plane using the so-called *Venn diagrams* (see Fig. 1.1, left).

A subset can be described by listing the elements of X which belong to it

$$A = \{x, y, \dots, z\};$$

the order in which elements appear is not essential. This clearly restricts the use of such notation to subsets with few elements. More often the notation

$$A = \{x \in X \mid p(x)\} \qquad \text{or} \qquad A = \{x \in X \ : \ p(x)\}$$

will be used (read 'A is the subset of elements x of X such that the condition $p(x)$ holds'); $p(x)$ denotes the *characteristic property* of the elements of the subset, i.e., the condition that is valid for the elements of the subset only, and not for other elements. For example, the subset A of natural numbers smaller or equal than 4 may be denoted

$$A = \{0, 1, 2, 3, 4\} \qquad \text{or} \qquad A = \{x \in \mathbb{N} \mid x \le 4\}.$$

The expression $p(x) =$'$x \le 4$' is an example of *predicate*, which we will return to in the following section.

The collection of all subsets of a given set X forms the **power set** of X, and is denoted by $\mathcal{P}(X)$. Obviously $X \in \mathcal{P}(X)$. Among the subsets of X there is the **empty set**, the set containing no elements. It is usually denoted by the symbol $\emptyset$, so $\emptyset \in \mathcal{P}(X)$. All other subsets of X are proper and non-empty.

Consider for instance $X = \{1, 2, 3\}$ as ambient set. Then

$$\mathcal{P}(X) = \{\,\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, X\}.$$

Note that X contains 3 elements (it has *cardinality* 3), while $\mathcal{P}(X)$ has $8 = 2^3$ elements, hence has cardinality 8. In general if a finite set (a set with a finite number of elements) has cardinality n, the power set of X has cardinality 2^n.

Starting from one or more subsets of X, one can define new subsets by means of set-theoretical operations. The simplest operation consists in taking the complement: if A is a subset of X, one defines the **complement** of A (in X) to be the subset

$$\mathcal{C}A = \{x \in X \mid x \notin A\}$$

made of all elements of X not belonging to A (Fig. 1.1, right).

Sometimes, in order to underline that complements are taken with respect to the ambient space X, one uses the more precise notation $\mathcal{C}_X A$. The following properties are immediate:

$$\mathcal{C}X = \emptyset, \qquad \mathcal{C}\emptyset = X, \qquad \mathcal{C}(\mathcal{C}A) = A.$$

For example, if $X = \mathbb{N}$ and A is the subset of *even* numbers (multiples of 2), then $\mathcal{C}A$ is the subset of *odd* numbers.

Given two subsets A and B of X, one defines **intersection** of A and B the subset

$$A \cap B = \{x \in X \mid x \in A \text{ and } x \in B\}$$

containing the elements of X that belong to both A and B, and **union** of A and B the subset

$$A \cup B = \{x \in X \mid x \in A \text{ or } x \in B\}$$

made of the elements that are either in A or in B (this is meant non-exclusively, so it includes elements of $A \cap B$), see Fig. 1.2.

We recall some properties of these operations.

i) Boolean properties:

$$A \cap \mathcal{C}A = \emptyset, \qquad A \cup \mathcal{C}A = X;$$

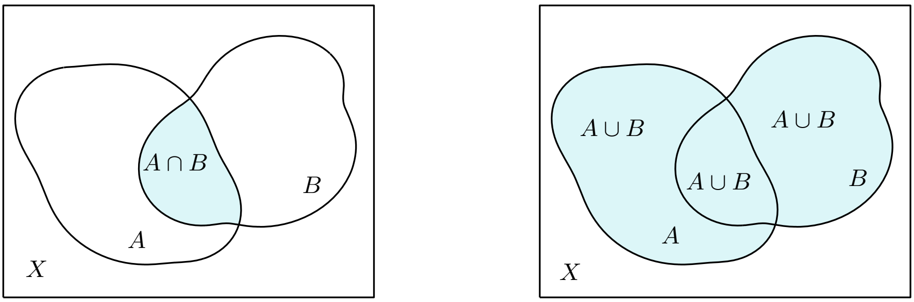


Figure 1.2. Intersection and union of sets

ii) commutative, associative and distributive properties:

$$
\begin{aligned}
A \cap B &= B \cap A, & A \cup B &= B \cup A,\\
(A \cap B) \cap C &= A \cap (B \cap C), & (A \cup B) \cup C &= A \cup (B \cup C),\\
(A \cap B) \cup C &= (A \cup C) \cap (B \cup C), & (A \cup B) \cap C &= (A \cap C) \cup (B \cap C);
\end{aligned}
$$

iii) De Morgan laws:

$$\mathcal{C}(A \cap B) = \mathcal{C}A \cup \mathcal{C}B, \qquad \mathcal{C}(A \cup B) = \mathcal{C}A \cap \mathcal{C}B.$$

Notice that the condition $A \subseteq B$ is equivalent to $A \cap B = A$, or $A \cup B = B$.

There are another couple of useful operations. The first is the **difference** between a subset A and a subset B, sometimes called **relative complement of B in A**

$$A \setminus B = \{x \in A \mid x \notin B\} = A \cap \mathcal{C}B$$

(read 'A minus B'), which selects the elements of A that do not belong to B. The second operation is the **symmetric difference** of the subsets A and B

$$A \,\Delta\, B = (A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B),$$

which picks out the elements belonging either to A or B, but not both (Fig. 1.3).

For example, let $X = \mathbb{N}$, A be the set of even numbers and $B = \{n \in \mathbb{N} \mid n \leq 10\}$ the set of natural numbers smaller or equal than 10. Then $B \setminus A = \{1, 3, 5, 7, 9\}$ is the set of odd numbers smaller than 10, $A \setminus B$ is the set of even numbers larger than 10, and $A \Delta B$ is the union of the latter two.

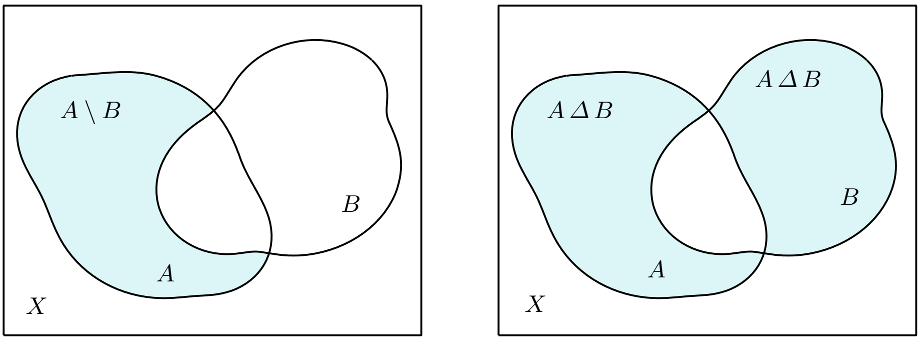


Figure 1.3. The difference $A \setminus B$ (left) and the symmetric difference $A \,\Delta\, B$ (right) of two sets

1.2 Elements of mathematical logic

In Mathematical Logic a **formula** is a declarative sentence, or statement, the truth or falsehood of which can be established. Thus within a certain context a formula carries a *truth value*: True or False. The truth value can be variously represented, for instance using the binary value of a memory bit (1 or 0), or by the state of an electric circuit (open or close). Examples of formulas are: '7 is an odd number' (True), '$3 > \sqrt{12}$' (False), 'Venus is a star' (False), 'This text is written in english' (True), et cetera. The statement 'Milan is far from Rome' is not a formula, at least without further specifications on the notion of distance; in this respect 'Milan is farther from Rome than Turin' is a formula. We shall indicate formulas by lower case letters p,q,r,

1.2.1 Connectives

New formulas can be built from old ones using logic operations expressed by certain formal symbols, called *connectives*.

The simplest operation is called **negation**: by the symbol $\neg p$ (spoken 'not p') one indicates the formula whose truth value is True if p is False, and False if p is True. For example if p='7 is a rational number', then $\neg p$ ='7 is an irrational number'.

The **conjunction** of two formulas p and q is the formula $p \wedge q$ ('p and q'), which is true if both p and q are true, false otherwise. The **disjunction** of p and q is the formula $p \vee q$ ('p or q'); the disjunction is false if p and q are both false, true in all other cases. Let for example p ='7 is a rational number' and q = '7 is an even number'; the formula $p \wedge q$ = '7 is an even rational number' is false since q is false, and $p \vee q$ = '7 is rational or even' is true because p is true.

Many statements in Mathematics are of the kind 'If p is true, then q is true', also read as 'sufficient condition for q to be true is that p be true', or 'necessary condition for p to be true is that q be true'. Such statements are different ways of expressing the same formula $p \Rightarrow q$ ('p implies q', or 'if p, then q'), called **implication**, where p is the 'hypothesis' or 'assumption', q the 'consequence' or 'conclusion'. By definition, the formula $p \Rightarrow q$ is false if p is true and q false, otherwise it is always true. In other words the implication does not allow to deduce a false conclusion from a true assumption, yet does not exclude a true conclusion being implied by a false hypothesis. Thus the statement 'if it rains, I'll take the umbrella' prevents me from going out without umbrella when it rains, but will not interfere with my decision if the sky is clear.

Using p and q it is easy to check that the formula $p \Rightarrow q$ has the same truth value of $\neg p \vee q$. Therefore the connective $\Rightarrow$ can be expressed in terms of the basic connectives $\neg$ and $\vee$.

Other frequent statements are structured as follows: 'the conclusion q is true if and only if the assumption p is true', or 'necessary and sufficient condition for a true q is a true p'. Statements of this kind correspond to the formula $p \Leftrightarrow q$ ('p is (logically) equivalent to q'), called **logic equivalence**. A logic equivalence is true if p and q are simultaneously true or simultaneously false, and false if the truth values of p and q differ. An example is the statement 'a natural number is odd if and only if its square is odd'. The formula $p \Leftrightarrow q$ is the conjuction of $p \Rightarrow q$ and $q \Rightarrow p$, in other words $p \Leftrightarrow q$ and $(p \Rightarrow q) \wedge (q \Rightarrow p)$ have the same truth value. Thus the connective $\Leftrightarrow$ can be expressed by means of the basic connectives $\neg$, $\vee$ and $\wedge$.

The formula $p \Rightarrow q$ (a statement like 'if p, then q') can be expressed in various other forms, all logically equivalent. These represent *rules of inference* to attain the truth of the implication. For example, $p \Rightarrow q$ is logically equivalent to the formula $\neg q \Rightarrow \neg p$, called *contrapositive* formula; symbolically

$$(p \Rightarrow q) \iff (\neg q \Rightarrow \neg p).$$

This is an easy check: $p \Rightarrow q$ is by definition false only when p is true and q false, i.e., when $\neg q$ is true and $\neg p$ false. But this corresponds precisely to the falsehood of $\neg q \Rightarrow \neg p$. Therefore we have established the following inference rule: in order to prove that the truth of p implies the truth of q, one may assume that the conclusion q is false and deduce from it the falsehood of the assumption p. To prove for instance the implication 'if a natural number is odd, then 10 does not divide it', we may suppose that the given number is a multiple of 10 and (easily) deduce that the number must be even.

A second inference rule is the so-called *proof by contradiction,* which we will sometimes use in the textbook. This is expressed by

$$(p \Rightarrow q) \iff (p \wedge \neg q \Rightarrow \neg p).$$

In order to prove the implication $p \Rightarrow q$ one can proceed as follows: suppose p is true and the conclusion q is false, and try to prove the initial hypothesis p false. Since p is also true, we obtain a self-contradictory statement.

A more general form of the proof by contradiction is given by the formula

$$(p \Rightarrow q) \iff (p \wedge \neg q \Rightarrow r \wedge \neg r),$$

where r is an additional formula: the implication $p \Rightarrow q$ is equivalent to assuming p true and q false, then deducing a simultaneously true and false statement r (note that the formula $r \wedge \neg r$ is always false, whichever the truth value of r).

At last, we mention a further rule of inference, called Principle of Mathematical Induction, for which we refer to Appendix A.1, p. 427.

1.2.2 Predicates

Let us now introduce a central concept. A **predicate** is an assertion or property $p(x, \ldots)$ that depends upon one or more variables $x, \ldots$ belonging to suitable sets, and which becomes a formula (hence true or false) whenever the variables are fixed. Let us consider an example. If x is an element of the set of natural numbers, the assertion $p(x)$ = 'x is an odd number' is a predicate: $p(7)$ is true, $p(10)$ false et c. If x and y denote students of the Polytechnic of Turin, the statement $p(x, y)$ = 'x and y follow the same lectures' is a predicate.

Observe that the aforementioned logic operations can be applied to predicates as well, and give rise to new predicates (e.g., $\neg p(x)$, $p(x) \vee q(x)$ and so on). This fact, by the way, establishes a precise relation among the essential connectives $\neg, \wedge, \vee$ and the set-theoretical operations of taking complements, intersection and union. In fact, recalling the definition $A = \{x \in X \mid p(x)\}$ of subset of a given set X, the 'characteristic property' $p(x)$ of the elements of A is nothing else but a predicate, which is true precisely for the elements of A. The complement $\mathcal{C}A$ is thus obtained by negating the characteristic property

$$\mathcal{C}A = \{x \in X \mid \neg p(x)\},$$

while the intersection and union of A with another subset $B = \{x \in X \mid q(x)\}$ are described respectively by the conjuction and the disjunction of the corresponding characteristic properties:

$$A \cap B = \{x \in X \mid p(x) \wedge q(x)\}, \qquad A \cup B = \{x \in X \mid p(x) \vee q(x)\}.$$

The properties of the set-theoretical operations recalled in the previous section translate into similar properties enjoyed by the logic operations, which the reader can easily write down.

1.2.3 Quantifiers

Given a predicate $p(x)$, with the variable x belonging to a certain set X, one is naturally lead to ask whether $p(x)$ is true *for all* elements x, or if *there exists at least one* element x making $p(x)$ true. When posing such questions we are actually considering the formulas

$$\forall x,\ p(x) \quad \text{(read \textit{`for all x, $p(x)$ holds'})}$$

and

$$\exists x,\ p(x) \quad \text{(read \textit{`there exists at least one x, such that $p(x)$ holds'})}.$$

If indicating the set to which x belongs becomes necessary, one writes '$\forall x \in X,\ p(x)$' and '$\exists x \in X,\ p(x)$'. The symbol $\forall$ ('for all') is called **universal quantifier**, and the symbol $\exists$ ('there exists at least') is said **existential quantifier**. (Sometimes a third quantifier is used, $\exists!$, which means 'there exists one and only one element' or 'there exists a unique'.)

We wish to stress that putting a quantifier in front of a predicate transforms the latter in a formula, whose truth value may be then determined. The predicate

$p(x) =$ 'x is strictly less than 7' for example, yields the false formula '$\forall x \in \mathbb{N},\ p(x)$' (since $p(8)$ is false, for example), while '$\exists x \in \mathbb{N},\ p(x)$' is true (e.g., $x = 6$ satisfies the assertion).

The effect of negation on a quantified predicate must be handled with attention. Suppose for instance x indicates the generic student of the Polytechnic, and let $p(x)$ = 'x is an Italian citizen'. The formula '$\forall x,\ p(x)$' ('every student of the Polytechnic has Italian citizenship') is false. Therefore its negation '$\neg(\forall x,\ p(x))$' is true, but beware: the latter does not state that all students are foreign, rather that 'there is at least one student who is not Italian'. Thus the negation of '$\forall x,\ p(x)$' is '$\exists x,\ \neg p(x)$'. We can symbolically write

$$\neg(\forall x,\ p(x)) \quad \Longleftrightarrow \quad \exists x,\ \neg p(x).$$

Similarly, it is not hard to convince oneself of the logic equivalence

$$\neg(\exists x,\ p(x)) \quad \Longleftrightarrow \quad \forall x,\ \neg p(x).$$

If a predicate depends upon two or more arguments, each of them may be quantified. Yet the *order* in which the quantifiers are written can be essential. Namely, two quantifiers of the same type (either universal or existential) can be swapped without modifying the truth value of the formula; in other terms

$$\begin{aligned} \forall x\, \forall y,\ p(x,y) \quad &\Longleftrightarrow \quad \forall y\, \forall x,\ p(x,y), \\ \exists x\, \exists y,\ p(x,y) \quad &\Longleftrightarrow \quad \exists y\, \exists x,\ p(x,y). \end{aligned}$$

On the contrary, exchanging the places of different quantifiers usually leads to different formulas, so one should be very careful when ordering quantifiers.

As an example, consider the predicate $p(x,y) =$ '$x \geq y$', with x, y varying in the set of natural numbers. The formula '$\forall x\, \forall y,\ p(x,y)$' means 'given any two natural numbers, each one is greater or equal than the other', clearly a false statement. The formula '$\forall x\, \exists y,\ p(x,y)$', meaning 'given any natural number x, there is a natural number y smaller or equal than x', is true, just take $y = x$ for instance. The formula '$\exists x\, \forall y,\ p(x,y)$' means 'there is a natural number x greater or equal than each natural number', and is false: each natural number x admits a successor $x + 1$ which is strictly bigger than x. Eventually, '$\exists x\, \exists y,\ p(x,y)$' ('there are at least two natural numbers such that one is bigger or equal than the other') holds trivially.

1.3 Sets of numbers

Let us briefly examine the main sets of numbers used in the book. The discussion is on purpose not exhaustive, since the main properties of these sets should already be known to the reader.

The set $\mathbb{N}$ of natural numbers. This set has the numbers $0, 1, 2, \ldots$ as elements. The operations of sum and product are defined on $\mathbb{N}$ and enjoy the well-known commutative, associative and distributive properties. We shall indicate by $\mathbb{N}_+$ the set of natural numbers different from 0

$$\mathbb{N}_+ = \mathbb{N} \setminus \{0\}.$$

A natural number n is usually represented in *base 10* by the expansion $n = c_k 10^k + c_{k-1}10^{k-1} + \cdots + c_1 10 + c_0$, where the c_i's are natural numbers from 0 to 9 called *decimal digits*; the expression is unique if one assumes $c_k \neq 0$ when $n \neq 0$. We shall write $n = (c_k c_{k-1} \ldots c_1 c_0)_{10}$, or more easily $n = c_k c_{k-1} \ldots c_1 c_0$. Any natural number ≥ 2 may be taken as base, instead of 10; a rather common alternative is 2, known as *binary base.*

Natural numbers can also be represented geometrically as points on a straight line. For this it is sufficient to fix a first point O on the line, called *origin*, and associate it to the number 0, and then choose another point P different from O, associated to the number 1. The direction of the line going from O to P is called *positive direction*, while the length of the segment OP is taken as *unit* for measurements. By marking multiples of OP on the line in the positive direction we obtain the points associated to the natural numbers (see Fig. 1.4).

The set $\mathbb{Z}$ of integer numbers. This set contains the numbers $0, +1, -1, +2, -2, \ldots$ (called integers). The set $\mathbb{N}$ can be identified with the subset of $\mathbb{Z}$ consisting of $0, +1, +2, \ldots$ The numbers $+1, +2, \ldots$ $(-1, -2, \ldots)$ are said *positive integers* (resp. *negative integers*). Sum and product are defined in $\mathbb{Z}$, together with the difference, which is the inverse operation to the sum.

An integer can be represented in decimal base $z = \pm c_k c_{k-1} \ldots c_1 c_0$. The geometric picture of negative integers extends that of the natural numbers to the left of the origin (Fig. 1.4).

The set $\mathbb{Q}$ of rational numbers. A rational number is the quotient, or ratio, of two integers, the second of which (denominator) is non-zero. Without loss of generality one can assume that the denominator is positive, whence each rational number, or rational for simplicity, is given by

$$r = \frac{z}{n}, \qquad \text{with } z \in \mathbb{Z} \text{ and } n \in \mathbb{N}_+.$$

Moreover, one may also suppose the fraction is reduced, that is, z and n have no common divisors. In this way the set $\mathbb{Z}$ is identified with the subset of rationals

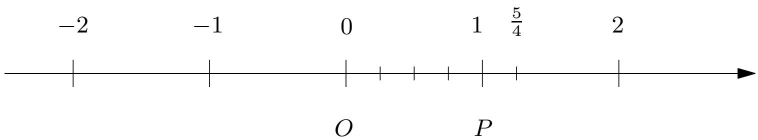


Figure 1.4. Geometric representation of numbers

whose denominator is 1. Besides sum, product and difference, the operation of division between two rationals is defined on $\mathbb{Q}$, so long as the second rational is other than 0. This is the inverse to the product.

A rational number admits a representation in base 10 of the kind $r = \pm c_k c_{k-1} \cdots c_1 c_0 . d_1 d_2 \cdots$, corresponding to

$$r = \pm(c_k 10^k + c_{k-1} 10^{k-1} + \cdots + c_1 10 + c_0 + d_1 10^{-1} + d_2 10^{-2} + \cdots).$$

The sequence of digits $d_1, d_2, \ldots$ written after the dot satisfies one and only one of the following properties: i) all digits are 0 from a certain subscript $i \geq 1$ onwards (in which case one has a *finite decimal expansion*; usually the zeroes are not written), or ii) starting from a certain point, a finite sequence of numbers not all zero – called *period* – repeats itself over and over (*infinite periodic decimal expansion*; the period is written once with a line drawn on top). For example the following expressions are decimal expansions of rational numbers

$$-\frac{35163}{100} = -351.6300 \cdots = -371.63 \quad \text{and} \quad \frac{11579}{925} = 12.51783783 \cdots = 12.51\overline{783}.$$

The expansion of certain rationals is not unique. If a rational number has a finite expansion in fact, then it also has a never-ending periodic one obtained from the former by reducing the right-most non-zero decimal digit by one unit, and adding the period 9. The expansions 1.0 and $0.\overline{9}$ define the same rational number 1; similarly, 8.357 and $8.356\overline{9}$ are equivalent representations of $\frac{4120}{493}$.

The geometric representation of a rational $r = \pm\frac{m}{n}$ is obtained by subdividing the segment OP in n equal parts and copying the subsegment m times in the positive or negative direction, according to the sign of r (see again Fig. 1.4).

The set $\mathbb{R}$ of real numbers. Not every point on the line corresponds to a rational number in the above picture. This means that not all segments can be measured by multiples and sub-multiples of the unit of length, irrespective of the choice of this unit.

It has been known since the ancient times that the diagonal of a square is not *commensurable* with the side, meaning that the length d of the diagonal is not a rational multiple of the side's length ℓ. To convince ourselves about this fact recall Pythagoras's Theorem. It considers any of the two triangles in which the diagonal splits the square (Fig. 1.5), and states that

$$d^2 = \ell^2 + \ell^2, \qquad \text{i.e.,} \qquad d^2 = 2\ell^2.$$

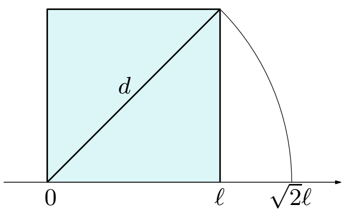


Figure 1.5. Square with side ℓ and its diagonal

Calling p the ratio between the lengths of diagonal and side, we square $d = p\ell$ and substitute in the last relation to obtain $p^2 = 2$. The number p is called the *square root* of 2 and it is indicated by the symbol $\sqrt{2}$.

Property 1.1 *If the number p satisfies $p^2 = 2$, it must be non-rational.*

Proof. By contradiction: suppose there exist two integers m and n, necessarily non-zero, such that $p = \frac{m}{n}$. Assume m, n are relatively prime. Taking squares we obtain $\frac{m^2}{n^2} = 2$, hence $m^2 = 2n^2$. Thus m^2 is even, which is to say that m is even. For a suitable natural number k then, $m = 2k$. Using this in the previous relation yields $4k^2 = 2n^2$, i.e., $n^2 = 2k^2$. Then n^2, whence also n, is even. But this contradicts the fact that m and n have no common factor, which comes from the assumption that p is rational. □

Another relevant example of incommensurable lengths, known for centuries, pertains to the length of a circle measured with respect to the diameter. In this case as well, one can prove that the lengths of circumference and diameter are not commensurable because the proportionality factor, known by the symbol π, cannot be a rational number.

The set of real numbers is an extension of the rationals and provides a *mathematical model of the straight line*, in the sense that each real number x can be associated to a point P on the line uniquely, and vice versa. The former is called the *coordinate* of P. There are several equivalent ways of constructing such extension. Without going into details, we merely recall that real numbers give rise to any possible decimal expansion. Real numbers that are not rational, called *irrational*, are characterised by having a *non-periodic infinite* decimal expansion, like

$$\sqrt{2} = 1.4142135623731\cdots \qquad \text{and} \qquad \pi = 3.1415926535897\cdots$$

Rather than the actual construction of the set $\mathbb{R}$, what is more interesting to us are the properties of real numbers, which allow one to work with the reals. Among these properties, we recall some of the most important ones.

i) *The arithmetic operations defined on the rationals extend to the reals* with similar properties.
ii) *The order relation $x < y$ of the rationals extends to the reals*, again with similar features. We shall discuss this matter more deeply in the following Sect. 1.3.1.
iii) *Rational numbers are dense in the set of real numbers.* This means there are infinitely many rationals sitting between any two real numbers. It also implies that each real number can be approximated by a rational number as well as we please. If for example $r = c_k c_{k-1} \cdots c_1 c_0.d_1 d_2 \cdots d_i d_{i+1} \cdots$ has a non-periodic infinite decimal expansion, we can approximate it by the rational $q_i = c_k c_{k-1} \cdots c_1 c_0.d_1 d_2 \cdots d_i$ obtained by ignoring all decimal digits past the ith one; as i increases, the approximation of r will get better and better.

iv) *The set of real numbers is complete.* Geometrically speaking, this is equivalent to asking that each point on the line is associated to a unique real number, as already mentioned. Completeness guarantees for instance the existence of the square root of 2, i.e., the solvability in $\mathbb{R}$ of the equation $x^2 = 2$, as well as of infinitely many other equations, algebraic or not. We shall return to this point in Sect. 1.3.2.

1.3.1 The ordering of real numbers

Non-zero real numbers are either positive or negative. Positive reals form the subset $\mathbb{R}_+$, negative reals the subset $\mathbb{R}_-$. We are thus in presence of a partition $\mathbb{R} = \mathbb{R}_- \cup \{0\} \cup \mathbb{R}_+$. The set

$$\mathbb{R}_* = \{0\} \cup \mathbb{R}_+$$

of non-negative reals will also be needed. Positive numbers correspond to points on the line lying at the right – with respect to the positive direction – of the origin.

Instead of $x \in \mathbb{R}_+$, one simply writes $x > 0$ ('x is bigger, or larger, than 0'); similarly, $x \in \mathbb{R}_*$ will be expressed by $x \geq 0$ ('x is bigger or equal than 0'). Therefore an order relation is defined by

$$x < y \qquad \Longleftrightarrow \qquad y - x > 0.$$

This is a *total* ordering, i.e., given any two distinct reals x and y, one (and only one) of the following holds: either $x < y$ or $y < x$. From the geometrical point of view the relation $x < y$ tells that the point with coordinate x is placed at the left of the point with coordinate y. Let us also define

$$x \leq y \qquad \Longleftrightarrow \qquad x < y \quad \text{or} \quad x = y.$$

Clearly, $x < y$ implies $x \leq y$. For example the relations $3 \leq 7$ and $7 \leq 7$ are true, whereas $3 \leq 2$ is not.

The order relation $\leq$ (or $<$) interacts with the algebraic operations of sum and product as follows:

$$\text{if } x \leq y \text{ and } z \text{ is any real number, then } x + z \leq y + z$$

(adding the same real number to both sides of an inequality leaves the latter unchanged);

$$\text{if } x \leq y \;\text{ and }\; \text{if} \quad \begin{array}{ll} z \geq 0, & \text{then} \;\; xz \leq yz, \\ z < 0, & \text{then} \;\; xz \geq yz \end{array}$$

(multiplying by a non-negative number both sides of an inequality does not alter it, while if the number is negative it inverts the inequality). Example: multiplying by -1 the inequality $-3 \leq 2$ gives $-2 \leq 3$. The latter property implies the well-known *sign rule*: the product of two numbers with alike signs is positive, the product of two numbers of different sign is negative.

Absolute value. Let us introduce now a simple yet important notion. Given a real number x, one calls *absolute value* of x the real number

$$|x| = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$$

Thus $|x| \geq 0$ for any x in $\mathbb{R}$. For instance $|5| = 5$, $|0| = 0$, $|-5| = 5$. Geometrically, $|x|$ represents the distance from the origin of the point with coordinate x; thus, $|x-y| = |y-x|$ is the distance between the two points of coordinates x and y.

The following relations, easy to prove, will be useful

$$|x+y| \leq |x| + |y|, \qquad \text{for all } x, y \in \mathbb{R} \tag{1.1}$$

(called *triangle inequality*) and

$$|xy| = |x||y|, \qquad \text{for all } x, y \in \mathbb{R}.$$

Throughout the text we shall solve equations and inequalities involving absolute values. Let us see the simplest ones. According to the definition,

$$|x| = 0$$

has the unique solution $x = 0$. If a is any number > 0, the equation

$$|x| = a$$

has two solutions $x = a$ and $x = -a$, so

$$|x| = a \qquad \iff \qquad x = \pm a, \qquad \forall a \geq 0.$$

In order to solve

$$|x| \leq a, \qquad \text{where } a \geq 0,$$

consider first the solutions $x \geq 0$, for which $|x| = x$, so that now the inequality reads $x \leq a$; then consider $x < 0$, in which case $|x| = -x$, and solve $-x \leq a$, or $-a \leq x$. To summarise, the solutions are real numbers x satisfying $0 \leq x \leq a$ or $-a \leq x < 0$, which may be written in a shorter way as

$$|x| \leq a \qquad \iff \qquad -a \leq x \leq a. \tag{1.2}$$

Similarly, it is easy to see that if $b \geq 0$,

$$|x| \geq b \qquad \Longleftrightarrow \qquad x \leq -b \;\text{ or }\; x \geq b. \tag{1.3}$$

The slightly more general inequality

$$|x - x_0| \leq a,$$

where $x_0 \in \mathbb{R}$ is fixed and $a \geq 0$, is equivalent to $-a \leq x - x_0 \leq a$; adding x_0 gives

$$|x - x_0| \leq a \qquad \Longleftrightarrow \qquad x_0 - a \leq x \leq x_0 + a. \tag{1.4}$$

In all examples we can replace the symbol $\leq$ by $<$ and the conclusions hold.

Intervals. The previous discussion shows that Mathematical Analysis often deals with subsets of $\mathbb{R}$ whose elements lie between two fixed numbers. They are called intervals.

Definition 1.2 *Let a and b be real numbers such that $a \leq b$. The* **closed interval** *with end-points a, b is the set*

$$[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}.$$

If $a < b$, one defines **open interval** *with end-points a, b the set*

$$(a, b) = \{x \in \mathbb{R} \mid a < x < b\}.$$

An equivalent notation is $]a, b[$.

If one includes only one end-point, then the interval with end-points a, b

$$[a, b) = \{x \in \mathbb{R} \mid a \leq x < b\}$$

is called *half-open on the right*, while

$$(a, b] = \{x \in \mathbb{R} \mid a < x \leq b\}.$$

is *half-open on the left*.

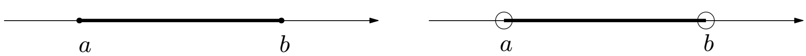


Figure 1.6. Geometric representation of the closed interval $[a, b]$ (left) and of the open interval (a, b) (right)

Example 1.3

Describe the set A of elements $x \in \mathbb{R}$ such that

$$2 \leq |x| < 5.$$

Because of (1.2) and (1.3), we easily have

$$A = (-5, -2] \cup [2, 5).$$

□

Intervals defined by a single inequality are useful, too. Define

$$[a, +\infty) = \{x \in \mathbb{R} \mid a \leq x\}, \qquad (a, +\infty) = \{x \in \mathbb{R} \mid a < x\},$$

and

$$(-\infty, b] = \{x \in \mathbb{R} \mid x \leq b\}, \qquad (-\infty, b) = \{x \in \mathbb{R} \mid x < b\}.$$

The symbols $-\infty$ and $+\infty$ do not indicate real numbers; they allow to extend the ordering of the reals with the convention that $-\infty < x$ and $x < +\infty$ for all $x \in \mathbb{R}$. Otherwise said, the condition $a \leq x$ is the same as $a \leq x < +\infty$, so the notation $[a, +\infty)$ is consistent with the one used for real end-points. Sometimes it is convenient to set

$$(-\infty, +\infty) = \mathbb{R}.$$

In general one says that an interval I is **closed** if it contains its end-points, **open** if the end-points are not included. All points of an interval, apart from the end-points, are called **interior points**.

Bounded sets. Let us now discuss the notion of boundedness of a set.

Definition 1.4 *A subset A of $\mathbb{R}$ is called* **bounded from above** *if there exists a real number b such that*

$$x \leq b, \qquad \textit{for all } x \in A.$$

Any b with this property is called **an upper bound** *of A.*
The set A is **bounded from below** *if there is a real number a with*

$$a \leq x, \qquad \textit{for all } x \in A.$$

Every a satisfying this relation is said **a lower bound** *of A.*
At last, one calls A **bounded** *if it is bounded from above and below.*

In terms of intervals, a set is bounded from above if it is contained in an interval of the sort $(-\infty, b]$ with $b \in \mathbb{R}$, and bounded if it is contained in an interval $[a, b]$ for some $a, b \in \mathbb{R}$. It is not difficult to show that A is bounded if and only if there exists a real $c > 0$ such that

$$|x| \leq c, \qquad \text{for all } x \in A.$$

Examples 1.5

i) The set $\mathbb{N}$ is bounded from below (each number $a \le 0$ is a lower bound), but not from above: in fact, the so-called **Archimedean property** holds: *for any real $b > 0$, there exists a natural number n with*

$$n > b. \tag{1.5}$$

ii) The interval $(-\infty, 1]$ is bounded from above, not from below. The interval $(-5, 12)$ is bounded.

iii) The set

$$A = \left\{ \frac{n}{n+1} \mid n \in \mathbb{N} \right\} = \left\{ 0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots \right\} \tag{1.6}$$

is bounded, in fact $0 \le \dfrac{n}{n+1} < 1$ for any $n \in \mathbb{N}$.

iv) The set $B = \{x \in \mathbb{Q} \mid x^2 < 2\}$ is bounded. Taking x such that $|x| > \frac{3}{2}$ for example, then $x^2 > \frac{9}{4} > 2$, so $x \notin B$. Thus $B \subset [-\frac{3}{2}, \frac{3}{2}]$. □

Definition 1.6 *A set $A \subset \mathbb{R}$* **admits a maximum** *if an element $x_M \in A$ exists such that*

$$x \le x_M, \qquad \text{for any } x \in A.$$

The element x_M (necessarily unique) is the **maximum of the set** *A and one denotes it by $x_M = \max A$.*
The **minimum** *of a set A, denoted by $x_m = \min A$, is defined in a similar way.*

A set admitting a maximum must be bounded from above: the maximum is an upper bound for the set, actually *the smallest of all possible upper bounds*, as we shall prove. The opposite is not true: a set can be bounded from above but not admit a maximum, like the set A of (1.6). We know already that 1 is an upper bound for A. Among all upper bounds, 1 is privileged, being *the smallest upper bound.* To convince ourselves of this fact, let us show that each real number $r < 1$ is not an upper bound, i.e., there is a natural number n such that

$$\frac{n}{n+1} > r.$$

The inequality is equivalent to $\dfrac{n+1}{n} < \dfrac{1}{r}$, hence $1 + \dfrac{1}{n} < \dfrac{1}{r}$, or $\dfrac{1}{n} < \dfrac{1-r}{r}$. This is to say $n > \dfrac{r}{1-r}$, and the existence of such n follows from property (1.5). So, 1 is the smallest upper bound of A, yet not the maximum, for $1 \notin A$: there is no natural number n such that $\dfrac{n}{n+1} = 1$. One calls 1 the *supremum*, or *least upper bound*, of A and writes $1 = \sup A$.

Analogously, 2 is the smallest of upper bounds of the interval $I = (0, 2)$, but it does not belong to I. Thus 2 is the supremum, or least upper bound, of I, $2 = \sup I$.

Definition 1.7 *Let $A \subset \mathbb{R}$ be bounded from above. The* **supremum** *or* **least upper bound of** A *is the smallest of all upper bounds of A, denoted by* $\sup A$. *If $A \subset \mathbb{R}$ is bounded from below, one calls* **infimum** *or* **greatest lower bound of** A *the largest of all lower bounds of A. This is denoted by* $\inf A$.

The number $s = \sup A$ is characterised by two conditions:

$$\begin{array}{ll} i) & x \le s \text{ for all } x \in A; \\ ii) & \text{for any real } r < s, \text{ there is an } x \in A \text{ with } x > r. \end{array} \tag{1.7}$$

While *i)* tells that s is an upper bound for A, according to *ii)* each number smaller than s *is not* an upper bound for A, rendering s the smallest among all upper bounds.

The two conditions (1.7) must be fulfilled in order to show that a given number is the supremum of a set. That is precisely what we did to claim that 1 was the supremum of (1.6).

The notion of supremum generalises that of maximum of a set. It is immediate to see that if a set admits a maximum, this maximum must be the supremum as well.

If a set A is not bounded from above, one says that its supremum is $+\infty$, i.e., one defines

$$\sup A = +\infty.$$

Similarly, $\inf A = -\infty$ for a set A not bounded from below.

1.3.2 Completeness of $\mathbb{R}$

The property of completeness of $\mathbb{R}$ may be formalised in several equivalent ways. The reader should have already come across (*Dedekind's*) *separability axiom*: decomposing $\mathbb{R}$ into the union of two disjoint subsets C_1 and C_2 (the pair (C_1, C_2) is called a *cut*) so that each element of C_1 is smaller or equal than every element in C_2, there exists a (unique) separating element $s \in \mathbb{R}$:

$$x_1 \le s \le x_2, \qquad \forall x_1 \in C_1,\ \forall x_2 \in C_2.$$

An alternative formulation of completeness involves the notion of supremum of a set: *every bounded set from above admits a supremum in* $\mathbb{R}$, i.e., there is a real number smaller or equal than all upper bounds of the set.

With the help of this property one can prove, for example, the existence in $\mathbb{R}$ of the square root of 2, hence of a number p (> 0) such that $p^2 = 2$. Going back to Example 1.5 iv), the completeness of the reals ensures that the bounded set $B = \{x \in \mathbb{Q} \mid x^2 < 2\}$ has a supremum, say p. Using the properties of $\mathbb{R}$ it is possible to show that $p^2 < 2$ cannot occur, otherwise p would not be an upper bound for B, and neither $p^2 > 2$ holds, for p would not be the least of all upper bounds. Thus necessarily $p^2 = 2$. Note that B, albeit contained in $\mathbb{Q}$, is not allowed to have a rational upper bound, because $p^2 = 2$ prevents p from being rational (Property 1.1).

This example explains why the completeness of $\mathbb{R}$ lies at the core of the possibility to solve in $\mathbb{R}$ many remarkable equations. We are thinking in particular about the family of algebraic equations

$$x^n = a, \tag{1.8}$$

where $n \in \mathbb{N}_+$ and $a \in \mathbb{R}$, for which it is worth recalling the following known fact.

Property 1.8 *i) Let $n \in \mathbb{N}_+$ be odd. Then for any $a \in \mathbb{R}$ equation* (1.8) *has exactly one solution in $\mathbb{R}$, denoted by $x = \sqrt[n]{a}$ or $x = a^{1/n}$ and called the* n**th root of** a.

ii) Let $n \in \mathbb{N}_+$ be even. For any $a > 0$ equation (1.8) *has two real solutions with the same absolute value but opposite signs; when $a = 0$ there is one solution $x = 0$ only; for $a < 0$ there are no solutions in $\mathbb{R}$. The non-negative solution is indicated by $x = \sqrt[n]{a}$ or $x = a^{1/n}$, and called the* n**th (arithmetic) root of** a.

1.4 Factorials and binomial coefficients

We introduce now some noteworthy integers that play a role in many areas of Mathematics.

Given a natural number $n \geq 1$, the product of all natural numbers between 1 and n goes under the name of **factorial of** n and is indicated by $n!$ (read 'n factorial'). Out of conveniency one sets $0! = 1$. Thus

$$0! = 1, \qquad 1! = 1, \qquad n! = 1 \cdot 2 \cdot \ldots \cdot n = (n-1)!\,n \quad \text{for} \quad n \geq 2. \tag{1.9}$$

Factorials grow extremely rapidly as n increases; for instance $5! = 120$, $10! = 3628800$ and $100! > 10^{157}$.

Example 1.9

Suppose we have $n \geq 2$ balls of different colours in a box. In how many ways can we extract the balls from the box?

When taking the first ball we are making a choice among the n balls in the box; the second ball will be chosen among the $n-1$ balls left, the third one among $n-2$ and so on. Altogether we have $n(n-1)\cdot\ldots\cdot 2\cdot 1 = n!$ different ways to extract the balls: $n!$ represents the number of arrangements of n distinct objects in a sequence, called **permutations** of n ordered objects.
If we stop after k extractions, $0 < k < n$, we end up with $n(n-1)\ldots(n-k+1)$ possible outcomes. The latter expression, also written as $\dfrac{n!}{(n-k)!}$, is the number of possible **permutations of n distinct objects in sequences of k objects**.
If we allow repeated colours, for instance by reintroducing in the box a ball of the same colour as the one just extracted, each time we choose among n. After $k > 0$ choices there are then n^k possible sequences of colours: n^k is the number of **permutations of n objects in sequences of k, with repetitions** (i.e., allowing an object to be chosen more than once). □

Given two natural numbers n and k such that $0 \le k \le n$, one calls **binomial coefficient** the number

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \tag{1.10}$$

(the symbol $\binom{n}{k}$ is usually read 'n choose k'). Notice that if $0 < k < n$

$$n! = 1\cdot\ldots\cdot n = 1\cdot\ldots\cdot(n-k)(n-k+1)\cdot\ldots\cdot(n-1)n = (n-k)!(n-k+1)\cdot\ldots\cdot(n-1)n,$$

so simplifying and rearranging the order of factors at the numerator, (1.10) becomes

$$\binom{n}{k} = \frac{n(n-1)\cdot\ldots\cdot(n-k+1)}{k!}, \tag{1.11}$$

another common expression for the binomial coefficient. From definition (1.10) it follows directly that

$$\binom{n}{k} = \binom{n}{n-k}$$

and

$$\binom{n}{0} = \binom{n}{n} = 1, \qquad \binom{n}{1} = \binom{n}{n-1} = n.$$

Moreover, it is easy to prove that for any $n \ge 1$ and any k with $0 < k < n$

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}, \tag{1.12}$$

which provides a convenient means for computing binomial coefficients *recursively*; the coefficients relative to n objects are easily determined once those involving $n-1$ objects are computed. The same formula suggests to write down binomial coefficients in a triangular pattern, known as *Pascal's triangle*[1] (Fig. 1.7): each coefficient of a given row, except for the 1's on the boundary, is the sum of the two numbers that lie above it in the preceding row, precisely as (1.12) prescribes. The construction of Pascal's triangle shows that the binomial coefficients are natural numbers.

$$\begin{array}{ccccccccccc} & & & & & 1 & & & & & \\ & & & & 1 & & 1 & & & & \\ & & & 1 & & 2 & & 1 & & & \\ & & 1 & & 3 & & 3 & & 1 & & \\ & 1 & & 4 & & 6 & & 4 & & 1 & \\ 1\ldots & & & & & & & & & & \ldots 1 \end{array}$$

Figure 1.7. Pascal's triangle

The term 'binomial coefficient' originates from the power expansion of the polynomial $a + b$ in terms of powers of a and b. The reader will remember the important identities

$$(a+b)^2 = a^2 + 2ab + b^2 \qquad \text{and} \qquad (a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3.$$

The coefficients showing up are precisely the binomial coefficients for $n = 2$ and $n = 3$. In general, for any $n \geq 0$, the formula

$$\begin{aligned} (a+b)^n &= a^n + na^{n-1}b + \ldots + \binom{n}{k} a^{n-k}b^k + \ldots + nab^{n-1} + b^n \\ &= \sum_{k=0}^{n} \binom{n}{k} a^{n-k}b^k \end{aligned} \tag{1.13}$$

holds, known as **(Newton's) binomial expansion**. This formula is proven with (1.12) using a *proof by induction* (see Appendix A.1, p. 428).

Example 1.9 (continuation)

Given n balls of different colours, let us fix k with $0 \leq k \leq n$. How many different sets of k balls can we form?

Extracting one ball at a time for k times, we already know that there are $n(n-1)\ldots(n-k+1)$ outcomes. On the other hand the same k balls, extracted in a different order, will yield the same set. Since the possible orderings of k elements are $k!$, we see that the number of distinct sets of k balls chosen from n is $\dfrac{n(n-1)\cdot\ldots\cdot(n-k+1)}{k!} = \dbinom{n}{k}$. This coefficient represents the number of **combinations of n objects taken k at a time**. Equivalently, the number of subsets of k elements of a set of cardinality n.

[1] Sometimes the denomination *Tartaglia's triangle* appears.

Formula (1.13) with $a = b = 1$ shows that the sum of all binomial coefficients with n fixed equals 2^n, non-incidentally also the total number of subsets of a set with n elements. □

1.5 Cartesian product

Let X, Y be non-empty sets. Given elements x in X and y in Y, we construct the **ordered pair** of numbers

$$(x, y),$$

whose *first component* is x and *second component* is y. An ordered pair is conceptually other than a set of two elements. As the name says, in an ordered pair the order of the components is paramount. This is not the case for a set. If $x \neq y$ the ordered pairs (x, y) and (y, x) are distinct, while $\{x, y\}$ and $\{y, x\}$ coincide as sets.

The set of all ordered pairs (x, y) when x varies in X and y varies in Y is the **Cartesian product** of X and Y, which is indicated by $X \times Y$. Mathematically,

$$X \times Y = \{(x, y) \mid x \in X,\ y \in Y\}.$$

The Cartesian product is represented using a rectangle, whose basis corresponds to the set X and whose height is Y (as in Fig. 1.8).

If the sets X, Y are different, the product $X \times Y$ will not be equal to $Y \times X$, in other words the Cartesian product is not commutative.

But if $Y = X$, it is customary to put $X \times X = X^2$ for brevity. In this case the subset of X^2

$$\Delta = \{(x, y) \in X^2 \mid x = y\}$$

of pairs with equal components is called the *diagonal* of the Cartesian product.

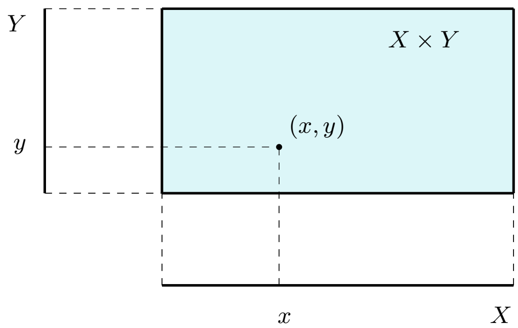


Figure 1.8. Cartesian product of sets

The most significant example of Cartesian product stems from $X = Y = \mathbb{R}$. The set $\mathbb{R}^2$ consists of ordered pairs of real numbers. Just as the set $\mathbb{R}$ mathematically represents a straight line, so $\mathbb{R}^2$ *is a model of the plane* (Fig. 1.9, left). In order to define this correspondence, choose a straight line in the plane and fix on it an origin O, a positive direction and a length unit. This shall be the *x-axis*. Rotating this line counter-clockwise around the origin by 90^o generates the *y-axis*. In this way we have now an orthonormal *frame* (we only mention that it is sometimes useful to consider frames whose axes are not orthogonal, and/or the units on the axes are different).

Given any point P on the plane, let us draw the straight lines parallel to the axes passing through the point. Denote by x the real number corresponding to the intersection of the x-axis with the parallel to the y-axis, and by y the real number corresponding to the intersection of the y-axis with the parallel to the x-axis. An ordered pair $(x, y) \in \mathbb{R}^2$ is thus associated to each point P on the plane, and vice versa. The components of the pair are called (*Cartesian*) *coordinates* of P in the chosen frame.

The notion of Cartesian product can be generalised to the product of more sets. Given n non-empty sets $X_1, X_2, \ldots, X_n$, one considers ordered $n-$tuples

$$(x_1, x_2, \ldots, x_n)$$

where, for every $i = 1, 2, \ldots, n$, each component x_i lives in the set X_i. The Cartesian product $X_1 \times X_2 \times \ldots \times X_n$ is then the set of all such $n-$tuples.

When $X_1 = X_2 = \ldots = X_n = X$ one simply writes $X \times X \times \ldots \times X = X^n$. In particular, $\mathbb{R}^3$ is the set of triples (x, y, z) of real numbers, and *represents a mathematical model of three-dimensional space* (Fig. 1.9, right).

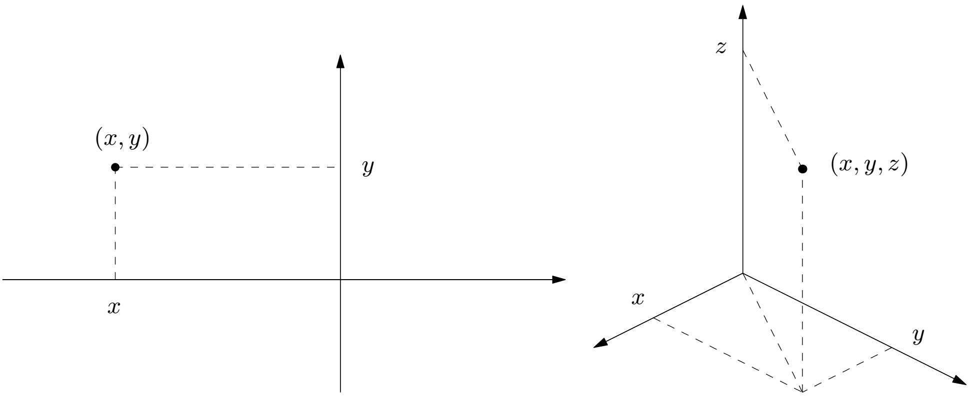


Figure 1.9. Models of the plane (left) and of space (right)

1.6 Relations in the plane

We call *Cartesian plane* a plane equipped with an orthonormal Cartesian frame built as above, which we saw can be identified with the product $\mathbb{R}^2$.

Every non-empty subset R of $\mathbb{R}^2$ defines a **relation** between real numbers; precisely, one says *x is R-related to y*, or *x is related to y by R*, if the ordered pair (x, y) belongs to R. The *graph* of the relation is the set of points in the plane whose coordinates belong to R.

A relation is commonly defined by one or more (in)equalities involving the variables x and y. The subset R is then defined as the set of pairs (x, y) such that x and y satisfy the constraints. Finding R often means determining its graph in the plane. Let us see some examples.

Examples 1.10

i) An equation like

$$ax + by = c,$$

with a, b constant and not both vanishing, defines a straight line. If $b = 0$, the line is parallel to the y-axis, whereas $a = 0$ yields a parallel to the x-axis. Assuming $b \neq 0$ we can write the equation as

$$y = mx + q,$$

where $m = -\frac{a}{b}$ and $q = \frac{c}{b}$. The number m is called *slope* of the line. The line can be plotted by finding the coordinates of two points that belong to it, hence two distinct pairs (x, y) solving the equation. In particular $c = 0$ (or $q = 0$) if and only if the origin belongs to the line. The equation $x - y = 0$ for example defines the bisectrix of the first and third quadrants of the plane.

ii) Replacing the '$=$' sign by '$<$' above, consider the inequality

$$ax + by < c.$$

It defines one of the half-planes in which the straight line of equation $ax+by = c$ divides the plane (Fig. 1.10). If $b > 0$ for instance, the half-plane below the line is obtained. This set is open, i.e., it does not contain the straight line, since the inequality is strict. The inequality $ax + by \leq c$ defines instead a closed set, i.e., including the line.

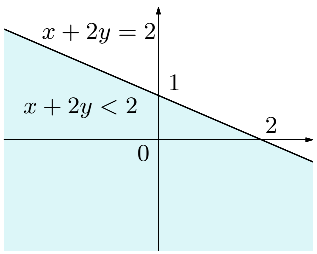


Figure 1.10. Graph of the relation of Example 1.10 ii)

iii) The system

$$\begin{cases} y > 0, \\ x - y \geq 0, \end{cases}$$

defines the intersection between the open half-plane above the x-axis and the closed half-plane lying below the bisectrix of the first and third quadrants. Thus the system describes (Fig. 1.11, left) the wedge between the positive x-axis and the bisectrix (the points on the x-axis are excluded).

iv) The inequality

$$|x - y| < 2$$

is equivalent, recall (1.2), to

$$-2 < x - y < 2.$$

The inequality on the left is in turn equivalent to $y < x+2$, so it defines the open half-plane below the line $y = x + 2$; similarly, the inequality on the right is the same as $y > x-2$ and defines the open half-plane above the line $y = x-2$. What we get is therefore the strip between the two lines, these excluded (Fig. 1.11, right).

v) By Pythagoras's Theorem, the equation

$$x^2 + y^2 = 1$$

defines the set of points P in the plane with distance 1 from the origin of the axes, that is, the circle centred at the origin with radius 1 (in trigonometry it goes under the name of *unit circle*). The inequality

$$x^2 + y^2 \leq 1$$

then defines the disc bounded by the unit circle (Fig. 1.12, left).

vi) The equation

$$y = x^2$$

yields the parabola with vertical axis, vertex at the origin and passing through the point P of coordinates $(1, 1)$.

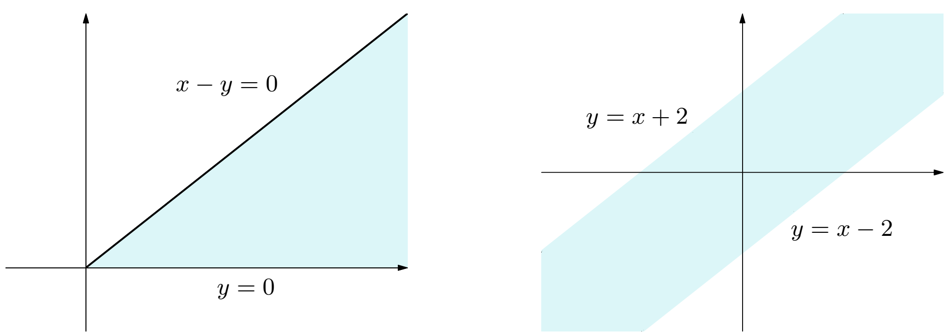


Figure 1.11. Graphs of the relations of Examples 1.10 iii) (left) and 1.10 iv) (right)

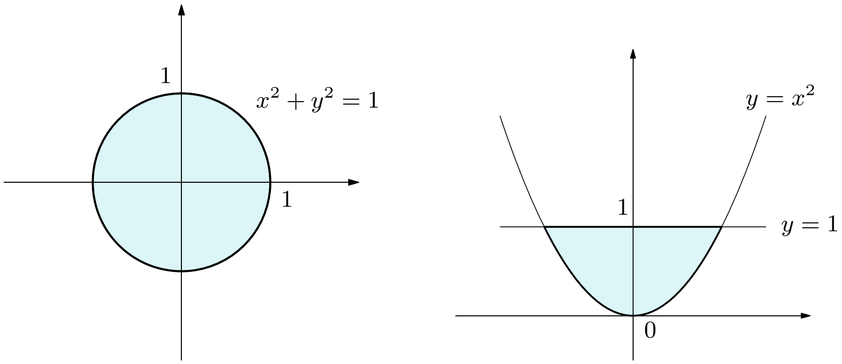


Figure 1.12. Graphs of the relations in Examples 1.10 v) (left) and 1.10 vi) (right)

Thus the inequalities

$$x^2 \le y \le 1$$

define the region enclosed by the parabola and by the straight line given by $y = 1$ (Fig. 1.12, right). □

1.7 Exercises

1. *Solve the following inequalities:*

a) $\dfrac{2x-1}{x-3} > 0$

b) $\dfrac{1-7x}{3x+5} > 0$

c) $\dfrac{x-1}{x-2} > \dfrac{2x-3}{x-3}$

d) $\dfrac{|x|}{x-1} > \dfrac{x+1}{2x-1}$

e) $\dfrac{2x+3}{x+5} \le \dfrac{x+1}{|x-1|}$

f) $\sqrt{x^2-6x} > x+2$

g) $x - 3 \le \sqrt{x^2-2x}$

h) $\dfrac{x+3}{(x+1)^2\sqrt{x^2-3}} \ge 0$

i) $\sqrt{|x^2-4|} - x \ge 0$

ℓ) $\dfrac{x\sqrt{|x^2-4|}}{x^2-4} - 1 > 0$

2. *Describe the following subsets of* $\mathbb{R}$*:*

a) $A = \{x \in \mathbb{R} : x^2 + 4x + 13 < 0\} \cap \{x \in \mathbb{R} : 3x^2 + 5 > 0\}$

b) $B = \{x \in \mathbb{R} : (x+2)(x-1)(x-5) < 0\} \cap \{x \in \mathbb{R} : \dfrac{3x+1}{x-2} \ge 0\}$

c) $C = \{x \in \mathbb{R} : \dfrac{x^2-5x+4}{x^2-9} < 0\} \cup \{x \in \mathbb{R} : \sqrt{7x+1} + x = 17\}$

d) $D = \{x \in \mathbb{R} : x - 4 \ge \sqrt{x^2-6x+5}\} \cup \{x \in \mathbb{R} : x+2 > \sqrt{x-1}\}$

3. *Determine and draw a picture of the following subsets of* $\mathbb{R}^2$*:*

$\boxed{\text{a)}}$ $A = \{(x,y) \in \mathbb{R}^2 : xy \geq 0\}$ b) $B = \{(x,y) \in \mathbb{R}^2 : x^2 - y^2 > 0\}$

$\boxed{\text{c)}}$ $C = \{(x,y) \in \mathbb{R}^2 : |y - x^2| < 1\}$ d) $D = \{(x,y) \in \mathbb{R}^2 : x^2 + \dfrac{y^2}{4} \geq 1\}$

$\boxed{\text{e)}}$ $E = \{(x,y) \in \mathbb{R}^2 : 1 + xy > 0\}$ f) $F = \{(x,y) \in \mathbb{R}^2 : x - y \neq 0\}$

4. *Tell whether the following subsets of* $\mathbb{R}$ *are bounded from above and/or below, specifying upper and lower bounds, plus maximum and minimum (if existent):*

$\boxed{\text{a)}}$ $A = \{x \in \mathbb{R} : x = n \text{ or } x = \dfrac{1}{n^2},\ n \in \mathbb{N} \setminus \{0\}\}$

b) $B = \{x \in \mathbb{R} : -1 < x \leq 1 \text{ or } x = 20\}$

$\boxed{\text{c)}}$ $C = \{x \in \mathbb{R} : 0 \leq x < 1 \text{ or } x = \dfrac{2n-3}{n-1},\ n \in \mathbb{N} \setminus \{0, 1\}\}$

d) $D = \{z \in \mathbb{R} : z = xy \text{ with } x, y \in \mathbb{R},\ -1 \leq x \leq 2,\ -3 \leq y < -1\}$

1.7.1 Solutions

1. *Inequalities:*

a) This is a fractional inequality. A fraction is positive if and only if numerator and denominator have the same sign. As $N(x) = 2x - 1 > 0$ if $x > 1/2$, and $D(x) = x - 3 > 0$ for $x > 3$, the inequality holds when $x < 1/2$ or $x > 3$.

b) $-\frac{5}{3} < x < \frac{1}{7}$.

c) Shift all terms to the left and simplify:

$$\frac{x-1}{x-2} - \frac{2x-3}{x-3} > 0\,, \qquad \text{i.e.,} \qquad \frac{-x^2+3x-3}{(x-2)(x-3)} > 0\,.$$

The roots of the numerator are not real, so $N(x) < 0$ always. The inequality thus holds when $D(x) < 0$, hence $2 < x < 3$.

d) Moving terms to one side and simplifying yields:

$$\frac{|x|}{x-1} - \frac{x+1}{2x-1} > 0\,, \qquad \text{i.e.,} \qquad \frac{|x|(2x-1) - x^2 + 1}{(x-1)(2x-1)} > 0\,.$$

Since $|x| = x$ for $x \geq 0$ and $|x| = -x$ for $x < 0$, we study the two cases separately.
When $x \geq 0$ the inequality reads

$$\frac{2x^2 - x - x^2 + 1}{(x-1)(2x-1)} > 0\,, \qquad \text{or} \qquad \frac{x^2 - x + 1}{(x-1)(2x-1)} > 0\,.$$

The numerator has no real roots, hence $x^2 - x + 1 > 0$ for all x. Therefore the inequality is satisfied if the denominator is positive. Taking the constrain $x \geq 0$ into account, this means $0 \leq x < 1/2$ or $x > 1$.
When $x < 0$ we have

$$\frac{-2x^2 + x - x^2 + 1}{(x-1)(2x-1)} > 0\,, \qquad \text{i.e.,} \qquad \frac{-3x^2 + x + 1}{(x-1)(2x-1)} > 0\,.$$

$N(x)$ is annihilated by $x_1 = \frac{1-\sqrt{13}}{6}$ and $x_2 = \frac{1+\sqrt{13}}{6}$, so $N(x) > 0$ for $x_1 < x < x_2$ (notice that $x_1 < 0$ and $x_2 \in (\frac{1}{2}, 1)$). As above the denominator is positive when $x < 1/2$ and $x > 1$. Keeping $x < 0$ in mind, we have $x_1 < x < 0$.
The initial inequality is therefore satisfied by any $x \in (x_1, \frac{1}{2}) \cup (1, +\infty)$.

e) $-5 < x \leq -2$, $-\frac{1}{3} \leq x < 1$, $1 < x \leq \frac{5+\sqrt{57}}{2}$; f) $x < -\frac{2}{5}$.

g) First of all observe that the right-hand side is always ≥ 0 where defined, hence when $x^2 - 2x \geq 0$, i.e., $x \leq 0$ or $x \geq 2$. The inequality is certainly true if the left-hand side $x - 3$ is ≤ 0, so for $x \leq 3$.
If $x - 3 > 0$, we take squares to obtain

$$x^2 - 6x + 9 \leq x^2 - 2x\,, \qquad \text{i.e.,} \qquad 4x \geq 9\,, \qquad \text{whence} \qquad x \geq \frac{9}{4}\,.$$

Gathering all information we conclude that the starting inequality holds wherever it is defined, that is for $x \leq 0$ and $x \geq 2$.

h) $x \in [-3, -\sqrt{3}) \cup (\sqrt{3}, +\infty)$.

i) As $|x^2 - 4| \geq 0$, $\sqrt{|x^2-4|}$ is well defined. Let us write the inequality in the form

$$\sqrt{|x^2-4|} \geq x\,.$$

If $x \leq 0$ the inequality is always true, for the left-hand side is positive. If $x > 0$ we square:

$$|x^2 - 4| \geq x^2\,.$$

Note that

$$|x^2-4| = \begin{cases} x^2 - 4 & \text{if } x \leq -2 \text{ or } x \geq 2, \\ -x^2 + 4 & \text{if } -2 < x < 2\,. \end{cases}$$

Consider the case $x \geq 2$ first; the inequality becomes $x^2 - 4 \geq x^2$, which is never true.
Let now $0 < x < 2$; then $-x^2 + 4 \geq x^2$, hence $x^2 - 2 \leq 0$. Thus $0 < x \leq \sqrt{2}$ must hold.
In conclusion, the inequality holds for $x \leq \sqrt{2}$.

ℓ) $x \in (-2, -\sqrt{2}) \cup (2, +\infty)$.

2. *Subsets of* $\mathbb{R}$*:*

a) Because $x^2 + 4x + 13 = 0$ cannot be solved over the reals, the condition $x^2 + 4x + 13 < 0$ is never satisfied and the first set is empty. On the other hand, $3x^2 + 5 > 0$ holds for every $x \in \mathbb{R}$, therefore the second set is the whole $\mathbb{R}$. Thus $A = \emptyset \cap \mathbb{R} = \emptyset$.

b) $B = (-\infty, -2) \cup (2, 5)$.

c) We can write
$$\frac{x^2 - 5x + 4}{x^2 - 9} = \frac{(x-4)(x-1)}{(x-3)(x+3)},$$
whence the first set is $(-3, 1) \cup (3, 4)$.
To find the second set, let us solve the irrational equation $\sqrt{7x+1} + x = 17$, which we write as $\sqrt{7x+1} = 17 - x$. The radicand must necessarily be positive, hence $x \geq -\frac{1}{7}$. Moreover, a square root is always ≥ 0, so we must impose $17 - x \geq 0$, i.e., $x \leq 17$. Thus for $-\frac{1}{7} \leq x \leq 17$, squaring yields
$$7x + 1 = (17 - x)^2, \qquad x^2 - 41x + 288 = 0.$$
The latter equation has two solutions $x_1 = 9$, $x_2 = 32$ (which fails the constraint $x \leq 17$, and as such cannot be considered). The second set then contains only $x = 9$.
Therefore $C = (-3, 1) \cup (3, 4) \cup \{9\}$.

d) $D = [1, +\infty)$.

3. *Subsets of* $\mathbb{R}^2$*:*

a) The condition holds if x and y have equal signs, thus in the first and third quadrants including the axes (Fig. 1.13, left).

b) See Fig. 1.13, right.

c) We have
$$|y - x^2| = \begin{cases} y - x^2 & \text{if } y \geq x^2, \\ x^2 - y & \text{if } y \leq x^2. \end{cases}$$
Demanding $y \geq x^2$ means looking at the region in the plane bounded from below by the parabola $y = x^2$. There, we must have
$$y - x^2 < 1, \qquad \text{i.e.,} \qquad y < x^2 + 1,$$

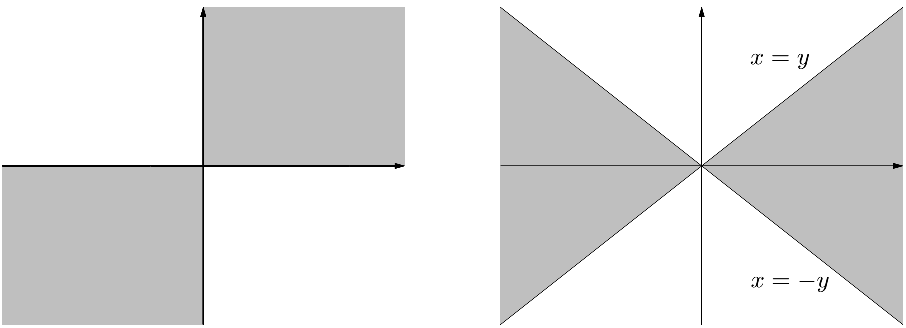
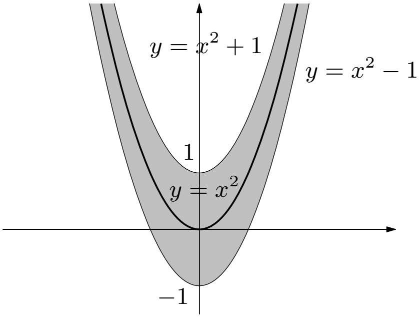


Figure 1.13. The sets A and B of Exercise 3



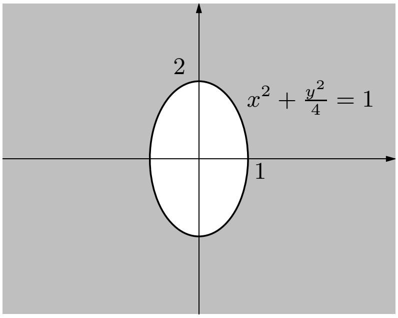


Figure 1.14. The sets C and D of Exercise 3

that is $x^2 \le y < x^2 + 1$.
Vice versa if $y < x^2$,

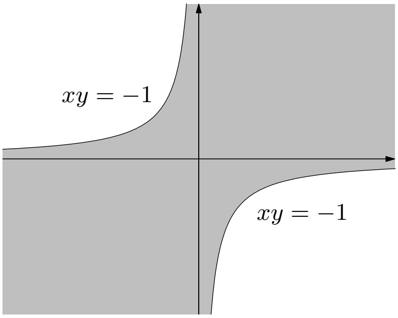
$$x^2 - y < 1\,, \qquad \text{i.e.,} \qquad y > x^2 - 1\,,$$

hence $x^2 - 1 < y \le x^2$.
Eventually, the required region is confined by (though does not include) the parabolas $y = x^2 - 1$ and $y = x^2 + 1$ (Fig. 1.14, left).

d) See Fig. 1.14, right.

e) For $x > 0$ the condition $1 + xy > 0$ is the same as $y > -\frac{1}{x}$. Thus we consider all points of the first and third quadrants above the hyperbola $y = -\frac{1}{x}$.
For $x < 0$, $1 + xy > 0$ means $y < -\frac{1}{x}$, satisfied by the points in the second and fourth quadrants this time, lying below the hyperbola $y = -\frac{1}{x}$.
At last, if $x = 0$, $1 + xy > 0$ holds for any y, implying that the y-axis belongs to the set E.
Therefore: the region lies between the two branches of the hyperbola (these are not part of E) $y = -\frac{1}{x}$, including the y-axis (Fig. 1.15, left).

f) See Fig. 1.15, right.



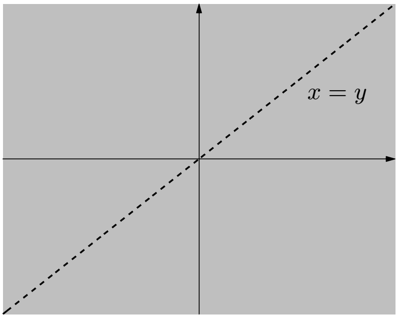


Figure 1.15. The sets E and F of Exercise 3

4. *Bounded and unbounded sets:*

a) We have $A = \{1, 2, 3, \ldots, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \ldots\}$. Since $\mathbb{N} \setminus \{0\} \subset A$, the set A is not bounded from above, hence $\sup A = +\infty$ and there is no maximum.
In addition, the fact that every element of A is positive makes A bounded from below. We claim that 0 is the greatest lower bound of A. In fact, if $r > 0$ were a lower bound of A, then $\frac{1}{n^2} > r$ for any non-zero $n \in \mathbb{N}$. This is the same as $n^2 < \frac{1}{r}$, hence $n < \frac{1}{\sqrt{r}}$. But the last inequality is absurd since natural numbers are not bounded from above. Finally $0 \notin A$, so we conclude $\inf A = 0$ and A has no minimum.

b) $\inf B = -1$, $\sup B = \max B = 20$, and $\min B$ does not exist.

c) $C = [0, 1] \cup \{\frac{3}{2}, \frac{5}{3}, \frac{7}{4}, \frac{9}{5}, \ldots\} \subset [0, 2)$; then C is bounded, and $\inf C = \min C = 0$. Since $\dfrac{2n-3}{n-1} = 2 - \dfrac{1}{n-1}$, it is not hard to show that $\sup C = 2$, although there is no maximum in C.

d) $\inf C = \min C = -6$, $\sup B = \max B = 3$.

2

Functions

Functions crop up regularly in everyday life (for instance: each student of the Polytechnic of Turin has a unique identification number), in physics (to each point of a region in space occupied by a fluid we may associate the velocity of the particle passing through that point at a given moment), in economy (each working day at Milan's stock exchange is tagged with the Mibtel index), and so on.

The mathematical notion of a function subsumes all these situations.

2.1 Definitions and first examples

Let X and Y be two sets. A **function** f **defined on** X **with values in** Y is a correspondence associating to each element $x \in X$ *at most* one element $y \in Y$. This is often shortened to 'a function from X to Y'. A synonym for function is **map**. The set of $x \in X$ to which f associates an element in Y is the **domain** of f; the domain is a subset of X, indicated by $\operatorname{dom} f$. One writes

$$f : \operatorname{dom} f \subseteq X \to Y.$$

If $\operatorname{dom} f = X$, one says that f is defined **on** X and writes simply $f : X \to Y$.

The element $y \in Y$ associated to an element $x \in \operatorname{dom} f$ is called the **image of** x **by** or **under** f and denoted $y = f(x)$. Sometimes one writes

$$f : x \mapsto f(x).$$

The set of images $y = f(x)$ of all points in the domain constitutes the **range of** f, a subset of Y indicated by $\operatorname{im} f$.

The **graph** of f is the subset $\Gamma(f)$ of the Cartesian product $X \times Y$ made of pairs $(x, f(x))$ when x varies in the domain of f, i.e.,

$$\Gamma(f) = \big\{(x, f(x)) \in X \times Y \ : \ x \in \operatorname{dom} f\big\}. \tag{2.1}$$

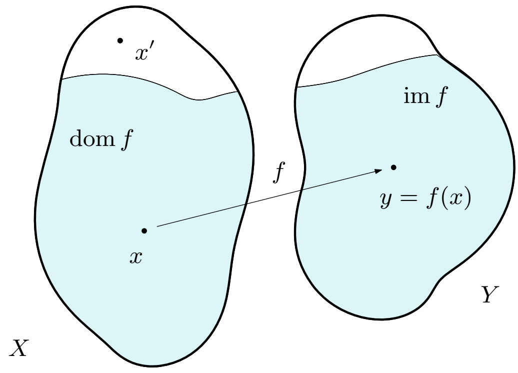


Figure 2.1. Naive representation of a function using Venn diagrams

In the sequel we shall consider maps between sets of numbers most of the time. If $Y = \mathbb{R}$, the function f is said **real** or **real-valued**. If $X = \mathbb{R}$, the function is of **one real variable**. Therefore the graph of a real function is a subset of the Cartesian plane $\mathbb{R}^2$.

A remarkable special case of map arises when $X = \mathbb{N}$ and the domain contains a set of the type $\{n \in \mathbb{N} \ : \ n \geq n_0\}$ for a certain natural number $n_0 \geq 0$. Such a function is called **sequence**. Usually, indicating by a the sequence, it is preferable to denote the image of the natural number n by the symbol a_n rather than $a(n)$; thus we shall write $a : n \mapsto a_n$. A common way to denote sequences is $\{a_n\}_{n \geq n_0}$ (ignoring possible terms with $n < n_0$) or even $\{a_n\}$.

Examples 2.1

Let us consider examples of real functions of real variable.

i) $f : \mathbb{R} \to \mathbb{R},\ f(x) = ax + b$ (a, b real coefficients), whose graph is a straight line (Fig. 2.2, top left).

ii) $f : \mathbb{R} \to \mathbb{R},\ f(x) = x^2$, whose graph is a parabola (Fig. 2.2, top right).

iii) $f : \mathbb{R} \setminus \{0\} \subset \mathbb{R} \to \mathbb{R},\ f(x) = \frac{1}{x}$, has a rectangular hyperbola in the coordinate system of its asymptotes as graph (Fig. 2.2, bottom left).

iv) A real function of a real variable can be defined by multiple expressions on different intervals, in which case is it called a **piecewise function**. An example is given by $f : [0, 3] \to \mathbb{R}$

$$f(x) = \begin{cases} 3x & \text{if } 0 \leq x \leq 1, \\ 4 - x & \text{if } 1 < x \leq 2, \\ x - 1 & \text{if } 2 < x \leq 3, \end{cases} \tag{2.2}$$

drawn in Fig. 2.2, bottom right.

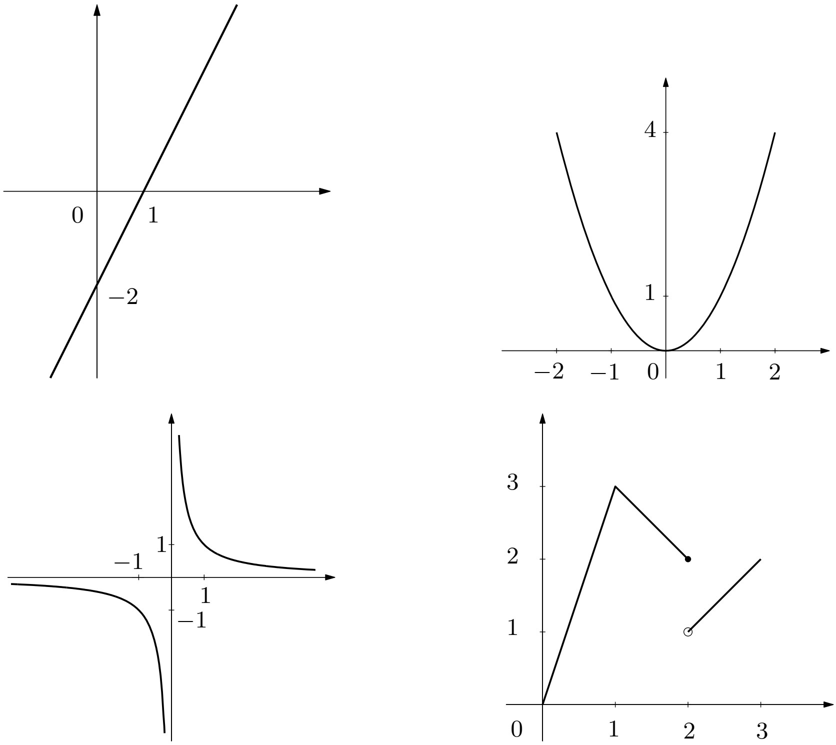


Figure 2.2. Graphs of the maps $f(x) = 2x-2$ (top left), $f(x) = x^2$ (top right), $f(x) = \dfrac{1}{x}$ (bottom left) and of the piecewise function (2.2) (bottom right)

Among piecewise functions, the following are particularly important:

v) the **absolute value** (Fig. 2.3, top left)

$$f : \mathbb{R} \to \mathbb{R}, \quad f(x) = |x| = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0; \end{cases}$$

vi) the **sign** (Fig. 2.3, top right)

$$f : \mathbb{R} \to \mathbb{Z}, \quad f(x) = \text{sign}(x) = \begin{cases} +1 & \text{if } x > 0, \\ \ 0 & \text{if } x = 0, \\ -1 & \text{if } x < 0; \end{cases}$$

vii) the **integer part** (Fig. 2.3, bottom left), also known as **floor function**,

$$f : \mathbb{R} \to \mathbb{Z}, \quad f(x) = [x] = \text{ the greatest integer } \leq x$$

(for example, $[4] = 4$, $[\sqrt{2}] = 1$, $[-1] = -1$, $[-\frac{3}{2}] = -2$); notice that

$$[x] \leq x < [x] + 1\,, \qquad \forall x \in \mathbb{R}\,;$$

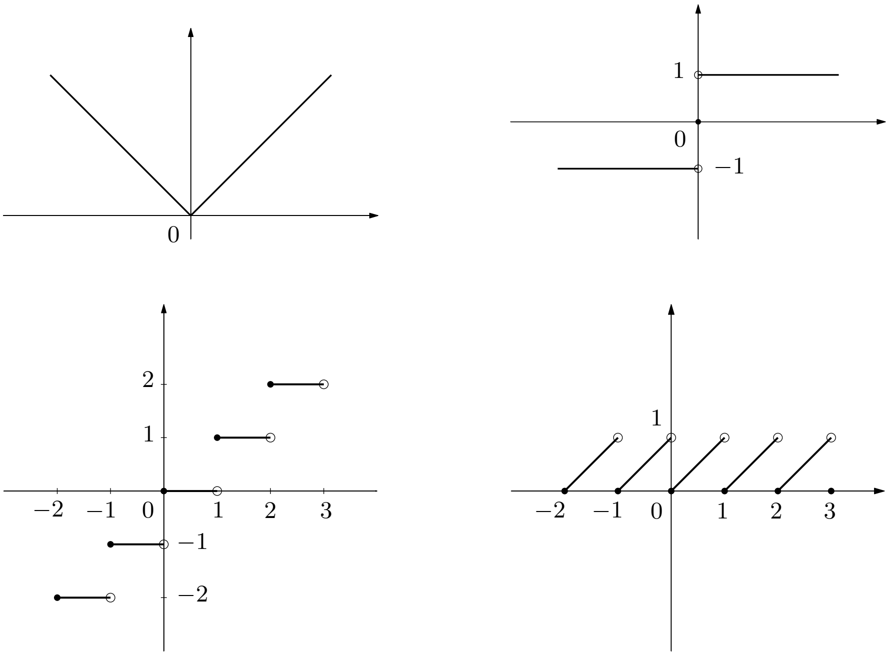


Figure 2.3. Clockwise from top left: graphs of the functions: absolute value, sign, mantissa and integer part

viii) the **mantissa** (Fig. 2.3, bottom right)

$$f : \mathbb{R} \to \mathbb{R}, \quad f(x) = M(x) = x - [x]$$

(the property of the floor function implies $0 \le M(x) < 1$).

Let us give some examples of sequences now.

ix) The sequence

$$a_n = \frac{n}{n+1} \tag{2.3}$$

is defined for all $n \ge 0$. The first few terms read

$$a_0 = 0\,, \quad a_1 = \frac{1}{2} = 0.5\,, \quad a_2 = \frac{2}{3} = 0.\overline{6}\,, \quad a_3 = \frac{3}{4} = 0.75\,.$$

Its graph is shown in Fig. 2.4 (top left).

x) The sequence

$$a_n = \left(1 + \frac{1}{n}\right)^n \tag{2.4}$$

is defined for $n \ge 1$. The first terms are

$$a_1 = 2\,, \quad a_2 = \frac{9}{4} = 2.25\,, \quad a_3 = \frac{64}{27} = 2.37\overline{037}\,, \quad a_4 = \frac{625}{256} = 2.44140625\,.$$

Fig. 2.4 (top right) shows the graph of such sequence.

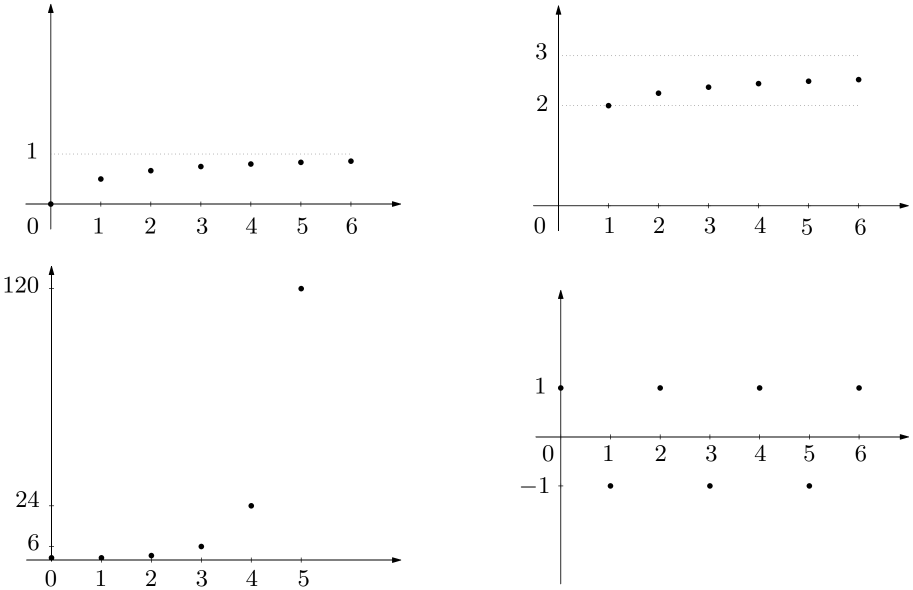


Figure 2.4. Clockwise: graphs of the sequences (2.3), (2.4), (2.6), (2.5)

xi) The sequence

$$a_n = n! \tag{2.5}$$

associates to each natural number its factorial, defined in (1.9). The graph of this sequence is shown in Fig. 2.4 (bottom left); as the values of the sequence grow rapidly as n increases, we used different scalings on the coordinate axes.

xii) The sequence

$$a_n = (-1)^n = \begin{cases} +1 & \text{if } n \text{ is even,} \\ -1 & \text{if } n \text{ is odd,} \end{cases} \qquad (n \geq 0) \tag{2.6}$$

has alternating values $+1$ and -1, according to the parity of n. The graph of the sequence is shown in Fig. 2.4 (bottom right).

At last, here are two maps defined on $\mathbb{R}^2$ (functions of *two real variables*).

xiii) The function

$$f : \mathbb{R}^2 \to \mathbb{R}, \quad f(x, y) = \sqrt{x^2 + y^2}$$

maps a generic point P of the plane with coordinates (x, y) to its distance from the origin.

xiv) The map

$$f : \mathbb{R}^2 \to \mathbb{R}^2, \quad f(x, y) = (y, x)$$

associates to a point P the point P' symmetric to P with respect to the bisectrix of the first and third quadrants. □

Consider a map from X to Y. One should take care in noting that the symbol for an element of X (to which one refers as the *independent variable*) and the symbol for an element in Y (*dependent variable*), are completely arbitary. What really determines the function is the way of associating each element of the domain to its corresponding image. For example, if x, y, z, t are symbols for real numbers, the expressions $y = f(x) = 3x$, $x = f(y) = 3y$, or $z = f(t) = 3t$ denote the *same* function, namely the one mapping each real number to its triple.

2.2 Range and pre-image

Let A be a subset of X. The **image of A under f** is the set

$$f(A) = \{f(x) \ : \ x \in A\} \subseteq \operatorname{im} f$$

of all the images of elements of A. Notice that $f(A)$ is empty if and only if A contains no elements of the domain of f. The image $f(X)$ of the whole set X is the range of f, already denoted by $\operatorname{im} f$.

Let y be any element of Y; the **pre-image of y by f** is the set

$$f^{-1}(y) = \{x \in \operatorname{dom} f \ : \ f(x) = y\}$$

of elements in X whose image is y. This set is empty precisely when y does not belong to the range of f. If B is a subset of Y, the **pre-image of B under f** is defined as the set

$$f^{-1}(B) = \{x \in \operatorname{dom} f \ : \ f(x) \in B\},$$

union of all pre-images of elements of B.

It is easy to check that $A \subseteq f^{-1}(f(A))$ for any subset A of $\operatorname{dom} f$, and $f(f^{-1}(B)) = B \cap \operatorname{im} f \subseteq B$ for any subset B of Y.

Example 2.2

Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$. The image under f of the interval $A = [1, 2]$ is the interval $B = [1, 4]$. Yet the pre-image of B under f is the union of the intervals $[-2, -1]$ and $[1, 2]$, namely, the set

$$f^{-1}(B) = \{x \in \mathbb{R} \ : \ 1 \le |x| \le 2\}$$

(see Fig. 2.5). □

The notions of infimum, supremum, maximum and minimum, introduced in Sect. 1.3.1, specialise in the case of images of functions.

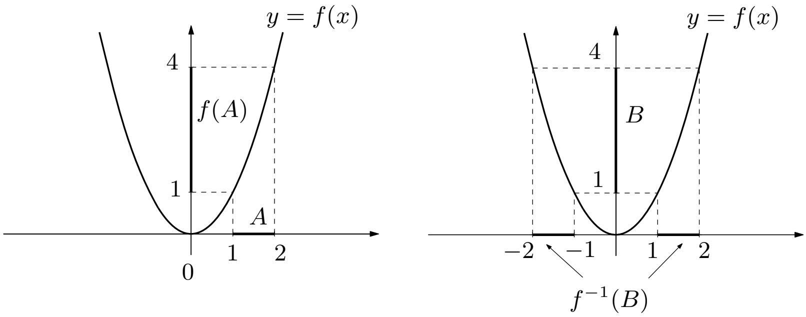


Figure 2.5. Image (left) and pre-image (right) of an interval relative to the function $f(x) = x^2$

Definition 2.3 *Let f be a real map and A a subset of* $\operatorname{dom} f$*. One calls* **supremum of** f **on** A (*or* **in** A) *the supremum of the image of A under f*

$$\sup_{x\in A} f(x) = \sup f(A) = \sup\{f(x) \mid x \in A\}.$$

Then f is **bounded from above on** A *if the set $f(A)$ is bounded from above, or equivalently, if* $\sup_{x\in A} f(x) < +\infty$*.*
If $\sup_{x\in A} f(x)$ *is finite and belongs to $f(A)$, then it is the maximum of this set. This number is the* **maximum value** (*or simply, the* **maximum**) **of** f **on** A *and is denoted by* $\max_{x\in A} f(x)$*.*
The concepts of **infimum** *and of* **minimum** *of f on A are defined similarly. Eventually, f is said* **bounded on** A *if the set $f(A)$ is bounded.*

At times, the shorthand notations $\sup_A f$, $\max_A f$, et c. are used.
The maximum value $M = \max_A f$ of f on the set A is characterised by the conditions:

i) M is a value assumed by the function on A, i.e.,

$$\text{there exists } x_M \in A \text{ such that } f(x_M) = M;$$

ii) M is greater or equal than any other value of the map on A, so

$$\text{for any } x \in A,\ f(x) \le M.$$

Example 2.4

Consider the function $f(x)$ defined in (2.2). One verifies easily

$$\max_{x\in[0,2]} f(x) = 3, \qquad \min_{x\in[0,2]} f(x) = 0, \qquad \max_{x\in[1,3]} f(x) = 3, \qquad \inf_{x\in[1,3]} f(x) = 1.$$

The map does not assume the value 1 anywhere in the interval $[1, 3]$, so there is no minimum on that set. □

2.3 Surjective and injective functions; inverse function

A map with values in Y is called **onto** if $\operatorname{im} f = Y$. This means that each $y \in Y$ is the image of one element $x \in X$ at least. The term **surjective** (**on** Y) has the same meaning. For instance, $f : \mathbb{R} \to \mathbb{R},\ f(x) = ax + b$ with $a \neq 0$ is surjective on $\mathbb{R}$, or onto: the real number y is the image of $x = \frac{y-b}{a}$. On the contrary, the function $f : \mathbb{R} \to \mathbb{R},\ f(x) = x^2$ is not onto, because its range coincides with the interval $[0, +\infty)$.

A function f is called **one-to-one** (or **1-1**) if every $y \in \operatorname{im} f$ is the image of a unique element $x \in \operatorname{dom} f$. Otherwise put, if $y = f(x_1) = f(x_2)$ for some elements x_1, x_2 in the domain of f, then necessarily $x_1 = x_2$. This, in turn, is equivalent to

$$x_1 \neq x_2 \quad \Rightarrow \quad f(x_1) \neq f(x_2)$$

for all $x_1, x_2 \in \operatorname{dom} f$ (see Fig. 2.6). Again, the term **injective** may be used. If a map f is one-to-one, we can associate to each element y in the range the unique x in the domain with $f(x) = y$. Such correspondence determines a function defined on Y and with values in X, called **inverse function** of f and denoted by the symbol f^{-1}. Thus

$$x = f^{-1}(y) \quad \Longleftrightarrow \quad y = f(x)$$

(the notation mixes up deliberately the pre-image of y under f with the unique element this set contains). The inverse function f^{-1} has the image of f as its domain, and the domain of f as range:

$$\operatorname{dom} f^{-1} = \operatorname{im} f, \qquad \operatorname{im} f^{-1} = \operatorname{dom} f.$$

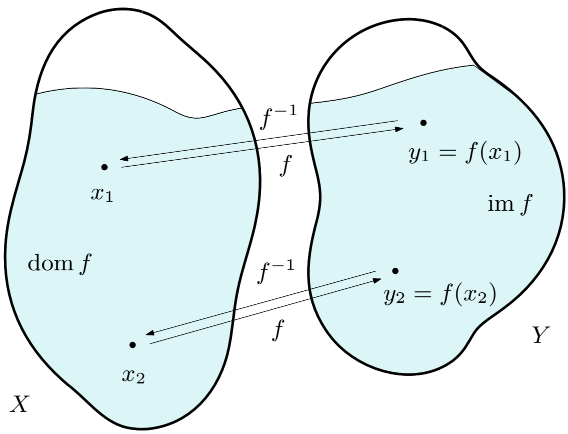


Figure 2.6. Representation of a one-to-one function and its inverse

A one-to-one map is therefore **invertible**; the two notions (injectivity and invertibility) coincide.

What is the link between the graphs of f, defined in (2.1), and of the inverse function f^{-1}? One has

$$\begin{aligned}\Gamma(f^{-1}) &= \{\big(y, f^{-1}(y)\big) \in Y \times X \ : \ y \in \operatorname{dom} f^{-1}\} \\ &= \{(f(x), x) \in Y \times X \ : \ x \in \operatorname{dom} f\}.\end{aligned}$$

Therefore, the graph of the inverse map may be obtained from the graph of f by *swapping* the components in each pair. For real functions of one real variable, this corresponds to a reflection in the Cartesian plane with respect to the bisectrix $y = x$ (see Fig. 2.7: a) is reflected into b)). On the other hand, finding the explicit expression $x = f^{-1}(y)$ of the inverse function could be hard, if possible at all.

Provided that the inverse map in the form $x = f^{-1}(y)$ can be determined, often one prefers to denote the independent variable (of f^{-1}) by x, and the dependent variable by y, thus obtaining the expression $y = f^{-1}(x)$. This is merely a change of notation (see the remark at the end of Sect. 2.1). The procedure allows to draw the graph of the inverse function in the same frame system of f (see Fig. 2.7, from b) to c)).

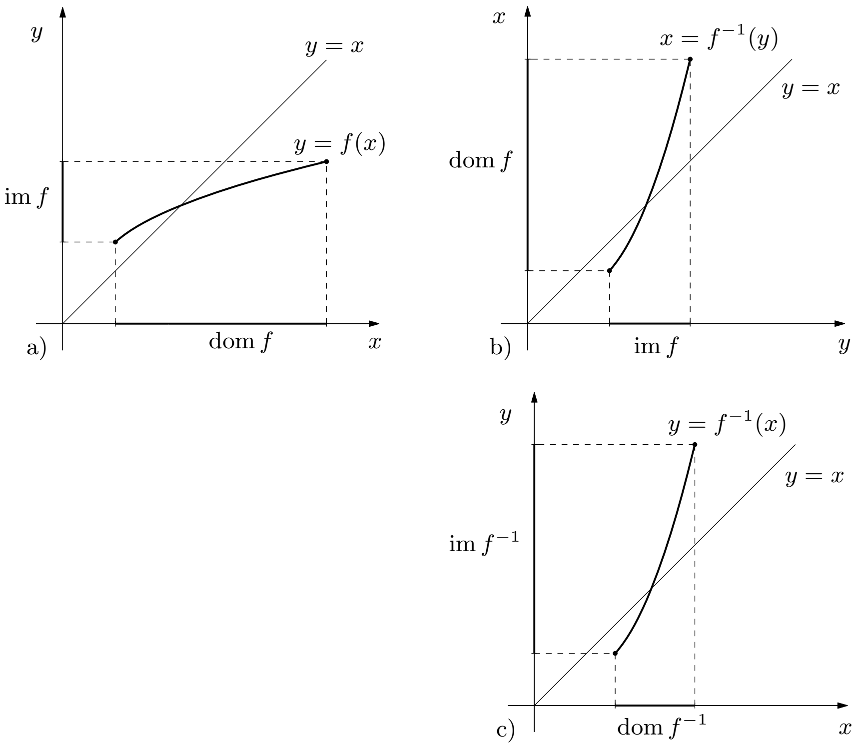


Figure 2.7. From the graph of a function to the graph of its inverse

Examples 2.5

i) The function $f : \mathbb{R} \to \mathbb{R},\ f(x) = ax + b$ is one-to-one for all $a \neq 0$ (in fact, $f(x_1) = f(x_2) \ \Rightarrow\ ax_1 = ax_2 \ \Rightarrow\ x_1 = x_2$). Its inverse is $x = f^{-1}(y) = \frac{y-b}{a}$, or $y = f^{-1}(x) = \frac{x-b}{a}$.

ii) The map $f : \mathbb{R} \to \mathbb{R},\ f(x) = x^2$ is not one-to-one because $f(x) = f(-x)$ for any real x. Yet if we consider only values ≥ 0 for the independent variable, i.e., if we **restrict** f to the interval $[0, +\infty)$, then the function becomes 1-1 (in fact, $f(x_1) = f(x_2) \ \Rightarrow\ x_1^2 - x_2^2 = (x_1 - x_2)(x_1 + x_2) = 0 \ \Rightarrow\ x_1 = x_2$). The inverse function $x = f^{-1}(y) = \sqrt{y}$ is also defined on $[0, +\infty)$. Conventionally one says that the 'squaring' map $y = x^2$ has the function 'square root' $y = \sqrt{x}$ for inverse (on $[0, +\infty)$). Notice that the restriction of f to the interval $(-\infty, 0]$ is 1-1, too; the inverse in this case is $y = -\sqrt{x}$.

iii) The map $f : \mathbb{R} \to \mathbb{R},\ f(x) = x^3$ is one-to-one. In fact $f(x_1) = f(x_2) \ \Rightarrow\ x_1^3 - x_2^3 = (x_1 - x_2)(x_1^2 + x_1x_2 + x_2^2) = 0 \ \Rightarrow\ x_1 = x_2$ since $x_1^2 + x_1x_2 + x_2^2 = \frac{1}{2}[x_1^2 + x_2^2 + (x_1 + x_2)^2] > 0$ for any $x_1 \neq x_2$. The inverse function is the 'cubic root' $y = \sqrt[3]{x}$, defined on all $\mathbb{R}$. □

As in Example ii) above, if a function f is not injective over the whole domain, it might be so on a subset $A \subseteq \operatorname{dom} f$. The **restriction of** f **to** A is the function

$$f_{|A} : A \to Y \qquad \text{such that} \qquad f_{|A}(x) = f(x)\,, \quad \forall x \in A\,,$$

and is therefore invertible.

Let f be defined on X with values Y. If f is one-to-one and onto, it is called a **bijection** (or **bijective function**) from X to Y. If so, the inverse map f^{-1} is defined on Y, and is one-to-one and onto (on X); thus, f^{-1} is a bijection from Y to X.

For example, the functions $f(x) = ax + b$ ($a \neq 0$) and $f(x) = x^3$ are bijections from $\mathbb{R}$ to itself. The function $f(x) = x^2$ is a bijection on $[0, +\infty)$ (i.e., from $[0, +\infty)$ to $[0, +\infty)$).

If f is a bijection between X and Y, the sets X and Y are in **bijective correspondence** through f: each element of X is assigned to one and only one element of Y, and vice versa. The reader should notice that two *finite* sets (i.e., containing a finite number of elements) are in bijective correspondence if and only if they have the same number of elements. On the contrary, an infinite set can correspond bijectively to a proper subset; the function (sequence) $f : \mathbb{N} \to \mathbb{N},\ f(n) = 2n$, for example, establishes a bijection between $\mathbb{N}$ and the subset of even numbers.

To conclude the section, we would like to mention a significant interpretation of the notions of 1-1, onto, and bijective maps just introduced. Both in pure Mathematics and in applications one is frequently interested in solving a problem, or an equation, of the form

$$f(x) = y\,,$$

where f is a suitable function between two sets X and Y. The quantity y represents the *datum* of the problem, while x stands for the *solution* to the problem, or *the unknown* of the equation. For instance, given the real number y, find the real number x solution of the algebraic equation

$$x^3 + x^2 - \sqrt[3]{x} = y.$$

Well, to say that f is an onto function on Y is the same as saying that the problem or equation of concern admits at least one solution for each given y in Y; asking f to be 1-1 is equivalent to saying the solution, if it exists at all, is unique. Eventually, f bijection from X to Y means that for any given y in Y there is one, and only one, solution $x \in X$.

2.4 Monotone functions

Let f be a real map of one real variable, and I the domain of f or an interval contained in the domain. We would like to describe precisely the situation in which the dependent variable increases or decreases as the independent variable grows. Examples are the increase in the pressure of a gas inside a sealed container as we raise its temperature, or the decrease of the level of fuel in the tank as a car proceeds on a highway. We have the following definition.

Definition 2.6 *The function f is* **increasing on** I *if, given elements x_1, x_2 in I with $x_1 < x_2$, one has $f(x_1) \le f(x_2)$; in symbols*

$$\forall x_1, x_2 \in I, \qquad x_1 < x_2 \quad \Rightarrow \quad f(x_1) \le f(x_2). \tag{2.7}$$

The function f is **strictly increasing on** I *if*

$$\forall x_1, x_2 \in I, \qquad x_1 < x_2 \quad \Rightarrow \quad f(x_1) < f(x_2). \tag{2.8}$$

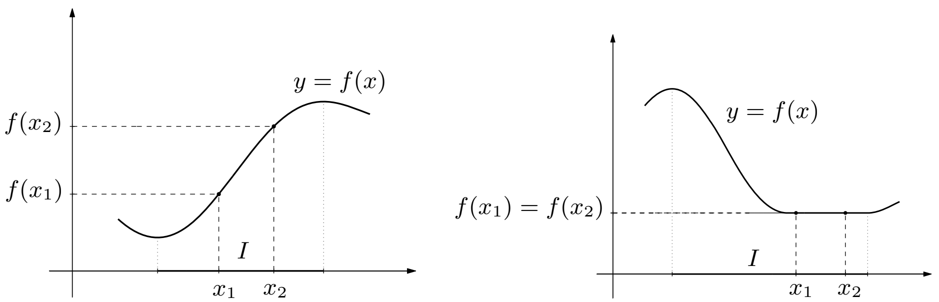


Figure 2.8. Strictly increasing (left) and decreasing (right) functions on an interval I

If a map is strictly increasing then it is increasing as well, hence condition (2.8) is stronger than (2.7).

The definitions of **decreasing** and **strictly decreasing** functions on I are obtained from the previous definitions by reverting the inequality between $f(x_1)$ and $f(x_2)$.

The function f is (**strictly**) **monotone on** I if it is either (strictly) increasing or (strictly) decreasing on I. An interval I where f is monotone is said **interval of monotonicity** of f.

Examples 2.7

i) The map $f : \mathbb{R} \to \mathbb{R},\ f(x) = ax + b$, is strictly increasing on $\mathbb{R}$ for $a > 0$, constant on $\mathbb{R}$ for $a = 0$ (hence increasing as well as decreasing), and strictly decreasing on $\mathbb{R}$ when $a < 0$.

ii) The map $f : \mathbb{R} \to \mathbb{R},\ f(x) = x^2$ is strictly increasing on $I = [0, +\infty)$. Taking in fact two arbitrary numbers $x_1, x_2 \geq 0$ with $x_1 < x_2$, we have $x_1^2 \leq x_1 x_2 < x_2^2$. Similarly, f is strictly decreasing on $(-\infty, 0]$. It is not difficult to check that all functions of the type $y = x^n$, with $n \geq 4$ even, have the same monotonic behaviour as f (Fig. 2.9, left).

iii) The function $f : \mathbb{R} \to \mathbb{R},\ f(x) = x^3$ strictly increases on $\mathbb{R}$. All functions like $y = x^n$ with n odd have analogous behaviour (Fig. 2.9, right).

iv) Referring to Examples 2.1, the maps $y = [x]$ and $y = \text{sign}(x)$ are increasing (though not strictly increasing) on $\mathbb{R}$.
The mantissa $y = M(x)$ of x, instead, is not monotone on $\mathbb{R}$; but it is nevertheless strictly increasing on each interval $[n, n+1)$, $n \in \mathbb{Z}$. □

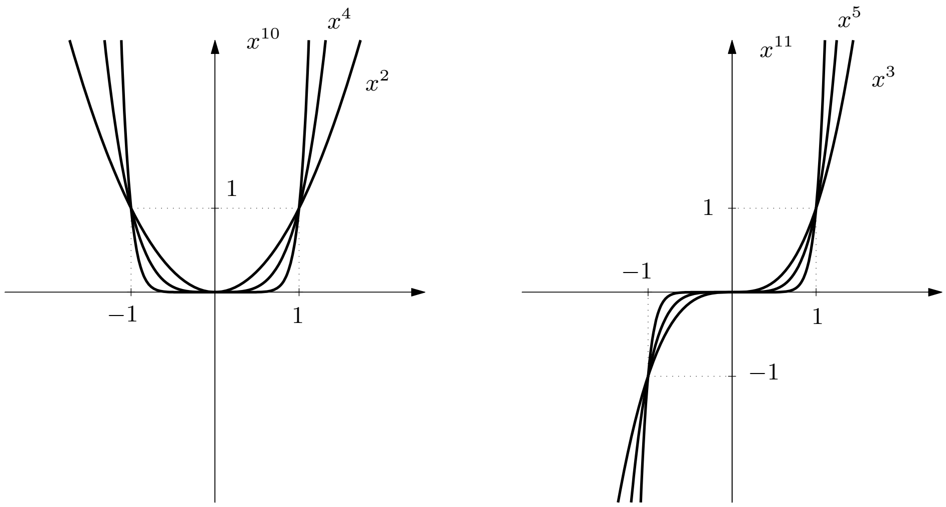


Figure 2.9. Graphs of some functions $y = x^n$ with n even (left) and n odd (right)

Now to a simple yet crucial result.

Proposition 2.8 *If f is strictly monotone on its domain, then f is one-to-one.*

Proof. To fix ideas, let us suppose f is strictly increasing. Given $x_1, x_2 \in \operatorname{dom} f$ with $x_1 \neq x_2$, then either $x_1 < x_2$ or $x_2 < x_1$. In the former case, using (2.8) we obtain $f(x_1) < f(x_2)$, hence $f(x_1) \neq f(x_2)$. In the latter case the same conclusion holds by swapping the roles of x_1 and x_2. □

Under the assumption of the above proposition, there exists the inverse function f^{-1} then; one can comfortably check that f^{-1} is also strictly monotone, and in the same way as f (both are strictly increasing or strictly decreasing). For instance, the strictly increasing function $f : [0, +\infty) \to [0, +\infty),\ f(x) = x^2$ has, as inverse, the strictly increasing function $f^{-1} : [0, +\infty) \to [0, +\infty),\ f^{-1}(x) = \sqrt{x}$.

The logic implication

$$f \text{ is strictly monotone on its domain} \qquad \Rightarrow \qquad f \text{ is one-to-one}$$

cannot be reversed. In other words, a map f may be one-to-one without increasing strictly on its domain. For instance $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} \dfrac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$$

is one-to-one, actually bijective on $\mathbb{R}$, but it is not strictly increasing, nor strictly decreasing or $\mathbb{R}$. We shall return to this issue in Sect. 4.3.

A useful remark is the following. The sum of functions that are similarly monotone (i.e., all increasing or all decreasing) is still a monotone function of the same kind, and turns out to be strictly monotone if one at least of the summands is. The map $f(x) = x^5 + x$, for instance, is strictly increasing on $\mathbb{R}$, being the sum of two functions with the same property. According to Proposition 2.8 f is then invertible; note however that the relation $f(x) = y$ cannot be made explicit in the form $x = f^{-1}(y)$.

2.5 Composition of functions

Let X, Y, Z be sets. Suppose f is a function from X to Y, and g a function from Y to Z. We can manifacture a new function h from X to Z by setting

$$h(x) = g(f(x)). \tag{2.9}$$

The function h is called **composition of f and g**, sometimes **composite map**, and is indicated by the symbol $h = g \circ f$ (read 'g composed (with) f').

Example 2.9

Consider the two real maps $y = f(x) = x - 3$ and $z = g(y) = y^2 + 1$ of one real variable. The composition of f and g reads $z = h(x) = g \circ f(x) = (x-3)^2 + 1$. □

Bearing in mind definition (2.9), the domain of the composition $g \circ f$ is determined as follows: in order for x to belong to the domain of $g \circ f$, $f(x)$ must be defined, so x must be in the domain of f; moreover, $f(x)$ has to be a element of the domain of g. Thus

$$x \in \operatorname{dom} g \circ f \qquad \Longleftrightarrow \qquad x \in \operatorname{dom} f \quad \text{and} \quad f(x) \in \operatorname{dom} g.$$

The domain of $g \circ f$ is then a *subset* of the domain of f (see Fig. 2.10).

Examples 2.10

i) The domain of $f(x) = \dfrac{x+2}{|x-1|}$ is $\mathbb{R} \setminus \{1\}$, while $g(y) = \sqrt{y}$ is defined on the interval $[0, +\infty)$. The domain of $g \circ f(x) = \sqrt{\dfrac{x+2}{|x-1|}}$ consists of the $x \neq 1$ such that $\dfrac{x+2}{|x-1|} \geq 0$; hence, $\operatorname{dom} g \circ f = [-2, +\infty) \setminus \{1\}$.

ii) Sometimes the composition $g \circ f$ has an empty domain. This happens for instance for $f(x) = \dfrac{1}{1+x^2}$ (notice $f(x) \leq 1$) and $g(y) = \sqrt{y-5}$ (whose domain is $[5, +\infty)$). □

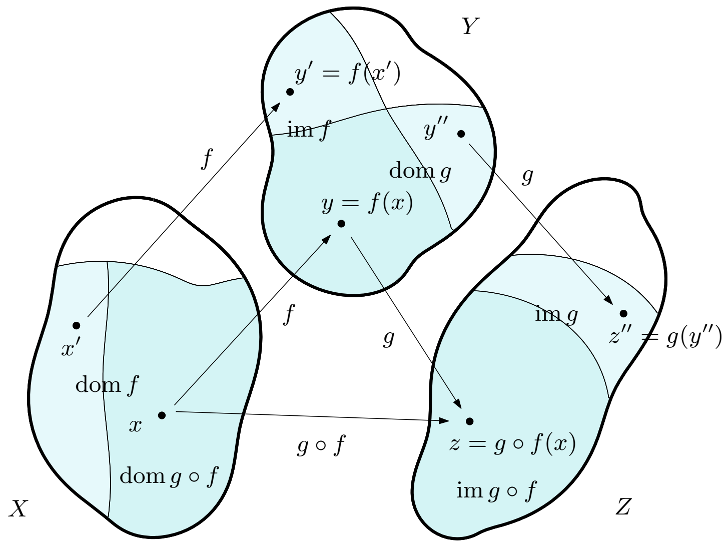


Figure 2.10. Representation of a composite function via Venn diagrams.

The operation of composition is not commutative: if $g \circ f$ and $f \circ g$ are both defined (for instance, when $X = Y = Z$), the two composites do not coincide in general. Take for example $f(x) = \dfrac{1}{x}$ and $g(x) = \dfrac{1}{1+x}$, for which $g \circ f(x) = \dfrac{x}{1+x}$, but $f \circ g(x) = 1 + x$.

If f and g are both one-to-one (or both onto, or both bijective), it is not difficult to verify that $g \circ f$ has the same property. In the first case in particular, the formula

$$(g \circ f)^{-1} = f^{-1} \circ g^{-1}$$

holds.

Moreover, if f and g are real monotone functions of real variable, $g \circ f$ too will be monotone, or better: $g \circ f$ is increasing if both f and g are either increasing or decreasing, and decreasing otherwise. Let us prove only one of these properties. Let for example f increase and g decrease; if $x_1 < x_2$ are elements in $\operatorname{dom} g \circ f$, the monotone behaviour of f implies $f(x_1) \le f(x_2)$; now the monotonicity of g yields $g(f(x_1)) \ge g(f(x_2))$, so $g \circ f$ is decreasing.

We observe finally that if f is a one-to-one function (and as such it admits inverse f^{-1}), then

$$\begin{aligned} f^{-1} \circ f(x) = f^{-1}(f(x)) = x, &\qquad \forall x \in \operatorname{dom} f, \\ f \circ f^{-1}(y) = f(f^{-1}(y)) = y, &\qquad \forall y \in \operatorname{im} f. \end{aligned}$$

Calling **identity map** on a set X the function $\mathrm{id}_X : X \to X$ such that $\mathrm{id}_X(x) = x$ for all $x \in X$, we have $f^{-1} \circ f = \mathrm{id}_{\operatorname{dom} f}$ and $f \circ f^{-1} = \mathrm{id}_{\operatorname{im} f}$.

2.5.1 Translations, rescalings, reflections

Let f be a real map of one real variable (for instance, the function of Fig. 2.11). Fix a real number $c \neq 0$, and denote by $t_c : \mathbb{R} \to \mathbb{R}$ the function $t_c(x) = x + c$. Composing f with t_c results in a **translation** of the graph of f: precisely, the

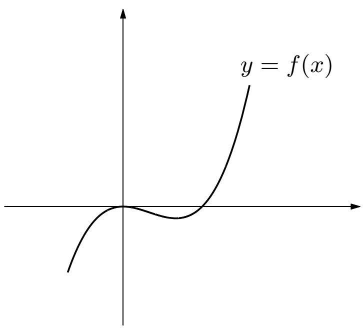


Figure 2.11. Graph of a function $f(x)$

graph of the function $f \circ t_c(x) = f(x+c)$ is shifted horizontally with respect to the graph of f: towards the left if $c > 0$, to the right if $c < 0$. Similarly, the graph of $t_c \circ f(x) = f(x)+c$ is translated vertically with respect to the graph of f, towards the top for $c > 0$, towards the bottom if $c < 0$. Fig. 2.12 provides examples of these situations.

Fix a real number $c > 0$ and denote by $s_c : \mathbb{R} \to \mathbb{R}$ the map $s_c(x) = cx$. The composition of f with s_c has the effect of **rescaling** the graph of f. Precisely, if $c > 1$ the graph of the function $f \circ s_c(x) = f(cx)$ is 'compressed' horizontally towards the y-axis, with respect to the graph of f; if $0 < c < 1$ instead, the graph 'stretches' away from the y-axis. The analogue effect, though in the vertical direction, is seen for the function $s_c \circ f(x) = cf(x)$: here $c > 1$ 'spreads out' the graph away from the x-axis, while $0 < c < 1$ 'squeezes' it towards the axis, see Fig. 2.13.

Notice also that the graph of $f(-x)$ is obtained by **reflecting** the graph of $f(x)$ along the y-axis, like in front of a mirror. The graph of $f(|x|)$ instead coincides with that of f for $x \geq 0$, and for $x < 0$ it is the mirror image of the latter with respect to the vertical axis. At last, the graph of $|f(x)|$ is the same as the graph of f when $f(x) \geq 0$, and is given by reflecting the latter where $f(x) < 0$, see Fig. 2.14.

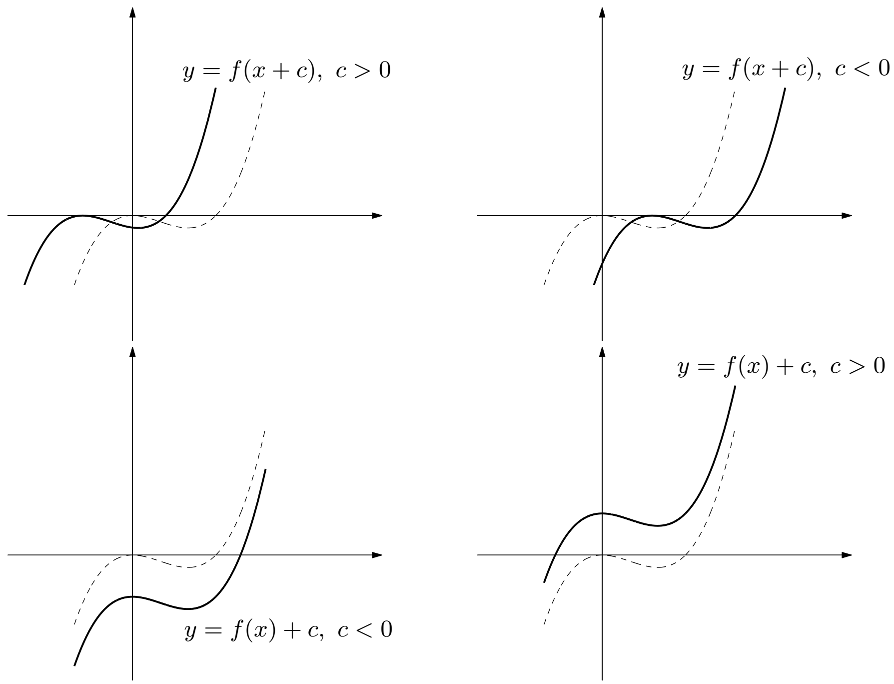


Figure 2.12. Graphs of the functions $f(x+c)$ ($c > 0$: top left, $c < 0$: top right), and $f(x)+c$ ($c < 0$: bottom left, $c > 0$: bottom right), where $f(x)$ is the map of Fig. 2.11

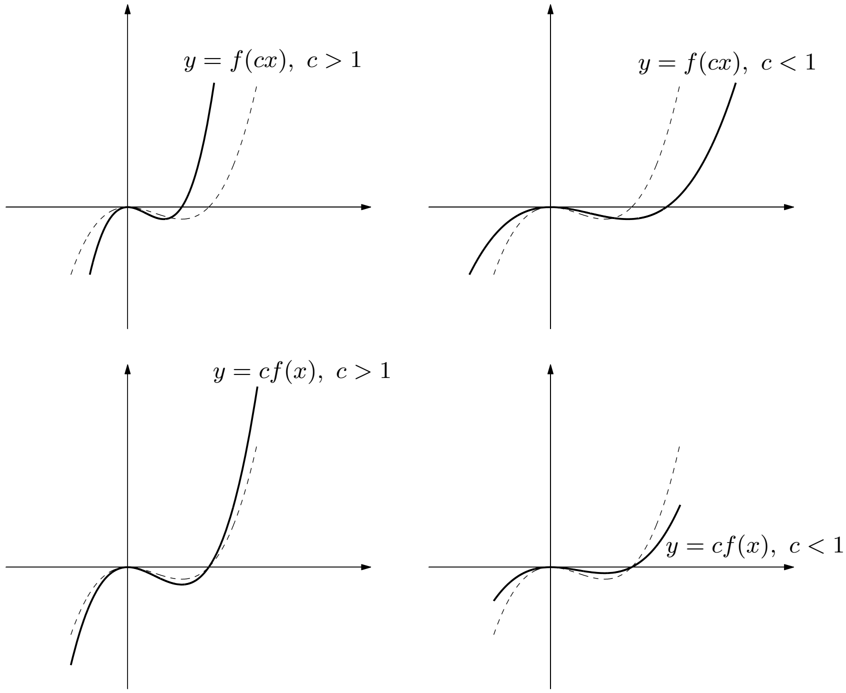


Figure 2.13. Graph of $f(cx)$ with $c > 1$ (top left), $0 < c < 1$ (top right), and of $cf(x)$ with $c > 1$ (bottom left), $0 < c < 1$ (bottom right)

2.6 Elementary functions and properties

We start with a few useful definitions.

Definition 2.11 *Let* $f : \mathrm{dom}\, f \subseteq \mathbb{R} \to \mathbb{R}$ *be a map with a symmetric domain with respect to the origin, hence such that* $x \in \mathrm{dom}\, f$ *forces* $-x \in \mathrm{dom}\, f$ *as well. The function* f *is said* **even** *if* $f(-x) = f(x)$ *for all* $x \in \mathrm{dom}\, f$*,* **odd** *if* $f(-x) = -f(x)$ *for all* $x \in \mathrm{dom}\, f$*.*

The graph of an even function is symmetric with respect to the y-axis, and that of an odd map symmetric with respect to the origin. If f is odd and defined in the origin, necessarily it must vanish at the origin, for $f(0) = -f(0)$.

Definition 2.12 *A function* $f : \mathrm{dom}\, f \subseteq \mathbb{R} \to \mathbb{R}$ *is said* **periodic of period** p *(with* $p > 0$ *real) if* $\mathrm{dom}\, f$ *is invariant under translations by* $\pm p$ *(i.e., if* $x \pm p \in \mathrm{dom}\, f$ *for all* $x \in \mathrm{dom}\, f$*) and if* $f(x + p) = f(x)$ *holds for any* $x \in \mathrm{dom}\, f$*.*

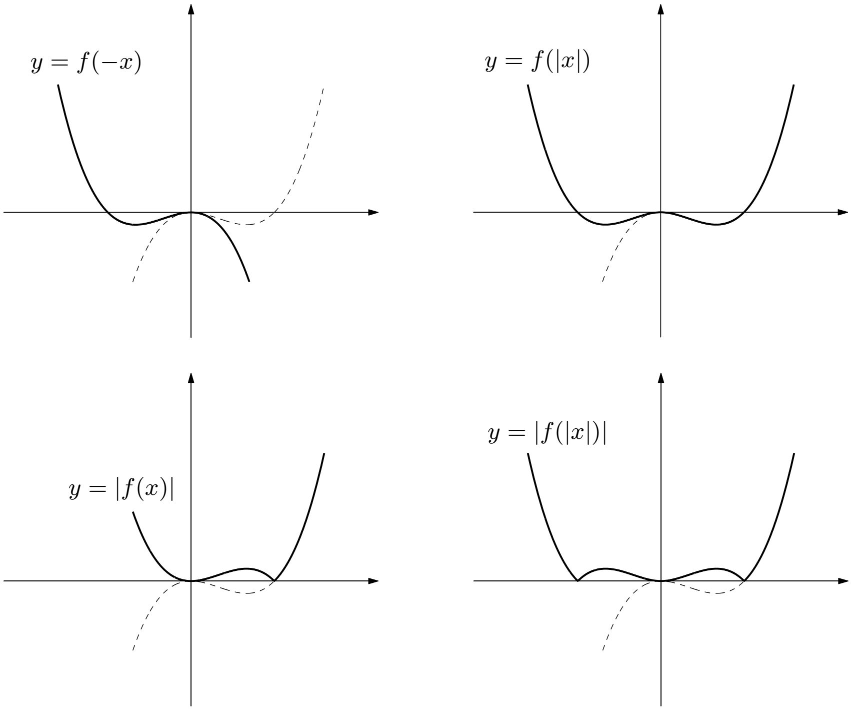


Figure 2.14. Clockwise: graph of the functions $f(-x)$, $f(|x|)$, $|f(|x|)|$, $|f(x)|$

One easily sees that an f periodic of period p is also periodic of any multiple mp ($m \in \mathbb{N} \setminus \{0\}$) of p. If the smallest period exists, it goes under the name of **minimum period** of the function. A constant map is clearly periodic of any period $p > 0$ and thus has no minimum period.

Let us review now the main elementary functions.

2.6.1 Powers

These are functions of the form $y = x^\alpha$. The case $\alpha = 0$ is trivial, giving rise to the constant function $y = x^0 = 1$. Suppose then $\alpha > 0$. For $\alpha = n \in \mathbb{N} \setminus \{0\}$, we find the monomial functions $y = x^n$ defined on $\mathbb{R}$, already considered in Example 2.7 ii) and iii). When n is odd, the maps are odd, strictly increasing on $\mathbb{R}$ and with range $\mathbb{R}$ (recall Property 1.8). When n is even, the functions are even, strictly decreasing on $(-\infty, 0]$ and strictly increasing on $[0, +\infty)$; their range is the interval $[0, +\infty)$.

Consider now the case $\alpha > 0$ rational. If $\alpha = \frac{1}{m}$ where $m \in \mathbb{N} \setminus \{0\}$, we define a function, called mth root of x and denoted $y = x^{1/m} = \sqrt[m]{x}$, inverting $y = x^m$. It has domain $\mathbb{R}$ if m is odd, $[0, +\infty)$ if m is even. The mth root is strictly increasing and ranges over $\mathbb{R}$ or $[0, +\infty)$, according to whether m is even or odd respectively.

In general, for $\alpha = \frac{n}{m} \in \mathbb{Q}$, $n, m \in \mathbb{N} \setminus \{0\}$ with no common divisors, the function $y = x^{n/m}$ is defined as $y = (x^n)^{1/m} = \sqrt[m]{x^n}$. As such, it has domain $\mathbb{R}$

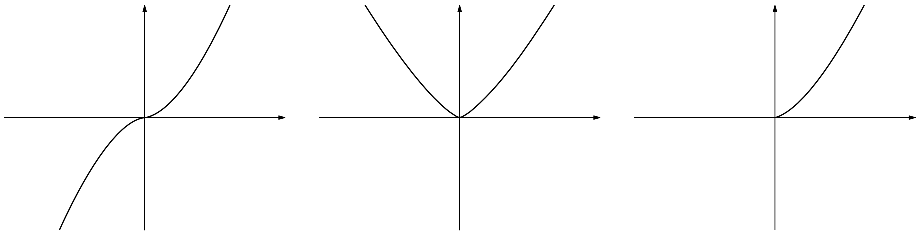


Figure 2.15. Graphs of the functions $y = x^{5/3}$ (left), $y = x^{4/3}$ (middle) and $y = x^{3/2}$ (right)

if m is odd, $[0, +\infty)$ if m is even. It is strictly increasing on $[0, +\infty)$ for any n, m, while if m is odd it strictly increases or decreases on $(-\infty, 0]$ according to the parity of n.

Let us consider some examples (Fig. 2.15). The map $y = x^{5/3}$, defined on $\mathbb{R}$, is strictly increasing and has range $\mathbb{R}$. The map $y = x^{4/3}$ is defined on $\mathbb{R}$, strictly decreases on $(-\infty, 0]$ and strictly increases on $[0, +\infty)$, which is also its range. To conclude, $y = x^{3/2}$ is defined only on $[0, +\infty)$, where it is strictly increasing and has $[0, +\infty)$ as range.

Let us introduce now the generic function $y = x^\alpha$ with irrational $\alpha > 0$. To this end, note that if a is a non-negative real number we can define the power a^α with $\alpha \in \mathbb{R}_+ \setminus \mathbb{Q}$, starting from powers with rational exponent and exploiting the density of rationals inside $\mathbb{R}$. If $a \geq 1$, we can in fact define $a^\alpha = \sup\{a^{n/m} \mid \frac{n}{m} \leq \alpha\}$, while for $0 \leq a < 1$ we set $a^\alpha = \inf\{a^{n/m} \mid \frac{n}{m} \leq \alpha\}$. Thus the map $y = x^\alpha$ with $\alpha \in \mathbb{R}_+ \setminus \mathbb{Q}$ is defined on $[0, +\infty)$, and one proves it is there strictly increasing and its range is $[0, +\infty)$.

Summarising, we have defined $y = x^\alpha$ for every value $\alpha > 0$. They are all defined a least on $[0, +\infty)$, interval on which they are strictly increasing; moreover, they satisfy $y(0) = 0$, $y(1) = 1$. It will turn out useful to remark that if $\alpha < \beta$,

$$0 < x^\beta < x^\alpha < 1, \quad \text{for} \;\; 0 < x < 1, \qquad 1 < x^\alpha < x^\beta, \quad \text{for} \;\; x > 1 \tag{2.10}$$

(see Fig. 2.16).

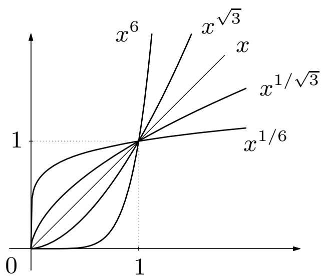


Figure 2.16. Graphs of $y = x^\alpha$, $x \geq 0$ for some $\alpha > 0$

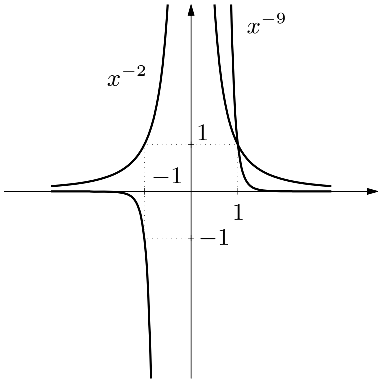


Figure 2.17. Graphs of $y = x^\alpha$ for a two values $\alpha < 0$

At last, consider the case of $\alpha < 0$. Set $y = x^\alpha = \dfrac{1}{x^{-\alpha}}$ by definition. Its domain coincides with the domain of $y = x^{-\alpha}$ minus the origin. All maps are strictly decreasing on $(0, +\infty)$, while on $(-\infty, 0)$ the behaviour is as follows: writing $\alpha = -\frac{n}{m}$ with m odd, the map is strictly increasing if n is even, strictly decreasing if n is odd, as shown in Fig. 2.17. In conclusion, we observe that for every $\alpha \neq 0$, the inverse function of $y = x^\alpha$, where defined, is $y = x^{1/\alpha}$.

2.6.2 Polynomial and rational functions

A **polynomial function**, or simply, a **polynomial**, is a map of the form $P(x) = a_n x^n + \cdots + a_1 x + a_0$ with $a_n \neq 0$; n is the **degree** of the polynomial. Such a map is defined over all $\mathbb{R}$; it is even (resp. odd) if and only if all coefficients indexed by even (odd) subscripts vanish (recall that 0 is an even number).

A **rational function** is of the kind $R(x) = \dfrac{P(x)}{Q(x)}$, where P and Q are polynomials. If these have no common factor, the domain of the rational function will be $\mathbb{R}$ without the zeroes of the denominator.

2.6.3 Exponential and logarithmic functions

Let a be a positive real number. According to what we have discussed previously, the **exponential function** $y = a^x$ is defined for any real number x; it satisfies $y(0) = a^0 = 1$.

If $a > 1$, the exponential is strictly increasing; if $a = 1$, this is the constant map 1, while if $a < 1$, the function is strictly decreasing. When $a \neq 1$, the range is $(0, +\infty)$ (Fig. 2.18). Recalling a few properties of powers is useful at this point: for any $x, y \in \mathbb{R}$

$$a^{x+y} = a^x a^y\,, \qquad a^{x-y} = \frac{a^x}{a^y}\,, \qquad (a^x)^y = a^{xy}\,.$$

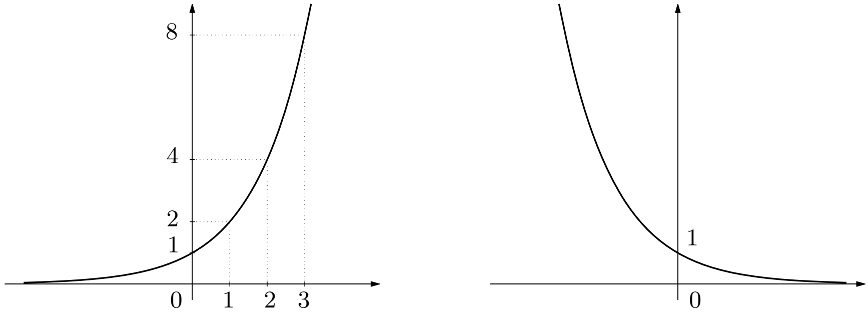


Figure 2.18. Graphs of the exponential functions $y = 2^x$ (left) and $y = (\frac{1}{2})^x$ (right)

When $a \neq 1$, the exponential function is strictly monotone on $\mathbb{R}$, hence invertible. The inverse $y = \log_a x$ is called **logarithm**, is defined on $(0, +\infty)$ and ranges over $\mathbb{R}$; it satisfies $y(1) = \log_a 1 = 0$. The logarithm is strictly increasing if $a > 1$, strictly decreasing if $a < 1$ (Fig. 2.19). The previous properties translate into the following:

$$
\begin{aligned}
&\log_a(xy) = \log_a x + \log_a y, \quad \forall x, y > 0\,, \\
&\log_a \frac{x}{y} = \log_a x - \log_a y, \quad \forall x, y > 0\,, \\
&\log_a(x^y) = y \log_a x, \quad \forall x > 0\,,\ \forall y \in \mathbb{R}\,.
\end{aligned}
$$

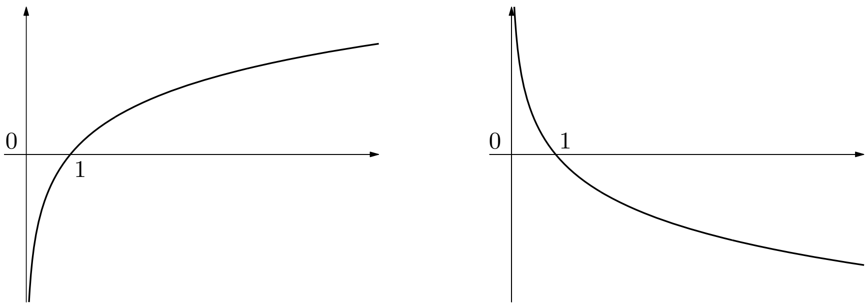


Figure 2.19. Graphs of $y = \log_2 x$ (left) and $y = \log_{1/2} x$ (right)

2.6.4 Trigonometric functions and inverses

Denote here by X, Y the coordinates on the Cartesian plane $\mathbb{R}^2$, and consider the **unit circle**, i.e., the circle of unit radius centred at the origin $O = (0, 0)$, whose equation reads $X^2 + Y^2 = 1$. Starting from the point $A = (1, 0)$, intersection of the circle with the positive x-axis, we go around the circle. More precisely, given any real x we denote by $P(x)$ the point on the circle reached by turning counter-clockwise along an arc of length x if $x \geq 0$, or clockwise by an arc of length $-x$ if $x < 0$. The point $P(x)$ determines an *angle* in the plane with vertex O and delimited by the outbound rays from O through the points A and $P(x)$ respectively (Fig. 2.20). The number x represents the measure of the angle in *radians*. The one-radian angle is determined by an arc of length 1. This angle measures $\frac{360}{2\pi} = 57.2957795\cdots$ degrees. Table 2.1 provides the correspondence between degrees and radians for important angles. Henceforth all angles shall be expressed in radians without further mention.

degrees	0	30	45	60	90	120	135	150	180	270	360
radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{3\pi}{2}$	2π

Table 2.1. Degrees versus radians

Increasing or decreasing by 2π the length x has the effect of going around the circle once, counter-clockwise or clockwise respectively, and returning to the initial point $P(x)$. In other words, there is a periodicity

$$P(x \pm 2\pi) = P(x), \qquad \forall x \in \mathbb{R}. \tag{2.11}$$

Denote by $\cos x$ ('cosine of x') and $\sin x$ ('sine of x') the X- and Y-coordinates, respectively, of the point $P(x)$. Thus $P(x) = (\cos x, \sin x)$. Hence the **cosine function** $y = \cos x$ and the **sine function** $y = \sin x$ are defined on $\mathbb{R}$ and assume all

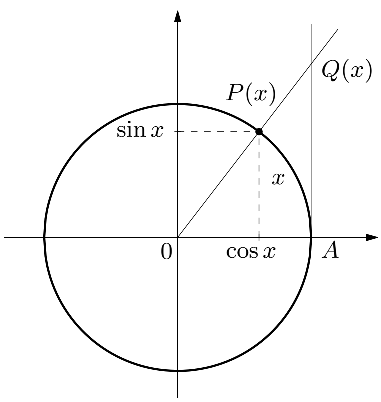


Figure 2.20. The unit circle

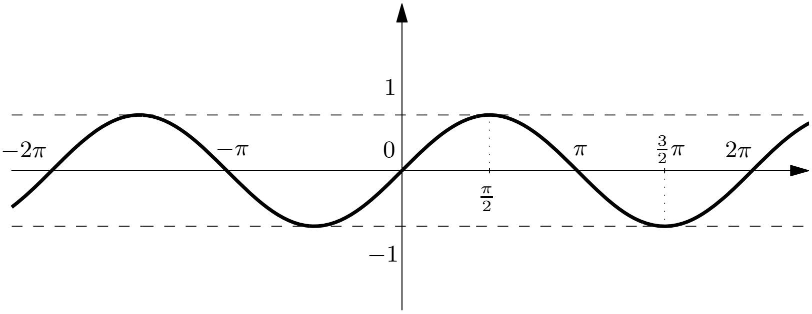


Figure 2.21. Graph of the map $y = \sin x$

values of the interval $[-1, 1]$; by (2.11), they are periodic maps of minimum period 2π. They satisfy the crucial trigonometric relation

$$\cos^2 x + \sin^2 x = 1, \qquad \forall x \in \mathbb{R}. \tag{2.12}$$

It is rather evident from the geometric interpretation that the sine function is odd, while the cosine function is even. Their graphs are represented in Figures 2.21 and 2.22.

Important values of these maps are listed in the following table (where k is any integer):

$$\begin{array}{llll}
\sin x = 0 & \text{for} \ \ x = k\pi\,, & \cos x = 0 & \text{for} \ \ x = \dfrac{\pi}{2} + k\pi\,, \\
\sin x = 1 & \text{for} \ \ x = \dfrac{\pi}{2} + 2k\pi\,, & \cos x = 1 & \text{for} \ \ x = 2k\pi\,, \\
\sin x = -1 & \text{for} \ \ x = -\dfrac{\pi}{2} + 2k\pi\,, & \cos x = -1 & \text{for} \ \ x = \pi + 2k\pi\,.
\end{array}$$

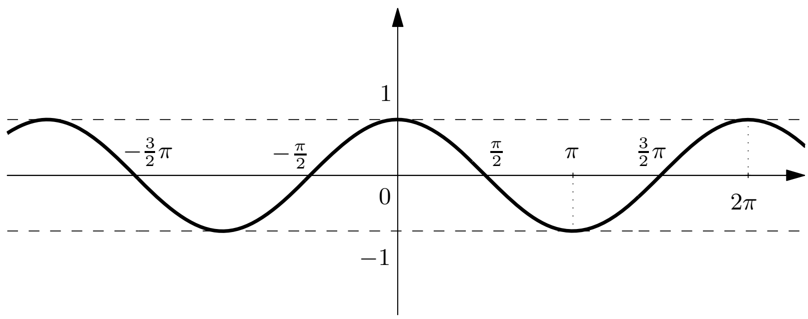


Figure 2.22. Graph of the map $y = \cos x$

Concerning monotonicity, one has

$$y = \sin x \quad \text{is} \quad \begin{cases} \text{strictly increasing on} & \left[-\dfrac{\pi}{2} + 2k\pi, \dfrac{\pi}{2} + 2k\pi\right] \\ \text{strictly decreasing on} & \left[\dfrac{\pi}{2} + 2k\pi, \dfrac{3\pi}{2} + 2k\pi\right], \end{cases}$$

$$y = \cos x \quad \text{is} \quad \begin{cases} \text{strictly decreasing on} & [2k\pi, \pi + 2k\pi] \\ \text{strictly increasing on} & [\pi + 2k\pi, 2\pi + 2k\pi]. \end{cases}$$

The addition and subtraction formulas are relevant

$$\begin{aligned} \sin(\alpha \pm \beta) &= \sin\alpha\cos\beta \pm \cos\alpha\sin\beta \\ \cos(\alpha \pm \beta) &= \cos\alpha\cos\beta \mp \sin\alpha\sin\beta. \end{aligned}$$

Suitable choices of the arguments allow to infer from these the duplication formulas

$$\sin 2x = 2\sin x\cos x, \qquad \cos 2x = 2\cos^2 x - 1, \tag{2.13}$$

rather than

$$\sin x - \sin y = 2\sin\frac{x-y}{2}\cos\frac{x+y}{2}, \tag{2.14}$$

$$\cos x - \cos y = -2\sin\frac{x-y}{2}\sin\frac{x+y}{2}, \tag{2.15}$$

or the following

$$\sin(x+\pi) = -\sin x, \qquad \cos(x+\pi) = -\cos x, \tag{2.16}$$

$$\sin(x+\frac{\pi}{2}) = \cos x, \qquad \cos(x+\frac{\pi}{2}) = -\sin x. \tag{2.17}$$

In the light of Sect. 2.5.1, the first of (2.17) tells that the graph of the cosine is obtained by left-translating the sine's graph by $\pi/2$ (compare Figures 2.21 and 2.22).

The **tangent function** $y = \tan x$ (sometimes $y = \mathrm{tg}\, x$) and the **cotangent function** $y = \mathrm{cotan}\, x$ (also $y = \mathrm{ctg}\, x$) are defined by

$$\tan x = \frac{\sin x}{\cos x}, \qquad \mathrm{cotan}\, x = \frac{\cos x}{\sin x}.$$

Because of (2.16), these maps are periodic of minimum period π, and not 2π. The tangent function is defined on $\mathbb{R} \setminus \{\frac{\pi}{2} + k\pi \ : \ k \in \mathbb{Z}\}$, it is strictly increasing on the

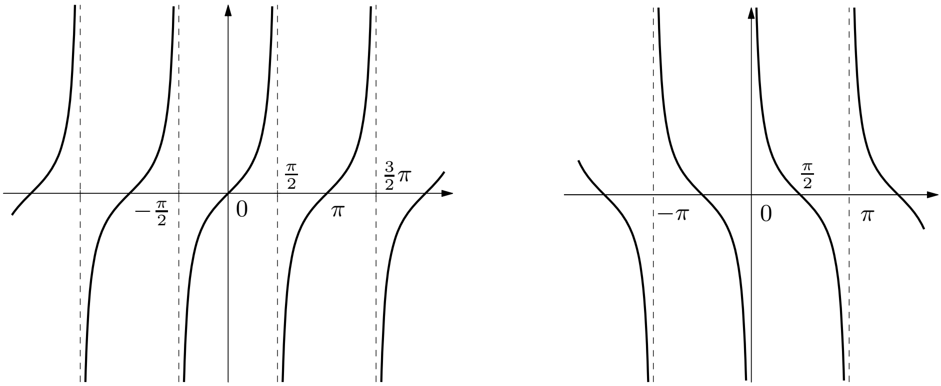


Figure 2.23. Graphs of the functions $y = \tan x$ (left) and $y = \operatorname{cotan} x$ (right)

intervals $(-\frac{\pi}{2} + k\pi, \frac{\pi}{2} + k\pi)$ where it assumes every real number as value. Similarly, the cotangent function is defined on $\mathbb{R} \setminus \{k\pi \ : \ k \in \mathbb{Z}\}$, is strictly decreasing on the intervals $(k\pi, \pi + k\pi)$, on which it assumes every real value. Both maps are odd. Their respective graphs are found in Fig. 2.23.

Recall that $\tan x$ expresses geometrically the Y-coordinate of the intersection point $Q(x)$ between the ray from the origin through $P(x)$ and the vertical line containing A (Fig. 2.20).

The trigonometric functions, being periodic, cannot be invertible on their whole domains. In order to invert them, one has to restrict to a maximal interval of strict monotonicity; in each case one such interval is chosen.

The map $y = \sin x$ is strictly increasing on $[-\frac{\pi}{2}, \frac{\pi}{2}]$. The inverse function on this particular interval is called **inverse sine** or **arcsine** and denoted $y = \arcsin x$

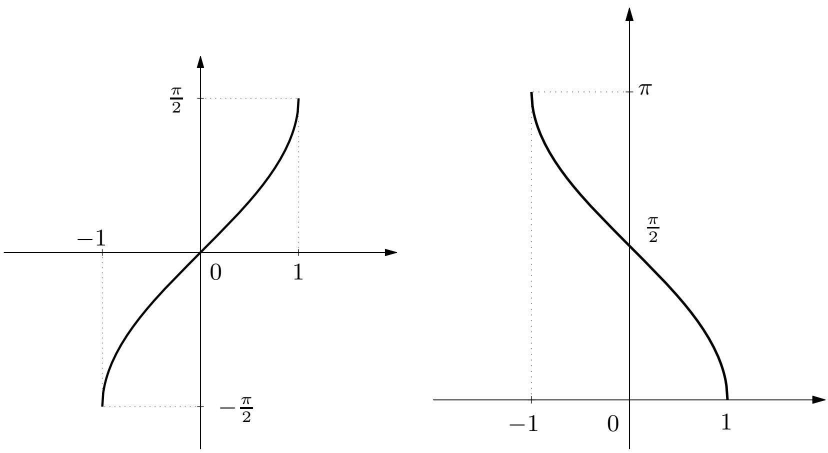


Figure 2.24. Graphs of $y = \arcsin x$ (left) and $y = \arccos x$ (right)

or $y = \operatorname{asin} x$; it is defined on $[-1,1]$, everywhere strictly increasing and ranging over the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$. This function is odd (Fig. 2.24, left).

Similarly, the function $y = \cos x$ is strictly decreasing on the interval $[0, \pi]$. By restricting it to this interval one can define the **inverse cosine**, or **arccosine**, $y = \arccos x$ or $y = \operatorname{acos} x$ on $[-1,1]$, which is everywhere strictly decreasing and has $[0, \pi]$ for range (Fig. 2.24, right).

The function $y = \tan x$ is strictly increasing on $(-\frac{\pi}{2}, \frac{\pi}{2})$. There, the inverse is called **inverse tangent**, or **arctangent**, and denoted $y = \arctan x$ or $y = \operatorname{atan} x$ (also $\operatorname{arctg} x$). It is strictly increasing on its entire domain $\mathbb{R}$, and has range $(-\frac{\pi}{2}, \frac{\pi}{2})$. Also this is an odd map (Fig. 2.25, left).

In the analogous way the **inverse cotangent**, or **arccotangent**, $y = \operatorname{arccotan} x$ is the inverse of the cotangent on $(0, \pi)$ (Fig. 2.25, right).

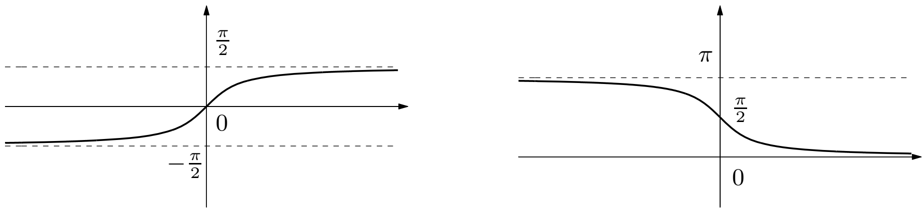


Figure 2.25. Graphs of $y = \arctan x$ (left) and $y = \operatorname{arccotan} x$ (right)

2.7 Exercises

1. *Determine the domains of the following functions:*

a) $f(x) = \dfrac{3x+1}{x^2+x-6}$

$\boxed{\text{b)}}$ $f(x) = \dfrac{\sqrt{x^2-3x-4}}{x+5}$

c) $f(x) = \log(x^2 - x)$

$\boxed{\text{d)}}$ $f(x) = \begin{cases} \dfrac{1}{2x+1} & \text{if } x \geq 0\,, \\ e^{\sqrt{x+1}} & \text{if } x < 0 \end{cases}$

2. *Determine the range of the following functions:*

$\boxed{\text{a)}}$ $f(x) = \dfrac{1}{x^2+1}$

$\boxed{\text{b)}}$ $f(x) = \sqrt{x+2} - 1$

c) $f(x) = e^{5x+3}$

d) $f(x) = \begin{cases} \log x & \text{if } x \geq 1\,, \\ -2x-5 & \text{if } x < 1 \end{cases}$

$\boxed{3.}$ *Find domain and range for the map $f(x) = \sqrt{\cos x - 1}$ and plot its graph.*

4. *Let* $f(x) = -\log(x-1)$*; determine* $f^{-1}([0,+\infty))$ *and* $f^{-1}((-\infty,-1])$.

5. *Sketch the graph of the following functions indicating the possible symmetries and/or periodicity:*

   a) $f(x) = \sqrt{1-|x|}$ b) $f(x) = 1 + \cos 2x$

   c) $f(x) = \tan\left(x + \dfrac{\pi}{2}\right)$ d) $f(x) = \begin{cases} x^2 - x - 1 & \text{if } x \le 1\,, \\ -x & \text{if } x > 1 \end{cases}$

6. *Using the map* $f(x)$ *in Fig. 2.26, draw the graphs of*

$$f(x)-1, \quad f(x+3), \quad f(x-1), \quad -f(x), \quad f(-x), \quad |f(x)|.$$

7. *Check that the function* $f:\mathbb{R}\to\mathbb{R}$ *defined by* $f(x) = x^2 - 2x + 5$ *is not invertible. Determine suitable invertible restrictions of* f *and write down the inverses explicitly.*

8. *Determine the largest interval* I *where the map*

$$f(x) = \sqrt{|x-2| - |x| + 2}$$

is invertible, and plot a graph. Write the expression of the inverse function of f *restricted to* I.

9. *Verify that* $f(x) = (1+3x)(2x - |x-1|)$*, defined on* $[0,+\infty)$*, is one-to-one. Determine its range and inverse function.*

10. *Let* f *and* g *be the functions below. Write the expressions for* $g\circ f$, $f\circ g$*, and determine the composites' domains.*

    a) $f(x) = x^2 - 3$ and $g(x) = \log(1+x)$

    b) $f(x) = \dfrac{7x}{x+1}$ and $g(x) = \sqrt{2-x}$

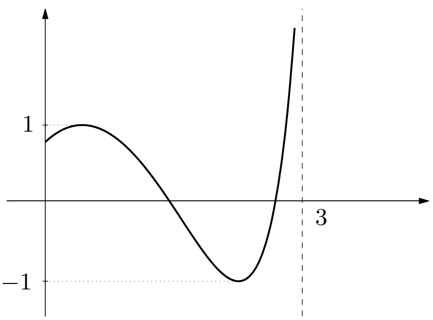


Figure 2.26. Graph of the function f in Exercise 6

11. *Write* $h(x) = \dfrac{2e^x + 1}{e^{2x} + 2}$ *as composition of the map* $f(x) = e^x$ *with some other function.*

12. *Given* $f(x) = x^2 - 3x + 2$ *and* $g(x) = x^2 - 5x + 6$, *find the expressions and graphs of*

$$h(x) = \min(f(x), g(x)) \qquad \text{and} \qquad k(x) = \max(h(x), 0).$$

2.7.1 Solutions

1. *Domains:*

a) $\operatorname{dom} f = \mathbb{R} \setminus \{-3, 2\}$.

b) The conditions $x^2 - 3x - 4 \geq 0$ and $x + 5 \neq 0$ are necessary. The first is tantamount to $(x+1)(x-4) \geq 0$, hence $x \in (-\infty, -1] \cup [4, +\infty)$; the second to $x \neq -5$. The domain of f is then

$$\operatorname{dom} f = (-\infty, -5) \cup (-5, -1] \cup [4, +\infty).$$

c) $\operatorname{dom} f = (-\infty, 0) \cup (1, +\infty)$.

d) In order to study the domain of this piecewise function, we treat the cases $x \geq 0$, $x < 0$ separately.
For $x \geq 0$, we must impose $2x+1 \neq 0$, i.e., $x \neq -\frac{1}{2}$. Since $-\frac{1}{2} < 0$, the function is well defined on $x \geq 0$.
For $x < 0$, we must have $x + 1 \geq 0$, or $x \geq -1$. For negative x then, the function is defined on $[-1, 0)$.
All in all, $\operatorname{dom} f = [-1, +\infty)$.

2. *Ranges:*

a) The map $y = x^2$ has range $[0, +\infty)$; therefore the range of $y = x^2 + 1$ is $[1, +\infty)$. Passing to reciprocals, the given function ranges over $(0, 1]$.

b) The map is obtained by translating the elementary function $y = \sqrt{x}$ (whose range is $[0, +\infty)$) to the left by -2 (yielding $y = \sqrt{x+2}$) and then downwards by 1 (which gives $y = \sqrt{x+2} - 1$). The graph is visualised in Fig. 2.27, and clearly $\operatorname{im} f = [-1, +\infty)$.
Alternatively, one can observe that $0 \leq \sqrt{x+2} < +\infty$ implies $-1 \leq \sqrt{x+2} - 1 < +\infty$, whence $\operatorname{im} f = [-1, +\infty)$.

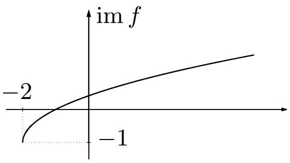


Figure 2.27. Graph of $y = \sqrt{x+2} - 1$

c) $\operatorname{im} f = (0, +\infty)$; d) $\operatorname{im} f = (-7, +\infty)$.

3. Imposing $\cos x - 1 \geq 0$ tells that $\cos x \geq 1$. Such constraint is true only for $x = 2k\pi$, $k \in \mathbb{Z}$, where the cosine equals 1; thus $\operatorname{dom} f = \{x \in \mathbb{R} : x = 2k\pi,\ k \in \mathbb{Z}\}$ and $\operatorname{im} f = \{0\}$. Fig. 2.28 provides the graph.

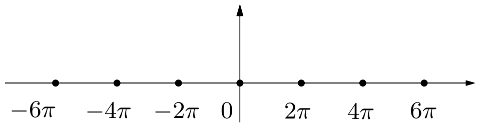


Figure 2.28. Graph of $y = \sqrt{\cos x - 1}$

4. $f^{-1}([0, +\infty)) = (1, 2]$ and $f^{-1}((-\infty, -1]) = [\mathrm{e} + 1, +\infty)$.

5. *Graphs and symmetries/periodicity:*

a) The function is even, not periodic and its graph is shown in Fig. 2.29 (top left).

b) The map is even and periodic of period π, with graph in Fig. 2.29 (top right).

c) This function is odd and periodic with period π, see Fig. 2.29 (bottom left).

d) The function has no symmetries nor a periodic behaviour, as shown in Fig. 2.29 (bottom right).

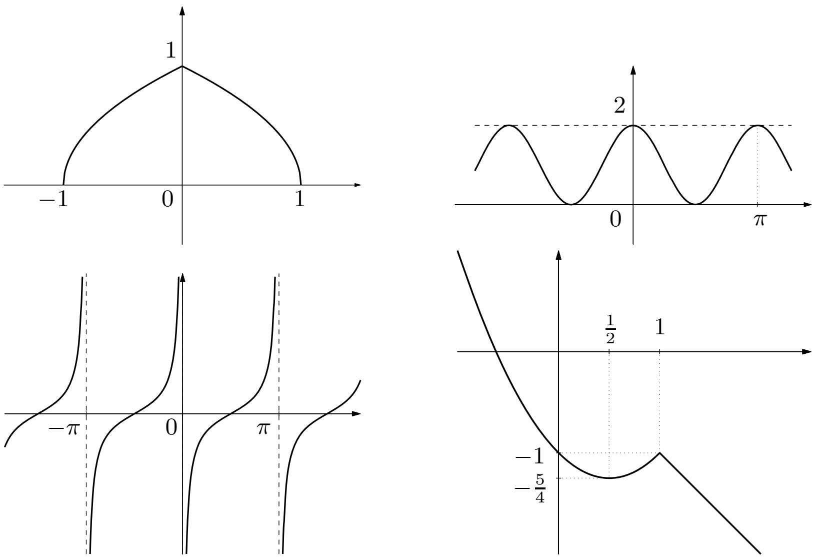


Figure 2.29. Graphs relative to Exercises 5.a) (top left), 5.b) (top right), 5.c) (bottom left) and 5.d) (bottom right)

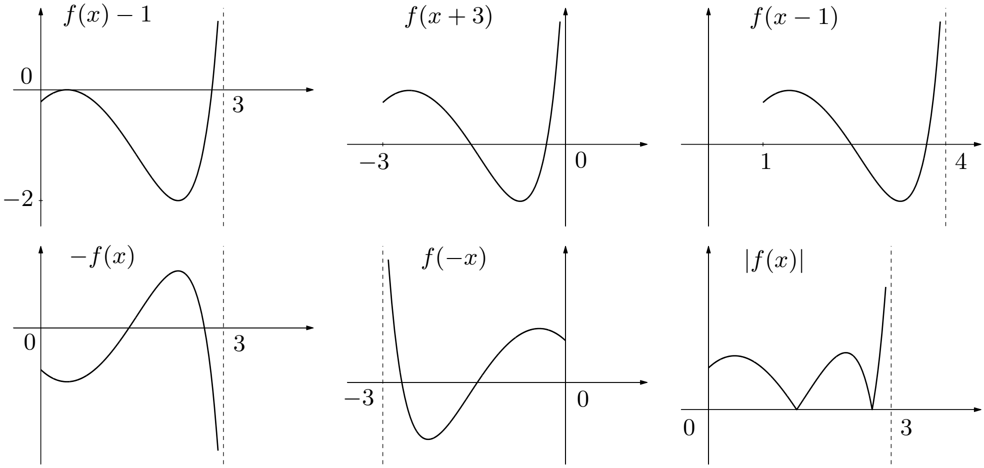


Figure 2.30. Graphs of Exercise 6

6. See Fig. 2.30.

7. The function represents a parabola with vertex $(1,4)$, and as such it is not invertible on $\mathbb{R}$, not being one-to-one (e.g., $f(0)=f(2)=5$). But restricted to the intervals $(-\infty,1]$ and $[1,+\infty)$ separately, it becomes invertible. Setting

$$f_1 = f_{|(-\infty,1]} : (-\infty,1] \to [4,+\infty)\ , \qquad f_2 = f_{|[1,+\infty)} : [1,+\infty) \to [4,+\infty)\,,$$

we can compute

$$f_1^{-1} : [4,+\infty) \to (-\infty,1]\ , \qquad f_2^{-1} : [4,+\infty) \to [1,+\infty)$$

explicitly. In fact, from $x^2-2x+5-y=0$ we obtain

$$x = 1 \pm \sqrt{y-4}.$$

With the ranges of f_1^{-1} and f_2^{-1} in mind, swapping the variables x, y yields

$$f_1^{-1}(x) = 1-\sqrt{x-4}\ , \qquad f_2^{-1}(x) = 1+\sqrt{x-4}.$$

8. Since

$$f(x) = \begin{cases} 2 & \text{if } x \le 0\,, \\ \sqrt{4-2x} & \text{if } 0 < x \le 2\,, \\ 0 & \text{if } x > 2\,, \end{cases}$$

the required interval I is $[0,2]$, and the graph of f is shown in Fig. 2.31.
In addition $f([0,2])=[0,2]$, so $f^{-1} : [0,2] \to [0,2]$. By putting $y=\sqrt{4-2x}$ we obtain $x=\frac{4-y^2}{2}$, which implies $f^{-1}(x) = 2-\frac{1}{2}x^2$.

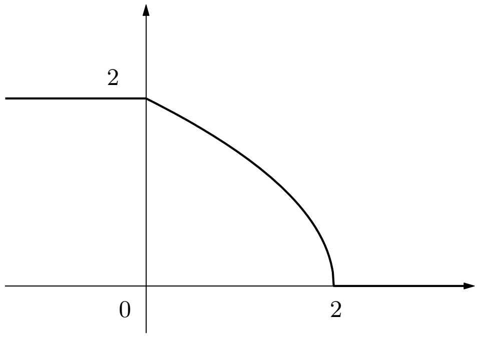


Figure 2.31. Graph of $y = \sqrt{|x-2| - |x| + 2}$

9. We have

$$f(x) = \begin{cases} 9x^2 - 1 & \text{if } 0 \le x \le 1\,, \\ 3x^2 + 4x + 1 & \text{if } x > 1 \end{cases}$$

and the graph of f is in Fig. 2.32.
The range of f is $[-1, +\infty)$. To determine f^{-1} we discuss the cases $0 \le x \le 1$ and $x > 1$ separately. For $0 \le x \le 1$, we have $-1 \le y \le 8$ and

$$y = 9x^2 - 1 \qquad \Longleftrightarrow \qquad x = \sqrt{\frac{y+1}{9}}.$$

For $x > 1$, we have $y > 8$ and

$$y = 3x^2 + 4x + 1 \qquad \Longleftrightarrow \qquad x = \frac{-2 + \sqrt{3y+1}}{3}.$$

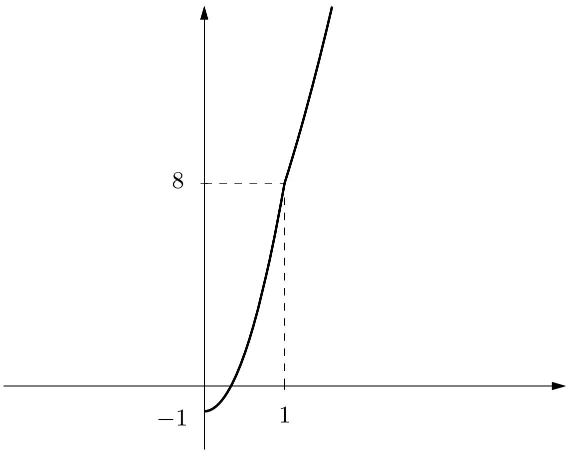


Figure 2.32. Graph of $y = (1+3x)(2x - |x-1|)$

Thus

$$f^{-1}(x) = \begin{cases} \sqrt{\dfrac{x+1}{9}} & \text{if } -1 \le x \le 8\,, \\ \dfrac{-2+\sqrt{3x+1}}{3} & \text{if } x > 8. \end{cases}$$

10. *Composite functions:*

a) As $g \circ f(x) = g(f(x)) = g(x^2-3) = \log(1+x^2-3) = \log(x^2-2)$, it follows $\operatorname{dom} g \circ f = \{x \in \mathbb{R} : x^2 - 2 > 0\} = (-\infty, -\sqrt{2}) \cup (\sqrt{2}, +\infty)$.
We have $f \circ g(x) = f(g(x)) = f(\log(1+x)) = (\log(1+x))^2 - 3$, so $\operatorname{dom} f \circ g = \{x \in \mathbb{R} : 1 + x > 0\} = (-1, +\infty)$.

b) $g \circ f(x) = \sqrt{\dfrac{2-5x}{x+1}}$ and $\operatorname{dom} g \circ f = (-1, \frac{2}{5}]$;

$f \circ g(x) = \dfrac{7\sqrt{2-x}}{\sqrt{2-x}+1}$ and $\operatorname{dom} f \circ g = (-\infty, 2]$.

11. $g(x) = \dfrac{2x+1}{x^2+2}$ and $h(x) = g \circ f(x)$.

12. After drawing the parabolic graphs $f(x)$ and $g(x)$ (Fig. 2.33), one sees that

$$h(x) = \begin{cases} x^2 - 3x + 2 & \text{if } x \le 2\,, \\ x^2 - 5x + 6 & \text{if } x > 2\,, \end{cases}$$

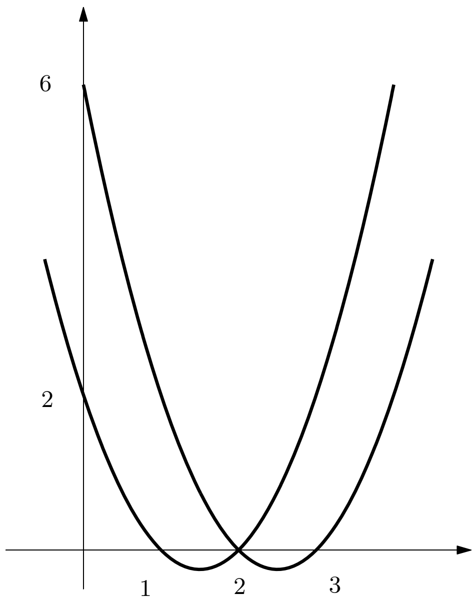


Figure 2.33. Graphs of the parabolas $f(x) = x^2 - 3x + 2$ and $g(x) = x^2 - 5x + 6$

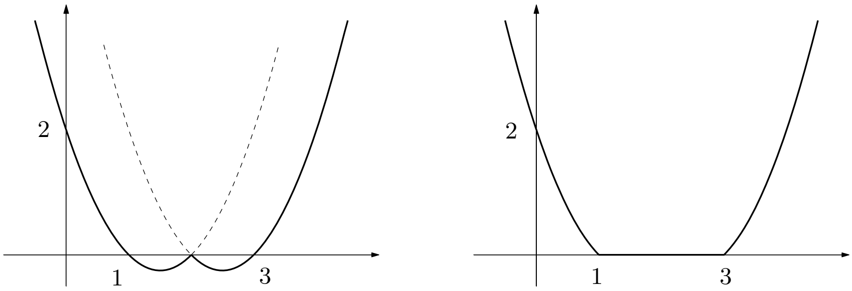


Figure 2.34. Graphs of the maps h (left) and k (right) relative to Exercise 12

and the graph of h is that of Fig. 2.34, left.
Proceeding as above,

$$k(x) = \begin{cases} x^2 - 3x + 2 & \text{if } x \le 1\,, \\ 0 & \text{if } 1 < x < 3\,, \\ x^2 - 5x + 6 & \text{if } x \ge 3\,, \end{cases}$$

and k has a graph as in Fig. 2.34, right.

3

Limits and continuity I

This chapter tackles the limit behaviour of a real sequence or a function of one real variable, and studies the continuity of such a function.

3.1 Neighbourhoods

The process of defining limits and continuity leads to consider real numbers which are 'close' to a certain real number. In equivalent geometrical jargon, one considers points on the real line 'in the proximity' of a given point. Let us begin by making mathematical sense of the notion of neighbourhood of a point.

> **Definition 3.1** *Let $x_0 \in \mathbb{R}$ be a point on the real line, and $r > 0$ a real number. We call* **neighbourhood of x_0 of radius** r *the open and bounded interval*
> $$I_r(x_0) = (x_0 - r, x_0 + r) = \{x \in \mathbb{R} \ : \ |x - x_0| < r\}.$$

Hence, the neighbourhood of 2 of radius 10^{-1}, denoted $I_{10^{-1}}(2)$, is the set of real numbers lying between 1.9 and 2.1, these excluded. By understanding the quantity $|x - x_0|$ as the Euclidean **distance** between the points x_0 and x, we can then say that $I_r(x_0)$ consists of the points on the real line whose distance from x_0 is less than r. If we interpret $|x - x_0|$ as the **tolerance** in the approximation of x_0 by x, then $I_r(x_0)$ becomes the set of real numbers approximating x_0 with a better margin of precision than r.

Figure 3.1. Neighbourhood of x_0 of radius r

Varying r in the set of positive real numbers, while mantaining x_0 in $\mathbb{R}$ fixed, we obtain a **family of neighbourhoods** of x_0. Each neighbourhood is a proper subset of any other in the family that has bigger radius, and in turn it contains all neighbourhoods of lesser radius.

Remark 3.2 The notion of neighbourhood of a point $x_0 \in \mathbb{R}$ is nothing but a particular case of the analogue for a point in the Cartesian product $\mathbb{R}^d$ (hence the plane if $d = 2$, space if $d = 3$), presented in Definition 8.11.

The upcoming definitions of limit and continuity, based on the idea of neighbourhood, can be stated directly for functions on $\mathbb{R}^d$, by considering functions of one real variable as subcases for $d = 1$. We prefer to follow a more gradual approach, so we shall examine first the one-dimensional case. Sect. 8.5 will be devoted to explaining how all this generalises to several dimensions. □

It is also convenient to include the case where x_0 is one of the points $+\infty$ or $-\infty$.

Definition 3.3 *For any real* $a \geq 0$*, we call* **neighbourhood of** $+\infty$ **with end-point** a *the open, unbounded interval*

$$I_a(+\infty) = (a, +\infty).$$

Similarly, a **neighbourhood of** $-\infty$ **with end-point** $-a$ *will be defined as*

$$I_a(-\infty) = (-\infty, -a).$$

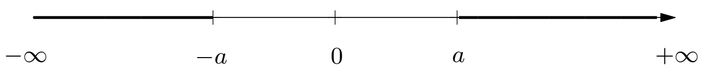


Figure 3.2. Neighbourhoods of $-\infty$ (left) and $+\infty$ (right)

The following convention will be useful in the sequel. We shall say that the property $P(x)$ holds 'in a neighbourhood' of a point c (c being a real number x_0 or $+\infty$, $-\infty$) if there is a certain neighbourhood of c such that for each of its points x, $P(x)$ holds. Colloquially, one also says '$P(x)$ holds around c', especially when the neighbourhood needs not to be specified. For example, the map $f(x) = 2x - 1$ is positive in a neighbourhood of $x_0 = 1$; in fact, $f(x) > 0$ for any $x \in I_{1/2}(1)$.

3.2 Limit of a sequence

Consider a real sequence $a : n \mapsto a_n$. We are interested in studying the behaviour of the values a_n as n increases, and we do so by looking first at a couple of examples.

Examples 3.4

i) Let $a_n = \dfrac{n}{n+1}$. The first terms of this sequence are presented in Table 3.1. We see that the values approach 1 as n increases. More precisely, the real number 1 can be approximated as well as we like by a_n for n sufficiently large. This clause is to be understood in the following sense: however small we fix $\varepsilon > 0$, from a certain point n_ε onwards all values a_n approximate 1 with a margin smaller that ε.
The condition $|a_n - 1| < \varepsilon$, in fact, is tantamount to $\dfrac{1}{n+1} < \varepsilon$, i.e., $n+1 > \dfrac{1}{\varepsilon}$; thus defining $n_\varepsilon = \left[\dfrac{1}{\varepsilon}\right]$ and taking any natural number $n > n_\varepsilon$, we have $n+1 > \left[\dfrac{1}{\varepsilon}\right] + 1 > \dfrac{1}{\varepsilon}$, hence $|a_n - 1| < \varepsilon$. In other words, for every $\varepsilon > 0$, there exists an n_ε such that

$$n > n_\varepsilon \qquad \Rightarrow \qquad |a_n - 1| < \varepsilon.$$

Looking at the graph of the sequence (Fig. 3.3), one can say that for all $n > n_\varepsilon$ the points (n, a_n) of the graph lie between the horizontal lines $y = 1 - \varepsilon$ and $y = 1 + \varepsilon$.

n	a_n
0	0.00000000000000
1	0.50000000000000
2	0.66666666666667
3	0.75000000000000
4	0.80000000000000
5	0.83333333333333
6	0.85714285714286
7	0.87500000000000
8	0.88888888888889
9	0.90000000000000
10	0.90909090909090
100	0.99009900990099
1000	0.99900099900100
10000	0.99990000999900
100000	0.99999000010000
1000000	0.99999900000100
10000000	0.99999990000001
100000000	0.99999999000000

n	a_n
1	2.0000000000000
2	2.2500000000000
3	2.3703703703704
4	2.4414062500000
5	2.4883200000000
6	2.5216263717421
7	2.5464996970407
8	2.5657845139503
9	2.5811747917132
10	2.5937424601000
100	2.7048138294215
1000	2.7169239322355
10000	2.7181459268244
100000	2.7182682371975
1000000	2.7182804691564
10000000	2.7182816939804
100000000	2.7182817863958

Table 3.1. Values, estimated to the 14th digit, of the sequences $a_n = \frac{n}{n+1}$ (left) and $a_n = \left(1 + \frac{1}{n}\right)^n$ (right)

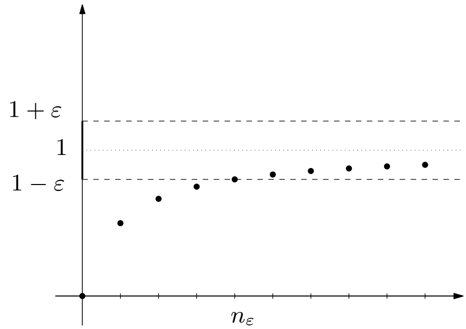


Figure 3.3. Convergence of the sequence $a_n = \frac{n}{n+1}$

ii) The first values of the sequence $a_n = \left(1+\dfrac{1}{n}\right)^n$ are shown in Table 3.1. One could imagine, even expect, that as n increases the values a_n get closer to a certain real number, whose decimal expansion starts as $2.718\ldots$ This is actually the case, and we shall return to this important example later. □

We introduce the notion of converging sequence. For simplicity we shall assume the sequence is defined on the set $\{n \in \mathbb{N} : n \geq n_0\}$ for a suitable $n_0 \geq 0$.

Definition 3.5 *A sequence $a : n \mapsto a_n$* **converges to the limit** $\ell \in \mathbb{R}$ *(or* **converges to** ℓ *or* **has limit** ℓ*), in symbols*

$$\lim_{n\to\infty} a_n = \ell,$$

if for any real number $\varepsilon > 0$ there exists an integer n_ε such that

$$\forall n \geq n_0, \qquad n > n_\varepsilon \quad \Rightarrow \quad |a_n - \ell| < \varepsilon.$$

Using the language of neighbourhoods, the condition $n > n_\varepsilon$ can be written $n \in I_{n_\varepsilon}(+\infty)$, while $|a_n - \ell| < \varepsilon$ becomes $a_n \in I_\varepsilon(\ell)$. Therefore, the definition of convergence to a limit is equivalent to: for any neighbourhood $I_\varepsilon(\ell)$ of ℓ, there exists a neighbourhood $I_{n_\varepsilon}(+\infty)$ of $+\infty$ such that

$$\forall n \geq n_0, \qquad n \in I_{n_\varepsilon}(+\infty) \quad \Rightarrow \quad a_n \in I_\varepsilon(\ell).$$

Examples 3.6

i) Referring to Example 3.4 i), we can say

$$\lim_{n\to\infty} \frac{n}{n+1} = 1.$$

ii) Let us check that

$$\lim_{n\to\infty} \frac{3n}{2+5n^2} = 0.$$

Given $\varepsilon > 0$, we must show

$$\left|\frac{3n}{2+5n^2}\right| < \varepsilon$$

for all n greater than a suitable natural number n_ε. Observing that for $n \geq 1$

$$\left|\frac{3n}{2+5n^2}\right| = \frac{3n}{2+5n^2} < \frac{3n}{5n^2} = \frac{3}{5n},$$

we have

$$\frac{3}{5n} < \varepsilon \quad \Rightarrow \quad \left|\frac{3n}{2+5n^2}\right| < \varepsilon.$$

But

$$\frac{3}{5n} < \varepsilon \quad \Longleftrightarrow \quad n > \frac{3}{5\varepsilon},$$

so we can set $n_\varepsilon = \left[\frac{3}{5\varepsilon}\right]$. □

Let us examine now a different behaviour as n increases. Consider for instance the sequence

$$a : n \mapsto a_n = n^2.$$

Its first few values are written in Table 3.2. Not only the values seem not to converge to any finite limit ℓ, they are not even bounded from above: however large we choose a real number $A > 0$, if n is big enough (meaning larger than a suitable n_A), a_n will be bigger than A. In fact, it is sufficient to choose $n_A = [\sqrt{A}]$ and note

$$n > n_A \ \Rightarrow n > \sqrt{A} \ \Rightarrow n^2 > A.$$

One says that the sequence diverges to $+\infty$ when that happens.

n	a_n
0	0
1	1
2	4
3	9
4	16
5	25
6	36
7	49
8	64
9	81
10	100
100	10000
1000	1000000
10000	100000000
100000	10000000000

Table 3.2. Values of $a_n = n^2$

In general the notion of divergent sequence is defined as follows.

Definition 3.7 *The sequence* $a : n \mapsto a_n$ **tends to** $+\infty$ (*or* **diverges to** $+\infty$, *or* **has limit** $+\infty$), *written*

$$\lim_{n\to\infty} a_n = +\infty,$$

if for any real $A > 0$ *there exists an* n_A *such that*

$$\forall n \geq n_0, \qquad n > n_A \quad \Rightarrow \quad a_n > A. \tag{3.1}$$

Using neighbourhoods, one can also say that for any neighbourhood $I_A(+\infty)$ of $+\infty$, there is a neighbourhood $I_{n_A}(+\infty)$ of $+\infty$ satisfying

$$\forall n \geq n_0, \qquad n \in I_{n_A}(+\infty) \quad \Rightarrow \quad a_n \in I_A(+\infty).$$

The definition of

$$\lim_{n\to\infty} a_n = -\infty$$

is completely analogous, with the proviso that the implication of (3.1) is changed to

$$\forall n \geq n_0, \qquad n > n_A \quad \Rightarrow \quad a_n < -A.$$

Examples 3.8

i) From what we have seen it is clear that

$$\lim_{n\to\infty} n^2 = +\infty.$$

ii) The sequence $a_n = 0 + 1 + 2 + \ldots + n = \displaystyle\sum_{k=0}^{n} k$ associates to n the sum of the natural numbers up to n. To determine the limit we show first of all that

$$\sum_{k=0}^{n} k = \frac{n(n+1)}{2}\,, \tag{3.2}$$

a relation with several uses in Mathematics. For that, note that a_n can also be written as $a_n = n + (n-1) + \ldots + 2 + 1 + 0 = \displaystyle\sum_{k=0}^{n} (n-k)$, hence

$$2a_n = \sum_{k=0}^{n} k + \sum_{k=0}^{n} (n-k) = \sum_{k=0}^{n} n = n\sum_{k=0}^{n} 1 = n(n+1)\,,$$

and the claim follows. Let us verify $\displaystyle\lim_{n\to\infty} \frac{n(n+1)}{2} = +\infty$. Since $\dfrac{n(n+1)}{2} > \dfrac{n^2}{2}$, we can proceed as in the example above, so for a given $A > 0$, we may choose $n_A = [\sqrt{2A}]$ □

The previous examples show that some sequences are **convergent**, other **divergent** (to $+\infty$ or $-\infty$). But if neither of these cases occurs, one says that the sequence is **indeterminate**. Such are for instance the sequence $a_n = (-1)^n$, which we have already met, or

$$a_n = \big(1 + (-1)^n\big)\, n = \begin{cases} 2n & \text{for } n \text{ even}, \\ 0 & \text{for } n \text{ odd}. \end{cases}$$

A sufficient condition to avoid an indeterminate behaviour is **monotonicity**. The definitions concerning monotone functions, given in Sect. 2.4, apply to sequences, as well, which are nothing but particular functions defined over the natural numbers. For them they become particularly simple: it will be enough to compare the values for *all* pairs of subscripts n, $n+1$ belonging to the domain of the sequence. So, a sequence is **monotone increasing** if

$$\forall n \geq n_0, \qquad a_n \leq a_{n+1},$$

the other definitions being analogous. The following result holds.

Theorem 3.9 *A monotone sequence $a : n \mapsto a_n$ is either convergent or divergent. Precisely, in case a_n is increasing:*

i) if the sequence is bounded from above, i.e., there is an upper bound $b \in \mathbb{R}$ such that $a_n \leq b$ for all $n \geq n_0$, then the sequence **converges** *to the supremum ℓ of its image:*

$$\lim_{n\to\infty} a_n = \ell = \sup\{a_n \ : \ n \geq n_0\};$$

ii) if the sequence is not bounded from above, then it diverges to $+\infty$.

In case the sequence is decreasing, the assertions modify in the obvious way.

Proof. Assume first that $\{a_n\}$ is bounded from above, which is to say that $\ell = \sup\{a_n \, : \, n \geq n_0\} \in \mathbb{R}$. Due to conditions (1.7), for any $\varepsilon > 0$ there exists an element a_{n_ε} such that $\ell - \varepsilon < a_{n_\varepsilon} \leq \ell$. As the sequence is monotone, $a_{n_\varepsilon} \leq a_n$, $\forall n \geq n_\varepsilon$; moreover, $a_n \leq \ell$, $\forall n \geq n_0$ by definition of the supremum. Therefore

$$\ell - \varepsilon < a_n \leq \ell < \ell + \varepsilon\,, \qquad \forall n \geq n_\varepsilon\,,$$

hence each term a_n with $n \geq n_\varepsilon$ belongs to the neighbourhood of ℓ of radius ε. But this is precisely the meaning of

$$\lim_{n\to\infty} a_n = \ell\,.$$

Let now $\ell = +\infty$. Put differently, for any $A > 0$ there exists an element a_{n_A} so that $a_{n_A} > A$. Monotonicity implies $a_n \geq a_{n_A} > A$, $\forall n \geq n_A$. Thus every a_n with $n \geq n_A$ belongs to the neighbourhood $I_A(+\infty) = (A, +\infty)$ of $+\infty$, i.e.,

$$\lim_{n\to\infty} a_n = +\infty\,.$$

□

Example 3.10

Let us go back to Example 3.4 i). The sequence $a_n = \dfrac{n}{n+1}$ is strictly increasing, for $a_n < a_{n+1}$, i.e., $\dfrac{n}{n+1} < \dfrac{n+1}{n+2}$, is equivalent to $n(n+2) < (n+1)^2$, hence $n^2 + 2n < n^2 + 2n + 1$, which is valid for any n.

Moreover, $a_n < 1$ for all $n \geq 0$; actually, 1 is the supremum of the set $\{a_n : n \in \mathbb{N}\}$, as remarked in Sect. 1.3.1. Theorem 3.9 recovers the already known result $\lim\limits_{n\to\infty} a_n = 1$. □

The number e

Consider the sequence $a_n = \left(1 + \dfrac{1}{n}\right)^n$ introduced in Example 3.4 ii). It is possible to prove that it is a strictly increasing sequence (hence in particular $a_n > 2 = a_1$ for any $n > 1$) and that it is bounded from above ($a_n < 3$ for all n). Thus Theorem 3.9 ensures that the sequence converges to a limit between 2 and 3, which traditionally is indicated by the symbol e:

$$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = \mathrm{e}. \tag{3.3}$$

This number, sometimes called **Napier's number** or **Euler's number**, plays a role of the foremost importance in Mathematics. It is an irrational number, whose first decimal digits are

$$\mathrm{e} = 2.71828182845905\cdots$$

Proofs of the stated properties are given in Appendix A.2.3, p. 437.

The number e is one of the most popular bases for exponentials and logarithms. The exponential function $y = \mathrm{e}^x$ shall sometimes be denoted by $y = \exp x$. The logarithm in base e is called **natural logarithm** and denoted by log or ln, instead of $\log_{\mathrm{e}}$ (for the base-10 logarithm, or decimal logarithm, one uses the capitalised symbol Log).

3.3 Limits of functions; continuity

Let f be a real function of real variable. We wish to describe the behaviour of the dependent variable $y = f(x)$ when the independent variable x 'approaches' a certain point $x_0 \in \mathbb{R}$, or one of the points at infinity $-\infty$, $+\infty$. We start with the latter case for conveniency, because we have already studied what sequences do at infinity.

3.3.1 Limits at infinity

Suppose f is defined around $+\infty$. In analogy to sequences we have some definitions.

Definition 3.11 *The function f* **tends to the limit $\ell \in \mathbb{R}$ for x going to** $+\infty$*, in symbols*

$$\lim_{x \to +\infty} f(x) = \ell,$$

if for any real number $\varepsilon > 0$ there is a real $B \geq 0$ such that

$$\forall x \in \operatorname{dom} f, \qquad x > B \quad \Rightarrow \quad |f(x) - \ell| < \varepsilon. \tag{3.4}$$

This condition requires that for any neighbourhood $I_\varepsilon(\ell)$ of ℓ, there exists a neighbourhood $I_B(+\infty)$ of $+\infty$ such that

$$\forall x \in \operatorname{dom} f, \qquad x \in I_B(+\infty) \quad \Rightarrow \quad f(x) \in I_\varepsilon(\ell).$$

Definition 3.12 *The function f* **tends to $+\infty$ for x going to** $+\infty$*, in symbols*

$$\lim_{x \to +\infty} f(x) = +\infty,$$

if for each real $A > 0$ there is a real $B \geq 0$ such that

$$\forall x \in \operatorname{dom} f, \qquad x > B \quad \Rightarrow \quad f(x) > A. \tag{3.5}$$

For functions tending to $-\infty$ one should replace $f(x) > A$ by $f(x) < -A$. The expression

$$\lim_{x \to +\infty} f(x) = \infty$$

means $\lim\limits_{x \to +\infty} |f(x)| = +\infty$.

If f is defined around $-\infty$, Definitions 3.11 and 3.12 modify to become definitions of limit (L, finite or infinite) for x going to $-\infty$, by changing $x > B$ to $x < -B$:

$$\lim_{x \to -\infty} f(x) = L.$$

At last, by

$$\lim_{x \to \infty} f(x) = L$$

one intends that f has limit L (finite or not) both for $x \to +\infty$ and $x \to -\infty$.

Examples 3.13

i) Let us check that

$$\lim_{x\to+\infty} \frac{x^2+2x}{2x^2+1} = \frac{1}{2}.$$

Given $\varepsilon > 0$, the condition $|f(x) - \frac{1}{2}| < \varepsilon$ is equivalent to

$$\left| \frac{4x-1}{2(2x^2+1)} \right| < \varepsilon.$$

Without loss of generality we assume $x > \frac{1}{4}$, so that the absolute value sign can be removed. Using simple properties of fractions

$$\frac{4x-1}{2(2x^2+1)} < \frac{2x}{2x^2+1} < \frac{2x}{2x^2} = \frac{1}{x} < \varepsilon \qquad \text{if } x > \frac{1}{\varepsilon}.$$

Thus (3.4) holds for $B = \max\left(\frac{1}{4}, \frac{1}{\varepsilon}\right)$.

ii) We prove

$$\lim_{x\to+\infty} \sqrt{x} = +\infty.$$

Let $A > 0$ be fixed. Since $\sqrt{x} > A$ implies $x > A^2$, putting $B = A^2$ fulfills (3.5).

iii) Consider

$$\lim_{x\to-\infty} \frac{1}{\sqrt{1-x}} = 0.$$

With $\varepsilon > 0$ fixed,

$$\left| \frac{1}{\sqrt{1-x}} \right| = \frac{1}{\sqrt{1-x}} < \varepsilon$$

is tantamount to $\sqrt{1-x} > \frac{1}{\varepsilon}$, that is $1 - x > \frac{1}{\varepsilon^2}$, or $x < 1 - \frac{1}{\varepsilon^2}$. So taking $B = \max\left(0, \frac{1}{\varepsilon^2} - 1\right)$, we have

$$x < -B \quad \Rightarrow \quad \left| \frac{1}{\sqrt{1-x}} \right| < \varepsilon. \qquad \square$$

3.3.2 Continuity. Limits at real points

We now investigate the behaviour of the values $y = f(x)$ of a function f when x 'approaches' a point $x_0 \in \mathbb{R}$. Suppose f is defined in a neighbourhood of x_0, but not necessarily at the point x_0 itself. Two examples will let us capture the essence of the notions of continuity and finite limit. Fix $x_0 = 0$ and consider the real functions of real variable $f(x) = x^3 + 1$, $g(x) = x + [1 - x^2]$ and $h(x) = \dfrac{\sin x}{x}$ (recall that $[z]$ indicates the integer part of z); their respective graphs, at least in a neighbourhood of the origin, are presented in Fig. 3.4 and 3.5.

As far as g is concerned, we observe that $|x| < 1$ implies $0 < 1 - x^2 \le 1$ and g assumes the value 1 only at $x = 0$; in the neighbourhood of the origin of unit radius then,

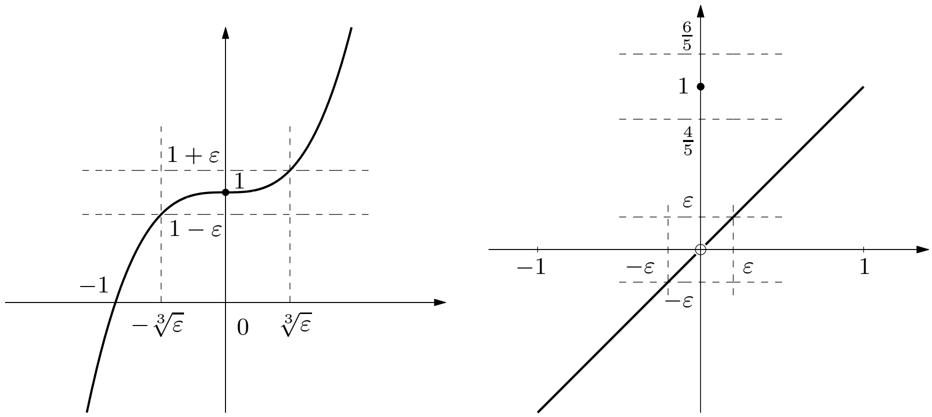


Figure 3.4. Graphs of $f(x) = x^3 + 1$ (left) and $g(x) = x + [1 - x^2]$ (right), in a neighbourhood of the origin

$$g(x) = \begin{cases} 1 & \text{if } x = 0\,, \\ x & \text{if } x \neq 0\,, \end{cases}$$

as the picture shows. Note the function h is not defined in the origin.

For each of f and g, let us compare the values at points x near the origin with the actual value at the origin. The two functions behave rather differently. The value $f(0) = 1$ can be approximated as well as we like by any $f(x)$, provided x is close enough to 0. Precisely, having fixed an (arbitrarily small) 'error' $\varepsilon > 0$ in advance, we can make $|f(x) - f(0)|$ smaller than ε for *all* x such that $|x - 0| = |x|$ is smaller than a suitable real $\delta > 0$. In fact $|f(x) - f(0)| = |x^3| = |x|^3 < \varepsilon$ means $|x| < \sqrt[3]{\varepsilon}$, so it is sufficient to choose $\delta = \sqrt[3]{\varepsilon}$. We shall say that the function f is continuous at the origin.

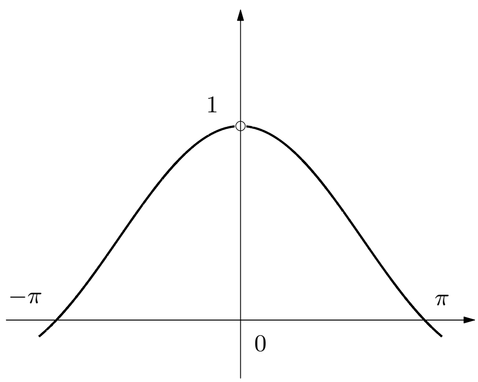


Figure 3.5. Graph of $h(x) = \dfrac{\sin x}{x}$ around the origin

On the other hand, $g(0) = 1$ cannot be approximated well by any $g(x)$ with x close to 0. For instance, let $\varepsilon = \frac{1}{5}$. Then $|g(x) - g(0)| < \varepsilon$ is equivalent to $\frac{4}{5} < g(x) < \frac{6}{5}$; but all x *different from* 0 and such that, say, $|x| < \frac{1}{2}$, satisfy $-\frac{1}{2} < g(x) = x < \frac{1}{2}$, in violation to the constraint for $g(x)$. The function g is not continuous at the origin.

At any rate, we can specify the behaviour of g around 0: for x closer and closer to 0, *yet different from* 0, the images $g(x)$ approximate not the value $g(0)$, but rather $\ell = 0$. In fact, with $\varepsilon > 0$ fixed, if $x \neq 0$ satisfies $|x| < \min(\varepsilon, 1)$, then $g(x) = x$ and $|g(x) - \ell| = |g(x)| = |x| < \varepsilon$. We say that g has limit 0 for x going to 0.

As for the function h, it cannot be continuous at the origin, since comparing the values $h(x)$, for x near 0, with the value at the origin simply makes no sense, for the latter is not even defined. Neverthless, the graph allows to 'conjecture' that these values might estimate $\ell = 1$ increasingly better, the closer we choose x to the origin. We are lead to say h has a limit for x going to 0, and this limit is 1. We shall substantiate this claim later on.

The examples just seen introduce us to the definition of continuity and of (finite) limit.

Definition 3.14 *Let x_0 be a point in the domain of a function f. This function is called* **continuous at** x_0 *if for any $\varepsilon > 0$ there is a $\delta > 0$ such that*

$$\forall x \in \operatorname{dom} f, \qquad |x - x_0| < \delta \quad \Rightarrow \quad |f(x) - f(x_0)| < \varepsilon. \tag{3.6}$$

In neighbourhood-talk: for any neighbourhood $I_\varepsilon(f(x_0))$ of $f(x_0)$ there exists a neighbourhood $I_\delta(x_0)$ of x_0 such that

$$\forall x \in \operatorname{dom} f, \qquad x \in I_\delta(x_0) \quad \Rightarrow \quad f(x) \in I_\varepsilon(f(x_0)). \tag{3.7}$$

Definition 3.15 *Let f be a function defined on a neighbourhood of $x_0 \in \mathbb{R}$, except possibly at x_0. Then f* **has limit** $\ell \in \mathbb{R}$ (*or* **tends to** ℓ *or* **converges to** ℓ) **for** x **approaching** x_0, *written*

$$\lim_{x \to x_0} f(x) = \ell,$$

if given any $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\forall x \in \operatorname{dom} f, \qquad 0 < |x - x_0| < \delta \quad \Rightarrow \quad |f(x) - \ell| < \varepsilon. \tag{3.8}$$

Alternatively: for any given neighbourhood $I_\varepsilon(\ell)$ of ℓ there is a neighbourhood $I_\delta(x_0)$ of x_0 such that

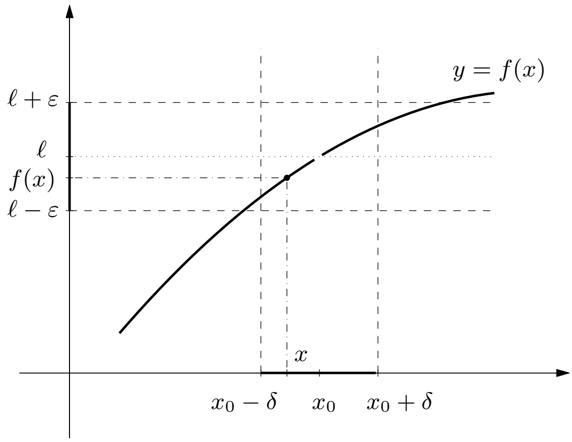


Figure 3.6. Definition of finite limit of a function

$$\forall x \in \operatorname{dom} f, \qquad x \in I_\delta(x_0) \setminus \{x_0\} \quad \Rightarrow \quad f(x) \in I_\varepsilon(\ell).$$

The definition of limit is represented in Fig. 3.6.

Let us compare the notions just seen. To have continuity one looks at the values $f(x)$ from the point of view of $f(x_0)$, whereas for limits these $f(x)$ are compared to ℓ, which *could* be different from $f(x_0)$, provided f is defined in x_0. To test the limit, moreover, the comparison with $x = x_0$ is excluded: requiring $0 < |x - x_0|$ means exactly $x \neq x_0$; on the contrary, the implication (3.6) is obviously true for $x = x_0$.

Let f be defined in a neighbourhood of x_0. If f is continuous at x_0, then (3.8) is certainly true with $\ell = f(x_0)$; vice versa if f has limit $\ell = f(x_0)$ for x going to x_0, then (3.6) holds. Thus the continuity of f at x_0 is tantamount to

$$\lim_{x \to x_0} f(x) = f(x_0). \tag{3.9}$$

In both definitions, after fixing an arbitrary $\varepsilon > 0$, one is asked to find *at least* one positive number δ ('there is a δ') for which (3.6) or (3.8) holds. If either implication holds for a certain δ, it will also hold for every $\delta' < \delta$. The definition does not require to find the biggest possible δ satisfying the implication. With this firmly in mind, testing continuity or verifying a limit can become much simpler.

Returning to the functions f, g, h of the beginning, we can now say that f is continuous at $x_0 = 0$,

$$\lim_{x \to 0} f(x) = 1 = f(0),$$

whereas g, despite having limit 0 for $x \to 0$, is not continuous:

$$\lim_{x \to 0} g(x) = 0 \neq g(0).$$

We shall prove in Example 4.6 i) that h admits a limit for x going to 0, and actually

$$\lim_{x\to 0} h(x) = 1.$$

The functions g and h suggest the following definition.

Definition 3.16 *Let f be defined on a neighbourhood of x_0, excluding the point x_0. If f admits limit $\ell \in \mathbb{R}$ for x approaching x_0, and if a) f is defined in x_0 but $f(x_0) \neq \ell$, or b) f is not defined in x_0, then we say x_0 is a* **(point of) removable discontinuity for** *f.*

The choice of terminology is justified by the fact that one can *modify* the function at x_0 by *defining* it in x_0, so that to obtain a continuous map at x_0. More precisely, the function

$$\tilde{f}(x) = \begin{cases} f(x) & \text{if } x \neq x_0, \\ \ell & \text{if } x = x_0, \end{cases}$$

is such that

$$\lim_{x\to x_0} \tilde{f}(x) = \lim_{x\to x_0} f(x) = \ell = \tilde{f}(x_0),$$

hence it is continuous at x_0.

For the above functions we have $\tilde{g}(x) = x$ in a neighbourhood of the origin, while

$$\tilde{h}(x) = \begin{cases} \dfrac{\sin x}{x} & \text{if } x \neq 0, \\ 1 & \text{if } x = 0. \end{cases}$$

In the latter case, we have defined the **continuous prolongation** of $y = \dfrac{\sin x}{x}$, by assigning the value that renders it continuous at the origin. From now on when referring to the function $y = \dfrac{\sin x}{x}$, we will always understand it as continuously prolonged in the origin.

Examples 3.17

We show that the main elementary functions are continuous.

i) Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = ax + b$ and $x_0 \in \mathbb{R}$ be given. For any $\varepsilon > 0$, $|f(x) - f(x_0)| < \varepsilon$ if and only if $|a|\,|x - x_0| < \varepsilon$. When $a = 0$, the condition holds for any $x \in \mathbb{R}$; if $a \neq 0$ instead, it is equivalent to $|x - x_0| < \dfrac{\varepsilon}{|a|}$, and we can put $\delta = \dfrac{\varepsilon}{|a|}$ in (3.6). The map f is thus continuous at every $x_0 \in \mathbb{R}$.

ii) The function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$ is continuous at $x_0 = 2$. We shall prove this fact in two different ways. Given $\varepsilon > 0$, $|f(x) - f(2)| < \varepsilon$, or $|x^2 - 4| < \varepsilon$, means

$$4 - \varepsilon < x^2 < 4 + \varepsilon. \tag{3.10}$$

We can suppose $\varepsilon \le 4$ (for if $|f(x) - f(2)| < \varepsilon$ for a certain ε, the same will be true for all $\varepsilon' > \varepsilon$); as we are looking for x in a neighbourhood of 2, we can furthermore assume $x > 0$. Under such assumptions (3.10) yields

$$\sqrt{4-\varepsilon} < x < \sqrt{4+\varepsilon},$$

hence

$$-(2-\sqrt{4-\varepsilon}) < x - 2 < \sqrt{4+\varepsilon} - 2. \tag{3.11}$$

This suggests to take $\delta = \min(2-\sqrt{4-\varepsilon}, \sqrt{4+\varepsilon}-2)$ ($= \sqrt{4+\varepsilon}-2$, easy to verify). If $|x-2| < \delta$, then (3.11) holds, which was equivalent to $|x^2-4| < \varepsilon$. With a few algebraic computations, this furnishes the *greatest* δ for which the inequality $|x^2-4| < \varepsilon$ is true.

We have already said that the largest value of δ is not required by the definitions, so we can also proceed alternatively. Since

$$|x^2-4| = |(x-2)(x+2)| = |x-2|\,|x+2|\,,$$

by restricting x to a neighbourhood of 2 of radius < 1, we will have $-1 < x-2 < 1$, hence $1 < x < 3$. The latter will then give $3 < x+2 = |x+2| < 5$. Thus

$$|x^2-4| < 5|x-2|. \tag{3.12}$$

To obtain $|x^2-4| < \varepsilon$ it will suffice to demand $|x-2| < \dfrac{\varepsilon}{5}$; since (3.12) holds when $|x-2| < 1$, we can set $\delta = \min\left(1, \dfrac{\varepsilon}{5}\right)$ and the condition (3.6) will be satisfied. The neighbourhood of radius < 1 was arbitrary: we could have chosen any other sufficiently small neighbourhood and obtain another δ, still respecting the continuity requirement.

Note at last that a similar reasoning tells f is continuous at every $x_0 \in \mathbb{R}$.

iii) We verify that $f : \mathbb{R} \to \mathbb{R},\ f(x) = \sin x$ is continuous at every $x_0 \in \mathbb{R}$. We establish first a simple but fundamental inequality.

Lemma 3.18 *For any $x \in \mathbb{R}$,*

$$|\sin x| \le |x|, \tag{3.13}$$

with equality holding if and only if $x = 0$.

Proof. Let us start assuming $0 < x \le \frac{\pi}{2}$ and look at the right-angled triangle PHA of Fig. 3.7. The vertical side PH is shorter than the hypotenuse PA, whose length is in turn less than the length of the arc PA (the shortest distance between two points is given by the straight line joining them):

$$\overline{PH} < \overline{PA} < \overset{\frown}{PA}\ .$$

By definition $\overline{PH} = \sin x > 0$, and $\overset{\frown}{PA} = x > 0$ (angles being in radians). Thus (3.13) is true. The case $-\frac{\pi}{2} \le x < 0$ is treated with the same

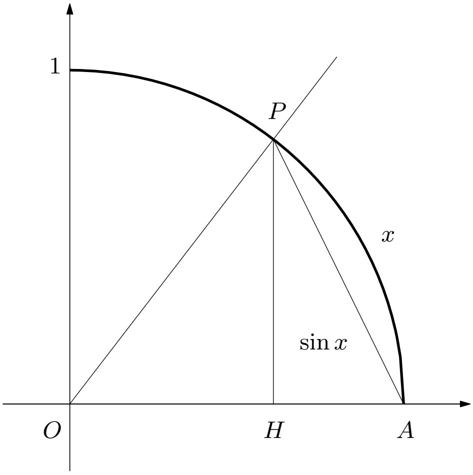


Figure 3.7. $|\sin x| \le |x|$

argument observing $|\sin x| = \sin|x|$ for $0 < |x| \le \frac{\pi}{2}$. At last, when $|x| > \frac{\pi}{2}$ one has $|\sin x| \le 1 < \frac{\pi}{2} < |x|$, ending the proof. □

Thanks to (3.13) we can prove that sine is a continuous function. Recalling formula (2.14),

$$\sin x - \sin x_0 = 2 \sin \frac{x - x_0}{2} \cos \frac{x + x_0}{2},$$

(3.13) and the fact that $|\cos t| \le 1$ for all $t \in \mathbb{R}$, imply

$$\begin{aligned} |\sin x - \sin x_0| &= 2 \left| \sin \frac{x - x_0}{2} \right| \left| \cos \frac{x + x_0}{2} \right| \\ &\le 2 \left| \frac{x - x_0}{2} \right| \cdot 1 = |x - x_0|. \end{aligned}$$

Therefore, given an $\varepsilon > 0$, if $|x - x_0| < \varepsilon$ we have $|\sin x - \sin x_0| < \varepsilon$; in other words, condition (3.6) is satisfied by $\delta = \varepsilon$.
Similarly, formula (2.15) allows to prove $g(x) = \cos x$ is continuous at every $x_0 \in \mathbb{R}$. □

Definition 3.19 *Let I be a subset of* $\operatorname{dom} f$*. The function f is called* **continuous on** I (*or* **over** I) *if f is continuous at every point of I.*

We remark that the use of the term ‘map’ (or ‘mapping’) is very different from author to author; in some books a map is simply a function (we have adopted this convention), for others the word ‘map’ automatically assumes continuity, so attention is required when browsing the literature.

The following result is particularly relevant and will be used many times without explicit mention. For its proof, see Appendix A.2.2, p. 436.

Proposition 3.20 *All elementary functions (polynomials, rational functions, powers, trigonometric functions, exponentials and their inverses) are continuous over their entire domains.*

Let us point out that there exists a notion of continuity of a function on a subset of its domain, that is stronger than the one given in Definition 3.19; it is called *uniform continuity*. We refer to Appendix A.3.3, p. 447, for its definition and main properties.

Now back to limits. A function f defined in a neighbourhood of x_0, x_0 excluded, may assume bigger and bigger values as the independent variable x gets closer to x_0. Consider for example the function

$$f(x) = \frac{1}{(x-3)^2}$$

on $\mathbb{R} \setminus \{3\}$, and fix an arbitrarily large real number $A > 0$. Then $f(x) > A$ for all $x \neq x_0$ such that $|x - 3| < \dfrac{1}{\sqrt{A}}$. We would like to say that f *tends to* $+\infty$ for x approaching x_0; the precise definition is as follows.

Definition 3.21 *Let f be defined in a neighbourhood of $x_0 \in \mathbb{R}$, except possibly at x_0. The function f* **has limit** $+\infty$ *(or* **tends to** $+\infty$*)* **for** x **approaching** x_0*, in symbols*

$$\lim_{x \to x_0} f(x) = +\infty,$$

if for any $A > 0$ there is a $\delta > 0$ such that

$$\forall x \in \operatorname{dom} f, \qquad 0 < |x - x_0| < \delta \quad \Rightarrow \quad f(x) > A. \tag{3.14}$$

Otherwise said, for any neighbourhood $I_A(+\infty)$ of $+\infty$ there exists a neighbourhood $I_\delta(x_0)$ di x_0 such that

$$\forall x \in \operatorname{dom} f, \qquad x \in I_\delta(x_0) \setminus \{x_0\} \quad \Rightarrow \quad f(x) \in I_A(+\infty).$$

The definition of

$$\lim_{x \to x_0} f(x) = -\infty$$

follows by changing $f(x) > A$ to $f(x) < -A$.

One also writes

$$\lim_{x \to x_0} f(x) = \infty$$

to indicate $\lim_{x \to x_0} |f(x)| = +\infty$. For instance the hyperbola $f(x) = \frac{1}{x}$, with graph in Fig. 2.2, *does not admit limit* for $x \to 0$, because on each neighbourhood $I_\delta(0)$ of the origin the function assumes both arbitrarily large positive and negative values together. On the other hand, $|f(x)|$ tends to $+\infty$ when x nears 0. In fact, for fixed $A > 0$

$$\forall x \in \mathbb{R} \setminus \{0\}, \qquad |x| < \frac{1}{A} \quad \Rightarrow \quad \frac{1}{|x|} > A.$$

Hence $\lim_{x \to 0} \frac{1}{x} = \infty$.

3.3.3 One-sided limits; points of discontinuity

The previous example shows that a map may have different limit behaviours at the left and right of a point x_0. The function $f(x) = \dfrac{1}{x}$ grows indefinitely as x takes *positive* values tending to 0; at the same time it becomes smaller as x goes to 0 assuming *negative* values. Consider the graph of the mantissa $y = M(x)$ (see Fig. 2.3, p. 34) on a neighbourhood of $x_0 = 1$ of radius < 1. Then

$$M(x) = \begin{cases} x & \text{if } x < 1, \\ x - 1 & \text{if } x \geq 1. \end{cases}$$

When x approaches 1, M tends to 0 if x takes values > 1 (i.e., at the right of 1), and tends to 1 if x assumes values < 1 (at the left).

The notions of *right-hand limit* and *left-hand limit* (or simply *right limit* and *left limit*) arise from the need to understand these cases. For that, we define **right neighbourhood of** x_0 **of radius** $r > 0$ the bounded half-open interval

$$I_r^+(x_0) = [x_0, x_0 + r) = \{x \in \mathbb{R} \ : \ 0 \leq x - x_0 < r\}.$$

The **left neighbourhood of** x_0 **of radius** $r > 0$ will be, similarly,

$$I_r^-(x_0) = (x_0 - r, x_0] = \{x \in \mathbb{R} \ : \ 0 \leq x_0 - x < r\}.$$

Substituting the condition $0 < |x - x_0| < \delta$ (i.e., $x \in I_\delta(x_0) \setminus \{x_0\}$) with $0 < x - x_0 < \delta$ (i.e., $x \in I_\delta^+(x_0) \setminus \{x_0\}$) in Definitions 3.15 and 3.21 produces the corresponding definitions for **right limit of** f **for** x **tending to** x_0, otherwise said **limit of** f **for** x **approaching** x_0 **from the right** or **limit on the right**; such will be denoted by

$$\lim_{x \to x_0^+} f(x).$$

For a finite limit, this reads as follows.

Definition 3.22 *Let f be defined on a right neighbourhood of $x_0 \in \mathbb{R}$, except possibly at x_0. The function f* **has right limit** $\ell \in \mathbb{R}$ **for** $x \to x_0$, *if for every $\varepsilon > 0$ there is a $\delta > 0$ such that*

$$\forall x \in \operatorname{dom} f, \qquad 0 < x - x_0 < \delta \quad \Rightarrow \quad |f(x) - \ell| < \varepsilon.$$

Alternatively, for any neighbourhood $I_\varepsilon(\ell)$ di ℓ there exists a right neighbourhood $I_\delta^+(x_0)$ of x_0 such that

$$\forall x \in \operatorname{dom} f, \qquad x \in I_\delta^+(x_0) \setminus \{x_0\} \quad \Rightarrow \quad f(x) \in I_\varepsilon(\ell).$$

The notion of continuity on the right is analogous.

Definition 3.23 *A function f defined on a right neighbourhood of $x_0 \in \mathbb{R}$ is called* **continuous on the right at** x_0 (*or* **right-continuous**) *if*

$$\lim_{x \to x_0^+} f(x) = f(x_0).$$

If a function is only defined on a right neighbourhood of x_0, right-continuity coincides with the earlier Definition (3.6). The function $f(x) = \sqrt{x}$ for example is defined on $[0, +\infty)$, and is continuous at 0.

Limits of f from the left and left-continuity are completely similar: now one has to use left neighbourhoods of x_0; the left limit shall be denoted by

$$\lim_{x \to x_0^-} f(x).$$

The following easy-to-prove property provides a handy criterion to study limits and continuity.

Proposition 3.24 *Let f be defined in a neighbourhood of $x_0 \in \mathbb{R}$, with the possible exception of x_0. The function f has limit L (finite or infinite) for $x \to x_0$ if and only if the right and left limits of f, for $x \to x_0$, exist and equal L.*

A function f defined in a neighbourhood of x_0 is continuous at x_0 if and only if it is continuous on the right and on the left at x_0.

Returning to the previous examples, it is not hard to see

$$\lim_{x\to 0^+} \frac{1}{x} = +\infty; \qquad \lim_{x\to 0^-} \frac{1}{x} = -\infty$$

and

$$\lim_{x\to 1^+} M(x) = 0; \qquad \lim_{x\to 1^-} M(x) = 1.$$

Note $M(1) = 0$, so $\lim\limits_{x\to 1^+} M(x) = M(1)$. All this means the function $M(x)$ is continuous on the right at $x_0 = 1$ (but not left-continuous, hence neither continuous, at $x_0 = 1$).

Definition 3.25 *Let f be defined on a neighbourhood of $x_0 \in \mathbb{R}$, except possibly at x_0. If the left and right limits of f for x going to x_0 are different, we say that x_0 is a* (**point of**) **discontinuity of the first kind** (*or a* **jump point**) *for f. The* **gap** *value of f at x_0 is the difference*

$$[f]_{x_0} = \lim_{x\to x_0^+} f(x) - \lim_{x\to x_0^-} f(x).$$

Thus the mantissa has a gap $= -1$ at $x_0 = 1$ and, in general, at each point $x_0 = n \in \mathbb{Z}$.

Also the floor function $y = [x]$ jumps, at each $x_0 = n \in \mathbb{Z}$, with gap $= 1$, for

$$\lim_{x\to n^+} [x] = n; \qquad \lim_{x\to n^-} [x] = n - 1.$$

The sign function $y = \operatorname{sign}(x)$ has a jump point at $x_0 = 0$, with gap $= 2$:

$$\lim_{x\to 0^+} \operatorname{sign}(x) = 1; \qquad \lim_{x\to 0^-} \operatorname{sign}(x) = -1.$$

Definition 3.26 *A discontinuity point which is not removable, nor of the first kind is said* **of the second kind**.

This occurs for instance when f does not admit limit (neither on the left nor on the right) for $x \to x_0$. The function $f(x) = \sin\frac{1}{x}$ has no limit for $x \to 0$ (see Fig. 3.8 and the explanation in Remark 4.19).

3.3.4 Limits of monotone functions

Monotonicity affects the possible limit behaviour of a map, as the following results explain.

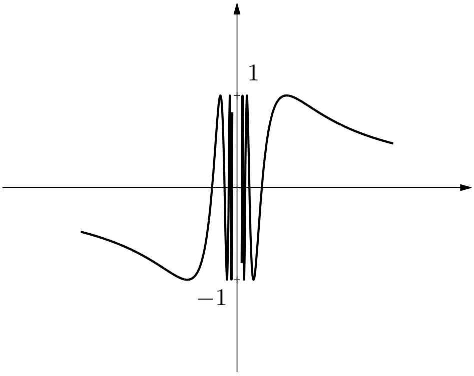


Figure 3.8. Graph of $f(x) = \sin \frac{1}{x}$

Theorem 3.27 *Let f be a monotone function defined on a right neighbourhood $I^+(c)$ of the point c (where c is real or $-\infty$), possibly without the point c itself. Then the right limit for $x \to c$ exists (finite or infinite), and precisely*

$$\lim_{x\to c^+} f(x) = \begin{cases} \inf\{f(x) : x \in I^+(c),\, x > c\} & \textit{if } f \textit{ is increasing,} \\ \sup\{f(x) : x \in I^+(c),\, x > c\} & \textit{if } f \textit{ is decreasing.} \end{cases}$$

In the same way, f monotone on a left neighbourhood $I^-(c)\setminus\{c\}$ of c (c real or $+\infty$) satisfies

$$\lim_{x\to c^-} f(x) = \begin{cases} \sup\{f(x) : x \in I^-(c),\, x < c\} & \textit{if } f \textit{ is increasing,} \\ \inf\{f(x) : x \in I^-(c),\, x < c\} & \textit{if } f \textit{ is decreasing.} \end{cases}$$

Proof. We shall prove that if f increases in the right neighbourhood $I^+(c)$ of c then

$$\lim_{x\to c^+} f(x) = \inf\{f(x) : x \in I^+(c),\, x > c\}\,.$$

The other cases are similar.

Let $\ell = \inf\{f(x) : x \in I^+(c),\, x > c\} \in \mathbb{R}$. The infimum is characterised, in analogy with (1.7), by:

i) *for all $x \in I^+(c) \setminus \{c\}$, $f(x) \geq \ell$;*

ii) *for any $\varepsilon > 0$, there exists an element $x_\varepsilon \in I^+(c)\setminus\{c\}$ such that $f(x_\varepsilon) < \ell + \varepsilon$.*

By monotonicity we have

$$f(x) \leq f(x_\varepsilon)\,, \qquad \forall x \in I^+(c) \setminus \{c\},\, x < x_\varepsilon\,,$$

therefore

$$\ell - \varepsilon < \ell \le f(x) < \ell + \varepsilon\,, \qquad \forall x \in I^+(c) \setminus \{c\},\, x < x_\varepsilon\,.$$

So, each $f(x)$ belongs to the neighbourhood of ℓ of radius ε if $x \neq c$ is in the right neighbourhood of c with supremum x_ε. Thus we have

$$\lim_{x\to c^+} f(x) = \ell\,.$$

Let now $\ell = -\infty$; this means that for any $A > 0$ there is an $x_A \in I^+(c) \setminus \{c\}$ such that $f(x_A) < -A$. Using monotonicity again we obtain $f(x) \le f(x_A) < -A$, $\forall x \in I^+(c) \setminus \{c\}$ and $x < x_A$. Hence $f(x)$ belongs to the neighbourhood of $-\infty$ with supremum $-A$ provided $x \neq c$ is in the right neighbourhood of c of supremum x_A. We conclude

$$\lim_{x\to c^+} f(x) = -\infty\,. \qquad \square$$

A straightforward consequence is that a monotone function can have only a discontinuity of the first kind.

Corollary 3.28 *Let f be monotone on a neighbourhood $I(x_0)$ of $x_0 \in \mathbb{R}$. Then the right and left limits for $x \to x_0$ exist and are finite. More precisely,*

i) if f is increasing

$$\lim_{x\to x_0^-} f(x) \le f(x_0) \le \lim_{x\to x_0^+} f(x);$$

ii) if f is decreasing

$$\lim_{x\to x_0^-} f(x) \ge f(x_0) \ge \lim_{x\to x_0^+} f(x).$$

Proof. Let f be increasing. Then for all $x \in I(x_0)$ with $x < x_0$, $f(x) \le f(x_0)$. The above theorem guarantees that

$$\lim_{x\to x_0^-} f(x) = \sup\{f(x) : x \in I(x_0),\, x < x_0\} \le f(x_0).$$

Similarly, for $x \in I(x_0)$ with $x > x_0$,

$$f(x_0) \le \inf\{f(x) : x \in I(x_0),\, x > x_0\} = \lim_{x\to x_0^+} f(x),$$

from which *i)* follows. The second implication is alike. $\square$

3.4 Exercises

1. *Using the definition prove that*

$\boxed{\text{a)}}$ $\displaystyle\lim_{n\to+\infty} n! = +\infty$ $\boxed{\text{b)}}$ $\displaystyle\lim_{n\to+\infty} \frac{n^2}{1-2n} = -\infty$

$\boxed{\text{c)}}$ $\displaystyle\lim_{x\to 1} (2x^2+3) = 5$ d) $\displaystyle\lim_{x\to 2^\pm} \frac{1}{x^2-4} = \pm\infty$

e) $\displaystyle\lim_{x\to-\infty} \frac{x}{\sqrt{x^2-1}} = -1$ f) $\displaystyle\lim_{x\to+\infty} \frac{x^2}{1-x} = -\infty$

$\boxed{2.}$ *Let* $f(x) = \operatorname{sign}(x^2-x)$. *Discuss the existence of the limits*

$$\lim_{x\to 0} f(x) \qquad \text{and} \qquad \lim_{x\to 1} f(x)$$

and study the function's continuity.

3. *Determine the values of the real parameter* α *for which the following maps are continuous on their respective domains:*

$\boxed{\text{a)}}$ $f(x) = \begin{cases} \alpha \sin(x+\frac{\pi}{2}) & \text{if } x>0\,, \\ 2x^2+3 & \text{if } x \le 0 \end{cases}$ b) $f(x) = \begin{cases} 3\mathrm{e}^{\alpha x-1} & \text{if } x \ge 1\,, \\ x+2 & \text{if } x<1 \end{cases}$

3.4.1 Solutions

1. *Limits:*

a) Let a real number $A>0$ be given; it is sufficient to choose any natural number $n_A \ge A$ and notice that if $n > n_A$ then

$$n! = n(n-1)\cdots 2\cdot 1 \ge n > n_A \ge A.$$

Thus $\displaystyle\lim_{n\to+\infty} n! = +\infty$.

b) Fix a real $A>0$ and note $\frac{n^2}{1-2n} < -A$ is the same as $\frac{n^2}{2n-1} > A$. For $n\ge 1$, that means $n^2-2An+A>0$. If we consider a natural number $n_A \ge A+\sqrt{A(A+1)}$, the inequality holds for all $n > n_A$.

c) Fix $\varepsilon>0$ and study the condition $|f(x)-\ell| < \varepsilon$:

$$|2x^2+3-5| = 2|x^2-1| = 2|x-1|\,|x+1| < \varepsilon\,.$$

Without loss of generality we assume x belongs to the neighbourhood of 1 of radius 1, i.e.,

$$-1 < x-1 < 1\,, \quad \text{whence} \quad 0 < x < 2 \quad \text{and} \quad 1 < x+1 = |x+1| < 3.$$

Therefore

$$|2x^2 + 3 - 5| < 2 \cdot 3|x-1| = 6|x-1|.$$

The expression on the right is $< \varepsilon$ if $|x-1| < \frac{\varepsilon}{6}$. It will be enough to set $\delta = \min(1, \frac{\varepsilon}{6})$ to prove the claim.

2. Since $x^2 - x > 0$ when $x < 0$ or $x > 1$, the function $f(x)$ is thus defined:

$$f(x) = \begin{cases} 1 & \text{if } x < 0 \text{ and } x > 1\,, \\ 0 & \text{if } x = 0 \text{ and } x = 1\,, \\ -1 & \text{if } 0 < x < 1\,. \end{cases}$$

So f is constant on the intervals $(-\infty, 0)$, $(0,1)$, $(1,+\infty)$ and

$$\lim_{x\to 0^-} f(x) = 1, \qquad \lim_{x\to 0^+} f(x) = -1,$$
$$\lim_{x\to 1^-} f(x) = -1, \qquad \lim_{x\to 1^+} f(x) = 1.$$

The required limits do not exist. The function is continuous on all $\mathbb{R}$ with the exception of the jump points $x = 0$ and $x = 1$.

3. *Continuity:*

a) The domain of f is $\mathbb{R}$ and the function is continuous for $x \neq 0$, irrespective of α. As for the continuity at $x = 0$, observe that

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} (2x^2 + 3) = 3 = f(0)\,,$$
$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \alpha \sin(x + \frac{\pi}{2}) = \alpha.$$

These imply f is continuous also in $x = 0$ if $\alpha = 3$.

b) $\alpha = 1$.

4

Limits and continuity II

The study of limits continues with the discussion of tools that facilitate computations and avoid having to resort to the definition each time. We introduce the notion of indeterminate form, and infer some remarkable limits. The last part of the chapter is devoted to continuous functions on real intervals.

4.1 Theorems on limits

A bit of notation to begin with: the symbol c will denote any of x_0, x_0^+, x_0^-, $+\infty, -\infty, \infty$ introduced previously. Correspondingly, $I(c)$ will be a neighbourhood $I_\delta(x_0)$ of $x_0 \in \mathbb{R}$ of radius δ, a right neighbourhood $I_\delta^+(x_0)$, a left neighbourhood $I_\delta^-(x_0)$, a neighbourhood $I_B(+\infty)$ of $+\infty$ with end-point $B > 0$, a neighbourhood $I_B(-\infty)$ of $-\infty$ with end-point $-B$, or a neighbourhood $I_B(\infty) = I_B(-\infty) \cup I_B(+\infty)$ of ∞.

We shall suppose from now on $f, g, h, \dots$ are functions *defined on a neighbourhood of* c with the point c deleted, unless otherwise stated. In accordance with the meaning of c, the expression $\lim\limits_{x\to c} f(x)$ will stand for the limit of f for $x \to x_0 \in \mathbb{R}$, the right or left limit, the limit for x tending to $+\infty$, $-\infty$, or for $|x| \to +\infty$.

4.1.1 Uniqueness and sign of the limit

We start with the uniqueness of a limit, which justifies having so far said 'the limit of f', in place of 'a limit of f'.

Theorem 4.1 (**Uniqueness of the limit**) *Suppose f admits* (*finite or infinite*) *limit ℓ for $x \to c$. Then f admits no other limit for $x \to c$.*

C. Canuto, A. Tabacco: *Mathematical Analysis I,* 2nd Ed.,
UNITEXT – La Matematica per il 3+2 84, DOI 10.1007/978-3-319-12772-9_4,

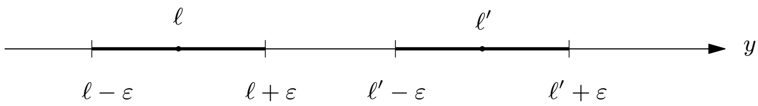


Figure 4.1. The neighbourhoods of ℓ, ℓ' of radius $\varepsilon \le \frac{1}{2}|\ell-\ell'|$ are disjoint

Proof. We assume there exist two limits $\ell' \neq \ell$ and infer a contradiction. We consider only the case where ℓ and ℓ' are both finite, for the other situations can be easily deduced adapting the same argument. First of all, since $\ell' \neq \ell$ there exist disjoint neighbourhoods $I(\ell)$ of ℓ and $I(\ell')$ of ℓ'

$$I(\ell) \cap I(\ell') = \emptyset. \tag{4.1}$$

To see this fact, it is enough to consider neighbourhoods of radius ε smaller or equal than half the distance of ℓ and ℓ', $\varepsilon \le \frac{1}{2}|\ell-\ell'|$ (Fig. 4.1). Taking $I(\ell)$, the hypothesis $\lim\limits_{x\to c} f(x) = \ell$ implies the existence of a neighbourhood $I(c)$ of c such that

$$\forall x \in \operatorname{dom} f, \qquad x \in I(c) \setminus \{c\} \quad \Rightarrow \quad f(x) \in I(\ell).$$

Similarly for $I(\ell')$, from $\lim\limits_{x\to c} f(x) = \ell'$ it follows there is $I'(c)$ with

$$\forall x \in \operatorname{dom} f, \qquad x \in I'(c) \setminus \{c\} \quad \Rightarrow \quad f(x) \in I(\ell').$$

The intersection of $I(c)$ and $I'(c)$ is itself a neighbourhood of c: it contains infinitely many points of the domain of f since we assumed f was defined in a neighbourhood of c (possibly minus c). Therefore if $\bar{x} \in \operatorname{dom} f$ is any point in the intersection, different from c,

$$f(\bar{x}) \in I(\ell) \cap I(\ell'),$$

hence the intervals $I(\ell)$ and $I(\ell')$ do have non-empty intersection, contradicting (4.1). □

The second property we present concerns the sign of a limit around a point c.

Theorem 4.2 *Suppose f admits limit ℓ (finite or infinite) for $x \to c$. If $\ell > 0$ or $\ell = +\infty$, there exists a neighbourhood $I(c)$ of c such that f is strictly positive on $I(c) \setminus \{c\}$. A similar assertion holds when $\ell < 0$ or $\ell = -\infty$.*

Proof. Assume ℓ is finite, positive, and consider the neighbourhood $I_\varepsilon(\ell)$ of ℓ of radius $\varepsilon = \ell/2 > 0$. According to the definition, there is a neighbourhood $I(c)$ of c satisfying

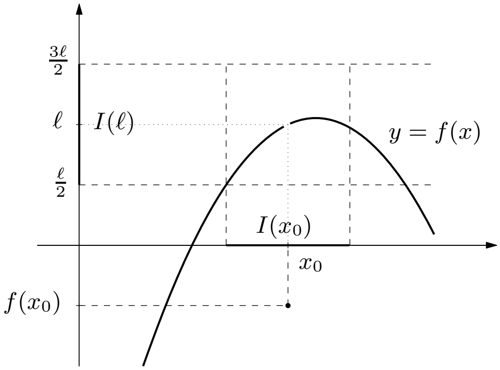


Figure 4.2. Around a limit value, the sign of a map does not change

$$\forall x \in \operatorname{dom} f, \qquad x \in I(c) \setminus \{c\} \quad \Rightarrow \quad f(x) \in I_\varepsilon(\ell).$$

As $I_\varepsilon(\ell) = (\frac{\ell}{2}, \frac{3\ell}{2}) \subset (0, +\infty)$, all values $f(x)$ are positive.
If $\ell = +\infty$ it suffices to take a neighbourhood $I_A(+\infty) = (A, +\infty)$ of $+\infty$ ($A > 0$) and use the corresponding definition of limit. □

The next result explains in which sense the implication in Theorem 4.2 can be 'almost' reversed.

Corollary 4.3 *Assume f admits limit ℓ (finite or infinite) for x tending to c. If there is a neighbourhood $I(c)$ of c such that $f(x) \geq 0$ in $I(c) \setminus \{c\}$, then $\ell \geq 0$ or $\ell = +\infty$. A similar assertion holds for a 'negative' limit.*

Proof. By contradiction, if $\ell = -\infty$ or $\ell < 0$, Theorem 4.2 would provide a neighbourhood $I'(c)$ of c such that $f(x) < 0$ on $I'(c) \setminus \{c\}$. On the intersection of $I(c)$ and $I'(c)$ we would then simultaneously have $f(x) < 0$ and $f(x) \geq 0$, which is not possible. □

Note that even assuming the stronger inequality $f(x) > 0$ on $I(c)$, we would not be able to exclude ℓ might be zero. For example, the map

$$f(x) = \begin{cases} x^2 & \text{if } x \neq 0, \\ 1 & \text{if } x = 0, \end{cases}$$

is strictly positive in every neighbourhood of the origin, yet $\lim_{x \to 0} f(x) = 0$.

4.1.2 Comparison theorems

A few results are known that allow to compare the behaviour of functions, the first of which generalises the above corollary.

Corollary 4.4 (**First comparison theorem**) *Let a function f have limit ℓ and a function g limit m (ℓ, m finite or not) for $x \to c$. If there is a neighbourhood $I(c)$ of c such that $f(x) \leq g(x)$ in $I(c) \setminus \{c\}$, then $\ell \leq m$.*

Proof. If $\ell = -\infty$ or $m = +\infty$ there is nothing to prove. Otherwise, consider the map $h(x) = g(x) - f(x)$. By assumption $h(x) \geq 0$ on $I(c) \setminus \{c\}$. Besides, Theorem 4.10 on the algebra of limits guarantees

$$\lim_{x \to c} h(x) = \lim_{x \to c} g(x) - \lim_{x \to c} f(x) = m - \ell.$$

The previous corollary applied to h forces $m - \ell \geq 0$, hence the claim. □

We establish now two useful criteria on the existence of limits based on comparing a given function with others whose limit is known.

Theorem 4.5 (**Second comparison theorem – finite case**, also known as "**Squeeze rule**") *Let functions f, g and h be given, and assume f and h have the same finite limit for $x \to c$, precisely*

$$\lim_{x \to c} f(x) = \lim_{x \to c} h(x) = \ell.$$

If there is a neighbourhood $I(c)$ of c where the three functions are defined (except possibly at c) and such that

$$f(x) \leq g(x) \leq h(x), \qquad \forall x \in I(c) \setminus \{c\}, \tag{4.2}$$

then

$$\lim_{x \to c} g(x) = \ell.$$

Proof. We follow the definition of limit for g. Fix a neighbourhood $I_\varepsilon(\ell)$ of ℓ; by the hypothesis $\lim\limits_{x \to c} f(x) = \ell$ we deduce the existence of a neighbourhood $I'(c)$ of c such that

$$\forall x \in \operatorname{dom} f, \qquad x \in I'(c) \setminus \{c\} \quad \Rightarrow \quad f(x) \in I_\varepsilon(\ell).$$

The condition $f(x) \in I_\varepsilon(\ell)$ can be written as $|f(x) - \ell| < \varepsilon$, or

$$\ell - \varepsilon < f(x) < \ell + \varepsilon, \tag{4.3}$$

recalling (1.4). Similarly, $\lim\limits_{x \to c} h(x) = \ell$ implies there is a neighbourhood $I''(c)$ of c such that

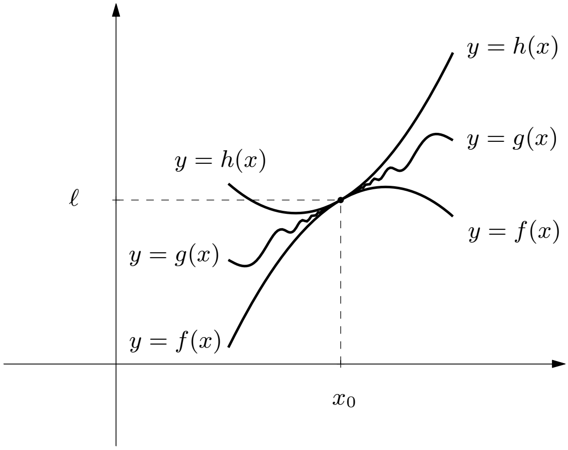


Figure 4.3. The squeeze rule

$$\forall x \in \operatorname{dom} h, \qquad x \in I''(c) \setminus \{c\} \quad \Rightarrow \quad \ell - \varepsilon < h(x) < \ell + \varepsilon. \qquad (4.4)$$

Define then $I'''(c) = I(c) \cap I'(c) \cap I''(c)$. On $I'''(c) \setminus \{c\}$ the constraints (4.2), (4.3) and (4.4) all hold, hence in particular

$$x \in I'''(c) \setminus \{c\} \quad \Rightarrow \quad \ell - \varepsilon < f(x) \le g(x) \le h(x) < \ell + \varepsilon.$$

This means $g(x) \in I_\varepsilon(\ell)$, concluding the proof. □

Examples 4.6

i) Let us prove the fundamental limit

$$\lim_{x\to 0} \frac{\sin x}{x} = 1. \qquad (4.5)$$

Observe first that $y = \dfrac{\sin x}{x}$ is even, for $\dfrac{\sin(-x)}{-x} = \dfrac{-\sin x}{-x} = \dfrac{\sin x}{x}$. It is thus sufficient to consider a positive x tending to 0, i.e., prove that $\displaystyle\lim_{x\to 0^+} \frac{\sin x}{x} = 1$. Recalling (3.13), for all $x > 0$ we have $\sin x < x$, or $\dfrac{\sin x}{x} < 1$. To find a lower bound, suppose $x < \frac{\pi}{2}$ and consider points on the unit circle: let A have coordinates $(1, 0)$, P coordinates $(\cos x, \sin x)$ and let Q be defined by $(1, \tan x)$ (Fig. 4.4). The circular sector OAP is a proper subset of the triangle OAQ, so

$$\text{Area } OAP < \text{ Area } OAQ.$$

Since

$$\text{Area } OAP = \frac{\overline{OA} \cdot \widehat{AP}}{2} = \frac{x}{2} \qquad \text{and} \qquad \text{Area } OAQ = \frac{\overline{OA} \cdot \overline{AQ}}{2} = \frac{\tan x}{2},$$

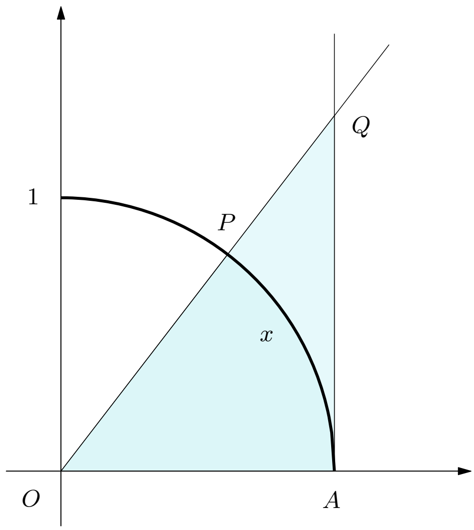


Figure 4.4. The sector OAP is properly contained in OAQ

it follows

$$\frac{x}{2} < \frac{\sin x}{2\cos x}, \qquad \text{i.e.,} \qquad \cos x < \frac{\sin x}{x}.$$

Eventually, on $0 < x < \frac{\pi}{2}$ one has

$$\cos x < \frac{\sin x}{x} < 1.$$

The continuity of the cosine ensures $\lim\limits_{x \to 0^+} \cos x = 1$. Now the claim follows from the Second comparison theorem.

ii) We would like to study how the function $g(x) = \dfrac{\sin x}{x}$ behaves for x tending to $+\infty$. Remember that

$$-1 \le \sin x \le 1 \tag{4.6}$$

for any real x. Dividing by $x > 0$ will not alter the inequalities, so in every neighbourhood $I_A(+\infty)$ of $+\infty$

$$-\frac{1}{x} \le \frac{\sin x}{x} \le \frac{1}{x}.$$

Now set $f(x) = -\dfrac{1}{x}$, $h(x) = \dfrac{1}{x}$ and note $\lim\limits_{x \to +\infty} \dfrac{1}{x} = 0$. By the previous theorem

$$\lim_{x \to +\infty} \frac{\sin x}{x} = 0. \qquad \square$$

The latter example is part of a more general result which we state next (and both are consequences of Theorem 4.5).

Corollary 4.7 *Let f be a* **bounded** *function around c, i.e., there exist a neighbourhood $I(c)$ and a constant $C > 0$ such that*

$$|f(x)| \leq C, \qquad \forall x \in I(c) \setminus \{c\}\,. \tag{4.7}$$

Let g be such that

$$\lim_{x \to c} g(x) = 0.$$

Then it follows

$$\lim_{x \to c} f(x)g(x) = 0.$$

Proof. By definition $\lim\limits_{x \to c} g(x) = 0$ if and only if $\lim\limits_{x \to c} |g(x)| = 0$, and (4.7) implies

$$0 \leq |f(x)g(x)| \leq C|g(x)|, \qquad \forall x \in I(c) \setminus \{c\}.$$

The claim follows by applying Theorem 4.5. □

Theorem 4.8 (**Second comparison theorem – infinite case**) *Let f, g be given functions and*

$$\lim_{x \to c} f(x) = +\infty.$$

If there exists a neighbourhood $I(c)$ of c, where both functions are defined (except possibly at c), such that

$$f(x) \leq g(x), \qquad \forall x \in I(c) \setminus \{c\}, \tag{4.8}$$

then

$$\lim_{x \to c} g(x) = +\infty.$$

A result of the same kind for f holds when the limit of g is $-\infty$.

Proof. The proof is, with the necessary changes, like that of Theorem 4.5, hence left to the reader. □

Example 4.9

Compute the limit of $g(x) = x + \sin x$ when $x \to +\infty$. Using (4.6) we have

$$x - 1 \leq x + \sin x, \qquad \forall x \in \mathbb{R}.$$

Set $f(x) = x - 1$; since $\lim\limits_{x \to +\infty} f(x) = +\infty$, the theorem tells us

$$\lim_{x \to +\infty} (x + \sin x) = +\infty.$$

□

4.1.3 Algebra of limits. Indeterminate forms of algebraic type

This section is devoted to the interaction of limits with the algebraic operations of sum, difference, product and quotient of functions.

First though, we must extend arithmetic operations to treat the symbols $+\infty$ and $-\infty$. Let us set:

$$\begin{aligned}
+\infty + s &= +\infty && (\text{if } s \in \mathbb{R} \text{ or } s = +\infty)\\
-\infty + s &= -\infty && (\text{if } s \in \mathbb{R} \text{ or } s = -\infty)\\
\pm\infty \cdot s &= \pm\infty && (\text{if } s > 0 \text{ or } s = +\infty)\\
\pm\infty \cdot s &= \mp\infty && (\text{if } s < 0 \text{ or } s = -\infty)\\
\frac{\pm\infty}{s} &= \pm\infty && (\text{if } s > 0)\\
\frac{\pm\infty}{s} &= \mp\infty && (\text{if } s < 0)\\
\frac{s}{0} &= \infty && (\text{if } s \in \mathbb{R} \setminus \{0\} \text{ or } s = \pm\infty)\\
\frac{s}{\pm\infty} &= 0 && (\text{if } s \in \mathbb{R})
\end{aligned}$$

Instead, the following expressions are *not* defined

$$\pm\infty + (\mp\infty), \qquad \pm\infty - (\pm\infty), \qquad \pm\infty \cdot 0, \qquad \frac{\pm\infty}{\pm\infty}, \qquad \frac{0}{0}.$$

A result of the foremost importance comes next.

Theorem 4.10 *Suppose f admits limit ℓ (finite or infinite) and g admits limit m (finite or infinite) for $x \to c$. Then*

$$\begin{aligned}
\lim_{x\to c} \big(f(x) \pm g(x)\big) &= \ell \pm m,\\
\lim_{x\to c} \big(f(x)\, g(x)\big) &= \ell\, m,\\
\lim_{x\to c} \frac{f(x)}{g(x)} &= \frac{\ell}{m},
\end{aligned}$$

provided the right-hand-side expressions make sense. (In the last case one assumes $g(x) \neq 0$ on some $I(c)\setminus\{c\}$.)

Proof. We shall prove two relations only, referring the reader to Appendix A.2.1, p. 433, for the ones left behind. The first we concentrate upon is

$$\lim_{x \to c} \big(f(x) + g(x)\big) = \ell + m$$

when ℓ and m are finite. Fix $\varepsilon > 0$, and consider the neighbourhood of ℓ of radius $\varepsilon/2$. By assumption there is a neighbourhood $I'(c)$ of c such that

$$\forall x \in \operatorname{dom} f, \qquad x \in I'(c) \setminus \{c\} \quad \Rightarrow \quad |f(x) - \ell| < \varepsilon/2.$$

For the same reason there is also an $I''(c)$ with

$$\forall x \in \operatorname{dom} g, \qquad x \in I''(c) \setminus \{c\} \quad \Rightarrow \quad |g(x) - m| < \varepsilon/2.$$

Put $I(c) = I'(c) \cap I''(c)$. Then if $x \in \operatorname{dom} f \cap \operatorname{dom} g$ belongs to $I(c) \setminus \{c\}$, both inequalities hold; the triangle inequality (1.1) yields

$$\begin{aligned} |(f(x) + g(x)) - (\ell + m)| &= |(f(x) - \ell) + (g(x) - m)| \\ &\leq |f(x) - \ell| + |g(x) - m| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \end{aligned}$$

proving the assertion.

The second relation is

$$\lim_{x \to c} \big(f(x)\, g(x)\big) = +\infty$$

with $\ell = +\infty$ and $m > 0$ finite. For a given real $A > 0$, consider the neighbourhood of $+\infty$ with end-point $B = 2A/m > 0$. We know there is a neighbourhood $I'(c)$ such that

$$\forall x \in \operatorname{dom} f, \qquad x \in I'(c) \setminus \{c\} \quad \Rightarrow \quad f(x) > B.$$

On the other hand, considering the neighbourhood of m of radius $m/2$, there exists an $I''(c)$ such that

$$\forall x \in \operatorname{dom} g, \qquad x \in I'(c) \setminus \{c\} \quad \Rightarrow \quad |g(x) - m| < m/2,$$

i.e., $m/2 < g(x) < 3m/2$. Set $I(c) = I'(c) \cap I''(c)$. If $x \in \operatorname{dom} f \cap \operatorname{dom} g$ is in $I(c) \setminus \{c\}$, the previous relations will be both fulfilled, whence

$$f(x)\, g(x) > f(x)\, \frac{m}{2} > B \frac{m}{2} = A.$$

□

Corollary 4.11 *If f and g are continuous maps at a point $x_0 \in \mathbb{R}$, then also $f(x) \pm g(x)$, $f(x)\, g(x)$ and $\dfrac{f(x)}{g(x)}$ (provided $g(x_0) \neq 0$) are continuous at x_0.*

Proof. The condition that f and g are continuous at x_0 is equivalent to $\lim_{x \to x_0} f(x) = f(x_0)$ and $\lim_{x \to x_0} g(x) = g(x_0)$ (recall (3.9)). The previous theorem allows to conclude. □

Corollary 4.12 *Rational functions are continuous on their domain. In particular, polynomials are continuous on* $\mathbb{R}$.

Proof. We verified in Example 3.17, part i), that the constants $y = a$ and the linear function $y = x$ are continuous on $\mathbb{R}$. Consequently, maps like $y = ax^n$ ($n \in \mathbb{N}$) are continuous. But then so are polynomials, being sums of the latter. Rational functions, as quotients of polynomials, inherit the property wherever the denominator does not vanish. □

Examples 4.13

i) Calculate

$$\lim_{x \to 0} \frac{2x - 3\cos x}{5 + x \sin x} = \ell.$$

The continuity of numerator and denominator descends from algebraic operations on continuous maps, and the denominator is not zero at $x = 0$. The substitution of 0 to x produces $\ell = -3/5$.

ii) Discuss the limit behaviour of $y = \tan x$ when $x \to \frac{\pi}{2}$. Since

$$\lim_{x \to \frac{\pi}{2}} \sin x = \sin \frac{\pi}{2} = 1 \qquad \text{and} \qquad \lim_{x \to \frac{\pi}{2}} \cos x = \cos \frac{\pi}{2} = 0,$$

the above theorem tells

$$\lim_{x \to \frac{\pi}{2}} \tan x = \lim_{x \to \frac{\pi}{2}} \frac{\sin x}{\cos x} = \frac{1}{0} = \infty.$$

But one can be more precise by looking at the sign of the tangent around $\frac{\pi}{2}$. Since $\sin x > 0$ in a neighbourhood of $\frac{\pi}{2}$, while $\cos x > 0$ (< 0) in a left (resp. right) neighbourhood of $\frac{\pi}{2}$, it follows

$$\lim_{x \to \frac{\pi}{2}^{\pm}} \tan x = \mp\infty.$$

iii) Let $R(x) = \dfrac{P(x)}{Q(x)}$ be rational and reduced, meaning the polynomials P, Q have no common factor. Call $x_0 \in \mathbb{R}$ a zero of Q, i.e., a point such that $Q(x_0) = 0$. Clearly $P(x_0) \neq 0$, otherwise P and Q would be both divisible by $(x - x_0)$. Then

$$\lim_{x \to x_0} R(x) = \infty$$

follows. In this case too, the sign of $R(x)$ around of x_0 retains some information. For instance, $y = \dfrac{x^2 - 3x + 1}{x^2 - x}$ is positive on a left neighbourhood of $x_0 = 1$ and negative on a right neighbourhood, so

$$\lim_{x\to 1^\pm} \frac{x^2-3x+1}{x^2-x} = \mp\infty.$$

In contrast, the function $y = \dfrac{x-2}{x^2-2x+1}$ is negative in a whole neighbourhood of $x_0 = 1$, hence

$$\lim_{x\to 1} \frac{x-2}{x^2-2x+1} = -\infty. \qquad \square$$

Theorem 4.10 gives no indication about the limit behaviour of an algebraic expression in three cases, listed below. The expressions in question are called **indeterminate forms** of algebraic type.

i) Consider $f(x)+g(x)$ (resp. $f(x)-g(x)$) when both f, g tend to ∞ with different (resp. same) signs. This gives rise to the indeterminate form denoted by the symbol
$$\infty - \infty.$$

ii) The product $f(x)\,g(x)$, when one function tends to ∞ and the other to 0, is the indeterminate form with symbol
$$\infty \cdot 0.$$

iii) Relatively to $\dfrac{f(x)}{g(x)}$, in case both functions tend to ∞ or 0, the indeterminate forms are denoted with
$$\frac{\infty}{\infty} \qquad \text{or} \qquad \frac{0}{0}.$$

In presence of an indeterminate form, the limit behaviour cannot be told a priori, and there are examples for each possible limit: infinite, finite non-zero, zero, even non-existing limit. Every indeterminate form should be treated singularly and requires often a lot of attention.

Later we shall find the actual limit behaviour of many important indeterminate forms. With those and this section's theorems we will discuss more complicated indeterminate forms. Additional tools to analyse this behaviour will be provided further on: they are the local comparison of functions by means of the Landau symbols (Sect. 5.1), de l'Hôpital's Theorem (Sect. 6.11), the Taylor expansion (Sect. 7.1).

Examples 4.14

i) Let x tend to $+\infty$ and define functions $f_1(x) = x + x^2$, $f_2(x) = x + 1$, $f_3(x) = x + \frac{1}{x}$, $f_4(x) = x + \sin x$. Set $g(x) = x$. Using Theorem 4.10, or Example 4.9, one verifies easily that all maps tend to $+\infty$. One has

$$\lim_{x\to+\infty} [f_1(x) - g(x)] = \lim_{x\to+\infty} x^2 = +\infty,$$
$$\lim_{x\to+\infty} [f_2(x) - g(x)] = \lim_{x\to+\infty} 1 = 1,$$
$$\lim_{x\to+\infty} [f_3(x) - g(x)] = \lim_{x\to+\infty} \frac{1}{x} = 0\,,$$

whereas the limit of $f_4(x) - g(x) = \sin x$ does not exist: the function $\sin x$ is periodic and assumes each value between -1 and 1 infinitely many times as $x \to +\infty$.

ii) Consider now $x \to 0$. Let $f_1(x) = x^3$, $f_2(x) = x^2$, $f_3(x) = x$, $f_4(x) = x^2 \sin \frac{1}{x}$, and $g(x) = x^2$. All functions converge to 0 (for f_4 apply Corollary 4.7). Now

$$\lim_{x\to 0} \frac{f_1(x)}{g(x)} = \lim_{x\to 0} x = 0,$$
$$\lim_{x\to 0} \frac{f_2(x)}{g(x)} = \lim_{x\to 0} 1 = 1,$$
$$\lim_{x\to 0} \frac{f_3(x)}{g(x)} = \lim_{x\to 0} \frac{1}{x}, = \infty,$$

but $\dfrac{f_4(x)}{g(x)} = \sin \dfrac{1}{x}$ does not admit limit for $x \to 0$ (Remark 4.19 furnishes a proof of this).

iii) Let us consider a polynomial

$$P(x) = a_n x^n + \ldots + a_1 x + a_0 \qquad (a_n \neq 0)$$

for $x \to \pm\infty$. A function of this sort can give rise to an indeterminate form $\infty - \infty$ according to the coefficients' signs and the degree of the monomials involved. The problem is sorted by factoring out the leading term (monomial of maximal degree) x^n

$$P(x) = x^n \left(a_n + \frac{a_{n-1}}{x} + \ldots + \frac{a_1}{x^{n-1}} + \frac{a_0}{x^n} \right).$$

The part in brackets converges to a_n when $x \to \pm\infty$, so

$$\lim_{x\to\pm\infty} P(x) = \lim_{x\to\pm\infty} a_n x^n = \infty$$

The sign of the limit is easily found. For instance,

$$\lim_{x\to-\infty} (-5x^3 + 2x^2 + 7) = \lim_{x\to-\infty} (-5x^3) = +\infty.$$

Take now a reduced rational function

$$R(x) = \frac{P(x)}{Q(x)} = \frac{a_n x^n + \ldots + a_1 x + a_0}{b_m x^m + \ldots + b_1 x + b_0} \qquad (a_n,\, b_m \neq 0,\ m > 0).$$

When $x \to \pm\infty$, an indeterminate form $\frac{\infty}{\infty}$ arises. With the same technique as before,

$$\lim_{x\to\pm\infty} \frac{P(x)}{Q(x)} = \lim_{x\to\pm\infty} \frac{a_n x^n}{b_m x^m} = \frac{a_n}{b_m} \lim_{x\to\pm\infty} x^{n-m} = \begin{cases} \infty & \text{if } n > m\,, \\ \dfrac{a_n}{b_m} & \text{if } n = m\,, \\ 0 & \text{if } n < m\,. \end{cases}$$

For example:

$$\lim_{x\to+\infty} \frac{3x^3 - 2x + 1}{x - x^2} = \lim_{x\to+\infty} \frac{3x^3}{-x^2} = -\infty\,,$$
$$\lim_{x\to-\infty} \frac{-4x^5 + 2x^3 - 7}{8x^5 - x^4 + 5x} = \lim_{x\to-\infty} \frac{-4x^5}{8x^5} = -\frac{1}{2}\,,$$
$$\lim_{x\to-\infty} \frac{6x^2 - x + 5}{-x^3 + 9} = \lim_{x\to-\infty} \frac{6x^2}{-x^3} = 0.$$

iv) The function $y = \dfrac{\sin x}{x}$ becomes indeterminate $\frac{0}{0}$ for $x \to 0$; we proved in part i), Examples 4.6 that y converges to 1. From this, we can deduce the behaviour of $y = \dfrac{1-\cos x}{x^2}$ as $x \to 0$, another indeterminate form of the type $\frac{0}{0}$. In fact,

$$\lim_{x\to 0} \frac{1-\cos x}{x^2} = \lim_{x\to 0} \frac{(1-\cos x)(1+\cos x)}{x^2(1+\cos x)} = \lim_{x\to 0} \frac{1-\cos^2 x}{x^2} \cdot \lim_{x\to 0} \frac{1}{1+\cos x}.$$

The fundamental trigonometric equation $\cos^2 x + \sin^2 x = 1$ together with Theorem 4.10 gives

$$\lim_{x\to 0} \frac{\sin^2 x}{x^2} = \lim_{x\to 0} \left(\frac{\sin x}{x}\right)^2 = \left(\lim_{x\to 0} \frac{\sin x}{x}\right)^2 = 1.$$

The same theorem tells also that the second limit is $\frac{1}{2}$, so we conclude

$$\lim_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}.$$

□

With these examples we have taken the chance to look at the behaviour of elementary functions at the boundary points of their domains. For completeness we gather the most significant limits relative to the elementary functions of Sect. 2.6, their proofs may be found in Appendix A.2.2, p. 435.

$$\lim_{x\to+\infty} x^\alpha = +\infty\,, \qquad \lim_{x\to 0^+} x^\alpha = 0 \qquad \alpha > 0$$

$$\lim_{x\to+\infty} x^\alpha = 0\,, \qquad \lim_{x\to 0^+} x^\alpha = +\infty \qquad \alpha < 0$$

$$\lim_{x\to\pm\infty} \frac{a_n x^n + \ldots + a_1 x + a_0}{b_m x^m + \ldots + b_1 x + b_0} = \frac{a_n}{b_m} \lim_{x\to\pm\infty} x^{n-m}$$

$$\lim_{x\to+\infty} a^x = +\infty\,, \qquad \lim_{x\to-\infty} a^x = 0 \qquad a > 1$$

$$\lim_{x\to+\infty} a^x = 0\,, \qquad \lim_{x\to-\infty} a^x = +\infty \qquad a < 1$$

$$\lim_{x\to+\infty} \log_a x = +\infty\,, \qquad \lim_{x\to 0^+} \log_a x = -\infty \quad a > 1$$

$$\lim_{x\to+\infty} \log_a x = -\infty\,, \qquad \lim_{x\to 0^+} \log_a x = +\infty \quad a < 1$$

$$\lim_{x\to\pm\infty} \sin x\,, \qquad \lim_{x\to\pm\infty} \cos x\,, \qquad \lim_{x\to\pm\infty} \tan x \qquad \text{do not exist}$$

$$\lim_{x\to\left(\frac{\pi}{2}+k\pi\right)^{\pm}} \tan x = \mp\infty\,, \quad \forall k \in \mathbb{Z}$$

$$\lim_{x\to\pm 1} \arcsin x = \pm\frac{\pi}{2} = \arcsin(\pm 1)$$

$$\lim_{x\to+1} \arccos x = 0 = \arccos 1\,, \qquad \lim_{x\to-1} \arccos x = \pi = \arccos(-1)$$

$$\lim_{x\to\pm\infty} \arctan x = \pm\frac{\pi}{2}$$

4.1.4 Substitution theorem

The so-called Substitution theorem is important in itself for theoretical reasons, besides providing a very useful method to compute limits.

Theorem 4.15 *Suppose a map f admits limit*

$$\lim_{x\to c} f(x) = \ell, \tag{4.9}$$

finite or not. Let g be defined on a neighbourhood of ℓ (excluding possibly the point ℓ) and such that

i) if $\ell \in \mathbb{R}$, g is continuous at ℓ;

ii) if $\ell = +\infty$ or $\ell = -\infty$, the limit $\lim\limits_{y\to\ell} g(y)$ exists, finite or not.

Then the composition $g \circ f$ admits limit for $x \to c$ and

$$\lim_{x\to c} g(f(x)) = \lim_{y\to\ell} g(y). \tag{4.10}$$

Proof. Set $m = \lim\limits_{y\to\ell} g(y)$ (noting that under *i)*, $m = g(\ell)$). Given any neighbourhood $I(m)$ of m, by *i)* or *ii)* there will be a neighbourhood $I(\ell)$ of ℓ such that

$$\forall y \in \operatorname{dom} g, \qquad y \in I(\ell) \quad \Rightarrow \quad g(y) \in I(m).$$

Note that in case *i)* we can use $I(\ell)$ instead of $I(\ell) \setminus \{\ell\}$ because g is continuous at ℓ (recall (3.7)), while ℓ does not belong to $I(\ell)$ for case *ii)*. With such $I(\ell)$, assumption (4.9) implies the existence of a neighbourhood $I(c)$ of c with

$$\forall x \in \operatorname{dom} f, \qquad x \in I(c) \setminus \{c\} \quad \Rightarrow \quad f(x) \in I(\ell).$$

Since $x \in \operatorname{dom} g \circ f$ means $x \in \operatorname{dom} f$ plus $y = f(x) \in \operatorname{dom} g$, the previous two implications now give

$$\forall x \in \operatorname{dom} g \circ f, \qquad x \in I(c) \setminus \{c\} \quad \Rightarrow \quad g(f(x)) \in I(m).$$

But $I(m)$ was arbitrary, so

$$\lim_{x \to c} g(f(x)) = m.$$

□

Remark 4.16 An alternative condition that yields the same conclusion is the following:

i') if $\ell \in \mathbb{R}$, there is a neighbourhood $I(c)$ of c where $f(x) \neq \ell$ for all $x \neq c$, and the limit $\lim_{y \to \ell} g(y)$ exists, finite or infinite.

The proof is analogous. □

In case $\ell \in \mathbb{R}$ and g is continuous at ℓ (case *i)*), then $\lim_{y \to \ell} g(y) = g(\ell)$, so (4.10) reads

$$\lim_{x \to c} g(f(x)) = g(\lim_{x \to c} f(x)). \tag{4.11}$$

An imprecise but effective way to put (4.11) into words is to say that a continuous function *commutes* (exchanges places) with the symbol of limit.

Theorem 4.15 implies that continuity is inherited by composite functions, as we discuss hereby.

Corollary 4.17 *Let f be continuous at x_0, and define $y_0 = f(x_0)$. Let furthermore g be defined around y_0 and continuous at y_0. Then the composite $g \circ f$ is continuous at x_0.*

Proof. From (4.11)

$$\lim_{x \to x_0} (g \circ f)(x) = g(\lim_{x \to x_0} f(x)) = g(f(x_0)) = (g \circ f)(x_0),$$

which is equivalent to the claim. □

A few practical examples will help us understand how the Substitution theorem and its corollary are employed.

Examples 4.18

i) The map $h(x) = \sin(x^2)$ is continuous on $\mathbb{R}$, being the composition of the continuous functions $f(x) = x^2$ and $g(y) = \sin y$.

ii) Let us determine

$$\lim_{x\to 0} \frac{\sin(x^2)}{x^2}.$$

Set $f(x) = x^2$ and

$$g(y) = \begin{cases} \dfrac{\sin y}{y} & \text{if } y \neq 0, \\ 1 & \text{if } y = 0. \end{cases}$$

Then $\lim\limits_{x\to 0} f(x) = 0$, and we know that g is continuous at the origin. Thus

$$\lim_{x\to 0} \frac{\sin(x^2)}{x^2} = \lim_{y\to 0} \frac{\sin y}{y} = 1.$$

iii) We study the behaviour of $h(x) = \arctan\left(\dfrac{1}{x-1}\right)$ around the point 1. Defining $f(x) = \dfrac{1}{x-1}$, we have $\lim\limits_{x\to 1^\pm} f(x) = \pm\infty$. If we call $g(y) = \arctan y$, $\lim\limits_{y\to\pm\infty} g(y) = \pm\dfrac{\pi}{2}$ (see the Table on page 101). Therefore

$$\lim_{x\to 1^\pm} \arctan\left(\frac{1}{x-1}\right) = \lim_{y\to\pm\infty} g(y) = \pm\frac{\pi}{2}.$$

iv) Determine

$$\lim_{x\to+\infty} \log\sin\frac{1}{x}.$$

Setting $f(x) = \sin\frac{1}{x}$ has the effect that $\ell = \lim\limits_{x\to+\infty} f(x) = 0$. Note that $f(x) > 0$ for all $x > \frac{1}{\pi}$. With $g(y) = \log y$ we have $\lim\limits_{y\to 0^+} g(y) = -\infty$, so Remark 4.16 yields

$$\lim_{x\to+\infty} \log\sin\frac{1}{x} = \lim_{y\to 0^+} g(y) = -\infty. \qquad \square$$

Remark 4.19 Theorem 4.15 extends easily to cover the case where the role of f is played by a sequence $a : n \mapsto a_n$ with limit

$$\lim_{n\to\infty} a_n = \ell.$$

Namely, under the same assumptions on g,

$$\lim_{n\to\infty} g(a_n) = \lim_{y\to\ell} g(y).$$

This result is often used to disprove the existence of a limit, in that it provides a **Criterion of non-existence for limits**: *if two sequences $a : n \mapsto a_n$, $b : n \mapsto b_n$ have the same limit ℓ and*

$$\lim_{n\to\infty} g(a_n) \neq \lim_{n\to\infty} g(b_n),$$

then g does not admit limit when its argument tends to ℓ.

For example we can prove, with the aid of the criterion, that $y = \sin x$ has no limit when $x \to +\infty$: define the sequences $a_n = 2n\pi$ and $b_n = \frac{\pi}{2} + 2n\pi$, $n \in \mathbb{N}$, so that

$$\lim_{n\to\infty} \sin a_n = \lim_{n\to\infty} 0 = 0, \qquad \text{and at the same time} \qquad \lim_{n\to\infty} \sin b_n = \lim_{n\to\infty} 1 = 1.$$

Similarly, the function $y = \sin \frac{1}{x}$ has neither left nor right limit for $x \to 0$. □

4.2 More fundamental limits. Indeterminate forms of exponential type

Consider the paramount limit (3.3). Instead of the sequence $a_n = \left(1 + \dfrac{1}{n}\right)^n$, we look now at the function of real variable

$$h(x) = \left(1 + \frac{1}{x}\right)^x.$$

It is defined when $1 + \frac{1}{x} > 0$, hence on $(-\infty, -1) \cup (0, +\infty)$. The following result states that h and the sequence resemble each other closely when x tends to infinity. Its proof is given in Appendix A.2.3, p. 439.

Property 4.20 *The following limit holds*

$$\lim_{x\to\pm\infty} \left(1 + \frac{1}{x}\right)^x = \mathrm{e}.$$

By manipulating this formula we achieve a series of new fundamental limits. The substitution $y = \frac{x}{a}$, with $a \neq 0$, gives

$$\lim_{x\to\pm\infty} \left(1 + \frac{a}{x}\right)^x = \lim_{y\to\pm\infty} \left(1 + \frac{1}{y}\right)^{ay} = \left[\lim_{y\to\pm\infty} \left(1 + \frac{1}{y}\right)^y\right]^a = \mathrm{e}^a.$$

In terms of the variable $y = \frac{1}{x}$ then,

$$\lim_{x\to 0} (1 + x)^{1/x} = \lim_{y\to\pm\infty} \left(1 + \frac{1}{y}\right)^y = \mathrm{e}.$$

The continuity of the logarithm together with (4.11) furnish

$$\lim_{x\to 0} \frac{\log_a(1 + x)}{x} = \lim_{x\to 0} \log_a (1 + x)^{1/x} = \log_a \lim_{x\to 0} (1 + x)^{1/x} = \log_a \mathrm{e} = \frac{1}{\log a}$$

for any $a > 0$. In particular, taking $a = \mathrm{e}$:

$$\lim_{x\to 0} \frac{\log(1+x)}{x} = 1.$$

Note by the way $a^x - 1 = y$ is equivalent to $x = \log_a(1+y)$, and $y \to 0$ if $x \to 0$. With this substitution,

$$\lim_{x\to 0} \frac{a^x - 1}{x} = \lim_{y\to 0} \frac{y}{\log_a(1+y)} = \left[\lim_{y\to 0} \frac{\log_a(1+y)}{y}\right]^{-1} = \log a. \tag{4.12}$$

Taking $a = \mathrm{e}$ produces

$$\lim_{x\to 0} \frac{\mathrm{e}^x - 1}{x} = 1.$$

Eventually, let us set $1 + x = \mathrm{e}^y$. Since $y \to 0$ when $x \to 0$,

$$\begin{aligned} \lim_{x\to 0} \frac{(1+x)^\alpha - 1}{x} &= \lim_{y\to 0} \frac{\mathrm{e}^{\alpha y} - 1}{\mathrm{e}^y - 1} = \lim_{y\to 0} \frac{\mathrm{e}^{\alpha y} - 1}{y} \frac{y}{\mathrm{e}^y - 1} \\ &= \lim_{y\to 0} \frac{(\mathrm{e}^\alpha)^y - 1}{y} \lim_{y\to 0} \frac{y}{\mathrm{e}^y - 1} = \log \mathrm{e}^\alpha = \alpha \end{aligned} \tag{4.13}$$

for any $\alpha \in \mathbb{R}$.

For the reader's conveniency, all fundamental limits found so far are gathered below.

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

$$\lim_{x\to 0} \frac{1 - \cos x}{x^2} = \frac{1}{2}$$

$$\lim_{x\to \pm\infty} \left(1 + \frac{a}{x}\right)^x = \mathrm{e}^a \qquad (a \in \mathbb{R})$$

$$\lim_{x\to 0} (1+x)^{1/x} = \mathrm{e}$$

$$\lim_{x\to 0} \frac{\log_a(1+x)}{x} = \frac{1}{\log a} \ \ (a > 0); \ \text{ in particular, } \ \lim_{x\to 0} \frac{\log(1+x)}{x} = 1$$

$$\lim_{x\to 0} \frac{a^x - 1}{x} = \log a \quad (a > 0); \qquad \text{in particular, } \quad \lim_{x\to 0} \frac{\mathrm{e}^x - 1}{x} = 1$$

$$\lim_{x\to 0} \frac{(1+x)^\alpha - 1}{x} = \alpha \qquad (\alpha \in \mathbb{R}).$$

Let us return to the map $h(x) = \left(1+\frac{1}{x}\right)^x$. By setting $f(x) = \left(1+\frac{1}{x}\right)$ and $g(x) = x$, we can write

$$h(x) = [f(x)]^{g(x)}.$$

In general such an expression may give rise to **indeterminate forms** for x tending to a certain c. Suppose f, g are functions defined in a neighbourhood of c, except possibly at c, and that they admit limit for $x \to c$. Assume moreover $f(x) > 0$ around c, so that h is well defined in a neighbourhood of c (except possibly at c). To understand h it is convenient to use the identity

$$f(x) = \mathrm{e}^{\log f(x)}.$$

From this in fact we obtain

$$h(x) = \mathrm{e}^{g(x)\log f(x)}.$$

By continuity of the exponential and (4.11), we have

$$\lim_{x\to c}[f(x)]^{g(x)} = \exp\left(\lim_{x\to c}\left[g(x)\log f(x)\right]\right).$$

In other words, $h(x)$ can be studied by looking at the exponent $g(x)\log f(x)$. An indeterminate form of the latter will thus develop an **indeterminate form** of exponential type for $h(x)$. Namely, we might find ourselves in one of these situations:

i) g tends to ∞ and f to 1 (so $\log f$ tends to 0): the exponent is an indeterminate form of type $\infty \cdot 0$, whence we say that h presents an indeterminate form of type
$$1^\infty.$$

ii) g and f both tend to 0 (so $\log f$ tends to $-\infty$): once again the exponent is of type $\infty \cdot 0$, and the function h is said to have an indeterminate form of type
$$0^0.$$

iii) g tends to 0 and f tends to $+\infty$ ($\log f \to +\infty$): the exponent is of type $\infty \cdot 0$, and h becomes indeterminate of type
$$\infty^0.$$

Examples 4.21

i) The map $h(x) = \left(1+\frac{1}{x}\right)^x$ is an indeterminate form of type 1^∞ when $x \to \pm\infty$, whose limit equals e.

ii) The function $h(x) = x^x$, for $x \to 0^+$, is an indeterminate form of type 0^0. We shall prove in Chap. 6 that $\lim_{x\to 0^+} x\log x = 0$, therefore $\lim_{x\to 0^+} h(x) = 1$.

iii) The function $h(x) = x^{1/x}$ is for $x \to +\infty$ an indeterminate form of type ∞^0. Substituting $y = \frac{1}{x}$, and recalling that $\log \frac{1}{y} = -\log y$, we obtain $\lim\limits_{x\to+\infty} \dfrac{\log x}{x} = -\lim\limits_{y\to 0^+} y \log y = 0$, hence $\lim\limits_{x\to+\infty} h(x) = 1$. □

When dealing with $h(x) = [f(x)]^{g(x)}$, a rather common mistake – with tragic consequences – is to calculate first the limit of f and/or g, substitute the map with this value and compute the limit of the expression thus obtained. This is to emphasize that **it might be incorrect** to calculate the limit for $x \to c$ of the indeterminate form $h(x) = [f(x)]^{g(x)}$ by finding first

$$m = \lim_{x\to c} g(x), \qquad \text{and from this proceed to} \quad \lim_{x\to c} [f(x)]^m .$$

Equally incorrect might be to determine

$$\lim_{x\to c} \ell^{\,g(x)}, \qquad \text{already knowing} \quad \ell = \lim_{x\to c} f(x).$$

For example, suppose we are asked to find the limit of $h(x) = \left(1 + \dfrac{1}{x}\right)^x$ for $x \to \pm\infty$; we might think of finding first $\ell = \lim\limits_{x\to\pm\infty} \left(1 + \dfrac{1}{x}\right) = 1$ and from this $\lim\limits_{x\to\pm\infty} 1^x = \lim\limits_{x\to\pm\infty} 1 = 1$. This would lead us to believe, wrongly, that h converges to 1, in spite of the fact the correct limit is e.

4.3 Global features of continuous maps

Hitherto the focus has been on several local properties of functions, whether in the neighbourhood of a real point or a point at infinity, and limits have been discussed in that respect. Now we turn our attention to continuous functions defined on a real interval, and establish properties of global nature, i.e., those relative to the behaviour on the entire domain.

Let us start with a plain definition.

Definition 4.22 *A* **zero** *of a real-valued function f is a point $x_0 \in \operatorname{dom} f$ at which the function vanishes.*

For instance, the zeroes of $y = \sin x$ are the multiples of π, i.e., the elements of the set $\{m\pi \mid m \in \mathbb{Z}\}$.

The problem of solving an equation like

$$f(x) = 0$$

is equivalent to determining the zeroes of the function $y = f(x)$. That is why it becomes crucial to have methods, both analytical and numerical, that allow to find the zeroes of a function, or at least their approximate position.

A simple condition to have a zero inside an interval goes as follows.

Theorem 4.23 (Existence of zeroes) *Let f be a continuous map on a closed, bounded interval $[a, b]$. If $f(a)f(b) < 0$, i.e., if the images of the end-points under f have different signs, f admits a zero within the open interval (a, b).*
If moreover f is strictly monotone on $[a, b]$, the zero is unique.

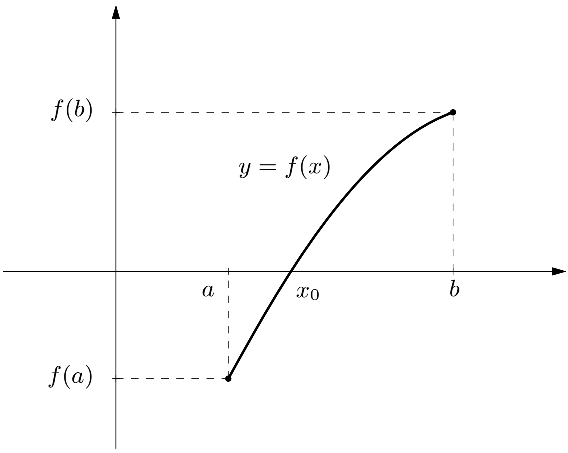


Figure 4.5. Theorem of existence of zeroes

Proof. Throughout the proof we shall use properties of sequences, for which we refer to the following Sect. 5.4. Assuming $f(a) < 0 < f(b)$ is not restrictive. Define $a_0 = a$, $b_0 = b$ and let $c_0 = \frac{a_0+b_0}{2}$ be the middle point of the interval $[a_0, b_0]$. There are three possibilities for $f(c_0)$. If $f(c_0) = 0$, the point $x_0 = c_0$ is a zero and the proof ends. If $f(c_0) > 0$, we set $a_1 = a_0$ and $b_1 = c_0$, so to consider the left half of the original interval. If $f(c_0) < 0$, let $a_1 = c_0$, $b_1 = b_0$ and take the right half of $[a_0, b_0]$ this time. In either case we have generated a sub-interval $[a_1, b_1] \subset [a_0, b_0]$ such that

$$f(a_1) < 0 < f(b_1) \quad \text{and} \quad b_1 - a_1 = \frac{b_0 - a_0}{2}.$$

Repeating the procedure we either reach a zero of f after a finite number of steps, or we build a sequence of nested intervals $[a_n, b_n]$ satisfying:

$$[a_0, b_0] \supset [a_1, b_1] \supset \ldots \supset [a_n, b_n] \supset \ldots\,,$$
$$f(a_n) < 0 < f(b_n) \quad \text{and} \quad b_n - a_n = \frac{b_0 - a_0}{2^n}$$

(the rigorous proof of the existence of such a sequence relies on the Principle of Induction; details are provided in Appendix A.1, p. 429). In this second situation, we claim that there is a unique point x_0 belonging to every interval of the sequence, and this point is a zero of f. For this, observe that the sequences $\{a_n\}$ and $\{b_n\}$ satisfy

$$a_0 \le a_1 \le \ldots \le a_n \le \ldots \le b_n \le \ldots \le b_1 \le b_0.$$

Therefore $\{a_n\}$ is monotone increasing and bounded, while $\{b_n\}$ is monotone decreasing and bounded. By Theorem 3.9 there exist $x_0^-, x_0^+ \in [a,b]$ such that

$$\lim_{n\to\infty} a_n = x_0^- \quad \text{and} \quad \lim_{n\to\infty} b_n = x_0^+.$$

On the other hand, Example 5.18 i) tells

$$x_0^+ - x_0^- = \lim_{n\to\infty} (b_n - a_n) = \lim_{n\to\infty} \frac{b-a}{2^n} = 0,$$

so $x_0^- = x_0^+$. Let x_0 denote this number. Since f is continuous, and using the Substitution theorem (Theorem 9, p. 138), we have

$$\lim_{n\to\infty} f(a_n) = \lim_{n\to\infty} f(b_n) = f(x_0).$$

But $f(a_n) < 0 < f(b_n)$, so the First comparison theorem (Theorem 4, p. 137) for $\{f(a_n)\}$ and $\{f(b_n)\}$ gives

$$\lim_{n\to\infty} f(a_n) \le 0 \quad \text{and} \quad \lim_{n\to\infty} f(b_n) \ge 0.$$

As $0 \le f(x_0) \le 0$, we obtain $f(x_0) = 0$.
In conclusion, if f is strictly monotone on $[a,b]$ it must be injective by Proposition 2.8, which forces the zero to be unique. □

Some comments on this theorem might prove useful. We remark first that without the hypothesis of continuity on the closed interval $[a,b]$, the condition $f(a)f(b) < 0$ would not be enough to ensure the presence of a zero. The function $f: [0,1] \to \mathbb{R}$

$$f(x) = \begin{cases} -1 & \text{for } x = 0, \\ +1 & \text{for } 0 < x \le 1 \end{cases}$$

takes values of discordant sign at the end-points but never vanishes; it has a jump point at $a = 0$.

Secondly, $f(a)f(b) < 0$ is a sufficient requirement only, and not a necessary one, to have a zero. The continuous map $f(x) = (2x-1)^2$ vanishes on $[0,1]$ despite being positive at both ends of the interval.

Thirdly, the halving procedure used in the proof can be transformed into an algorithm of approximation, known in Numerical Analysis under the name *Bisection method*.

A first application of the Theorem of existence of zeroes comes next.

Example 4.24

The function $f(x) = x^4 + x^3 - 1$ on $[0,1]$ is a polynomial, hence continuous. As $f(0) = -1$ and $f(1) = 1$, f must vanish somewhere on $[0,1]$. The zero is unique because the map is strictly increasing (it is sum of the strictly increasing functions $y = x^4$ and $y = x^3$, and of the constant function $y = -1$). □

Our theorem can be generalised usefully as follows.

Corollary 4.25 *Let f be continuous on the interval I and suppose it admits non-zero limits (finite or infinite) that are different in sign for x tending to the end-points of I. Then f has a zero in I, which is unique if f is strictly monotone on I.*

Proof. The result is a consequence of Theorems 4.2 and 4.23 (Existence of zeroes). For more details see Appendix A.3.2, p. 444. □

Example 4.26

Consider the map $f(x) = x + \log x$, defined on $I = (0, +\infty)$. The functions $y = x$ and $y = \log x$ are continuous and strictly increasing on I, and so is f. Since $\lim\limits_{x\to 0^+} f(x) = -\infty$ and $\lim\limits_{x\to+\infty} f(x) = +\infty$, f has exactly one zero on its domain. □

Corollary 4.27 *Consider f and g continuous maps on the closed bounded interval $[a,b]$. If $f(a) < g(a)$ and $f(b) > g(b)$, there exists at least one point x_0 in the open interval (a,b) with*

$$f(x_0) = g(x_0). \tag{4.14}$$

Proof. Consider the auxiliary function $h(x) = f(x) - g(x)$, which is continuous in $[a,b]$ as sum of continuous maps. By assumption, $h(a) = f(a) - g(a) < 0$ and $h(b) = f(b) - g(b) > 0$. So, h satisfies the Theorem of existence of zeroes and admits in (a,b) a point x_0 such that $h(x_0) = 0$. But this is precisely (4.14).

Note that if h is strictly increasing on $[a,b]$, the solution of (4.14) has to be unique in the interval. □

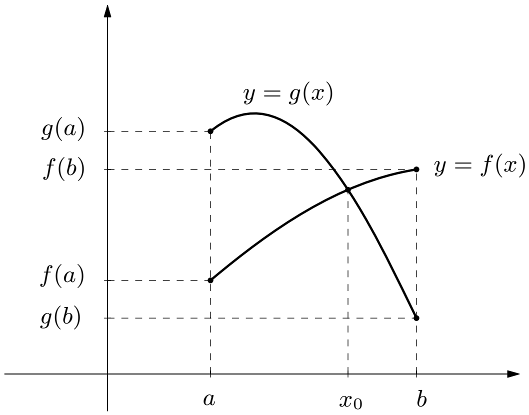


Figure 4.6. Illustration of Corollary 4.27

Example 4.28

Solve the equation

$$\cos x = x. \tag{4.15}$$

For any real x, $-1 \leq \cos x \leq 1$, so the equation cannot be solved when $x < -1$ or $x > 1$. Similarly, no solution exists on $[-1, 0)$, because $\cos x$ is positive while x is negative on that interval. Therefore the solutions, if any, must hide in $[0, 1]$: there the functions $f(x) = x$ and $g(x) = \cos x$ are continuous and $f(0) = 0 < 1 = g(0)$, $f(1) = 1 > \cos 1 = g(1)$ (cosine is 1 only for multiples of 2π). The above corollary implies that equation (4.15) has a solution in $(0, 1)$. There can be no other solution, for f is strictly increasing and g strictly decreasing on $[0, 1]$, making $h(x) = f(x) - g(x)$ strictly increasing. □

When one of the functions is a constant, the corollary implies this result.

Theorem 4.29 (**Intermediate value theorem**) *If a function f is continuous on the closed and bounded interval $[a, b]$, it assumes all values between $f(a)$ and $f(b)$.*

Proof. When $f(a) = f(b)$ the statement is trivial, so assume first $f(a) < f(b)$. Call z an arbitrary value between $f(a)$ and $f(b)$ and define the constant map $g(x) = z$. From $f(a) < z < f(b)$ we have $f(a) < g(a)$ and $f(b) > g(b)$. Corollary 4.27, applied to f and g in the interval $[a, b]$, yields a point x_0 in $[a, b]$ such that $f(x_0) = g(x_0) = z$.

If $f(a) > f(b)$, we just swap the roles of f and g. □

The Intermediate value theorem has, among its consequences, the remarkable fact that a continuous function maps intervals to intervals. This is the content of the next result.

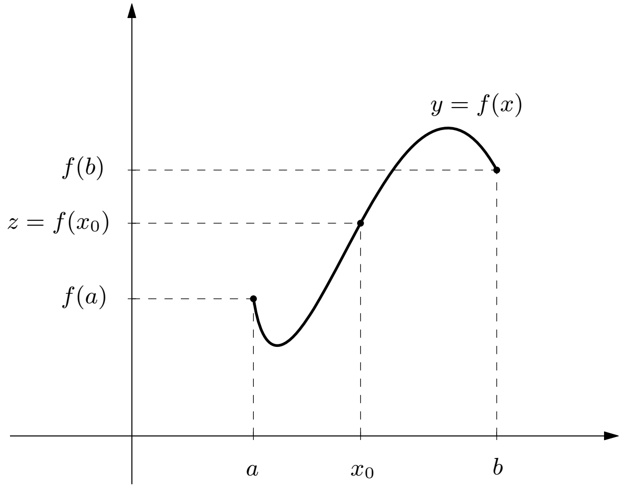


Figure 4.7. Intermediate value theorem

Corollary 4.30 *Let f be continuous on an interval I. The range $f(I)$ of I under f is an interval delimited by* $\inf_I f$ *and* $\sup_I f$.

Proof. A subset of $\mathbb{R}$ is an interval if and only if it contains the interval $[\alpha, \beta]$ as subset, for any $\alpha < \beta$.

Let then $y_1 < y_2$ be points of $f(I)$. There exist in I two (necessarily distinct) pre-images x_1 and x_2, i.e., $f(x_1) = y_1$, $f(x_2) = y_2$. If $J \subseteq I$ denotes the closed interval between x_1 and x_2, we need only to apply the Intermediate value theorem to f restricted to J, which yields $[y_1, y_2] \subseteq f(J) \subseteq f(I)$. The range $f(I)$ is then an interval, and according to Definition 2.3 its end-points are $\inf_I f$ and $\sup_I f$. □

Either one of $\inf_I f$, $\sup_I f$ may be finite or infinite, and may or not be an element of the interval itself. If, say, $\inf_I f$ belongs to the range, the function admits minimum on I (and the same for $\sup_I f$).

In case I is open or half-open, its image $f(I)$ can be an interval of any kind. Let us see some examples. Regarding $f(x) = \sin x$ on the open bounded $I = (-\frac{\pi}{2}, \frac{\pi}{2})$, the image $f(I) = (-1, 1)$ is open and bounded. Yet under the same map, the image of the open bounded set $(0, 2\pi)$ is $[-1, 1]$, bounded but closed. Take now $f(x) = \tan x$: it maps the bounded interval $(-\frac{\pi}{2}, \frac{\pi}{2})$ to the unbounded one $(-\infty, +\infty)$. Simple examples can be built also for unbounded I.

But if I is a closed bounded interval, its image under a continuous map cannot be anything but a closed bounded interval. More precisely, the following fundamental result holds, whose proof is given in Appendix A.3.2, p. 443.

Theorem 4.31 (**Weierstrass**) *A continuous map f on a closed and bounded interval $[a,b]$ is bounded and admits minimum and maximum*

$$m = \min_{x\in[a,b]} f(x) \qquad \text{and} \qquad M = \max_{x\in[a,b]} f(x).$$

Consequently,

$$f([a,b]) = [m,M]. \tag{4.16}$$

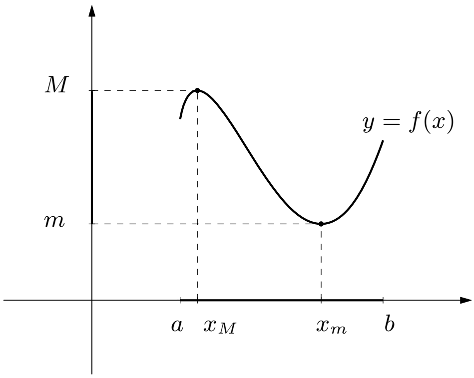


Figure 4.8. The Theorem of Weierstrass

In conclusion to this section, we present two results about invertibility (their proofs may be found in Appendix A.3.2, p. 445). We saw in Sect. 2.4 that a strictly monotone function is also one-to-one (invertible), and in general the opposite implication does not hold. Nevertheless, when speaking of continuous functions the notions of strict monotonicity and injectivity coincide. Moreover, the inverse function is continuous on its domain of definition.

Theorem 4.32 *A continuous function f on an interval I is one-to-one if and only if it is strictly monotone.*

Theorem 4.33 *Let f be continuous and invertible on an interval I. Then the inverse f^{-1} is continuous on the interval $J = f(I)$.*

Theorem 4.33 guarantees, by the way, the continuity of the inverse trigonometric functions $y = \arcsin x$, $y = \arccos x$ and $y = \arctan x$ on their domains, and of the logarithm $y = \log_a x$ on $\mathbb{R}_+$ as well, as inverse of the exponential $y = a^x$. These facts were actually already known from Proposition 3.20.

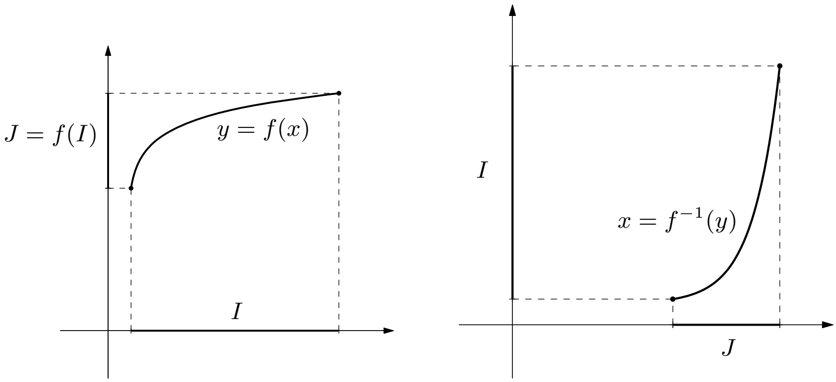


Figure 4.9. Graph of a continuous invertible map (left) and its inverse (right)

4.4 Exercises

1. *Compute the following limits using the Comparison theorems:*

a) $\displaystyle\lim_{x\to+\infty} \frac{\cos x}{\sqrt{x}}$ b) $\displaystyle\lim_{x\to+\infty} \left(\sqrt{x}+\sin x\right)$

c) $\displaystyle\lim_{x\to-\infty} \frac{2x-\sin x}{3x+\cos x}$ d) $\displaystyle\lim_{x\to+\infty} \frac{[x]}{x}$

e) $\displaystyle\lim_{x\to0} \sin x \cdot \sin\frac{1}{x}$ f) $\displaystyle\lim_{x\to0} \frac{x-\tan x}{x^2}$

2. *Determine the limits:*

a) $\displaystyle\lim_{x\to0} \frac{x^4-2x^3+5x}{x^5-x}$ b) $\displaystyle\lim_{x\to+\infty} \frac{x+3}{x^3-2x+5}$

c) $\displaystyle\lim_{x\to-\infty} \frac{x^3+x^2+x}{2x^2-x+3}$ d) $\displaystyle\lim_{x\to+\infty} \frac{2x^2+5x-7}{5x^2-2x+3}$

e) $\displaystyle\lim_{x\to-1} \frac{x+1}{\sqrt{6x^2+3}+3x}$ f) $\displaystyle\lim_{x\to2} \frac{\sqrt[3]{10-x}-2}{x-2}$

g) $\displaystyle\lim_{x\to+\infty} \left(\sqrt{x+1}-\sqrt{x}\right)$ h) $\displaystyle\lim_{x\to+\infty} \frac{\sqrt{x}+x}{x}$

i) $\displaystyle\lim_{x\to-\infty} \left(\sqrt[3]{x+1}-\sqrt[3]{x-1}\right)$ ℓ) $\displaystyle\lim_{x\to-\infty} \frac{\sqrt{2x^2+3}}{4x+2}$

3. *Relying on the fundamental limits, compute:*

a) $\displaystyle\lim_{x\to0} \frac{\sin^2 x}{x}$ b) $\displaystyle\lim_{x\to0} \frac{x\tan x}{1-\cos x}$

c) $\displaystyle \lim_{x\to 0} \frac{\sin 2x - \sin 3x}{4x}$

d) $\displaystyle \lim_{x\to 0^+} \frac{1-\cos\sqrt{x}}{2x^2}$

e) $\displaystyle \lim_{x\to 0} \frac{\tan x - \sin x}{x^3}$

f) $\displaystyle \lim_{x\to 0} \frac{\cos(\tan x) - 1}{\tan x}$

g) $\displaystyle \lim_{x\to 1} \frac{\cos \frac{\pi}{2}x}{1-x}$

h) $\displaystyle \lim_{x\to \frac{\pi}{2}} \frac{\sin x - 1}{\left(\frac{\pi}{2}-x\right)^2}$

i) $\displaystyle \lim_{x\to \pi} \frac{\cos x + 1}{\cos 3x + 1}$

ℓ) $\displaystyle \lim_{x\to 0} \frac{\sqrt{1+\tan x} - \sqrt{1-\tan x}}{\sin x}$

4. *Calculate:*

a) $\displaystyle \lim_{x\to 0} \frac{\log(1+x)}{3^x - 1}$

b) $\displaystyle \lim_{x\to 0} \frac{\mathrm{e}^{2x}-1}{\mathrm{e}^{3x}-1}$

c) $\displaystyle \lim_{x\to \mathrm{e}} \frac{\log x - 1}{x - \mathrm{e}}$

d) $\displaystyle \lim_{x\to +\infty} \frac{\mathrm{e}^{x}}{\mathrm{e}^{x}-1}$

e) $\displaystyle \lim_{x\to 0^+} \frac{2\mathrm{e}^{2x}-1}{2x}$

f) $\displaystyle \lim_{x\to 1} \frac{\log x}{\mathrm{e}^{x}-\mathrm{e}}$

g) $\displaystyle \lim_{x\to 0} \frac{\sqrt[5]{1+3x}-1}{x}$

h) $\displaystyle \lim_{x\to -1} \frac{x+1}{\sqrt[4]{x+17}-2}$

5. *Compute the limits:*

a) $\displaystyle \lim_{x\to +\infty} \frac{x^{5/2} - 2x\sqrt{x} + 1}{2\sqrt{x^5} - 1}$

b) $\displaystyle \lim_{x\to 0} \frac{\mathrm{e}^{x} - \mathrm{e}^{-x}}{\sin x}$

c) $\displaystyle \lim_{x\to 0} \left(\operatorname{cotan} x - \frac{1}{\sin x}\right)$

d) $\displaystyle \lim_{x\to +\infty} \sqrt{x}\left(\sqrt{x+1} - \sqrt{x-1}\right)$

e) $\displaystyle \lim_{x\to +\infty} \left(\frac{x-1}{x+3}\right)^{x-2}$

f) $\displaystyle \lim_{x\to 0} (1+x)^{\operatorname{cotan} x}$

g) $\displaystyle \lim_{x\to 5} \frac{x-5}{\sqrt{x}-\sqrt{5}}$

h) $\displaystyle \lim_{x\to -\infty} \frac{3^x - 3^{-x}}{3^x + 3^{-x}}$

i) $\displaystyle \lim_{x\to 0} \left(\frac{1}{x\tan x} - \frac{1}{x\sin x}\right)$

ℓ) $\displaystyle \lim_{x\to +\infty} x\mathrm{e}^{x} \sin\left(\mathrm{e}^{-x}\sin\frac{2}{x}\right)$

m) $\displaystyle \lim_{x\to +\infty} x(2+\sin x)$

n) $\displaystyle \lim_{x\to -\infty} x\mathrm{e}^{\sin x}$

6. *Determine the domain of the functions below and their limit behaviour at the end-points of the domain:*

a) $\displaystyle f(x) = \frac{x^3 - x^2 + 3}{x^2 + 3x + 2}$

b) $\displaystyle f(x) = \frac{\mathrm{e}^{x}}{1+x^4}$

c) $\displaystyle f(x) = \log\left[1 + \exp\left(\frac{x^2+1}{x}\right)\right]$

d) $f(x) = \sqrt[3]{x}\,\mathrm{e}^{-x^2}$

4.4.1 Solutions

1. *Limits:*

a) 0; b) $+\infty$.

c) We have

$$\lim_{x\to-\infty} \frac{2x - \sin x}{3x + \cos x} = \lim_{x\to-\infty} \frac{x\left(2 - \frac{\sin x}{x}\right)}{x\left(3 + \frac{\cos x}{x}\right)} = \frac{2}{3}$$

because $\displaystyle\lim_{x\to-\infty} \frac{\sin x}{x} = \lim_{x\to-\infty} \frac{\cos x}{x} = 0$ by Corollary 4.7.

d) From $[x] \le x < [x] + 1$ (Example 2.1 vii)) one deduces straightaway $x - 1 < [x] \le x$, whence

$$\frac{x-1}{x} < \frac{[x]}{x} \le 1$$

for $x > 0$. Therefore, the Second comparison theorem 4.5 gives

$$\lim_{x\to+\infty} \frac{[x]}{x} = 1.$$

e) 0.

f) First of all $f(x) = \dfrac{x - \tan x}{x^2}$ is an odd map, so $\displaystyle\lim_{x\to0^+} f(x) = -\lim_{x\to0^-} f(x)$. Let now $0 < x < \frac{\pi}{2}$. From

$$\sin x < x < \tan x$$

(see Example 4.6 i) for a proof) it follows

$$\sin x - \tan x < x - \tan x < 0, \qquad \text{that is,} \qquad \frac{\sin x - \tan x}{x^2} < \frac{x - \tan x}{x^2} < 0.$$

Secondly,

$$\lim_{x\to0^+} \frac{\sin x - \tan x}{x^2} = \lim_{x\to0^+} \frac{\sin x\,(\cos x - 1)}{x^2 \cos x} = \lim_{x\to0^+} \frac{\sin x}{\cos x}\,\frac{\cos x - 1}{x^2} = 0\,.$$

Thus the Second comparison theorem 4.5 makes us conclude that

$$\lim_{x\to0^+} \frac{x - \tan x}{x^2} = 0,$$

therefore the required limit is 0.

2. *Limits:*

a) -5; b) 0.

c) Simple algebraic operations give

$$\lim_{x\to-\infty} \frac{x^3+x^2+x}{2x^2-x+3} = \lim_{x\to-\infty} \frac{x^3\left(1+\frac{1}{x}+\frac{1}{x^2}\right)}{x^2\left(2-\frac{1}{x}+\frac{3}{x^2}\right)} = \lim_{x\to-\infty} \frac{x}{2} = -\infty\,.$$

d) $\frac{2}{5}$.

e) Rationalising the denominator we see

$$\begin{aligned}\lim_{x\to-1} \frac{x+1}{\sqrt{6x^2+3}+3x} &= \lim_{x\to-1} \frac{(x+1)(\sqrt{6x^2+3}-3x)}{6x^2+3-9x^2}\\ &= \lim_{x\to-1} \frac{(x+1)(\sqrt{6x^2+3}-3x)}{3(1-x)(1+x)} = 1\,.\end{aligned}$$

f) Use the relation $a^3-b^3=(a-b)(a^2+ab+b^2)$ in

$$\begin{aligned}\lim_{x\to2} \frac{\sqrt[3]{10-x}-2}{x-2} &= \lim_{x\to2} \frac{10-x-8}{(x-2)(\sqrt[3]{(10-x)^2}+2\sqrt[3]{10-x}+4)}\\ &= \lim_{x\to2} \frac{-1}{\sqrt[3]{(10-x)^2}+2\sqrt[3]{10-x}+4} = -\frac{1}{12}\,.\end{aligned}$$

g) 0; h) 1; i) 0.

ℓ) We have

$$\lim_{x\to-\infty} \frac{\sqrt{2x^2+3}}{4x+2} = \lim_{x\to-\infty} \frac{|x|\sqrt{2+\frac{3}{x^2}}}{x\left(4+\frac{2}{x}\right)} = \frac{\sqrt{2}}{4} \lim_{x\to-\infty} \frac{-x}{x} = -\frac{\sqrt{2}}{4}\,.$$

3. *Limits:*

a) 0; b) 2.

c) We manipulate the expression so to obtain a fundamental limit:

$$\lim_{x\to0} \frac{\sin 2x-\sin 3x}{4x} = \lim_{x\to0} \frac{\sin 2x}{4x} - \lim_{x\to0} \frac{\sin 3x}{4x} = \frac{1}{2}-\frac{3}{4} = -\frac{1}{4}\,.$$

d) We use the cosine's fundamental limit:

$$\lim_{x\to0^+} \frac{1-\cos\sqrt{x}}{2x^2} = \lim_{x\to0^+} \frac{1-\cos\sqrt{x}}{x} \lim_{x\to0^+} \frac{1}{2x} = \frac{1}{2} \lim_{x\to0^+} \frac{1}{2x} = +\infty\,.$$

e) $\frac{1}{2}$.

f) Putting $y=\tan x$ and substituting,

$$\lim_{x\to0} \frac{\cos(\tan x)-1}{\tan x} = \lim_{y\to0} \frac{\cos y-1}{y} = \lim_{y\to0} \frac{\cos y-1}{y^2}\cdot y = 0\,.$$

g) Letting $y = 1 - x$ transforms the limit into:

$$\lim_{x\to 1} \frac{\cos\frac{\pi}{2}x}{1-x} = \lim_{y\to 0} \frac{\cos\frac{\pi}{2}(1-y)}{y} = \lim_{y\to 0} \frac{\sin\frac{\pi}{2}y}{y} = \frac{\pi}{2}.$$

h) $-\frac{1}{2}$; i) $\frac{1}{9}$.

ℓ) One has

$$\begin{aligned}\lim_{x\to 0} \frac{\sqrt{1+\tan x} - \sqrt{1-\tan x}}{\sin x} &= \lim_{x\to 0} \frac{1+\tan x - 1 + \tan x}{\sin x\left(\sqrt{1+\tan x} + \sqrt{1-\tan x}\right)} \\ &= \frac{1}{2}\lim_{x\to 0} \frac{2\tan x}{\sin x} = \lim_{x\to 0} \frac{1}{\cos x} = 1.\end{aligned}$$

4. *Limits:*

a) $\frac{1}{\log 3}$; b) $\frac{2}{3}$.

c) By defining $y = x - \mathrm{e}$ we recover a known fundamental limit:

$$\begin{aligned}\lim_{x\to \mathrm{e}} \frac{\log x - 1}{x - \mathrm{e}} &= \lim_{y\to 0} \frac{\log(y+\mathrm{e}) - 1}{y} = \lim_{y\to 0} \frac{\log \mathrm{e}\,(1 + y/\mathrm{e}) - 1}{y} \\ &= \lim_{y\to 0} \frac{\log(1+y/\mathrm{e})}{y} = \frac{1}{\mathrm{e}}.\end{aligned}$$

Another possibility is to set $z = x/\mathrm{e}$:

$$\lim_{x\to \mathrm{e}} \frac{\log x - 1}{x - \mathrm{e}} = \lim_{z\to 1} \frac{\log(\mathrm{e}z) - 1}{\mathrm{e}(z-1)} = \frac{1}{\mathrm{e}} \lim_{z\to 1} \frac{\log z}{z-1} = \frac{1}{\mathrm{e}}.$$

d) 1.

e) We have

$$\begin{aligned}\lim_{x\to 0^+} \frac{2\mathrm{e}^{2x} - 1}{2x} &= \lim_{x\to 0^+} \frac{2(\mathrm{e}^{2x} - 1) + 1}{2x} \\ &= \lim_{x\to 0^+} 2\frac{\mathrm{e}^{2x} - 1}{2x} + \lim_{x\to 0^+} \frac{1}{2x} = 2 + \lim_{x\to 0^+} \frac{1}{2x} = +\infty.\end{aligned}$$

f) Substitute $y = x - 1$, so that

$$\begin{aligned}\lim_{x\to 1} \frac{\log x}{\mathrm{e}^x - \mathrm{e}} &= \lim_{x\to 1} \frac{\log x}{\mathrm{e}(\mathrm{e}^{x-1} - 1)} \\ &= \lim_{y\to 0} \frac{\log(1+y)}{\mathrm{e}(\mathrm{e}^y - 1)} = \frac{1}{\mathrm{e}} \lim_{y\to 0} \frac{\log(1+y)}{y} \cdot \frac{y}{\mathrm{e}^y - 1} = \frac{1}{\mathrm{e}}.\end{aligned}$$

g) $\frac{3}{5}$.

h) The new variable $y = x + 1$ allows to recognize (4.13), so

$$\lim_{x\to -1} \frac{x+1}{\sqrt[4]{x+17}-2} = \lim_{y\to 0} \frac{y}{\sqrt[4]{y+16}-2} = \lim_{y\to 0} \frac{y}{2\left(\sqrt[4]{1+\frac{y}{16}}-1\right)}$$
$$= \frac{16}{2} \lim_{y\to 0} \frac{y/16}{\sqrt[4]{1+\frac{y}{16}}-1} = 8 \cdot 4 = 32\,.$$

5. *Limits:*

a) $\frac{1}{2}$.

b) We have

$$\lim_{x\to 0} \frac{\mathrm{e}^x - \mathrm{e}^{-x}}{\sin x} = \lim_{x\to 0} \frac{\mathrm{e}^{-x}\left(\mathrm{e}^{2x}-1\right)}{\sin x} = \lim_{x\to 0} \mathrm{e}^{-x} \cdot \frac{\mathrm{e}^{2x}-1}{2x} \cdot 2 \cdot \frac{x}{\sin x} = 2\,.$$

c) One has

$$\lim_{x\to 0} \left(\operatorname{cotan} x - \frac{1}{\sin x}\right) = \lim_{x\to 0} \frac{\cos x - 1}{\sin x} = \lim_{x\to 0} \frac{\cos x - 1}{x^2} \cdot \frac{x}{\sin x} \cdot x = 0\,.$$

d) 1.

e) Start with

$$\lim_{x\to +\infty} \left(\frac{x-1}{x+3}\right)^{x-2} = \exp\left(\lim_{x\to +\infty} (x-2) \log \frac{x-1}{x+3}\right)$$
$$= \exp\left(\lim_{x\to +\infty} (x-2) \log \left(1 - \frac{4}{x+3}\right)\right) = \mathrm{e}^L\,.$$

Now define $y = \dfrac{1}{x+3}$, and substitute $x = \dfrac{1}{y} - 3$ at the exponent:

$$L = \lim_{y\to 0^+} \left(\frac{1}{y} - 5\right) \log(1-4y) = \lim_{y\to 0^+} \left(\frac{\log(1-4y)}{y} - 5\log(1-4y)\right) = -4\,.$$

The required limit equals e^{-4}.

f) e; g) $2\sqrt{5}$.

h) We have

$$\lim_{x\to -\infty} \frac{3^x - 3^{-x}}{3^x + 3^{-x}} = \lim_{x\to -\infty} \frac{3^{-x}\left(3^{2x}-1\right)}{3^{-x}\left(3^{2x}+1\right)} = -1\,.$$

i) $-\frac{1}{2}$.

ℓ) Start by multiplying numerator and denominator by the same function:

$$\lim_{x\to +\infty} x\mathrm{e}^x\mathrm{e}^{-x} \sin\frac{2}{x} \cdot \frac{\sin\left(\mathrm{e}^{-x}\sin\frac{2}{x}\right)}{\mathrm{e}^{-x}\sin\frac{2}{x}} = \lim_{x\to +\infty} x\sin\frac{2}{x} \cdot \lim_{x\to +\infty} \frac{\sin\left(\mathrm{e}^{-x}\sin\frac{2}{x}\right)}{\mathrm{e}^{-x}\sin\frac{2}{x}}$$
$$= L_1 \cdot L_2\,.$$

Now put $y = \frac{1}{x}$ in the first factor to get

$$L_1 = \lim_{y\to 0^+} \frac{\sin 2y}{y} = 2\,;$$

next, let $t = \mathrm{e}^{-x}\sin\frac{2}{x}$. Since $t \to 0$ for $x \to +\infty$, by Corollary 4.7, the second factor is

$$L_2 = \lim_{t\to 0} \frac{\sin t}{t} = 1\,,$$

and eventually the limit is 2.

m) The fact that $-1 \le \sin x \le 1$ implies $1 \le 2 + \sin x \le 3$, so $x \le x(2+\sin x)$ when $x > 0$. Since $\lim\limits_{x\to+\infty} x = +\infty$, the Second comparison theorem 4.8 gives $+\infty$ for an answer.

n) $-\infty$.

6. *Domains and limits:*

a) $\operatorname{dom} f = \mathbb{R} \setminus \{-2, -1\}$,
$\lim\limits_{x\to-2^\pm} f(x) = \pm\infty$, $\lim\limits_{x\to-1^\pm} f(x) = \pm\infty$, $\lim\limits_{x\to\pm\infty} f(x) = \pm\infty$.

b) The function is defined on the entire $\mathbb{R}$ and

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{\mathrm{e}^x}{x^4}\cdot\frac{x^4}{1+x^4} = \lim_{x\to+\infty}\frac{\mathrm{e}^x}{x^4} = +\infty\,,$$
$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \mathrm{e}^x \cdot \lim_{x\to-\infty}\frac{1}{1+x^4} = 0\,.$$

c) This function makes sense when $x \neq 0$ (because $1 + \exp\left(\frac{x^2+1}{x}\right) > 0$ for any non-zero x). As for the limits:

$$\lim_{x\to-\infty} f(x) = \log \lim_{x\to-\infty}\left(1+\exp\left(\frac{x^2+1}{x}\right)\right) = \log 1 = 0\,,$$
$$\lim_{x\to+\infty} f(x) = \log \lim_{x\to+\infty}\left(1+\exp\left(\frac{x^2+1}{x}\right)\right) = +\infty\,,$$
$$\lim_{x\to0^-} f(x) = \log \lim_{x\to0^-}\left(1+\exp\left(\frac{x^2+1}{x}\right)\right) = \log 1 = 0\,,$$
$$\lim_{x\to0^+} f(x) = \log \lim_{x\to0^+}\left(1+\exp\left(\frac{x^2+1}{x}\right)\right) = +\infty\,.$$

d) $\operatorname{dom} f = \mathbb{R}$; $\lim\limits_{x\to\pm\infty} f(x) = 0$.

5

Local comparison of functions. Numerical sequences and series

In the first part of this chapter we learn how to compare the behaviour of two functions in the neighbourhood of a point. To this aim, we introduce suitable symbols – known as Landau symbols – that make the description of the possible types of behaviour easier. Of particular importance is the comparison between functions tending to 0 or ∞.

In the second part, we revisit some results on limits which we discussed in general for functions, and adapt them to the case of sequences. We present specific techniques for the analysis of the limiting behaviour of sequences. At last, numerical series are introduced and the main tools for the study of their convergence are provided.

5.1 Landau symbols

As customary by now, we denote by c one of the symbols x_0 (real number), x_0^+, x_0^-, or $+\infty$, $-\infty$. By 'neighbourhood of c' we intend a neighbourhood – previously defined – of one of these symbols.

Let f and g be two functions defined in a neighbourhood of c, with the possible exception of the point c itself. Let also $g(x) \neq 0$ for $x \neq c$. Assume the limit

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \ell \tag{5.1}$$

exists, finite or not. We introduce the following definition.

Definition 5.1 *If ℓ is finite, we say that f is* **controlled by** *g for x tending to c, and we shall use the notation*

$$f = O(g), \qquad x \to c\,,$$

read as 'f is **big o** *of g for x tending to c'.*

This property can be made more precise by distinguishing three cases:

a) If ℓ is finite and non-zero, we say that f has **the same order of magnitude as** *g (or* **is of the same order of magnitude***) for x tending to c; if so, we write*

$$f \asymp g, \qquad x \to c.$$

As sub-case we have:

b) If $\ell = 1$, we call f **equivalent to** *g for x tending to c; in this case we use the notation*

$$f \sim g, \qquad x \to c.$$

c) Eventually, if $\ell = 0$, we say that f is **negligible with respect to** *g when x goes to c; for this situation the symbol*

$$f = o(g), \qquad x \to c,$$

will be used, spoken 'f is **little o** *of g for x tending to c'.*

Not included in the previous definition is the case in which ℓ is infinite. But in such a case

$$\lim_{x \to c} \frac{g(x)}{f(x)} = \frac{1}{\ell} = 0,$$

so we can say that $g = o(f)$ for $x \to c$.

The symbols O, $\asymp$, $\sim$, o are called **Landau symbols**.

Remark 5.2 The Landau symbols can be defined under more general assumptions than those considered at present, i.e., the mere existence of the limit (5.1). For instance the expression $f = O(g)$ as $x \to c$ could be extended to mean that there is a constant $C > 0$ such that in a suitable neighbourhood I of c

$$|f(x)| \le C|g(x)|, \qquad \forall x \in I,\ x \ne c.$$

The given definition is nevertheless sufficient for our purposes. □

Examples 5.3

i) Keeping in mind Examples 4.6, we have

$$\sin x \sim x,\ x \to 0, \qquad \text{in fact} \qquad \lim_{x \to 0} \frac{\sin x}{x} = 1,$$

$$\sin x = o(x),\ x \to +\infty, \qquad \text{since} \qquad \lim_{x \to +\infty} \frac{\sin x}{x} = 0;$$

ii) We have $\sin x = o(\tan x)$, $x \to \frac{\pi}{2}$ since

$$\lim_{x \to \frac{\pi}{2}} \frac{\sin x}{\tan x} = \lim_{x \to \frac{\pi}{2}} \cos x = 0.$$

iii) One has $\cos x \asymp 2x - \pi$, $x \to \frac{\pi}{2}$, because

$$\lim_{x \to \frac{\pi}{2}} \frac{\cos x}{2x - \pi} = \lim_{t \to 0} \frac{\cos(t + \frac{\pi}{2})}{2t} = -\lim_{t \to 0} \frac{\sin t}{2t} = -\frac{1}{2}. \qquad \square$$

Properties of the Landau symbols

i) It is clear from the definitions that the symbols $\asymp$, $\sim$, o are particular instances of O, in the sense that

$$f \asymp g \Rightarrow f = O(g), \qquad f \sim g \Rightarrow f = O(g), \qquad f = o(g) \Rightarrow f = O(g)$$

for $x \to c$. Moreover the symbol $\sim$ is a subcase of $\asymp$

$$f \sim g \quad \Rightarrow \quad f \asymp g.$$

Observe that if $f \asymp g$, then (5.1) implies

$$\lim_{x \to c} \frac{f(x)}{\ell g(x)} = 1, \qquad \text{hence} \quad f \sim \ell g.$$

ii) The following property is useful

$$f \sim g \qquad \Longleftrightarrow \qquad f = g + o(g). \tag{5.2}$$

By defining $h(x) = f(x) - g(x)$ in fact, so that $f(x) = g(x) + h(x)$, we have

$$\begin{aligned} f \sim g \quad &\Longleftrightarrow \quad \lim_{x \to c} \frac{f(x)}{g(x)} = 1 \quad \Longleftrightarrow \quad \lim_{x \to c} \left(\frac{f(x)}{g(x)} - 1 \right) = 0 \\ &\Longleftrightarrow \quad \lim_{x \to c} \frac{h(x)}{g(x)} = 0 \quad \Longleftrightarrow \quad h = o(g). \end{aligned}$$

iii) Computations are simplified once we notice that for any constant $\lambda \neq 0$

$$o(\lambda f) = o(f) \qquad \text{and} \qquad \lambda\, o(f) = o(f). \tag{5.3}$$

In fact $g = o(\lambda f)$ means that $\lim_{x \to c} \frac{g(x)}{\lambda f(x)} = 0$, otherwise said $\lim_{x \to c} \frac{g(x)}{f(x)} = 0$, or $g = o(f)$. The remaining identity is proved in a similar way. Analogous properties to (5.3) hold for the symbol O.

Note that $o(f)$ and $O(f)$ do not indicate one specific function, rather a precise property of any map represented by one of the two symbols.

iv) Prescribing $f = o(1)$ amounts to asking that f converge to 0 when $x \to c$. Namely

$$\lim_{x \to c} f(x) = \lim_{x \to c} \frac{f(x)}{1} = 0.$$

Similarly $f = O(1)$ means f converges to a finite limit for x tending to c. More generally (compare Remark 5.2), $f = O(1)$ means that f is *bounded* in a neighbourhood of c: that is to say, there exists a constant $C > 0$ such that

$$|f(x)| \le C, \qquad \forall x \in I,\ x \ne c,$$

I being a suitable neighbourhood of c.

v) The continuity of a function f at a point x_0 can be expressed by means of the symbol o in the equivalent form

$$f(x) = f(x_0) + o(1), \quad x \to x_0. \tag{5.4}$$

Recalling (3.9) in fact, we have

$$\begin{aligned} \lim_{x\to x_0} f(x) = f(x_0) \qquad &\Longleftrightarrow \qquad \lim_{x\to x_0} \big(f(x) - f(x_0)\big) = 0 \\ &\Longleftrightarrow \qquad f(x) - f(x_0) = o(1), \quad x \to x_0. \end{aligned}$$

The algebra of "little o's"

i) Let us compare the behaviour of the monomials x^n as $x \to 0$:

$$x^n = o(x^m), \quad x \to 0, \qquad \Longleftrightarrow \qquad n > m.$$

In fact

$$\lim_{x\to 0} \frac{x^n}{x^m} = \lim_{x\to 0} x^{n-m} = 0 \ \text{ if and only if } \ n - m > 0.$$

Therefore *when $x \to 0$, the bigger of two powers of x is negligible.*

ii) Now consider the limit when $x \to \pm\infty$. Proceeding as before we obtain

$$x^n = o(x^m), \quad x \to \pm\infty, \qquad \Longleftrightarrow \qquad n < m.$$

So, *for $x \to \pm\infty$, the lesser power of x is negligible.*

iii) The symbols of Landau allow to simplify algebraic formulas quite a lot when studying limits. Consider for example the limit for $x \to 0$. The following properties, which define a special *"algebra of little o's"*, hold. Their proof is left to the reader as an exercise:

$$\begin{aligned} &\text{a)} \quad o(x^n) \pm o(x^n) = o(x^n)\,; \\ &\text{b)} \quad o(x^n) \pm o(x^m) = o(x^p)\,, \quad \text{with } p = \min(n, m)\,; \\ &\text{c)} \quad o(\lambda x^n) = o(x^n), \qquad \text{for each } \lambda \in \mathbb{R} \setminus \{0\}\,; \end{aligned} \tag{5.5}$$

d) $\varphi(x)o(x^n) = o(x^n)$ if φ is bounded in a neighbourhood of $x = 0$;

e) $x^m o(x^n) = o(x^{m+n})$;

f) $o(x^m)o(x^n) = o(x^{m+n})$;

g) $[o(x^n)]^k = o(x^{kn})$.

Fundamental limits

The fundamental limits in the Table of p. 106 can be reformulated using the symbols of Landau:

$$\begin{array}{llll}
\sin x \sim x, & x \to 0; & & \\
1-\cos x \asymp x^2, & x \to 0; \ \text{precisely,} & 1-\cos x \sim \frac{1}{2}x^2, & x \to 0; \\
\log(1+x) \sim x, & x \to 0; \ \text{equivalently,} & \log x \sim x-1, & x \to 1; \\
\mathrm{e}^x - 1 \sim x, & x \to 0; & & \\
(1+x)^\alpha - 1 \sim \alpha x, & x \to 0. & &
\end{array}$$

With (5.2), and taking property (5.5) c) into account, these relations read:

$$\begin{array}{llll}
\sin x = x + o(x), & x \to 0; & & \\
1-\cos x = \frac{1}{2}x^2 + o(x^2), & x \to 0, \ \text{or} & \cos x = 1 - \frac{1}{2}x^2 + o(x^2), & x \to 0; \\
\log(1+x) = x + o(x), & x \to 0, \ \text{or} & \log x = x - 1 + o(x-1), & x \to 1; \\
\mathrm{e}^x = 1 + x + o(x), & x \to 0; & & \\
(1+x)^\alpha = 1 + \alpha x + o(x), & x \to 0. & &
\end{array}$$

Besides, we shall prove in Sect. 6.11 that:

$$\begin{array}{llll}
\text{a)} & x^\alpha = o(\mathrm{e}^x), & x \to +\infty, & \forall \alpha \in \mathbb{R}; \\
\text{b)} & \mathrm{e}^x = o(|x|^\alpha), & x \to -\infty, & \forall \alpha \in \mathbb{R}; \\
\text{c)} & \log x = o(x^\alpha), & x \to +\infty, & \forall \alpha > 0; \\
\text{d)} & \log x = o\left(\dfrac{1}{x^\alpha}\right), & x \to 0^+, & \forall \alpha > 0.
\end{array} \tag{5.6}$$

Examples 5.4

i) From $\mathrm{e}^t = 1 + t + o(t)$, $t \to 0$, by setting $t = 5x$ we have $\mathrm{e}^{5x} = 1 + 5x + o(5x)$, i.e., $\mathrm{e}^{5x} = 1 + 5x + o(x)$, $x \to 0$. In other words $\mathrm{e}^{5x} - 1 \sim 5x$, $x \to 0$.

ii) Setting $t = -3x^2$ in $(1+t)^{1/2} = 1 + \frac{1}{2}t + o(t)$, $t \to 0$, we obtain $(1-3x^2)^{1/2} = 1 - \frac{3}{2}x^2 + o(-3x^2) = 1 - \frac{3}{2}x^2 + o(x^2)$, $x \to 0$. Thus $(1-3x^2)^{1/2} - 1 \sim -\frac{3}{2}x^2$, $x \to 0$.

iii) The relation $\sin t = t + o(t)$, $t \to 0$, implies, by putting $t = 2x$, $x \sin 2x = x(2x + o(2x)) = 2x^2 + o(x^2)$, $x \to 0$. Then $x \sin 2x \sim 2x^2$, $x \to 0$. □

We explain now how to use the symbols of Landau for calculating limits. All maps dealt with below are supposed to be defined, and not to vanish, on a neighbourhood of c, except possibly at c.

Proposition 5.5 *Let us consider the limits*

$$\lim_{x\to c} f(x)g(x) \qquad \text{and} \qquad \lim_{x\to c} \frac{f(x)}{g(x)}.$$

Given functions $\tilde{f}$ and $\tilde{g}$ such that $\tilde{f} \sim f$ and $\tilde{g} \sim g$ for $x \to c$, then

$$\lim_{x\to c} f(x)g(x) = \lim_{x\to c} \tilde{f}(x)\tilde{g}(x), \tag{5.7}$$

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{\tilde{f}(x)}{\tilde{g}(x)}. \tag{5.8}$$

Proof. Start with (5.7). Then

$$\begin{aligned} \lim_{x\to c} f(x)g(x) &= \lim_{x\to c} \frac{f(x)}{\tilde{f}(x)}\tilde{f}(x)\frac{g(x)}{\tilde{g}(x)}\tilde{g}(x) \\ &= \lim_{x\to c} \frac{f(x)}{\tilde{f}(x)} \lim_{x\to c} \frac{g(x)}{\tilde{g}(x)} \lim_{x\to c} \tilde{f}(x)\tilde{g}(x). \end{aligned}$$

From the definition of $\tilde{f} \sim f$ and $\tilde{g} \sim g$ the result follows. The proof of (5.8) is completely analogous. □

Corollary 5.6 *Consider the limits*

$$\lim_{x\to c} \big(f(x)+f_1(x)\big)\big(g(x)+g_1(x)\big) \qquad \text{and} \qquad \lim_{x\to c} \frac{f(x)+f_1(x)}{g(x)+g_1(x)}.$$

If $f_1 = o(f)$ and $g_1 = o(g)$ when $x \to c$, then

$$\lim_{x\to c} \big(f(x)+f_1(x)\big)\big(g(x)+g_1(x)\big) = \lim_{x\to c} f(x)g(x), \tag{5.9}$$

$$\lim_{x\to c} \frac{f(x)+f_1(x)}{g(x)+g_1(x)} = \lim_{x\to c} \frac{f(x)}{g(x)}. \tag{5.10}$$

Proof. Set $\tilde{f} = f + f_1$; by assumption $\tilde{f} = f + o(f)$, so from (5.2) one has $\tilde{f} \sim f$. Similarly, putting $\tilde{g} = g + g_1$ yields $\tilde{g} \sim g$. The claim follows from the previous Proposition. □

The meaning of these properties is clear: when computing the limit of a product, we may substitute each factor with an equivalent function. Alternatively, one may ignore negligible summands with respect to others within one factor. In a similar way one can handle the limit of a quotient, numerator and denominator now being the 'factors'.

Examples 5.7

i) Compute

$$\lim_{x\to 0} \frac{1-\cos 2x}{\sin^2 3x}.$$

From the equivalence $1-\cos t \sim \frac{1}{2}t^2$, $t \to 0$, the substitution $t = 2x$ gives

$$1-\cos 2x \sim 2x^2, \quad x \to 0.$$

Putting $t = 3x$ in $\sin t \sim t$, $t \to 0$, we obtain $\sin 3x \sim 3x$, $x \to 0$, hence

$$\sin^2 3x \sim 9x^2, \quad x \to 0.$$

Therefore (5.8) implies

$$\lim_{x\to 0} \frac{1-\cos 2x}{\sin^2 3x} = \lim_{x\to 0} \frac{2x^2}{9x^2} = \frac{2}{9}.$$

ii) Evaluate

$$\lim_{x\to 0} \frac{\sin 2x + x^3}{4x + 5\log(1+x^2)}.$$

We shall show that for $x \to 0$, x^3 is negligible with respect to $\sin 2x$, and similarly $5\log(1+x^2)$ is negligible with respect to $4x$. With that, we can use the previous corollary and conclude

$$\lim_{x\to 0} \frac{\sin 2x + x^3}{4x + 5\log(1+x^2)} = \lim_{x\to 0} \frac{\sin 2x}{4x} = \frac{1}{2}.$$

Recall $\sin 2x \sim 2x$ for $x \to 0$; thus

$$\lim_{x\to 0} \frac{x^3}{\sin 2x} = \lim_{x\to 0} \frac{x^3}{2x} = 0,$$

that is to say $x^3 = o(\sin 2x)$ for $x \to 0$. On the other hand, since $\log(1+t) \sim t$ for $t \to 0$, writing $t = x^2$ yields $\log(1+x^2) \sim x^2$ when $x \to 0$. Then

$$\lim_{x\to 0} \frac{5\log(1+x^2)}{4x} = \lim_{x\to 0} \frac{5x^2}{4x} = 0,$$

i.e., $5\log(1+x^2) = o(4x)$ for $x \to 0$. □

These 'simplification' rules hold only in the case of products and quotients. *They do not apply to limits of sums or differences of functions.* Otherwise put, the fact that $\tilde{f} \sim f$ and $\tilde{g} \sim g$ when $x \to c$, does not allow to conclude that

$$\lim_{x\to c}[f(x) \pm g(x)] = \lim_{x\to c}[\tilde{f}(x) \pm \tilde{g}(x)].$$

For example set $f(x) = \sqrt{x^2+2x}$ and $g(x) = \sqrt{x^2-1}$ and consider the limit

$$\lim_{x\to+\infty} \left(\sqrt{x^2+2x} - \sqrt{x^2-1}\right).$$

Rationalisation turns this limit into

$$\lim_{x\to+\infty} \frac{(x^2+2x)-(x^2-1)}{\sqrt{x^2+2x}+\sqrt{x^2-1}} = \lim_{x\to+\infty} \frac{2x+1}{x\left(\sqrt{1+\frac{2}{x}}+\sqrt{1-\frac{1}{x^2}}\right)} = 1.$$

Had we substituted to $f(x)$ the function $\tilde{f}(x) = x$, equivalent to f for $x \to +\infty$, we would have obtained a different limit, actually a *wrong* one. In fact,

$$\lim_{x\to+\infty} \left(x - \sqrt{x^2-1}\right) = \lim_{x\to+\infty} \frac{x^2-(x^2-1)}{x+\sqrt{x^2-1}} = \lim_{x\to+\infty} \frac{1}{x(1+\sqrt{1-\frac{1}{x^2}})} = 0.$$

The reason for the mismatch lies in the *cancellation* of the leading term x^2 appearing in the numerator after rationalisation, which renders the terms of lesser degree important for the limit, even though they are negligible with respect to x^2 for $x \to +\infty$.

5.2 Infinitesimal and infinite functions

Definition 5.8 *Let f be a function defined in a neighbourhood of c, except possibly at c. Then f is said* **infinitesimal** (*or* **an infinitesimal**) *at c if*

$$\lim_{x\to c} f(x) = 0,$$

i.e., if $f = o(1)$ for $x \to c$. The function f is said **infinite** *at c if*

$$\lim_{x\to c} f(x) = \infty.$$

Let us introduce the following terminology to compare two infinitesimal or infinite maps.

Definition 5.9 *Let f, g be two infinitesimals at c.*
If $f \asymp g$ for $x \to c$, f and g are said **infinitesimals of the same order**.
If $f = o(g)$ for $x \to c$, f is called **infinitesimal of bigger order than** *g.*
If $g = o(f)$ for $x \to c$, f is called **infinitesimal of smaller order than** *g.*
If none of the above are satisfied, f and g are said **non-comparable** *infinitesimals.*

Definition 5.10 *Let f and g be two infinite maps at c.*
If $f \asymp g$ for $x \to c$, f and g are said to be **infinite of the same order**.
If $f = o(g)$ for $x \to c$, f is called **infinite of smaller order than** g.
If $g = o(f)$ for $x \to c$, f is called **infinite of bigger order than** g.
If none of the above are satisfied, the infinite functions f and g are said **non-comparable**.

Examples 5.11

Bearing in mind the fundamental limits seen above, it is immediate to verify the following facts:

i) $e^x - 1$ is an infinitesimal of the same order as x at the origin.

ii) $\sin x^2$ is an infinitesimal of bigger order than x at the origin.

iii) $\dfrac{\sin x}{(1-\cos x)^2}$ is infinite of bigger order than $\dfrac{1}{x}$ at the origin.

iv) For every $\alpha > 0$, e^x is infinite of bigger order than x^α for $x \to +\infty$.

v) For every $\alpha > 0$, $\log x$ is infinite of smaller order than $\dfrac{1}{x^\alpha}$ for $x \to 0^+$.

vi) The functions $f(x) = x \sin \frac{1}{x}$ and $g(x) = x$ are infinitesimal for x tending to 0 (for f recall Corollary 4.7). But the quotient $\dfrac{f(x)}{g(x)} = \sin \frac{1}{x}$ does not admit limit for $x \to 0$, for in any neighbourhood of 0 it attains every value between -1 and 1 infinitely many times. Therefore none of the conditions $f \asymp g$, $f = o(g)$, $g = o(f)$ hold for $x \to 0$. The two functions f and g are thus not comparable. □

Using a non-rigorous yet colourful language, we shall express the fact that f is infinitesimal (or infinite) of bigger order than g by saying that f tends to 0 (or ∞) *faster* than g. This suggests to measure the speed at which an infinitesimal (or infinite) map converges to its limit value.

For that purpose, let us fix an infinitesimal (or infinite) map φ defined in a neighbourhood of c and particularly easy to compute. We shall use it as term of comparison ('test function') and in fact call it an **infinitesimal test function** (or **infinite test function**) at c. When the limit behaviour is clear, we refer to φ as test function for brevity. The most common test functions (certainly *not* the only ones) are the following. If $c = x_0 \in \mathbb{R}$, we choose

$$\varphi(x) = x - x_0 \qquad \text{or} \qquad \varphi(x) = |x - x_0|$$

as infinitesimal test functions (the latter in case we need to consider non-integer powers of φ, see later), and

$$\varphi(x) = \frac{1}{x - x_0} \qquad \text{or} \qquad \varphi(x) = \frac{1}{|x - x_0|}$$

as infinite test functions. For $c = x_0^+$ ($c = x_0^-$), we will choose as infinitesimal test function

$$\varphi(x) = x - x_0 \qquad (\varphi(x) = x_0 - x)$$

and as infinite test function

$$\varphi(x) = \frac{1}{x - x_0} \qquad (\varphi(x) = \frac{1}{x_0 - x}\ ).$$

For $c = +\infty$, the infinitesimal and infinite test functions will respectively be

$$\varphi(x) = \frac{1}{x} \qquad \text{and} \qquad \varphi(x) = x,$$

while for $c = -\infty$, we shall take

$$\varphi(x) = \frac{1}{|x|} \qquad \text{and} \qquad \varphi(x) = |x|.$$

The definition of 'speed of convergence' of an infinitesimal or infinite f depends on how f compares to the powers of the infinitesimal or infinite test function. To be precise, we have the following definition

Definition 5.12 *Let f be infinitesimal (or infinite) at c. If there exists a real number $\alpha > 0$ such that*

$$f \asymp \varphi^\alpha, \quad x \to c, \tag{5.11}$$

the constant α is called the **order of f at c with respect to the infinitesimal (infinite) test function** *φ.*

Notice that if condition (5.11) holds, it determines the order uniquely. In the first case in fact, it is immediate to see that for any $\beta < \alpha$ one has $f = o(\varphi^\beta)$, while $\beta > \alpha$ implies $\varphi^\beta = o(f)$. A similar argument holds for infinite maps.

If f has order α at c with respect to the test function φ, then there is a real number $\ell \neq 0$ such that

$$\lim_{x \to c} \frac{f(x)}{\varphi^\alpha(x)} = \ell.$$

Rephrasing:

$$f \sim \ell\varphi^\alpha, \quad x \to c,$$

which is to say – recalling (5.2) – $f = \ell\varphi^\alpha + o(\ell\varphi^\alpha)$, for $x \to c$. For the sake of simplicity we can omit the constant ℓ in the symbol o, because if a function h satisfies $h = o(\ell\varphi^\alpha)$, then $h = o(\varphi^\alpha)$ as well. Therefore

$$f = \ell\varphi^\alpha + o(\varphi^\alpha), \quad x \to c.$$

Definition 5.13 *The function*

$$p(x) = \ell\varphi^\alpha(x) \tag{5.12}$$

is called the **principal part of the infinitesimal** (**infinite**) **map** f **at** c **with respect to the infinitesimal** (**infinite**) **test function** φ.

From the qualitative point of view the behaviour of the function f in a small enough neighbourhood of c coincides with the behaviour of its principal part (in geometrical terms, the two graphs resemble each other). With a suitable choice of test function φ, like one of those mentioned above, the behaviour of the function $\ell\varphi^\alpha(x)$ becomes immediately clear. So if one is able to determine the principal part of a function, even a complicated one, at a given point c, the local behaviour around that point is easily described.

We wish to stress that to find the order and the principal part of a function f at c, one must start from the limit

$$\lim_{x\to c} \frac{f(x)}{\varphi^\alpha(x)}$$

and understand if there is a number α for which such limit – say ℓ – is finite and different from zero. If so, α is the required order, and the principal part of f is given by (5.12).

Examples 5.14

i) The function $f(x) = \sin x - \tan x$ is infinitesimal for $x \to 0$. Using the basic equivalences of p. 127 and Proposition 5.5, we can write

$$\sin x - \tan x = \frac{\sin x\,(\cos x - 1)}{\cos x} \sim \frac{x \cdot \left(-\frac{1}{2}x^2\right)}{1} = -\frac{1}{2}x^3, \qquad x \to 0.$$

It follows that $f(x)$ is infinitesimal of order 3 at the origin with respect to the test function $\varphi(x) = x$; its principal part is $p(x) = -\frac{1}{2}x^3$.

ii) The function

$$f(x) = \sqrt{x^2+3} - \sqrt{x^2-1}$$

is infinitesimal for $x \to +\infty$. Rationalising the expression we get

$$f(x) = \frac{(x^2+3)-(x^2-1)}{\sqrt{x^2+3}+\sqrt{x^2-1}} = \frac{4}{x\left(\sqrt{1+\frac{3}{x^2}}+\sqrt{1-\frac{1}{x^2}}\right)}.$$

The right-hand side shows that if one chooses $\varphi(x) = \frac{1}{x}$ then

$$\lim_{x\to+\infty} \frac{f(x)}{\varphi(x)} = 2.$$

Therefore f is infinitesimal of first order for $x \to +\infty$ with respect to the test function $\frac{1}{x}$, with principal part $p(x) = \frac{2}{x}$.

iii) The function

$$f(x) = \sqrt{9x^5 + 7x^3 - 1}$$

is infinite when $x \to +\infty$. To determine its order with respect to $\varphi(x) = x$, we consider the limit

$$\lim_{x\to+\infty} \frac{f(x)}{x^\alpha} = \lim_{x\to+\infty} \frac{x^{\frac{5}{2}}\sqrt{9 + \frac{7}{x^2} - \frac{1}{x^5}}}{x^\alpha}.$$

By choosing $\alpha = \frac{5}{2}$ the limit becomes 3. So f has order $\frac{5}{2}$ for $x \to +\infty$ with respect to the test function $\varphi(x) = x$. The principal part is $p(x) = 3x^{5/2}$. □

Remark 5.15 The previous are typical instances of how to determine the order of a function with respect to some test map. The reader should not be mislead to believe that this is always possible. Given an infinitesimal or an infinite f at c, and having chosen a corresponding test map φ, it may well happen that there is no real number $\alpha > 0$ satisfying $f \asymp \varphi^\alpha$ for $x \to c$. In such a case it is convenient to make a different choice of test function, one more suitable to describe the behaviour of f around c. We shall clarify this fact with two examples.

Start by taking the function $f(x) = \mathrm{e}^{2x}$ for $x \to +\infty$. Using (5.6) a), it follows immediately that $x^\alpha = o(\mathrm{e}^{2x})$, whichever $\alpha > 0$ is considered. So it is not possible to determine an order for f with respect to $\varphi(x) = x$: the exponential map grows too quickly for any polynomial function to keep up with it. But if we take as test function $\varphi(x) = \mathrm{e}^x$ then clearly f has order 2 with respect to φ.

Consider now $f(x) = x \log x$ for $x \to 0^+$. In (5.6) d) we claimed that

$$\lim_{x\to 0^+} \frac{\log x}{\frac{1}{x^\beta}} = 0, \qquad \forall \beta > 0.$$

So in particular $f(x) = \dfrac{\log x}{1/x}$ is infinitesimal when $x \to 0^+$. Using the test function $\varphi(x) = x$ one sees that

$$\lim_{x\to 0^+} \frac{x \log x}{x^\alpha} = \lim_{x\to 0^+} \frac{\log x}{x^{\alpha-1}} = \begin{cases} 0 & \text{if } \alpha < 1, \\ -\infty & \text{otherwise.} \end{cases}$$

Definition 5.9 yields that f is an infinitesimal of *bigger* order than any power of x with exponent less than one. At the same time it has *smaller* order than x and all powers with exponent greater than one. In this case too, it is not possible to determine the order of f with respect to x. The function $|f(x)| = x|\log x|$ goes to zero more slowly than x, yet faster than x^α for any $\alpha < 1$. Thus it can be used as alternative infinitesimal test map when $x \to 0^+$. □

5.3 Asymptotes

We now consider a function f defined in a neighbourhood of $+\infty$ and wish to study its behaviour for $x \to +\infty$. A remarkable case is that in which f behaves as a polynomial of first degree. Geometrically speaking, this corresponds to the fact that the graph of f will more and more look like a straight line. Precisely, we suppose there exist two real numbers m and q such that

$$\lim_{x\to+\infty} \big(f(x) - (mx+q)\big) = 0, \tag{5.13}$$

or, using the symbols of Landau,

$$f(x) = mx + q + o(1)\,, \qquad x \to +\infty.$$

We then say that the line $g(x) = mx + q$ is a **right asymptote** of the function f. The asymptote is called **oblique** if $m \neq 0$, **horizontal** if $m = 0$. In geometrical terms condition (5.13) tells that the vertical distance $d(x) = |f(x) - g(x)|$ between the graph of f and the asymptote tends to 0 as $x \to +\infty$ (Fig. 5.1).

The asymptote's coefficients can be recovered using limits:

$$m = \lim_{x\to+\infty} \frac{f(x)}{x} \qquad \text{and} \qquad q = \lim_{x\to+\infty} \big(f(x) - mx\big)\,. \tag{5.14}$$

The first relation comes from (5.13) noting that

$$0 = \lim_{x\to+\infty} \frac{f(x)-mx-q}{x} = \lim_{x\to+\infty} \frac{f(x)}{x} - \lim_{x\to+\infty} \frac{mx}{x} - \lim_{x\to+\infty} \frac{q}{x} = \lim_{x\to+\infty} \frac{f(x)}{x} - m,$$

while the second one follows directly from (5.13). The conditions (5.14) furnish the means to find the possible asymptote of a function f. If in fact both limits exist

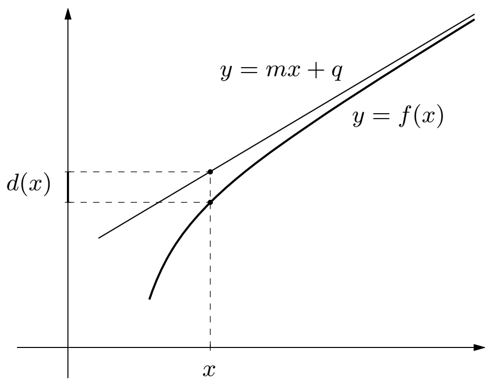


Figure 5.1. Graph of a function with its right asymptote

and are finite, f admits $y = mx + q$ as a right asymptote. If only one of (5.14) is not finite instead, then f will not have an asymptote.

Notice that if f has an oblique asymptote, i.e., if $m \neq 0$, the first of (5.14) tells us that f is infinite of order 1 with respect to the test function $\varphi(x) = x$ for $x \to +\infty$. The reader should beware that not all functions satisfying the latter condition do admit an oblique asymptote: the function $f(x) = x + \sqrt{x}$ for example is equivalent to x for $x \to +\infty$, but has no asymptote since the second limit in (5.14) is $+\infty$.

Remark 5.16 The definition of (linear) asymptote given above is a particular instance of the following. The function f is called **asymptotic** to a function g for $x \to +\infty$ if

$$\lim_{x \to +\infty} (f(x) - g(x)) = 0.$$

If (5.13) holds one can then say that f is asymptotic to the line $g(x) = mx + q$. The function $f(x) = x^2 + \frac{1}{x}$ instead has no line as asymptote for $x \to +\infty$, but is nevertherless asymptotic to the parabola $g(x) = x^2$. □

In a similar fashion one defines oblique or horizontal asymptotes for $x \to -\infty$ (that is oblique or horizontal left asymptotes).

If the line $y = mx + q$ is an oblique or horizontal asymptote both for $x \to +\infty$ and $x \to -\infty$, we shall say that it is a **complete oblique** or **complete horizontal asymptote** for f.

Eventually, if at a point $x_0 \in \mathbb{R}$ one has $\lim\limits_{x \to x_0} f(x) = \infty$, the line $x = x_0$ is called a **vertical asymptote** for f at x_0. The distance between points on the graph of f and on a vertical asymptote with the same y-coordinate converges to zero for $x \to x_0$. If the limit condition holds only for $x \to x_0^+$ or $x \to x_0^-$ we talk about a vertical **right** or **left** asymptote respectively.

Examples 5.17

i) Let $f(x) = \dfrac{x}{x+1}$. As

$$\lim_{x \to \pm\infty} f(x) = 1 \qquad \text{and} \qquad \lim_{x \to -1^{\pm}} f(x) = \mp\infty,$$

the function has a horizontal asymptote $y = 1$ and a vertical asymptote $x = -1$.

ii) The map $f(x) = \sqrt{1 + x^2}$ satisfies

$$\lim_{x \to \pm\infty} f(x) = +\infty, \qquad \lim_{x \to \pm\infty} \frac{f(x)}{x} = \lim_{x \to \pm\infty} \frac{|x|\sqrt{1 + x^{-2}}}{x} = \pm 1$$

and

$$\lim_{x \to +\infty} \left(\sqrt{1 + x^2} - x\right) = \lim_{x \to +\infty} \frac{1 + x^2 - x^2}{\sqrt{1 + x^2} + x} = 0\,,$$

$$\lim_{x \to -\infty} \left(\sqrt{1 + x^2} + x\right) = \lim_{x \to -\infty} \frac{1 + x^2 - x^2}{\sqrt{1 + x^2} - x} = 0\,.$$

Therefore f has an oblique asymptote for $x \to +\infty$ given by $y = x$, plus another one of equation $y = -x$ for $x \to -\infty$.

iii) Let $f(x) = x + \log x$. Since

$$\lim_{x\to 0^+} (x + \log x) = -\infty, \qquad \lim_{x\to +\infty} (x + \log x) = +\infty,$$

$$\lim_{x\to +\infty} \frac{x + \log x}{x} = 1, \qquad \lim_{x\to +\infty} (x + \log x - x) = +\infty,$$

the function has a vertical right asymptote $x = 0$ but no horizontal nor oblique asymptotes. □

5.4 Further properties of sequences

We return to the study of the limit behaviour of sequences begun in Sect. 3.2. General theorems concerning functions apply to sequences as well (the latter being particular functions defined over the integers, after all). For the sake of completeness those results will be recalled, and adapted to the case of concern. We shall also state and prove other specific properties of sequences.

We say that a sequence $\{a_n\}_{n\geq n_0}$ satisfies a given property **eventually**, if there exists an integer $N \geq n_0$ such that the sequence $\{a_n\}_{n\geq N}$ satisfies that property. This definition allows for a more flexible study of sequences.

Theorems on sequences

1. *Uniqueness of the limit*: the limit of a sequence, when defined, is unique.
2. *Boundedness*: a converging sequence is bounded.
3. *Existence of limit for monotone sequences*: if an eventually monotone sequence is bounded, then it converges; if not bounded then it diverges (to $+\infty$ if increasing, to $-\infty$ if decreasing).
4. *First comparison theorem*: let $\{a_n\}$ and $\{b_n\}$ be sequences with finite or infinite limits $\lim\limits_{n\to\infty} a_n = \ell$ and $\lim\limits_{n\to\infty} b_n = m$. If $a_n \leq b_n$ eventually, then $\ell \leq m$.
5. *Second comparison theorem ("Squeeze rule")*: let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be sequences with $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} c_n = \ell$. If $a_n \leq b_n \leq c_n$ eventually, then $\lim\limits_{n\to\infty} b_n = \ell$.
6. *Theorem*: a sequence $\{a_n\}$ is infinitesimal, that is $\lim\limits_{n\to\infty} a_n = 0$, if and only if the sequence $\{|a_n|\}$ is infinitesimal.
7. *Theorem*: let $\{a_n\}$ be an infinitesimal sequence and $\{b_n\}$ a bounded one. Then the sequence $\{a_n b_n\}$ is infinitesimal.

8. *Algebra of limits*: let $\{a_n\}$ and $\{b_n\}$ be such that $\lim\limits_{n\to\infty} a_n = \ell$ and $\lim\limits_{n\to\infty} b_n = m$ (ℓ, m finite or infinite). Then

$$\lim_{n\to\infty} (a_n \pm b_n) = \ell \pm m\,,$$
$$\lim_{n\to\infty} a_n\, b_n = \ell\, m\,,$$
$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\ell}{m}, \qquad \text{if } b_n \neq 0 \text{ eventually}\,,$$

each time the right-hand sides are defined according to the Table on p. 96.

9. *Substitution theorem*: let $\{a_n\}$ be a sequence with $\lim\limits_{n\to\infty} a_n = \ell$ and suppose g is a function defined in a neighbourhood of ℓ:
a) if $\ell \in \mathbb{R}$ and g is continuous at ℓ, then $\lim\limits_{n\to\infty} g(a_n) = g(\ell)$;
b) if $\ell \notin \mathbb{R}$ and $\lim\limits_{x\to\ell} g(x) = m$ exists, then $\lim\limits_{n\to\infty} g(a_n) = m$.

Proof. We shall only prove Theorem 2 since the others are derived adapting the similar proofs given for functions.
Let the sequence $\{a_n\}_{n\geq n_0}$ be given, and suppose it converges to $\ell \in \mathbb{R}$. With $\varepsilon = 1$ fixed, there exists an integer $n_1 \geq n_0$ so that $|a_n - \ell| < 1$ for all $n > n_1$. For such n's then the triangle inequality (1.1) yields

$$|a_n| = |a_n - \ell + \ell| \leq |a_n - \ell| + |\ell| < 1 + |\ell|.$$

By putting $M = \max\{|a_{n_0}|, \ldots, |a_{n_1}|, 1 + |\ell|\}$ one obtains $|a_n| \leq M$, $\forall n \geq n_0$. □

Examples 5.18

i) Consider the sequence $a_n = q^n$,where q is a fixed number in $\mathbb{R}$. It goes under the name of *geometric sequence*. We claim that

$$\lim_{n\to\infty} q^n = \begin{cases} 0 & \text{if } |q| < 1, \\ 1 & \text{if } q = 1, \\ +\infty & \text{if } q > 1, \\ \text{does not exist} & \text{if } q \leq -1. \end{cases}$$

If either $q = 0$ or $q = 1$, the sequence is constant and thus trivially convergent to 0 or 1 respectively. When $q = -1$ the sequence is indeterminate.
Let $q > 1$: the sequence is now strictly increasing and so admits a limit. In order to show that the limit is indeed $+\infty$ we write $q = 1 + r$ with $r > 0$ and apply the binomial formula (1.13):

$$q^n = (1+r)^n = \sum_{k=0}^{n} \binom{n}{k} r^k = 1 + nr + \sum_{k=2}^{n} \binom{n}{k} r^k.$$

As all terms in the last summation are positive, we obtain

$$(1+r)^n \geq 1 + nr\,, \qquad \forall n \geq 0\,, \tag{5.15}$$

called **Bernoulli inequality**[1]. Therefore $q^n \geq 1 + nr$; passing to the limit for $n \to \infty$ and using the First comparison theorem we can conclude.

Let us examine the case $|q| < 1$ with $q \neq 0$. We just saw that $\dfrac{1}{|q|} > 1$ implies $\lim\limits_{n\to\infty} \left(\dfrac{1}{|q|}\right)^n = +\infty$. The sequence $\{|q|^n\}$ is thus infinitesimal, and so is $\{q^n\}$.

At last, take $q < -1$. Since

$$\lim_{k\to\infty} q^{2k} = \lim_{k\to\infty} (q^2)^k = +\infty, \qquad \lim_{k\to\infty} q^{2k+1} = q \lim_{k\to\infty} q^{2k} = -\infty,$$

the sequence q^n is indeterminate.

ii) Let p be a fixed positive number and consider the sequence $\sqrt[n]{p}$. Applying the Substitution theorem with $g(x) = p^x$ we have

$$\lim_{n\to\infty} \sqrt[n]{p} = \lim_{n\to\infty} p^{1/n} = p^0 = 1\,.$$

iii) Consider the sequence $\sqrt[n]{n}$; using once again the Substitution theorem together with (5.6) c), it follows that

$$\lim_{n\to\infty} \sqrt[n]{n} = \lim_{n\to\infty} \exp \frac{\log n}{n} = \mathrm{e}^0 = 1.$$

□

There are easy criteria to decide whether a sequence is infinitesimal or infinite. Among them, the following is the most widely employed.

Theorem 5.19 (Ratio test) *Let $\{a_n\}$ be a sequence for which $a_n > 0$ eventually. Suppose the limit*

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = q$$

exists, finite or infinite. If $q < 1$ then $\lim\limits_{n\to\infty} a_n = 0$; if $q > 1$ then $\lim\limits_{n\to\infty} a_n = +\infty$.

[1] By the Principle of Induction, one can prove that (5.15) actually holds for any $r \geq -1$; see Appendix A.1, p. 427.

Proof. Suppose $a_n > 0$, $\forall n \geq n_0$. Take $q < 1$ and set $\varepsilon = 1 - q$. By definition of limit there exists an integer $n_\varepsilon \geq n_0$ such that for all $n > n_\varepsilon$

$$\frac{a_{n+1}}{a_n} < q + \varepsilon = 1\,, \quad \text{i.e.,} \quad a_{n+1} < a_n.$$

So the sequence $\{a_n\}$ is monotone decreasing eventually, and as such it admits a finite non-negative limit ℓ. Now if ℓ were different from zero, the fact that

$$q = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{\ell}{\ell} = 1$$

would contradict the assumption $q < 1$.
If $q > 1$, it is enough to consider the sequence $\{1/a_n\}$. □

Nothing can be said if $q = 1$.

Remark 5.20 The previous theorem has another proof, which emphasizes the speed at which a sequence converges to 0 or $+\infty$. Take for example the case $q < 1$. The definition of limit tells that for all r with $q < r < 1$, if one puts $\varepsilon = r - q$ there is a $n_\varepsilon \geq n_0$ such that

$$\frac{a_{n+1}}{a_n} < r \quad \text{that is,} \quad a_{n+1} < r a_n$$

for each $n > n_\varepsilon$. Repeating the argument leads to

$$a_{n+1} < r a_n < r^2 a_{n-1} < \ldots < r^{n-n_\varepsilon} a_{n_\varepsilon+1} \tag{5.16}$$

(a precise proof of which requires the Principle of Induction; see Appendix A.1, p. 430). The First comparison test and the limit behaviour of the geometric sequence (Example 5.18 i)) allow to conclude. Formula (5.16) shows that the smaller q is, the faster the sequence $\{a_n\}$ goes to 0.
Similar considerations hold when $q > 1$. □

At last we consider a few significant sequences converging to $+\infty$. We compare their limit behaviour using Definition 5.10. To be precise we examine the sequences

$$\log n\,, \ n^\alpha\,, \ q^n\,, \ n!\,, \ n^n \quad (\alpha > 0, \ q > 1)$$

and show that each sequence is infinite of order bigger than the one preceding it. Comparing the first two is immediate, for the Substitution theorem and (5.6) c) yield $\log n = o(n^\alpha)$ for $n \to \infty$.
The remaining cases are tackled by applying the Ratio test 5.19 to the quotient of two nearby sequences. Precisely, let us set $a_n = \dfrac{n^\alpha}{q^n}$. Then

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^\alpha}{q^{n+1}} \frac{q^n}{n^\alpha} = \left(\frac{n+1}{n}\right)^\alpha \frac{1}{q} \quad \to \quad \frac{1}{q} < 1\,, \qquad n \to \infty.$$

Thus $\lim_{n\to\infty} a_n = 0$, or $n^\alpha = o(q^n)$ for $n \to \infty$.

Now take $a_n = \dfrac{q^n}{n!}$, so

$$\frac{a_{n+1}}{a_n} = \frac{q^{n+1}}{(n+1)!}\,\frac{n!}{q^n} = \frac{q}{(n+1)n!}\,n! = \frac{q}{n+1} \quad \to \quad 0 < 1\,, \qquad n \to \infty\,,$$

and then $q^n = o(n!)$ per $n \to \infty$.
Eventually, let $a_n = \dfrac{n!}{n^n}$. Then

$$\begin{aligned}\frac{a_{n+1}}{a_n} &= \frac{(n+1)!}{(n+1)^{n+1}}\,\frac{n^n}{n!} = \frac{(n+1)n!}{(n+1)(n+1)^n}\,\frac{n^n}{n!} = \left(\frac{n}{n+1}\right)^n \\ &= \frac{1}{\left(\frac{n+1}{n}\right)^n} = \frac{1}{\left(1+\frac{1}{n}\right)^n} \quad \to \quad \frac{1}{\mathrm{e}} < 1\,, \qquad n \to \infty\,,\end{aligned}$$

and so $n! = o(n^n)$ for $n \to \infty$. To be more precise, one could actually prove the so-called **Stirling formula**,

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{\mathrm{e}}\right)^n\,, \qquad n \to \infty\,,$$

a helpful approximation of the factorial of large natural numbers.

5.5 Numerical series

Consider a segment of length $\ell = 2$ (Fig. 5.2). The middle point splits it into two parts of length $a_0 = \ell/2 = 1$. While keeping the left half fixed, we further subdivide the right one in two parts of length $a_1 = \ell/4 = 1/2$. Iterating the process indefinitely one can think of the initial segment as the union of infinitely many 'left' segments of lengths $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$ Correspondingly, the total length of the starting segment can be thought of as sum of the lengths of all sub-segments, in other words

$$2 = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots \tag{5.17}$$

On the right we have a sum of infinitely many terms. The notion of infinite sum can be defined properly using sequences, and leads to numerical series.

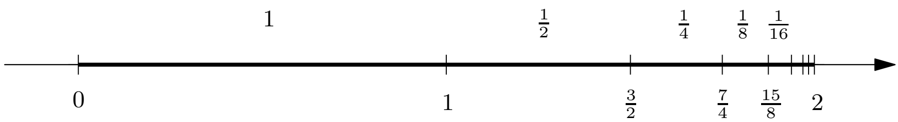


Figure 5.2. Successive splittings of the interval $[0, 2]$. The coordinates of the subdivision points are indicated below the blue line, while the lengths of subintervals lie above it

Given the sequence $\{a_k\}_{k\geq 0}$, one constructs the so-called **sequence of partial sums** $\{s_n\}_{n\geq 0}$ in the following manner:

$$s_0 = a_0\,, \qquad s_1 = a_0 + a_1\,, \qquad s_2 = a_0 + a_1 + a_2,$$

and in general

$$s_n = a_0 + a_1 + \ldots + a_n = \sum_{k=0}^{n} a_k\,.$$

Note that $s_n = s_{n-1} + a_n$. Then it is only natural to study the limit behaviour of such a sequence. Let us (formally) define

$$\sum_{k=0}^{\infty} a_k = \lim_{n\to\infty} \sum_{k=0}^{n} a_k = \lim_{n\to\infty} s_n\,.$$

The symbol $\sum_{k=0}^{\infty} a_k$ is called (numerical) **series**, and a_k is the **general term** of the series.

Definition 5.21 *Given the sequence* $\{a_k\}_{k\geq 0}$ *and* $s_n = \sum_{k=0}^{n} a_k$, *consider the limit* $\lim_{n\to\infty} s_n$.

i) If the limit exists and is finite, we say that **the series** $\sum_{k=0}^{\infty} a_k$ **converges**. *The value* s *of the limit is called* **sum of the series** *and one writes* $s = \sum_{k=0}^{\infty} a_k$.

ii) If the limit exists and is infinite, we say that **the series** $\sum_{k=0}^{\infty} a_k$ **diverges**.

iii) If the limit does not exist, we say that **the series** $\sum_{k=0}^{\infty} a_k$ **is indeterminate**.

Examples 5.22

i) Let us go back to the interval split infinitely many times. The length of the shortest segment obtained after $k+1$ subdivisions is $a_k = \frac{1}{2^k}$, $k \geq 0$. Thus, we consider the series $\sum_{k=0}^{\infty} \dfrac{1}{2^k}$. Its partial sums read

$$s_0 = 1\,, \qquad s_1 = 1 + \frac{1}{2} = \frac{3}{2}\,, \qquad s_2 = 1 + \frac{1}{2} + \frac{1}{4} = \frac{7}{4}\,,$$
$$\vdots$$
$$s_n = 1 + \frac{1}{2} + \ldots + \frac{1}{2^n}\,.$$

Using the fact that $a^{n+1} - b^{n+1} = (a-b)(a^n + a^{n-1}b + \ldots + ab^{n-1} + b^n)$, and choosing $a = 1$ and $b = x$ arbitrary but different from one, we obtain the identity

$$1 + x + \ldots + x^n = \frac{1 - x^{n+1}}{1 - x}\,. \tag{5.18}$$

Therefore

$$s_n = 1 + \frac{1}{2} + \ldots + \frac{1}{2^n} = \frac{1 - \frac{1}{2^{n+1}}}{1 - \frac{1}{2}} = 2\left(1 - \frac{1}{2^{n+1}}\right) = 2 - \frac{1}{2^n},$$

and so

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \left(2 - \frac{1}{2^n}\right) = 2\,.$$

The series converges and its sum is 2. This provides solid ground for having written (5.17) earlier.

ii) Consider the series $\sum_{k=0}^{\infty} k$. Recalling (3.2), we have

$$s_n = \sum_{k=0}^{n} k = \frac{n(n+1)}{2}\,.$$

Then

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{n(n+1)}{2} = +\infty\,,$$

and the series diverges (to $+\infty$).

iii) The partial sums of the series $\sum_{k=0}^{\infty} (-1)^k$ satisfy

$$\begin{array}{ll} s_0 = 1\,, & s_1 = 1 - 1 = 0 \\ s_2 = s_1 + 1 = 1 & s_3 = s_2 - 1 = 0 \\ & \vdots \\ s_{2n} = 1 & s_{2n+1} = 0\,. \end{array}$$

The terms with even index are all equal to 1 while the odd ones are 0. Therefore $\lim_{n\to\infty} s_n$ cannot exist and the series is indeterminate. □

Sometimes the sequence $\{a_k\}$ is only defined for $k \geq k_0$ with $k_0 > 0$; Definition 5.21 then modifies in the obvious way. The following fact holds, whose rather immediate proof is left to the reader.

Property 5.23 *The behaviour of a series does not change by adding, changing or removing a finite number of terms.*

This property does not tell anything about the sum of a converging series, which in general changes by manipulating the terms. For instance

$$\sum_{k=1}^{\infty} \frac{1}{2^k} = \sum_{k=0}^{\infty} \frac{1}{2^k} - 1 = 2 - 1 = 1\,.$$

Examples 5.24

i) The series $\displaystyle\sum_{k=2}^{\infty} \frac{1}{(k-1)k}$ is called **series of Mengoli**. As

$$a_k = \frac{1}{(k-1)k} = \frac{1}{k-1} - \frac{1}{k},$$

it follows that

$$\begin{aligned} s_2 &= a_2 = \frac{1}{1\cdot 2} = 1 - \frac{1}{2} \\ s_3 &= a_2 + a_3 = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) = 1 - \frac{1}{3}, \end{aligned}$$

and in general

$$s_n = a_2 + a_3 + \ldots + a_n = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \ldots + \left(\frac{1}{n-1} - \frac{1}{n}\right) = 1 - \frac{1}{n}\,.$$

Thus

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \left(1 - \frac{1}{n}\right) = 1$$

and the series converges to 1.

ii) For the series $\displaystyle\sum_{k=1}^{\infty} \log\left(1 + \frac{1}{k}\right)$ one has

$$a_k = \log\left(1 + \frac{1}{k}\right) = \log\frac{k+1}{k} = \log(k+1) - \log k$$

so

$$\begin{aligned} s_1 &= \log 2 \\ s_2 &= \log 2 + (\log 3 - \log 2) = \log 3 \\ &\ \vdots \\ s_n &= \log 2 + (\log 3 - \log 2) + \ldots + (\log(n+1) - \log n) = \log(n+1)\,. \end{aligned}$$

Then

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \log(n+1) = +\infty$$

and the series diverges (to $+\infty$). □

The two instances just considered belong to the larger class of **telescopic series**. These are defined by $a_k = b_{k+1} - b_k$ for a suitable sequence $\{b_k\}_{k \geq k_0}$. Since $s_n = b_{n+1} - b_{k_0}$, the behaviour of a telescopic series is the same as that of the sequence $\{b_k\}$.

We shall now present a simple yet useful necessary condition for a numerical series to converge.

Property 5.25 *Let* $\sum_{k=0}^{\infty} a_k$ *be a converging series. Then*

$$\lim_{k \to \infty} a_k = 0 \,. \tag{5.19}$$

Proof. Let $s = \lim_{n \to \infty} s_n$. Since $a_k = s_k - s_{k-1}$, then

$$\lim_{k \to \infty} a_k = \lim_{k \to \infty} (s_k - s_{k-1}) = s - s = 0 \,,$$

i.e., $\{a_k\}$ is infinitesimal. □

Observe that condition (5.19) is not sufficient to guarantee that the series converge. The general term of a series may tend to 0 without the series having to converge. For example we saw that the series $\sum_{k=1}^{\infty} \log\left(1 + \frac{1}{k}\right)$ diverges, but at the same time $\lim_{k \to \infty} \log\left(1 + \frac{1}{k}\right) = 0$ (Example 5.24 ii)).

If a series converges to s, the quantity

$$r_n = s - s_n = \sum_{k=n+1}^{\infty} a_k \,.$$

is called n**th remainder**.

Property 5.26 *Take a converging series* $\sum_{k=0}^{\infty} a_k$. *Then the remainder satisfies*

$$\lim_{n \to \infty} r_n = 0 \,.$$

Proof. Indeed,

$$\lim_{n \to \infty} r_n = \lim_{n \to \infty} (s - s_n) = s - s = 0 \,.$$

□

Example 5.27

Consider the **geometric series**

$$\sum_{k=0}^{\infty} q^k ,$$

where q is a fixed number in $\mathbb{R}$.
If $q = 1$ then $s_n = a_0 + a_1 + \ldots + a_n = 1 + 1 + \ldots + 1 = n + 1$ and $\lim_{n\to\infty} s_n = +\infty$, whence the series diverges to $+\infty$.
If $q \neq 1$ instead, (5.18) implies

$$s_n = 1 + q + q^2 + \ldots + q^n = \frac{1 - q^{n+1}}{1 - q} .$$

Example 5.18 gives

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{1 - q^{n+1}}{1 - q} = \begin{cases} \dfrac{1}{1-q} & \text{if } |q| < 1 , \\ +\infty & \text{if } q > 1 , \\ \text{does not exist} & \text{if } q \leq -1 . \end{cases}$$

In conclusion,

$$\sum_{k=0}^{\infty} q^k \quad \begin{cases} \text{converges to } \dfrac{1}{1-q} & \text{if } |q| < 1 , \\ \text{diverges to } \; +\infty & \text{if } q \geq 1 , \\ \text{is indeterminate} & \text{if } q \leq -1 . \end{cases}$$

□

That said, it is not always possible to predict the behaviour of a series $\sum_{k=0}^{\infty} a_k$ using merely the definition. It may well happen that the sequence of partial sums cannot be computed explicitly, so it becomes important to have other ways to establish whether the series converges or not. Only in case of convergence, it could be necessary to determine the actual sum. This may require using more sophisticated techniques, which go beyond the scopes of this text.

5.5.1 Positive-term series

We deal with series $\sum_{k=0}^{\infty} a_k$ for which $a_k \geq 0$ for any $k \in \mathbb{N}$. The following result holds.

Proposition 5.28 *A series* $\sum_{k=0}^{\infty} a_k$ *with positive terms either converges or diverges to* $+\infty$.

Proof. The sequence s_n is monotonically increasing since

$$s_{n+1} = s_n + a_n \geq s_n\,, \qquad \forall n \geq 0\,.$$

It is then sufficient to use Theorem 3.9 to conclude that $\lim_{n\to\infty} s_n$ exists, and is either finite or $+\infty$. □

We list now a few tools for studying the convergence of positive-term series.

Theorem 5.29 (**Comparison test**) *Let* $\sum_{k=0}^{\infty} a_k$ *and* $\sum_{k=0}^{\infty} b_k$ *be positive-term series such that* $0 \leq a_k \leq b_k$*, for any* $k \geq 0$.

i) If the series $\sum_{k=0}^{\infty} b_k$ *converges, then also the series* $\sum_{k=0}^{\infty} a_k$ *converges and* $\sum_{k=0}^{\infty} a_k \leq \sum_{k=0}^{\infty} b_k$.

ii) If $\sum_{k=0}^{\infty} a_k$ *diverges, then* $\sum_{k=0}^{\infty} b_k$ *diverges as well.*

Proof. *i)* Denote by $\{s_n\}$ and $\{t_n\}$ the sequences of partial sums of $\sum_{k=0}^{\infty} a_k$, $\sum_{k=0}^{\infty} b_k$ respectively. Since $a_k \leq b_k$ for all k,

$$s_n \leq t_n\,, \qquad \forall n \geq 0\,.$$

By assumption, the series $\sum_{k=0}^{\infty} b_k$ converges, so $\lim_{n\to\infty} t_n = t \in \mathbb{R}$. Proposition 5.28 implies the limit $\lim_{n\to\infty} s_n = s$ exists, finite or infinite. By the First comparison theorem (Theorem 4, p. 137) we have

$$s = \lim_{n\to\infty} s_n \leq \lim_{n\to\infty} t_n = t \in \mathbb{R}\,.$$

Therefore $s \in \mathbb{R}$, and the series $\sum_{k=0}^{\infty} a_k$ converges. Furthermore $s \leq t$.

ii) If the series $\sum_{k=0}^{\infty} b_k$ converged, part *i)* of this proof would force $\sum_{k=0}^{\infty} a_k$ to converge, too. □

Examples 5.30

i) Consider $\sum_{k=1}^{\infty} \frac{1}{k^2}$. Since

$$\frac{1}{k^2} < \frac{1}{(k-1)k} \qquad \forall k \geq 2\,,$$

and the series of Mengoli $\sum_{k=2}^{\infty} \frac{1}{(k-1)k}$ converges (Example 5.24 i)), we conclude that our series converges and its sum is smaller or equal than 2. One could prove that the precise value of the sum is $\pi^2/6$.

ii) The series $\sum_{k=1}^{\infty} \frac{1}{k}$ is known as **harmonic series**. In Chap. 6 (Exercise 12) we shall prove the inequality $\log(1+x) \leq x$, for all $x > -1$, whereby

$$\log\Big(1+\frac{1}{k}\Big) \leq \frac{1}{k}\,, \qquad \forall k \geq 1.$$

Since the series $\sum_{k=1}^{\infty} \log\Big(1+\frac{1}{k}\Big)$ diverges (Example 5.24 ii)), then also the harmonic series must diverge. □

Here is a useful criterion that generalizes the Comparison test.

Theorem 5.31 (**Asymptotic comparison test**) *Let $\sum_{k=0}^{\infty} a_k$ and $\sum_{k=0}^{\infty} b_k$ be positive-term series and suppose the sequences $\{a_k\}_{k\geq 0}$ and $\{b_k\}_{k\geq 0}$ have the same order of magnitude for $k \to \infty$. Then the series have the same behaviour.*

Proof. Having the same order of magnitude for $k \to \infty$ is equivalent to

$$\lim_{k\to\infty} \frac{a_k}{b_k} = \ell \in \mathbb{R} \setminus \{0\}\,.$$

Therefore the sequences $\Big\{\frac{a_k}{b_k}\Big\}_{k\geq 0}$ and $\Big\{\frac{b_k}{a_k}\Big\}_{k\geq 0}$ are both convergent, hence both bounded (Theorem 2, p. 137). So, there must exist constants $M_1, M_2 > 0$ such that

$$\left|\frac{a_k}{b_k}\right| \leq M_1 \qquad \text{and} \qquad \left|\frac{b_k}{a_k}\right| \leq M_2$$

for any $k > 0$, i.e.,

$$|a_k| \le M_1 |b_k| \qquad \text{and} \qquad |b_k| \le M_2 |a_k|\,.$$

Now it suffices to use Theorem 5.29 to finish the proof. □

Examples 5.32

i) Consider $\sum_{k=0}^{\infty} a_k = \sum_{k=0}^{\infty} \frac{k+3}{2k^2+5}$ and let $b_k = \frac{1}{k}$. Then

$$\lim_{k\to\infty} \frac{a_k}{b_k} = \frac{1}{2}$$

and the given series behaves as the harmonic series, hence diverges.

ii) Take the series $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \sin\frac{1}{k^2}$. As $\sin\frac{1}{k^2} \sim \frac{1}{k^2}$ for $k \to \infty$, the series has the same behaviour of $\sum_{k=1}^{\infty} \frac{1}{k^2}$, so it converges. □

Eventually, here are two more results – of algebraic flavour and often easy to employ – which provide sufficient conditions for a series to converge or diverge.

Theorem 5.33 (Ratio test) *Let $\sum_{k=0}^{\infty} a_k$ have $a_k > 0$, $\forall k \ge 0$. Assume the limit*

$$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \ell$$

exists, finite or infinite. If $\ell < 1$ the series converges; if $\ell > 1$ it diverges.

Proof. First take ℓ finite. By definition of limit we know that for any $\varepsilon > 0$, there is an integer $k_\varepsilon \ge 0$ such that for all $k > k_\varepsilon$ one has

$$\left| \frac{a_{k+1}}{a_k} - \ell \right| < \varepsilon \quad \text{i.e.,} \quad \ell - \varepsilon < \frac{a_{k+1}}{a_k} < \ell + \varepsilon\,.$$

Assume $\ell < 1$. Choose $\varepsilon = \frac{1-\ell}{2}$ and set $q = \frac{1+\ell}{2}$, so

$$0 < \frac{a_{k+1}}{a_k} < \ell + \varepsilon = q\,, \qquad \forall k > k_\varepsilon\,.$$

Repeating the argument we obtain

$$a_{k+1} < q a_k < q^2 a_{k-1} < \ldots < q^{k-k_\varepsilon} a_{k_\varepsilon+1}$$

hence

$$a_{k+1} < \frac{a_{k_\varepsilon+1}}{q^{k_\varepsilon}} q^k, \quad \forall k > k_\varepsilon.$$

The claim follows by Theorem 5.29 and from the fact that the geometric series, with $q < 1$, converges (Example 5.27).
Now consider $\ell > 1$. Choose $\varepsilon = \ell - 1$, and notice

$$1 = \ell - \varepsilon < \frac{a_{k+1}}{a_k}, \quad \forall k > k_\varepsilon.$$

Thus $a_{k+1} > a_k > \ldots > a_{k_\varepsilon+1} > 0$, so the necessary condition for convergence fails, for $\lim_{k\to\infty} a_k \neq 0$.
Eventually, if $\ell = +\infty$, we put $A = 1$ in the condition of limit, and there exists $k_A \geq 0$ with $a_k > 1$, for any $k > k_A$. Once again the necessary condition to have convergence does not hold. □

Theorem 5.34 (**Root test**) *Given a series* $\sum_{k=0}^{\infty} a_k$ *with non-negative terms, suppose*

$$\lim_{k\to\infty} \sqrt[k]{a_k} = \ell$$

exists, finite or infinite. If $\ell < 1$ *the series converges, if* $\ell > 1$ *it diverges.*

Proof. Since this proof is essentially identical to the previous one, we leave it to the reader. □

Examples 5.35

i) For $\sum_{k=0}^{\infty} \frac{k}{3^k}$ we have $a_k = \frac{k}{3^k}$ and $a_{k+1} = \frac{k+1}{3^{k+1}}$, therefore

$$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lim_{k\to\infty} \frac{1}{3}\frac{k+1}{k} = \frac{1}{3} < 1.$$

The given series converges by the Ratio test 5.33.

ii) The series $\sum_{k=1}^{\infty} \frac{1}{k^k}$ has

$$\lim_{k\to\infty} \sqrt[k]{a_k} = \lim_{k\to\infty} \frac{1}{k} = 0 < 1.$$

The Root test 5.34 ensures that the series converges. □

We remark that the Ratio and Root tests do not allow to conclude anything if $\ell = 1$. For example, $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges and $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges, yet they both satisfy Theorems 5.33 and 5.34 with $\ell = 1$.

5.5.2 Alternating series

These are series of the form

$$\sum_{k=0}^{\infty}(-1)^k b_k \qquad \text{with} \qquad b_k > 0\,, \quad \forall k \geq 0.$$

For them the following result due to Leibniz holds.

Theorem 5.36 (**Leibniz's alternating series test**) *An alternating series* $\sum_{k=0}^{\infty}(-1)^k b_k$ *converges if the following conditions hold*

i) $\lim_{k\to\infty} b_k = 0$ *;*
ii) the sequence $\{b_k\}_{k\geq 0}$ *decreases monotonically.*

Denoting by s *its sum, for all* $n \geq 0$

$$|r_n| = |s - s_n| \leq b_{n+1} \qquad \textit{and} \qquad s_{2n+1} \leq s \leq s_{2n}\,.$$

Proof. As $\{b_k\}_{k\geq 0}$ is a decreasing sequence, one has

$$s_{2n} = s_{2n-2} - b_{2n-1} + b_{2n} = s_{2n-2} - (b_{2n-1} - b_{2n}) \leq s_{2n-2}$$

and

$$s_{2n+1} = s_{2n-1} + b_{2n} - b_{2n+1} \geq s_{2n-1}\,.$$

Thus the subsequence of partial sums made by the terms with even index decreases, whereas the subsequence of terms with odd index increases. For any $n \geq 0$, moreover,

$$s_{2n} = s_{2n-1} + b_{2n} \geq s_{2n-1} \geq \ldots \geq s_1$$

and

$$s_{2n+1} = s_{2n} - b_{2n+1} \leq s_{2n} \leq \ldots \leq s_0.$$

Thus $\{s_{2n}\}_{n\geq 0}$ is bounded from below and $\{s_{2n+1}\}_{n\geq 0}$ from above. By Theorem 3.9 both sequences converge, so let us put

$$\lim_{n\to\infty} s_{2n} = \inf_{n\geq 0} s_{2n} = s^* \qquad \text{and} \qquad \lim_{n\to\infty} s_{2n+1} = \sup_{n\geq 0} s_{2n+1} = s_*\,.$$

However, the two limits coincide, since

$$s^* - s_* = \lim_{n\to\infty}\big(s_{2n} - s_{2n+1}\big) = \lim_{n\to\infty} b_{2n+1} = 0\,;$$

we conclude that the series $\sum_{k=0}^{\infty}(-1)^k b_k$ has sum $s = s^* = s_*$. In addition,

$$s_{2n+1} \le s \le s_{2n}\,, \qquad \forall n \ge 0\,,$$

in other words the sequence $\{s_{2n}\}_{n\ge 0}$ approximates s from above, while $\{s_{2n+1}\}_{n\ge 0}$ approximates s from below.
For any $n \ge 0$ we have

$$0 \le s - s_{2n+1} \le s_{2n+2} - s_{2n+1} = b_{2n+2}$$

and

$$0 \le s_{2n} - s \le s_{2n} - s_{2n+1} = b_{2n+1},$$

i.e., $|r_n| = |s - s_n| \le b_{n+1}$. □

Example 5.37

Consider the alternating harmonic series $\sum_{k=1}^{\infty}(-1)^k \frac{1}{k}$. Given that

$$\lim_{k\to\infty} b_k = \lim_{k\to\infty} \frac{1}{k} = 0$$

and the sequence $\left\{\frac{1}{k}\right\}_{k\ge 1}$ is strictly monotone decreasing, the series converges.

□

In order to study series with arbitrary signs it is useful to introduce the notion of absolute convergence.

Definition 5.38 *The series* $\sum_{k=0}^{\infty} a_k$ **converges absolutely** *if the positive-term series* $\sum_{k=0}^{\infty} |a_k|$ *converges.*

Example 5.39

The series $\sum_{k=0}^{\infty}(-1)^k \frac{1}{k^2}$ converges absolutely because $\sum_{k=0}^{\infty} \frac{1}{k^2}$ converges. □

The next fact ensures that absolute convergence implies convergence.

Theorem 5.40 (**Absolute convergence test**) *If* $\sum_{k=0}^{\infty} a_k$ *converges absolutely then it also converges and*

$$\left|\sum_{k=0}^{\infty} a_k\right| \le \sum_{k=0}^{\infty} |a_k| \,.$$

Proof. Let us introduce the sequences

$$a_k^+ = \begin{cases} a_k & \text{if } a_k \ge 0 \\ 0 & \text{if } a_k < 0 \end{cases} \qquad \text{and} \qquad a_k^- = \begin{cases} 0 & \text{if } a_k \ge 0 \\ -a_k & \text{if } a_k < 0 \,. \end{cases}$$

Notice $a_k^+, a_k^- \ge 0$ for any $k \ge 0$, and

$$a_k = a_k^+ - a_k^- \,, \qquad |a_k| = a_k^+ + a_k^- \,.$$

Since $0 \le a_k^+, a_k^- \le |a_k|$, for any $k \ge 0$, the Comparison test (Theorem 5.29) says that the series $\sum_{k=0}^{\infty} a_k^+$ and $\sum_{k=0}^{\infty} a_k^-$ converge. Observing that

$$\sum_{k=0}^{\infty} a_k = \sum_{k=0}^{\infty} \left(a_k^+ - a_k^-\right) = \sum_{k=0}^{\infty} a_k^+ - \sum_{k=0}^{\infty} a_k^- \,,$$

for any $n \ge 0$, we deduce that also the series $\sum_{k=0}^{\infty} a_k = \sum_{k=0}^{\infty} a_k^+ - \sum_{k=0}^{\infty} a_k^-$ converges.

Finally, passing to the limit $n \to \infty$ the relation

$$\left|\sum_{k=0}^{n} a_k\right| \le \sum_{k=0}^{n} |a_k|$$

yields the desired inequality. □

Remark 5.41 There are series that converge, but not absolutely. The alternating harmonic series $\sum_{k=1}^{\infty} (-1)^k \frac{1}{k}$ is one such example, for it has a finite sum, but does not converge absolutely, since the harmonic series $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges. In such a situation one speaks about **conditional convergence**. □

The previous criterion allows to study alternating series by their absolute convergence. As the series of absolute values has positive terms, the criteria seen in Sect. 5.5.1 apply.

5.6 Exercises

1. *Compare the infinitesimals:*

   $\boxed{\text{a)}}$ $x-1,\ \sqrt[3]{\dfrac{1}{x}}-1,\ (\sqrt{x}-1)^2$ for $x\to 1$

   b) $\dfrac{1}{x^3},\ \mathrm{e}^{-x},\ x^2\mathrm{e}^{-x},\ x^2 3^{-x}$ for $x\to+\infty$

2. *Compare the infinite maps:*

   $\boxed{\text{a)}}$ $x^4,\ \sqrt[3]{x^{11}-2x^2},\ \dfrac{x^4}{\log(1+x)}$ when $x\to+\infty$

   b) $\dfrac{x^2}{\log x},\ x\log x,\ x^2 3^x,\ 3^x\log x$ when $x\to+\infty$

$\boxed{3.}$ *Verify that* $f(x)=\sqrt{x+3}-\sqrt{3}$ *and* $g(x)=\sqrt{x+5}-\sqrt{5}$ *are infinitesimals of the same order for* $x\to 0$ *and determine* $\ell\in\mathbb{R}$ *such that* $f(x)\sim\ell g(x)$ *for* $x\to 0$.

4. *Verify that* $f(x)=\sqrt[3]{x^3-2x^2+1}$ *and* $g(x)=2x+1$ *are infinite of the same order for* $x\to-\infty$ *and determine* $\ell\in\mathbb{R}$ *with* $f(x)\sim\ell g(x)$ *when* $x\to-\infty$.

5. *Determine the order and the principal part with respect to* $\varphi(x)=\frac{1}{x}$, *for* $x\to+\infty$, *of the infinitesimal functions:*

   $\boxed{\text{a)}}$ $f(x)=\dfrac{2x^2+\sqrt[5]{x}}{x^4}$

   b) $f(x)=\sqrt{\dfrac{x}{x+3}}-1$

   $\boxed{\text{c)}}$ $f(x)=\sin\left(\sqrt{x^2+1}-x\right)$

   $\boxed{\text{d)}}$ $f(x)=\log\left(9+\sin\dfrac{2}{x}\right)-2\log 3$

6. *Determine the order and the principal part with respect to* $\varphi(x)=x$, *for* $x\to+\infty$, *of the infinite functions:*

   $\boxed{\text{a)}}$ $f(x)=x-\sqrt{x^2+x^4}$

   b) $f(x)=\dfrac{1}{\sqrt{x^2+2}-\sqrt{x^2+1}}$

7. *Find the order and the principal part with respect to* $\varphi(x)=x$, *for* $x\to 0$, *of the infinitesimal functions:*

   $\boxed{\text{a)}}$ $f(x)=\left(\sqrt{1+3x}-1\right)\sin 2x^2$

   b) $f(x)=\sqrt[3]{\cos x}-1$

   c) $f(x)=\dfrac{\sqrt{1+3x^3}}{1+2x^3}-1$

   $\boxed{\text{d)}}$ $f(x)=\dfrac{\mathrm{e}^x}{1+x^2}-1$

   e) $f(x)=\log\cos x$

   $\boxed{\text{f)}}$ $f(x)=\mathrm{e}^{\cos x}-\mathrm{e}^{\sqrt{x^3+1}}$

8. *Find the order and the principal part with respect to $\varphi(x) = x - x_0$, for $x \to x_0$, of the infinitesimals:*

a) $f(x) = \log x - \log 3\,, \qquad x_0 = 3$

b) $f(x) = \sqrt{x} - \sqrt{2}\,, \qquad x_0 = 2$

c) $f(x) = \mathrm{e}^{x^2} - \mathrm{e}\,, \qquad x_0 = 1$

d) $f(x) = \sin x\,, \qquad x_0 = \pi$

e) $f(x) = 1 + \cos x\,, \qquad x_0 = \pi$

f) $f(x) = \sin(\pi \cos x)\,, \qquad x_0 = \pi$

9. *Compute the limits:*

a) $\displaystyle \lim_{x\to 0} \frac{1}{x^2}\left(\frac{\sqrt{1+3x^2}}{\cos x} - 1\right)$

b) $\displaystyle \lim_{x\to 2} \frac{(\sqrt{x}-\sqrt{2})^2}{x-2}$

c) $\displaystyle \lim_{x\to 3} \frac{\log(3-\sqrt{x+1})}{3-x}$

d) $\displaystyle \lim_{x\to 1} \frac{\mathrm{e}^{\sqrt{x+2}} - \mathrm{e}^{\sqrt{3}}}{(x-1)^2}$

10. *Determine domain and asymptotes of the following functions:*

a) $f(x) = \dfrac{x^2+1}{\sqrt{x^2-1}}$

b) $f(x) = x + 2\arctan x$

c) $f(x) = \dfrac{x^2 - (x+1)|x-2|}{2x+3}$

d) $f(x) = x\mathrm{e}^{1/|x^2-1|}$

e) $f(x) = \left(1 + \dfrac{1}{x}\right)^x$

f) $f(x) = \log(x + \mathrm{e}^x)$

11. *Study the behaviour of the sequences:*

a) $a_n = n - \sqrt{n}$

b) $a_n = (-1)^n \dfrac{n^2+1}{\sqrt{n^2+2}}$

c) $a_n = \dfrac{3^n - 4^n}{1+4^n}$

d) $a_n = \dfrac{(2n)!}{n!}$

e) $a_n = \dfrac{(2n)!}{(n!)^2}$

f) $a_n = \dbinom{n}{3}\dfrac{6}{n^3}$

g) $a_n = \left(\dfrac{n^2-n+1}{n^2+n+2}\right)^{\sqrt{n^2+2}}$

h) $a_n = 2^n \sin(2^{-n}\pi)$

i) $a_n = n\cos\dfrac{n+1}{n}\dfrac{\pi}{2}$

ℓ) $a_n = n!\left(\cos\dfrac{1}{\sqrt{n!}} - 1\right)$

12. *Compute the following limits:*

a) $\displaystyle \lim_{n\to\infty} \frac{n^2+1}{2^n+5^n}$

b) $\displaystyle \lim_{n\to\infty} n\left(\sqrt{1+\frac{1}{n}} - \sqrt{1-\frac{2}{n}}\right)$

c) $\lim_{n\to\infty} \frac{\cos n}{n}$

d) $\lim_{n\to\infty} (1 + (-1)^n)$

$\boxed{\text{e)}}$ $\lim_{n\to\infty} \sqrt[n]{3n^3 + 2}$

$\boxed{\text{f)}}$ $\lim_{n\to\infty} \frac{(n+3)! - n!}{n^2(n+1)!}$

$\boxed{\text{g)}}$ $\lim_{n\to\infty} n \left(\sqrt[3]{1 + \frac{1}{n}} - 1 \right)$

h) $\lim_{n\to\infty} \frac{n^2 + \sin n}{n^2 + 2n - 3}$

13. *Study the convergence of the following positive-term series:*

a) $\sum_{k=0}^{\infty} \frac{3}{2k^2 + 1}$

$\boxed{\text{b)}}$ $\sum_{k=2}^{\infty} \frac{2^k}{k^5 - 3}$

$\boxed{\text{c)}}$ $\sum_{k=0}^{\infty} \frac{3^k}{k!}$

$\boxed{\text{d)}}$ $\sum_{k=1}^{\infty} \frac{k!}{k^k}$

$\boxed{\text{e)}}$ $\sum_{k=1}^{\infty} k \arcsin \frac{7}{k^2}$

f) $\sum_{k=1}^{\infty} \log \left(1 + \frac{5}{k^2} \right)$

14. *Study the convergence of the following alternating series:*

a) $\sum_{k=1}^{\infty} (-1)^k \log \left(\frac{1}{k} + 1 \right)$

b) $\sum_{k=0}^{\infty} (-1)^k \sqrt{\frac{k^3 + 3}{2k^3 - 5}}$

$\boxed{\text{c)}}$ $\sum_{k=1}^{\infty} \sin \left(k\pi + \frac{1}{k} \right)$

$\boxed{\text{d)}}$ $\sum_{k=1}^{\infty} (-1)^k \left(\left(1 + \frac{1}{k^2} \right)^{\sqrt{2}} - 1 \right)$

15. *Study the convergence of:*

a) $\sum_{k=1}^{\infty} \left(1 - \cos \frac{1}{k^3} \right)$

$\boxed{\text{b)}}$ $\sum_{k=1}^{\infty} \frac{\sin k}{k^2}$

c) $\sum_{k=1}^{\infty} \frac{1}{k^3} \binom{k}{2}$

$\boxed{\text{d)}}$ $\sum_{k=0}^{\infty} (-1)^k \left(\sqrt[k]{2} - 1 \right)$

16. *Verify that the following series converge and determine their sum:*

a) $\sum_{k=1}^{\infty} (-1)^k \frac{2^{k-1}}{5^k}$

$\boxed{\text{b)}}$ $\sum_{k=0}^{\infty} \frac{3^k}{2 \cdot 4^{2k}}$

$\boxed{\text{c)}}$ $\sum_{k=1}^{\infty} \frac{2k+1}{k^2(k+1)^2}$

d) $\sum_{k=0}^{\infty} \frac{1}{(2k+1)(2k+3)}$

5.6.1 Solutions

1. *Comparing infinitesimals:*

a) Since

$$\lim_{x\to 1} \frac{x-1}{\sqrt[3]{\frac{1}{x}-1}} = \lim_{x\to 1} \sqrt[3]{\frac{x}{1-x}}(x-1) = -\lim_{x\to 1} \sqrt[3]{x}(x-1)^{2/3} = 0$$

$$\lim_{x\to 1} \frac{(\sqrt{x}-1)^2}{x-1} = \lim_{x\to 1} \frac{(x-1)^2}{(x-1)(\sqrt{x}+1)^2} = \lim_{x\to 1} \frac{x-1}{(\sqrt{x}+1)^2} = 0\,,$$

we have, for $x \to 1$,

$$x-1 = o\left(\sqrt[3]{\frac{1}{x}-1}\right), \qquad (\sqrt{x}-1)^2 = o(x-1)\,.$$

Thus we can order the three infinitesimals by increasing order from left to right:

$$\sqrt[3]{\frac{1}{x}-1}\,, \quad x-1\,, \quad (\sqrt{x}-1)^2\,.$$

The same result can be attained observing that for $x \to 1$

$$\sqrt[3]{\frac{1}{x}-1} = \sqrt[3]{\frac{1-x}{x}} \sim -(x-1)^{1/3}$$

and

$$\sqrt{x}-1 = \sqrt{1+(x-1)}-1 \sim \frac{1}{2}(x-1),$$

so $(\sqrt{x}-1)^2 \sim \frac{1}{4}(x-1)^2$.

b) Putting in increasing order we have: $\dfrac{1}{x^3}$, $x^2 \mathrm{e}^{-x}$, e^{-x}, $x^2 3^{-x}$.

2. *Comparison of infinite maps:*

a) As

$$\lim_{x\to+\infty} \frac{x^4}{\sqrt[3]{x^{11}-2x^2}} = \lim_{x\to+\infty} \frac{x^4}{x^{11/3}\sqrt[3]{1-2x^{-9}}} = \lim_{x\to+\infty} \frac{x^{1/3}}{\sqrt[3]{1-2x^{-9}}} = +\infty\,,$$

it follows $\sqrt[3]{x^{11}-2x^2} = o(x^4)$ for $x \to +\infty$, so $\sqrt[3]{x^{11}-2x^2}$ is infinite of smaller order than x^4.

It is immediate to see that $\dfrac{x^4}{\log(1+x)} = o(x^4)$. Moreover

$$\lim_{x\to+\infty} \frac{\sqrt[3]{x^{11}-2x^2}\,\log(1+x)}{x^4} = \lim_{x\to+\infty} \frac{\log(1+x)\sqrt[3]{1-2x^{-9}}}{x^{1/3}}$$
$$= \lim_{x\to+\infty} \frac{\log(1+x)}{x^{1/3}} = 0\,,$$

that is, $\sqrt[3]{x^{11}-2x^2} = o\left(\dfrac{x^4}{\log(1+x)}\right)$. Therefore the order increases from left to right in

$$\sqrt[3]{x^{11}-2x^2}\,, \qquad \frac{x^4}{\log(1+x)}\,, \qquad x^4\,.$$

b) Following the increasing order we have $x\log x$, $\dfrac{x^2}{\log x}$, $3^x\log x$, $x^2 3^x$.

3. Since

$$\lim_{x\to0} \frac{\sqrt{x+3}-\sqrt{3}}{\sqrt{x+5}-\sqrt{5}} = \lim_{x\to0} \frac{(x+3-3)(\sqrt{x+5}+\sqrt{5})}{(x+5-5)(\sqrt{x+3}+\sqrt{3})}$$
$$= \lim_{x\to0} \frac{\sqrt{x+5}+\sqrt{5}}{\sqrt{x+3}+\sqrt{3}} = \sqrt{\frac{5}{3}}$$

we conclude that $f(x) \sim \sqrt{\frac{5}{3}}\, g(x)$ as $x \to 0$.

4. The result is $f(x) \sim \frac{1}{2}\, g(x)$ for $x \to -\infty$.

5. *Order of infinitesimal and principal part:*

a) We have

$$\lim_{x\to+\infty} \frac{f(x)}{1/x^\alpha} = \lim_{x\to+\infty} x^\alpha \frac{2x^2+\sqrt[5]{x}}{x^4} = \lim_{x\to+\infty} x^\alpha \frac{2+x^{-9/5}}{x^2} = \lim_{x\to+\infty} 2x^{\alpha-2}\,.$$

This limit is finite and equals 2 if $\alpha = 2$. Therefore the order of $f(x)$ is 2 and its principal part $p(x) = \frac{2}{x^2}$.
Alternatively one could remark that for $x \to +\infty$, $\sqrt[5]{x} = o(x^2)$, so $2x^2 + \sqrt[5]{x} \sim 2x^2$ and then $f(x) \sim \frac{2x^2}{x^4} = \frac{2}{x^2}$.

b) This is an infinitesimal of first order with principal part $p(x) = -\frac{3}{2x}$.

c) Note first of all that

$$\lim_{x\to+\infty} \left(\sqrt{x^2-1}-x\right) = \lim_{x\to+\infty} \frac{x^2+1-x^2}{\sqrt{x^2-1}+x} = 0,$$

hence the function $f(x)$ is infinitesimal for $x \to +\infty$. In addition

$$\lim_{x\to+\infty} \frac{\sin\left(\sqrt{x^2-1}-x\right)}{\sqrt{x^2-1}-x} = \lim_{y\to0} \frac{\sin y}{y} = 1\,.$$

Then

$$\lim_{x\to+\infty} x^\alpha \sin\left(\sqrt{x^2-1}-x\right) = \lim_{x\to+\infty} x^\alpha \left(\sqrt{x^2-1}-x\right) \frac{\sin\left(\sqrt{x^2-1}-x\right)}{\sqrt{x^2-1}-x}$$
$$= \lim_{x\to+\infty} x^\alpha \left(\sqrt{x^2-1}-x\right) .$$

One can otherwise observe that $\sin g(x) \sim g(x)$ for $x \to x_0$ if the function $g(x)$ is infinitesimal for $x \to x_0$. For $x \to +\infty$ then,

$$\sin\left(\sqrt{x^2-1}-x\right) \sim \sqrt{x^2-1}-x$$

and Proposition 5.5 yields directly

$$\lim_{x\to+\infty} x^\alpha \sin\left(\sqrt{x^2-1}-x\right) = \lim_{x\to+\infty} x^\alpha \left(\sqrt{x^2-1}-x\right) .$$

Computing the right-hand-side limit yields

$$\lim_{x\to+\infty} x^\alpha \left(\sqrt{x^2-1}-x\right) = \lim_{x\to+\infty} \frac{x^\alpha}{\sqrt{x^2-1}+x} = \lim_{x\to+\infty} \frac{x^{\alpha-1}}{\sqrt{1+\frac{1}{x^2}}+1} = \frac{1}{2}$$

if $\alpha = 1$. Therefore the order is 1 and the principal part reads $p(x) = \frac{1}{2x}$.

d) Consider

$$\log\left(9+\sin\frac{2}{x}\right) - 2\log 3 = \log 9\left(1+\frac{1}{9}\sin\frac{2}{x}\right) - \log 9$$
$$= \log\left(1+\frac{1}{9}\sin\frac{2}{x}\right) .$$

For $x \to +\infty$ we have $\frac{1}{9}\sin\frac{2}{x} \sim \frac{2}{9x}$ (see the previous exercise) and $\log(1+y) \sim y$ for $y \to 0$. So

$$\lim_{x\to+\infty} x^\alpha f(x) = \lim_{x\to+\infty} x^\alpha \frac{1}{9}\sin\frac{2}{x} = \lim_{x\to+\infty} \frac{2x^\alpha}{9x} = \frac{2}{9}$$

if $\alpha = 1$. Thus the order of f is 1 and its principal part $p(x) = \frac{2}{9x}$.

6. *Order of infinite and principal part:*

a) A computation shows

$$\lim_{x\to+\infty} \frac{f(x)}{x^\alpha} = \lim_{x\to+\infty} \frac{x^2\left(\frac{1}{x}-\sqrt{\frac{1}{x^2}+1}\right)}{x^\alpha} = -\lim_{x\to+\infty} x^{2-\alpha} = -1$$

when $\alpha = 2$. Then f has order 2 and principal part $p(x) = -x^2$.

b) The order of f is 1 and the principal part is $p(x) = 2x$.

7. *Order of infinitesimal and principal part:*

a) First, $\sqrt{1+3x}-1 \sim \frac{3}{2}x$ for $x \to 0$, in fact

$$\lim_{x\to 0} \frac{\sqrt{1+3x}-1}{\frac{3}{2}x} = \lim_{x\to 0} \frac{2}{3}\frac{1+3x-1}{x(\sqrt{1+3x}+1)} = \lim_{x\to 0} \frac{2}{\sqrt{1+3x}+1} = 1\,.$$

But $\sin 2x^2 \sim 2x^2$ for $x \to 0$, so

$$f(x) \sim \frac{3}{2}x \cdot 2x^2, \qquad \text{i.e.,} \qquad f(x) \sim 3x^3\,, \quad x \to 0\,.$$

Therefore the order of f is 3 and the principal part is $p(x) = 3x^3$.

b) The order of f is 2 and the principal part is $p(x) = -\frac{1}{6}x^2$.

c) The function f has order 3 and principal part $p(x) = -\frac{1}{2}x^3$.

d) Using the relation $\mathrm{e}^x = 1 + x + o(x)$ for $x \to 0$ we have

$$\begin{aligned}\lim_{x\to 0} \frac{f(x)}{x^\alpha} &= \lim_{x\to 0} \frac{\mathrm{e}^x - 1 - x^2}{x^\alpha(1+x^2)} = \lim_{x\to 0} \frac{\mathrm{e}^x - 1 - x^2}{x^\alpha} \\ &= \lim_{x\to 0}\left(\frac{\mathrm{e}^x - 1}{x^\alpha} - x^{2-\alpha}\right) = 1\end{aligned}$$

for $\alpha = 1$. The order of f is 1 and the principal part is $p(x) = x$.

e) The function f has order 2 with principal part $p(x) = -\frac{1}{2}x^2$.

f) Recalling that

$$\begin{aligned}&\cos x = 1 - \frac{1}{2}x^2 + o(x^2) \qquad x \to 0\,, \\ &\sqrt{x^3+1} = (1+x^3)^{1/2} = 1 + \frac{1}{2}x^3 + o(x^3) \qquad x \to 0\,, \\ &\mathrm{e}^t = 1 + t + o(t) \qquad t \to 0\,,\end{aligned}$$

we have

$$\begin{aligned}f(x) &= \mathrm{e}^{1-\frac{1}{2}x^2+o(x^2)} - \mathrm{e}^{1+\frac{1}{2}x^3+o(x^3)} = \mathrm{e}\left(\mathrm{e}^{-\frac{1}{2}x^2+o(x^2)} - \mathrm{e}^{\frac{1}{2}x^3+o(x^3)}\right) \\ &= \mathrm{e}\left(1 - \frac{1}{2}x^2 + o(x^2) - 1 - \frac{1}{2}x^3 + o(x^3)\right) \\ &= \mathrm{e}\left(-\frac{1}{2}x^2 + o(x^2)\right) = -\frac{\mathrm{e}}{2}x^2 + o(x^2)\,, \quad x \to 0\,.\end{aligned}$$

This means f is infinitesimal of order 2 and has principal part $p(x) = -\frac{\mathrm{e}}{2}x^2$.

8. *Order of infinitesimal and principal part:*

a) Set $t = x - 3$ so that $t \to 0$ when $x \to 3$. Then

$$\log x - \log 3 = \log(3+t) - \log 3 = \log 3\left(1 + \frac{t}{3}\right) - \log 3 = \log\left(1 + \frac{t}{3}\right).$$

Since $\log\left(1 + \frac{t}{3}\right) \sim \frac{t}{3}$ for $t \to 0$, it follows

$$f(x) = \log x - \log 3 \sim \frac{1}{3}(x-3)\,, \quad x \to 3,$$

hence f is infinitesimal of order 1 and has principal part $p(x) = \frac{1}{3}(x-3)$.

b) The order of f is 1 and the principal part is $p(x) = \frac{\sqrt{2}}{4}(x-2)$.

c) Remembering that $\mathrm{e}^t - 1 \sim t$ as $t \to 0$,

$$\begin{aligned} f(x) &= \mathrm{e}(\mathrm{e}^{x^2-1} - 1) \sim \mathrm{e}(x^2-1) \\ &= \mathrm{e}(x+1)(x-1) \sim 2\mathrm{e}(x-1) \quad \text{for} \quad x \to 1\,. \end{aligned}$$

Thus f is infinitesimal of order 1 and has principal part $p(x) = 2\mathrm{e}(x-1)$.

d) The order of f is 1 and the principal part $p(x) = -(x-\pi)$.

e) By setting $t = x - \pi$ it follows that

$$1 + \cos x = 1 + \cos(t+\pi) = 1 - \cos t\,.$$

But $t \to 0$ for $x \to \pi$, so $1 - \cos t \sim \frac{1}{2}t^2$ and

$$f(x) = 1 + \cos x \sim \frac{1}{2}(x-\pi)^2\,, \quad x \to \pi\,.$$

Therefore f ha order 2 and principal part $p(x) = \frac{1}{2}(x-\pi)^2$.

f) The order of f is 2 and the principal part reads $p(x) = \frac{\pi}{2}(x-\pi)^2$.

9. *Limits:*

a) We remind that when $x \to 0$,

$$\sqrt{1+3x^2} = 1 + \frac{3}{2}x^2 + o(x^2) \qquad \text{and} \qquad \cos x = 1 - \frac{1}{2}x^2 + o(x^2)\,,$$

so we have

$$\begin{aligned} \lim_{x\to 0} \frac{\sqrt{1+3x^2} - \cos x}{x^2 \cos x} &= \lim_{x\to 0} \frac{1 + \frac{3}{2}x^2 - 1 + \frac{1}{2}x^2 + o(x^2)}{x^2} \\ &= \lim_{x\to 0} \frac{2x^2 + o(x^2)}{x^2} = 2\,. \end{aligned}$$

b) 0.

c) Let $y = 3 - x$, so that

$$L = \lim_{x\to 3^-} \frac{\log(3-\sqrt{x+1})}{3-x} = \lim_{y\to 0^+} \frac{\log(3-\sqrt{4-y})}{y}$$
$$= \lim_{y\to 0^+} \frac{\log(3-2\sqrt{1-y/4})}{y}\,.$$

But since $\sqrt{1-y/4} = 1 - \frac{1}{8}y + o(y)$, $y \to 0$, we have

$$L = \lim_{y\to 0^+} \frac{\log(3-2+\frac{y}{4}+o(y))}{y} = \lim_{y\to 0^+} \frac{\log(1+\frac{y}{4}+o(y))}{y}$$
$$= \lim_{y\to 0^+} \frac{\frac{y}{4}+o(y)}{y} = \frac{1}{4}\,.$$

d) Albeit the limit does not exist, the right limit is $+\infty$ and the left one $-\infty$.

10. *Domain and asymptotes:*

a) The function is defined for $x^2 - 1 > 0$, that is to say $x < -1$ and $x > 1$; thus $\operatorname{dom} f = (-\infty,-1) \cup (1,+\infty)$. It is even, so its behaviour on $x < 0$ can be deduced from $x > 0$. We have

$$\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} \frac{x^2\left(1+\frac{1}{x^2}\right)}{|x|\sqrt{1-\frac{1}{x^2}}} = \lim_{x\to\pm\infty} \frac{x^2}{|x|} = +\infty$$

$$\lim_{x\to -1^-} f(x) = \frac{2}{0^+} = +\infty\,, \qquad \lim_{x\to 1^+} f(x) = \frac{2}{0^+} = +\infty\,.$$

The line $x = -1$ is a vertical left asymptote and $x = 1$ is a vertical right asymptote; there are no horizontal asymptotes. Let us search for an oblique asymptote for $x \to +\infty$:

$$\lim_{x\to+\infty} \frac{f(x)}{x} = \lim_{x\to+\infty} \frac{x^2\left(1+\frac{1}{x^2}\right)}{x^2\sqrt{1-\frac{1}{x^2}}} = 1$$

$$\lim_{x\to+\infty} (f(x) - x) = \lim_{x\to+\infty} \frac{x^2+1-x\sqrt{x^2-1}}{\sqrt{x^2-1}}$$
$$= \lim_{x\to+\infty} \frac{(x^2+1)^2 - x^4 + x^2}{\sqrt{x^2-1}(x^2+1+x\sqrt{x^2-1})}$$
$$= \lim_{x\to+\infty} \frac{3x^2+1}{x^3\sqrt{1-\frac{1}{x^2}}\left(1+\frac{1}{x^2}+\sqrt{1-\frac{1}{x^2}}\right)} = \lim_{x\to+\infty} \frac{3x^2}{2x^3} = 0\,,$$

showing that the line $y = x$ is a oblique right asymptote.
For $x \to -\infty$ we proceed in a similar way to obtain that the line $y = -x$ is an oblique left asymptote.

b) $\operatorname{dom} f = \mathbb{R}$; $y = x + \pi$ is an oblique right asymptote, $y = x - \pi$ an oblique left asymptote.

c) The function is defined for $x \neq -\frac{3}{2}$, hence $\operatorname{dom} f = \mathbb{R} \setminus \{-\frac{3}{2}\}$. Moreover

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{x^2 - (x+1)(2-x)}{2x+3} = \lim_{x\to-\infty} \frac{2x^2 - x - 2}{2x+3} = -\infty$$
$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{x^2 - (x+1)(x-2)}{2x+3} = \lim_{x\to+\infty} \frac{x+2}{2x+3} = \frac{1}{2}$$
$$\lim_{x\to-\frac{3}{2}^\pm} f(x) = \lim_{x\to-\frac{3}{2}^\pm} \frac{x^2 - (x+1)(2-x)}{2x+3} = \frac{4}{0^\pm} = \pm\infty\,;$$

making the line $y = \frac{1}{2}$ a horizontal right asymptote and $x = -\frac{3}{2}$ a vertical asymptote. Computing

$$\lim_{x\to-\infty} \frac{f(x)}{x} = \lim_{x\to-\infty} \frac{2x^2 - x - 2}{x(2x+3)} = 1$$
$$\lim_{x\to-\infty} (f(x) - x) = \lim_{x\to+\infty} \frac{-4x-2}{2x+3} = -2$$

tells that $y = x - 2$ is an oblique left asymptote.

d) $\operatorname{dom} f = \mathbb{R}\setminus\{\pm 1\}$; $x = \pm 1$ are vertical asymptotes; the line $y = x$ is a complete oblique asymptote.

e) $\operatorname{dom} f = (-\infty, -1) \cup (0, +\infty)$; horizontal asymptote $y = \mathrm{e}$, vertical left asymptote $x = -1$.

f) the function f is defined for $x + \mathrm{e}^x > 0$. To solve this inequality, note that $g(x) = x + \mathrm{e}^x$ is a strictly increasing function on $\mathbb{R}$ (as sum of two such maps) satisfying $g(-1) = -1 + \frac{1}{\mathrm{e}} < 0$ and $g(0) = 1 > 0$. The Theorem of existence of zeroes 4.23 implies that there is a unique point $x_0 \in (-1, 0)$ with $g(x_0) = 0$. Thus $g(x) > 0$ for $x > x_0$ and $\operatorname{dom} f = (x_0, +\infty)$. Moreover

$$\lim_{x\to x_0^+} f(x) = \log \lim_{x\to x_0^+} (x + \mathrm{e}^x) = -\infty \qquad \text{and} \qquad \lim_{x\to+\infty} f(x) = +\infty\,,$$

so $x = x_0$ is a vertical right asymptote and there are no horizontal ones for $x \to +\infty$. For oblique asymptotes consider

$$\lim_{x\to+\infty} \frac{f(x)}{x} = \lim_{x\to+\infty} \frac{\log \mathrm{e}^x(1 + x\mathrm{e}^{-x})}{x} = \lim_{x\to+\infty} \frac{x + \log(1 + x\mathrm{e}^{-x})}{x}$$
$$= 1 + \lim_{x\to+\infty} \frac{\log(1 + x\mathrm{e}^{-x})}{x} = 1\,,$$
$$\lim_{x\to+\infty} (f(x) - x) = \lim_{x\to+\infty} \log(1 + x\mathrm{e}^{-x}) = 0$$

because $\lim\limits_{x\to+\infty} x\mathrm{e}^{-x} = 0$ (recalling (5.6) a)). Thus the line $y = x$ is an oblique right asymptote.

11. *Sequences behaviour:*

a) Diverges to $+\infty$; b) indeterminate.

c) The geometric sequence (Example 5.18 i)) suggests to consider

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{4^n\left(\left(\frac{3}{4}\right)^n - 1\right)}{4^n(4^{-n}+1)} = -1\,,$$

hence the given sequence converges to -1.

d) Diverges to $+\infty$.

e) Let us write

$$a_n = \frac{2n(2n-1)\cdots(n+2)(n+1)}{n(n-1)\cdots 2\cdot 1} = \frac{2n}{n}\cdot\frac{2n-1}{n-1}\cdots\frac{n+2}{2}\cdot\frac{n+1}{1} > n+1\,.$$

As $\lim\limits_{n\to\infty}(n+1) = +\infty$, the Second comparison theorem (with infinite limits) forces the sequence to diverge to $+\infty$.

f) Converges to 1.

g) Since

$$a_n = \exp\left(\sqrt{n^2+2}\,\log\frac{n^2-n+1}{n^2+n+2}\right)\,,$$

we consider the sequence

$$b_n = \sqrt{n^2+2}\,\log\frac{n^2-n+1}{n^2+n+2} = \sqrt{n^2+2}\,\log\left(1-\frac{2n+1}{n^2+n+2}\right)\,.$$

Note that

$$\lim_{n\to\infty}\frac{2n+1}{n^2+n+2} = 0$$

implies

$$\log\left(1-\frac{2n+1}{n^2+n+2}\right) \sim -\frac{2n+1}{n^2+n+2}\,,\qquad n\to\infty\,.$$

Then

$$\lim_{n\to\infty} b_n = -\lim_{n\to\infty}\frac{\sqrt{n^2+2}\,(2n+1)}{n^2+n+2} = -\lim_{n\to\infty}\frac{2n^2}{n^2} = -2$$

and the sequence $\{a_n\}$ converges to e^{-2}.

h) Call $x = 2^{-n}\pi$, so that $x\to 0^+$ for $n\to\infty$. Then

$$\lim_{n\to\infty} a_n = \lim_{x\to 0^+}\pi\,\frac{\sin x}{x} = \pi$$

and $\{a_n\}$ converges to π.

i) Because

$$\cos \frac{n+1}{n}\frac{\pi}{2} = \cos\left(\frac{\pi}{2} + \frac{\pi}{2n}\right) = -\sin\frac{\pi}{2n}\,,$$

putting $x = \frac{\pi}{2n}$ has the effect that

$$\lim_{n\to\infty} a_n = -\lim_{n\to\infty} n\sin\frac{\pi}{2n} = -\lim_{x\to 0^+} \frac{\pi}{2}\frac{\sin x}{x} = -\frac{\pi}{2}\,,$$

thus $\{a_n\}$ converges to $-\frac{\pi}{2}$.

ℓ) Converges to $-\frac{1}{2}$.

12. *Limits:*

a) 0.

b) Since $\frac{1}{n} \to 0$ for $n \to \infty$,

$$\sqrt{1+\frac{1}{n}} = 1 + \frac{1}{2n} + o\left(\frac{1}{n}\right) \qquad \text{and} \qquad \sqrt{1-\frac{2}{n}} = 1 - \frac{1}{n} + o\left(\frac{1}{n}\right)$$

so

$$\lim_{n\to\infty} n\left(\sqrt{1+\frac{1}{n}} - \sqrt{1-\frac{2}{n}}\right) = \lim_{n\to\infty} n\left(\frac{3}{2n} + o\left(\frac{1}{n}\right)\right) = \frac{3}{2}\,.$$

c) 0; d) does not exist.

e) Let us write

$$\sqrt[n]{3n^3+2} = \exp\left(\frac{1}{n}\log(3n^3+2)\right)$$

and observe

$$\frac{1}{n}\log(3n^3+2) = \frac{\log\left(3n^3\left(1+\frac{2}{3n^3}\right)\right)}{n} = \frac{\log 3}{n} + \frac{3\log n}{n} + \frac{\log\left(1+\frac{2}{3n^3}\right)}{n}\,.$$

In addition

$$\log\left(1+\frac{2}{3n^3}\right) \sim \frac{2}{3n^3}\,, \qquad n\to\infty\,.$$

Thus

$$\lim_{n\to\infty}\frac{1}{n}\log(3n^3+2) = 0$$

and the required limit is $e^0 = 1$.

f) From

$$\frac{(n+3)!-n!}{n^2(n+1)!} = \frac{n!((n+3)(n+2)(n+1)-1)}{n^2(n+1)n!} = \frac{(n+3)(n+2)(n+1)-1}{n^2(n+1)}$$

it follows that

$$\lim_{n\to\infty}\frac{(n+3)!-n!}{n^2(n+1)!} = \lim_{n\to\infty}\frac{(n+3)(n+2)(n+1)-1}{n^2(n+1)} = 1\,.$$

g) As

$$\sqrt[3]{1+\frac{1}{n}} = 1 + \frac{1}{3n} + o\left(\frac{1}{n}\right), \qquad n \to \infty,$$

we have

$$\lim_{n\to\infty} n\left(\sqrt[3]{1+\frac{1}{n}} - 1\right) = \lim_{n\to\infty} n\left(\frac{1}{3n} + o\left(\frac{1}{n}\right)\right) = \frac{1}{3}.$$

h) 1.

13. *Convergence of positive-term series:*

a) Converges.

b) The general term a_k tends to $+\infty$ for $k \to \infty$. By Property 5.25 the series diverges. Alternatively, the Root test 5.34 can be used.

c) By the Ratio test 5.33 one obtains

$$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lim_{k\to\infty} \frac{3^{k+1}}{(k+1)!}\frac{k!}{3^k};$$

writing $(k+1)! = (k+1)k!$ and simplifying we see that

$$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lim_{k\to\infty} \frac{3}{k+1} = 0$$

and the series converges.

d) Again with the help of the Ratio test 5.33, we have

$$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lim_{k\to\infty} \frac{(k+1)!}{(k+1)^{k+1}} \cdot \frac{k^k}{k!} = \lim_{k\to\infty} \left(\frac{k}{k+1}\right)^k = \frac{1}{\mathrm{e}} < 1$$

and the series converges.

e) Notice

$$a_k \sim k\frac{7}{k^2} = \frac{7}{k} \qquad \text{for} \qquad k \to \infty.$$

By the Asymptotic behaviour test 5.31, and remembering that the harmonic series does not converge, we conclude that the given series must diverge too.

f) Converges.

14. *Convergence of alternating series:*

a) Converges; b) does not converge.

c) Since

$$\sin\left(k\pi + \frac{1}{k}\right) = \cos(k\pi)\,\sin\frac{1}{k} = (-1)^k \sin\frac{1}{k}\,,$$

the given series has alternating sign, with $b_k = \sin\frac{1}{k}$. As

$$\lim_{k\to\infty} b_k = 0 \qquad \text{and} \qquad b_{k+1} < b_k\,,$$

Leibniz's test 5.36 guarantees convergence. The series does not converge absolutely since $\sin\frac{1}{k} \sim \frac{1}{k}$ for $k \to \infty$, so the series of absolute values behaves as the (diverging) harmonic series.

d) By using one of the equivalences of p. 127 one sees that

$$\left|(-1)^k \left(\left(1 + \frac{1}{k^2}\right)^{\sqrt{2}} - 1\right)\right| \sim \frac{\sqrt{2}}{k^2}\,, \qquad k \to \infty\,.$$

Example 5.30 i) suggests to apply the Asymptotic comparison test 5.31 to the series of absolute values. We conclude that the given series converges absolutely.

15. *Study of convergence:*

a) Converges.

b) Observe first that

$$\left|\frac{\sin k}{k^2}\right| \le \frac{1}{k^2}\,, \qquad \text{for all } k > 0\,;$$

the series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges and the Comparison test 5.29 tells that the series of absolute values converges. Thus the given series converges absolutely.

c) Diverges.

d) This is an alternating series where $b_k = \sqrt[k]{2} - 1$. The first term $b_0 = 0$ apart, the sequence $\{b_k\}_{k\ge1}$ decreases because $\sqrt[k]{2} > \sqrt[k+1]{2}$ for all $k \ge 1$. Thus Leibniz's test 5.36 allows us to conclude that the series converges. Notice that convergence is not absolute, as

$$\sqrt[k]{2} - 1 = \mathrm{e}^{\frac{\log 2}{k}} - 1 \sim \frac{\log 2}{k}\,, \qquad k \to \infty,$$

confirming that the series of absolute values is like the harmonic series, which diverges.

16. *Computing the sum of a converging series:*

a) $-\frac{1}{7}$.

b) Apart from a constant, this is essentially a geometric series; by Example 5.27 then,

$$\sum_{k=1}^{\infty} \frac{3^k}{2 \cdot 4^{2k}} = \frac{1}{2} \sum_{k=1}^{\infty} \left(\frac{3}{16}\right)^k = \frac{1}{2} \left(\frac{1}{1-\frac{3}{16}} - 1\right) = \frac{3}{26}$$

(note that the first index in the sum is 1).

c) The series is telescopic since

$$\frac{2k+1}{k^2(k+1)^2} = \frac{1}{k^2} - \frac{1}{(k+1)^2},$$

so

$$s_n = 1 - \frac{1}{(n+1)^2}.$$

This implies $s = \lim_{n\to\infty} s_n = 1$.

d) $\frac{1}{2}$.

6

Differential calculus

The precise definition of the notion of derivative, studying a function's differentiability and computing its successive derivatives, the use of derivatives to analyse the local and global behaviours of functions are all constituents of Differential Calculus.

6.1 The derivative

We start by defining the derivative of a function.

Let $f : \mathrm{dom}\, f \subseteq \mathbb{R} \to \mathbb{R}$ be a real function of one real variable, take $x_0 \in \mathrm{dom}\, f$ and suppose f is defined in a neighbourhood $I_r(x_0)$ of x_0. With $x \in I_r(x_0),\ x \neq x_0$ fixed, denote by

$$\Delta x = x - x_0$$

the (positive or negative) **increment of the independent variable between x_0 and x**, and by

$$\Delta f = f(x) - f(x_0)$$

the corresponding **increment of the dependent variable**. Note that $x = x_0 + \Delta x$, $f(x) = f(x_0) + \Delta f$.

The ratio

$$\frac{\Delta f}{\Delta x} = \frac{f(x) - f(x_0)}{x - x_0} = \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

is called **difference quotient of f between x_0 and x**.

In this manner Δf represents the *absolute increment* of the dependent variable f when passing from x_0 to $x_0 + \Delta x$, whereas the difference quotient detects the *rate of increment* (while $\Delta f / f$ is the *relative increment*). Multiplying the difference quotient by 100 we obtain the so-called *percentage increment*. Suppose a rise by $\Delta x = 0.2$ of the variable x prompts an increment $\Delta f = 0.06$ of f; the difference quotient $\frac{\Delta f}{\Delta x}$ equals $0.3 = \frac{30}{100}$, corresponding to a 30% increase.

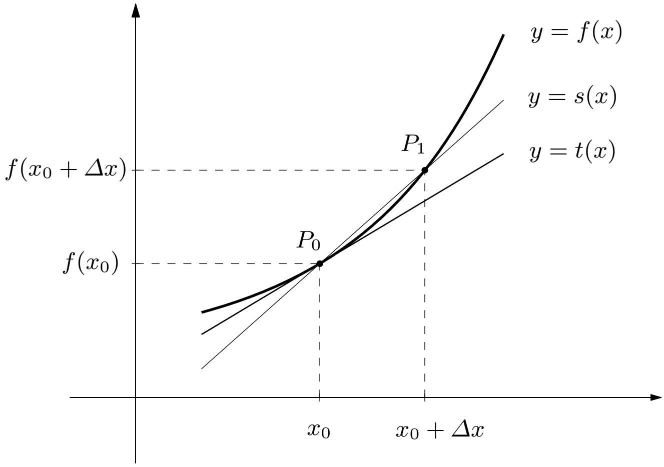


Figure 6.1. Secant and tangent lines to the graph of f at P_0

Graphically, the difference quotient between x_0 and a point x_1 around x_0 is the slope of the straight line s passing through $P_0 = \big(x_0, f(x_0)\big)$ and $P_1 = \big(x_1, f(x_1)\big)$, points that belong to the graph of the function; this line is called **secant** of the graph of f at P_0 and P_1 (Fig. 6.1). Putting $\Delta x = x_1 - x_0$ and $\Delta f = f(x_1) - f(x_0)$, the equation of the secant line reads

$$y = s(x) = f(x_0) + \frac{\Delta f}{\Delta x}(x - x_0), \qquad x \in \mathbb{R}. \tag{6.1}$$

A typical application of the difference quotient comes from physics. Let M be a point-particle moving along a straight line; call $s = s(t)$ the x-coordinate of the position of M at time t, with respect to a reference point O. Between the instants t_0 and $t_1 = t_0 + \Delta t$, the particle changes position by $\Delta s = s(t_1) - s(t_0)$. The difference quotient $\frac{\Delta s}{\Delta t}$ represents the *average velocity* of the particle in the given interval of time.

How does the difference quotient change, as Δx approaches 0? This is answered by the following notion.

Definition 6.1 *A map f defined on a neighbourhood of $x_0 \in \mathbb{R}$ is called* **differentiable at** x_0 *if the limit of the difference quotient* $\dfrac{\Delta f}{\Delta x}$ *between x_0 and x exists and is finite, as x approaches x_0. The real number*

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

is called (**first**) **derivative of** f **at** x_0.

The derivative at x_0 is variously denoted, for instance also by

$$y'(x_0), \qquad \frac{\mathrm{d}f}{\mathrm{d}x}(x_0), \qquad \mathrm{D}f(x_0).$$

The first symbol goes back to Newton, the second is associated to Leibniz.

From the geometric point of view $f'(x_0)$ is the slope of the **tangent line** at $P_0 = \big(x_0, f(x_0)\big)$ to the graph of f: such line t is obtained as the limiting position of the secant s at P_0 and $P = \big(x, f(x)\big)$, when P approaches P_0. From (6.1) and the previous definition we have

$$y = t(x) = f(x_0) + f'(x_0)(x - x_0), \qquad x \in \mathbb{R}.$$

In the physical example given above, the derivative $v(t_0) = s'(t_0) = \lim\limits_{\Delta t \to 0} \dfrac{\Delta s}{\Delta t}$ is the instantaneous *velocity* of the particle M at time t_0.

Let

$$\mathrm{dom}\, f' = \{x \in \mathrm{dom}\, f \ : \ f \text{ is differentiable at } x\}$$

and define the function $f' : \mathrm{dom}\, f' \subseteq \mathbb{R} \to \mathbb{R},\ f' : x \mapsto f'(x)$ mapping $x \in \mathrm{dom}\, f'$ to the value of the derivative of f at x. This map is called **(first) derivative of** f.

Definition 6.2 *Let I be a subset of* $\mathrm{dom}\, f$. *We say that f is* **differentiable on** I (*or* **in** I) *if f is differentiable at each point of I.*

A first yet significant property of differentiable maps is the following.

Proposition 6.3 *If f is differentiable at x_0, it is also continuous at x_0.*

Proof. Continuity at x_0 prescribes

$$\lim_{x \to x_0} f(x) = f(x_0), \qquad \text{that is} \qquad \lim_{x \to x_0} \big(f(x) - f(x_0)\big) = 0.$$

If f is differentiable at x_0, then

$$\begin{aligned}\lim_{x \to x_0} \big(f(x) - f(x_0)\big) &= \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}\,(x - x_0) \\ &= \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}\, \lim_{x \to x_0} (x - x_0) \\ &= f'(x_0) \cdot 0 = 0.\end{aligned}$$

$\square$

Not all continuous maps at a point are differentiable though. Consider the map $f(x) = |x|$: it is *continuous* at the origin, yet the difference quotient between the origin and a point $x \neq 0$ is

$$\frac{\Delta f}{\Delta x} = \frac{f(x) - f(0)}{x - 0} = \frac{|x|}{x} = \begin{cases} +1 & \text{if } x > 0, \\ -1 & \text{if } x < 0, \end{cases} \tag{6.2}$$

so the limit for $x \to 0$ does not exist. Otherwise said, f *is not differentiable* at the origin. This particular example shows that the implication of Proposition 6.3 can not be reversed: differentiability is thus a *stronger* property than continuity, an aspect to which Sect. 6.3 is entirely devoted.

6.2 Derivatives of the elementary functions. Rules of differentiation

We begin by tackling the issue of differentiability for elementary functions using Definition 6.1.

i) Consider the affine map $f(x) = ax + b$, and let $x_0 \in \mathbb{R}$ be arbitrary. Then

$$f'(x_0) = \lim_{\Delta x \to 0} \frac{\big(a(x_0 + \Delta x) + b\big) - (ax_0 + b)}{\Delta x} = \lim_{\Delta x \to 0} a = a,$$

in agreement with the fact that the graph of f is a straight line of slope a. The derivative of $f(x) = ax + b$ is then the constant map $f'(x) = a$.

In particular if f is constant ($a = 0$), its derivative is identically zero.

ii) Take $f(x) = x^2$ and $x_0 \in \mathbb{R}$. Since

$$f'(x_0) = \lim_{\Delta x \to 0} \frac{(x_0 + \Delta x)^2 - x_0^2}{\Delta x} = \lim_{\Delta x \to 0} (2x_0 + \Delta x) = 2x_0,$$

the derivative of $f(x) = x^2$ is the function $f'(x) = 2x$.

iii) Now let $f(x) = x^n$ with $n \in \mathbb{N}$. The binomial formula (1.13) yields

$$\begin{aligned} f'(x_0) &= \lim_{\Delta x \to 0} \frac{(x_0 + \Delta x)^n - x_0^n}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{x_0^n + nx_0^{n-1}\Delta x + \displaystyle\sum_{k=2}^{n} \binom{n}{k} x_0^{n-k} (\Delta x)^k - x_0^n}{\Delta x} \\ &= \lim_{\Delta x \to 0} \left(nx_0^{n-1} + \sum_{k=2}^{n} \binom{n}{k} x_0^{n-k} (\Delta x)^{k-1} \right) = nx_0^{n-1}. \end{aligned}$$

for all $x_0 \in \mathbb{R}$. Therefore, $f'(x) = nx^{n-1}$ is the derivative of $f(x) = x^n$.

iv) Even more generally, consider $f(x) = x^\alpha$ where $\alpha \in \mathbb{R}$, and let $x_0 \neq 0$ be a point of the domain. Then

$$f'(x_0) = \lim_{\Delta x \to 0} \frac{(x_0 + \Delta x)^\alpha - x_0^\alpha}{\Delta x} = \lim_{\Delta x \to 0} \frac{x_0^\alpha \left[\left(1 + \frac{\Delta x}{x_0}\right)^\alpha - 1\right]}{\Delta x}$$
$$= x_0^{\alpha-1} \lim_{\Delta x \to 0} \frac{\left(1 + \frac{\Delta x}{x_0}\right)^\alpha - 1}{\frac{\Delta x}{x_0}}.$$

Substituting $y = \frac{\Delta x}{x_0}$ brings the latter into the form of the fundamental limit (4.13), so

$$f'(x_0) = \alpha x_0^{\alpha-1}.$$

When $\alpha > 1$, f is differentiable at $x_0 = 0$ as well, and $f'(0) = 0$. The function $f(x) = x^\alpha$ is thus differentiable at all points where the expression $x^{\alpha-1}$ is well defined; its derivative is $f'(x) = \alpha x^{\alpha-1}$.

For example $f(x) = \sqrt{x} = x^{1/2}$, defined on $[0, +\infty)$, is differentiable on $(0, +\infty)$ with derivative $f'(x) = \dfrac{1}{2\sqrt{x}}$. The function $f(x) = \sqrt[3]{x^5} = x^{5/3}$ is defined on $\mathbb{R}$, where it is also differentiable, and $f'(x) = \frac{5}{3}x^{2/3} = \frac{5}{3}\sqrt[3]{x^2}$.

v) Now consider the trigonometric functions. Take $f(x) = \sin x$ and $x_0 \in \mathbb{R}$. Formula (2.14) gives

$$f'(x_0) = \lim_{\Delta x \to 0} \frac{\sin(x_0 + \Delta x) - \sin x_0}{\Delta x} = \lim_{\Delta x \to 0} \frac{2 \sin \frac{\Delta x}{2} \cos(x_0 + \frac{\Delta x}{2})}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{\sin \frac{\Delta x}{2}}{\frac{\Delta x}{2}} \lim_{\Delta x \to 0} \cos\left(x_0 + \frac{\Delta x}{2}\right).$$

The limit (4.5) and the cosine's continuity tell

$$f'(x_0) = \cos x_0.$$

Hence the derivative of $f(x) = \sin x$ is $f'(x) = \cos x$.

Using in a similar way formula (2.15), we can see that the derivative of $f(x) = \cos x$ is the function $f'(x) = -\sin x$.

vi) Eventually, consider the exponential function $f(x) = a^x$. By (4.12) we have

$$f'(x_0) = \lim_{\Delta x \to 0} \frac{a^{x_0 + \Delta x} - a^{x_0}}{\Delta x} = a^{x_0} \lim_{\Delta x \to 0} \frac{a^{\Delta x} - 1}{\Delta x} = a^{x_0} \log a,$$

showing that the derivative of $f(x) = a^x$ is $f'(x) = (\log a) a^x$.

As $\log \mathrm{e} = 1$, the derivative of $f(x) = \mathrm{e}^x$ is $f'(x) = \mathrm{e}^x = f(x)$, whence the derivative f' coincides at each point with the function f itself. This is a crucial fact, and a reason for choosing e as privileged base for the exponential map.

We next discuss differentiability in terms of operations (algebraic operations, composition, inversion) on functions. We shall establish certain *differentiation rules* to compute derivatives of functions that are built from the elementary ones, without resorting to the definition each time. The proofs may be found in Appendix A.4.1, p. 449.

Theorem 6.4 (Algebraic operations) *Let $f(x), g(x)$ be differentiable maps at $x_0 \in \mathbb{R}$. Then the maps $f(x) \pm g(x)$, $f(x)g(x)$ and, if $g(x_0) \neq 0$, $\dfrac{f(x)}{g(x)}$ are differentiable at x_0. To be precise,*

$$(f \pm g)'(x_0) = f'(x_0) \pm g'(x_0), \tag{6.3}$$

$$(f\,g)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0), \tag{6.4}$$

$$\left(\frac{f}{g}\right)'(x_0) = \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{[g(x_0)]^2}. \tag{6.5}$$

Corollary 6.5 (Linearity of the derivative) *If $f(x)$ and $g(x)$ are differentiable at $x_0 \in \mathbb{R}$, the map $\alpha f(x) + \beta g(x)$ is differentiable at x_0 for any $\alpha, \beta \in \mathbb{R}$ and*

$$(\alpha f + \beta g)'(x_0) = \alpha f'(x_0) + \beta g'(x_0). \tag{6.6}$$

Proof. Consider (6.4) and recall that differentiating a constant gives zero; then $(\alpha f)'(x_0) = \alpha f'(x_0)$ and $(\beta g)'(x_0) = \beta g'(x_0)$ follow. The rest is a consequence of (6.3). □

Examples 6.6

i) To differentiate a polynomial, we use the fact that $D\,x^n = nx^{n-1}$ and apply the corollary repeatedly. So, $f(x) = 3x^5 - 2x^4 - x^3 + 3x^2 - 5x + 2$ differentiates to

$$f'(x) = 3 \cdot 5x^4 - 2 \cdot 4x^3 - 3x^2 + 3 \cdot 2x - 5 = 15x^4 - 8x^3 - 3x^2 + 6x - 5.$$

ii) For rational functions, we compute the numerator and denominator's derivatives and then employ rule (6.5), to the effect that

$$f(x) = \frac{x^2 - 3x + 1}{2x - 1}$$

has derivative

$$f'(x) = \frac{(2x-3)(2x-1) - (x^2 - 3x + 1)2}{(2x-1)^2} = \frac{2x^2 - 2x + 1}{4x^2 - 4x + 1}.$$

iii) Consider $f(x) = x^3 \sin x$. The product rule (6.4) together with $(\sin x)' = \cos x$ yield

$$f'(x) = 3x^2 \sin x + x^3 \cos x.$$

iv) The function

$$f(x) = \tan x = \frac{\sin x}{\cos x}$$

can be differentiated with (6.5)

$$f'(x) = \frac{\cos x \, \cos x - \sin x \,(-\sin x)}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = 1 + \frac{\sin^2 x}{\cos^2 x} = 1 + \tan^2 x.$$

Another possibility is to use $\cos^2 x + \sin^2 x = 1$ to obtain

$$f'(x) = \frac{1}{\cos^2 x}.$$

□

Theorem 6.7 (**"Chain rule"**) *Let $f(x)$ be differentiable at $x_0 \in \mathbb{R}$ and $g(y)$ a differentiable map at $y_0 = f(x_0)$. Then the composition $g \circ f(x) = g\big(f(x)\big)$ is differentiable at x_0 and*

$$(g \circ f)'(x_0) = g'(y_0) f'(x_0) = g'\big(f(x_0)\big) f'(x_0). \tag{6.7}$$

Examples 6.8

i) The map $h(x) = \sqrt{1 - x^2}$ is the composite of $f(x) = 1 - x^2$, whose derivative is $f'(x) = -2x$, and $g(y) = \sqrt{y}$, for which $g'(y) = \dfrac{1}{2\sqrt{y}}$. Then (6.7) directly gives

$$h'(x) = \frac{1}{2\sqrt{1 - x^2}}(-2x) = -\frac{x}{\sqrt{1 - x^2}}.$$

ii) The function $h(x) = \mathrm{e}^{\cos 3x}$ is composed by $f(x) = \cos 3x$, $g(y) = \mathrm{e}^y$. But $f(x)$ is in turn the composite of $\varphi(x) = 3x$ and $\psi(y) = \cos y$; thus (6.7) tells $f'(x) = -3 \sin 3x$. On the other hand $g'(y) = \mathrm{e}^y$. Using (6.7) once again we conclude

$$h'(x) = -3\mathrm{e}^{\cos 3x} \sin 3x.$$

□

Theorem 6.9 (**Derivative of the inverse function**) *Suppose $f(x)$ is a continuous, invertible map on a neighbourhood of $x_0 \in \mathbb{R}$, and differentiable at x_0, with $f'(x_0) \neq 0$. Then the inverse map $f^{-1}(y)$ is differentiable at $y_0 = f(x_0)$, and*

$$(f^{-1})'(y_0) = \frac{1}{f'(x_0)} = \frac{1}{f'(f^{-1}(y_0))}. \tag{6.8}$$

Examples 6.10

i) The function $y = f(x) = \tan x$ has derivative $f'(x) = 1 + \tan^2 x$ and inverse $x = f^{-1}(y) = \arctan y$. By (6.8)

$$(f^{-1})'(y) = \frac{1}{1+\tan^2 x} = \frac{1}{1+y^2}.$$

Setting for simplicity $f^{-1} = g$ and denoting the independent variable with x, the derivative of $g(x) = \arctan x$ is the function $g'(x) = \dfrac{1}{1+x^2}$.

ii) We are by now acquainted with the function $y = f(x) = \sin x$: it is invertible on $[-\frac{\pi}{2}, \frac{\pi}{2}]$, namely $x = f^{-1}(y) = \arcsin y$. Moreover, f differentiates to $f'(x) = \cos x$. Using $\cos^2 x + \sin^2 x = 1$, and taking into account that on that interval $\cos x \geq 0$, one can write the derivative of f in the equivalent form $f'(x) = \sqrt{1 - \sin^2 x}$. Now (6.8) yields

$$(f^{-1})'(y) = \frac{1}{\sqrt{1-\sin^2 x}} = \frac{1}{\sqrt{1-y^2}}.$$

Put once again $f^{-1} = g$ and change names to the variables: the derivative of $g(x) = \arcsin x$ is $g'(x) = \dfrac{1}{\sqrt{1-x^2}}$.

In similar fashion $g(x) = \arccos x$ differentiates to $g'(x) = -\dfrac{1}{\sqrt{1-x^2}}$.

iii) Consider $y = f(x) = a^x$. It has derivative $f'(x) = (\log a)a^x$ and inverse $x = f^{-1}(y) = \log_a y$. The usual (6.8) gives

$$(f^{-1})'(y) = \frac{1}{(\log a)a^x} = \frac{1}{(\log a)y}.$$

Defining $f^{-1} = g$ and renaming x the independent variable gives $g'(x) = \dfrac{1}{(\log a)x}$ as derivative of $g(x) = \log_a x$ $(x > 0)$.
Take now $h(x) = \log_a(-x)$ (with $x < 0$), composition of $x \mapsto -x$ and $g(y)$: then $h'(x) = \dfrac{1}{(\log a)(-x)}(-1) = \dfrac{1}{(\log a)x}$. Putting all together shows that $g(x) = \log_a |x|$ $(x \neq 0)$ has derivative $g'(x) = \dfrac{1}{(\log a)x}$.

With the choice of base $a = \mathrm{e}$ the derivative of $g(x) = \log |x|$ is $g'(x) = \dfrac{1}{x}$. □

Remark 6.11 Let $f(x)$ be differentiable and strictly positive on an interval I. Due to the previous result and the Chain rule, the derivative of the composite map $g(x) = \log f(x)$ is

$$g'(x) = \frac{f'(x)}{f(x)}.$$

The expression $\dfrac{f'}{f}$ is said **logarithmic derivative** of the map f. □

The section ends with a useful corollary to the Chain rule 6.7.

Property 6.12 *If f is an even (or odd) differentiable function on all its domain, the derivative f' is odd (resp. even).*

Proof. Since f is even, $f(-x) = f(x)$ for any $x \in \operatorname{dom} f$. Let us differentiate both sides. As $f(-x)$ is the composition of $x \mapsto -x$ and $y \mapsto f(y)$, its derivative reads $-f'(-x)$. Then $f'(-x) = -f'(x)$ for all $x \in \operatorname{dom} f$, so f' is odd. Similarly if f is odd. □

We reckon it could be useful to collect the derivatives of the main elementary functions in one table, for reference.

$$\mathrm{D}\, x^\alpha = \alpha x^{\alpha-1} \qquad (\forall \alpha \in \mathbb{R})$$

$$\mathrm{D}\, \sin x = \cos x$$

$$\mathrm{D}\, \cos x = -\sin x$$

$$\mathrm{D}\, \tan x = 1 + \tan^2 x = \frac{1}{\cos^2 x}$$

$$\mathrm{D}\, \arcsin x = \frac{1}{\sqrt{1-x^2}}$$

$$\mathrm{D}\, \arccos x = -\frac{1}{\sqrt{1-x^2}}$$

$$\mathrm{D}\, \arctan x = \frac{1}{1+x^2}$$

$$\mathrm{D}\, a^x = (\log a)\, a^x \qquad \text{in particular,} \quad \mathrm{D}\, \mathrm{e}^x = \mathrm{e}^x$$

$$\mathrm{D}\, \log_a |x| = \frac{1}{(\log a)\, x} \qquad \text{in particular,} \quad \mathrm{D}\, \log |x| = \frac{1}{x}$$

6.3 Where differentiability fails

It was noted earlier that the function $f(x) = |x|$ is continuous but not differentiable at the origin. At each other point of the real line f is differentiable, for it coincides with the line $y = x$ when $x > 0$, and with $y = -x$ for $x < 0$. Therefore $f'(x) = +1$

for $x > 0$ and $f'(x) = -1$ on $x < 0$. The reader will recall the sign function (Example 2.1 iv)), for which

$$\mathrm{D}\,|x| = \mathrm{sign}(x), \qquad \text{for all } x \neq 0.$$

The origin is an *isolated point of non-differentiability* for $y = |x|$.

Returning to the expression (6.2) for the difference quotient at the origin, we observe that the one-sided limits exist and are finite:

$$\lim_{x\to 0^+} \frac{\Delta f}{\Delta x} = 1, \qquad \lim_{x\to 0^-} \frac{\Delta f}{\Delta x} = -1.$$

This fact suggests us to introduce the following notion.

Definition 6.13 *Suppose f is defined on a right neighbourhood of $x_0 \in \mathbb{R}$. It is called* **differentiable on the right at** x_0 *if the right limit of the difference quotient $\dfrac{\Delta f}{\Delta x}$ between x_0 and x exists finite, for x approaching x_0. The real number*

$$f_+(x_0) = \lim_{x\to x_0^+} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{\Delta x\to 0^+} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

is the **right** (*or* **backward**) **derivative of** f **at** x_0. *Similarly it goes for the* **left** (*or* **forward**) **derivative** $f'_-(x_0)$.

If f is defined only on a right (resp. left) neighbourhood of x_0 and is differentiable on the right (resp. the left) at x_0, we shall simply say that f is differentiable at x_0, and write $f'(x_0) = f'_+(x_0)$ (resp. $f'(x_0) = f'_-(x_0)$).

From Proposition 3.24 the following criterion is immediate.

Property 6.14 *A map f defined around a point $x_0 \in \mathbb{R}$ is differentiable at x_0 if and only if it is differentiable on both sides at x_0 and the left and right derivatives coincide, in which case*

$$f'(x_0) = f'_+(x_0) = f'_-(x_0).$$

Instead, if f is differentiable at x_0 on the left and on the right, but the two derivatives are different (as for $f(x) = |x|$ at the origin), x_0 is called **corner (point)** for f (Fig. 6.2). The term originates in the geometric observation that the right derivative of f at x_0 represents the slope of the *right tangent* to the graph of f at $P_0 = (x_0, f(x_0))$, i.e., the limiting position of the secant through P_0 and $P = (x, f(x))$ as $x > x_0$ approaches x_0. In case the right and left tangent (similarly defined) do not coincide, they form an *angle* at P_0.

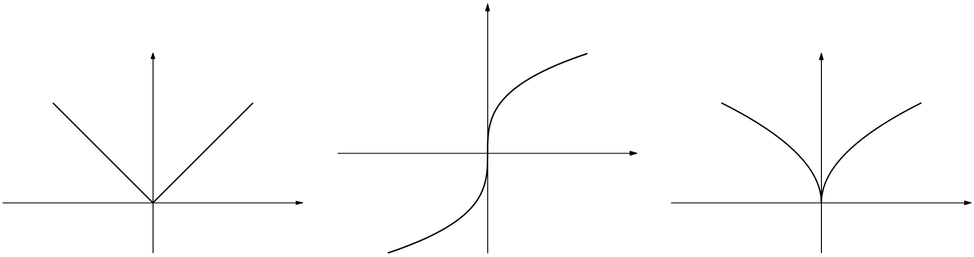


Figure 6.2. Non-differentiable maps: the origin is a corner point (left), a point with vertical tangent (middle), a cusp (right)

Other interesting cases occur when the right and left limits of the difference quotient of f at x_0 exist, but one at least is not finite. These will be still denoted by $f'_+(x_0)$ and $f'_-(x_0)$.

Precisely, if just one of $f'_+(x_0)$, $f'_-(x_0)$ is infinite, we still say that x_0 is a **corner point** for f.

If both $f'_+(x_0)$ and $f'_-(x_0)$ are infinite and with same sign (hence the limit of the difference quotient is $+\infty$ or $-\infty$), x_0 is a **point with vertical tangent** for f. This is the case for $f(x) = \sqrt[3]{x}$:

$$f'_\pm(0) = \lim_{x\to 0^\pm} \frac{\sqrt[3]{x}}{x} = \lim_{x\to 0^\pm} \frac{1}{\sqrt[3]{x^2}} = +\infty.$$

When $f'_+(x_0)$, $f'_-(x_0)$ are finite and have different signs, x_0 is called a **cusp** (**point**) of f. For instance the map $f(x) = \sqrt{|x|}$ has a cusp at the origin, for

$$f'_\pm(0) = \lim_{x\to 0^\pm} \frac{\sqrt{|x|}}{x} = \lim_{x\to 0^\pm} \frac{\sqrt{|x|}}{\operatorname{sign}(x)\,|x|} = \lim_{x\to 0^\pm} \frac{1}{\operatorname{sign}(x)\,\sqrt{|x|}} = \pm\infty.$$

Another criterion for differentiability at a point x_0 is up next. The proof is deferred to Sect. 6.11, for it relies on de l'Hôpital's Theorem.

Theorem 6.15 *Let f be continuous at x_0 and differentiable at all points $x \neq x_0$ in a neighbourhood of x_0. Then f is differentiable at x_0 provided that the limit of $f'(x)$ for $x \to x_0$ exists finite. If so,*

$$f'(x_0) = \lim_{x\to x_0} f'(x).$$

Example 6.16

We take the function

$$f(x) = \begin{cases} a\sin 2x - 4 & \text{if } x < 0, \\ b(x-1) + \mathrm{e}^x & \text{if } x \geq 0, \end{cases}$$

and ask ourselves whether there are real numbers a and b rendering f differentiable at the origin. The continuity at the origin (recall: differentiable implies continuous) forces the two values

$$\lim_{x\to 0^-} f(x) = -4, \qquad \lim_{x\to 0^+} f(x) = f(0) = -b+1$$

to agree, hence $b = 5$. With b fixed, we may impose the equality of the right and left limits of $f'(x)$ for $x \to 0$, to the effect that $f'(x)$ admits finite limit for $x \to 0$. Then we use Theorem 6.15, which prescribes that

$$\lim_{x\to 0^-} f'(x) = \lim_{x\to 0^-} 2a\cos 2x = 2a, \qquad \text{and} \qquad \lim_{x\to 0^+} f'(x) = \lim_{x\to 0^+} (5+\mathrm{e}^x) = 6$$

are the same, so $a = 3$. □

Remark 6.17 In using Theorem 6.15 one should not forget to impose continuity at the point x_0. The mere existence of the limit for f' is not enough to guarantee f will be differentiable at x_0. For example, $f(x) = x + \operatorname{sign} x$ is differentiable at every $x \neq 0$: since $f'(x) = 1$, it necessarily follows $\lim_{x\to 0} f'(x) = 1$. The function is nonetheless not differentiable, because not continuous, at $x = 0$. □

6.4 Extrema and critical points

Definition 6.18 *One calls* $x_0 \in \operatorname{dom} f$ *a* **relative** (*or* **local**) **maximum point for** f *if there is a neighbourhood* $I_r(x_0)$ *of* x_0 *such that*

$$\forall x \in I_r(x_0) \cap \operatorname{dom} f, \qquad f(x) \leq f(x_0).$$

Then $f(x_0)$ *is a* **relative** (*or* **local**) **maximum of** f*.*
One calls x_0 *an* **absolute maximum point** (*or* **global maximum point**) **for** f *if*

$$\forall x \in \operatorname{dom} f, \qquad f(x) \leq f(x_0),$$

and $f(x_0)$ *becomes the* (**absolute**) **maximum of** f*. In either case, the maximum is said* **strict** *if* $f(x) < f(x_0)$ *when* $x \neq x_0$*.*

Exchanging the symbols $\leq$ with $\geq$ one obtains the definitions of **relative** and **absolute minimum point**. A minimum or maximum point shall be referred to generically as an **extremum** (**point**) of f.

Examples 6.19

i) The parabola $f(x) = 1+2x-x^2 = 2-(x-1)^2$ has a strict absolute maximum point at $x_0 = 1$, and 2 is the function's absolute maximum. Notice the derivative $f'(x) = 2(1-x)$ is zero at that point. There are no minimum points (relative or absolute).

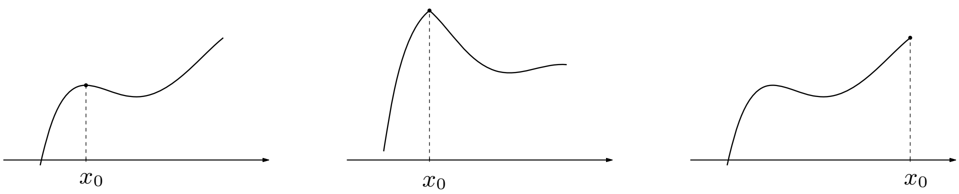


Figure 6.3. Types of maxima

ii) For $g(x) = \arcsin x$ (see Fig. 2.24), $x_0 = 1$ is a strict absolute maximum point, with maximum value $\frac{\pi}{2}$. The point $x_1 = -1$ is a strict absolute minimum, with value $-\frac{\pi}{2}$. At these extrema g is not differentiable. □

We are interested in finding the extremum points of a given function. Provided the latter is differentiable, it might be useful to look for the points where the first derivative vanishes.

Definition 6.20 *A* **critical point** (*or* **stationary point**) *of f is a point x_0 at which f is differentiable with derivative $f'(x_0) = 0$.*

The tangent at a critical point is horizontal.

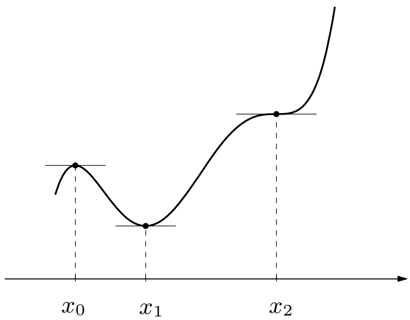


Figure 6.4. Types of critical points

Theorem 6.21 (**Fermat**) *Suppose f is defined in a full neighbourhood of a point x_0 and differentiable at x_0. If x_0 is an extremum point, then it is critical for f, i.e.,*

$$f'(x_0) = 0.$$

Proof. To fix ideas, assume x_0 is a relative maximum point and that $I_r(x_0)$ is a neighbourhood where $f(x) \leq f(x_0)$ for all $x \in I_r(x_0)$. On such neighbourhood then $\Delta f = f(x) - f(x_0) \leq 0$.

If $x > x_0$, hence $\Delta x = x - x_0 > 0$, the difference quotient $\dfrac{\Delta f}{\Delta x}$ is non-positive. Corollary 4.3 implies

$$\lim_{x \to x_0^+} \frac{f(x) - f(x_0)}{x - x_0} \leq 0.$$

Vice versa, if $x < x_0$, i.e., $\Delta x < 0$, then $\dfrac{\Delta f}{\Delta x}$ is non-negative, so

$$\lim_{x \to x_0^-} \frac{f(x) - f(x_0)}{x - x_0} \geq 0.$$

By Property 6.14,

$$f'(x_0) = \lim_{x \to x_0^+} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0^-} \frac{f(x) - f(x_0)}{x - x_0},$$

so $f'(x_0)$ is simultaneously ≤ 0 and ≥ 0, hence zero.
A similar argument holds for relative minima. □

Fermat's Theorem 6.21 ensures that the extremum points of a differentiable map which belong to the *interior* of the domain should be searched for among critical points.

A function can nevertheless have critical points that are not extrema, as in Fig. 6.4. The map $f(x) = x^3$ has the origin as a critical point ($f'(x) = 3x^2 = 0$ if and only if $x = 0$), but admits no extremum since it is strictly increasing on the whole $\mathbb{R}$.

At the same time though, a function may have non-critical extremum point (Fig. 6.3); this happens when a function is not differentiable at an extremum that lies inside the domain (e.g. $f(x) = |x|$, whose absolute minimum is attained at the origin), or when the extremum point is on the boundary (as in Example 6.19 ii)). The upshot is that in order to find *all* extrema of a function, browsing through the critical points might not be sufficient.

To summarise, extremum points are contained among the points of the domain at which either

i) the first derivative vanishes,

ii) or the function is not differentiable,

iii) or among the domain's boundary points (inside $\mathbb{R}$).

6.5 Theorems of Rolle, Lagrange, and Cauchy

We begin this section by presenting two theorems, Rolle's Theorem and Lagrange's or Mean Value Theorem, that are fundamental for the study of differentiable maps on an interval.

Theorem 6.22 (**Rolle**) *Let f be a function defined on a closed bounded interval $[a,b]$, continuous on $[a,b]$ and differentiable on (a,b) (at least). If $f(a)=f(b)$, there exists an $x_0 \in (a,b)$ such that*

$$f'(x_0)=0.$$

In other words, f admits at least one critical point in (a,b).

Proof. By the Theorem of Weierstrass the range $f([a,b])$ is the closed interval $[m,M]$ bounded by the minimum and maximum values m, M of the map:

$$m = \min_{x\in[a,b]} f(x) = f(x_m), \qquad M = \max_{x\in[a,b]} f(x) = f(x_M),$$

for suitable $x_m, x_M \in [a,b]$.
In case $m=M$, f is constant on $[a,b]$, so in particular $f'(x)=0$ for any $x\in(a,b)$ and the theorem follows.
Suppose then $m<M$. Since $m \le f(a) = f(b) \le M$, one of the strict inequalities $f(a)=f(b)<M$, $m<f(a)=f(b)$ will hold.
If $f(a)=f(b)<M$, the absolute maximum point x_M cannot be a nor b; thus, $x_M \in (a,b)$ is an interior extremum point at which f is differentiable. By Fermat's Theorem 6.21 we have that $x_M = x_0$ is a critical point.
If $m<f(a)=f(b)$, one proves analogously that x_m is the critical point x_0 of the claim. □

The theorem proves the existence of one critical point in (a,b); Fig. 6.5 shows that there could actually be more.

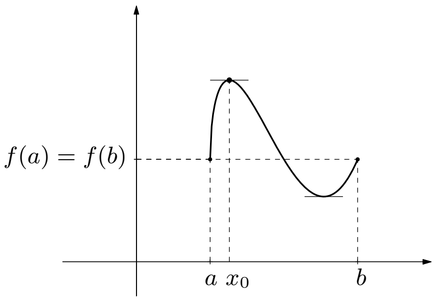


Figure 6.5. Rolle's Theorem

Theorem 6.23 (**Mean Value Theorem** or **Lagrange Theorem**) *Let f be defined on the closed and bounded interval $[a, b]$, continuous on $[a, b]$ and differentiable (at least) on (a, b). Then there is a point $x_0 \in (a, b)$ such that*

$$\frac{f(b) - f(a)}{b - a} = f'(x_0). \tag{6.9}$$

Every such point x_0 we shall call **Lagrange point for f in** (a, b).

Proof. Introduce an auxiliary map

$$g(x) = f(x) - \frac{f(b) - f(a)}{b - a}(x - a)$$

defined on $[a, b]$. It is continuous on $[a, b]$ and differentiable on (a, b), as difference of f and an affine map, which is differentiable on all of $\mathbb{R}$. Note

$$g'(x) = f'(x) - \frac{f(b) - f(a)}{b - a}.$$

It is easily seen that

$$g(a) = f(a), \qquad g(b) = f(a),$$

so Rolle's Theorem applies to g, with the consequence that there is a point $x_0 \in (a, b)$ satisfying

$$g'(x_0) = f'(x_0) - \frac{f(b) - f(a)}{b - a} = 0.$$

But this is exactly (6.9). □

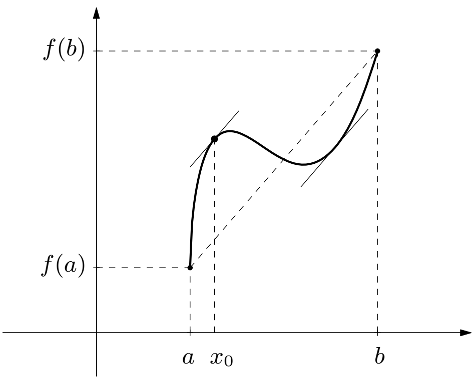


Figure 6.6. Lagrange point for f in (a, b)

The meaning of the Mean Value Theorem is clarified in Fig. 6.6. At each Lagrange point, the tangent to the graph of f is *parallel* to the secant line passing through the points $\big(a, f(a)\big)$ and $\big(b, f(b)\big)$.

Example 6.24

Consider $f(x) = 1 + x + \sqrt{1-x^2}$, a continuous map on its domain $[-1,1]$ as composite of elementary continuous functions. It is also differentiable on the open interval $(-1,1)$ (not at the end-points), in fact

$$f'(x) = 1 - \frac{x}{\sqrt{1-x^2}}.$$

Thus f fulfills the Mean Value Theorem's hypotheses, and must admit a Lagrange point in $(-1,1)$. Now (6.9) becomes

$$1 = \frac{f(1)-f(-1)}{1-(-1)} = f'(x_0) = 1 - \frac{x_0}{\sqrt{1-x_0^2}},$$

satisfied by $x_0 = 0$. □

The following result is a generalisation of the Mean Value Theorem 6.23 (which is recovered by $g(x) = x$ in its statement). It will be useful during the proofs of de l'Hôpital's Theorem 6.41 and Taylor's formula with Lagrange's remainder (Theorem 7.2).

Theorem 6.25 (Cauchy) *Let f and g be maps defined on the closed, bounded interval $[a,b]$, continuous on $[a,b]$ and differentiable (at least) on (a,b). Suppose $g'(x) \neq 0$ for all $x \in (a,b)$. Then there exists $x_0 \in (a,b)$ such that*

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(x_0)}{g'(x_0)}\,. \tag{6.10}$$

Proof. Note first that $g(a) \neq g(b)$, otherwise Rolle's Theorem would have $g'(x)$ vanish somewhere in (a,b), against the assumption.
Take the function

$$h(x) = f(x) - \frac{f(b)-f(a)}{g(b)-g(a)}\big(g(x)-g(a)\big)$$

defined on $[a,b]$. It is continuous on $[a,b]$ and differentiable on the open interval (a,b), because difference of maps with those properties. Moreover

$$h'(x) = f'(x) - \frac{f(b)-f(a)}{g(b)-g(a)}\,g'(x).$$

As

$$h(a) = f(a), \qquad h(b) = f(a)\,,$$

the map h satisfies Rolle's Theorem, so there must be a point $x_0 \in (a, b)$ with

$$h'(x_0) = f'(x_0) - \frac{f(b) - f(a)}{g(b) - g(a)}\, g'(x_0) = 0,$$

which is exactly (6.10). □

6.6 First and second finite increment formulas

We shall discuss a couple of useful relations to represent how a function varies when passing from one point to another of its domain.

Let us begin by assuming f is differentiable at x_0. By definition

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0),$$

that is to say

$$\lim_{x \to x_0} \left(\frac{f(x) - f(x_0)}{x - x_0} - f'(x_0) \right) = \lim_{x \to x_0} \frac{f(x) - f(x_0) - f'(x_0)(x - x_0)}{x - x_0} = 0.$$

Using the Landau symbols of Sect. 5.1, this becomes

$$f(x) - f(x_0) - f'(x_0)(x - x_0) = o(x - x_0), \quad x \to x_0.$$

An equivalent formulation is

$$f(x) - f(x_0) = f'(x_0)(x - x_0) + o(x - x_0), \quad x \to x_0, \tag{6.11}$$

or

$$\Delta f = f'(x_0)\Delta x + o(\Delta x), \quad \Delta x \to 0, \tag{6.12}$$

by putting $\Delta x = x - x_0$ and $\Delta f = f(x) - f(x_0)$.
Equations (6.11)-(6.12) are equivalent writings of what we call the **first formula of the finite increment**, the geometric interpretation of which can be found in Fig. 6.7. It tells that if $f'(x_0) \neq 0$, the increment Δf, corresponding to a change Δx, is proportional to Δx itself, if one disregards an infinitesimal which is negligible with respect to Δx. For Δx small enough, in practice, Δf can be treated as $f'(x_0)\Delta x$.

Now take f continuous on an interval I of $\mathbb{R}$ and differentiable on the interior points. Fix $x_1 < x_2$ in I and note that f is continuous on $[x_1, x_2]$ and differentiable on (x_1, x_2). Therefore f, restricted to $[x_1, x_2]$, satisfies the Mean Value Theorem, so there is $\bar{x} \in (x_1, x_2)$ such that

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(\bar{x}),$$

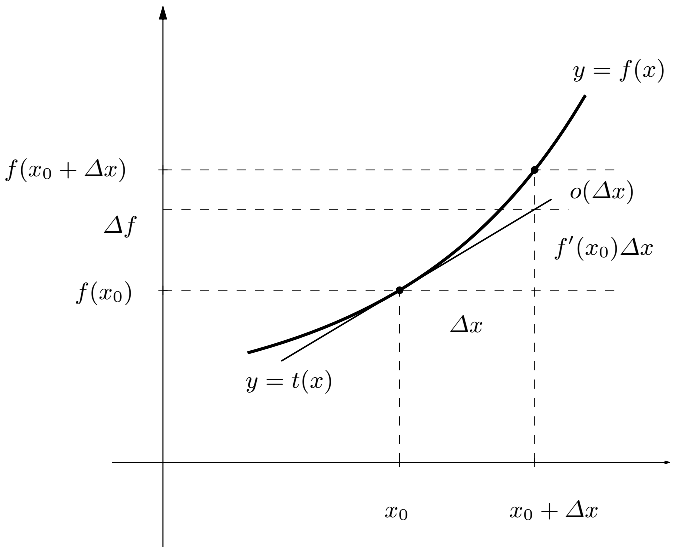


Figure 6.7. First formula of the finite increment

that is, a point $\bar{x} \in (x_1, x_2)$ with

$$f(x_2) - f(x_1) = f'(\bar{x})(x_2 - x_1). \tag{6.13}$$

We shall refer to this relation as the **second formula of the finite increment.** It has to be noted that the point $\bar{x}$ depends upon the choice of x_1 and x_2, albeit this dependency is in general not explicit. The formula's relevance derives from the possibility of gaining information about the increment $f(x_2) - f(x_1)$ from the behaviour of f' on the interval $[x_1, x_2]$.

The second formula of the finite increment may be used to describe the local behaviour of a map in the neighbourhood of a certain x_0 with more precision than that permitted by the first formula. Suppose f is continuous at x_0 and differentiable around x_0 except possibly at the point itself. If x is a point in the neighbourhood of x_0, (6.13) can be applied to the interval bounded by x_0 and x, to the effect that

$$\Delta f = f'(\bar{x})\Delta x, \tag{6.14}$$

where $\bar{x}$ lies between x_0 and x. This alternative formulation of (6.13) expresses the increment of the dependent variable Δf as if it were a multiple of Δx; at closer look though, one realises that the proportionality coefficient, i.e., the derivative evaluated at a point near x_0, depends upon Δx (and on x_0), besides being usually not known.

A further application of (6.13) is described in the next result. This will be useful later.

Property 6.26 *A function defined on a real interval I and everywhere differentiable is constant on I if and only if its first derivative vanishes identically.*

Proof. Let f be the map. Suppose first f is constant, therefore for every $x_0 \in I$, the difference quotient $\dfrac{f(x) - f(x_0)}{x - x_0}$, with $x \in I$, $x \neq x_0$, is zero. Then $f'(x_0) = 0$ by definition of derivative.
Vice versa, suppose f has zero derivative on I and let us prove that f is constant on I. This would be equivalent to demanding

$$f(x_1) = f(x_2), \qquad \forall x_1, x_2 \in I.$$

Take $x_1, x_2 \in I$ and use formula (6.13) on f. For a suitable $\bar{x}$ between x_1, x_2, we have

$$f(x_2) - f(x_1) = f'(\bar{x})(x_2 - x_1) = 0,$$

thus $f(x_1) = f(x_2)$. □

6.7 Monotone maps

In the light of the results on differentiability, we tackle the issue of monotonicity.

Theorem 6.27 *Let I be an interval upon which the map f is differentiable. Then:*

a) If f is increasing on I, then $f'(x) \geq 0$ for all $x \in I$.

b1) If $f'(x) \geq 0$ for any $x \in I$, then f is increasing on I;

b2) if $f'(x) > 0$ for all $x \in I$, then f is strictly increasing on I.

Proof. Let us prove claim a). Suppose f increasing on I and consider an interior point x_0 of I. For all $x \in I$ such that $x < x_0$, we have

$$f(x) - f(x_0) \leq 0 \qquad \text{and} \qquad x - x_0 < 0.$$

Thus, the difference quotient $\dfrac{\Delta f}{\Delta x}$ between x_0 and x is non-negative. On the other hand, for any $x \in I$ with $x > x_0$,

$$f(x) - f(x_0) \geq 0 \qquad \text{and} \qquad x - x_0 > 0.$$

Here too the difference quotient $\dfrac{\Delta f}{\Delta x}$ between x_0 and x is positive or zero. Altogether,

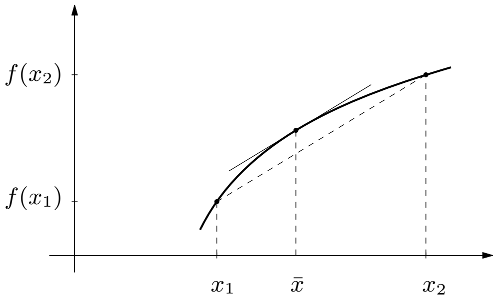


Figure 6.8. Proof of Theorem 6.27, *b*)

$$\frac{\Delta f}{\Delta x} = \frac{f(x) - f(x_0)}{x - x_0} \geq 0, \qquad \forall x \neq x_0;$$

Corollary 4.3 on

$$\lim_{x \to x_0} \frac{\Delta f}{\Delta x} = f'(x_0)$$

yields $f'(x_0) \geq 0$. As for the possible extremum points in I, we arrive at the same conclusion by considering one-sided limits of the difference quotient, which is always ≥ 0.
Now to the implications in parts *b*). Take f with $f'(x) \geq 0$ for all $x \in I$. The idea is to fix points $x_1 < x_2$ in I and prove that $f(x_1) \leq f(x_2)$. For that we use (6.13) and note that $f'(\bar{x}) \geq 0$ by assumption. But since $x_2 - x_1 > 0$, we have

$$f(x_2) - f(x_1) = f'(\bar{x})(x_2 - x_1) \geq 0,$$

proving *b*1). Considering f such that $f'(x) > 0$ for all $x \in I$ instead, (6.13) implies $f(x_2) - f(x_1) > 0$, hence also *b*2) holds. □

The theorem asserts that if f is differentiable on I, the following logic equivalence holds:

$$f'(x) \geq 0, \quad \forall x \in I \qquad \Longleftrightarrow \qquad f \text{ is increasing on } I.$$

Furthermore,

$$f'(x) > 0, \quad \forall x \in I \qquad \Longrightarrow \qquad f \text{ is strictly increasing on } I.$$

The latter implication is not reversible: f *strictly* increasing on I does not imply $f'(x) > 0$ for all $x \in I$. We have elsewhere observed that $f(x) = x^3$ is everywhere strictly increasing, despite having vanishing derivative at the origin.

A similar statement to the above holds if we change the word ‘increasing’ with ‘decreasing’ and the symbols $\geq$, $>$ with $\leq$, $<$.

Corollary 6.28 *Let f be differentiable on I and x_0 an interior critical point. If $f'(x) \geq 0$ at the left of x_0 and $f'(x) \leq 0$ at its right, then x_0 is a maximum point for f. Similarly, $f'(x) \leq 0$ at the left, and ≥ 0 at the right of x_0 implies x_0 is a minimum point.*

Theorem 6.27 and Corollary 6.28 justify the search for extrema among the zeroes of f', and explain why the derivative's sign affects monotonicity intervals.

Example 6.29

The map $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x\mathrm{e}^{2x}$ differentiates to $f'(x) = (2x+1)\mathrm{e}^{2x}$, whence $x_0 = -\frac{1}{2}$ is the sole critical point. As $f'(x) > 0$ if and only if $x > -\frac{1}{2}$, $f(x_0)$ is an absolute minimum. The function is strictly decreasing on $(-\infty, -\frac{1}{2}]$ and strictly increasing on $[-\frac{1}{2}, +\infty)$. □

6.8 Higher-order derivatives

Let f be differentiable around x_0 and let its first derivative f' be also defined around x_0.

Definition 6.30 *If f' is a differentiable function at x_0, one says f* **is twice differentiable at** x_0. *The expression*

$$f''(x_0) = (f')'(x_0)$$

is called **second derivative of f at x_0**. *The* **second derivative of** f, *denoted f'', is the map associating to x the number $f''(x)$, provided the latter is defined.*

Other notations commonly used for the second derivative include

$$y''(x_0), \qquad \frac{\mathrm{d}^2 f}{\mathrm{d}x^2}(x_0), \qquad \mathrm{D}^2 f(x_0).$$

The third derivative, where defined, is the derivative of the second derivative:

$$f'''(x_0) = (f'')'(x_0).$$

In general, for any $k \geq 1$, the **derivative of order k** (**kth derivative**) **of f at** x_0 is the first derivative, where defined, of the derivative of order $(k-1)$ of f at x_0:

$$f^{(k)}(x_0) = (f^{(k-1)})'(x_0).$$

Alternative symbols are:

$$y^{(k)}(x_0), \qquad \frac{\mathrm{d}^k f}{\mathrm{d}x^k}(x_0), \qquad \mathrm{D}^k f(x_0)\,.$$

For conveniency one defines $f^{(0)}(x_0) = f(x_0)$ as well.

Examples 6.31

We compute the derivatives of all orders for three elementary functions.

i) Choose $n \in \mathbb{N}$ and consider $f(x) = x^n$. Then

$$\begin{aligned} f'(x) &= nx^{n-1} = \frac{n!}{(n-1)!}x^{n-1} \\ f''(x) &= n(n-1)x^{n-2} = \frac{n!}{(n-2)!}x^{n-2} \\ &\ \vdots \quad \vdots \\ f^{(n)}(x) &= n(n-1)\cdots 2\cdot 1\, x^{n-n} = n!\,. \end{aligned}$$

More concisely,

$$f^{(k)}(x) = \frac{n!}{(n-k)!}x^{n-k}$$

with $0 \le k \le n$. Furthermore, $f^{(n+1)}(x) = 0$ for any $x \in \mathbb{R}$ (the derivative of the constant function $f^{(n)}(x)$ is 0), and consequently all derivatives $f^{(k)}$ of order $k > n$ exist and vanish identically.

ii) The sine function $f(x) = \sin x$ satisfies $f'(x) = \cos x$, $f''(x) = -\sin x$, $f'''(x) = -\cos x$ and $f^{(4)}(x) = \sin x$. Successive derivatives of f clearly reproduce this cyclical pattern. The same phenomenon occurs for $y = \cos x$.

iii) Because $f(x) = \mathrm{e}^x$ differentiates to $f'(x) = \mathrm{e}^x$, it follows that $f^{(k)}(x) = \mathrm{e}^x$ for every $k \ge 0$, proving the remarkable fact that all higher-order derivatives of the exponential function are equal to e^x. □

A couple of definitions wrap up the section.

Definition 6.32 *A map f is* **of class** $\mathcal{C}^k$ *($k \ge 0$) on an interval I if f is differentiable k times everywhere on I and its kth derivative $f^{(k)}$ is continuous on I. The collection of all $\mathcal{C}^k$ maps on I is denoted by $\mathcal{C}^k(I)$.*
A map f is **of class** $\mathcal{C}^\infty$ *on I if it is arbitrarily differentiable everywhere on I. One indicates by $\mathcal{C}^\infty(I)$ the collection of such maps.*

In virtue of Proposition 6.3, if $f \in \mathcal{C}^k(I)$ all derivatives of order smaller or equal than k are continuous on I. Similarly, if $f \in \mathcal{C}^\infty(I)$, all its derivatives are continuous on I.

Moreover, the elementary functions are differentiable any number of times (so they are of class $\mathcal{C}^\infty$) at every interior point of their domains.

6.9 Convexity and inflection points

Let f be differentiable at the point x_0 of the domain. As customary, we indicate by $y = t(x) = f(x_0) + f'(x_0)(x - x_0)$ the equation of the tangent to the graph of f at x_0.

Definition 6.33 *The map f is* **convex at** x_0 *if there is a neighbourhood $I_r(x_0) \subseteq \operatorname{dom} f$ such that*

$$\forall x \in I_r(x_0), \qquad f(x) \geq t(x);$$

f is **strictly convex** *if $f(x) > t(x)$, $\forall\, x \neq x_0$.*

The definitions for **concave** and **strictly concave** functions are alike (just change $\geq, >$ to $\leq, <$).

What does this say geometrically? A map is convex at a point if around that point the graph lies 'above' the tangent line, concave if its graph is 'below' the tangent (Fig. 6.9).

Example 6.34

We claim that $f(x) = x^2$ is strictly convex at $x_0 = 1$. The tangent at the given point has equation

$$t(x) = 1 + 2(x - 1) = 2x - 1\,.$$

Since $f(x) > t(x)$ means $x^2 > 2x - 1$, hence $x^2 - 2x + 1 = (x - 1)^2 > 0$, t lies below the graph except at the touching point $x = 1$. □

Definition 6.35 *A differentiable map f on an interval I is* **convex on** *I if it is convex at each point of I.*

For understanding convexity, inflection points play a role reminiscent of extremum points for the study of monotone functions.

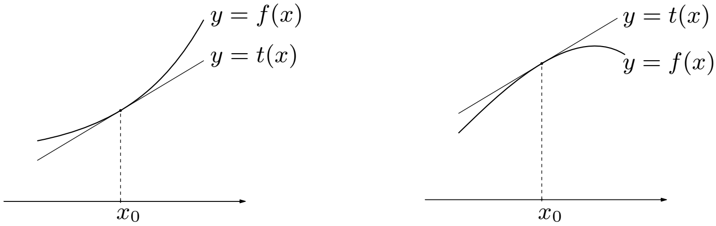


Figure 6.9. Strictly convex (left) and strictly concave (right) maps at x_0

Definition 6.36 *The point x_0 is an* **inflection point for** *f if there is a neighbourhood $I_r(x_0) \subseteq \operatorname{dom} f$ where one of the following conditions holds: either*

$$\forall x \in I_r(x_0), \qquad \begin{cases} \textit{if } x < x_0, & f(x) \le t(x), \\ \textit{if } x > x_0, & f(x) \ge t(x), \end{cases}$$

or

$$\forall x \in I_r(x_0), \qquad \begin{cases} \textit{if } x < x_0, & f(x) \ge t(x), \\ \textit{if } x > x_0, & f(x) \le t(x) \end{cases}$$

In the former case we speak of an **ascending** *inflection, in the latter the inflection is* **descending**.

In the plane, the graph of f 'cuts through' the inflectional tangent at an inflection point (Fig. 6.10).

The analysis of convexity and inflections of a function is helped a great deal by the next results.

Theorem 6.37 *Given a differentiable map f on the interval I,*

a) if f is convex on I, then f' is increasing on I.

b1) If f' is increasing on I, then f is convex on I;

b2) if f' is strictly increasing on I, then f is strictly convex on I.

Proof. See Appendix A.4.3, p. 455. □

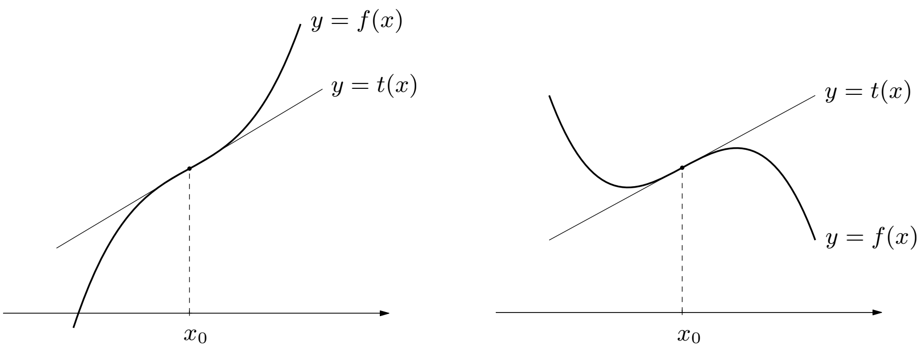


Figure 6.10. Ascending (left) and descending (right) inflections at x_0

Corollary 6.38 *If f is differentiable twice on I, then*

a) *f convex on I implies $f''(x) \geq 0$ for all $x \in I$.*
b1) *$f''(x) \geq 0$ for all $x \in I$ implies f convex on I;*
b2) *$f''(x) > 0$ for all $x \in I$ implies f strictly convex on I.*

Proof. This follows directly from Theorem 6.37 by applying Theorem 6.27 to the function f'. □

There is a second formulation for this, namely: under the same hypothesis, the following formulas are true:

$$f''(x) \geq 0, \quad \forall x \in I \qquad \Longleftrightarrow \qquad f \text{ is convex on } I$$

and

$$f''(x) > 0, \quad \forall x \in I \qquad \Longrightarrow \qquad f \text{ is strictly convex on } I.$$

Here, as in the characterisation of monotone functions, the last implication has no reverse. For instance, $f(x) = x^4$ is strictly convex on $\mathbb{R}$, but has vanishing second derivative at the origin.

Analogies clearly exist concerning concave functions.

Corollary 6.39 *Let f be twice differentiable around x_0.*

a) *If x_0 is an inflection point, then $f''(x_0) = 0$.*
b) *Assume $f''(x_0) = 0$. If f'' changes sign when crossing x_0, then x_0 is an inflection point (ascending if $f''(x) \leq 0$ at the left of x_0 and $f''(x) \geq 0$ at its right, descending otherwise). If f'' does not change sign, x_0 is not an inflection point.*

The proof relies on Taylor's formula, and will be given in Sect. 7.4.

The reader ought to beware that $f''(x_0) = 0$ does not warrant x_0 is a point of inflection for f. The function $f(x) = x^4$ has second derivative $f''(x) = 12x^2$ which vanishes at $x_0 = 0$. The origin is nonetheless not an inflection point, for the tangent at x_0 is the axis $y = 0$, and the graph of f stays always above it. In addition, f'' *does not change sign* around x_0.

Example 6.29 (continuation)

For $f(x) = x\mathrm{e}^{2x}$ we have $f''(x) = 4(x+1)\mathrm{e}^{2x}$ vanishing at $x_1 = -1$. As $f''(x) > 0$ if and only if $x > -1$, f is strictly concave on $(-\infty, -1)$ and strictly convex on $(-1, +\infty)$. The point $x_1 = -1$ is an ascending inflection. The graph of $f(x)$ is shown in Fig. 6.11. □

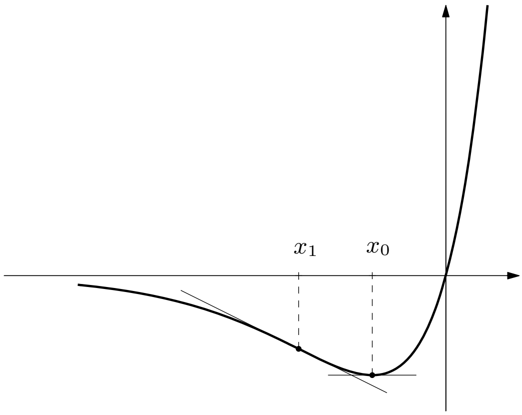


Figure 6.11. Example 6.29

6.9.1 Extension of the notion of convexity

The geometrical nature of convex maps manifests itself by considering a generalisation of the notion given in Sect. 6.9. Recall a subset C of the plane is said **convex** if the segment $\overline{P_1P_2}$ between any two points $P_1, P_2 \in C$ is all contained in C.

Given a function $f : I \subseteq \mathbb{R} \to \mathbb{R}$, we denote by

$$E_f = \{(x, y) \in \mathbb{R}^2 : x \in I,\ y \geq f(x)\}$$

the set of points of the plane lying above the graph of f (as in Fig. 6.12, left).

Definition 6.40 *The map $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is called* **convex on** I *if the set E_f is a convex subset of the plane.*

It is easy to convince oneself that the convexity of E_f can be checked by considering points P_1, P_2 belonging to the graph of f only. In other words, given

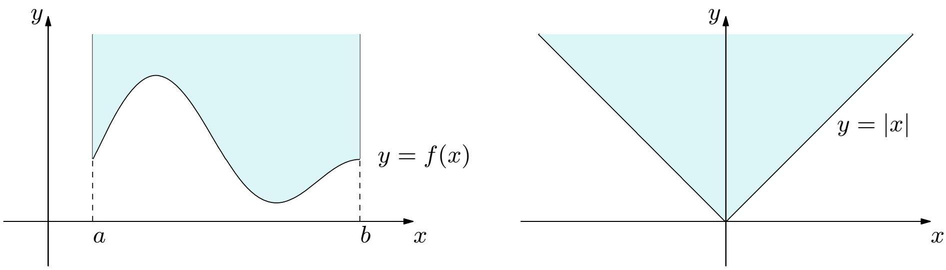


Figure 6.12. The set E_f for a generic f defined on I (left) and for $f(x) = |x|$ (right)

x_1, x_2 in I, the segment S_{12} between $(x_1, f(x_1))$ and $(x_2, f(x_2)$ should lie above the graph.

Since one can easily check that any x between x_1 and x_2 can be represented as

$$x = (1-t)x_1 + tx_2 \qquad \text{with} \quad t = \frac{x - x_1}{x_2 - x_1} \in [0,1]\,,$$

the convexity of f reads

$$f\big((1-t)x_1 + tx_2\big) \leq (1-t)f(x_1) + tf(x_2) \qquad \forall x_1, x_2 \in I\,, \forall t \in [0,1]\,.$$

If the inequality is strict for $x_1 \neq x_2$ and $t \in (0,1)$, the function is called **strictly convex on** I.

For *differentiable* functions on the interval I, Definitions 6.40, 6.33 can be proven to be equivalent. But a function may well be convex according to Definition 6.40 without being differentiable on I, like $f(x) = |x|$ on $I = \mathbb{R}$ (Fig. 6.12, right). Note, however, that convexity implies continuity at all interior points of I, although discontinuities may occur at the end-points.

6.10 Qualitative study of a function

We have hitherto supplied the reader with several analytical tools to study a map f on its domain and draw a relatively thorough – qualitatively speaking – graph. This section describes a step-by-step procedure for putting together all the information acquired.

Domain and symmetries

It should be possible to determine the domain of a generic function starting from the elementary functions that build it via algebraic operations and composition.
The study is greatly simplified if one detects the map's possible symmetries and periodicity at the very beginning (see Sect. 2.6). For instance, an even or odd map can be studied only for positive values of the variable. We point out that a function might present different kinds of symmetries, like the symmetry with respect to a vertical line other than the y-axis: the graph of $f(x) = \mathrm{e}^{-|x-2|}$ is symmetric with respect to $x = 2$ (Fig. 6.13).

For the same reason the behaviour of a periodic function is captured by its restriction to an interval as wide as the period.

Behaviour at the end-points of the domain

Assuming the domain is a union of intervals, as often happens, one should find the one-sided limits at the end-points of each interval. Then the existence of asymptotes should be discussed, as in Sect. 5.3.
For instance, consider

$$f(x) = \frac{\log(2-x)}{\sqrt{x^2 - 2x}}\,.$$

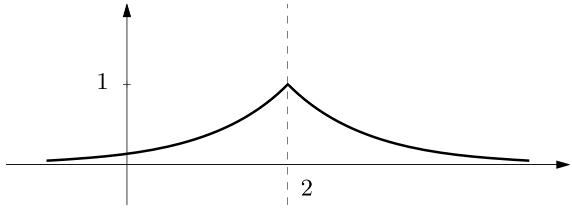


Figure 6.13. The function $f(x) = \mathrm{e}^{-|x-2|}$

Now, $\log(2-x)$ is defined for $2-x>0$, or $x<2$; in addition, $\sqrt{x^2-2x}$ has domain $x^2-2x \geq 0$, so $x \leq 0$ or $x \geq 2$, and being a denominator, $x \neq 0, 2$. Thus $\operatorname{dom} f = (-\infty, 0)$. Since $\lim_{x\to 0^-} f(x) = +\infty$, the line $x = 0$ is a vertical left asymptote, while $\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{\log(2-x)}{|x|} = 0$ yields the horizontal left asymptote $y = 0$.

Monotonicity and extrema

The first step consists in computing the derivative f' and its domain $\operatorname{dom} f'$. Even if the derivative's analytical expression might be defined on a larger interval, one should in any case have $\operatorname{dom} f' \subseteq \operatorname{dom} f$. For example $f(x) = \log x$ has $f'(x) = \frac{1}{x}$ and $\operatorname{dom} f = \operatorname{dom} f' = (0, +\infty)$, despite $g(x) = \frac{1}{x}$ makes sense for any $x \neq 0$. After that, the zeroes and sign of f' should be determined. They allow to find the intervals where f is monotone and discuss the nature of critical points (the zeroes of f'), in the light of Sect. 6.7.

A careless analysis might result in wrong conclusions. Suppose a map f is differentiable on the union $(a,b) \cup (b,c)$ of two bordering intervals where $f' > 0$. If f is not differentiable at the point b, deducing from that that f is increasing on $(a,b) \cup (b,c)$ **is wrong**. The function $f(x) = -\frac{1}{x}$ satisfies $f'(x) = \frac{1}{x^2} > 0$ on $(-\infty,0) \cup (0,+\infty)$, but it is not globally increasing therein (e.g. $f(-1) > f(1)$); we can only say f is increasing on $(-\infty, 0)$ and on $(0, +\infty)$ *separately.*

Recall that extremum points need not only be critical points. The function $f(x) = \sqrt{\frac{x}{1+x^2}}$, defined on $x \geq 0$, has a critical point $x = 1$ giving an absolute maximum. At the other extremum $x = 0$, the function is not differentiable, although $f(0)$ is the absolute minimum.

Convexity and inflection points

Along the same lines one determines the intervals upon which the function is convex or concave, and its inflections. As in Sect. 6.9, we use the second derivative for this.

Sign of the function and its higher derivatives

When sketching the graph of f we might find useful (not compulsory) to establish the sign of f and its vanishing points (the x-coordinates of the intersections of the graph with the horizontal axis). The roots of $f(x) = 0$ are not always easy to find analytically. In such cases one may resort to the Theorem of existence of zeroes 4.23, and deduce the presence of a unique zero within a certain interval. Likewise can be done for the sign of the first or second derivatives.

The function $f(x) = x \log x - 1$ is defined for $x > 0$. One has $f(x) < 0$ when $x \le 1$. On $x \ge 1$ the map is strictly increasing (in fact $f'(x) = \log x + 1 > 0$ for $x > 1/\mathrm{e}$); besides, $f(1) = -1 < 0$ and $f(\mathrm{e}) = \mathrm{e} - 1 > 0$. Therefore there is exactly one zero somewhere in $(1, \mathrm{e})$, f is negative to the left of said zero and positive to the right.

6.10.1 Hyperbolic functions

An exemplary application of what seen so far is the study of a family of functions, called **hyperbolic**, that show up in various concrete situations.

We introduce the maps $f(x) = \sinh x$ and $g(x) = \cosh x$ by

$$\sinh x = \frac{\mathrm{e}^x - \mathrm{e}^{-x}}{2} \qquad \text{and} \qquad \cosh x = \frac{\mathrm{e}^x + \mathrm{e}^{-x}}{2}.$$

They are respectively called **hyperbolic sine** and **hyperbolic cosine.** The terminology stems from the fundamental relation

$$\cosh^2 x - \sinh^2 x = 1\,, \qquad \forall x \in \mathbb{R}\,,$$

whence the point P of coordinates $(X, Y) = (\cosh x, \sinh x)$ runs along the right branch of the rectangular hyperbola $X^2 - Y^2 = 1$ as x varies.

The first observation is that $\operatorname{dom} f = \operatorname{dom} g = \mathbb{R}$; moreover, $f(x) = -f(-x)$ and $g(x) = g(-x)$, hence the hyperbolic sine is an odd map, whereas the hyperbolic cosine is even. Concerning the limit behaviour,

$$\lim_{x\to\pm\infty} \sinh x = \pm\infty\,, \qquad \lim_{x\to\pm\infty} \cosh x = +\infty\,.$$

This implies that there are no vertical nor horizontal asymptotes. No oblique asymptotes exist either, because these functions behave like exponentials for $x \to \infty$. More precisely

$$\sinh x \sim \pm\frac{1}{2}\mathrm{e}^{|x|}\,, \qquad \cosh x \sim \frac{1}{2}\mathrm{e}^{|x|}\,, \qquad x \to \pm\infty\,.$$

It is clear that $\sinh x = 0$ if and only if $x = 0$, $\sinh x > 0$ when $x > 0$, while $\cosh x > 0$ everywhere on $\mathbb{R}$. The monotonic features follow easily from

$$\mathrm{D} \sinh x = \cosh x \qquad \text{and} \qquad \mathrm{D} \cosh x = \sinh x\,, \qquad \forall x \in \mathbb{R}\,.$$

Thus the hyperbolic sine is increasing on the entire $\mathbb{R}$. The hyperbolic cosine is strictly increasing on $[0, +\infty)$ and strictly decreasing on $(-\infty, 0]$, has an absolute minimum $\cosh 0 = 1$ at $x = 0$ (so $\cosh x \ge 1$ on $\mathbb{R}$).

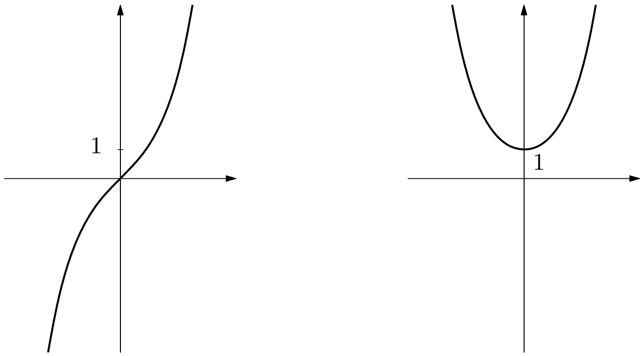


Figure 6.14. Hyperbolic sine (left) and hyperbolic cosine (right)

Differentiating once more gives

$$\mathrm{D}^2 \sinh x = \sinh x \qquad \text{and} \qquad \mathrm{D}^2 \cosh x = \cosh x\,, \qquad \forall x \in \mathbb{R}\,,$$

which says that the hyperbolic sine is strictly convex on $(0, +\infty)$ and strictly concave on $(-\infty, 0)$. The origin is an ascending inflection point. The hyperbolic cosine is strictly convex on the whole $\mathbb{R}$. The graphs are drawn in Fig. 6.14.

In analogy to the ordinary trigonometric functions, there is a **hyperbolic tangent** defined as

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{\mathrm{e}^{2x} - 1}{\mathrm{e}^{2x} + 1}\,.$$

Its domain is $\mathbb{R}$, it is odd, strictly increasing and ranges over the open interval $(-1, 1)$ (Fig. 6.15).

The inverse map to the hyperbolic sine, appropriately called **inverse hyperbolic sine**, is defined on all of $\mathbb{R}$, and can be made explicit by means of the logarithm (inverse of the exponential)

$$\sinh^{-1} x = \log(x + \sqrt{x^2 + 1})\,, \qquad x \in \mathbb{R}\,. \tag{6.15}$$

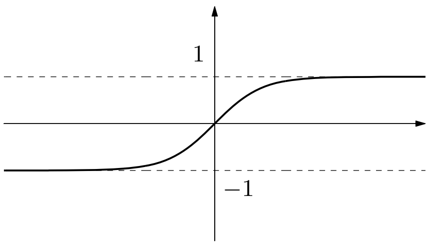


Figure 6.15. Hyperbolic tangent

There normally is no confusion with the reciprocal $1/\sinh x$, whence the use of notation[1]. The **inverse hyperbolic cosine** is obtained by inversion of the hyperbolic cosine restricted to $[0, +\infty)$

$$\cosh^{-1} x = \log(x + \sqrt{x^2 - 1}), \qquad x \in [1, +\infty). \tag{6.16}$$

To conclude, the **inverse hyperbolic tangent** inverts the corresponding hyperbolic map on $\mathbb{R}$

$$\tanh^{-1} x = \frac{1}{2} \log \frac{1+x}{1-x}, \qquad x \in (-1, 1). \tag{6.17}$$

The inverse hyperbolic functions have first derivatives

$$\begin{gathered} \mathrm{D} \sinh^{-1} x = \frac{1}{\sqrt{x^2+1}}, \qquad \mathrm{D} \cosh^{-1} x = \frac{1}{\sqrt{x^2-1}}, \\ \mathrm{D} \tanh^{-1} x = \frac{1}{1-x^2}. \end{gathered} \tag{6.18}$$

6.11 The Theorem of de l'Hôpital

This final section is entirely devoted to a single result, due to its relevance in computing the limits of indeterminate forms. Its proof can be found in Appendix A.4.2, p. 452. As always, c is one of x_0, x_0^+, x_0^-, $+\infty$, $-\infty$.

Theorem 6.41 *Let f, g be maps defined on a neighbourhood of c, except possibly at c, and such that*

$$\lim_{x \to c} f(x) = \lim_{x \to c} g(x) = L,$$

where $L = 0, +\infty$ or $-\infty$. If f and g are differentiable around c, except possibly at c, with $g' \neq 0$, and if

$$\lim_{x \to c} \frac{f'(x)}{g'(x)}$$

exists (finite or not), then also

$$\lim_{x \to c} \frac{f(x)}{g(x)} \tag{6.19}$$

exists and equals the previous limit.

[1] Some authors also like the symbol Arc sinh.

Under said hypotheses the results states that

$$\lim_{x\to c}\frac{f(x)}{g(x)}=\lim_{x\to c}\frac{f'(x)}{g'(x)}. \tag{6.20}$$

Examples 6.42

i) The limit

$$\lim_{x\to 0}\frac{\mathrm{e}^{2x}-\mathrm{e}^{-2x}}{\sin 5x}$$

gives rise to an indeterminate form of type $\frac{0}{0}$. Since numerator and denominator are differentiable functions,

$$\lim_{x\to 0}\frac{2\mathrm{e}^{2x}+2\mathrm{e}^{-2x}}{5\cos 5x}=\frac{4}{5}.$$

Therefore

$$\lim_{x\to 0}\frac{\mathrm{e}^{2x}-\mathrm{e}^{-2x}}{\sin 5x}=\frac{4}{5}.$$

ii) When the ratio $f'(x)/g'(x)$ is still an indeterminate form, supposing f and g are twice differentiable around c, except maybe at c, we can iterate the recipe of (6.20) by studying the limit of $f''(x)/g''(x)$, and so on.
Consider for instance the indeterminate form $0/0$

$$\lim_{x\to 0}\frac{1+3x-\sqrt{(1+2x)^3}}{x\sin x}.$$

Differentiating numerator and denominator, we are lead to

$$\lim_{x\to 0}\frac{3-3\sqrt{1+2x}}{\sin x+x\cos x},$$

still of the form $0/0$. Thus we differentiate again

$$\lim_{x\to 0}\frac{-\frac{3}{\sqrt{1+2x}}}{2\cos x-x\sin x}=-\frac{3}{2}.$$

Applying (6.20) twice allows to conclude

$$\lim_{x\to 0}\frac{1+3x-\sqrt{(1+2x)^3}}{\sin^2 x}=-\frac{3}{2}.$$ □

Remark 6.43 De l'Hôpital's Theorem is a sufficient condition only, for the existence of (6.19). Otherwise said, it might happen that the limit of the derivatives' difference quotient does not exist, whereas we have the limit of the functions' difference quotient. For example, set $f(x)=x+\sin x$ and $g(x)=2x+\cos x$. While the ratio f'/g' does not admit limit as $x\to+\infty$ (see Remark 4.19), the limit of f/g exists:

$$\lim_{x\to+\infty}\frac{x+\sin x}{2x+\cos x}=\lim_{x\to+\infty}\frac{x+o(x)}{2x+o(x)}=\frac{1}{2}.$$ □

6.11.1 Applications of de l'Hôpital's theorem

We survey some situations where the result of de l'Hôpital lends a helping hand.

Fundamental limits

By means of Theorem 6.41 we recover the important limits

$$\lim_{x\to+\infty}\frac{e^x}{x^\alpha}=+\infty, \qquad \lim_{x\to-\infty}|x|^\alpha e^x=0, \qquad \forall\alpha\in\mathbb{R}, \tag{6.21}$$

$$\lim_{x\to+\infty}\frac{\log x}{x^\alpha}=0, \qquad \lim_{x\to0^+}x^\alpha\log x=0, \qquad \forall\alpha>0. \tag{6.22}$$

These were presented in (5.6) in the equivalent formulation of the Landau symbols.
Let us begin with the first of (6.21) when $\alpha=1$. From (6.20)

$$\lim_{x\to+\infty}\frac{e^x}{x}=\lim_{x\to+\infty}\frac{e^x}{1}=+\infty.$$

For any other $\alpha>0$, we have

$$\lim_{x\to+\infty}\frac{e^x}{x^\alpha}=\lim_{x\to+\infty}\left(\frac{1}{\alpha}\frac{e^{\frac{x}{\alpha}}}{\frac{x}{\alpha}}\right)^\alpha=\frac{1}{\alpha^\alpha}\left(\lim_{y\to+\infty}\frac{e^y}{y}\right)^\alpha=+\infty.$$

At last, for $\alpha\le0$ the result is rather trivial because there is no indeterminacy. As for the second formula of (6.21)

$$\lim_{x\to-\infty}|x|^\alpha e^x=\lim_{x\to-\infty}\frac{|x|^\alpha}{e^{-x}}=\lim_{x\to-\infty}\frac{|x|^\alpha}{e^{|x|}}=\lim_{y\to+\infty}\frac{y^\alpha}{e^y}=0.$$

Now to (6.22):

$$\lim_{x\to+\infty}\frac{\log x}{x^\alpha}=\lim_{x\to+\infty}\frac{\frac{1}{x}}{\alpha x^{\alpha-1}}=\frac{1}{\alpha}\lim_{x\to+\infty}\frac{1}{x^\alpha}=0$$

and

$$\lim_{x\to0^+}x^\alpha\log x=\lim_{x\to0^+}\frac{\log x}{x^{-\alpha}}=\lim_{x\to0^+}\frac{\frac{1}{x}}{(-\alpha)x^{-\alpha-1}}=-\frac{1}{\alpha}\lim_{x\to0^+}x^\alpha=0.$$

Proof of Theorem 6.15

We are now in a position to prove this earlier claim.

Proof. By definition only,

$$f'(x_0)=\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0};$$

but this is an indeterminate form, since

$$\lim_{x\to x_0}\big(f(x)-f(x_0)\big)=\lim_{x\to x_0}(x-x_0)=0,$$

hence de l'Hôpital implies

$$f'(x_0)=\lim_{x\to x_0}\frac{f'(x)}{1}.$$

□

Computing the order of magnitude of a map

Through examples we explain how de l'Hôpital's result detects the order of magnitude of infinitesimal or infinite functions, and their principal parts.

The function

$$f(x) = \mathrm{e}^x - 1 - \sin x$$

is infinitesimal for $x \to 0$. With infinitesimal test function $\varphi(x) = x$ we apply the theorem twice (supposing for a moment this is possible)

$$\lim_{x\to 0} \frac{\mathrm{e}^x - 1 - \sin x}{x^\alpha} = \lim_{x\to 0} \frac{\mathrm{e}^x - \cos x}{\alpha x^{\alpha-1}} = \lim_{x\to 0} \frac{\mathrm{e}^x + \sin x}{\alpha(\alpha-1)x^{\alpha-2}}.$$

When $\alpha = 2$ the right-most limit exists and is in fact $\frac{1}{2}$. This fact alone justifies the use of de l'Hôpital's Theorem. Thus $f(x)$ is infinitesimal of order 2 at the origin with respect to $\varphi(x) = x$; its principal part is $p(x) = \frac{1}{2}x^2$.

Next, consider

$$f(x) = \tan x,$$

an infinite function for $x \to \frac{\pi}{2}^-$. Setting $\varphi(x) = \dfrac{1}{\frac{\pi}{2} - x}$, we have

$$\lim_{x\to\frac{\pi}{2}^-} \frac{\tan x}{\left(\frac{1}{\frac{\pi}{2}-x}\right)^\alpha} = \lim_{x\to\frac{\pi}{2}^-} \sin x \lim_{x\to\frac{\pi}{2}^-} \frac{\left(\frac{\pi}{2} - x\right)^\alpha}{\cos x}.$$

While the first limit is 1, for the second we apply de l'Hôpital's Theorem

$$\lim_{x\to\frac{\pi}{2}^-} \frac{\left(\frac{\pi}{2} - x\right)^\alpha}{\cos x} = \lim_{x\to\frac{\pi}{2}^-} \frac{-\alpha(\frac{\pi}{2} - x)^{\alpha-1}}{-\sin x}.$$

The latter equals 1 when $\alpha = 1$, so $\tan x$ is infinite of first order, for $x \to \frac{\pi}{2}^-$, with respect to $\varphi(x) = \dfrac{1}{\frac{\pi}{2} - x}$. The principal part is indeed $\varphi(x)$.

6.12 Exercises

1. *Discuss differentiability at the point x_0 indicated:*

   a) $f(x) = x + |x-1|\,, \quad x_0 = 1$

   $\boxed{\text{b)}}$ $f(x) = \sin|x|\,, \quad x_0 = 0$

   $\boxed{\text{c)}}$ $f(x) = \begin{cases} \mathrm{e}^{-1/x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}\,, \quad x_0 = 0$

   d) $f(x) = \sqrt{1+x^3}\,, \quad x_0 = -1$

2. *Say where the following maps are differentiable and find the derivatives:*

$\boxed{\text{a)}}$ $f(x) = x\sqrt{|x|}$

b) $f(x) = \cos|x|$

c) $f(x) = \begin{cases} x^2+1 & \text{if } x \geq 0, \\ e^x - x & \text{if } x < 0 \end{cases}$

$\boxed{\text{d)}}$ $f(x) = \begin{cases} x^2+x-5 & \text{if } x \geq 1, \\ x-4 & \text{if } x < 1 \end{cases}$

3. *Compute, where defined, the first derivative of:*

a) $f(x) = 3x\sqrt[3]{1+x^2}$

b) $f(x) = \log|\sin x|$

c) $f(x) = \cos\left(e^{x^2+1}\right)$

d) $f(x) = \dfrac{1}{x\log x}$

4. *On the given interval, find maximum and minimum of:*

a) $f(x) = \sin x + \cos x\,, \qquad [0, 2\pi]$

$\boxed{\text{b)}}$ $f(x) = x^2 - |x+1| - 2\,, \qquad [-2, 1]$

5. *Write the equation of the tangent at x_0 to the graph of the following maps:*

$\boxed{\text{a)}}$ $f(x) = \log(3x-2)\,, \qquad x_0 = 2$

b) $f(x) = \dfrac{x}{1+x^2}\,, \qquad x_0 = 1$

$\boxed{\text{c)}}$ $f(x) = e^{\sqrt{2x+1}}\,, \qquad x_0 = 0$

d) $f(x) = \sin\dfrac{1}{x}\,, \qquad x_0 = \dfrac{1}{\pi}$

$\boxed{6.}$ *Verify that $f(x) = 5x + x^3 + 2x^5$ is invertible on $\mathbb{R}$, f^{-1} is differentiable on the same set, and compute $(f^{-1})'(0)$ and $(f^{-1})'(8)$.*

$\boxed{7.}$ *Prove that $f(x) = (x-1)e^{x^2} + \arctan(\log x) + 2$ is invertible on its domain and find the range.*

$\boxed{8.}$ *Verify that $f(x) = \log(2+x) + 2\dfrac{x+1}{x+2}$ has no zeroes apart from $x_0 = -1$.*

$\boxed{9.}$ *Determine the number of zeroes and critical points of*

$$f(x) = \frac{x\log x - 1}{x^2}.$$

$\boxed{10.}$ *Discuss relative and absolute minima of the map*

$$f(x) = 2\sin x + \frac{1}{2}\cos 2x$$

on $[0,\, 2\pi]$.

$\boxed{11.}$ *Find the largest interval containing $x_0 = \frac{1}{2}$ on which the function*

$$f(x) = \log x - \frac{1}{\log x}$$

has an inverse, which is also explicitly required. Calculate the derivative of the inverse at the origin.

$\boxed{12.}$ *Verify that*

$$\log(1+x) \le x\,, \qquad \forall x > -1\,.$$

$\boxed{13.}$ *Sketch a graph for $f(x) = 3x^5 - 50x^3 + 135x$. Then find the largest and smallest possible numbers of real roots of $f(x) + k$, as k varies in the reals.*

$\boxed{14.}$ *Consider $f(x) = x^4 - 2\sqrt{\log x}$ and*
a) *find its domain;*
b) *discuss monotonicity;*
c) *prove the point $(\mathrm{e}^4 - 2, \mathrm{e})$ belongs to the graph of f^{-1}, then compute the derivative of f^{-1} at $\mathrm{e}^4 - 2$.*

$\boxed{15.}$ *Regarding*

$$f(x) = \frac{\sqrt{x^2-3}}{x+1},$$

a) *find domain, limits at the domain's boundary and possible asymptotes;*
b) *study the intervals of monotonicity, the maximum and minimum points, specifying which are relative, which absolute;*
c) *sketch a graph;*
d) *define*

$$g(x) = \begin{cases} f(x+\sqrt{3}) & \text{if } x \ge 0\,, \\ f(x-\sqrt{3}) & \text{if } x < 0\,. \end{cases}$$

Relying on the results found for f draw a picture of g, and study its continuity and differentiability at the origin.

$\boxed{16.}$ *Given*

$$f(x) = \sqrt{|x^2-4|} - x,$$

a) *find domain, limits at the domain's boundary and asymptotes;*
b) *determine the sign of f;*
c) *study the intervals of monotonicity and list the extrema;*
d) *detect the points of discontinuity and of non-differentiability;*
e) *sketch the graph of f.*

$\boxed{17.}$ *Consider*

$$f(x) = \sqrt[3]{\mathrm{e}^{2x} - 1}.$$

a) *What does $f(x)$ do at the boundary of the domain?*
b) *Where is f monotone, where not differentiable?*
c) *Discuss convexity and find the inflection points.*
d) *Sketch a graph.*

18. *Let*

$$f(x) = 1 - e^{-|x|} + \frac{x}{e}$$

be given.
a) *Find domain and asymptotes, if any;*
b) *discuss differentiability and monotonic properties;*
c) *determine maxima, minima, saying whether global or local;*
d) *sketch the graph.*

19. *Given*

$$f(x) = e^x(x^2 - 8|x-3| - 8),$$

determine
a) *the monotonicity;*
b) *the relative extrema and range* $\operatorname{im} f$*;*
c) *the points where f is not continuous, or not differentiable;*
d) *a rough graph;*
e) *whether there is a real α such that*

$$g(x) = f(x) - \alpha|x-3|$$

is of class $\mathcal{C}^1$ over the whole real line.

20. *Given*

$$f(x) = \frac{\log|1+x|}{(1+x)^2},$$

find
a) *domain, behaviour at the boundary, asymptotes,*
b) *monotonicity intervals, relative or absolute maxima and minima,*
c) *convexity and inflection points,*
d) *and sketch a graph.*

21. *Let*

$$f(x) = \frac{x\log|x|}{1+\log^2|x|}.$$

a) *Prove f can be prolonged with continuity to $\mathbb{R}$ and discuss the differentiability of the prolongation g;*
b) *determine the number of stationary points g has;*
c) *draw a picture for g that takes monotonicity and asymptotes into account.*

22. *Determine for*

$$f(x) = \arctan \frac{|x|+3}{x-3}$$

a) *domain, limits at the boundary, asymptotes;*
b) *monotonicity, relative and absolute extremum points,* $\inf f$ *and* $\sup f$*;*
c) *differentiability;*
d) *concavity and convexity;*
e) *a graph that highlights the previous features.*

23. *Consider the map*

$$f(x) = \arcsin \sqrt{2e^x - e^{2x}}$$

and say
a) *what are the domain, the boundary limits, the asymptotes of* $f(x)$*;*
b) *at which points the function is differentiable;*
c) *where* f *is monotone, where it reaches a maximum or a minimum;*
d) *what the graph of* $f(x)$ *looks like, using the information so far collected.*
e) *Define a map* $\tilde{f}$ *continuously prolonging* f *to the entire* $\mathbb{R}$.

6.12.1 Solutions

1. *Differentiability:*

a) Not differentiable.

b) The right and left limits of the difference quotient, for $x \to 0$, are:

$$\lim_{x\to 0^+} \frac{\sin x - 0}{x - 0} = 1\,, \qquad \lim_{x\to 0^-} \frac{\sin(-x) - 0}{x - 0} = -1\,.$$

Consequently, the function is not differentiable at $x_0 = 0$.

c) For $x \neq 0$ the map is differentiable and

$$f'(x) = \frac{2}{x^3} e^{-1/x^2}\,.$$

Moreover $\lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0} f'(x) = 0$, so f is continuous at $x_0 = 0$. By Theorem 6.15, it is also differentiable at that point.

d) Not differentiable.

2. *Differentiability:*

a) Because

$$f(x) = \begin{cases} x\sqrt{x} & \text{if } x \geq 0\,, \\ x\sqrt{-x} & \text{if } x < 0\,, \end{cases}$$

f' is certainly differentiable at $x \neq 0$ with

$$f'(x) = \begin{cases} \frac{3}{2}\sqrt{x} & \text{if } x > 0\,, \\ \frac{3}{2}\sqrt{-x} & \text{if } x < 0\,. \end{cases}$$

The map is continuous on $\mathbb{R}$ (composites and products preserve continuity), hence in particular also at $x = 0$. Furthermore, $\lim_{x\to 0^+} f'(x) = \lim_{x\to 0^-} f'(x) = 0$, making f differentiable at $x = 0$, with $f'(0) = 0$.

b) Differentiable on $\mathbb{R}$, $f'(x) = -\sin x$.

c) Differentiable everywhere, $f'(x) = \begin{cases} 2x & \text{if } x \geq 0\,, \\ \mathrm{e}^x - 1 & \text{if } x < 0. \end{cases}$

d) The map is clearly continuous for $x \neq 1$; but also at $x = 1$, since

$$\lim_{x\to 1^+} (x^2 + x - 5) = f(1) = -3 = \lim_{x\to 1^-} (x - 4).$$

The derivative is

$$f'(x) = \begin{cases} 2x + 1 & \text{if } x > 1\,, \\ 1 & \text{if } x < 1\,, \end{cases}$$

so f is differentiable at least on $\mathbb{R} \setminus \{1\}$. Using Theorem 6.15 on the right- and left-hand derivatives independently, gives

$$f_+(1) = \lim_{x\to 1^+} f'(x) = 3\,, \quad f'_-(1) = \lim_{x\to 1^-} f'(x) = 1\,.$$

At the point $x = 1$, a corner, the function is not differentiable.

3. *Derivatives:*

a) $f'(x) = \dfrac{5x^2 + 3}{(1 + x^2)^{2/3}}$

b) $f'(x) = \operatorname{cotan} x$

c) $f'(x) = -2x\mathrm{e}^{x^2+1} \sin \mathrm{e}^{x^2+1}$

d) $f'(x) = -\dfrac{\log x + 1}{x^2 \log^2 x}$

4. *Maxima and minima:*

Both functions are continuous so the existence of maxima and minima is guaranteed by Weierstrass's theorem.

a) Maximum value $\sqrt{2}$ at the point $x = \frac{\pi}{4}$; minimum $-\sqrt{2}$ at $x = \frac{5}{4}\pi$. (The interval's end-points are relative minimum and maximum points, not absolute.)

b) One has

$$f(x) = \begin{cases} x^2 + x - 1 & \text{if } x < -1\,, \\ x^2 - x - 3 & \text{if } x \geq -1\,. \end{cases}$$

The function coincides with the parabola $y = (x + \frac{1}{2})^2 - \frac{5}{4}$ for $x < -1$. The latter has vertex in $(-\frac{1}{2}, -\frac{5}{4})$ and is convex, so on the interval $[-2, -1]$ of concern it decreases; its maximum is 1 at $x = -2$ and minimum -1 at $x = -1$.

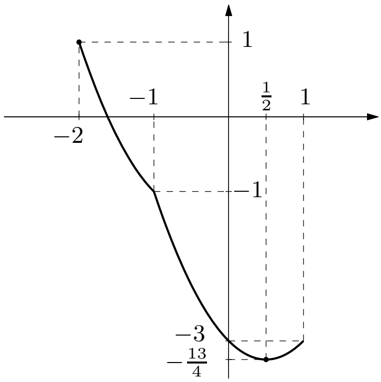


Figure 6.16. Graph of $f(x) = x^2 - |x+1| - 2$

For $x \geq -1$, we have the convex parabola $y = (x - \frac{1}{2})^2 - \frac{13}{4}$ with vertex $(\frac{1}{2}, -\frac{13}{4})$. Thus on $[-1, 1]$, there is a minimum point $x = \frac{1}{2}$ with image $f(\frac{1}{2}) = -\frac{13}{4}$. Besides, $f(-1) = -1$ and $f(1) = -3$, so the maximum -1 is reached at $x = -1$. In conclusion, f has minimum $-\frac{13}{4}$ (for $x = \frac{1}{2}$) and maximum 1 (at $x = -2$); see Fig. 6.16.

5. *Tangent lines:*

a) Since

$$f'(x) = \frac{3}{3x-2}\,, \qquad f(2) = \log 4\,, \qquad f'(2) = \frac{3}{4}\,,$$

the equation of the tangent is

$$y = f(2) + f'(2)(x-2) = \log 4 + \frac{3}{4}(x-2)\,.$$

b) $y = \frac{1}{2}$.

c) As

$$f'(x) = \frac{\mathrm{e}^{\sqrt{2x+1}}}{\sqrt{2x+1}}\,, \qquad f(0) = f'(0) = \mathrm{e}\,,$$

the tangent has equation

$$y = f(0) + f'(0)x = \mathrm{e} + \mathrm{e}x\,.$$

d) $y = \pi^2(x - \frac{1}{\pi})$.

6. As sum of strictly increasing elementary functions on $\mathbb{R}$, so is our function. Therefore invertibility follows. By continuity and because $\lim\limits_{x\to\pm\infty} f(x) = \pm\infty$, Corollary 4.30 implies $\operatorname{im} f = \mathbb{R}$. The function is differentiable on the real line, $f'(x) = 5 + 3x^2 + 10x^4 > 0$ for all $x \in \mathbb{R}$; Theorem 6.9 tells that f^{-1} is differentiable on $\mathbb{R}$. Eventually $f(0) = 0$ and $f(1) = 8$, so

$$(f^{-1})'(0) = \frac{1}{f'(0)} = \frac{1}{5} \qquad \text{and} \qquad (f^{-1})'(8) = \frac{1}{f'(1)} = \frac{1}{18}\,.$$

7. On the domain $(0, +\infty)$ the map is strictly increasing (as sum of strictly increasing maps), hence invertible. Monotonicity follows also from the positivity of

$$f'(x) = (2x^2 - 2x + 1)\mathrm{e}^{x^2} + \frac{1}{x(1 + \log^2 x)}.$$

In addition, f is continuous, so Corollary 4.30 ensures that the range is an interval bounded by $\inf f$ and $\sup f$:

$$\inf f = \lim_{x \to 0^+} f(x) = -1 - \frac{\pi}{2} + 2 = 1 - \frac{\pi}{2}, \qquad \sup f = \lim_{x \to +\infty} f(x) = +\infty\,.$$

Therefore $\operatorname{im} f = (1 - \frac{\pi}{2}, +\infty)$.

8. The map is defined only for $x > -2$, and continuous, strictly increasing on the whole domain as

$$f'(x) = \frac{1}{x+2} + \frac{2}{(x+2)^2} > 0\,, \qquad \forall x > -2\,.$$

Therefore $f(x) < f(1) = 0$ for $x < 1$ and $f(x) > f(1) = 0$ for $x > 1$.

9. The domain is $x > 0$. The zeroes solve

$$x \log x - 1 = 0 \qquad \text{i.e.} \qquad \log x = \frac{1}{x}\,.$$

If we set $h(x) = \log x$ and $g(x) = \frac{1}{x}$, then

$$h(1) = 0 < 1 = g(1) \qquad \text{and} \qquad h(\mathrm{e}) = 1 > \frac{1}{\mathrm{e}} = g(\mathrm{e})\,;$$

Corollary 4.27 says there is an $x_0 \in (1, \mathrm{e})$ such that $h(x_0) = g(x_0)$. Such a point has to be unique because h is strictly increasing and g strictly decreasing. Thus f has only one vanishing point, confined inside $(1, \mathrm{e})$.
For the critical points, we compute the first derivative:

$$f'(x) = \frac{x^2(\log x + 1) - 2x(x \log x - 1)}{x^4} = \frac{x + 2 - x\log x}{x^3}\,.$$

The zeroes of f' are then the roots of

$$x + 2 - x \log x = 0 \qquad \text{i.e.} \qquad \log x = \frac{2+x}{x}\,.$$

Let $\bar{g}(x) = \frac{2+x}{x} = 1 + \frac{2}{x}$, whence

$$h(\mathrm{e}) = 1 < 1 + \frac{2}{\mathrm{e}} = \bar{g}(\mathrm{e}) \qquad \text{and} \qquad h(\mathrm{e}^2) = 2 > 1 + \frac{2}{\mathrm{e}^2} = \bar{g}(\mathrm{e}^2)\,;$$

again, Corollary 4.27 indicates a unique $\bar{x}_0 \in (\mathrm{e}, \mathrm{e}^2)$ with $h(\bar{x}_0) = \bar{g}(\bar{x}_0)$ (uniqueness follows from the monotonicity of h and $\bar{g}$). In conclusion, f has precisely one critical point, lying in $(\mathrm{e}, \mathrm{e}^2)$.

10. In virtue of the duplication formulas (2.13),

$$f'(x) = 2\cos x - \sin 2x = 2\cos x(1 - \sin x).$$

Thus $f'(x) = 0$ when $x = \frac{\pi}{2}$ and $x = \frac{3}{2}\pi$, $f'(x) > 0$ for $0 < x < \frac{\pi}{2}$ or $\frac{3}{2}\pi < x < 2\pi$. This says $x = \frac{\pi}{2}$ is an absolute maximum point, where $f(\frac{\pi}{2}) = \frac{3}{2}$, while $x = \frac{3}{2}\pi$ gives an absolute minimum $f(\frac{3}{2}\pi) = -\frac{5}{2}$. Additionally, $f(0) = f(2\pi) = \frac{1}{2}$ so the boundary of $[0, 2\pi]$ are extrema: more precisely, $x = 0$ is a minimum point and $x = 2\pi$ a maximum point.

11. Since f is defined on $x > 0$ with $x \neq 1$, the maximal interval containing $x_0 = \frac{1}{2}$ where f is invertible must be a subset of $(0, 1)$. On the latter, we study the monotonicity, or equivalently the invertibility, of f which is, remember, continuous everywhere on the domain. Since

$$f'(x) = \frac{1}{x} + \frac{1}{x\log^2 x} = \frac{\log^2 x + 1}{x\log^2 x},$$

it is immediate to see $f'(x) > 0$ for any $x \in (0, 1)$, meaning f is strictly increasing on $(0, 1)$. Therefore the largest interval of invertibility is indeed $(0, 1)$.
To write the inverse explicitly, put $t = \log x$ so that

$$y = t - \frac{1}{t}, \qquad t^2 - ty - 1 = 0, \qquad t = \frac{y \pm \sqrt{y^2 + 4}}{2},$$

and changing variable back to x,

$$x = \mathrm{e}^{\frac{y \pm \sqrt{y^2+4}}{2}}.$$

Being interested in $x \in (0, 1)$ only, we have

$$x = f^{-1}(y) = \mathrm{e}^{\frac{y - \sqrt{y^2+4}}{2}},$$

or, in the more customary notation,

$$y = f^{-1}(x) = \mathrm{e}^{\frac{x - \sqrt{x^2+4}}{2}}.$$

Eventually $f^{-1}(0) = \mathrm{e}^{-1}$, so

$$(f^{-1})'(0) = \frac{1}{f'(\mathrm{e}^{-1})} = \frac{1}{2\mathrm{e}}.$$

12. The function $f(x) = \log(1+x) - x$ is defined on $x > -1$, and

$$\lim_{x\to -1^+} f(x) = -\infty\,, \qquad \lim_{x\to +\infty} f(x) = \lim_{x\to +\infty} \big(-x + o(x)\big) = -\infty\,.$$

As

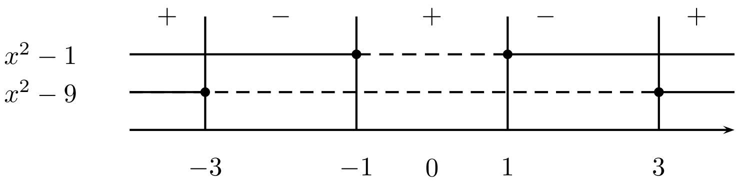
$$f'(x) = \frac{1}{1+x} - 1 = -\frac{x}{1+x}\,,$$

$x = 0$ is critical, plus $f'(x) > 0$ on $x < 0$ and $f'(x) < 0$ for $x > 0$. Thus f increases on $(-1, 0]$ and decreases on $[0, +\infty)$; $x = 0$ is the point where the absolute maximum $f(0) = 0$ is reached. In conclusion $f(x) \le f(0) = 0$, for all $x > -1$.

13. One checks f is odd, plus

$$\begin{aligned} f'(x) &= 15x^4 - 150x^2 + 135 = 15(x^4 - 10x^2 + 9) \\ &= 15(x^2-1)(x^2-9) = 15(x+1)(x-1)(x+3)(x-3)\,. \end{aligned}$$

The sign of f' is summarised in the diagram:



What this tells is that f is increasing on $(-\infty, -3]$, $[-1, 1]$ and $[3, +\infty)$, while decreasing on $[-3, -1]$ and $[1, 3]$. The points $x = -1$, $x = 3$ are relative minima, $x = 1$ and $x = -3$ relative maxima:

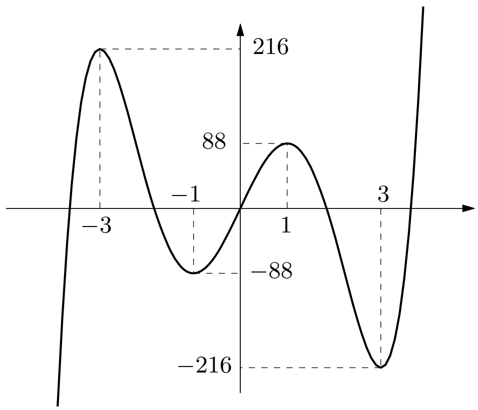


Figure 6.17. The function $f(x) = 3x^5 - 50x^3 + 135x$

$$f(1) = -f(-1) = 88 \qquad \text{and} \qquad f(3) = -f(-3) = -216.$$

Besides,

$$\lim_{x\to-\infty} f(x) = -\infty, \qquad \lim_{x\to+\infty} f(x) = +\infty.$$

The graph of f is in Fig. 6.17.

The second problem posed is equivalent to studying the number of solutions of $f(x) = -k$ as k varies: this is the number of intersections between the graph of f and the line $y = -k$. Indeed,

if $k > 216$ or $k < -216$	one solution
if $k = \pm 216$	two solutions
if $k \in (-216, -88) \cup (88, 216)$	three solutions
if $k = \pm 88$	four solutions
if $k \in (-88, 88)$	five solutions.

This gives the maximum (5) and minimum (1) number of roots of the polynomial $3x^5 - 50x^3 + 135x + k$.

14. *Study of the function* $f(x) = x^4 - 2\sqrt{\log x}$*:*

a) Necessarily $x > 0$ and $\log x \ge 0$, i.e., $x \ge 1$, so $\operatorname{dom} f = [1, +\infty)$.

b) From

$$f'(x) = \frac{4x^4\sqrt{\log x} - 1}{x\sqrt{\log x}}$$

we have

$$f'(x) = 0 \quad \iff \quad 4x^4\sqrt{\log x} = 1 \quad \iff \quad g_1(x) = \log x = \frac{1}{16x^8} = g_2(x)\,.$$

On $x \ge 1$ there is an intersection x_0 between the graphs of g_1, g_2 (Fig. 6.18). Hence $f'(x) > 0$ for $x > x_0$, f is decreasing on $[1, x_0]$, increasing on $[x_0, +\infty)$.

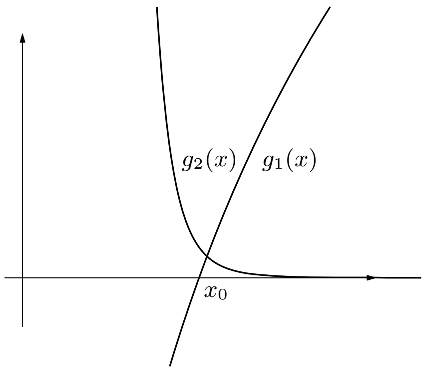


Figure 6.18. Graphs of $g_1(x) = \log x$ and $g_2(x) = \dfrac{1}{16x^8}$

This makes x_0 a minimum point, and monotonicity gives f invertible on $[1, x_0]$ and $[x_0, +\infty)$. In addition, $g_1(1) = \log 1 = 0 < \frac{1}{16} = g_2(1)$ and $g_1(2) = \log 2 > \frac{1}{2^{12}} = g_2(2)$, which implies $1 < x_0 < 2$.

c) As $f(\mathrm{e}) = \mathrm{e}^4 - 2$, the point $(\mathrm{e}^4 - 2, \mathrm{e})$ belongs to the graph of f^{-1} and

$$(f^{-1})'(\mathrm{e}^4 - 2) = \frac{1}{f'(\mathrm{e})} = \frac{\mathrm{e}}{4\mathrm{e}^4 - 1}.$$

15. *Study of* $f(x) = \frac{\sqrt{x^2-3}}{x+1}$:

a) The domain is determined by $x^2 - 3 \geq 0$ together with $x \neq -1$, hence $\operatorname{dom} f = (-\infty, -\sqrt{3}] \cup [\sqrt{3}, +\infty)$. At the boundary points:

$$\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} \frac{|x|\sqrt{1-\frac{3}{x^2}}}{x(1+\frac{1}{x})} = \lim_{x\to\pm\infty} \frac{|x|}{x} = \pm 1\,,$$
$$\lim_{x\to-\sqrt{3}^-} f(x) = \lim_{x\to\sqrt{3}^+} f(x) = 0,$$

so $y = 1$ is the horizontal right asymptote, $y = -1$ the horizontal left asymptote.

b) The derivative

$$f'(x) = \frac{x+3}{(x+1)^2\sqrt{x^2-3}}$$

vanishes at $x = -3$ and is positive for $x \in (-3, -\sqrt{3}) \cup (\sqrt{3}, +\infty)$. Thus f is increasing on $[-3, -\sqrt{3}]$ and $[\sqrt{3}, +\infty)$, decreasing on $(-\infty, -3]$; $x = -3$ is an absolute minimum with $f(-3) = -\frac{\sqrt{6}}{2} < -1$. Furthermore, the points $x = \pm\sqrt{3}$ are extrema too, namely $x = -\sqrt{3}$ is a relative maximum, $x = \sqrt{3}$ a relative minimum: $f(\pm\sqrt{3}) = 0$.

c) Fig. 6.19 (left) shows the graph of f.

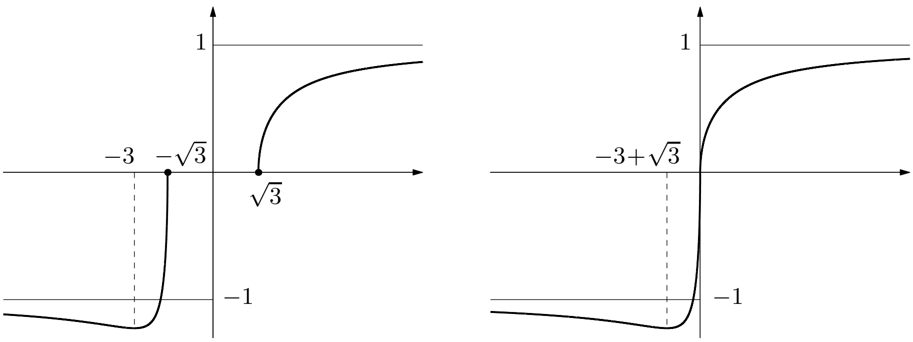


Figure 6.19. Graphs of f (left) and g (right) of Exercise 15

d) Right-translating the negative branch of f by $\sqrt{3}$ gives $g(x)$ for $x < 0$, whereas shifting to the left the branch on $x > 0$ gives the positive part of g. The final result is shown in Fig. 6.19 (right).
The map g is continuous on $\mathbb{R}$, in particular

$$\lim_{x\to 0^-} g(x) = \lim_{x\to 0^-} f(x-\sqrt{3}) = f(-\sqrt{3}) = 0 = f(\sqrt{3}) = \lim_{x\to 0^+} g(x).$$

Since

$$\lim_{x\to 0^\pm} g'(x) = \lim_{x\to \sqrt{3}^+} f'(x) = \lim_{x\to -\sqrt{3}^-} f'(x) = +\infty$$

g is not differentiable at $x = 0$.

16. *Study of* $f(x) = \sqrt{|x^2-4|} - x$*:*

a) The domain is $\mathbb{R}$ and

$$\lim_{x\to +\infty} f(x) = \lim_{x\to +\infty} \frac{x^2-4-x^2}{\sqrt{x^2-4}+x} = 0^-, \qquad \lim_{x\to -\infty} f(x) = +\infty.$$

Thus $y = 0$ is a horizontal right asymptote. Let us search for oblique asymptotic directions. As

$$\lim_{x\to -\infty} \frac{f(x)}{x} = \lim_{x\to -\infty} \left(-\sqrt{1-\frac{4}{x^2}}-1\right) = -2\,,$$

$$\lim_{x\to -\infty} (f(x)+2x) = \lim_{x\to -\infty} \left(\sqrt{x^2-4}+x\right) = \lim_{x\to -\infty} \frac{x^2-4-x^2}{\sqrt{x^2-4}-x} = 0,$$

the line $y = -2x$ is an oblique left asymptote.

b) It suffices to solve $\sqrt{|x^2-4|} - x \ge 0$. First, $\sqrt{|x^2-4|} \ge x$ for any $x < 0$. When $x \ge 0$, we distinguish two cases: $x^2-4 < 0$ (so $0 \le x < 2$) and $x^2-4 \ge 0$ (i.e., $x \ge 2$).
On $0 \le x < 2$, squaring gives

$$4-x^2 \ge x^2 \qquad \Longleftrightarrow \qquad x^2-2 \le 0 \qquad \Longleftrightarrow \qquad 0 \le x \le \sqrt{2}\,.$$

For $x \ge 2$, squaring implies $x^2-4 \ge x^2$, which holds nowhere. The function then vanishes only at $x = \sqrt{2}$, is positive on $x < \sqrt{2}$ and strictly negative for $x > \sqrt{2}$.

c) Since

$$f(x) = \begin{cases} \sqrt{4-x^2}-x & \text{if } -2 < x < 2\,, \\ \sqrt{x^2-4}-x & \text{if } x \le -2\,,\ x \ge 2\,, \end{cases}$$

we have

$$f'(x) = \begin{cases} \dfrac{-x}{\sqrt{4-x^2}} - 1 & \text{if } -2 < x < 2\,, \\ \dfrac{x}{\sqrt{x^2-4}} - 1 & \text{if } x < -2\,,\ x > 2\,. \end{cases}$$

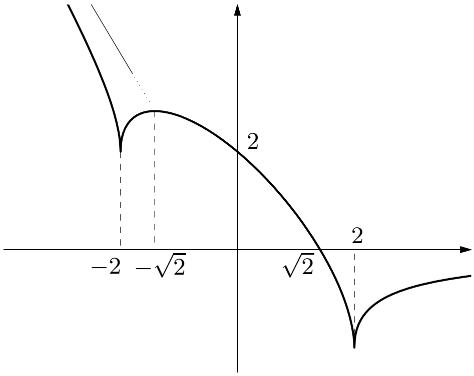


Figure 6.20. The function $f(x) = \sqrt{|x^2-4|} - x$

When $-2 < x < 2$, $f'(x) \geq 0$ if $x + \sqrt{4-x^2} \leq 0$, that is $\sqrt{4-x^2} \leq -x$. The inequality does not hold for $x \geq 0$; on $-2 < x < 0$ we square, so that

$$4 - x^2 \leq x^2 \qquad \Longleftrightarrow \qquad x^2 - 2 \geq 0 \qquad \Longleftrightarrow \qquad -2 \leq x \leq -\sqrt{2}\,.$$

Hence $f'(x) = 0$ for $x = -\sqrt{2}$, $f'(x) > 0$ for $-2 < x < -\sqrt{2}$ and $f'(x) < 0$ when $-\sqrt{2} < x < 2$.
If $x < -2$ or $x > 2$, $f'(x) \geq 0$ if $x - \sqrt{x^2-4} \geq 0$, i.e., $\sqrt{x^2-4} \leq x$. The latter is never true for $x < -2$; for $x > 2$, $x^2 \geq x^2 - 4$ is a always true. Therefore $f'(x) > 0$ per $x > 2$ e $f'(x) < 0$ per $x < -2$.
Summary: f decreases on $(-\infty, -2]$ and $[-\sqrt{2}, 2]$, increases on $[-2, -\sqrt{2}]$ and $[2, +\infty)$. The points $x = \pm 2$ are relative minima, $x = -\sqrt{2}$ a relative maximum. The corresponding values are $f(-2) = 2$, $f(2) = -2$, $f(-\sqrt{2}) = 2\sqrt{2}$, so $x = 2$ is actually a global minimum.

d) As composite of continuous elementary maps, f is continuous on its domain. To study differentiability it is enough to examine f' for $x \to \pm 2$. Because

$$\lim_{x \to \pm 2} f'(x) = \infty,$$

at $x = \pm 2$ there is no differentiability.

e) The graph is shown in Fig. 6.20.

17. *Study of* $f(x) = \sqrt[3]{e^{2x} - 1}$*:*

a) The function is defined everywhere

$$\lim_{x \to +\infty} f(x) = +\infty \qquad \text{and} \qquad \lim_{x \to -\infty} f(x) = -1.$$

b) The first derivative

$$f'(x) = \frac{2}{3} \frac{e^{2x}}{(e^{2x} - 1)^{2/3}}$$

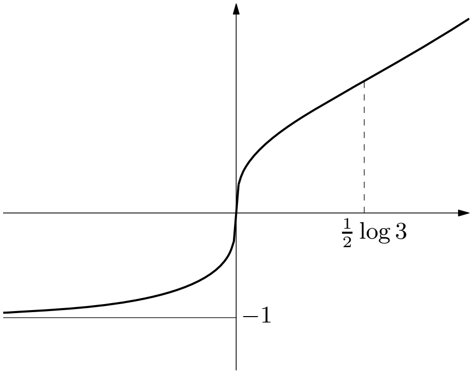


Figure 6.21. The map $f(x) = \sqrt[3]{e^{2x} - 1}$

is positive for $x \in \mathbb{R} \setminus \{0\}$, and f is not differentiable at $x = 0$, for $\lim_{x\to 0} f'(x) = +\infty$. Therefore f increases everywhere on $\mathbb{R}$.

c) The second derivative (for $x \neq 0$)

$$f''(x) = \frac{4}{9}e^{2x} \frac{e^{2x} - 3}{(e^{2x} - 1)^{5/3}}$$

vanishes at $x = \frac{1}{2}\log 3$; it is positive when $x \in (-\infty, 0) \cup (\frac{1}{2}\log 3, +\infty)$. This makes $x = \frac{1}{2}\log 3$ an ascending inflection, plus f convex on $(-\infty, 0]$ and $[\frac{1}{2}\log 3, +\infty)$, concave on $[0, \frac{1}{2}\log 3]$. Suitably extending the definition, the point $x = 0$ may be acknowledged as an inflection (with vertical tangent).

d) See Fig. 6.21.

18. *Study of* $f(x) = 1 - e^{-|x|} + \frac{x}{e}$:

a) Clearly $\operatorname{dom} f = \mathbb{R}$. As

$$\lim_{x\to\pm\infty} e^{-|x|} = 0\,,$$

we immediately obtain

$$\lim_{x\to\pm\infty} f(x) = \pm\infty\,,$$
$$\lim_{x\to\pm\infty} \frac{f(x)}{x} = \lim_{x\to\pm\infty} \left(\frac{1}{x} - \frac{e^{-|x|}}{x} + \frac{1}{e}\right) = \frac{1}{e}\,,$$
$$\lim_{x\to\pm\infty} \left(f(x) - \frac{x}{e}\right) = \lim_{x\to\pm\infty} (1 - e^{-|x|}) = 1.$$

This makes $y = \frac{1}{e}x + 1$ a complete oblique asymptote.

b) The map is continuous on $\mathbb{R}$, and certainly differentiable for $x \neq 0$. As

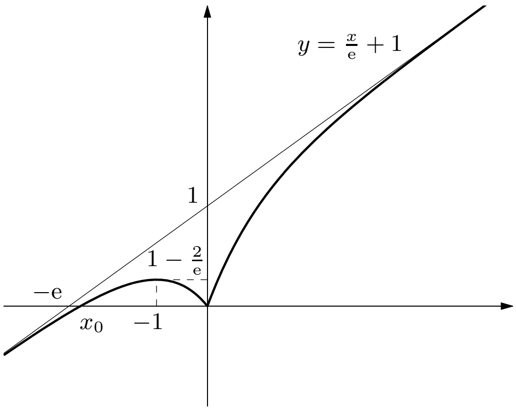


Figure 6.22. Graph of $f(x) = 1 - e^{-|x|} + \frac{x}{e}$

$$f'(x) = \begin{cases} e^{-x} + \dfrac{1}{e} & \text{if } x > 0\,, \\ -e^{x} + \dfrac{1}{e} & \text{if } x < 0\,, \end{cases}$$

it follows

$$\begin{aligned} \lim_{x\to 0^-} f'(x) &= \lim_{x\to 0^-} \left(-e^{x} + \frac{1}{e}\right) = \frac{1}{e} - 1 \\ &\neq \lim_{x\to 0^+} f'(x) = \lim_{x\to 0^+} \left(e^{-x} + \frac{1}{e}\right) = \frac{1}{e} + 1, \end{aligned}$$

preventing differentiability at $x = 0$.
Moreover, for $x > 0$ we have $f'(x) > 0$. On $x < 0$, $f'(x) > 0$ if $e^x < \frac{1}{e}$, i.e., $x < -1$. The map is increasing on $(-\infty, -1]$ and $[0, +\infty)$, decreasing on $[-1, 0]$.

c) The previous considerations imply $x = -1$ is a local maximum with $f(-1) = 1 - \frac{2}{e}$, $x = 0$ a local minimum where $f(0) = 0$.

d) See Fig. 6.22.

19. *Study of* $f(x) = e^x(x^2 - 8|x - 3| - 8)$*:*

a) The domain covers $\mathbb{R}$. Since

$$f(x) = \begin{cases} e^x(x^2 + 8x - 32) & \text{if } x < 3\,, \\ e^x(x^2 - 8x + 16) & \text{if } x \geq 3\,, \end{cases}$$

we have

$$f'(x) = \begin{cases} e^x(x^2 + 10x - 24) & \text{if } x < 3\,, \\ e^x(x^2 - 6x + 8) & \text{if } x > 3\,. \end{cases}$$

On $x < 3$: $f'(x) = 0$ if $x^2 + 10x - 24 = 0$, so $x = -12$ or $x = 2$, while $f'(x) > 0$ if $x \in (-\infty, -12) \cup (2, 3)$. On $x > 3$: $f'(x) = 0$ if $x^2 - 6x + 8 = 0$, i.e., $x = 4$ ($x = 2$ is a root, but lying outside the interval $x > 3$ we are considering), while $f'(x) > 0$ if $x \in (4, +\infty)$.
Therefore f is increasing on the intervals $(-\infty, -12]$, $[2, 3]$ and $[4, +\infty)$, decreasing on $[-12, 2]$ and $[3, 4]$.

b) From part a) we know $x = -12$ and $x = 3$ are relative maxima, $x = 2$ and $x = 4$ relative minima: $f(-12) = 16e^{-12}$, $f(2) = -12e^2$, $f(3) = e^3$ and $f(4) = 0$. For the range, let us determine

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} e^x(x^2 + 8x - 32) = 0\,,$$
$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} e^x(x^2 - 8x + 16) = +\infty.$$

Continuity implies

$$\operatorname{im} f = [\min f(x), \sup f(x)) = [f(2), +\infty) = [-12e^2, +\infty).$$

c) No discontinuities are present, for the map is the composite of continuous functions. As for the differentiability, the only unclear point is $x = 3$. But

$$\lim_{x\to3^-} f'(x) = \lim_{x\to3^-} e^x(x^2 + 10x - 24) = 15e^3\,,$$
$$\lim_{x\to3^+} f'(x) = \lim_{x\to3^+} e^x(x^2 - 6x + 8) = -e^3,$$

so$f$ is not differentiable at $x = 3$.

d) See Fig. 6.23; a neighbourhood of $x = -12$ is magnified.

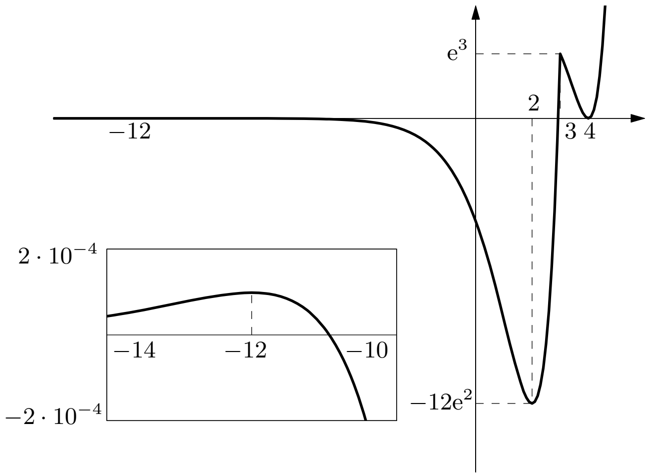


Figure 6.23. Graph of $f(x) = e^x(x^2 - 8|x-3| - 8)$

e) The function g is continuous on the real axis and

$$g'(x) = \begin{cases} e^x(x^2+10x-24)+\alpha & \text{if } x<3\,, \\ e^x(x^2-6x+8)-\alpha & \text{if } x>3\,. \end{cases}$$

In order for g to be differentiable at $x=3$, we must have

$$\lim_{x\to 3^-} g'(x) = 15e^3+\alpha = \lim_{x\to 3^+} g'(x) = -e^3-\alpha\,;$$

the value $\alpha = -8e^3$ makes g of class $\mathcal{C}^1$ on the whole real line.

20. *Study of* $f(x) = \frac{\log|1+x|}{(1+x)^2}$*:*

a) $\operatorname{dom} f = \mathbb{R}\setminus\{-1\}$. By (5.6) c)

$$\lim_{x\to\pm\infty} f(x) = 0^+$$

while

$$\lim_{x\to -1^\pm} f(x) = \frac{\infty}{0^+} = -\infty\,.$$

From this, $x=-1$ is a vertical asymptote, and $y=0$ is a complete oblique asymptote.

b) The derivative

$$f'(x) = \frac{1-2\log|x+1|}{(x+1)^3}$$

tells that $f(x)$ is differentiable on the domain; $f'(x)=0$ if $|x+1|=\sqrt{e}$, hence for $x=-1\pm\sqrt{e}$; $f'(x)>0$ if $x\in(-\infty,-\sqrt{e}-1)\cup(-1,\sqrt{e}-1)$. All this says f increases on $(-\infty,-\sqrt{e}-1]$ and $(-1,-1+\sqrt{e}]$, decreases on $[-\sqrt{e}-1,-1)$ and $[-1+\sqrt{e},+\infty)$, has (absolute) maxima at $x=-1\pm\sqrt{e}$, with $f(-1\pm\sqrt{e})=\frac{1}{2e}$.

c) From

$$f''(x) = \frac{-5+6\log|x+1|}{(x+1)^4}$$

the second derivative is defined at each point of $\operatorname{dom} f$, and vanishes at $|x+1| = e^{5/6}$, so $x=-1\pm e^{5/6}$. Since $f''(x)>0$ on $x\in(-\infty,-1-e^{5/6})\cup(e^{5/6}-1,+\infty)$, f is convex on $(-\infty,-1-e^{5/6}]$ and $[e^{5/6}-1,+\infty)$, while is concave on $[-1-e^{5/6},-1)$ and $(-1,e^{5/6}-1]$. The points $x=-1\pm e^{5/6}$ are inflections.

d) See Fig. 6.24.

21. *Study of* $f(x) = \frac{x\log|x|}{1+\log^2|x|}$*:*

a) The domain is clear: $\operatorname{dom} f = \mathbb{R}\setminus\{0\}$. Since $\lim\limits_{x\to 0} f(x) = 0$ (x 'wins' against the logarithm) we can extend f to $\mathbb{R}$ with continuity, by defining $g(0)=0$. The function is odd, so we shall restrict the study to $x>0$.

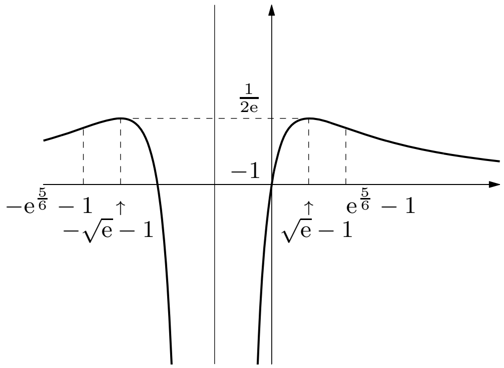


Figure 6.24. Graph of $f(x) = \frac{\log|1+x|}{(1+x)^2}$

As far as the differentiability is concerned, when $x > 0$

$$f'(x) = \frac{\log^3 x - \log^2 x + \log x + 1}{(1+\log^2 x)^2};$$

with $t = \log x$, the limit reads

$$\lim_{x\to 0} f'(x) = \lim_{t\to-\infty} \frac{t^3 - t^2 + t + 1}{(1+t^2)^2} = \lim_{t\to-\infty} \frac{t^3}{t^4} = 0.$$

Therefore the map g, prolongation of f, is not only continuous but also differentiable, due to Theorem 6.15, on the entire $\mathbb{R}$. In particular $g'(0) = 0$.

b) Part a) is also telling that $x = 0$ is stationary for g. To find other critical points, we look at the zeroes of the map $h(t) = t^3 - t^2 + t + 1$, where $t = \log x$ $(x > 0)$. Since

$$\lim_{t\to-\infty} h(t) = -\infty, \qquad \lim_{t\to\infty} h(t) = +\infty,$$
$$h(0) = 1, \qquad h'(t) = 3t^2 - 2t + 1 > 0, \quad \forall t \in \mathbb{R},$$

h is always increasing and has one negative zero t_0. Its graph is represented in Fig. 6.25 (left).
As $t_0 = \log x_0 < 0$, $0 < x_0 = \mathrm{e}^{t_0} < 1$. But the function is odd, so g has two more stationary points, x_0 and $-x_0$ respectively.

c) By the previous part $g'(x) > 0$ on $(x_0, +\infty)$ and $g'(x) < 0$ on $(0, x_0)$. To summarise then, g (odd) is increasing on $(-\infty, -x_0]$ and $[x_0, +\infty)$, decreasing on $[-x_0, x_0]$. Because

$$\lim_{x\to+\infty} g(x) = +\infty$$

and

$$\lim_{x\to+\infty} \frac{g(x)}{x} = \lim_{x\to+\infty} \frac{\log x}{1+\log^2 x} = \lim_{t\to+\infty} \frac{t}{1+t^2} = 0,$$

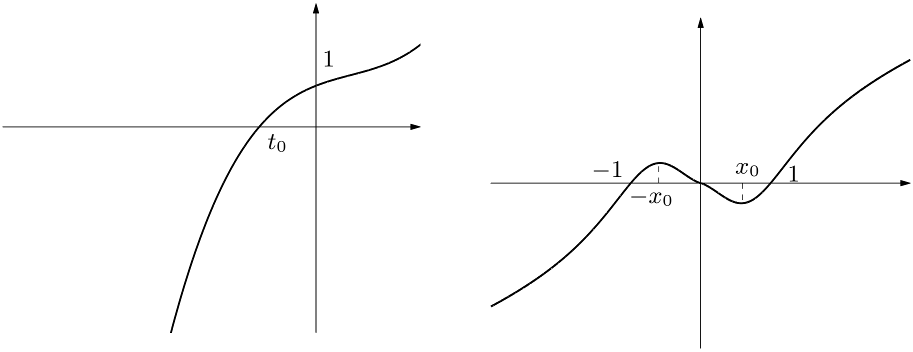


Figure 6.25. The functions h (left) and g (right) of Exercise 21

there are no asymptotes.
For the graph see Fig. 6.25 (right).

22. *Study of* $f(x) = \arctan \frac{|x|+3}{x-3}$:

a) $\operatorname{dom} f = \mathbb{R} \setminus \{3\}$. The function is more explicitly given by

$$f(x) = \begin{cases} \arctan \dfrac{-x+3}{x-3} = \arctan(-1) = -\dfrac{\pi}{4} & \text{if } x \le 0\,, \\ \arctan \dfrac{x+3}{x-3} & \text{if } x > 0, \end{cases}$$

whence

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} -\frac{\pi}{4} = -\frac{\pi}{4}, \qquad \lim_{x\to+\infty} f(x) = \arctan 1 = \frac{\pi}{4},$$
$$\lim_{x\to3^-} f(x) = \arctan \frac{6}{0^-} = \arctan(-\infty) = -\frac{\pi}{2},$$
$$\lim_{x\to3^+} f(x) = \arctan \frac{6}{0^+} = \arctan(+\infty) = \frac{\pi}{2}.$$

Then the straight lines $y = -\frac{\pi}{4}$, $y = \frac{\pi}{4}$ are horizontal asymptotes (left and right respectively).

b) The map

$$f'(x) = \begin{cases} 0 & \text{if } x < 0\,, \\ -\dfrac{3}{x^2+9} & \text{if } x > 0, \quad x \ne 3, \end{cases}$$

is negative on $x > 0$, $x \ne 3$, so f is strictly decreasing on $[0, 3)$ and $(3, +\infty)$, but only non-increasing on $(-\infty, 3)$. The reader should take care that f is not strictly decreasing on the whole $[0, 3) \cup (3, +\infty)$ (recall the remarks of p. 197). The interval $(-\infty, 0)$ consists of points of relative non-strict maxima and minima, for $f(x) = -\frac{\pi}{4}$, whereas $x = 0$ is a relative maximum.

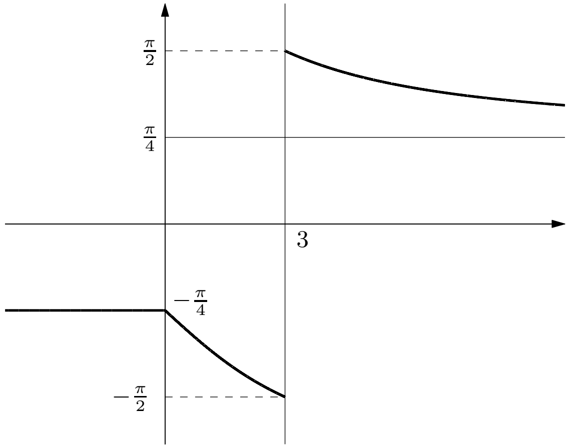


Figure 6.26. The function $f(x) = \arctan \dfrac{|x|+3}{x-3}$

Eventually, $\inf f(x) = -\frac{\pi}{2}$, $\sup f(x) = \frac{\pi}{2}$ (the map admits no maximum, nor minimum).

c) Our map is certainly differentiable on $\mathbb{R} \setminus \{0, 3\}$. At $x = 3$, f is not defined; at $x = 0$, f is continuous but

$$\lim_{x \to 0^-} f'(x) = 0 \neq \lim_{x \to 0^+} f'(x) = \lim_{x \to 0^+} -\frac{3}{x^2+9} = -\frac{1}{3},$$

showing that differentiability does not extend beyond $\mathbb{R} \setminus \{0, 3\}$.

d) Computing

$$f''(x) = \begin{cases} 0 & \text{if } x < 0\,, \\ \dfrac{6x}{(x^2+9)^2} & \text{if } x > 0, \quad x \neq 3\,, \end{cases}$$

reveals that $f''(x) > 0$ for $x > 0$ with $x \neq 3$, so f is convex on $[0, 3)$ and $(3, +\infty)$.

e) See Fig. 6.26.

23. *Study of* $f(x) = \arcsin \sqrt{2e^x - e^{2x}}$*:*

a) We have to impose $2e^x - e^{2x} \geq 0$ and $-1 \leq \sqrt{2e^x - e^{2x}} \leq 1$ for the domain; the first constraint is equivalent to $2 - e^x \geq 0$, hence $x \leq \log 2$. Having assumed that square roots are always positive, the second inequality reduces to $2e^x - e^{2x} \leq 1$. With $y = e^x$, we can write $y^2 - 2y + 1 = (y-1)^2 \geq 0$, which is always true. Thus $\operatorname{dom} f = (-\infty, \log 2]$. Moreover,

$$\lim_{x \to -\infty} f(x) = 0\,, \qquad f(\log 2) = 0,$$

and $y = 0$ is a horizontal left asymptote.

b) From

$$f'(x) = \frac{e^x(1-e^x)}{\sqrt{e^x(2-e^x)(1-2e^x+e^{2x})}} = \frac{e^x(1-e^x)}{\sqrt{e^x(2-e^x)(1-e^x)^2}}$$

$$= \begin{cases} -\dfrac{e^x}{\sqrt{e^x(2-e^x)}} & \text{if } 0 < x < \log 2\,, \\ \dfrac{e^x}{\sqrt{e^x(2-e^x)}} & \text{if } x < 0\,, \end{cases}$$

we see that

$$\lim_{x\to(\log 2)^-} f'(x) = -\infty\,, \qquad \lim_{x\to 0^+} f'(x) = -1\,, \qquad \lim_{x\to 0^-} f'(x) = 1\,.$$

In this way f is not differentiable at $x = \log 2$, where the tangent is vertical, and at the corner point $x = 0$.

c) The sign of f' is positive for $x < 0$ and negative for $0 < x < \log 2$, meaning that $x = 0$ is a global maximum point, $f(0) = \frac{\pi}{2}$, while at $x = \log 2$ the absolute minimum $f(\log 2) = 0$ is reached; f is monotone on $(-\infty, 0]$ (increasing) and $[0, \log 2]$ (decreasing).

d) See Fig. 6.27.

e) A possible choice to extend f with continuity is

$$\tilde{f}(x) = \begin{cases} f(x) & \text{if } x \le \log 2\,, \\ 0 & \text{if } x > \log 2\,. \end{cases}$$

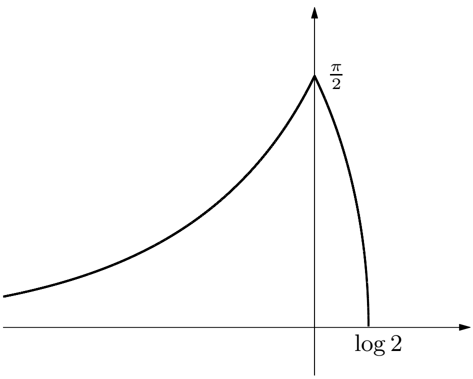


Figure 6.27. The map $f(x) = \arcsin\sqrt{2e^x - e^{2x}}$

7

Taylor expansions and applications

The Taylor expansion of a function around a real point x_0 is the representation of the map as sum of a polynomial of a certain degree and an infinitesimal function of order bigger than the degree. It provides an extremely effective tool both from the qualitative and the quantitative point of view. In a small enough neighbourhood of x_0 one can approximate the function, however complicated, using the polynomial; the qualitative features of the latter are immediate, and polynomials are easy to handle. The expansions of the main elementary functions can be aptly combined to produce more involved expressions, in a way not dissimilar to the algebra of polynomials.

7.1 Taylor formulas

We wish to tackle the problem of approximating a function f, around a given point $x_0 \in \mathbb{R}$, by polynomials of increasingly higher degree.

We begin by assuming f be continuous at x_0. Introducing the constant polynomial (degree zero)

$$Tf_{0,x_0}(x) = f(x_0), \qquad \forall x \in \mathbb{R},$$

formula (5.4) prompts us to write

$$f(x) = Tf_{0,x_0}(x) + o(1), \quad x \to x_0. \tag{7.1}$$

Put in different terms, we may approximate f around x_0 using a zero-degreepolynomial, in such a way that the difference $f(x) - T\,f_{0,x_0}(x)$ (called *error of approximation*, or *remainder*), is infinitesimal at x_0 (Fig. 7.1). The above relation is the first instance of a Taylor formula.

Suppose now f is not only continuous but also differentiable at x_0: then the first formula of the finite increment (6.11) holds. By defining the polynomial in x of degree one

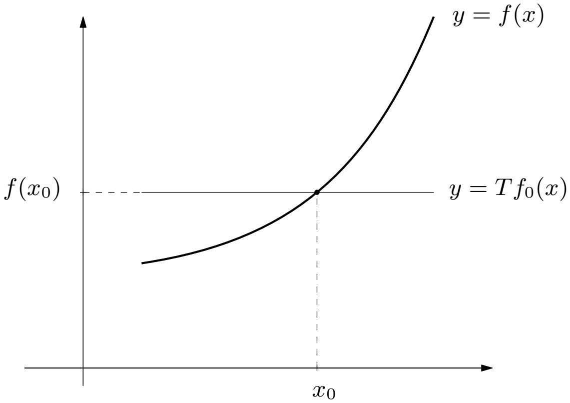


Figure 7.1. Local approximation of f by the polynomial $Tf_0 = Tf_{0,x_0}$

$$Tf_{1,x_0}(x) = f(x_0) + f'(x_0)(x - x_0),$$

whose graph is the tangent line to f at x_0 (Fig. 7.2), relation (6.11) reads

$$f(x) = Tf_{1,x_0}(x) + o(x - x_0), \quad x \to x_0. \tag{7.2}$$

This is another Taylor formula: it says that a differentiable map at x_0 can be locally approximated by a linear function, with an error of approximation that not only tends to 0 as $x \to x_0$, but is infinitesimal of order bigger than one.

In case f is differentiable in a neighbourhood of x_0, except perhaps at x_0, the second formula of the finite increment (6.13) is available: putting $x_1 = x_0$, $x_2 = x$ we write the latter as

$$f(x) = Tf_{0,x_0}(x) + f'(\bar{x})(x - x_0), \tag{7.3}$$

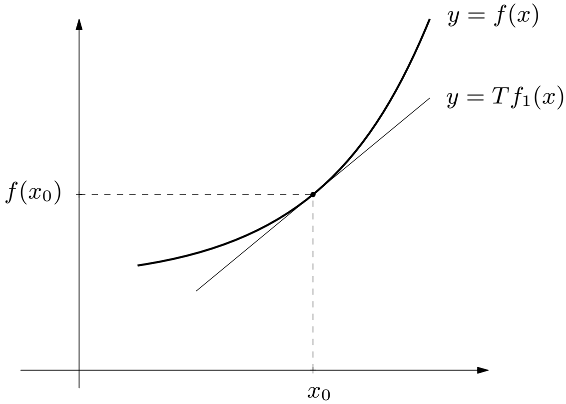


Figure 7.2. Local approximation of f by the polynomial $Tf_1 = Tf_{1,x_0}$

where $\bar{x}$ denotes a suitable point between x_0 and x. Compare this with (7.1): now we have a more accurate expression for the remainder. This allows to appraise numerically the accuracy of the approximation, once the increment $x - x_0$ and an estimate of f' around x_0 are known. Formula (7.3) is of Taylor type as well, and the remainder is called *Lagrange's remainder*. In (7.1), (7.2) we call it *Peano's remainder*, instead.

Now that we have approximated f with polynomials of degrees 0 or 1, as $x \to x_0$, and made errors $o(1) = o\big((x-x_0)^0\big)$ or $o(x-x_0)$ respectively, the natural question is whether it is possible to approximate the function by a quadratic polynomial, with an error $o\big((x-x_0)^2\big)$ as $x \to x_0$. Equivalently, we seek for a real number a such that

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + a(x-x_0)^2 + o\big((x-x_0)^2\big), \quad x \to x_0 . \qquad (7.4)$$

This means

$$\lim_{x \to x_0} \frac{f(x) - f(x_0) - f'(x_0)(x-x_0) - a(x-x_0)^2}{(x-x_0)^2} = 0.$$

By de l'Hôpital's Theorem, such limit holds if

$$\lim_{x \to x_0} \frac{f'(x) - f'(x_0) - 2a(x-x_0)}{2(x-x_0)} = 0,$$

i.e.,

$$\lim_{x \to x_0} \left(\frac{1}{2} \frac{f'(x) - f'(x_0)}{x - x_0} - a \right) = 0,$$

or

$$\frac{1}{2} \lim_{x \to x_0} \frac{f'(x) - f'(x_0)}{x - x_0} = a.$$

We conclude that (7.4) is valid when the right-hand-side limit exists and is finite: in other words, when f is twice differentiable at x_0. If so, the coefficient a is $\frac{1}{2}f''(x_0)$. In this way we have obtained the Taylor formula (with Peano's remainder)

$$f(x) = Tf_{2,x_0}(x) + o\big((x-x_0)^2\big), \quad x \to x_0, \qquad (7.5)$$

where

$$Tf_{2,x_0}(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{1}{2} f''(x_0)(x-x_0)^2$$

is the Taylor polynomial of f at x_0 with degree 2 (Fig. 7.3).

The recipe just described can be iterated, and leads to polynomial approximations of increasing order. The final result is the content of the next theorem.

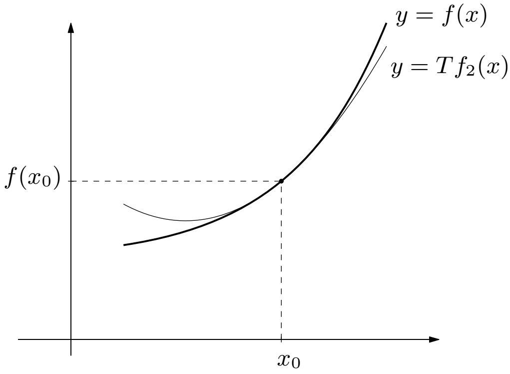


Figure 7.3. Local approximation of f by $Tf_2 = Tf_{2,x_0}$

Theorem 7.1 (**Taylor formula with Peano's remainder**) *Let $n \geq 0$ and f be n times differentiable at x_0. Then the* **Taylor formula** *holds*

$$f(x) = Tf_{n,x_0}(x) + o\big((x - x_0)^n\big), \quad x \to x_0, \tag{7.6}$$

where

$$\begin{aligned} Tf_{n,x_0}(x) &= \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(x_0)(x - x_0)^k \\ &= f(x_0) + f'(x_0)(x - x_0) + \ldots + \frac{1}{n!} f^{(n)}(x_0)(x - x_0)^n. \end{aligned} \tag{7.7}$$

The term $Tf_{n,x_0}(x)$ is the **Taylor polynomial of f at x_0 of order** (or **degree**) n, while $o\big((x - x_0)^n\big)$ as in (7.6) is **Peano's remainder of order** n. The representation of f given by (7.6) is called **Taylor expansion** of f at x_0 of order n, with remainder in Peano's form.

Under stronger hypotheses on f we may furnish a preciser formula for the remainder, thus extending (7.3).

Theorem 7.2 (**Taylor formula with Lagrange's remainder**) *Let $n \geq 0$ and f differentiable n times at x_0, with continuous nth derivative, be given; suppose f is differentiable $n + 1$ times around x_0, except possibly at x_0. Then the Taylor formula*

$$f(x) = Tf_{n,x_0}(x) + \frac{1}{(n+1)!} f^{(n+1)}(\bar{x})(x - x_0)^{n+1}, \tag{7.8}$$

holds, for a suitable $\bar{x}$ between x_0 and x.

This remainder is said **Lagrange's remainder of order** n, and (7.8) is the Taylor expansion of f at x_0 of order n with Lagrange's remainder.

Theorems 7.1 and 7.2 are proven in Appendix A.4.4, p. 456.

An additional form of the remainder of order n in a Taylor formula, called **integral remainder**, will be provided in Theorem 9.44.

A Taylor expansion centred at the origin ($x_0 = 0$) is sometimes called **Maclaurin expansion**. A useful relation to simplify the computation of a Maclaurin expansion goes as follows.

Property 7.3 *The Maclaurin polynomial of an even* (*respectively, odd*) *map involves only even* (*odd*) *powers of the independent variable.*

Proof. If f is even and n times differentiable around the origin, the claim follows from (7.7) with $x_0 = 0$, provided we show all derivatives of odd order vanish at the origin.

Recalling Property 6.12, f even implies f' odd, f'' even, f''' odd et cetera. In general, even-order derivatives $f^{(2k)}$ are even functions, whereas $f^{(2k+1)}$ are odd. But an odd map g must necessarily vanish at the origin (if defined there), because $x = 0$ in $g(-x) = -g(x)$ gives $g(0) = -g(0)$, whence $g(0) = 0$.

The argument is the same for f odd. □

7.2 Expanding the elementary functions

The general results permit to expand simple elementary functions. Other functions will be discussed in Sect. 7.3.

The exponential function

Let $f(x) = \mathrm{e}^x$. Since all derivatives are identical with e^x, we have $f^{(k)}(0) = 1$ for any $k \geq 0$. Maclaurin's expansion of order n with Peano's remainder is

$$\mathrm{e}^x = 1 + x + \frac{x^2}{2} + \ldots + \frac{x^k}{k!} + \ldots + \frac{x^n}{n!} + o(x^n) = \sum_{k=0}^{n} \frac{x^k}{k!} + o(x^n)\,. \tag{7.9}$$

Using Lagrange's remainder, we have

$$\mathrm{e}^x = \sum_{k=0}^{n} \frac{x^k}{k!} + \frac{\mathrm{e}^{\bar{x}}}{(n+1)!} x^{n+1}, \qquad \text{for a certain } \bar{x} \text{ between } 0 \text{ and } x. \tag{7.10}$$

Maclaurin's polynomials for e^x of order $n = 1, 2, 3, 4$ are shown in Fig. 7.4.

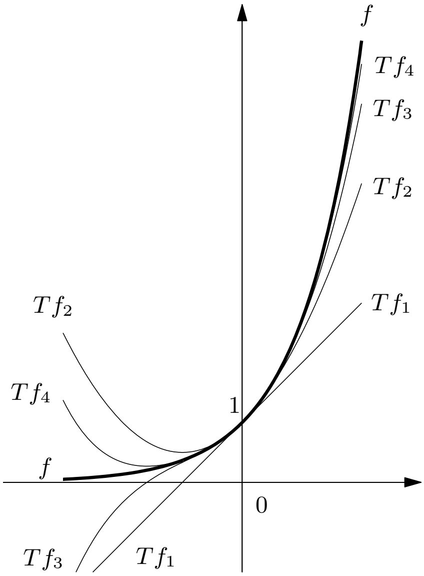


Figure 7.4. Local approximation of $f(x) = \mathrm{e}^x$ by $Tf_n = Tf_{n,0}$ for $n = 1, 2, 3, 4$

Remark 7.4 Set $x = 1$ in the previous formula:

$$\mathrm{e} = \sum_{k=0}^{n} \frac{1}{k!} + \frac{\mathrm{e}^{\bar{x}}}{(n+1)!} \qquad (\text{con } 0 < \bar{x} < 1).$$

For any $n \geq 0$, we obtain an estimate (from below) of the number e, namely

$$\mathrm{e}_n = \sum_{k=0}^{n} \frac{1}{k!}; \tag{7.11}$$

because $1 < \mathrm{e}^{\bar{x}} < \mathrm{e} < 3$ moreover, the following is an estimate of the error:

$$\frac{1}{(n+1)!} < \mathrm{e} - \mathrm{e}_n < \frac{3}{(n+1)!}.$$

In contrast to the sequence $\{a_n = \left(1 + \frac{1}{n}\right)^n\}$ used to define the constant e, the sequence $\{\mathrm{e}_n\}$ converges at the rate of a factorial, hence very rapidly (compare Tables 7.1 and 3.1). Formula (7.11) gives therefore an excellent numerical approximation of the number e. □

The expansion of $f(x) = \mathrm{e}^x$ at a generic x_0 follows from the fact that $f^{(k)}(x_0) = \mathrm{e}^{x_0}$

$$\begin{aligned}
\mathrm{e}^x &= \mathrm{e}^{x_0} + \mathrm{e}^{x_0}(x - x_0) + \mathrm{e}^{x_0}\frac{(x-x_0)^2}{2} + \ldots + \mathrm{e}^{x_0}\frac{(x-x_0)^n}{n!} + o\big((x-x_0)^n\big) \\
&= \sum_{k=0}^{n} \mathrm{e}^{x_0}\frac{(x-x_0)^k}{k!} + o\big((x-x_0)^n\big).
\end{aligned}$$

n	e_n
0	1.0000000000000
1	2.0000000000000
2	2.5000000000000
3	2.6666666666667
4	2.7083333333333
5	2.7166666666667
6	2.7180555555556
7	2.7182539682540
8	2.7182787698413
9	2.7182815255732
10	2.7182818011464

Table 7.1. Values of the sequence $\{\mathrm{e}_n\}$ of (7.11)

The logarithm

The derivatives of the function $f(x) = \log x$ are

$$f'(x) = \frac{1}{x} = x^{-1}\,, \qquad f''(x) = (-1)x^{-2}\,, \qquad f'''(x) = (-1)(-2)x^{-3}\,,$$

and in general,

$$f^{(k)}(x) = (-1)^{k-1}(k-1)!\, x^{-k}.$$

Thus for $k \geq 1$,

$$\frac{f^{(k)}(1)}{k!} = (-1)^{k-1}\frac{1}{k}$$

and the Taylor expansion of order n at $x_0 = 1$ is

$$\begin{aligned} \log x &= (x-1) - \frac{(x-1)^2}{2} + \ldots + (-1)^{n-1}\frac{(x-1)^n}{n} + o\big((x-1)^n\big) \\ &= \sum_{k=1}^{n}(-1)^{k-1}\frac{(x-1)^k}{k} + o\big((x-1)^n\big). \end{aligned} \tag{7.12}$$

Let us change the independent variable $x - 1 \to x$, to obtain the Maclaurin expansion of order n of $\log(1+x)$

$$\begin{aligned} \log(1+x) &= x - \frac{x^2}{2} + \ldots + (-1)^{n-1}\frac{x^n}{n} + o(x^n) \\ &= \sum_{k=1}^{n}(-1)^{k-1}\frac{x^k}{k} + o(x^n). \end{aligned} \tag{7.13}$$

The Maclaurin polynomials of order $n = 1, 2, 3, 4$ for $y = \log(1+x)$ are represented in Fig. 7.5.

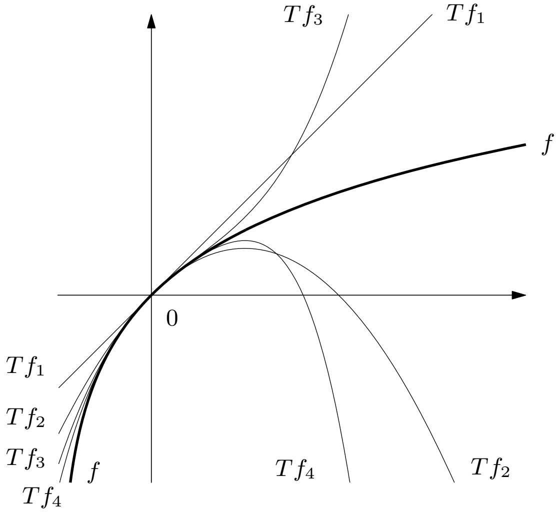


Figure 7.5. Local approximation of $f(x) = \log(1+x)$ by $Tf_n = Tf_{n,0}$ for $n = 1, 2, 3, 4$

The trigonometric functions

The function $f(x) = \sin x$ is odd, so by Property 7.3 its Maclaurin expansion contains just odd powers of x. We have $f'(x) = \cos x$, $f'''(x) = -\cos x$ and in general $f^{(2k+1)}(x) = (-1)^k \cos x$, whence $f^{(2k+1)}(0) = (-1)^k$. Maclaurin's expansion up to order $n = 2m+2$ reads

$$\begin{aligned} \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^m \frac{x^{2m+1}}{(2m+1)!} + o(x^{2m+2}) \\ &= \sum_{k=0}^{m} (-1)^k \frac{x^{2k+1}}{(2k+1)!} + o(x^{2m+2}). \end{aligned} \tag{7.14}$$

The typical structure of the expansion of an odd map should be noticed. Maclaurin's polynomial $Tf_{2m+2,0}$ of even order $2m+2$ coincides with the polynomial $Tf_{2m+1,0}$ of odd degree $2m+1$, for $f^{(2m+2)}(0) = 0$. Stopping at order $2m+1$ would have rendered

$$\sin x = \sum_{k=0}^{m} (-1)^k \frac{x^{2k+1}}{(2k+1)!} + o(x^{2m+1}),$$

to which (7.14) is preferable, because it contains more information on the remainder's behaviour when $x \to 0$. Figure 7.6 represents the Maclaurin polynomials of degree $2m+1$, $0 \le m \le 6$, of the sine.

As far as the even map $f(x) = \cos x$ is concerned, only even exponents appear. From $f''(x) = -\cos x$, $f^{(4)}(x) = \cos x$ and $f^{(2k)}(x) = (-1)^k \cos x$, it follows $f^{(2k)}(0) = (-1)^k$, so Maclaurin's expansion of order $n = 2m+1$ is

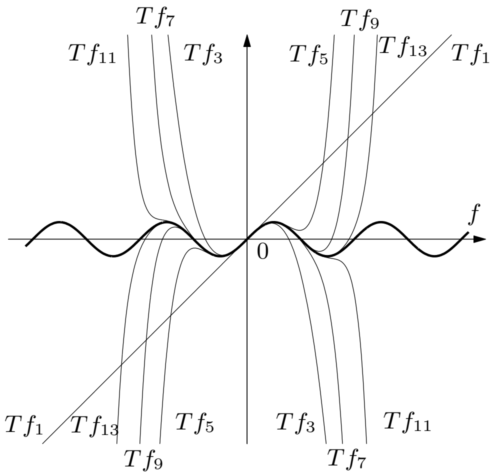


Figure 7.6. Local approximation of $f(x) = \sin x$ by polynomials $Tf_{2m+1} = Tf_{2m+1,0}$ with $0 \le m \le 6$

$$\begin{aligned} \cos x &= 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \ldots + (-1)^m \frac{x^{2m}}{(2m)!} + o(x^{2m+1}) \\ &= \sum_{k=0}^{m} (-1)^k \frac{x^{2k}}{(2k)!} + o(x^{2m+1}). \end{aligned} \tag{7.15}$$

The considerations made about the sine apply also here. Maclaurin's polynomials of order $2m$ ($1 \le m \le 6$) for $y = \cos x$ can be seen in Fig. 7.7.

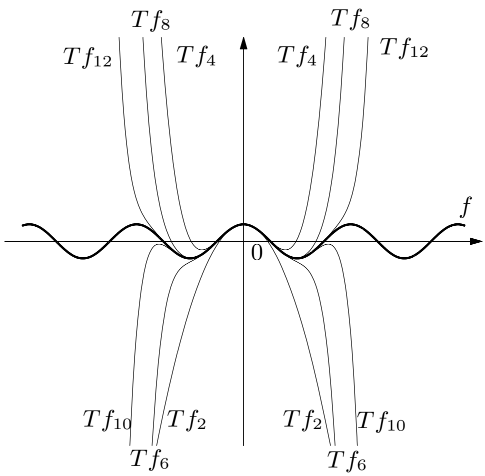


Figure 7.7. Local approximation of $f(x) = \cos x$ by $Tf_{2m} = Tf_{2m,0}$ when $1 \le m \le 6$

Power functions

Consider the family of maps $f(x) = (1+x)^\alpha$ for arbitrary $\alpha \in \mathbb{R}$. We have

$$\begin{aligned} f'(x) &= \alpha(1+x)^{\alpha-1} \\ f''(x) &= \alpha(\alpha-1)(1+x)^{\alpha-2} \\ f'''(x) &= \alpha(\alpha-1)(\alpha-2)(1+x)^{\alpha-3}. \end{aligned}$$

From the general relation $f^{(k)}(x) = \alpha(\alpha-1)\ldots(\alpha-k+1)(1+x)^{\alpha-k}$ we get

$$f(0) = 1\,, \qquad \frac{f^{(k)}(0)}{k!} = \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!} \quad \text{for } k \geq 1\,.$$

At this point it becomes convenient to extend the notion of binomial coefficient (1.10), and allow α to be any real number by putting, in analogy to (1.11),

$$\binom{\alpha}{0} = 1\,, \qquad \binom{\alpha}{k} = \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!} \quad \text{for } k \geq 1\,. \tag{7.16}$$

Maclaurin's expansion to order n is thus

$$\begin{aligned} (1+x)^\alpha &= 1 + \alpha x + \frac{\alpha(\alpha-1)}{2}x^2 + \ldots + \binom{\alpha}{n}x^n + o(x^n) \\ &= \sum_{k=0}^{n} \binom{\alpha}{k} x^k + o(x^n). \end{aligned} \tag{7.17}$$

Let us see in detail what happens for special values of the parameter. When $\alpha = -1$

$$\begin{aligned} &\binom{-1}{2} = \frac{(-1)(-2)}{2} = 1, \quad \binom{-1}{3} = \frac{(-1)(-2)(-3)}{3!} = -1, \ \ldots, \\ &\binom{-1}{k} = \frac{(-1)(-2)\cdots(-k)}{k!} = (-1)^k, \end{aligned}$$

so

$$\frac{1}{1+x} = 1 - x + x^2 - \ldots + (-1)^n x^n + o(x^n) = \sum_{k=0}^{n} (-1)^k x^k + o(x^n). \tag{7.18}$$

Choosing $\alpha = \frac{1}{2}$ gives

$$\binom{\frac12}{2} = \frac{\frac12(\frac12 - 1)}{2} = -\frac{1}{8}, \qquad \binom{\frac12}{3} = \frac{\frac12(\frac12-1)(\frac12-2)}{3!} = \frac{1}{16},$$

and the expansion of $f(x) = \sqrt{1+x}$ arrested to the third order is

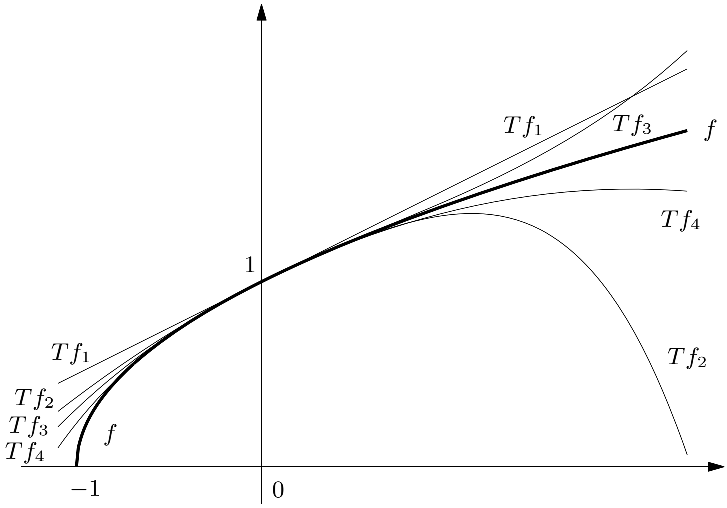


Figure 7.8. Local approximation of $f(x) = \sqrt{1+x}$ by $Tf_n = Tf_{n,0}$ for $n = 1, 2, 3, 4$

$$\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + o(x^3).$$

The polynomials of order $n = 1, 2, 3, 4$ are shown in Fig. 7.8.

For conveniency, the following table collects the expansions with Peano's remainder obtained so far. A more comprehensive list is found on p. 476.

$$e^x = 1 + x + \frac{x^2}{2} + \ldots + \frac{x^k}{k!} + \ldots + \frac{x^n}{n!} + o(x^n)$$

$$\log(1+x) = x - \frac{x^2}{2} + \ldots + (-1)^{n-1}\frac{x^n}{n} + o(x^n)$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^m \frac{x^{2m+1}}{(2m+1)!} + o(x^{2m+2})$$

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \ldots + (-1)^m \frac{x^{2m}}{(2m)!} + o(x^{2m+1})$$

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2}x^2 + \ldots + \binom{\alpha}{n}x^n + o(x^n)$$

$$\frac{1}{1+x} = 1 - x + x^2 - \ldots + (-1)^n x^n + o(x^n)$$

$$\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + o(x^3)$$

7.3 Operations on Taylor expansions

Consider the situation where a map f has a complicated analytic expression, that involves several elementary functions; it might not be that simple to find its Taylor expansion using the definition, because computing derivatives at a point up to a certain order n is no straighforward task. But with the expansions of the elementary functions at our avail, a more convenient strategy may be to start from these and combine them suitably to arrive at f. The techniques are explained in this section.

This approach is indeed justified by the following result.

Proposition 7.5 *Let $f : (a,b) \to \mathbb{R}$ be n times differentiable at $x_0 \in (a,b)$. If there exists a polynomial P_n, of degree $\leq n$, such that*

$$f(x) = P_n(x) + o\big((x-x_0)^n\big) \qquad \text{for } x \to x_0, \tag{7.19}$$

then P_n is the Taylor polynomial $T_n = Tf_{n,x_0}$ of order n for the map f at x_0.

Proof. Formula (7.19) is equivalent to

$$P_n(x) = f(x) + \varphi(x), \qquad \text{with } \ \varphi(x) = o\big((x-x_0)^n\big) \text{ for } x \to x_0.$$

On the other hand, Taylor's formula for f at x_0 reads

$$T_n(x) = f(x) + \psi(x), \qquad \text{with } \ \psi(x) = o\big((x-x_0)^n\big).$$

Therefore

$$P_n(x) - T_n(x) = \varphi(x) - \psi(x) = o\big((x-x_0)^n\big). \tag{7.20}$$

But the difference $P_n(x) - T_n(x)$ is a polynomial of degree lesser or equal than n, hence it may be written as

$$P_n(x) - T_n(x) = \sum_{k=0}^{n} c_k (x-x_0)^k.$$

The claim is that all coefficients c_k vanish. Suppose, by contradiction, there are some non-zero c_k, and let m be the smallest index between 0 and n such that $c_m \neq 0$. Then

$$P_n(x) - T_n(x) = \sum_{k=m}^{n} c_k (x-x_0)^k$$

so

$$\frac{P_n(x) - T_n(x)}{(x-x_0)^m} = c_m + \sum_{k=m+1}^{n} c_k (x-x_0)^{k-m},$$

by factoring out $(x-x_0)^m$. Taking the limit for $x \to x_0$ and recalling (7.20), we obtain

$$0 = c_m,$$

in contrast with the assumption. □

The proposition guarantees that however we arrive at an expression like (7.19) (in a mathematically correct way), this must be exactly the Taylor expansion of order n for f at x_0.

Example 7.6

Suppose the function $f(x)$ satisfies

$$f(x) = 2 - 3(x-2) + (x-2)^2 - \frac{1}{4}(x-2)^3 + o((x-2)^3) \qquad \text{for } x \to 2.$$

Then (7.7) implies

$$f(2) = 2, \quad f'(2) = -3, \quad \frac{f''(2)}{2} = 1, \quad \frac{f'''(2)}{3!} = -\frac{1}{4},$$

hence

$$f(2) = 2, \quad f'(2) = -3, \quad f''(2) = 2, \quad f'''(2) = -\frac{3}{2}.$$

□

For simplicity we shall assume henceforth $x_0 = 0$. This is always possible by a change of the variables, $x \to t = x - x_0$.

Let now

$$f(x) = a_0 + a_1 x + ... + a_n x^n + o(x^n) = p_n(x) + o(x^n)$$

and

$$g(x) = b_0 + b_1 x + ... + b_n x^n + o(x^n) = q_n(x) + o(x^n)$$

be the Maclaurin expansions of the maps f and g.

Sums

From (5.5) a), it follows

$$\begin{aligned} f(x) \pm g(x) &= [p_n(x) + o(x^n)] \pm [q_n(x) + o(x^n)] \\ &= [p_n(x) \pm q_n(x)] + [o(x^n) \pm o(x^n)] \\ &= p_n(x) \pm q_n(x) + o(x^n). \end{aligned}$$

The expansion of a sum is the sum of the expansions involved.

Example 7.7

Let us find the expansions at the origin of the hyperbolic sine and cosine, introduced in Sect. 6.10.1. Changing x to $-x$ in

$$\mathrm{e}^x = 1 + x + \frac{x^2}{2} + ... + \frac{x^{2n+2}}{(2n+2)!} + o(x^{2n+2})$$

gives

$$\mathrm{e}^{-x} = 1 - x + \frac{x^2}{2} - \ldots + \frac{x^{2n+2}}{(2n+2)!} + o(x^{2n+2}).$$

Thus

$$\sinh x = \frac{1}{2}(\mathrm{e}^x - \mathrm{e}^{-x}) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots + \frac{x^{2n+1}}{(2n+1)!} + o(x^{2n+2}).$$

Similarly,

$$\cosh x = \frac{1}{2}(\mathrm{e}^x + \mathrm{e}^{-x}) = 1 + \frac{x^2}{2} + \frac{x^4}{4!} + \ldots + \frac{x^{2n}}{(2n)!} + o(x^{2n+1}).$$

The analogies of these expansions to $\sin x$ and $\cos x$ should not go amiss. □

Note that when the expansions of f and g have the same monomial terms up to the exponent n, these *all cancel out* in the difference $f - g$. In order to find the first non-zero coefficient in the expansion of $f - g$ one has to look at an expansions of f and g of order $n' > n$. In general it is not possible to predict what the minimum n' will be, so one must proceed case by case. Using expansions 'longer' than necessary entails superfluous computations, but is no mistake, in principle. On the contrary, terminating an expansion 'too soon' leads to meaningless results or, in the worst scenario, to a wrong conclusion.

Example 7.8

Determine the order at 0 of

$$h(x) = \mathrm{e}^x - \sqrt{1+2x}$$

by means of Maclaurin's expansion (see Sect. 7.4 in this respect).
Using first order expansions,

$$f(x) = \mathrm{e}^x = 1 + x + o(x),$$
$$g(x) = \sqrt{1+2x} = 1 + x + o(x),$$

leads to the cancellation phenomenon just described. We may only say

$$h(x) = o(x),$$

which is clearly not enough for the order of h. Instead, if we expand to second order

$$f(x) = \mathrm{e}_x = 1 + x + \frac{x^2}{2} + o(x^2)$$

$$g(x) = \sqrt{1+2x} = 1 + x - \frac{x^2}{2} + o(x^2),$$

then

$$h(x) = x_2 + o(x_2)$$

shows $h(x)$ is infinitesimal of order two at the origin. □

Products

Using (5.5) d) and then (5.5) a) shows that

$$\begin{aligned}
f(x)g(x) &= [p_n(x)+o(x^n)][q_n(x)+o(x^n)] \\
&= p_n(x)q_n(x)+p_n(x)o(x^n)+q_n(x)o(x^n)+o(x^n)o(x^n) \\
&= p_n(x)q_n(x)+o(x^n)+o(x^n)+o(x^{2n}) \\
&= p_n(x)q_n(x)+o(x^n).
\end{aligned}$$

The product $p_n(x)q_n(x)$ contains powers of x larger than n; each of them is an $o(x^n)$, so we can eschew calculating it explicitly. We shall write

$$p_n(x)q_n(x) = r_n(x)+o(x^n),$$

intending that $r_n(x)$ gathers all powers of order $\leq n$, and nothing else, so in conclusion

$$f(x)g(x) = r_n(x)+o(x^n).$$

Example 7.9

Expand to second order

$$h(x) = \sqrt{1+x}\,\mathrm{e}^x$$

at the origin. Since

$$f(x) = \sqrt{1+x} = 1+\frac{x}{2}-\frac{x^2}{8}+o(x^2),$$

$$g(x) = \mathrm{e}^x = 1+x+\frac{x^2}{2}+o(x^2),$$

it follows

$$\begin{aligned}
h(x) &= \left(1+\frac{x}{2}-\frac{x^2}{8}\right)\left(1+x+\frac{x^2}{2}\right)+o(x^2) \\
&= \left(1+x+\frac{x^2}{2}\right)+\left(\frac{x}{2}+\frac{x^2}{2}+\boxed{\frac{x^3}{4}}\right)-\left(\frac{x^2}{8}+\boxed{\frac{x^3}{8}}+\boxed{\frac{x^4}{16}}\right)+o(x^2) \\
&= 1+\frac{3}{2}x+\frac{7}{8}x^2+o(x^2).
\end{aligned}$$

The boxed terms have order larger than two, and therefore are already accounted for by the symbol $o(x^2)$. Because of this, they need not have been computed explicitly, although no harm was done. □

Quotients

Suppose $g(0) \neq 0$ and let

$$h(x) = \frac{f(x)}{g(x)},$$

for which we search an expansion

$$h(x) = r_n(x)+o(x^n), \qquad \text{with } r_n(x) = \sum_{k=0}^{n} c_k x^k.$$

From $h(x)g(x) = f(x)$ we have

$$r_n(x)q_n(x) + o(x^n) = p_n(x) + o(x^n).$$

This means that the part of degree $\le n$ in the polynomial $r_n(x)q_n(x)$ (degree $2n$) must coincide with $p_n(x)$. By this observation we can determine the coefficients c_k of $r_n(x)$ starting from c_0. The practical computation may be carried out like the division algorithm for polynomials, so long as the latter are *ordered with respect to the increasing powers of* x:

$$\begin{array}{r|l}
a_0 + a_1x + a_2x^2 + \ldots + a_nx^n + o(x^n) & \underline{b_0 + b_1x + b_2x^2 + \ldots + b_nx^n + o(x^n)} \\
\underline{a_0 + a_1'x + a_2'x^2 + \ldots + a_n'x^n + o(x^n)} & c_0 + c_1x + \ldots + c_nx^n + o(x^n) \\
0 + \tilde{a}_1x + \tilde{a}_2x^2 + \ldots + \tilde{a}_nx^n + o(x^n) & \\
\underline{\tilde{a}_1x + \tilde{a}_2'x^2 + \ldots + \tilde{a}_n'x^n + o(x^n)} & \\
\vdots \qquad\qquad & \\
\overline{0 + o(x^n)} &
\end{array}$$

Examples 7.10

i) Let us compute the second order expansion of $h(x) = \dfrac{e^x}{3 + 2\log(1+x)}$. By (7.9), (7.13), we have $e^x = 1 + x + \frac{1}{2}x^2 + o(x^2)$, and $3 + 2\log(1+x) = 3 + 2x - x^2 + o(x^2)$; dividing

$$\begin{array}{r|l}
1 + x + \frac{1}{2}x^2 + o(x^2) & \underline{3 + 2x - x^2 + o(x^2)} \\
\underline{1 + \frac{2}{3}x - \frac{1}{3}x^2 + o(x^2)} & \frac{1}{3} + \frac{1}{9}x + \frac{11}{54}x^2 + o(x^2) \\
\frac{1}{3}x + \frac{5}{6}x^2 + o(x^2) & \\
\underline{\frac{1}{3}x + \frac{2}{9}x^2 + o(x^2)} & \\
\frac{11}{18}x^2 + o(x^2) & \\
\underline{\frac{11}{18}x^2 + o(x^2)} & \\
o(x^2) &
\end{array}$$

produces $h(x) = \frac{1}{3} + \frac{1}{9}x + \frac{11}{54}x^2 + o(x^2)$.

ii) Expand $h(x) = \tan x$ to the fourth order. The function being odd, it suffices to find Maclaurin's polynomial of degree three, which is the same as the one of order four. Since

$$\sin x = x - \frac{x^3}{6} + o(x^3) \qquad \text{and} \qquad \cos x = 1 - \frac{x^2}{2} + o(x^3)\,,$$

dividing

$$\begin{array}{r|l} x - \frac{x^3}{6} + o(x^3) & \underline{1 - \frac{x^2}{2} + o(x^3)} \\ \underline{x - \frac{x^3}{2} + o(x^3)} & x + \frac{x^3}{3} + o(x^3) \\ \frac{x^3}{3} + o(x^3) & \\ \underline{\frac{x^3}{3} + o(x^3)} & \\ o(x^3) & \end{array}$$

yields

$$\tan x = x + \frac{x^3}{3} + o(x^3) = x + \frac{x^3}{3} + o(x^4)\,. \qquad \square$$

Composite maps

Let

$$f(x) = a_1 x + a_2 x^2 + ... + a_n x^n + o(x^n)$$

be the Maclaurin expansion of an *infinitesimal* function for $x \to 0$ (hence $a_0 = 0$). Write

$$g(y) = b_0 + b_1 y + ... + b_n y^n + o(y^n)$$

for a second map $g(y)$. Recall

$o(y^n)$ stands for *an infinitesimal of bigger order than* y^n as $y \to 0$,

which can be written

$$o(y^n) = y^n o(1) \qquad \text{with } o(1) \to 0 \text{ for } y \to 0.$$

Now consider the composition $h(x) = g\big(f(x)\big)$ and substitute $y = f(x)$ in the expansion of $g(y)$:

$$g(f(x)) = b_0 + b_1 f(x) + b_2[f(x)]^2 + ... + b_n[f(x)]^n + [f(x)]^n o(1).$$

As $f(x)$ is continuous at 0, $y = f(x) \to 0$ for $x \to 0$, so $o(1) \to 0$ for $x \to 0$ as well. Furthermore, expanding

$$[f(x)]^n = a_1^n x^n + o(x^n)$$

yields

$$[f(x)]^n o(1) = o(x^n) \qquad \text{per } x \to 0.$$

The powers $[f(x)]^k$ $(1 \le k \le n)$, expanded with respect to x up to order n, provide the expression of $g\big(f(x)\big)$.

Examples 7.11

i) Calculate to order two the expansion at 0 of

$$h(x) = \mathrm{e}^{\sqrt{1+x}-1}.$$

Define

$$f(x) = \sqrt{1+x} - 1 = \frac{x}{2} - \frac{x^2}{8} + o(x^2),$$

$$g(y) = \mathrm{e}^y = 1 + y + \frac{y^2}{2} + o(y^2).$$

Then

$$\begin{aligned} h(x) &= 1 + \left(\frac{x}{2} - \frac{x^2}{8} + o(x^2)\right) + \frac{1}{2}\left(\frac{x}{2} - \frac{x^2}{8} + o(x^2)\right)^2 + o(x^2) \\ &= 1 + \left(\frac{x}{2} - \frac{x^2}{8} + o(x^2)\right) + \frac{1}{2}\left(\frac{x^2}{4} + o(x^2)\right) + o(x^2) \\ &= 1 + \frac{x}{2} + o(x^2). \end{aligned}$$

ii) Expand to order three in 0 the map

$$h(x) = \frac{1}{1 + \log(1+x)}.$$

We can view this map as a quotient, but also as the composition of

$$f(x) = \log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + o(x^3)$$

with

$$g(y) = \frac{1}{1+y} = 1 - y + y^2 - y^3 + o(y^3).$$

It follows

$$\begin{aligned} h(x) &= 1 - \left(x - \frac{x^2}{2} + \frac{x^3}{3} + o(x^3)\right) + \left(x - \frac{x^2}{2} + \frac{x^3}{3} + o(x^3)\right)^2 \\ &\qquad - \left(x - \frac{x^2}{2} + \frac{x^3}{3} + o(x^3)\right)^3 + o(x^3) \\ &= 1 - \left(x - \frac{x^2}{2} + \frac{x^3}{3} + o(x^3)\right) + \big(x^2 - x^3 + o(x^3)\big) - \big(x^3 + o(x^3)\big) + o(x^3) \\ &= 1 - x + \frac{3x^2}{2} - \frac{7x^3}{3} + o(x^3). \end{aligned}$$

□

Remark 7.12 If $f(x)$ is infinitesimal of order greater that one at the origin, we can spare ourselves some computations, in the sense that we might be able to infer the expansion of $h(x) = g\big(f(x)\big)$ of degree n from lower-order expansions of $g(y)$. For example, let f be infinitesimal of order 2 at the origin ($a_1 = 0,\ a_2 \neq 0$). Because $[f(x)]^k = a_2^k x^{2k} + o(x^{2k})$, an expansion for $g(y)$ of order $\frac{n}{2}$ (if n even) or $\frac{n+1}{2}$ (n odd) is sufficient to determine $h(x)$ up to degree n. (Note that $f(x)$ should be expanded to order n, in general.) □

Example 7.13

Expand to second order

$$h(x) = \sqrt{\cos x} = \sqrt{1 + (\cos x - 1)}.$$

Set

$$f(x) = \cos x - 1 = -\frac{x^2}{2} + o(x^2) \qquad \text{(2nd order)}$$
$$g(y) = \sqrt{1+y} = 1 + \frac{y}{2} + o(y) \qquad \text{(1st order).}$$

Then

$$h(x) = 1 + \frac{1}{2}\left(-\frac{x^2}{2} + o(x^2)\right) + o(x^2)$$
$$= 1 - \frac{x^2}{4} + o(x^2) \qquad \text{(2nd order).}$$

□

Asymptotic expansions (not of Taylor type)

In many situations where $f(x)$ is infinite for $x \to 0$ (or $x \to x_0$) it is possible to find an 'asymptotic' expansion of $f(x)$ in increasing powers of x $(x - x_0)$, by allowing negative powers in the expression:

$$f(x) = \frac{a_{-m}}{x^m} + \frac{a_{-m+1}}{x^{m-1}} + \ldots + \frac{a_{-1}}{x} + a_0 + a_1 x + \ldots + a_n x^n + o(x^n).$$

This form helps to understand better how f tends to infinity. In fact, if $a_{-m} \neq 0$, f will be infinite of order m with respect to the test function x^{-1}.

To a similar expansion one often arrives by means of the Taylor expansion of $\frac{1}{f(x)}$, which is infinitesimal for $x \to 0$.

We explain the procedure with an example.

Example 7.14

Let us expand 'asymptotically', for $x \to 0$, the function

$$f(x) = \frac{1}{e^x - 1}.$$

The exponential expansion arrested at order three gives

$$e^x - 1 = x + \frac{x^2}{2} + \frac{x^3}{6} + o(x^3)$$
$$= x\left(1 + \frac{x}{2} + \frac{x^2}{6} + o(x^2)\right),$$

so

$$f(x) = \frac{1}{x}\frac{1}{1 + \frac{x}{2} + \frac{x^2}{6} + o(x^2)}.$$

The latter ratio can be treated using Maclaurin's formula

$$\frac{1}{1+y} = 1 - y + y^2 + o(y^2);$$

by putting

$$y = \frac{x}{2} + \frac{x^2}{6} + o(x^2)$$

in fact, we obtain

$$f(x) = \frac{1}{x}\left(1 - \frac{x}{2} + \frac{x^2}{12} + o(x^2)\right) = \frac{1}{x} - \frac{1}{2} + \frac{x}{12} + o(x)\,,$$

the asymptotic expansion of f at the origin. Looking at such expression, we can deduce for instance that f is infinite of order 1 with respect to $\varphi(x) = \frac{1}{x}$, as $x \to 0$.

Ignoring the term $x/12$ and writing $f(x) = \frac{1}{x} - \frac{1}{2} + o(1)$ shows f is asymptotic to the hyperbola

$$g(x) = \frac{2-x}{2x}. \qquad \square$$

7.4 Local behaviour of a map via its Taylor expansion

Taylor expansions at a given point are practical tools for studying how a function locally behaves around that point. We examine in the sequel a few interesting applications of Taylor expansions.

Order and principal part of infinitesimal functions

Let

$$f(x) = a_0 + a_1(x - x_0) + \ldots + a_n(x - x_0)^n + o\big((x - x_0)^n\big)$$

be the Taylor expansion of order n at a point x_0, and suppose there is an index m with $1 \le m \le n$ such that

$$a_0 = a_1 = \ldots = a_{m-1} = 0, \qquad \text{but} \quad a_m \neq 0.$$

In a sufficiently small neighbourhood of x_0,

$$f(x) = a_m(x - x_0)^m + o\big((x - x_0)^m\big)$$

will behave like the polynomial

$$p(x) = a_m(x - x_0)^m,$$

which is the *principal part* of f with respect to the infinitesimal $y = x - x_0$. In particular, $f(x)$ has order m with respect to that test function.

Example 7.15

Compute the order of the infinitesimal $f(x) = \sin x - x\cos x - \frac{1}{3}x^3$ with respect to $\varphi(x) = x$ as $x \to 0$. Expanding sine and cosine with Maclaurin we have

$$f(x) = -\frac{1}{30}x^5 + o(x^5), \quad x \to 0.$$

Therefore f is infinitesimal of order 5 and has principal part $p(x) = -\frac{1}{30}x^5$. The same result descends from de l'Hôpital's Theorem, albeit differentiating five times is certainly more work than using well-known expansions. $\square$

Local behaviour of a function

The knowledge of the Taylor expansion of f to order two around a point x_0,

$$f(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + o\big((x - x_0)^2\big)\,, \qquad x \to x_0\,,$$

allows us to deduce from (7.7) that

$$f(x_0) = a_0\,, \qquad f'(x_0) = a_1\,, \qquad f''(x_0) = 2a_2\,.$$

Suppose f is differentiable twice with continuity around x_0. By Theorem 4.2 the signs of a_0, a_1, a_2 (when $\neq 0$) coincide with the signs of $f(x)$, $f'(x)$, $f''(x)$, respectively, in a neighbourhood of x_0. This fact permits, in particular, to detect local monotonicity and convexity, because of Theorem 6.27 *b2)* and Corollary 6.38 *b2)*.

Example 7.6 (continuation)

Return to Example 7.6: we have $f(2) > 0$, $f'(2) < 0$ and $f''(2) > 0$. Around $x_0 = 2$ then, f is strictly positive, strictly decreasing and strictly convex. □

We deal with the cases $a_1 = 0$ or $a_2 = 0$ below.

Nature of critical points

Let x_0 be a critical point for f, which is assumed differentiable around x_0. By Corollary 6.28, different signs of f' at the left and right of x_0 mean that the point is an extremum; if the sign stays the same instead, x_0 is an inflection point with horizontal tangent.

When f possesses higher derivatives at x_0, in alternative to the sign of f' around x_0 we can understand what sort of critical point x_0 is by looking at the first non-zero derivative of f evaluated at the point. In fact,

Theorem 7.16 *Let f be differentiable $n \geq 2$ times at x_0 and suppose*

$$f'(x_0) = \ldots = f^{(m-1)}(x_0) = 0, \qquad f^{(m)}(x_0) \neq 0 \tag{7.21}$$

for some $2 \leq m \leq n$.

i) When m is even, x_0 is an extremum, namely a maximum if $f^{(m)}(x_0) < 0$, a minimum if $f^{(m)}(x_0) > 0$.

ii) When m is odd, x_0 is an inflection point with horizontal tangent; more precisely the inflection is descending if $f^{(m)}(x_0) < 0$, ascending if $f^{(m)}(x_0) > 0$.

Proof. Compare $f(x)$ and $f(x_0)$ around x_0. From (7.6)-(7.7) and the assumption (7.21), we have

$$f(x) - f(x_0) = \frac{f^{(m)}(x_0)}{m!}(x-x_0)^m + o\big((x-x_0)^m\big).$$

But $o\big((x-x_0)^m\big) = (x-x_0)^m o(1)$, so

$$f(x) - f(x_0) = (x-x_0)^m \left[\frac{f^{(m)}(x_0)}{m!} + h(x)\right],$$

for a suitable $h(x)$, infinitesimal when $x \to x_0$. Therefore, in a sufficiently small neighbourhood of x_0, the term in square brackets has the same sign as $f^{(m)}(x_0)$, hence the sign of $f(x) - f(x_0)$, in that same neighbourhood, is determined by $f^{(m)}(x_0)$ and $(x-x_0)^m$. Examining all sign possibilities proves the claim. □

Example 7.17

Assume that around $x_0 = 1$ we have

$$f(x) = 2 - 15(x-1)^4 + 20(x-1)^5 + o\big((x-1)^5\big). \tag{7.22}$$

From this we deduce

$$f'(1) = f''(1) = f'''(1) = 0\,, \quad \text{but} \quad f^{(4)}(1) = -360 < 0\,.$$

Then x_0 is a relative maximum (Fig. 7.9, left).
Suppose now that in a neighbourhood of $x_1 = -2$ we can write

$$f(x) = 3 + 10(x+2)^5 - 35(x+2)^7 + o\big((x+2)^7\big). \tag{7.23}$$

Then

$$f'(-2) = f''(-2) = f'''(-2) = f^{(4)}(-2) = 0\,, \quad \text{and} \quad f^{(5)}(-2) = 10 \cdot 5! > 0\,,$$

telling x_1 is an ascending inflection with horizontal tangent (Fig. 7.9, right). □

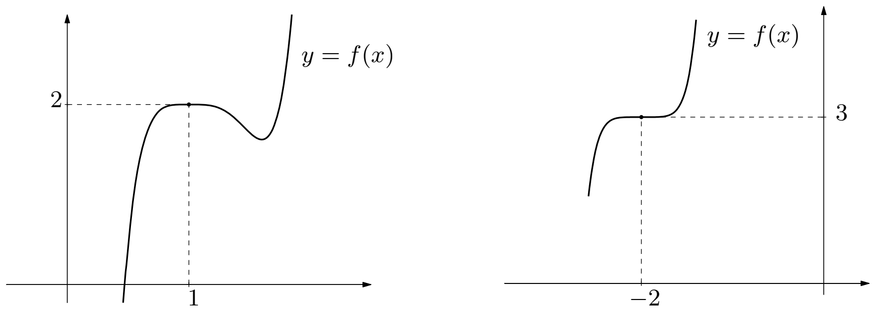


Figure 7.9. The map defined in (7.22), around $x_0 = 1$ (right), and the one defined in (7.23), around $x_0 = -2$ (left)

Points of inflection

Consider a twice differentiable f around x_0. By Taylor's formulas we can decide whether x_0 is an inflection point for f.

First though, we need to prove Corollary 6.39 stated in Chap. 6, whose proof we had to postpone to the present section.

Proof. a) Let x_0 be an inflection point for f. Denoting as usual by $y = t(x) = f(x_0) + f'(x_0)(x - x_0)$ the tangent line to f at x_0, Taylor's formula (7.6) $(n = 2)$ gives

$$f(x) - t(x) = \frac{1}{2} f''(x_0)(x - x_0)^2 + o\big((x - x_0)^2\big),$$

which we can write

$$f(x) - t(x) = (x - x_0)^2 \left[\frac{1}{2} f''(x_0) + h(x)\right]$$

for some infinitesimal h at x_0. By contradiction, if $f''(x_0) \neq 0$, in an arbitrarily small neighbourhood of x_0 the right-hand side would have constant sign at the left and right of x_0; this cannot be by hypothesis, as f is assumed to inflect at x_0.
b) In this case we use Taylor's formula (7.8) with $n = 2$. For any $x \neq x_0$, around x_0 there is a point $\bar{x}$, lying between x_0 and x, such that

$$f(x) - t(x) = \frac{1}{2} f''(\bar{x})(x - x_0)^2.$$

Analysing the sign of the right-hand side concludes the proof. □

Suppose, from now on, that $f''(x_0) = 0$ and f admits derivatives higher than the second. Instead of considering the sign of f'' around x_0, we may study the point x_0 by means of the first non-zero derivative of order > 2 evaluated at x_0.

Theorem 7.18 *Let f be n times differentiable ($n \geq 3$) at x_0, with*

$$f''(x_0) = \ldots = f^{(m-1)}(x_0) = 0, \qquad f^{(m)}(x_0) \neq 0 \tag{7.24}$$

for some m ($3 \leq m \leq n$).

i) When m is odd, x_0 is an inflection point: descending if $f^{(m)}(x_0) < 0$, ascending if $f^{(m)}(x_0) > 0$.
ii) When m is even, x_0 is not an inflection for f.

Proof. Just like in Theorem 7.16, we obtain

$$f(x) - t(x) = (x - x_0)^m \left[\frac{f^{(m)}(x_0)}{m!} + h(x)\right],$$

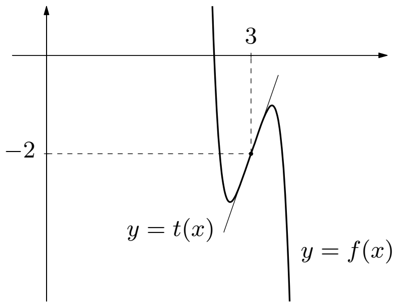


Figure 7.10. Local behaviour of the map (7.25)

where $h(x)$ is a suitable infinitesimal function for $x \to x_0$. The claim follows from a sign argument concerning the right-hand side. □

Example 7.19

Suppose that around $x_0 = 3$ we have

$$f(x) = -2 + 4(x-3) - 90(x-3)^5 + o\big((x-3)^5\big). \tag{7.25}$$

Then $f''(3) = f'''(3) = f^{(4)}(3) = 0$, $f^{(5)}(3) = -90 \cdot 5! < 0$. This implies that $x_0 = 3$ is a descending inflection for f (Fig. 7.10). □

7.5 Exercises

1. *Use the definition to write the Taylor polynomial, of order n and centred at x_0, for:*

   a) $f(x) = \mathrm{e}^x\,, \qquad n = 4\,, \qquad x_0 = 2$

   b) $f(x) = \sin x\,, \qquad n = 6\,, \qquad x_0 = \frac{\pi}{2}$

   c) $f(x) = \log x\,, \qquad n = 3\,, \qquad x_0 = 3$

   d) $f(x) = \sqrt{2x+1}\,, \qquad n = 3\,, \qquad x_0 = 4$

   e) $f(x) = 7 + x - 3x^2 + 5x^3\,, \qquad n = 2\,, \qquad x_0 = 1$

   f) $f(x) = 2 - 8x^2 + 4x^3 + 9x^4\,, \qquad n = 3\,, \qquad x_0 = 0$

2. *Determine the Taylor expansion of the indicated functions of the highest-possible order; the expansion should be centred around x_0 and have Peano's remainder:*

   a) $f(x) = x^2|x| + \mathrm{e}^{2x}\,, \qquad x_0 = 0$

   b) $f(x) = 2 + x + (x-1)\sqrt[3]{x^2-1}\,, \qquad x_0 = 1$

3. *With the aid of the elementary functions, write the Maclaurin expansion of the indicated functions of the given order, with Peano's remainder:*

a) $f(x) = x\cos 3x - 3\sin x\,, \qquad n = 2$

b) $f(x) = \log \dfrac{1+x}{1+3x}\,, \qquad n = 4$

c) $f(x) = \mathrm{e}^{x^2} \sin 2x\,, \qquad n = 5$

d) $f(x) = \mathrm{e}^{-x\cos x} + \sin x - \cos x\,, \qquad n = 2$

e) $f(x) = \sqrt[3]{\cos(3x - x^2)}\,, \qquad n = 4$

f) $f(x) = \dfrac{x}{\sqrt[6]{1+x^2}} - \sin x\,, \qquad n = 5$

g) $f(x) = \cosh^2 x - \sqrt{1+2x^2}\,, \qquad n = 4$

h) $f(x) = \dfrac{\mathrm{e}^{2x} - 1}{\sqrt{\cos 2x}}\,, \qquad n = 3$

i) $f(x) = \dfrac{1}{-\sqrt{8}\sin x - 2\cos x}\,, \qquad n = 3$

ℓ) $f(x) = \sqrt[3]{8 + \sin 24x^2} - 2(1 + x^2\cos x^2)\,, \qquad n = 4$

4. *Ascertain order and find principal part, for $x \to 0$, with respect to $\varphi(x) = x$ of the indicated functions:*

a) $f(x) = \mathrm{e}^{\cos 2x} - \mathrm{e}$

b) $f(x) = \dfrac{\cos 2x + \log(1 + 4x^2)}{\cosh 2x} - 1$

c) $f(x) = \dfrac{\sqrt{x^3} - \sin^3\sqrt{x}}{\mathrm{e}^{3\sqrt{x}} - 1}$

d) $f(x) = 2x + (x^2 - 1)\log \dfrac{1+x}{1-x}$

e) $f(x) = x - \arctan \dfrac{x}{\sqrt{1-4x^2}}$

f) $f(x) = \sqrt[3]{1-x^2} - \sqrt{1 - \dfrac{2}{3}x^2 + \sin\dfrac{x^4}{18}}$

5. *Calculate order and principal part, when $x \to +\infty$, with respect to $\varphi(x) = \frac{1}{x}$ of the indicated functions:*

a) $f(x) = \dfrac{1}{x-2} - \dfrac{1}{2(x-2) - \log(x-1)}$

b) $f(x) = \mathrm{e}^{-\frac{x}{4x^2+1}} - 1$

c) $f(x) = \sqrt[3]{1 + 3x^2 + x^3} - \sqrt[5]{2 + 5x^4 + x^5}$

d) $f(x) = \sqrt[3]{2 + \sinh\dfrac{2}{x^2}} - \sqrt[3]{2}$

6. *Compute the limits:*

a) $\lim\limits_{x\to 0} (1 + x^6)^{1/(x^4 \sin^2 3x)}$

b) $\lim\limits_{x\to 2} \dfrac{\cos\frac{3}{4}\pi x - \frac{3}{2}\pi\log\frac{x}{2}}{(4-x^2)^2}$

c) $\displaystyle\lim_{x\to 0}\frac{1}{x}\left(\frac{1}{\sin(\tan x)}-\frac{1}{x}\right)$ d) $\displaystyle\lim_{x\to 0}\left(e^{x^7}+\sin^2 x-\sinh^2 x\right)^{1/x^4}$

e) $\displaystyle\lim_{x\to 0}\frac{18x^4}{\sqrt[3]{\cos 6x}-1+6x^2}$ f) $\displaystyle\lim_{x\to 0}\frac{3x^4[\log(1+\sinh^2 x)]\cosh^2 x}{1-\sqrt{1+x^3}\,\cos\sqrt{x^3}}$

7. *As a varies in $\mathbb{R}$, determine the order of the infinitesimal map*

$$h(x)=\log\cos x+\log\cosh(ax)$$

as $x\to 0$.

8. *Compute the sixth derivative of*

$$h(x)=\frac{\sinh(x^2+2\sin^4 x)}{1+x^{10}}$$

evaluated at $x=0$.

9. *Let*

$$\varphi(x)=\log(1+4x)-\sinh 4x+8x^2.$$

Determine the sign of $y=\sin\varphi(x)$ on a left and on a right neighbourhood of $x_0=0$.

10. *Prove that there exists a neighbourhood of* 0 *where*

$$2\cos(x+x^2)\le 2-x^2-2x^3.$$

11. *Compute the limit*

$$\lim_{x\to 0^+}\frac{e^{x/2}-\cosh\sqrt{x}}{(x+\sqrt[5]{x})^\alpha}$$

for all values $\alpha\in\mathbb{R}^+$.

12. *Determine $\alpha\in\mathbb{R}$ so that*

$$f(x)=(\arctan 2x)^2-\alpha x\sin x$$

is infinitesimal of the fourth order as $x\to 0$.

7.5.1 Solutions

1. *Taylor's polynomials:*

a) All derivatives of $f(x)=e^x$ are identical with the function itself, so $f^{(k)}(2)=e^2$, $\forall k\ge 0$. Therefore

$$Tf_{4,2}(x)=e^2+e^2(x-2)+\frac{e^2}{2}(x-2)^2+\frac{e^2}{6}(x-2)^3+\frac{e^2}{24}(x-2)^4\,.$$

b) $Tf_{6,\frac{\pi}{2}}(x) = 1 - \frac{1}{2}(x - \frac{\pi}{2})^2 + \frac{1}{4!}(x - \frac{\pi}{2})^4 - \frac{1}{6!}(x - \frac{\pi}{2})^6$.

c) From $f'(x) = \frac{1}{x}$, $f''(x) = -\frac{1}{x^2}$, $f'''(x) = \frac{2}{x^3}$ it follows $f(3) = \log 3$, $f'(3) = \frac{1}{3}$, $f''(3) = -\frac{1}{9}$, $f'''(3) = \frac{2}{27}$. Then

$$Tf_{3,3}(x) = \log 3 + \frac{1}{3}(x-3) - \frac{1}{18}(x-3)^2 + \frac{1}{81}(x-3)^3.$$

d) $Tf_{3,4}(x) = 3 + \frac{1}{3}(x-4) - \frac{1}{54}(x-4)^2 + \frac{1}{486}(x-4)^3$.

e) As $f'(x) = 1 - 6x + 15x^2$, $f''(x) = -6 + 30x$, we have $f(1) = 10$, $f'(1) = 10$, $f''(1) = 24$ and

$$Tf_{2,1}(x) = 10 + 10(x-1) + 12(x-1)^2.$$

Alternatively, we may substitute $t = x-1$, i.e. $x = 1+t$. The polynomial $f(x)$, written in the variable t, reads

$$g(t) = f(1+t) = 7 + (1+t) - 3(1+t)^2 + 5(1+t)^3 = 10 + 10t + 12t^2 + 5t^3.$$

Therefore the Taylor polynomial of $f(x)$ centred at $x_0 = 1$ corresponds to the Maclaurin polynomial of $g(t)$, whence immediately

$$Tg_{2,0}(t) = 10 + 10t + 12t^2.$$

Returning to the variable x, we find the same result.

f) $Tf_{3,0}(x) = 2 - 8x^2 + 4x^3$.

2. *Taylor's expansions:*

a) We can write $f(x) = g(x) + h(x)$ using $g(x) = x^2|x|$ and $h(x) = e^{2x}$. The sum $h(x)$ is differentiable on $\mathbb{R}$ *ad libitum*, whereas $g(x)$ is continuous on $\mathbb{R}$ but arbitrarily differentiable only for $x \neq 0$. Additionally

$$g'(x) = \begin{cases} 3x^2 & \text{if } x > 0\,, \\ -3x^2 & \text{if } x < 0\,, \end{cases} \qquad g''(x) = \begin{cases} 6x & \text{if } x > 0\,, \\ -6x & \text{if } x < 0\,, \end{cases}$$

so

$$\lim_{x\to 0^+} g'(x) = \lim_{x\to 0^-} g'(x) = 0, \qquad \lim_{x\to 0^+} g''(x) = \lim_{x\to 0^-} g''(x) = 0.$$

By Theorem 6.15 we infer g is differentiable twice at the origin, with vanishing derivatives. Since $g''(x) = 6|x|$ is not differentiable at $x = 0$, g is not differentiable three times at 0, which makes f expandable only up to order 2. From $h'(x) = 2e^{2x}$ and $h''(x) = 4e^{2x}$, we have $f(0) = 1$, $f'(0) = 2$, $f''(0) = 4$, so Maclaurin's formula reads:

$$f(x) = 1 + 2x + 2x^2 + o(x^2)\,.$$

b) The map is differentiable only once at $x_0 = 1$, and the expansion is $f(x) = 3 + (x-1) + o(x-1)$.

3. *Maclaurin's expansions:*

a) $f(x) = -2x + o(x^2)$.

b) Writing $f(x) = \log(1+x) - \log(1+3x)$, we can use the expansion of $\log(1+t)$ with $t = x$ and $t = 3x$

$$\begin{aligned} f(x) &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} - 3x + \frac{(3x)^2}{2} - \frac{(3x)^3}{3} + \frac{(3x)^4}{4} + o(x^4) \\ &= -2x + 4x^2 - \frac{26}{3}x^3 + 20x^4 + o(x^4)\,. \end{aligned}$$

c) Combining the expansions of e^t with $t = x^2$, and of $\sin t$ with $t = 2x$:

$$\begin{aligned} f(x) &= \left(1 + x^2 + \frac{x^4}{2} + o(x^5)\right)\left(2x - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} + o(x^5)\right) \\ &= 2x + 2x^3 + x^5 - \frac{4}{3}x^3 - \frac{4}{3}x^5 + \frac{4}{15}x^5 + o(x^5) \\ &= 2x + \frac{2}{3}x^3 - \frac{1}{15}x^5 + o(x^5)\,. \end{aligned}$$

d) $f(x) = x^2 + o(x^2)$.

e) $f(x) = 1 - \frac{3}{2}x^2 + x^3 - \frac{31}{24}x^4 + o(x^4)$.

f) This is solved by expanding $(1+t)^\alpha$ and changing $\alpha = -\frac{1}{6}$ and $t = x^2$:

$$\begin{aligned} \frac{x}{\sqrt[6]{1+x^2}} &= x\left(1+x^2\right)^{-1/6} = x\left(1 - \frac{1}{6}x^2 + \binom{-\frac{1}{6}}{2}x^4 + o(x^4)\right) \\ &= x - \frac{1}{6}x^3 + \frac{7}{72}x^5 + o(x^5), \end{aligned}$$

from which

$$f(x) = x - \frac{1}{6}x^3 + \frac{7}{72}x^5 - x + \frac{1}{6}x^3 - \frac{1}{5!}x^5 + o(x^5) = \frac{4}{45}x^5 + o(x^5).$$

g) Referring to the expansions of $\cosh x$ and $(1+t)^\alpha$, with $\alpha = \frac{1}{2}$, $t = 2x^2$:

$$\begin{aligned} f(x) &= \left(1 + \frac{1}{2}x^2 + \frac{1}{4!}x^4 + o(x^4)\right)^2 - \left(1+2x^2\right)^{1/2} \\ &= 1 + x^2 + \frac{1}{4}x^4 + \frac{2}{4!}x^4 + o(x^4) - \left(1 + \frac{1}{2}2x^2 + \binom{1/2}{2}(2x^2)^2 + o(x^4)\right) \\ &= 1 + x^2 + \frac{1}{3}x^4 - 1 - x^2 + \frac{1}{2}x^4 + o(x^4) = \frac{5}{6}x^4 + o(x^4)\,. \end{aligned}$$

h) $f(x) = 2x + 2x^2 + \frac{10}{3}x^3 + o(x^3)$.

i) Substitute to $\sin x$, $\cos x$ the respective Maclaurin expansions, to the effect that
$$f(x) = \frac{1}{-2 - \sqrt{8}x + x^2 + \frac{\sqrt{8}}{3!}x^3 + o(x^3)}.$$
Expansion of the reciprocal eventually gives us
$$f(x) = -\frac{1}{2} + \frac{\sqrt{2}}{2}x - \frac{5}{4}x^2 + \frac{17}{12}\sqrt{2}\,x^3 + o(x^3)\,.$$

ℓ) $f(x) = -2x^4 + o(x^4)$.

4. *Order of infinitesimal and principal part for $x \to 0$:*

a) The order is 2 and $p(x) = -2\,\mathrm{e}\,x^2$ the principal part.

b) Write
$$h(x) = \frac{\cos 2x + \log(1+4x^2) - \cosh 2x}{\cosh 2x},$$
and note that the order for $x \to 0$ can be deduced from the numerator only, for the denominator converges to 1. The expansions of $\cos t$, $\log(1+t)$ and $\cosh t$ are known, so
$$\begin{aligned}&\cos 2x + \log(1+4x^2) - \cosh 2x\\ &\quad = 1 - \frac{1}{2}(2x)^2 + \frac{1}{4!}(2x)^4 + (2x)^2 - \frac{1}{2}(2x)^4 - 1 - \frac{1}{2}(2x)^2 - \frac{1}{4!}(2x)^4 + o(x^4)\\ &\quad = -8x^4 + o(x^4).\end{aligned}$$
Thus the order is 4, the principal part $p(x) = -8x^4$.

c) Expanding $\sin t$ and e^t, then putting $t = \sqrt{x}$, we have
$$g(t) = \frac{t^3 - \sin^3 t}{\mathrm{e}^{3t} - 1} = \frac{t^3 - \left(t - \frac{1}{6}t^3 + o(t^3)\right)^3}{1 + 3t + o(t) - 1} = \frac{\frac{1}{2}t^5 + o(t^5)}{3t + o(t)} = \frac{1}{6}t^4 + o(t^4)$$
for $t \to 0$. Hence
$$f(x) = \frac{1}{6}x^2 + o(x^2),$$
implying that the order is 2 and $p(x) = \frac{1}{6}x^2$.

d) The map has order 3 with principal part $p(x) = \frac{4}{3}x^3$.

e) Use the expansion of $(1+t)^\alpha$ (where $\alpha = -\frac{1}{2}$) and $\arctan t$:
$$(1-4x^2)^{-1/2} = 1 + 2x^2 + o(x^3)\,, \qquad \frac{x}{\sqrt{1-4x^2}} = x + 2x^3 + o(x^4)$$
$$\begin{aligned}\arctan\frac{x}{\sqrt{1-4x^2}} &= x + 2x^3 + o(x^4) - \frac{1}{3}(x - 2x^3 + o(x^4))^3 + o(x^3)\\ &= x + \frac{5}{3}x^3 + o(x^3).\end{aligned}$$

In conclusion,

$$f(x) = -\frac{5}{3}x^3 + o(x^3),,$$

so that the order is 3 and the principal part $p(x) = -\frac{5}{3}x^3$.

f) Order 6 and principal part $p(x) = (-\frac{5}{3^4} + \frac{1}{2\cdot 3^3})x^6$.

5. *Order of infinitesimal and principal part as $x \to +\infty$:*

a) When $x \to +\infty$ we write

$$\begin{aligned} f(x) &= \frac{x-2-\log(x-1)}{2(x-2)^2-(x-2)\log(x-1)} \\ &= \frac{x-2-\log(x-1)}{2x^2-8x+8-(x-2)\log(x-1)} \\ &= \frac{x+o(x)}{2x^2+o(x^2)} = \frac{1}{2x} + o\left(\frac{1}{x}\right), \end{aligned}$$

from which one can recognise the order 1 and the principal part $p(x) = \frac{1}{2x}$.

b) The map is infinitesimal of order one, with principal part $p(x) = -\frac{1}{4x}$.

c) Write

$$\begin{aligned} f(x) &= \sqrt[3]{x^3\left(1+\frac{3}{x}+\frac{1}{x^3}\right)} - \sqrt[5]{x^5\left(1+\frac{5}{x}+\frac{2}{x^5}\right)} \\ &= x\left(1+\frac{3}{x}+\frac{1}{x^3}\right)^{1/3} - x\left(1+\frac{5}{x}+\frac{2}{x^5}\right)^{1/5}. \end{aligned}$$

Using the expansion of $(1+t)^\alpha$ first with $\alpha = \frac{1}{3}$, $t = \frac{3}{x} + \frac{1}{x^3}$, then with $\alpha = \frac{1}{5}$, $t = \frac{5}{x} + \frac{2}{x^5}$, we get

$$\begin{aligned} f(x) &= x\left[1+\frac{1}{3}\left(\frac{3}{x}+\frac{1}{x^3}\right) - \binom{\frac{1}{3}}{2}\left(\frac{3}{x}+\frac{1}{x^3}\right)^2 + o\left(\frac{1}{x^2}\right) + \right. \\ &\qquad \left. -1 - \frac{1}{5}\left(\frac{5}{x}+\frac{2}{x^5}\right) - \binom{\frac{1}{5}}{2}\left(\frac{5}{x}+\frac{2}{x^5}\right)^2 + o\left(\frac{1}{x^2}\right)\right] \\ &= x\left(\frac{1}{x}+\frac{1}{3x^3}-\frac{1}{x^2}-\frac{1}{x}-\frac{2}{5x^5}+\frac{2}{x^2}+o\left(\frac{1}{x^2}\right)\right) \\ &= x\left(\frac{1}{x^2}+o\left(\frac{1}{x^2}\right)\right) = \frac{1}{x}+o\left(\frac{1}{x}\right). \end{aligned}$$

Therefore the order is 1, and $p(x) = \frac{1}{x}$.

d) The order is 2 and $p(x) = \frac{\sqrt[3]{2}}{3x^2}$.

6. *Limits:*

a) Let us rewrite as

$$\begin{aligned}\lim_{x\to 0}(1+x^6)^{1/(x^4\sin^2 3x)} &= \lim_{x\to 0}\exp\left(\frac{1}{x^4\sin^2 3x}\ \log(1+x^6)\right)\\ &= \exp\left(\lim_{x\to 0}\frac{\log(1+x^6)}{x^4\sin^2 3x}\right) = \mathrm{e}^L.\end{aligned}$$

To compute L, take the expansions of $\log(1+t)$ and $\sin t$:

$$L = \lim_{x\to 0}\frac{x^6+o(x^6)}{x^4(3x+o(x^2))^2} = \lim_{x\to 0}\frac{x^6+o(x^6)}{9x^6+o(x^6)} = \frac{1}{9}.$$

The required limit is $\mathrm{e}^{1/9}$.

b) $\frac{3}{256}\pi$.

c) Expanding the sine and tangent,

$$\begin{aligned}L &= \lim_{x\to 0}\frac{x-\sin(\tan x)}{x^2\sin(\tan x)} = \lim_{x\to 0}\frac{x-\tan x+\frac{1}{6}\tan^3 x+o(x^3)}{x^2(\tan x+o(x))}\\ &= \lim_{x\to 0}\frac{x-x-\frac{1}{3}x^3+\frac{1}{6}x^3+o(x^3)}{x^3+o(x^3)} = \lim_{x\to 0}\frac{-\frac{1}{6}x^3+o(x^3)}{x^3+o(x^3)} = -\frac{1}{6}.\end{aligned}$$

d) $\mathrm{e}^{-2/3}$; e) -1.

f) Observe that

$$3x^4[\log(1+\sinh^2 x)]\cosh^2 x \sim 3x^4\sinh^2 x \sim 3x^6\,.$$

for $x\to 0$. Moreover, the denominator can be written as

$$\begin{aligned}\text{Den}\ :\ & 1-(1+x^3)^{1/2}\cos x^{3/2}\\ &= 1-\left(1+\frac{1}{2}x^3+\binom{1/2}{2}x^6+o(x^6)\right)\left(1-\frac{1}{2}x^3+\frac{1}{4!}x^6+o(x^6)\right)\\ &= 1-\left(1+\frac{1}{2}x^3-\frac{1}{8}x^6-\frac{1}{2}x^3-\frac{1}{4}x^6+\frac{1}{24}x^6+o(x^6)\right) = \frac{1}{3}x^6+o(x^6).\end{aligned}$$

The limit is thus

$$\lim_{x\to 0}\frac{3x^6+o(x^6)}{\frac{1}{3}x^6+o(x^6)} = 9.$$

7. Expand $\log(1+t)$, $\cos t$, $\cosh t$, so that

$$h(x) = \log\left(1-\frac{1}{2}x^2+\frac{1}{4!}x^4+o(x^5)\right)+\log\left(1+\frac{1}{2}(ax)^2+\frac{1}{4!}(ax)^4+o(x^5)\right)$$

$$= -\frac{1}{2}x^2 + \frac{1}{4!}x^4 - \frac{1}{2}\left(-\frac{1}{2}x^2 + \frac{1}{4!}x^4\right)^2 + o(x^5) + \frac{a^2}{2}x^2 + \frac{a^4}{4!}x^4 - \\ -\frac{1}{2}\left(\frac{a^2}{2}x^2 + \frac{a^4}{4!}x^4\right)^2 + o(x^5)$$

$$= \frac{1}{2}(a^2-1)x^2 + \left(\frac{1}{4!} - \frac{1}{8}\right)(a^4+1)x^4 + o(x^5).$$

If $a \neq \pm 1$, $h(x)$ is infinitesimal of order 2 for $x \to 0$. If $a = \pm 1$ the first non-zero coefficient multiplies x^4, making h infinitesimal of order 4 for $x \to 0$.

8. In order to compute $h^{(6)}(x)$ at $x = 0$ we use the fact that the Maclaurin coefficient of x^6 is $a_6 = \frac{h^{(6)}(0)}{6!}$. Therefore we need the expansion up to order six. Working on $\sin t$ and $\sinh t$, the numerator of h becomes

$$\begin{aligned}\text{Num} \ : \ &\sinh\left(x^2 + 2\left(x^4 - \frac{4}{3!}x^6 + o(x^6)\right)\right) \\ &= \sinh\left(x^2 + 2x^4 - \frac{4}{3}x^6 + o(x^6)\right) = x^2 + 2x^4 - \frac{4}{3}x^6 + \frac{1}{3!}x^6 + o(x^6) \\ &= x^2 + 2x^4 - \frac{7}{6}x^6 + o(x^6).\end{aligned}$$

Dividing $x^2 + 2x^4 - \frac{7}{6}x^6 + o(x^6)$ by $1 + x^{10}$ one finds

$$h(x) = x^2 + 2x^4 - \frac{7}{6}x^6 + o(x^6),$$

so $h^{(6)}(0) = -\frac{7}{6} \cdot 6! = -840$.

9. Use the expansions of $\log(1+t)$ and $\sinh t$ to write

$$\varphi(x) = 4x - \frac{1}{2}(4x)^2 + \frac{1}{3}(4x)^3 - 4x - \frac{1}{3!}(4x)^3 + 8x^2 + o(x^3) = \frac{32}{3}x^3 + o(x^3).$$

Since the sine has the same sign as its argument around the origin, the function $y = \sin\varphi(x)$ is negative for $x < 0$ and positive for $x > 0$.

10. Using $\cos t$ in Maclaurin's form,

$$\begin{aligned}2\cos(x+x^2) &= 2\left(1 - \frac{1}{2}(x+x^2)^2 + \frac{1}{4!}(x+x^2)^4 + o\left((x+x^2)^4\right)\right) \\ &= 2 - (x^2 + 2x^3 + x^4) + \frac{1}{3\cdot 4}x^4 + o(x^4) \\ &= 2 - x^2 - 2x^3 - \frac{11}{12}x^4 + o(x^4)\end{aligned}$$

on some neighbourhood I of the origin. Then the given inequality holds on I, because the principal part of the difference between right- and left-hand side, clearly negative, equals $-\frac{11}{12}x^4$.

11. Expand numerator and denominator separately as

$$\begin{aligned}
\text{Num} \;:\; & 1+\frac{1}{2}x+\frac{1}{2}\left(\frac{x}{2}\right)^2+o(x^2)-\left(1+\frac{1}{2}x+\frac{1}{4!}x^2+o(x^2)\right) \\
&= \left(\frac{1}{8}-\frac{1}{4!}\right)x^2+o(x^2)=\frac{1}{12}x^2+o(x^2)\,, \\
\text{Den} \;:\; & \left[x^{1/5}\left(1+x^{4/5}\right)\right]^\alpha \\
&= x^{\alpha/5}\left(1+x^{4/5}\right)^\alpha = x^{\alpha/5}\left(1+\alpha x^{4/5}+o(x^{4/5})\right).
\end{aligned}$$

Then

$$\begin{aligned}
\lim_{x\to 0^+}\frac{e^{x/2}-\cosh\sqrt{x}}{(x+\sqrt[5]{x})^\alpha} &= \lim_{x\to 0^+}\frac{\frac{1}{12}x^2+o(x^2)}{x^{\alpha/5}\left(1+\alpha x^{4/5}+o(x^{4/5})\right)} \\
&= \begin{cases} \frac{1}{12} & \text{if } 2=\frac{\alpha}{5}\,, \\ 0 & \text{if } 2>\frac{\alpha}{5}\,, \\ +\infty & \text{if } 2<\frac{\alpha}{5} \end{cases} = \begin{cases} \frac{1}{12} & \text{if } \alpha=10\,, \\ 0 & \text{if } \alpha<10\,, \\ +\infty & \text{if } \alpha>10. \end{cases}
\end{aligned}$$

12. Writing $\arctan t$ and $\sin t$ in Maclaurin's form provides

$$\begin{aligned}
f(x) &= \left(2x-\frac{1}{3}(2x)^3+o(x^3)\right)^2-\alpha x\left(x-\frac{1}{6}x^3+o(x^3)\right) \\
&= 4x^2-\frac{32}{3}x^4+o(x^4)-\alpha x^2+\frac{\alpha}{6}x^4+o(x^4) \\
&= (4-\alpha)x^2-\left(\frac{32}{3}-\frac{\alpha}{6}\right)x^4+o(x^4).
\end{aligned}$$

This proves $f(x)$ infinitesimal of the fourth order at the origin if $\alpha=4$. For such value in fact,

$$f(x)=10x^4+o(x^4).$$

8

Geometry in the plane and in space

The chapter has two main goals. The first is to discuss the possibilities of representing objects in the plane and in three-dimensional space; in this sense we can think of this as an ideal continuation of Chap. 1. We shall introduce coordinate systems other than the Cartesian system, plus vectors and their elementary properties, and then the set $\mathbb{C}$ of complex numbers. Secondly, it is a good occasion for introducing concepts that will be dealt with in more depth during other lecture courses, for instance functions of several variables, or the theory of curves in space.

8.1 Polar, cylindrical, and spherical coordinates

A point P in the Cartesian plane can be described, apart from using the known coordinates (x, y), by **polar coordinates** (r,θ), which are defined as follows. Denote by r the distance of P from the origin O. If $r > 0$ we let θ be the angle, measured in radians up to multiples of 2π, between the positive x-axis and the half-line emanating from O and passing through P, as in Fig. 8.1. It is common to

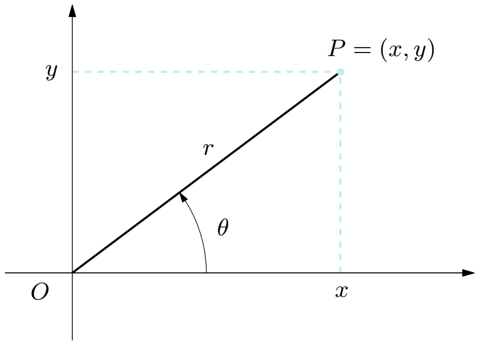


Figure 8.1. Polar and Cartesian coordinates in the plane

choose θ in $(-\pi, \pi]$, or in $[0, 2\pi)$. When $r = 0$, P coincides with the origin, and θ may be any number.

The passage from polar coordinates (r, θ) to Cartesian coordinates (x, y) is given by

$$x = r\cos\theta\,, \qquad\qquad y = r\sin\theta\,. \tag{8.1}$$

The inverse transformation, provided θ is chosen in the interval $(-\pi, \pi]$, is

$$r = \sqrt{x^2 + y^2}\,, \qquad\qquad \theta = \begin{cases} \arctan\dfrac{y}{x} & \text{if } x > 0\,, \\ \arctan\dfrac{y}{x} + \pi & \text{if } x < 0,\ y \geq 0\,, \\ \arctan\dfrac{y}{x} - \pi & \text{if } x < 0,\ y < 0\,, \\ \dfrac{\pi}{2} & \text{if } x = 0,\ y > 0\,, \\ -\dfrac{\pi}{2} & \text{if } x = 0,\ y < 0\ . \end{cases} \tag{8.2}$$

Examples 8.1

i) Let P have Cartesian coordinates $(x, y) = (6\sqrt{2}, 2\sqrt{6})$. Its distance from the origin is

$$r = \sqrt{72 + 24} = \sqrt{96} = 4\sqrt{6}\,.$$

As $x > 0$,

$$\theta = \arctan\frac{2\sqrt{6}}{6\sqrt{2}} = \arctan\frac{\sqrt{3}}{3} = \frac{\pi}{6}\,.$$

The polar coordinates of P are then $(r, \theta) = \big(4\sqrt{6}, \frac{\pi}{6}\big)$.

ii) Let now P have Cartesian coordinates $(x, y) = (-5, -5)$. Then $r = 5\sqrt{2}$, and since $x, y < 0$,

$$\theta = \arctan\frac{-5}{-5} - \pi = \arctan 1 - \pi = \frac{\pi}{4} - \pi = -\frac{3}{4}\pi$$

whence $(r, \theta) = \big(5\sqrt{2}, -\frac{3}{4}\pi\big)$.

iii) Take P of polar coordinates $(r, \theta) = \big(4, \frac{2}{3}\pi\big)$ this time; in the Cartesian system

$$x = 4\cos\frac{2}{3}\pi = 4\cos\big(\pi - \frac{\pi}{3}\big) = -4\cos\frac{\pi}{3} = -2\,,$$
$$y = 4\sin\frac{2}{3}\pi = 4\sin\big(\pi - \frac{\pi}{3}\big) = 4\sin\frac{\pi}{3} = 2\sqrt{3}\,. \qquad \square$$

Moving on to the representation of a point $P \in \mathbb{R}^3$ of coordinates (x, y, z), we shall introduce two new frame systems: cylindrical coordinates and spherical coordinates.

The cylindrical system is simply given by replacing the coordinates (x, y) of the point P', orthogonal projection of P on the xy-plane, by its polar ones (r', θ), and mantaining z as it is. Denoting (r', θ, t) the **cylindrical coordinates** of P, we have

$$x = r' \cos\theta\,, \qquad y = r' \sin\theta\,, \qquad z = t\,.$$

In this case too the angle θ is defined up to multiples of 2π; if we confine θ to the interval $(-\pi, \pi]$, as above, cylindrical coordinates are functions of the Cartesian ones by defining r' and θ with (8.2) (Fig. 8.2, left).

Spherical coordinates (r, φ, θ) are defined as follows. Let $r = \sqrt{x^2 + y^2 + z^2}$ be the distance of P from the origin, φ the angle between the positive z-axis and the ray from O through P, θ the angle between the positive x-axis and the line in the xy-plane passing through the origin and the projection P' of P on the same plane. This is probably better understood by looking at Fig. 8.2, right. Borrowing terms from geography, one calls θ the **longitude** and φ the **colatitude** of P (whereas $\frac{\pi}{2} - \varphi$ is the **latitude**, in radians).

Therefore $z = r\cos\varphi$, while the expressions $x = r' \cos\theta$ and $y = r' \sin\theta$ derive from noting that r' is the distance of P' from O, $r' = r\sin\varphi$. Then the Cartesian coordinates of P are, in terms of the spherical triple (r, φ, θ),

$$x = r\sin\varphi\cos\theta\,, \qquad y = r\sin\varphi\sin\theta\,, \qquad z = r\cos\varphi\,.$$

The inverse transformation is easily found by dimensional reduction. We just remark that it is enough to vary φ in an interval of width π, e.g. $[0, \pi]$. Instead, θ has freedom 2π, for instance $\theta \in (-\pi, \pi]$, as in the 2-dimensional case.

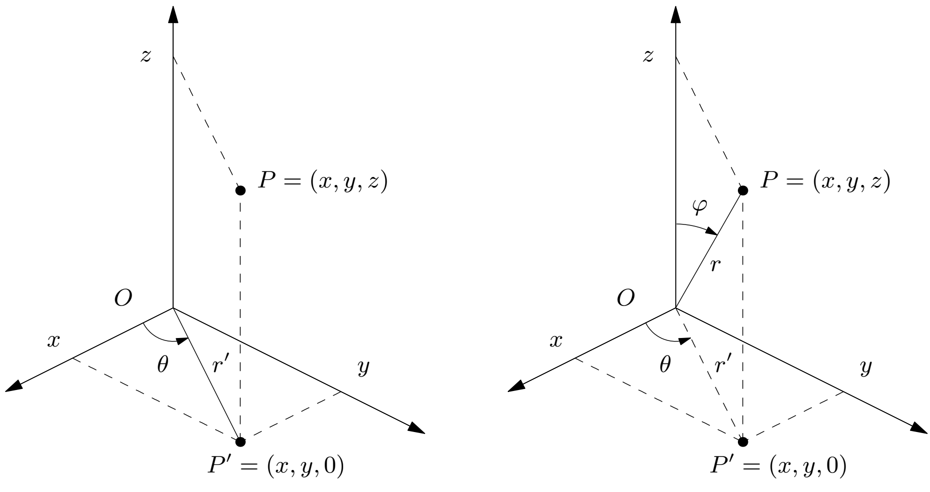


Figure 8.2. Cylindrical coordinates (left) and spherical coordinates (right)

Example 8.2

Consider the point P of Cartesian coordinates $(1, 1, \sqrt{6})$. The point $P' = (1, 1, 0)$ is the orthogonal projection of P onto the xy-plane, so its polar coordinates are $(r', \theta) = \left(\sqrt{2}, \frac{\pi}{4}\right)$ in that plane. The cylindrical coordinates of P are therefore $(r', \theta, t) = \left(\sqrt{2}, \frac{\pi}{4}, \sqrt{6}\right)$.

Now to spherical coordinates. First, $r = \sqrt{1+1+6} = 2\sqrt{2}$; moreover, $\sin\varphi = \frac{\sqrt{2}}{2\sqrt{2}} = \frac{1}{2}$ implies $\varphi = \pi/6$, because φ varies in $[0, \pi]$. Therefore P has coordinates $(r, \theta, \varphi) = \left(2\sqrt{2}, \frac{\pi}{4}, \frac{\pi}{6}\right)$. □

8.2 Vectors in the plane and in space

We discuss the basics of Vector Calculus, which focuses on vectors and how they add, multiply, and so on. We start with vectors whose initial point is the origin, and later generalise this situation to arbitrary initial points in the plane or in space.

8.2.1 Position vectors

Equip the plane with an orthogonal frame system. A pair $(x, y) \neq (0, 0)$ in $\mathbb{R}^2$ identifies a **position vector** (or just **vector**) $\boldsymbol{v}$ in the plane, given by the line segment with initial point $O = (0, 0)$ and end point $P = (x, y)$, see Fig. 8.3, left. (The orientation from O to P is indicated by an arrow with point at P.)

The coordinates x, y of P are said **components** of the vector $\boldsymbol{v}$ (in the chosen frame system); one writes $\boldsymbol{v} = (x, y)$, identifying the vector $\boldsymbol{v}$ with its end point P.

Position vectors in space are defined in a similar fashion: a vector $\boldsymbol{v}$ with components $(x, y, z) \neq (0, 0, 0)$ is drawn as the oriented segment going from $O = (0, 0, 0)$ to $P = (x, y, z)$ (Fig. 8.3, right), so one writes $\boldsymbol{v} = (x, y, z)$.

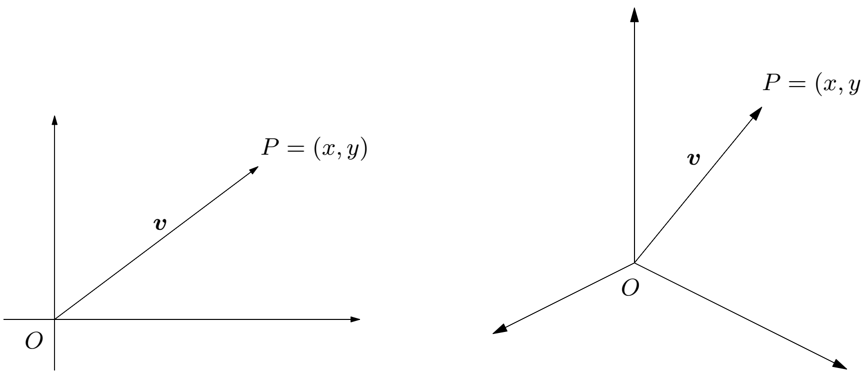


Figure 8.3. A vector in the plane (left), and in space (right)

In space or in the plane, the vector $\mathbf{0}$ with components all zero is called the **zero vector**; it is identified with the origin and has no arrow. In this way position vectors in the plane (or in space) are in bijective correspondence with points of $\mathbb{R}^2$ (resp. $\mathbb{R}^3$). Henceforth we shall not specify every time whether we are talking about planar or spatial vectors: the generic $\boldsymbol{v}$, of components (v_1, v_2) or (v_1, v_2, v_3), will be described with $(v_1, \ldots, v_d)$. The capital letter V will be the set of vectors of the plane or of space, with no distinction.

Having fixed the origin point O, a vector is intrinsically determined (irrespective of the chosen Cartesian frame) by a **direction**, the straight line through the origin and containing the vector, an **orientation**, the direction given by the arrow, and a **length** or **norm**, the actual length of the segment OP. Rather often the notion of direction tacitly includes an orientation as well.

Let us define operations. Take vectors $\boldsymbol{v} = (v_1, \ldots, v_d)$ and $\boldsymbol{w} = (w_1, \ldots, w_d)$. The **sum** of $\boldsymbol{v}$ and $\boldsymbol{w}$ is the vector $\boldsymbol{v} + \boldsymbol{w}$ whose components are given by the sum of the corresponding (i.e., with the same subscript) components of the two original vectors

$$\boldsymbol{v} + \boldsymbol{w} = (v_1 + w_1, \ldots, v_d + w_d)\,. \tag{8.3}$$

In Vector Calculus real numbers $\lambda \in \mathbb{R}$ are referred to as **scalars**. The **product of the vector $\boldsymbol{v}$ by (the scalar) λ** is the vector $\lambda\boldsymbol{v}$, whose jth component is the product of the jth component of $\boldsymbol{v}$ by λ

$$\lambda\boldsymbol{v} = (\lambda v_1, \ldots, \lambda v_d)\,. \tag{8.4}$$

The product $(-1)\boldsymbol{v}$ is denoted $-\boldsymbol{v}$ and said **opposite vector** to $\boldsymbol{v}$. The difference $\boldsymbol{v} - \boldsymbol{w}$ is defined as

$$\boldsymbol{v} - \boldsymbol{w} = \boldsymbol{v} + (-\boldsymbol{w}) = (v_1 - w_1, \ldots, v_d - w_d)\,. \tag{8.5}$$

The operations just introduced enjoy the familiar properties of the sum and the product (associative, commutative, distributive, ...), due to their component-wise nature.

These operations have also a neat geometric interpretation. If $\lambda > 0$, the vector $\lambda\boldsymbol{v}$ has the same direction (and orientation) as $\boldsymbol{v}$, i.e., it lies on the same (oriented) straight line, and its length is λ times the length of $\boldsymbol{v}$ (see Fig. 8.4); if $\lambda < 0$, $\lambda\boldsymbol{v} = -|\lambda|\boldsymbol{v} = |\lambda|(-\boldsymbol{v})$ so the same argument applies to $-\boldsymbol{v}$. Two vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ are **parallel**, or **collinear**, if $\boldsymbol{w} = \lambda\boldsymbol{v}$ for a $\lambda \neq 0$.

The sum of non-zero position vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ should be understood as follows. When the two vectors are collinear, $\boldsymbol{w} = \lambda\boldsymbol{v}$, then $\boldsymbol{v} + \boldsymbol{w} = (1 + \lambda)\boldsymbol{v}$, parallel to both of them. Otherwise, $\boldsymbol{v}$ and $\boldsymbol{w}$ lie on distinct straight lines, say r_v and r_w, that meet at the origin. Let Π be the plane determined by these lines (if $\boldsymbol{v}$ and $\boldsymbol{w}$ are vectors in the plane, clearly Π is the plane); $\boldsymbol{v}$ and $\boldsymbol{w}$ determine a parallelogram on Π (Fig. 8.5). Precisely, let P, Q be the end points of $\boldsymbol{v}$ and $\boldsymbol{w}$. The parallelogram in question is then enclosed by the lines r_v, r_w, the parallel to r_w through P and

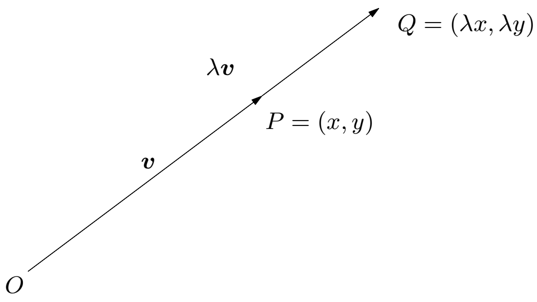


Figure 8.4. The vectors $\boldsymbol{v}$ and $\lambda \boldsymbol{v}$

the parallel to r_v through Q; its vertices are O, P, Q and R, the vertex 'opposite' the origin. The sum $\boldsymbol{v} + \boldsymbol{w}$ is then the diagonal $\overline{OR}$, oriented from O to R. The vertex R can be reached by 'moving' along the sides: for instance, we can start at P and draw a segment parallel to $\overline{OQ}$, having the same length, and lying on the same side with respect to r_v.

Figure 8.6 represents the difference $\boldsymbol{v}-\boldsymbol{w}$: the position vector $\boldsymbol{v}-\boldsymbol{w} = \boldsymbol{v}+(-\boldsymbol{w})$ is the diagonal of the parallelogram determined by vectors $\boldsymbol{v}$, $-\boldsymbol{w}$. Alternatively, we can take the diagonal $\overline{QP}$ and displace it 'rigidly' to the origin, i.e., keeping it parallel to itself, finding $\boldsymbol{v} - \boldsymbol{w}$.

The set V of vectors (in the plane or in space), equipped with the operations of sum and multiplication by a scalar, is an example of a **vector space** over $\mathbb{R}$. Any vector $\boldsymbol{v} = \lambda \boldsymbol{v}_1 + \mu \boldsymbol{v}_2$, with $\boldsymbol{v}_1, \boldsymbol{v}_2 \in V$ and $\lambda, \mu \in \mathbb{R}$ is called a **linear combination** of the two vectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$. This generalises to linear combinations of a finite number of vectors.

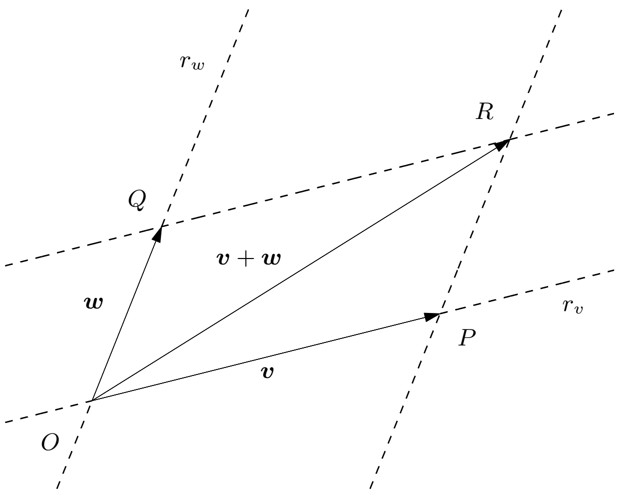


Figure 8.5. Sum of two vectors $\boldsymbol{v} + \boldsymbol{w}$

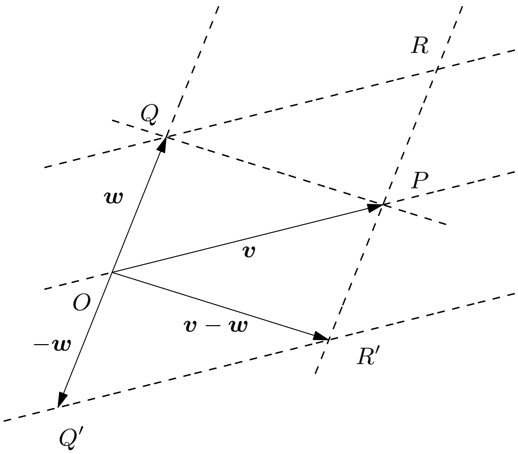


Figure 8.6. Difference vector $\boldsymbol{v} - \boldsymbol{w}$

Examples 8.3

i) Given vectors $\boldsymbol{v}_1 = (2, 5, -4)$ and $\boldsymbol{v}_2 = (-1, 3, 0)$, the sum $\boldsymbol{v} = 3\boldsymbol{v}_1 - 5\boldsymbol{v}_2$ is $\boldsymbol{v} = (11, 0, -12)$.

ii) The vectors $\boldsymbol{v} = (\sqrt{8}, -2, 2\sqrt{5})$ and $\boldsymbol{w} = (2, -\sqrt{2}, \sqrt{10})$ are parallel, since the ratios of the corresponding components is always the same:

$$\frac{\sqrt{8}}{2} = \frac{-2}{-\sqrt{2}} = \frac{2\sqrt{5}}{\sqrt{10}} = \sqrt{2}\,;$$

hence $\boldsymbol{v} = \sqrt{2}\,\boldsymbol{w}$. □

8.2.2 Norm and scalar product

The **norm** of a position vector $\boldsymbol{v}$ with end point P is defined, we recall, as the length of $\overline{OP}$, i.e., the Euclidean distance of P to the origin. It is denoted by the symbol $\|\boldsymbol{v}\|$ and can be expressed in terms of $\boldsymbol{v}$'s components like

$$\|\boldsymbol{v}\| = \sqrt{\sum_{i=1}^{d} v_i^2} = \begin{cases} \sqrt{v_1^2 + v_2^2} & \text{if } d = 2\,, \\ \sqrt{v_1^2 + v_2^2 + v_3^2} & \text{if } d = 3\,. \end{cases}$$

The norm of a vector is always non-negative, and moreover $\|\boldsymbol{v}\| = 0$ if and only if $\boldsymbol{v} = \boldsymbol{0}$. The following relations hold, proof of which will be given on p. 269:

$$\|\lambda \boldsymbol{v}\| = |\lambda|\, \|\boldsymbol{v}\|\,, \qquad \|\boldsymbol{v} + \boldsymbol{w}\| \le \|\boldsymbol{v}\| + \|\boldsymbol{w}\| \tag{8.6}$$

for any $v, w \in V$ and any $\lambda \in \mathbb{R}$.

A vector of norm 1 is called **unit vector**, and geometrically, it has end point P lying on the unit circle or unit sphere centred at the origin. Each vector $\boldsymbol{v}$ has a corresponding unit vector $\hat{\boldsymbol{v}} = \frac{\boldsymbol{v}}{\|\boldsymbol{v}\|}$, parallel to $\boldsymbol{v}$. Thus $\boldsymbol{v} = \|\boldsymbol{v}\|\,\hat{\boldsymbol{v}}$, showing that any vector can be represented as the product of a unit vector by its own length.

Let us introduce the operation known as **scalar product**, or **dot product** of two vectors. Given $\boldsymbol{v} = (v_1, \ldots, v_d)$ and $\boldsymbol{w} = (w_1, \ldots, w_d)$, their dot product is the real number

$$\boldsymbol{v} \cdot \boldsymbol{w} = \sum_{i=1}^{d} v_i w_i = \begin{cases} v_1 w_1 + v_2 w_2 & \text{if } d = 2\,, \\ \\ v_1 w_1 + v_2 w_2 + v_3 w_3 & \text{if } d = 3\,. \end{cases}$$

Easy-to-verify properties are:

$$\boldsymbol{v} \cdot \boldsymbol{w} = \boldsymbol{w} \cdot \boldsymbol{v}\,, \tag{8.7}$$

$$(\lambda \boldsymbol{v_1} + \mu \boldsymbol{v_2}) \cdot \boldsymbol{w} = \lambda(\boldsymbol{v_1} \cdot \boldsymbol{w}) + \mu(\boldsymbol{v_2} \cdot \boldsymbol{w})\,. \tag{8.8}$$

for any $\boldsymbol{v}, \boldsymbol{w}, \boldsymbol{v_1}, \boldsymbol{v_2} \in V, \lambda, \mu \in \mathbb{R}$.

A vector's norm may be defined from the scalar product, as

$$\|\boldsymbol{v}\| = \sqrt{\boldsymbol{v} \cdot \boldsymbol{v}} \tag{8.9}$$

for any $\boldsymbol{v} \in V$. Vice versa, for any $\boldsymbol{v}, \boldsymbol{w} \in V$, one has

$$\boldsymbol{v} \cdot \boldsymbol{w} = \frac{1}{2}\big(\|\boldsymbol{v} + \boldsymbol{w}\|^2 - \|\boldsymbol{v}\|^2 - \|\boldsymbol{w}\|^2\big)\,, \tag{8.10}$$

which allows to compute scalar products using norms (see p. 269 for the proof).

Furthermore, a fundamental relation, known as **Cauchy-Schwarz inequality**, holds: for every $\boldsymbol{v}, \boldsymbol{w} \in V$

$$|\boldsymbol{v} \cdot \boldsymbol{w}| \le \|\boldsymbol{v}\|\,\|\boldsymbol{w}\|\,. \tag{8.11}$$

Even more precisely,

$$\boldsymbol{v} \cdot \boldsymbol{w} = \|\boldsymbol{v}\|\,\|\boldsymbol{w}\|\,\cos\theta \tag{8.12}$$

where θ is the angle formed by $\boldsymbol{v}$ and $\boldsymbol{w}$ (whether θ is the clockwise, anti-clockwise, acute or obtuse angle is completely irrelevant, for $\cos\theta = \cos(-\theta) = \cos(2\pi - \theta)$). Formulas (8.11) and (8.12) as well will be proved later.

The dot product leads to the notion of orthogonality. Two vectors $\boldsymbol{v}$, $\boldsymbol{w}$ are said **orthogonal** (or **perpendicular**) if

$$\boldsymbol{v} \cdot \boldsymbol{w} = 0\,;$$

formula (8.12) tells that two vectors are orthogonal when either one is the zero vector, or the angle between them is a right angle. By (8.10), the orthogonality of $\boldsymbol{v}$ and $\boldsymbol{w}$ is equivalent with

$$\|\boldsymbol{v} + \boldsymbol{w}\|^2 = \|\boldsymbol{v}\|^2 + \|\boldsymbol{w}\|^2\,,$$

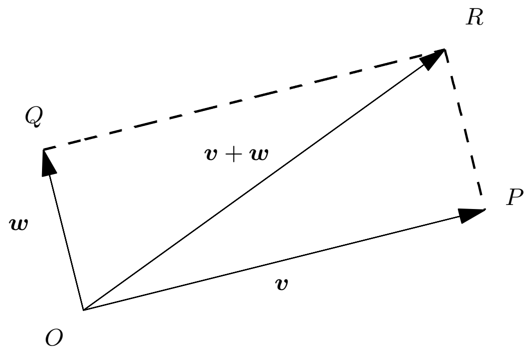


Figure 8.7. Pythagoras's Theorem

well known to the reader under the name Pythagoras's Theorem (Fig. 8.7).

Given a vector $\boldsymbol{v}$ and a unit vector $\boldsymbol{u}$, the **component** of $\boldsymbol{v}$ **along** $\boldsymbol{u}$ is the vector

$$\boldsymbol{v}_u = (\boldsymbol{v} \cdot \boldsymbol{u})\, \boldsymbol{u}\,,$$

while the **component** of $\boldsymbol{v}$ **orthogonal to** $\boldsymbol{u}$ is the complement

$$\boldsymbol{v}_{u^\perp} = \boldsymbol{v} - \boldsymbol{v}_u\,.$$

Therefore the vector $\boldsymbol{v}$ splits as a sum

$$\boldsymbol{v} = \boldsymbol{v}_u + \boldsymbol{v}_{u^\perp} \qquad \text{with} \qquad \boldsymbol{v}_u \cdot \boldsymbol{v}_{u^\perp} = 0\,, \tag{8.13}$$

a relation called **orthogonal decomposition** of $\boldsymbol{v}$ with respect to the unit vector $\boldsymbol{u}$ (Fig. 8.8).

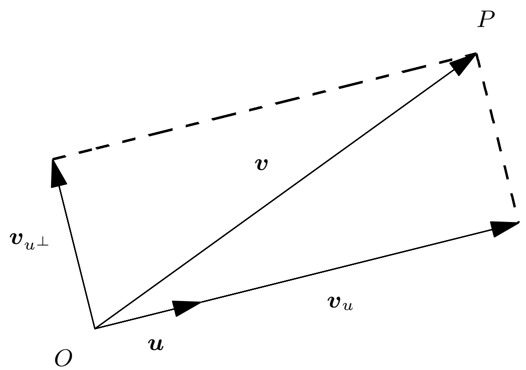


Figure 8.8. Orthogonal decomposition of $\boldsymbol{v}$ with respect to the unit vector $\boldsymbol{u}$

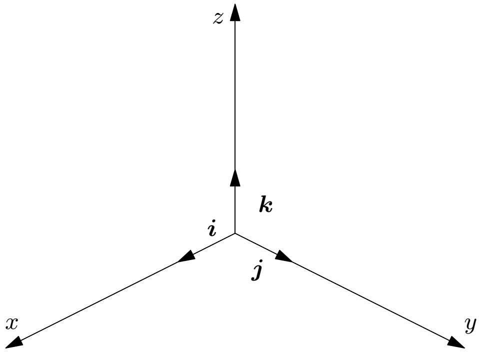


Figure 8.9. The unit vectors $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$

Examples 8.4

i) $\boldsymbol{v} = (1, 0, \sqrt{3})$ and $\boldsymbol{w} = (1, 2, \sqrt{3})$ have norm

$$\|\boldsymbol{v}\| = \sqrt{1+0+3} = 2\,, \qquad \|\boldsymbol{w}\| = \sqrt{1+4+3} = 2\sqrt{2}\,;$$

their scalar product is $\boldsymbol{v} \cdot \boldsymbol{w} = 1 + 0 + 3 = 4$.

To compute the angle θ they form, we recover from (8.12)

$$\cos\theta = \frac{\boldsymbol{v} \cdot \boldsymbol{w}}{\|\boldsymbol{v}\|\,\|\boldsymbol{w}\|} = \frac{\sqrt{2}}{2},$$

so $\theta = \frac{\pi}{4}$.

ii) The vectors $\boldsymbol{v} = (1, 2, -1)$, $\boldsymbol{w} = (-1, 1, 1)$ are orthogonal since $\boldsymbol{v} \cdot \boldsymbol{w} = -1 + 2 - 1 = 0$.

iii) Take the unit vector $\boldsymbol{u} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$. Given $\boldsymbol{v} = (3, 1, 1)$, we have

$$\boldsymbol{v} \cdot \boldsymbol{u} = \sqrt{3} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{3}} = \sqrt{3},$$

so the component of $\boldsymbol{v}$ along $\boldsymbol{u}$ is

$$\boldsymbol{v}_u = \sqrt{3}\Big(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\Big) = (1, 1, -1)\,,$$

while the orthogonal component reads

$$\boldsymbol{v}_{u^\perp} = (3, 1, 1) - (1, 1, -1) = (2, 0, 2)\,.$$

That (8.13) holds is now easy to check. □

We introduce the unit vectors $\boldsymbol{i} = (1, 0, 0)$, $\boldsymbol{j} = (0, 1, 0)$ and $\boldsymbol{k} = (0, 0, 1)$ of space, which are parallel to the axes of the Cartesian frame (Fig. 8.9); at times these unit vectors are denoted $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$. They are clearly pairwise orthogonal

$$\boldsymbol{i} \cdot \boldsymbol{j} = \boldsymbol{j} \cdot \boldsymbol{k} = \boldsymbol{i} \cdot \boldsymbol{k} = 0\,. \tag{8.14}$$

They form a so-called **orthonormal frame** for V (by definition, a set of pairwise orthogonal unit vectors).

Let $\boldsymbol{v} = (v_1, v_2, v_3)$ be arbitrary. Since

$$\begin{aligned}\boldsymbol{v} &= (v_1, 0, 0) + (0, v_2, 0) + (0, 0, v_3) \\ &= v_1(1, 0, 0) + v_2(0, 1, 0) + v_3(0, 0, 1)\end{aligned}$$

we write

$$\boldsymbol{v} = v_1\boldsymbol{i} + v_2\boldsymbol{j} + v_3\boldsymbol{k}\,.$$

This explains that any vector in space can be represented as a linear combination of the unit vectors $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$, whence the latter triple forms an **orthonormal basis** of V. The dot product of $\boldsymbol{v}$ with the orthonormal vectors $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ yields the components of $\boldsymbol{v}$

$$v_1 = \boldsymbol{v} \cdot \boldsymbol{i}\,, \qquad v_2 = \boldsymbol{v} \cdot \boldsymbol{j}\,, \qquad v_3 = \boldsymbol{v} \cdot \boldsymbol{k}\,.$$

Summarising, a generic vector $\boldsymbol{v} \in V$ admits the representation

$$\boldsymbol{v} = (\boldsymbol{v} \cdot \boldsymbol{i})\,\boldsymbol{i} + (\boldsymbol{v} \cdot \boldsymbol{j})\,\boldsymbol{j} + (\boldsymbol{v} \cdot \boldsymbol{k})\,\boldsymbol{k}\,. \tag{8.15}$$

Similarly, planar vectors can be represented by

$$\boldsymbol{v} = (\boldsymbol{v} \cdot \boldsymbol{i})\,\boldsymbol{i} + (\boldsymbol{v} \cdot \boldsymbol{j})\,\boldsymbol{j}$$

with respect to the orthonormal basis made by $\boldsymbol{i} = (1, 0)$ and $\boldsymbol{j} = (0, 1)$.

Proofs of some formulas above

Proof. We start from (8.6). The equality follows from the definition of norm. The inequality descends from the definition in case $\boldsymbol{v}$ and $\boldsymbol{w}$ are collinear; for generic $\boldsymbol{v}, \boldsymbol{w}$ instead, it states a known property of triangles, according to which any side is shorter than the sum of the other two. In the triangle OPR of Fig. 8.5 in fact, $\|\boldsymbol{v} + \boldsymbol{w}\| = |\overline{OR}|$, $\|\boldsymbol{v}\| = |\overline{OP}|$ and $\|\boldsymbol{w}\| = |\overline{PR}|$.

Formula (8.10) derives from expanding $\|\boldsymbol{v} + \boldsymbol{w}\|^2$ using (8.7)–(8.9) as follows:

$$\begin{aligned}\|\boldsymbol{v} + \boldsymbol{w}\|^2 &= (\boldsymbol{v} + \boldsymbol{w}) \cdot (\boldsymbol{v} + \boldsymbol{w}) \\ &= \boldsymbol{v} \cdot \boldsymbol{v} + \boldsymbol{w} \cdot \boldsymbol{v} + \boldsymbol{v} \cdot \boldsymbol{w} + \boldsymbol{w} \cdot \boldsymbol{w} \\ &= \|\boldsymbol{v}\|^2 + 2\boldsymbol{v} \cdot \boldsymbol{w} + \|\boldsymbol{w}\|^2\,.\end{aligned} \tag{8.16}$$

The Cauchy-Schwarz inequality (8.11) can be proved by writing the second of (8.6) as $\|\boldsymbol{v} + \boldsymbol{w}\|^2 \leq \big(\|\boldsymbol{v}\| + \|\boldsymbol{w}\|\big)^2$. For the left-hand side we use (8.16), so that $\boldsymbol{v} \cdot \boldsymbol{w} \leq \|\boldsymbol{v}\|\,\|\boldsymbol{w}\|$; but the latter is (8.11) in case $\boldsymbol{v} \cdot \boldsymbol{w} \geq 0$. When $\boldsymbol{v} \cdot \boldsymbol{w} < 0$, it suffices to flip the sign of $\boldsymbol{v}$, to the effect that

$$|\boldsymbol{v} \cdot \boldsymbol{w}| = -\boldsymbol{v} \cdot \boldsymbol{w} = (-\boldsymbol{v}) \cdot \boldsymbol{w} \leq \|-\boldsymbol{v}\|\,\|\boldsymbol{w}\| = \|\boldsymbol{v}\|\,\|\boldsymbol{w}\|\,.$$

Eventually, let us prove (8.12). Suppose $\boldsymbol{v}$ and $\boldsymbol{w}$ are non-zero vectors (for otherwise the relation is trivially satisfied by any θ). Without loss of generality we may assume $0 \leq \theta \leq \pi$. Let $\boldsymbol{u} = \hat{\boldsymbol{w}} = \frac{\boldsymbol{w}}{\|\boldsymbol{w}\|}$ be the unit vector corresponding to $\boldsymbol{w}$. Then the component of $\boldsymbol{v}$ along $\boldsymbol{u}$ is

$$\boldsymbol{v}_u = \frac{\boldsymbol{v} \cdot \boldsymbol{w}}{\|\boldsymbol{w}\|}\,\boldsymbol{u}\,. \tag{8.17}$$

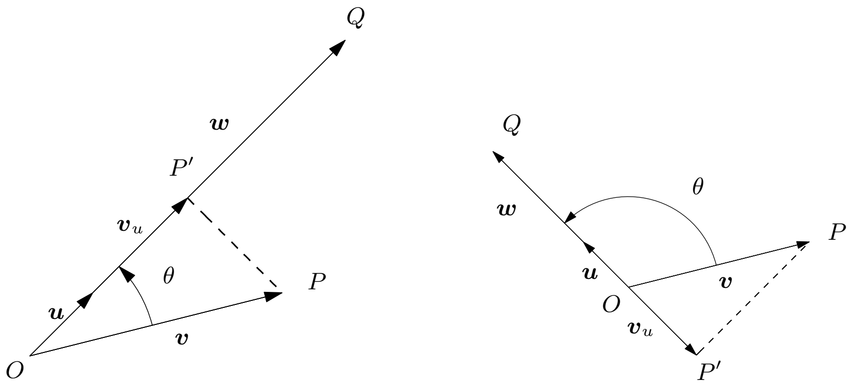


Figure 8.10. Projection of $\boldsymbol{v}$ along $\boldsymbol{w}$ (the angle formed by the vectors is acute on the left, obtuse on the right)

Suppose first that $0 < \theta < \pi/2$. In the triangle $OP'P$ (Fig. 8.10, left) $\|\boldsymbol{v}_u\| = |\overline{OP'}| = |\overline{OP}| \cos\theta = \|\boldsymbol{v}\| \cos\theta$; as $\boldsymbol{v}_u$ has the same orientation as $\boldsymbol{u}$, we have

$$\boldsymbol{v}_u = \|\boldsymbol{v}\| \cos\theta \, \boldsymbol{u} \,. \tag{8.18}$$

If θ is obtuse instead, in Fig. 8.10, right, we have $\|\boldsymbol{v}_u\| = \|\boldsymbol{v}\| \cos(\pi - \theta) = -\|\boldsymbol{v}\| \cos\theta$; precisely because now $\boldsymbol{v}_u$ has opposite sign to $\boldsymbol{u}$, (8.18) still holds. In the remaining cases $\theta = 0, \pi/2, \pi$ it is not hard to reach the same conclusion. Comparing (8.17) and (8.18), and noting $\lambda\boldsymbol{v} = \mu\boldsymbol{v}$ means $\lambda = \mu$ if $\boldsymbol{v} \neq 0$, we finally get to

$$\frac{\boldsymbol{v} \cdot \boldsymbol{w}}{\|\boldsymbol{w}\|} = \|\boldsymbol{v}\| \cos\theta,$$

whence (8.12). □

8.2.3 General vectors

Many applications involve vectors at points different from the origin, like forces in physics acting on a point-particle. The general notion of vector can be defined as follows.

Let $\boldsymbol{v}$ be a non-zero position vector of components (v_1, v_2), and P_0 an arbitrary point of the plane, with coordinates (x_{01}, x_{02}). Define P_1 by the coordinates $(x_{11}, x_{12}) = (x_{01} + v_1, x_{02} + v_2)$, as in Fig. 8.11. The line segment $\overline{P_0P_1}$ from P_0 to P_1 is parallel to $\boldsymbol{v}$ and has the same orientation. We say that it represents the **vector $\boldsymbol{v}$ at** P_0, and we write $(P_0, \boldsymbol{v})$. Vice versa, given any segment going from $P_0 = (x_{01}, x_{02})$ to $P_1 = (x_{11}, x_{12})$, we define the vector $\boldsymbol{v}$ of components $(v_1, v_2) = (x_{11} - x_{01}, x_{12} - x_{02})$. The segment identifies the vector $\boldsymbol{v}$ at P_0.

A general **vector** in the plane is mathematically speaking a pair $(P_0, \boldsymbol{v})$, whose first component is a point P_0 of the plane, and whose second component is a

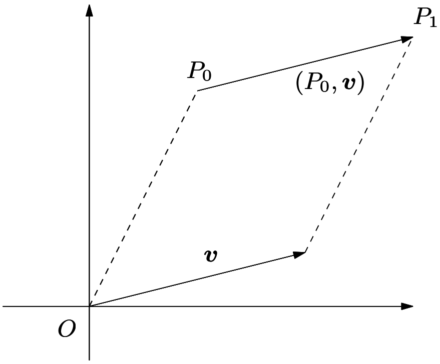


Figure 8.11. The position vector $\boldsymbol{v}$ and the same vector at P_0

position vector $\boldsymbol{v}$. Normally though, and from now onwards, the vector $(P_0, \boldsymbol{v})$ shall be denoted simply by $\boldsymbol{v}$; we will make the initial point P_0 explicit only if necessary. Analogous considerations are valid for vectors in space.

The operations on (position) vectors introduced earlier carry over to vectors *with the same initial point.* The vectors $(P_0, \boldsymbol{v})$ and $(P_0, \boldsymbol{w})$ add up to $(P_0, \boldsymbol{v}) + (P_0, \boldsymbol{w})$, equal to $(P_0, \boldsymbol{v} + \boldsymbol{w})$ by definition. Operations between vectors at different points are not defined, at least in this context.

8.3 Complex numbers

According to conventional wisdom, not every algebraic equation

$$p(x) = 0$$

(p being a polynomial of degree n in x) admits solutions in the field of real numbers. The simplest example is given by $p(x) = x^2 + 1$, i.e., the equation

$$x^2 = -1\,. \tag{8.19}$$

This would prescribe to take the square root of the negative number -1, and it is well known this is not possible in $\mathbb{R}$. The same happens for the generic quadratic equation

$$ax^2 + bx + c = 0 \tag{8.20}$$

when the discriminant $\Delta = b^2 - 4ac$ is less than zero. The existence of solutions of algebraic equations needs to be guaranteed both in pure and applied Mathematics. This apparent deficiency of real numbers is overcome by enlarging $\mathbb{R}$ to a set, called complex numbers, where adding and multiplying preserve the same formal properties of the reals. Obviously defining this extension-of-sorts so to contain the roots of every possible algebraic equations might seem daunting. The good news is that considering equation (8.19) only is sufficient in order to solve any algebraic equation, due to a crucial and deep result that goes under the name of Fundamental Theorem of Algebra.

8.3.1 Algebraic operations

A **complex number** z can be defined as an ordered pair $z = (x, y)$ of real numbers x, y. As such, the set of complex numbers $\mathbb{C}$ can be identified with $\mathbb{R}^2$. The reals x and y are the **real part** and the **imaginary part** of z

$$x = \mathcal{R}e\, z \qquad \text{and} \qquad y = \mathcal{I}m\, z$$

respectively. The subset of complex numbers of the form $(x, 0)$ is identified with $\mathbb{R}$, and with this in mind one is entitled to write $\mathbb{R} \subset \mathbb{C}$. Complex numbers of the form $(0, y)$ are called **purely imaginary**.

Two complex numbers $z_1 = (x_1, y_1)$, $z_2 = (x_2, y_2)$ are equal if they have coinciding real and imaginary parts

$$z_1 = z_2 \qquad \Longleftrightarrow \qquad x_1 = x_2 \quad \text{and} \quad y_1 = y_2\,.$$

Over $\mathbb{C}$, we define the sum and product of two numbers by

$$z_1 + z_2 = (x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2) \tag{8.21}$$

$$z_1\, z_2 = (x_1, y_1)\,(x_2, y_2) = (x_1\, x_2 - y_1\, y_2, x_1\, y_2 + x_2\, y_1)\,. \tag{8.22}$$

Notice

$$(x, 0) + (0, y) = (x, y)\,, \qquad (0, 1)\,(y, 0) = (0, y),$$

so

$$(x, y) = (x, 0) + (0, 1)\,(y, 0)\,. \tag{8.23}$$

Moreover, (8.21) and (8.22) are old acquaintances when restricted to the reals:

$$(x_1, 0) + (x_2, 0) = (x_1 + x_2, 0) \qquad \text{and} \qquad (x_1, 0)\,(x_2, 0) = (x_1\, x_2, 0)\,.$$

In this sense complex numbers are a natural extension of real numbers.

Introduce the symbol i to denote the purely imaginary number $(0, 1)$. By identifying $(r, 0)$ with the real number r, (8.23) reads

$$z = x + iy\,,$$

called **Cartesian form** or **algebraic form** of $z = (x, y)$.

Observe that

$$i^2 = (0, 1)\,(0, 1) = (-1, 0) = -1\,,$$

so the complex number i is a root of equation (8.19). The sum (8.21) and multiplication (8.22) of complex numbers in Cartesian form become

$$z_1 + z_2 = (x_1 + iy_1) + (x_2 + iy_2) = x_1 + x_2 + i(y_1 + y_2)\,, \tag{8.24}$$

$$z_1\, z_2 = (x_1 + iy_1)\,(x_2 + iy_2) = x_1\, x_2 - y_1\, y_2 + i(x_1\, y_2 + x_2\, y_1)\,. \tag{8.25}$$

The recipe is to use the familiar rules of algebra, taking the relation $i^2 = -1$ into account.

The next list of properties is left to the reader to check:

$$\begin{aligned} &z_1 + z_2 = z_2 + z_1\,, && z_1\, z_2 = z_2\, z_1\,,\\ &(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)\,, && (z_1\, z_2)\, z_3 = z_1\,(z_2\, z_3)\,,\\ &z_1\,(z_2 + z_3) = z_1\, z_2 + z_1\, z_3 \end{aligned}$$

for any $z_1, z_2, z_3 \in \mathbb{C}$. The numbers $0 = (0, 0)$ and $1 = (1, 0)$ are the additive and multiplicative units respectively, because

$$z + 0 = 0 + z = z \qquad \text{and} \qquad z\,1 = 1\,z = z\,, \quad \forall z \in \mathbb{C}\,.$$

The **opposite** or **negative** of $z = (x, y)$ is the complex number $-z = (-x, -y)$, in fact $z + (-z) = 0$. With this we can define, for any $z_1, z_2 \in \mathbb{C}$, the **difference**:

$$z_1 - z_2 = z_1 + (-z_2)$$

or, equivalently,

$$x_1 + iy_1 - (x_2 + iy_2) = x_1 - x_2 + i(y_1 - y_2)\,.$$

The **inverse** or **reciprocal** of a complex number $z \neq 0$, denoted $\frac{1}{z}$ or z^{-1}, is given by the relation $zz^{-1} = 1$, and it is easy to see

$$\frac{1}{z} = z^{-1} = \frac{x}{x^2 + y^2} + i\frac{-y}{x^2 + y^2}\,.$$

The formula

$$\frac{z_1}{z_2} = z_1\; z_2^{-1} = \frac{x_1\, x_2 + y_1\, y_2}{x_2^2 + y_2^2} + i\,\frac{x_2\, y_1 - x_1\, y_2}{x_2^2 + y_2^2}$$

defines the **ratio** or **quotient** of $z_1, z_2 \in \mathbb{C}$ with $z_2 \neq 0$.

At last, let us remark that the ordering of real numbers cannot be extended to $\mathbb{C}$ to preserve the compatibility properties of Sect. 1.3.1 in any way.

8.3.2 Cartesian coordinates

With the identification of $\mathbb{C}$ and $\mathbb{R}^2$, it becomes natural to associate the number $z = (x, y) = x + iy$ to the point of coordinates x and y in the Cartesian plane (Fig. 8.12). The point z can also be thought of as the position vector having end point at z. The horizontal axis of the plane is called **real axis** and the vertical axis **imaginary axis**. For any $z_1, z_2 \in \mathbb{C}$ the sum $z_1 + z_2$ corresponds to the vector obtained by the parallelogram rule (as in Fig. 8.13, left), while $z_1 - z_2$ is represented by the difference vector (same figure, right).

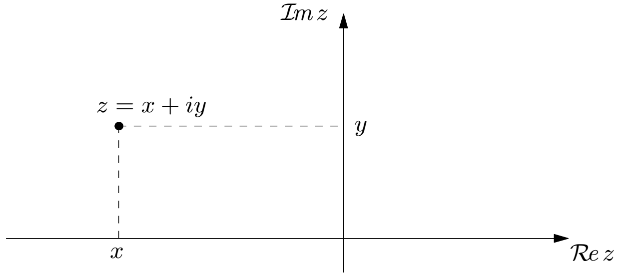


Figure 8.12. Cartesian coordinates of the complex number $z = x + iy$

The **modulus** (or **absolute value**) of $z = x + iy$, denoted $|z|$, is the non-negative number

$$|z| = \sqrt{x^2 + y^2}$$

representing the distance of (x, y) from the origin; non-incidentally, this definition is the same as that of norm of the vector $\boldsymbol{v}$ associated to z, $|z| = \|\boldsymbol{v}\|$. Moreover, if a complex number is real, its modulus is the absolute value as of Sect. 1.3.1. This justifies the choice of name, and explains why the absolute value of a real number is sometimes called modulus. We point out that, whereas the statement $z_1 < z_2$ has no meaning, the inequality $|z_1| < |z_2|$ does, indeed the point (corresponding to) z_1 is closer to the origin than the point z_2. The distance of the points corresponding to z_1 and z_2 is $|z_1 - z_2|$.

Given $z \in \mathbb{C}$, the following are easy:

$$\begin{aligned}
&|z| \geq 0\,; \quad |z| = 0 \text{ if and only if } z = 0\,;\\
&|z|^2 = (\mathcal{R}e\, z)^2 + (\mathcal{I}m\, z)^2\,;\\
&\mathcal{R}e\, z \leq |\mathcal{R}e\, z| \leq |z|\,, \quad \mathcal{I}m\, z \leq |\mathcal{I}m\, z| \leq |z|\,;\\
&\big||z_1| - |z_2|\big| \leq |z_1 + z_2| \leq |z_1| + |z_2|\,.
\end{aligned}$$

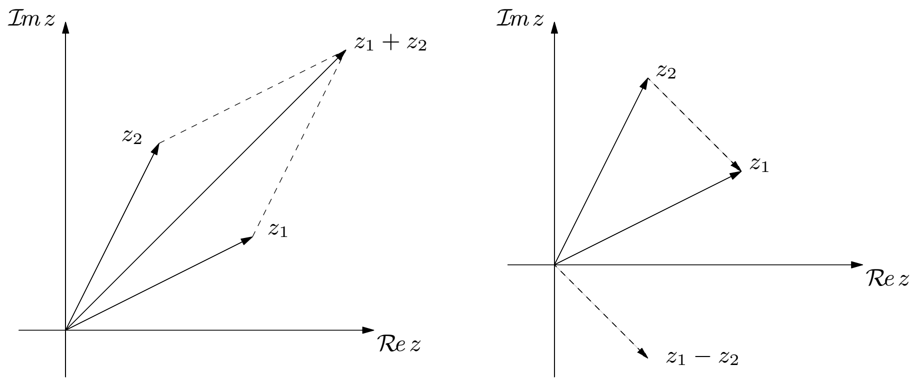


Figure 8.13. Sum (left) and difference (right) of complex numbers

The **complex conjugate**, or just conjugate, of $z = x + iy$ is the complex number

$$\bar{z} = x - iy\,. \tag{8.26}$$

On the plane, the conjugate $\bar{z}$ is the point $(x, -y)$ obtained by reflection of (x, y) with respect to the real axis. The following properties hold for any $z, z_1, z_2 \in \mathbb{C}$:

$$\begin{aligned}
&\overline{\bar{z}} = z\,, && |\bar{z}| = |z|\,, && z\,\bar{z} = |z|^2\,,\\
&\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2\,, && \overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2\,, &&\\
&\overline{z_1\,z_2} = \bar{z}_1\,\bar{z}_2\,, && \overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2} \quad (z_2 \neq 0)\,. &&
\end{aligned} \tag{8.27}$$

Of immediate proof is also

$$\mathcal{R}e\,z = \frac{z + \bar{z}}{2}\,, \qquad \mathcal{I}m\,z = \frac{z - \bar{z}}{2i}\,,$$

for all $z \in \mathbb{C}$.

8.3.3 Trigonometric and exponential form

Let r and θ be the polar coordinates of the point (x, y). Since

$$x = r\cos\theta \qquad \text{and} \qquad y = r\sin\theta\,,$$

the number $z = (x, y)$ has a **polar form**, also called **trigonometric**,

$$z = r\,(\cos\theta + i\sin\theta)\,. \tag{8.28}$$

First of all, $r = |z|$. The number θ, denoted by $\theta = \arg z$, is said **argument** of z (less often, but to some more suggestively, 'amplitude'). Geometrically $\arg z$ is an angle (in radians) delimited by the positive real axis and the direction of the position vector z (as in Fig. 8.14).

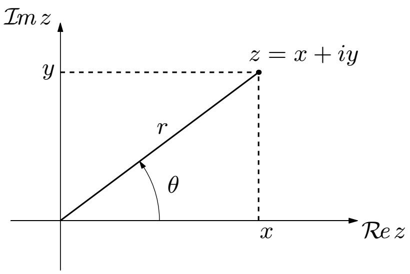


Figure 8.14. Polar coordinates of the number $z = x + iy$

The argument can assume infinitely many values, all differing by integer multiples of 2π. One calls **principal value** of $\arg z$, and denotes by the capitalised symbol $\operatorname{Arg} z$, the unique value θ of $\arg z$ such that $-\pi < \theta \le \pi$; the principal value is defined analytically by (8.2).

Two complex numbers $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ are equal if and only if $r_1 = r_2$ and θ_1, θ_2 differ by an integer multiple of 2π.

The representation in polar form is useful to multiply complex numbers, and consequently, to compute powers and nth roots. Let in fact

$$z_1 = r_1\,(\cos\theta_1 + i\sin\theta_1) \qquad \text{and} \qquad z_2 = r_2\,(\cos\theta_2 + i\sin\theta_2)\,;$$

the addition formulas for trigonometric functions tell us that

$$\begin{aligned} z_1\,z_2 &= r_1\,r_2\,\big[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \sin\theta_2\cos\theta_1)\big] \\ &= r_1\,r_2\,\big[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\big]\,. \end{aligned} \tag{8.29}$$

Therefore

$$\arg\,(z_1\,z_2) = \arg z_1 + \arg z_2\,. \tag{8.30}$$

Note that this identity is false when using Arg: take for instance $z_1 = -1 = \cos\pi + i\sin\pi$ and $z_2 = i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$, so

$$z_1\,z_2 = -i = \cos\big(-\frac{\pi}{2}\big) + i\sin\big(-\frac{\pi}{2}\big)\,,$$

i.e.,

$$\operatorname{Arg} z_1 = \pi\,, \qquad \operatorname{Arg} z_2 = \frac{\pi}{2}\,, \qquad \operatorname{Arg} z_1 + \operatorname{Arg} z_2 = \frac{3}{2}\pi \neq \operatorname{Arg} z_1\,z_2 = -\frac{\pi}{2}\,.$$

The so-called **exponential form** is also useful. To define it, let us extend the exponential function to the case where the exponent is purely imaginary, by putting

$$\mathrm{e}^{i\theta} = \cos\theta + i\sin\theta \tag{8.31}$$

for any $\theta \in \mathbb{R}$. Such a relation is sometimes called **Euler formula**, and can be actually proved within the theory of series over the complex numbers. We shall take it as definition without further mention. The expression (8.28) now becomes

$$z = r\mathrm{e}^{i\theta}\,, \tag{8.32}$$

the exponential form of z. The complex conjugate of z is

$$\bar{z} = r(\cos\theta - i\sin\theta) = r(\cos(-\theta) + i\sin(-\theta)) = r\mathrm{e}^{-i\theta}$$

in exponential form.

Then (8.29) immediately furnishes the product of $z_1 = r_1\mathrm{e}^{i\theta_1}$ and $z_2 = r_2\mathrm{e}^{i\theta_2}$

$$z_1\,z_2 = r_1\,r_2\,\mathrm{e}^{i(\theta_1+\theta_2)}\,. \tag{8.33}$$

Thus the moduli are multiplied, the arguments added. To divide complex numbers (8.29) gives, with $r_1 = r_2 = 1$,

$$e^{i\theta_1} e^{i\theta_2} = e^{i(\theta_1+\theta_2)} . \tag{8.34}$$

In particular,

$$e^{i\theta} e^{-i\theta} = 1$$

so $e^{-i\theta}$ is the inverse of $e^{i\theta}$. The reciprocal of $z = re^{i\theta} \neq 0$ is then

$$z^{-1} = \frac{1}{r} e^{-i\theta} . \tag{8.35}$$

Combining this formula with the product one shows that the ratio of $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2} \neq 0$ is

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)} . \tag{8.36}$$

8.3.4 Powers and *n*th roots

Re-iterating (8.33) and (8.35) we obtain, for any $n \in \mathbb{Z}$,

$$z^n = r^n e^{in\theta} . \tag{8.37}$$

For $r = 1$, this is the so-called **De Moivre's formula**

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta . \tag{8.38}$$

By (8.37) we can calculate nth roots of a complex number. Fix $n \geq 1$ and a complex number $w = \rho\, e^{i\varphi}$, and let us determine the numbers $z = r\, e^{i\theta}$ such that $z^n = w$. Relation (8.37) implies

$$z^n = r^n e^{in\theta} = \rho\, e^{i\varphi} = w,$$

which means

$$\begin{cases} r^n = \rho\,, \\ n\theta = \varphi + 2k\pi\,, \quad k \in \mathbb{Z}\,, \end{cases}$$

hence

$$\begin{cases} r = \sqrt[n]{\rho}\,, \\ \theta = \dfrac{\varphi + 2k\pi}{n}\,, \quad k \in \mathbb{Z}\,. \end{cases}$$

The expression of θ does not necessarily give the principal values of the roots' arguments. Nevertheless, as sine and cosine are periodic, we have n distinct solutions

$$z_k = \sqrt[n]{\rho}\, e^{i\frac{\varphi+2k\pi}{n}} = \sqrt[n]{\rho} \left(\cos\frac{\varphi+2k\pi}{n} + i\sin\frac{\varphi+2k\pi}{n} \right) , \quad k = 0, 1, \ldots, n-1$$

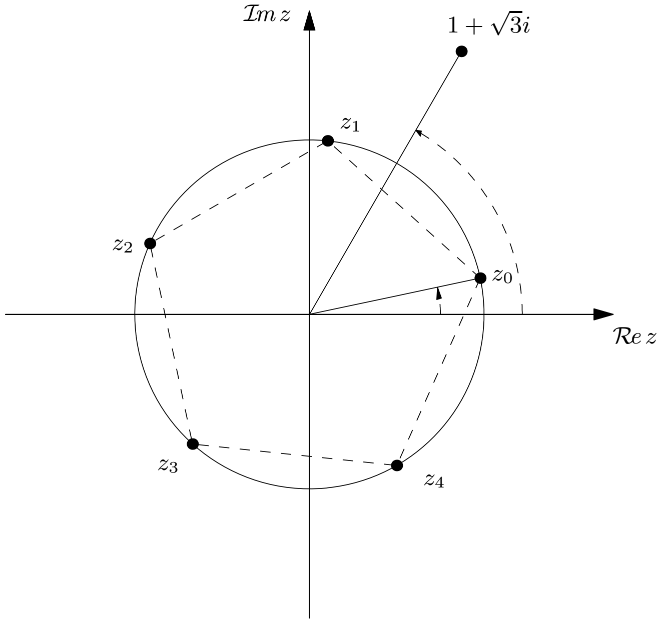


Figure 8.15. The point $1+\sqrt{3}i$ and its fifth roots z_j, $j=0,\dots,4$

to the problem. These points lie on the circle centred at the origin with radius $\sqrt[n]{\rho}$; they are precisely the vertices of a regular polygon of n sides (an 'n-gon', see Fig. 8.15).

Examples 8.5

i) For $n \geq 1$ consider the equation

$$z^n = 1\,.$$

Writing $1 = 1e^{i0}$ we obtain the n distinct roots

$$z_k = e^{i\frac{2k\pi}{n}}\,, \qquad k = 0, 1, \dots, n-1,$$

called n**th roots of unity**. When n is odd, only one of these is real, $z_0 = 1$, whilst for n even there are two real roots of unity $z_0 = 1$ and $z_{n/2} = -1$ (Fig. 8.16).

ii) Verify that

$$z^2 = -1$$

admits, as it should, the solutions $z_\pm = \pm i$. Write $-1 = 1e^{i\pi}$, from which

$$z_+ = z_0 = e^{i\frac{\pi}{2}} = i \quad \text{and} \quad z_- = z_1 = e^{i\frac{\pi+2\pi}{2}} = e^{-i\frac{\pi}{2}} = -i\,. \qquad \square$$

Note finally that (8.31) permits to define the exponential of arbitrary (not only imaginary) complex numbers $z = x + iy$, by letting

$$e^z = e^x e^{iy} = e^x(\cos y + i \sin y)\,. \tag{8.39}$$

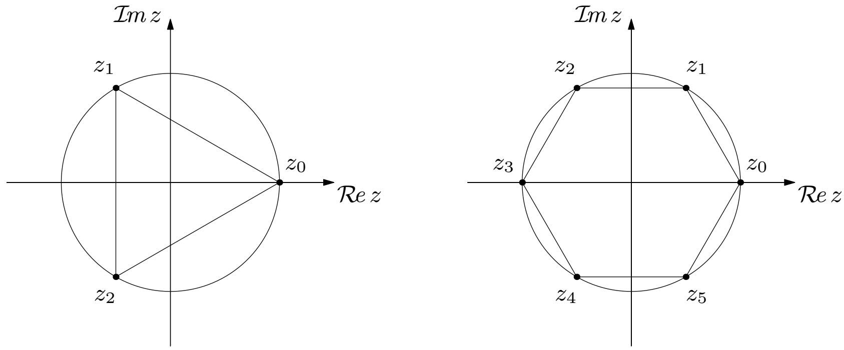


Figure 8.16. Roots of unity: cubic roots (left) and sixth roots (right)

Using (8.34) it is now an easy task to verify that the fundamental relation $e^{z_1+z_2} = e^{z_1}e^{z_2}$ is still valid in the realm of complex numbers. In addition to that,

$$|e^z| = e^{\mathcal{R}e\, z} > 0\,, \qquad\qquad \arg e^z = \mathcal{I}m\, z\,.$$

The first tells, amongst other things, that $e^z \neq 0$ for any $z \in \mathbb{C}$. The periodicity of the trigonometric functions implies

$$e^{z+2k\pi i} = e^z\,, \qquad \text{for all } k \in \mathbb{Z}\,.$$

8.3.5 Algebraic equations

We will show that the quadratic equation with real coefficients

$$az^2 + bz + c = 0$$

admits two complex-conjugate solutions in case the discriminant Δ is negative. We can suppose $a > 0$. Inspired by the square of a binomial we write

$$0 = z^2 + \frac{b}{a}z + \frac{c}{a} = \left(z^2 + 2\frac{b}{2a}z + \frac{b^2}{4a^2}\right) + \frac{c}{a} - \frac{b^2}{4a^2}\,,$$

that is

$$\left(z + \frac{b}{2a}\right)^2 = \frac{\Delta}{4a^2} < 0\,.$$

Therefore

$$z + \frac{b}{2a} = \pm i\frac{\sqrt{-\Delta}}{2a},$$

or

$$z = \frac{-b \pm i\sqrt{-\Delta}}{2a}\,.$$

We write this as $z = \dfrac{-b \pm \sqrt{\Delta}}{2a}$, in analogy to the case $\Delta \geq 0$.

The procedure may be applied when the coefficients $a \neq 0$, b and c are complex numbers, as well. Thus

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

are the two solutions of the equation $az^2 + bz + c = 0$ in the greatest possible generality.

Third- and fourth-degree algebraic equations have three and four solutions respectively (counted with multiplicity): these roots can be made explicit via algebraic operations, namely square and cubic roots[1]. There can be instead no analytic expression for solving an equation of fifth degree or higher. Despite all though, the Fundamental Theorem of Algebra warrants that every algebraic equation $p(z) = 0$, where p is a polynomial of degree n with real or complex coefficients, admits exactly n solutions in $\mathbb{C}$, each counted with its multiplicity. This is how it goes.

Theorem 8.6 *Let $p(z) = a_n z^n + \ldots + a_1 z + a_0$, with $a_n \neq 0$, be a polynomial of degree n with coefficients $a_k \in \mathbb{C}$, $0 \leq k \leq n$. There exist $m \leq n$ distinct complex numbers $z_1, \ldots, z_m$, and m non-zero natural numbers $\mu_1, \ldots, \mu_m$ with $\mu_1 + \ldots + \mu_m = n$, such that $p(z)$ factorises as*

$$p(z) = a_n (z - z_1)^{\mu_1} \ldots (z - z_m)^{\mu_m}.$$

The numbers z_k are the roots of the polynomial p, in other words the solutions of $p(z) = 0$; the exponent μ_k is the multiplicity of the root z_k. A root is simple if it has multiplicity one, double if the muliplicity is 2, and so on.

It is opportune to remark that if the coefficients of p are real and if z_0 is a complex root, then also $\bar{z}_0$ is a root of p. In fact, taking conjugates of $p(z_0) = 0$ and using known properties (see (8.27)), we obtain

$$0 = \bar{0} = \overline{p(z_0)} = \bar{a}_n \bar{z}_0^n + \ldots + \bar{a}_1 \bar{z}_0 + \bar{a}_0 = a_n \bar{z}_0^n + \ldots + a_1 \bar{z}_0 + a_0 = p(\bar{z}_0)\,.$$

The polynomial $p(z)$ is then divisible by $(z - z_0)(z - \bar{z}_0)$, a quadratic polynomial with real coefficients.

A version of the Fundamental Theorem of Algebra for real polynomials, that does not involve complex numbers, is stated in Theorem 9.15.

[1] The cubic equation $x^3 + ax^2 + bx + c = 0$ for example, reduces to the form $y^3 + py + q = 0$, by changing $x = y - \frac{a}{3}$; p and q are suitable coefficients, which are easy to find. The solutions of the reduced equation read

$$y = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} - \sqrt[3]{\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}\,,$$

a formula due to Cardano. Extracting a root yields as many solutions as the order of the root (here 2 or 3), yielding a maximum of 12 solutions, at least in principle: it is possible to prove that at most 3 of them are distinct.

8.4 Curves in the plane and in space

The second part of the chapter sees the return of functions, and the present section devotes itself in particular to the notion of a curve in Euclidean space or on the plane. A curve can describe the boundary of a planar region such as a polygon, or an ellipse; it is a good model for the trajectory of a point-particle moving in time under the effect of a force. In Chap. 10 we shall see how to perform integral calculus along curves, which enables to describe mathematically the notion of work, to stay with the physical analogy.

Let I be an arbitrary interval of the real line and $\boldsymbol{\gamma} : I \to \mathbb{R}^3$ a map. Denote by $\boldsymbol{\gamma}(t) = \big(x(t), y(t), z(t)\big)$ the point of $\mathbb{R}^3$ image of $t \in I$ under $\boldsymbol{\gamma}$. One says $\boldsymbol{\gamma}$ is a **continuous map on** I if the components $x, y, z : I \to \mathbb{R}$ are continuous functions.

Definition 8.7 *A continuous map $\boldsymbol{\gamma} : I \subseteq \mathbb{R} \to \mathbb{R}^3$ is called a* **curve** *(in space). The range of the map is called* **image** *and will be denoted by the letter $C = \boldsymbol{\gamma}(I) \subseteq \mathbb{R}^3$.*

If the image lies on a plane, one talks about a **plane curve**. A special case is that where $\boldsymbol{\gamma}(t) = \big(x(t), y(t), 0\big)$, that is, curves lying in the xy-plane which we indicate simply as $\boldsymbol{\gamma} : I \to \mathbb{R}^2$, $\boldsymbol{\gamma}(t) = \big(x(t), y(t)\big)$.

Thus a curve is a function of one real variable, whereas the image is a subset of space (or the plane). Curves furnish a way to parametrise their image by associating to each value of the parameter $t \in I$ exactly one point. The set C could be the image of many curves, by different parametrisations. For example, the plane curve

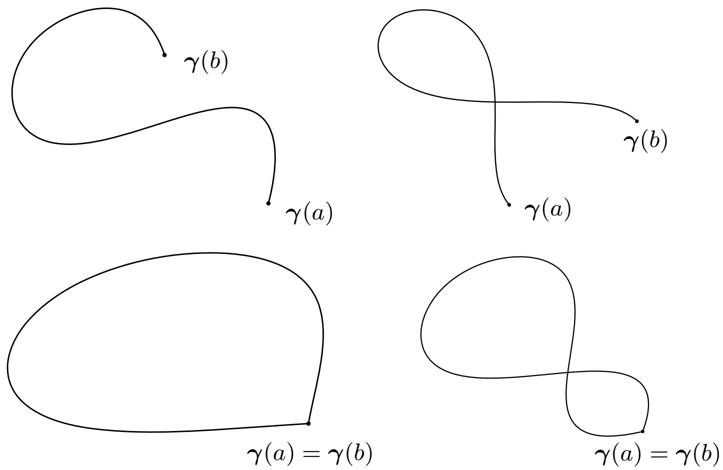


Figure 8.17. Clockwise from top left: images $C = \boldsymbol{\gamma}([a, b])$ of a simple arc, a non-simplearc, a closed arc which is not simple, a Jordan arc

$\boldsymbol{\gamma}(t) = (t,t)$ with $t \in [0,1]$ has the segment with endpoints $A = (0,0), B = (1,1)$ as image. But this is also the image of $\boldsymbol{\delta}(t) = (t^2,t^2)$, $t \in [0,1]$; the two curves $\boldsymbol{\gamma}$ and $\boldsymbol{\delta}$ are parametrisations of the segment AB. The middle point of AB is for example image of $t = \frac{1}{2}$ under $\boldsymbol{\gamma}$ and $t = \frac{\sqrt{2}}{2}$ under $\boldsymbol{\delta}$.

A **curve** $\boldsymbol{\gamma}$ is **simple** if $\boldsymbol{\gamma}$ is a one-to-one map, i.e., if different values of the parameter determine distinct points on the image.

Suppose the interval $I = [a,b]$ is closed and bounded, as in the previous examples, in which case the curve $\boldsymbol{\gamma}$ is called an **arc**. An arc is **closed** if $\boldsymbol{\gamma}(a) = \boldsymbol{\gamma}(b)$; clearly a closed arc is not simple. Nevertheless, one defines **simple closed arc** (or **Jordan arc**) a closed arc which is simple except for one single point $\boldsymbol{\gamma}(a) = \boldsymbol{\gamma}(b)$. Fig. 8.17 illustrates various types of situations.

The reader might encounter the word arc in the literature (as in 'arc of circumference') to denote a subset of $\mathbb{R}^2$ or $\mathbb{R}^3$, endowed with the most natural – hence implicitly understood – parametrisation.

Examples 8.8

i) The simple plane curve

$$\boldsymbol{\gamma}(t) = (at+b, ct+d)\,, \qquad t \in \mathbb{R}\,,\ a \neq 0\,,$$

has for an image the line $y = \dfrac{c}{a}x + \dfrac{ad-bc}{a}$. Setting $x = x(t) = at + b$ and $y = y(t) = ct + d$, in fact, gives $t = \dfrac{x-b}{a}$, so

$$y = \frac{c}{a}(x-b) + d = \frac{c}{a}x + \frac{ad-bc}{a}\,.$$

ii) The curve

$$\boldsymbol{\gamma}(t) = \big(x(t), y(t)\big) = (1+\cos t, 3+\sin t)\,, \qquad t \in [0, 2\pi]\,,$$

has the circle centred at $(1,3)$ with radius 1 as image; in fact $\big(x(t)-1\big)^2 + \big(y(t)-3\big)^2 = \cos^2 t + \sin^2 t = 1$. This is a simple closed curve and provides the most natural way to parametrise the circle that starts at $(2,3)$ and runs in the counter-clockwise direction.

In general, the image of the Jordan curve

$$\boldsymbol{\gamma}(t) = \big(x(t), y(t)\big) = (x_0 + r\cos t, y_0 + r\sin t)\,, \qquad t \in [0, 2\pi]\,,$$

is the circle with centre (x_0, y_0) and radius r.

If t varies in an interval $[0, 2k\pi]$, with $k \geq 2$ a positive integer, the curve has the same image seen as a set; but because we wind around the centre k times, the curve is not simple.

Instead, if t varies in $[0,\pi]$, the curve is an arc of circumference, simple but not closed.

iii) Given $a, b > 0$, the map

$$\boldsymbol{\gamma}(t) = \big(x(t), y(t)\big) = (a\cos t, b\sin t)\,, \qquad t \in [0, 2\pi]\,,$$

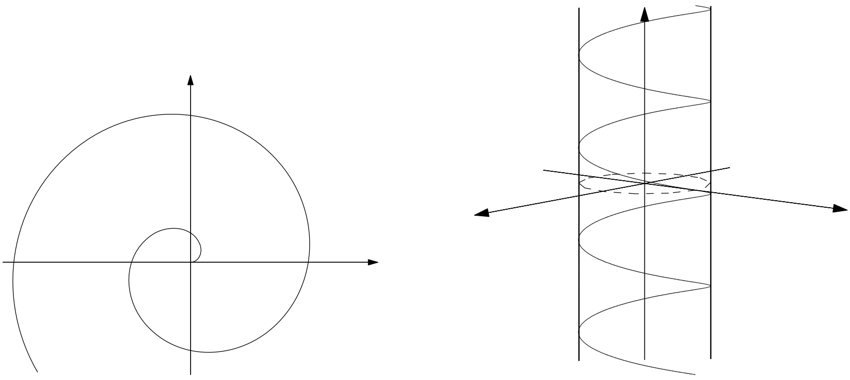


Figure 8.18. The spiral and helix of Examples 8.8 iv), vi)

is a simple closed curve parametrising the ellipse with centre in the origin and semi-axes a and b.

iv) The image of

$$\gamma(t) = \big(x(t), y(t)\big) = (t\cos t, t\sin t)\,, \qquad t \in [0, +\infty)\,,$$

is drawn in Fig. 8.18 (left); the spiral coils counter-clockwise around the origin. The generic point $\gamma(t)$ has distance $\sqrt{x^2(t)+y^2(t)} = t$ from the origin, so it moves always farther as t grows, making the spiral a simple curve.

v) Let $P = (x_P, y_P, z_P)$ and $Q = (x_Q, y_Q, z_Q)$ be distinct points in space. The image of the simple curve

$$\gamma(t) = P + (Q-P)t\,, \qquad t \in \mathbb{R}\,,$$

is the straight line through P and Q, because $\gamma(0) = P$, $\gamma(1) = Q$ and the vector $\gamma(t) - P$ has constant direction, being parallel to $Q - P$.

The same line can be parametrised more generally by

$$\gamma(t) = P + (Q-P)\frac{t-t_0}{t_1-t_0}\,, \qquad t \in \mathbb{R}\,, \tag{8.40}$$

where $t_0 \neq t_1$; in this case $\gamma(t_0) = P$, $\gamma(t_1) = Q$.

vi) Consider the simple curve

$$\gamma(t) = \big(x(t), y(t), z(t)\big) = (\cos t, \sin t, t)\,, \qquad t \in \mathbb{R}\,.$$

Its image is the circular helix (Fig. 8.18, right) resting on the infinite cylinder along the z-axis with radius one, i.e., the set $\{(x,y,z) \in \mathbb{R}^3 : x^2+y^2=1\}$. □

A curve $\gamma : I \to \mathbb{R}^3$ is **differentiable** if the components $x, y, z : I \to \mathbb{R}$ are differentiable maps on I (recall that differentiable on I means differentiable at every interior point, and differentiable on one side at the boundary, if this is included in I). Let $\gamma' : I \to \mathbb{R}^3$ be the derivative function $\gamma'(t) = \big(x'(t), y'(t), z'(t)\big)$.

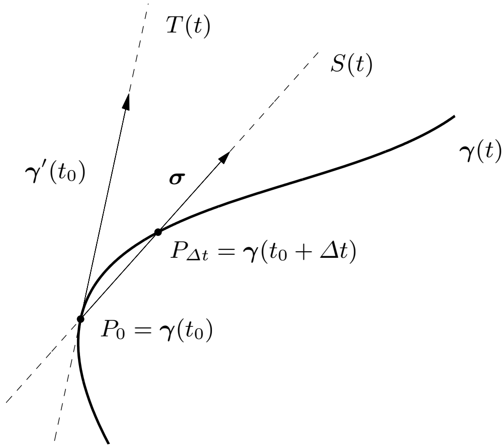


Figure 8.19. Tangent vector and secant at the point P_0

Definition 8.9 *The curve* $\gamma : I \to \mathbb{R}^3$ *is* **regular** *if it is differentiable over* I *with continuous derivative (i.e., the components are of class* $\mathcal{C}^1$ *on* I*) and if* $\gamma'(t) \neq (0,0,0)$*, for every* $t \in I$.

A curve $\gamma : I \to \mathbb{R}^3$ *is said* **piecewise regular** *if* I *is the union of finitely-many subintervals where* γ *is regular.*

When the curve γ is regular and $t_0 \in I$, the vector $\gamma'(t_0)$ is called **tangent vector** to (the image of) the curve at $P_0 = \gamma(t_0)$. The name comes from the geometric picture (Fig. 8.19). Let $t_0 + \Delta t \in I$ be such that the point $P_{\Delta t} = \gamma(t_0 + \Delta t)$ is different from P_0, and consider the straight line passing through P_0 and $P_{\Delta t}$. By (8.40) such line can be parametrised as

$$S(t) = P_0 + \big(P_{\Delta t} - P_0\big)\frac{t - t_0}{\Delta t} = \gamma(t_0) + \frac{\gamma(t_0 + \Delta t) - \gamma(t_0)}{\Delta t}(t - t_0)\,. \qquad (8.41)$$

As Δt goes to 0, the point $P_{\Delta t}$ approaches P_0 (component-wise). At the same time, the regularity assumption forces the vector $\boldsymbol{\sigma} = \boldsymbol{\sigma}(t_0, \Delta t) = \dfrac{\gamma(t_0 + \Delta t) - \gamma(t_0)}{\Delta t}$ to tend to $\gamma'(t_0)$. Therefore the limiting position of (8.41) is

$$T(t) = \gamma(t_0) + \gamma'(t_0)(t - t_0)\,, \qquad t \in \mathbb{R}\,,$$

the straight line tangent to the curve at P_0. To be very precise, the tangent vector at P_0 is the vector $\big(P_0, \gamma'(t_0)\big)$, but it is common to write it simply $\gamma'(t_0)$ (as discussed in Sect. 8.2.3). One can easily verify that the tangent line to a curve at a point is an intrinsic notion – independent of the chosen parametrisation, whereas the tangent vector does depend on the parametrisation, as far as length and orientation are concerned.

In kinematics, a curve represents a trajectory, i.e., the position $\boldsymbol{\gamma}(t)$ a particle occupies at time t. If the curve is regular, the tangent vector $\boldsymbol{\gamma}'(t)$ describes the velocity of the particle at time t.

Examples 8.10

i) All curves in Examples 8.8 are regular.

ii) Let $f : I \to \mathbb{R}$ be differentiable with continuity on I. The curve

$$\boldsymbol{\gamma}(t) = \big(t, f(t)\big)\,, \qquad t \in I\,,$$

is regular, and has image the graph of the function f. In fact,

$$\boldsymbol{\gamma}'(t) = \big(1, f'(t)\big) \neq (0,0)\,, \qquad \text{for any } t \in I\,.$$

iii) The arc $\boldsymbol{\gamma} : [0,2] \to \mathbb{R}^2$

$$\boldsymbol{\gamma}(t) = \begin{cases} (t,1)\,, & \text{if } t \in [0,1)\,, \\ (t,t)\,, & \text{if } t \in [1,2]\,, \end{cases}$$

parametrises the polygonal chain ABC (Fig. 8.20, left), while

$$\boldsymbol{\gamma}(t) = \begin{cases} (t,1)\,, & \text{if } t \in [0,1)\,, \\ (t,t)\,, & \text{if } t \in [1,2)\,, \\ \big(4-t, 2-\frac{1}{2}(t-2)\,, & \text{if } t \in [2,4]\,, \end{cases}$$

describes $ABCA$ (Fig. 8.20, right). Both are piecewise regular curves.

iv) The curves

$$\boldsymbol{\gamma}(t) = \big(1+\sqrt{2}\cos t, \sqrt{2}\sin t\big)\,, \qquad t \in [0, 2\pi]\,,$$
$$\widetilde{\boldsymbol{\gamma}}(t) = \big(1+\sqrt{2}\cos 2t, -\sqrt{2}\sin 2t\big)\,, \qquad t \in [0, \pi]\,,$$

parametrise the same circle C (counter-clockwise and clockwise respectively) with centre $(1,0)$ and radius $\sqrt{2}$.

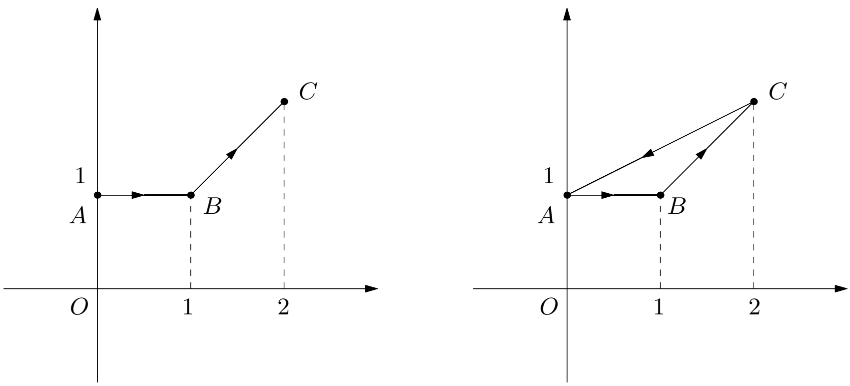


Figure 8.20. The polygonal chains ABC (left) and $ABCA$ (right) in Example 8.10 iii)

They are regular and differentiate to

$$\boldsymbol{\gamma}'(t) = \sqrt{2}\big(-\sin t, \cos t\big)\,, \qquad \widetilde{\boldsymbol{\gamma}}'(t) = 2\sqrt{2}\big(-\sin 2t, -\cos 2t\big)\,.$$

The point $P_0 = (0,1) \in C$ is the image under $\boldsymbol{\gamma}$ of $t_0 = \frac{3}{4}\pi$, under $\widetilde{\boldsymbol{\gamma}}$ of the value $\widetilde{t_0} = \frac{5}{8}\pi$, $P_0 = \boldsymbol{\gamma}(t_0) = \widetilde{\boldsymbol{\gamma}}(\widetilde{t_0})$. In the former case the tangent vector is $\boldsymbol{\gamma}'(t_0) = (-1,-1)$ and the tangent to C at P_0

$$T(t) = (0,1) - (1,1)\big(t - \frac{3}{4}\pi\big) = \big(-t + \frac{3}{4}\pi, 1 - t + \frac{3}{4}\pi\big)\,, \qquad t \in \mathbb{R}\,.$$

For the latter parametrisation, $\widetilde{\boldsymbol{\gamma}}'(\widetilde{t_0}) = (2,2)$ and

$$\widetilde{T}(t) = (0,1) + (2,2)\big(t - \frac{5}{8}\pi\big) = \big(2(t - \frac{5}{8}\pi), 1 + 2(t - \frac{5}{8}\pi)\big)\,, \qquad t \in \mathbb{R}\,.$$

The tangent vectors at P_0 have different lengths and orientations, but same direction. Recalling Example 8.8 in fact, in both cases the tangent line has equation $y = 1 + x$. □

8.5 Functions of several variables

Object of our investigation in the previous chapters have been functions of one real variable, that is, maps defined on a subset of the real line $\mathbb{R}$ (like an interval), with values in $\mathbb{R}$.

We would like now to extend some of those notions and introduce new ones, relative to real-valued functions of **two** or **three real variables**. These are defined on subsets of $\mathbb{R}^2$ or $\mathbb{R}^3$ and valued in $\mathbb{R}$

$$\begin{aligned} f : \operatorname{dom} f \subseteq \mathbb{R}^d &\to \mathbb{R} \qquad\qquad (d = 2 \text{ or } 3)\,, \\ \boldsymbol{x} &\mapsto f(\boldsymbol{x})\,. \end{aligned}$$

The symbol $\boldsymbol{x}$ indicates a generic element of $\mathbb{R}^d$, hence a pair $\boldsymbol{x} = (x_1, x_2)$ if $d = 2$ or a triple $\boldsymbol{x} = (x_1, x_2, x_3)$ if $d = 3$. For simplicity we might write $(x_1, x_2) = (x, y)$ and $(x_1, x_2, x_3) = (x, y, z)$, and the coordinates of $\boldsymbol{x}$ shall be $(x_1, \ldots, x_d)$ when it is not relevant whether $d = 2$ or 3. Each $\boldsymbol{x} \in \mathbb{R}^d$ is uniquely associated to a point P of the plane or space, whose coordinates in an orthogonal Cartesian frame are the components of $\boldsymbol{x}$. In turn, P determines a position vector of components $x_1, \ldots, x_d$, so the element $\boldsymbol{x} \in \mathbb{R}^d$ can be thought of as that vector. In this way, $\mathbb{R}^d$ inherits the operations of sum $\boldsymbol{x} + \boldsymbol{y} = (x_1 + y_1, \ldots, x_d + y_d)$, multiplication $\lambda\boldsymbol{x} = (\lambda x_1, \ldots, \lambda x_d)$ and dot product $\boldsymbol{x} \cdot \boldsymbol{y} = x_1 y_1 + \ldots + x_d y_d$. Furthermore, the Euclidean norm $\|\boldsymbol{x}\| = \sqrt{x_1^2 + \ldots + x_d^2}$ represents the Euclidean distance of P to O. Notice $\|\boldsymbol{x} - \boldsymbol{y}\| = \sqrt{(x_1 - y_1)^2 + \ldots + (x_d - y_d)^2}$ is the distance between the points P and Q of respective coordinates $\boldsymbol{x}$ and $\boldsymbol{y}$.

8.5.1 Continuity

By means of the norm we can define neighbourhoods of a point in $\mathbb{R}^d$ and extend the concepts of continuity, and limit, to functions of several variables.

Definition 8.11 *Let* $\boldsymbol{x}_0 \in \mathbb{R}^d$ *and* $r > 0$ *real. The set*

$$I_r(\boldsymbol{x}_0) = \{\boldsymbol{x} \in \mathbb{R}^d : \|\boldsymbol{x} - \boldsymbol{x}_0\| < r\}$$

of points $\mathbb{R}^d$ *whose distance from* $\boldsymbol{x}_0$ *is less than* r *is called* **neighbourhood** *of* $\boldsymbol{x}_0$ *of radius* r.

With $\boldsymbol{x}_0 = (x_{01}, \ldots, x_{0d})$, the condition $\|\boldsymbol{x} - \boldsymbol{x}_0\| < r$ is equivalent to

$$(x_1 - x_{01})^2 + (x_2 - x_{02})^2 < r^2 \qquad \text{if } d = 2\,,$$
$$(x_1 - x_{01})^2 + (x_2 - x_{02})^2 + (x_3 - x_{03})^2 < r^2 \qquad \text{if } d = 3\,.$$

Therefore $I_r(\boldsymbol{x}_0)$ is respectively the disc or the ball centred at $\boldsymbol{x}_0$ with radius r, without boundary.

Defining continuity is formally the same as for one real variable.

Definition 8.12 *A function* $f : \mathrm{dom}\, f \subseteq \mathbb{R}^d \to \mathbb{R}$ *is* **continuous at** $\boldsymbol{x}_0 \in \mathrm{dom}\, f$ *if for any* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that*

$$\forall x \in \mathrm{dom}\, f, \qquad \|\boldsymbol{x} - \boldsymbol{x}_0\| < \delta \quad \Rightarrow \quad |f(\boldsymbol{x}) - f(\boldsymbol{x}_0)| < \varepsilon\,.$$

Otherwise said, if

$$\forall x \in \mathrm{dom}\, f, \qquad x \in I_\delta(\boldsymbol{x}_0) \quad \Rightarrow \quad f(\boldsymbol{x}) \in I_\varepsilon\big(f(\boldsymbol{x}_0)\big).$$

Example 8.13

The map $f : \mathbb{R}^2 \to \mathbb{R}$, $f(\boldsymbol{x}) = 2x_1 + 5x_2$ is continuous at $\boldsymbol{x}_0 = (3, 1)$, for

$$\begin{aligned}|f(\boldsymbol{x}) - f(\boldsymbol{x}_0)| &= |2(x_1 - 3) + 5(x_2 - 1)| \\ &\le 2|x_1 - 3| + 5|x_2 - 1| \le 7\|\boldsymbol{x} - \boldsymbol{x}_0\|\,.\end{aligned}$$

Here we have used the fact that $|y_i| \le \|\boldsymbol{y}\|$ for any $i = 1, \ldots, d$ and any $\boldsymbol{y} \in \mathbb{R}^d$, a direct consequence of the definition of norm. Given $\varepsilon > 0$, it is sufficient to choose $\delta = \varepsilon/7$.

The same argument shows that f is continuous at every $\boldsymbol{x}_0 \in \mathbb{R}^2$. □

A map $f : \mathrm{dom}\, f \subseteq \mathbb{R}^d \to \mathbb{R}$ is continuous on the region $\Omega \subseteq \mathrm{dom}\, f$ if it is continuous at each point $\boldsymbol{x} \in \Omega$.

The limit for $\boldsymbol{x} \to \boldsymbol{x}_0 \in \mathbb{R}^d$ is defined in a completely similar way to the one given in Chap. 3.

8.5.2 Partial derivatives and gradient

Let $f : \operatorname{dom} f \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a function of two variables defined in a neighbourhood of $\boldsymbol{x}_0 = (x_0, y_0)$. Now fix the second variable to obtain a map of one real variable x defined around $x_0 \in \mathbb{R}$

$$x \mapsto f(x, y_0).$$

If this is differentiable at x_0, one says that the function f admits **partial derivative with respect to** x at $\boldsymbol{x}_0$, written

$$\frac{\partial f}{\partial x}(\boldsymbol{x}_0) = \frac{\mathrm{d}}{\mathrm{d}x} f(x, y_0)\Big|_{x=x_0}.$$

Similarly, if $y \mapsto f(x_0, y)$ is differentiable at y_0, one says that f admits partial derivative with respect to y at $\boldsymbol{x}_0$

$$\frac{\partial f}{\partial y}(\boldsymbol{x}_0) = \frac{\mathrm{d}}{\mathrm{d}y} f(x_0, y)\Big|_{y=y_0}.$$

If both conditions above hold, f admits (first) partial derivatives at $\boldsymbol{x}_0$, and therefore the **gradient** vector of f at $\boldsymbol{x}_0$ is well defined: this is denoted

$$\nabla f(\boldsymbol{x}_0) = \Big(\frac{\partial f}{\partial x}(\boldsymbol{x}_0), \frac{\partial f}{\partial y}(\boldsymbol{x}_0)\Big) \in \mathbb{R}^2 .$$

In the same fashion, let $f : \operatorname{dom} f \subseteq \mathbb{R}^3 \to \mathbb{R}$ be a function of three variables defined around $\boldsymbol{x}_0 = (x_0, y_0, z_0)$; the (first) partial derivatives at $\boldsymbol{x}_0$ with respect to x, y, z are

$$\frac{\partial f}{\partial x}(\boldsymbol{x}_0) = \frac{\mathrm{d}}{\mathrm{d}x} f(x, y_0, z_0)\Big|_{x=x_0},$$
$$\frac{\partial f}{\partial y}(\boldsymbol{x}_0) = \frac{\mathrm{d}}{\mathrm{d}y} f(x_0, y, z_0)\Big|_{y=y_0},$$
$$\frac{\partial f}{\partial z}(\boldsymbol{x}_0) = \frac{\mathrm{d}}{\mathrm{d}z} f(x_0, y_0, z)\Big|_{z=z_0},$$

assuming implicitly that the right-hand-side terms exist. The gradient of f at $\boldsymbol{x}_0$ is the vector

$$\nabla f(\boldsymbol{x}_0) = \Big(\frac{\partial f}{\partial x}(\boldsymbol{x}_0), \frac{\partial f}{\partial y}(\boldsymbol{x}_0), \frac{\partial f}{\partial z}(\boldsymbol{x}_0)\Big) \in \mathbb{R}^3 .$$

Examples 8.14

i) Let $f(x,y) = \sqrt{x^2+y^2}$ be the distance function from the origin. Considering $\boldsymbol{x}_0 = (2,-1)$ we have

$$\frac{\partial f}{\partial x}(2,-1) = \Big(\frac{\mathrm{d}}{\mathrm{d}x}\sqrt{x^2+1}\Big)(2) = \frac{x}{\sqrt{x^2+1}}_{|x=2} = \frac{2}{\sqrt{5}}$$

$$\frac{\partial f}{\partial y}(2,-1) = \Big(\frac{\mathrm{d}}{\mathrm{d}y}\sqrt{4+y^2}\Big)(-1) = \frac{y}{\sqrt{4+y^2}}\bigg|_{y=-1} = -\frac{1}{\sqrt{5}}\,.$$

so

$$\nabla f(2,-1) = \big(\frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}}\big) = \frac{1}{\sqrt{5}}(2,-1)\,.$$

ii) For $f(x,y,z) = y\log(2x-3z)$ we have, at $\boldsymbol{x}_0 = (2,3,1)$,

$$\frac{\partial f}{\partial x}(2,3,1) = \Big(\frac{\mathrm{d}}{\mathrm{d}x}\,3\log(2x-3)\Big)(2) = 3\frac{2}{2x-3}\bigg|_{x=2} = 6\,,$$

$$\frac{\partial f}{\partial y}(2,3,1) = \Big(\frac{\mathrm{d}}{\mathrm{d}y}\,y\log 1\Big)(3) = 0\,,$$

$$\frac{\partial f}{\partial z}(2,3,1) = \Big(\frac{\mathrm{d}}{\mathrm{d}z}\,3\log(4-3z)\Big)(1) = 3\frac{-3}{4-3z}\bigg|_{z=1} = -9\,,$$

thus

$$\nabla f(2,3,1) = (6,0,-9)\,.$$

□

Set $\boldsymbol{x} = (x_1,\ldots,x_d)$. The partial derivative of f at $\boldsymbol{x}_0$ with respect to the variable x_i, $i=1,\ldots,d$, is often indicated also by

$$\mathrm{D}_{x_i} f(\boldsymbol{x}_0) \quad \text{or} \quad f_{x_i}(\boldsymbol{x}_0)\,.$$

The function

$$\frac{\partial f}{\partial x_i} : \boldsymbol{x} \mapsto \frac{\partial f}{\partial x_i}(\boldsymbol{x})\,,$$

defined on a subset $\operatorname{dom}\frac{\partial f}{\partial x_i} \subseteq \operatorname{dom} f \subseteq \mathbb{R}^d$ with values in $\mathbb{R}$, is called **partial derivative** of f with respect to x_i. The **gradient function** of f,

$$\nabla f : \boldsymbol{x} \mapsto \nabla f(\boldsymbol{x}),$$

is defined on the intersection of the domains of the partial derivatives. The gradient is an example of a *vector field*, i.e., a function defined on a subset of $\mathbb{R}^d$ with values in $\mathbb{R}^d$ (thought of as a vector space).

Examples 8.15

Let us look at the previous examples.

i) The gradient of $f(x,y) = \sqrt{x^2+y^2}$ is

$$\nabla f(\boldsymbol{x}) = \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}} \right) = \frac{\boldsymbol{x}}{\|\boldsymbol{x}\|}$$

and $\operatorname{dom} \nabla f = \mathbb{R}^2 \setminus \{\mathbf{0}\}$.

ii) For the function $f(x,y,z) = y \log(2x-3z)$ we have

$$\nabla f(\boldsymbol{x}) = \left(\frac{2y}{2x-3z}, \log(2x-3z), \frac{-3y}{2x-3z} \right),$$

so $\operatorname{dom} \nabla f = \operatorname{dom} f = \{(x,y,z) \in \mathbb{R}^3 : 2x-3z > 0\}$. □

Partial derivatives with respect to x_i, $i = 1, \ldots, d$ are special directional derivatives, which we discuss hereby. Let f be a map defined around a point $\boldsymbol{x}_0 \in \mathbb{R}^d$ and suppose $\boldsymbol{v} \in \mathbb{R}^d$ is a given non-zero vector. By definition, f admits **(partial) derivative** at $\boldsymbol{x}_0$ **in the direction** $\boldsymbol{v}$ if the quantity

$$\frac{\partial f}{\partial \boldsymbol{v}}(\boldsymbol{x}_0) = \lim_{t \to 0} \frac{f(\boldsymbol{x}_0 + t\boldsymbol{v}) - f(\boldsymbol{x}_0)}{t}$$

exists and is finite. Another name is **directional derivative along** $\boldsymbol{v}$, written $\mathrm{D}_{\boldsymbol{v}} f(\boldsymbol{x}_0)$.

The condition expresses the differentiability at $t_0 = 0$ of the map $t \mapsto f(\boldsymbol{x}_0 + t\boldsymbol{v})$ defined around t_0 (because if t is small enough, $\boldsymbol{x}_0 + t\boldsymbol{v}$ is in the neighbourhood of $\boldsymbol{x}_0$ where f is well defined). The curve $t \mapsto \boldsymbol{x}_0 + t\boldsymbol{v} = \boldsymbol{\gamma}(t)$ is a parametrisation of the straight line passing through $\boldsymbol{x}_0$ with direction $\boldsymbol{v}$, and $(f \circ \boldsymbol{\gamma})(t) = f(\boldsymbol{x}_0 + t\boldsymbol{v})$. The directional derivative at $\boldsymbol{x}_0$ along $\boldsymbol{v}$ is therefore

$$\frac{\partial f}{\partial \boldsymbol{v}}(\boldsymbol{x}_0) = \left(\frac{\mathrm{d}}{\mathrm{d}t} f \circ \boldsymbol{\gamma} \right)(0).$$

Let $\boldsymbol{e}_i$ be the unit vector whose ith component is 1 and all others zero (so $\boldsymbol{e}_1 = \boldsymbol{i}$, $\boldsymbol{e}_2 = \boldsymbol{j}$, $\boldsymbol{e}_3 = \boldsymbol{k}$). Taking $\boldsymbol{v} = \boldsymbol{e}_i$ gives the partial derivative at $\boldsymbol{x}_0$ with respect to x_i

$$\frac{\partial f}{\partial \boldsymbol{e}_i}(\boldsymbol{x}_0) = \frac{\partial f}{\partial x_i}(\boldsymbol{x}_0), \qquad i = 1, \ldots, d.$$

For example, let $d = 2$ and $i = 1$: from

$$f(\boldsymbol{x}_0 + t\boldsymbol{e}_1) = f\big((x_0, y_0) + t(1,0)\big) = f(x_0 + t, y_0)$$

we obtain, substituting $x = x_0 + t$,

$$\begin{aligned}\frac{\partial f}{\partial \boldsymbol{e}_i}(x_0, y_0) &= \lim_{t \to 0} \frac{f(x_0 + t, y_0) - f(x_0, y_0)}{t} \\ &= \lim_{x \to x_0} \frac{f(x, y_0) - f(x_0, y_0)}{x - x_0} = \frac{\partial f}{\partial x}(x_0, y_0).\end{aligned}$$

It can be proved that if f admits partial derivatives with respect to every variable x_i in a whole neighbourhood of $\boldsymbol{x}_0$, and if such maps are in this neighbourhood continuous, then f admits at $\boldsymbol{x}_0$ derivatives along any vector $\boldsymbol{v} \neq \boldsymbol{0}$; these directional derivatives can be written using the gradient as follows

$$\frac{\partial f}{\partial \boldsymbol{v}}(\boldsymbol{x}_0) = \boldsymbol{v} \cdot \nabla f(\boldsymbol{x}_0) = v_1 \frac{\partial f}{\partial x_1}(\boldsymbol{x}_0) + \cdots + v_d \frac{\partial f}{\partial x_d}(\boldsymbol{x}_0)\,.$$

From this formula we also deduce the useful relations

$$\frac{\partial f}{\partial x_i}(\boldsymbol{x}_0) = \boldsymbol{e}_i \cdot \nabla f(\boldsymbol{x}_0), \qquad i = 1, \ldots, d\,.$$

Under the same assumptions on f, if $\boldsymbol{\gamma} : I \to \mathbb{R}^d$ is any differentiable curve at $t_0 \in I$ such that $\boldsymbol{\gamma}(t_0) = \boldsymbol{x}_0$, the composite map $(f \circ \boldsymbol{\gamma})(t) = f\big(\boldsymbol{\gamma}(t)\big)$ remains differentiable at t_0 and

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} f \circ \boldsymbol{\gamma}\right)(t) = \boldsymbol{\gamma}'(t_0) \cdot \nabla f(\boldsymbol{x}_0)\,; \tag{8.42}$$

this should be understood as a generalisation of the chain rule seen for one real variable.

Example 8.16

Consider the distance function $f(x,y) = \sqrt{x^2+y^2}$ and let $\boldsymbol{\gamma} : (0,+\infty) \to \mathbb{R}^2$ be the spiral $\boldsymbol{\gamma}(t) = (t\cos t, t\sin t)$. Since

$$f\big(\boldsymbol{\gamma}(t)\big) = \sqrt{t^2\cos^2 t + t^2 \sin^2 t} = t\,,$$

we see directly that $\dfrac{\mathrm{d}}{\mathrm{d}t} f\big(\boldsymbol{\gamma}(t)\big) = 1$ for any $t > 0$. Let us double-check the same result using (8.42). Define $\boldsymbol{x} = \boldsymbol{\gamma}(t)$ and the unit vector $\hat{\boldsymbol{x}} = \frac{\boldsymbol{x}}{\|\boldsymbol{x}\|} = (\cos t, \sin t)$. Then $\boldsymbol{\gamma}'(t) = (\cos t, \sin t) + t(-\sin t, \cos t) = \hat{\boldsymbol{x}} + t\hat{\boldsymbol{x}}^{\perp}$; the notation for the unit vector $\hat{\boldsymbol{x}}^{\perp} = (-\sin t, \cos t)$ is due to $\hat{\boldsymbol{x}}^{\perp} \cdot \hat{\boldsymbol{x}} = 0$. We already know though (Example 8.15) that $\nabla f(\boldsymbol{x}) = \hat{\boldsymbol{x}}$ for any $\boldsymbol{x} \neq \boldsymbol{0}$. Therefore

$$\boldsymbol{\gamma}'(t) \cdot \nabla f(\boldsymbol{x}) = (\hat{\boldsymbol{x}} + t\hat{\boldsymbol{x}}^{\perp}) \cdot \hat{\boldsymbol{x}} = \hat{\boldsymbol{x}} \cdot \hat{\boldsymbol{x}} + t\,\hat{\boldsymbol{x}}^{\perp} \cdot \hat{\boldsymbol{x}} = \|\hat{\boldsymbol{x}}\|^2 = 1\,,$$

as expected. □

8.6 Exercises

1. *Determine the polar coordinates of the following points in the plane:*

$$A = (5\sqrt{6}, 5\sqrt{2})\,, \qquad B = (5\sqrt{6}, -5\sqrt{2})\,,$$

$$C = (-5\sqrt{6}, 5\sqrt{2})\,, \qquad D = (-5\sqrt{6}, -5\sqrt{2})\,.$$

2. *Write the following points of the plane in polar coordinates:*

   a) $A = (-5, 0)$ b) $B = (0, 4)$ c) $C = (0, -3)$

3. *Determine the polar coordinates of the following points (without computing explicitly the angle):*

   a) $A = (2\sqrt{3} - 3\sqrt{2}, 1)$ b) $B = (3\sqrt{2} - 2\sqrt{3}, 3\sqrt{2} + 2\sqrt{3})$

4. *Determine the polar coordinates of the following points in the plane (leaving the argument written in terms of a trigonometric function):*

$$A = \big(\cos\frac{\pi}{9}, \sin\frac{\pi}{9}\big)\,, \qquad B = \big(-\cos\frac{\pi}{9}, \sin\frac{\pi}{9}\big)\,, \qquad C = \big(\sin\frac{\pi}{9}, \cos\frac{\pi}{9}\big)\,.$$

5. *Change to polar coordinates:*

   a) $A = \big(\frac{\sqrt{2}}{2}\cos\frac{\pi}{9} - \frac{\sqrt{2}}{2}\sin\frac{\pi}{9}, \frac{\sqrt{2}}{2}\cos\frac{\pi}{9} + \frac{\sqrt{2}}{2}\sin\frac{\pi}{9}\big)$

   b) $B = \big(2\cos\frac{28}{9}\pi, 2\sin\frac{28}{9}\pi\big)$

6. *Given* $\boldsymbol{v}_1 = (1, 0, -2)$ *and* $\boldsymbol{v}_2 = (0, 1, 1)$, *find a real number* λ *so that* $\boldsymbol{v}_1 + \lambda\boldsymbol{v}_2$ *is orthogonal to* $\boldsymbol{v}_3 = (-1, 1, 1)$.

7. *Describe the set of planar vectors orthogonal to* $\boldsymbol{v} = (2, -5)$.

8. *Determine the set of vectors in space orthogonal to* $\boldsymbol{v}_1 = (1, 0, 2)$ *and* $\boldsymbol{v}_2 = (2, -1, 3)$ *simultaneously.*

9. *Find the norm of the vectors:*

$$\boldsymbol{v}_1 = (0, \sqrt{3}, 7)\,, \quad \boldsymbol{v}_2 = (1, 5, -2)\,, \quad \boldsymbol{v}_3 = \big(\cos\frac{\pi}{5}, \sin\frac{\pi}{5}\cos\frac{\pi}{7}, -\sin\frac{\pi}{5}\sin\frac{\pi}{7}\big)\,.$$

10. *Determine the cosine of the angle formed by the following pairs:*

    a) $\boldsymbol{v} = (0, 1, 0)\,, \quad \boldsymbol{w} = (0, \frac{2}{\sqrt{3}}, 2)$ b) $\boldsymbol{v} = (1, 2, -1)\,, \quad \boldsymbol{w} = (-1, 1, 1)$

11. *Determine the unit vector* $\boldsymbol{u}$ *corresponding to* $\boldsymbol{w} = (5, -3, -\sqrt{2})$. *Then find the component of* $\boldsymbol{v} = (2, -1, 2\sqrt{2})$ *along* $\boldsymbol{u}$ *and the orthogonal one.*

12. *Write the following complex numbers in Cartesian form:*

    a) $(2 - 3i)(-2 + i)$ b) $(3 + i)(3 - i)\left(\frac{1}{5} + \frac{1}{10}i\right)$

    c) $\dfrac{1 + 2i}{3 - 4i} + \dfrac{2 - i}{5i}$ d) $\dfrac{5}{(1 - i)(2 - i)(3 - i)}$

13. *Determine the trigonometric and exponential forms of:*

a) $z = i$ b) $z = -1$

c) $z = 1 + i$ d) $z = i(1 + i)$

e) $z = \dfrac{1+i}{1-i}$ f) $z = \sin\alpha + i\cos\alpha$

14. *Compute the modulus of:*

a) $z = \dfrac{1}{1-i} + \dfrac{2i}{i-1}$ b) $z = 1 + i - \dfrac{i}{1-2i}$

$\boxed{15.}$ *Prove that* $\left|\dfrac{3z-i}{3+iz}\right| = 1$ *if* $|z| = 1$.

16. *Solve the following equations:*

a) $z^2 - 2z + 2 = 0$ $\boxed{\text{b)}}$ $z^2 + 3iz + 1 = 0$

$\boxed{\text{c)}}$ $z|z| - 2z + i = 0$ d) $|z|^2 z^2 = i$

e) $z^2 + i\bar{z} = 1$ $\boxed{\text{f)}}$ $z^3 = |z|^4$

$\boxed{17.}$ *Verify* $1 + i$ *is a root of the polynomial* $z^4 - 5z^3 + 10z^2 - 10z + 4$ *and then determine all remaining roots.*

18. *Compute* z^2, z^9, z^{20} *when:*

a) $z = \dfrac{1-i}{i}$ $\boxed{\text{b)}}$ $z = \dfrac{2}{\sqrt{3}-i} + \dfrac{1}{i}$

19. *Write explicitly the following numbers in one of the known forms and draw their position in the plane:*

a) $z = \sqrt[3]{-i}$ $\boxed{\text{b)}}$ $z = \sqrt[5]{1}$ c) $z = \sqrt{2-2i}$

20. *Determine the domains of the functions:*

$\boxed{\text{a)}}$ $f(x,y) = \dfrac{x - 3y + 7}{x - y^2}$

$\boxed{\text{b)}}$ $f(x,y) = \sqrt{1 - 3xy}$

$\boxed{\text{c)}}$ $f(x,y) = \sqrt{3x + y + 1} - \dfrac{1}{\sqrt{2y - x}}$

$\boxed{\text{d)}}$ $f(x,y,z) = \log(x^2 + y^2 + z^2 - 9)$

21. *Calculate all partial derivatives of:*

a) $f(x,y) = \sqrt{3x+y^2}$ *at* $(x_0, y_0) = (1,2)$

b) $f(x,y,z) = y\mathrm{e}^{x+yz}$ *at* $(x_0, y_0, z_0) = (0,1,-1)$

22. *Find the gradient for:*

a) $f(x,y) = \arctan \dfrac{x+y}{x-y}$

b) $f(x,y) = (x+y)\log(2x-y)$

c) $f(x,y,z) = \sin(x+y)\cos(y-z)$

d) $f(x,y,z) = (x+y)^z$

23. *Compute the directional derivatives of the following maps along the vector* $\boldsymbol{v}$ *and evaluate them at the point indicated:*

a) $f(x,y) = x\sqrt{y-3}$ $\quad \boldsymbol{v} = (-1,6)$ $\quad \boldsymbol{x}_0 = (2,12)$

b) $f(x,y,z) = \dfrac{1}{x+2y-3z}$ $\quad \boldsymbol{v} = (12,-9,-4)$ $\quad \boldsymbol{x}_0 = (1,1,-1)$

8.6.1 Solutions

1. All four points have modulus $r = \sqrt{25 \cdot 6 + 25 \cdot 2} = 5\sqrt{8}$. Formula (8.2) yields, for A,

$$\theta_A = \arctan \frac{5\sqrt{2}}{5\sqrt{6}} = \arctan \frac{1}{\sqrt{3}} = \frac{\pi}{6}$$

since $x > 0$. Similarly for B

$$\theta_B = \arctan\left(-\frac{1}{\sqrt{3}}\right) = -\arctan \frac{1}{\sqrt{3}} = -\frac{\pi}{6}\,.$$

For the point C, since $x < 0$ and $y > 0$,

$$\theta_C = \arctan\left(-\frac{1}{\sqrt{3}}\right) + \pi = -\frac{\pi}{6} + \pi = \frac{5}{6}\pi\,,$$

while $x < 0, y < 0$ for D, so

$$\theta_D = \arctan \frac{1}{\sqrt{3}} - \pi = \frac{\pi}{6} - \pi = -\frac{5}{6}\pi\,.$$

2. *Polar coordinates in the plane:*

a) $r = 5\,, \quad \theta = \pi\,;$ $\qquad$ b) $r = 4\,, \quad \theta = \frac{\pi}{2}\,;$ $\qquad$ c) $r = 3\,, \quad \theta = -\frac{\pi}{2}\,.$

3. *Polar coordinates:*

a) The modulus is $r = \sqrt{31 - 12\sqrt{6}}$. From $2\sqrt{3} < 3\sqrt{2}$ we have

$$\begin{aligned}\theta &= \arctan \frac{1}{2\sqrt{3} - 3\sqrt{2}} + \pi = \arctan \frac{2\sqrt{3} + 3\sqrt{2}}{-6} + \pi \\ &= -\arctan\Big(\frac{\sqrt{3}}{3} + \frac{\sqrt{2}}{2}\Big) + \pi\,.\end{aligned}$$

b) $r = 5\sqrt{6}\,, \quad \theta = \arctan(5 + 2\sqrt{6})\,.$

4. All points have unit modulus $r = 1$. For A

$$\theta_A = \arctan\tan\frac{\pi}{9} = \frac{\pi}{9}\,.$$

For B, $x < 0$ and $y > 0$, so

$$\theta_B = \arctan\Big(-\tan\frac{\pi}{9}\Big) + \pi = -\frac{\pi}{9} + \pi = \frac{8}{9}\pi\,.$$

As for C,

$$\theta_C = \arctan\frac{\cos\frac{\pi}{9}}{\sin\frac{\pi}{9}}\,;$$

by (2.17), and since the tangent function has period π, it follows

$$\frac{\cos\frac{\pi}{9}}{\sin\frac{\pi}{9}} = -\frac{\sin(\frac{\pi}{9} + \frac{\pi}{2})}{\cos(\frac{\pi}{9} + \frac{\pi}{2})} = -\tan\frac{11}{18}\pi = -\tan\Big(-\frac{7}{18}\pi\Big) = \tan\frac{7}{18}\pi\,,$$

hence $\theta_C = \frac{7}{18}\pi$.

5. *Polar coordinates:*

a) Just note $\frac{\sqrt{2}}{2} = \sin\frac{\pi}{4} = \cos\frac{\pi}{4}$ and apply the addition formulas for sine/cosine:

$$A = \Big(\cos\big(\frac{\pi}{4} + \frac{\pi}{9}\big), \sin\big(\frac{\pi}{4} + \frac{\pi}{9}\big)\Big) = \Big(\cos\frac{13}{36}\pi, \sin\frac{13}{36}\pi\Big)\,.$$

Because $\frac{13}{36}\pi < \frac{\pi}{2}$, we immediately have $r = 1$ and $\theta = \frac{13}{36}\pi$.

b) $r = 2\,, \quad \theta = -\frac{8}{9}\pi\,.$

6. The vectors $\boldsymbol{v}_1 + \lambda\boldsymbol{v}_2$ and $\boldsymbol{v}_3$ are orthogonal if $(\boldsymbol{v}_1 + \lambda\boldsymbol{v}_2)\cdot\boldsymbol{v}_3 = 0$. But

$$(\boldsymbol{v}_1 + \lambda\boldsymbol{v}_2)\cdot\boldsymbol{v}_3 = \boldsymbol{v}_1\cdot\boldsymbol{v}_3 + \lambda\boldsymbol{v}_2\cdot\boldsymbol{v}_3 = -3 + 2\lambda\,,$$

whence $\lambda = \frac{3}{2}$ follows.

7. A vector (x, y) is orthogonal to $\boldsymbol{v}$ if $(x, y)\cdot(2, -5) = 2x - 5y = 0$. The required set is then made by the vectors lying on the straight line $2x - 5y = 0$. One way to describe this set is $\{\lambda(5, 2) : \lambda \in \mathbb{R}\}$.

8. Imposing $\boldsymbol{w} = (x, y, z)$ orthogonal to $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ yields $\boldsymbol{w} \cdot \boldsymbol{v}_1 = x + 2z = 0$ plus $\boldsymbol{w} \cdot \boldsymbol{v}_2 = 2x - y + 3z = 0$, hence $x = -2z$ and $y = -z$. Put $z = \lambda$, and the set becomes $\{\lambda(-2, -1, 1) : \lambda \in \mathbb{R}\}$.

9. $\|\boldsymbol{v}_1\| = \sqrt{52}\,, \qquad \|\boldsymbol{v}_2\| = \sqrt{30}\,, \qquad \|\boldsymbol{v}_3\| = 1\,.$

10. *Angles between vectors:*

a) $\cos\theta = \frac{1}{2}$; b) $\cos\theta = 1$.

11. From $\|\boldsymbol{w}\| = 6$ it follows $\boldsymbol{u} = \left(\frac{5}{6}, -\frac{1}{2}, -\frac{\sqrt{2}}{6}\right)$. Since $\boldsymbol{v} \cdot \boldsymbol{w} = \frac{3}{2}$,

$$\boldsymbol{v}_u = \left(\frac{5}{4}, -\frac{3}{4}, -\frac{\sqrt{2}}{4}\right),$$
$$\boldsymbol{v}_{u^\perp} = (2, -1, 2\sqrt{2}) - \left(\frac{5}{4}, -\frac{3}{4}, -\frac{\sqrt{2}}{4}\right) = \left(\frac{3}{4}, -\frac{1}{4}, \frac{9}{4}\sqrt{2}\right).$$

12. *Cartesian form of complex numbers:*

a) $-1 + 8i$; b) $2 + i$; c) $-\frac{2}{5}$; d) $\frac{1}{2}i$.

13. *Exponential and trigonometric form:*

a) $z = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = e^{i\frac{\pi}{2}}$; b) $z = \cos\pi + i\sin\pi = e^{i\pi}$;

c) $z = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = \sqrt{2}e^{i\frac{\pi}{4}}$; d) $z = \sqrt{2}\left(\cos\frac{3}{4}\pi + i\sin\frac{3}{4}\pi\right) = \sqrt{2}e^{i\frac{3}{4}\pi}$;

e) $\cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = e^{i\frac{\pi}{2}}$; f) $\cos\left(\frac{\pi}{2} - \alpha\right) + i\sin\left(\frac{\pi}{2} - \alpha\right) = e^{i(\frac{\pi}{2} - \alpha)}$.

14. *Modulus of complex numbers:*

a) $\sqrt{\frac{5}{2}}$; b) $\sqrt{\frac{13}{5}}$.

15. We proceed indirectly and multiply first the denominator by $|\bar{z}|$ $(= 1)$ to get

$$\left|\frac{3z - i}{3 + iz}\right| = \left|\frac{3z - i}{3\bar{z} + i}\right| = \left|\frac{3z - i}{\overline{\overline{3z - i}}}\right| = \frac{|3z - i|}{|\overline{3z - i}|} = 1\,.$$

16. *Solving equations:*

a) $z = 1 \pm i$.

b) The formula for a quadratic equation gives

$$z = \frac{-3i \pm \sqrt{-9 - 4}}{2} = \frac{-3i \pm \sqrt{13}i}{2} = \frac{-3 \pm \sqrt{13}}{2}i\,.$$

c) Write $z = x + iy$, so that the equation reads

$$(x+iy)\sqrt{x^2+y^2} - 2x - 2iy + i = 0\,,$$

or equivalently,

$$x\sqrt{x^2+y^2} - 2x + i\left(y\sqrt{x^2+y^2} - 2y + 1\right) = 0\,.$$

The real and imaginary parts at the two sides of the equality must be the same, so

$$\begin{cases} x\left(\sqrt{x^2+y^2} - 2\right) = 0 \\ y\sqrt{x^2+y^2} - 2y + 1 = 0\,. \end{cases}$$

The first equation in the system implies either $x = 0$ or $\sqrt{x^2+y^2} = 2$. Substituting 2 to the square root in the second equation gives $1 = 0$, which cannot be. Therefore the only solutions are

$$\begin{cases} x = 0 \\ y|y| - 2y + 1 = 0\,. \end{cases}$$

Distinguishing the cases $y \geq 0$ and $y < 0$, we have

$$\begin{cases} x = 0 \\ y^2 - 2y + 1 = 0\,, \end{cases} \quad \text{and} \quad \begin{cases} x = 0 \\ -y^2 - 2y + 1 = 0 \end{cases}$$

so

$$\begin{cases} x = 0 \\ y = 1 \end{cases} \quad \text{and} \quad \begin{cases} x = 0 \\ y = -1 \pm \sqrt{2}\,. \end{cases}$$

In conclusion, the solutions are $z = i$, $z = i(-1-\sqrt{2})$ (because $y = -1+\sqrt{2} > 0$ must be discarded).

d) $z = \pm\dfrac{\sqrt{2}}{2}(1+i)\,;$ e) $z = \dfrac{\sqrt{7}}{2} - i\dfrac{1}{2}\,;\ z = -\dfrac{\sqrt{7}}{2} - i\dfrac{1}{2}\,.$

f) Using $|z|^2 = z\bar{z}$, the new equation is

$$z^3 = z^2\bar{z}^2 \qquad \Longleftrightarrow \qquad z^2(z - \bar{z}^2) = 0\,.$$

One solution is certainly $z = 0$, and the others satisfy $z - \bar{z}^2 = 0$. Write $z = x + iy$, so to obtain

$$\begin{cases} x^2 - y^2 - x = 0 \\ 2xy + y = 0\,. \end{cases}$$

The bottom relation factorises into $y(2x+1) = 0$, hence we have two subsystems

$$\begin{cases} y = 0 \\ x(x-1) = 0\,, \end{cases} \qquad \begin{cases} x = -\frac{1}{2} \\ y^2 = \frac{3}{4}\,. \end{cases}$$

Putting real and imaginary parts back together, the solutions read

$$z = 0\,; \qquad z = 1\,; \qquad z = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i\,.$$

17. The fact that the polynomial has real coefficients implies the existence of the complex-conjugate $\bar{z} = 1-i$ to $z = 1+i$ as root. This means $(z-1-i)(z-1+i) = z^2 - 2z + 2$ divides the polynomial, indeed

$$z^4 - 5z^3 + 10z^2 - 10z + 4 = (z^2-2z+2)(z^2-3z+2) = (z^2-2z+2)(z-1)(z-2)\,.$$

Thus the roots are

$$z = 1+i\,, \qquad z = 1-i\,, \qquad z = 1\,, \qquad z = 2\,.$$

18. *Powers of complex numbers:*

a) $z^2 = 2i\,, \qquad z^9 = -16(1+i)\,, \qquad z^{20} = -2^{10}\,.$

b) Rationalising the denominators yields

$$z = 2\,\frac{\sqrt{3}+i}{4} - i = \frac{1}{2}(\sqrt{3}-i)\,.$$

Now write the number in exponential form

$$z = \frac{1}{2}(\sqrt{3}-i) = \mathrm{e}^{-\frac{\pi}{6}i},$$

from which

$$\begin{aligned} z^2 &= \mathrm{e}^{-\frac{\pi}{3}i} = \cos\frac{\pi}{3} - i\sin\frac{\pi}{3} = \frac{1}{2}(1-\sqrt{3}i)\,; \\ z^9 &= \mathrm{e}^{-\frac{3}{2}\pi i} = \mathrm{e}^{\frac{\pi}{2}i} = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = i\,, \\ z^{20} &= \mathrm{e}^{-\frac{20}{6}\pi i} = \mathrm{e}^{\frac{2}{3}\pi i} = \frac{1}{2}(-1+\sqrt{3}i)\,. \end{aligned}$$

19. *Computing and drawing complex numbers:*

a) $z_0 = i\,, \qquad z_1 = -\frac{1}{2}\big(\sqrt{3}+i\big)\,, \qquad z_2 = \frac{1}{2}\big(\sqrt{3}-i\big)$
They are drawn in Fig. 8.21, left.

b) Write the number 1 as $1 = \mathrm{e}_{0\pi i}$. Then because $\mathrm{e}_{a+2\pi} = \mathrm{e}_a$, we have

$$z_0 = 1\,, \quad z_1 = \mathrm{e}^{\frac{2}{5}\pi i}\,, \quad z_2 = \mathrm{e}^{\frac{4}{5}\pi i}\,, \quad z_3 = \mathrm{e}^{-\frac{4}{5}\pi i}\,, \quad z_4 = \mathrm{e}^{-\frac{2}{5}\pi i}\,,$$

see Fig. 8.21, middle.

c) $z_0 = \sqrt[4]{8}\mathrm{e}^{\frac{7}{8}\pi i}\,, \qquad z_1 = \sqrt[4]{8}\mathrm{e}^{-\frac{1}{8}\pi i}$ are represented in Fig. 8.21, right.

20. *Domain of functions:*

a) The domain is $\{(x,y) \in \mathbb{R}^2 : x \neq y^2\}$, the set of all plane points off the parabola $x = y^2$.

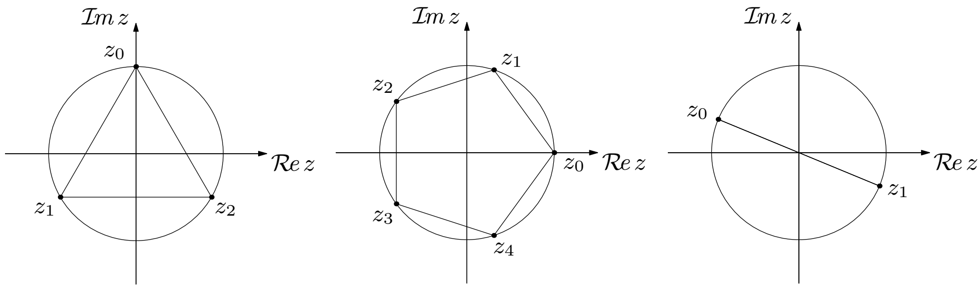


Figure 8.21. From left: cubic roots of $-i$, fifth roots of unity, square roots of $2 - 2i$

b) The map is well defined where the radicand is non-negative, so the domain is

$$\{(x, y) \in \mathbb{R}^2 : y \le \frac{1}{3x} \text{ if } x > 0,\ y \ge \frac{1}{3x} \text{ if } x < 0,\ y \in \mathbb{R} \text{ if } x = 0\},$$

the set of points lying between the branches of the hyperbola $y = \frac{1}{3x}$.

c) Only for $3x + y + 1 \ge 0$ and $2y - x > 0$ the function is defined, which makes

$$\{(x, y) \in \mathbb{R}^2 : y \ge -3x - 1\} \cap \{(x, y) \in \mathbb{R}^2 : y > \frac{x}{2}\}$$

the domain of the function, represented in Fig. 8.22.

d) The map is well defined where the logarithm's argument is positive, hence the domain is the subset of space

$$\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + x^2 > 9\}.$$

These are the points outside the sphere centred at the origin and with radius three.

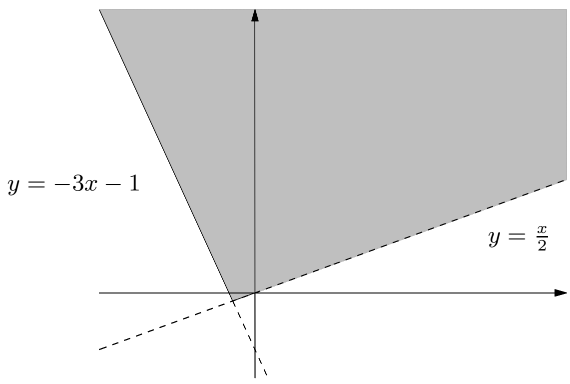


Figure 8.22. The domain of the map $f(x, y) = \sqrt{3x + y + 1} - \frac{1}{\sqrt{2y - x}}$

21. *Partial derivatives:*

a) $\dfrac{\partial f}{\partial x}(1,2) = \dfrac{3}{2\sqrt{7}}\,, \quad \dfrac{\partial f}{\partial y}(1,2) = \dfrac{2}{\sqrt{7}}$.

b) $\dfrac{\partial f}{\partial x}(0,1,-1) = \mathrm{e}^{-1}\,, \quad \dfrac{\partial f}{\partial y}(0,1,-1) = 0\,, \quad \dfrac{\partial f}{\partial z}(0,1,-1) = \mathrm{e}^{-1}\,.$

22. *Gradients:*

a) $\nabla f(x,y) = \left(-\dfrac{y}{x^2+y^2}\,, \dfrac{x}{x^2+y^2}\right)\,.$

b) $\nabla f(x,y) = \left(\log(2x-y) + \dfrac{2(x+y)}{2x-y}\,, \log(2x-y) - \dfrac{x+y}{2x-y}\right)\,.$

c) $\nabla f(x,y,z) = \big(\cos(x+y)\cos(y-z)\,, \cos(x+2y-z)\,, \sin(x+y)\sin(y-z)\big)\,.$

d) $\nabla f(x,y,z) = \big(z(x+y)^{z-1}\,, z(x+y)^{z-1}\,, (x+y)^z \log(x+y)\big)\,.$

23. *Directional derivatives:*

a) $\dfrac{\partial f}{\partial \boldsymbol{v}}(\boldsymbol{x}_0) = -1\,;$ b) $\dfrac{\partial f}{\partial \boldsymbol{v}}(\boldsymbol{x}_0) = \dfrac{1}{2}$.

9

Integral calculus I

Integral calculus tackles two rather different issues:

i) Find all functions that differentiate to a given map over an interval of the real line. This operation is essentially an anti-derivation of sorts, and goes by the name of **indefinite integration**.

ii) Define precisely and compute the area of a region in the plane bounded by graphs of maps defined on closed bounded intervals, known as **definite integration**.

The two problems seem to have little in common, at first sight. The outcome of indefinite integration is, as we shall soon see, an infinite set of functions. Definite integration produces instead a number, the surface area of a certain planar region. A cornerstone result, not casually called the *Fundamental Theorem of integral calculus* lest its importance goes amiss, states that the two problems are actually equivalent: if one can reconstruct a map knowing its derivative, then it is not hard to find the area of the region bounded by the derivative's graph and the lines parallel to the coordinate axes, and vice versa.

The beginning of the chapter is devoted to the former problem. Then, we explain two constructions of definite integrals, due to Cauchy and Riemann; albeit strongly related, these are presented as separate items for the didactic purpose of keeping the treatise as versatile as possible. Only in later sections we discuss the properties of integrals in a uniform manner. Eventually, we prove the Fundamental Theorem of integral calculus and show how it is employed to determine areas.

9.1 Primitive functions and indefinite integrals

Let f be a function defined on some interval I.

Definition 9.1 *Each function F, differentiable on I, such that*

$$F'(x) = f(x), \qquad \forall x \in I,$$

is called a **primitive (function)** *or an* **antiderivative** *of f on I.*

Not any map defined on a real interval admits primitives: not necessarily, in other words, will any function be the derivative of some map. Finding all maps that admit primitives on a real interval, which we call **integrable** on that interval, is too-far-reaching a problem for this book's aims. We limit ourselves to point out an important class of integrable maps, that of **continuous** maps on a real interval; the fact that continuity implies integrability will follow from the Fundamental Theorem of integral calculus.

Examples 9.2

i) Given the map $f(x) = x$ on $\mathbb{R}$, a primitive function is $F(x) = \frac{1}{2}x^2$. The latter is not the only primitive of f: each map $G(x) = \frac{1}{2}x^2 + c$, where c is an arbitrary constant, is a primitive of f, because differentiating a constant gives nought.

ii) Consider $f(x) = \frac{1}{x}$ over the interval $I = (-\infty, 0)$. The collection of maps $F(x) = \log|x| + c$ $(c \in \mathbb{R})$ consists of primitives of f on I. □

The previous examples should explain that if $F(x)$ is a primitive of $f(x)$ on the interval I, then also maps of type $F(x) + c$, with c constant, are primitives. It becomes therefore natural to ask whether there are other primitives at all. The answer is no, as shown in the next crucial result.

Proposition 9.3 *If F and G are both primitive maps of f on the interval I, there exists a constant c such that*

$$G(x) = F(x) + c, \qquad \forall x \in I.$$

Proof. Take the function $H(x) = G(x) - F(x)$ and differentiate it

$$H'(x) = G'(x) - F'(x) = f(x) - f(x) = 0, \qquad \forall x \in I.$$

Thus H has zero derivative at every point of I, and as such it must be constant by Property 6.26. □

Summarising, the following characterisation of the set of primitives of f holds.

Theorem 9.4 *Let f be an integrable map on I and F a primitive. Then any primitive of f is of the form $F(x) + c$, with the constant c varying in $\mathbb{R}$.*

That in turn motivates the following name.

Definition 9.5 *The set of all primitives of f on a real interval is indicated by*

$$\int f(x)\,\mathrm{d}x$$

(*called* **indefinite integral** *of f, and spoken 'integral of $f(x)\ \mathrm{d}x$'*).

If F is a primitive then,

$$\int f(x)\,\mathrm{d}x = \{F(x) + c \ : \ c \in \mathbb{R}\}.$$

It has to be clear that the indefinite integral of f is *not* a number; it stands rather for a set of infinitely many maps. It is just quicker to omit the curly brackets and write

$$\int f(x)\,\mathrm{d}x = F(x) + c,$$

which might be sloppy but is certainly effective.

Examples 9.6

i) The map $f(x) = x^4$ resembles the derivative $5x^4 = \mathrm{D}\,x^5$, so a primitive of f is given by $F(x) = \frac{1}{5}x^5$ and

$$\int x^4\,\mathrm{d}x = \frac{1}{5}x^5 + c.$$

ii) Let $f(x) = \mathrm{e}^{2x}$. Recalling that $\mathrm{D}\,\mathrm{e}^{2x} = 2\mathrm{e}^{2x}$, the map $F(x) = \frac{1}{2}\mathrm{e}^{2x}$ is one primitive, hence

$$\int \mathrm{e}^{2x}\,\mathrm{d}x = \frac{1}{2}\mathrm{e}^{2x} + c.$$

iii) As $\mathrm{D}\cos 5x = -5\sin 5x$, $f(x) = \sin 5x$ has primitive $F(x) = -\frac{1}{5}\cos 5x$, and

$$\int \sin 5x\,\mathrm{d}x = -\frac{1}{5}\cos 5x + c.$$

iv) Let

$$f(x) = \sin|x| = \begin{cases} -\sin x & \text{if } x < 0\,, \\ \sin x & \text{if } x \geq 0\,. \end{cases}$$

We adopt the following strategy to determine all primitive maps of $f(x)$ on $\mathbb{R}$. We split the real line in two intervals $I_1 = (-\infty, 0)$, $I_2 = (0, +\infty)$ and discuss the cases separately. On I_1, primitives of $f(x)$ are of the form

$$F_1(x) = \cos x + c_1 \qquad \text{with } c_1 \in \mathbb{R} \text{ arbitrary;}$$

similarly, on I_2, a primitive will look like

$$F_2(x) = -\cos x + c_2 \qquad c_2 \in \mathbb{R} \text{ arbitrary.}$$

The general primitive $F(x)$ on $\mathbb{R}$ will be written as

$$F(x) = \begin{cases} F_1(x) & \text{if } x < 0, \\ F_2(x) & \text{if } x > 0. \end{cases}$$

Moreover F will have to be *continuous* at $x = 0$, because a primitive is by mere definition differentiable – hence continuous a fortiori – at every point in $\mathbb{R}$. We should thus make sure that the two primitives *agree*, by imposing

$$\lim_{x \to 0^-} F(x) = \lim_{x \to 0^+} F(x).$$

As F_1 and F_2 are continuous at $x = 0$, the condition reads $F_1(0) = F_2(0)$, that is

$$1 + c_1 = -1 + c_2.$$

The relation between c_1, c_2 allows to determine one constant in terms of the other (coherently with the fact that each primitive depends on one, *and only one*, arbitrary real number). For example, putting $c_1 = c$ gives $c_2 = 2 + c$. The expression for the general primitive of $f(x)$ on $\mathbb{R}$ is then

$$F(x) = \begin{cases} \cos x + c & \text{if } x < 0, \\ -\cos x + 2 + c & \text{if } x \geq 0. \end{cases}$$ □

Theorem 9.4 states that the graph of a primitive of an integrable map is the vertical translate of any other (see Fig. 9.1).

How to select a particular map among all primitives of a given f then? One way is to assign a value y_0 at a given point x_0 on I. The knowledge of any one primitive $F(x)$ determines the primitive $G(x) = F(x) + c_0$ of $f(x)$ whose value at

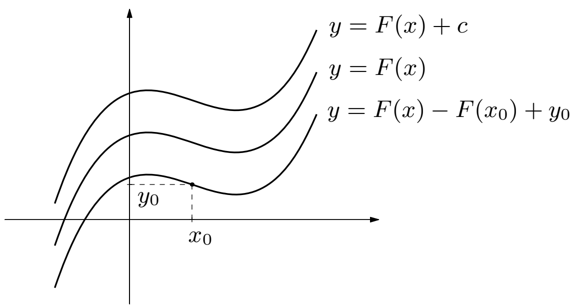


Figure 9.1. Primitives of a given map differ by an additive constant

x_0 is precisely y_0. In fact,

$$G(x_0) = F(x_0) + c_0 = y_0$$

yields $c_0 = y_0 - F(x_0)$ and so

$$G(x) = F(x) - F(x_0) + y_0.$$

The table of derivatives of the main elementary maps can be at this point read backwards, as a list of primitives. For instance,

$$
\begin{aligned}
&\text{a)} \qquad \int x^\alpha \,\mathrm{d}x = \frac{x^{\alpha+1}}{\alpha+1} + c \qquad && (\alpha \neq -1) \\
&\text{b)} \qquad \int \frac{1}{x} \,\mathrm{d}x = \log|x| + c && (\text{for } x > 0 \text{ or } x < 0) \\
&\text{c)} \qquad \int \sin x \,\mathrm{d}x = -\cos x + c && \\
&\text{d)} \qquad \int \cos x \,\mathrm{d}x = \sin x + c && \\
&\text{e)} \qquad \int \mathrm{e}^x \,\mathrm{d}x = \mathrm{e}^x + c && \\
&\text{f)} \qquad \int \frac{1}{1+x^2} \,\mathrm{d}x = \arctan x + c && \\
&\text{g)} \qquad \int \frac{1}{\sqrt{1-x^2}} \,\mathrm{d}x = \arcsin x + c &&
\end{aligned}
\tag{9.1}
$$

Examples 9.7

i) Determine the primitive of $f(x) = \cos x$ with value 5 at $x_0 = \frac{\pi}{2}$. The map $F(x) = \sin x$ is one primitive. We are then searching for a $G(x) = \sin x + c_0$. Imposing $G(\frac{\pi}{2}) = 5$ we see $c_0 = 4$, and the required primitive is

$$G(x) = \sin x + 4.$$

ii) Find the value at $x_1 = 3$ of the primitive of $f(x) = 6x^2 + 5x$ that vanishes at the point $x_0 = 1$. One primitive map of $f(x)$ is

$$F(x) = 2x^3 + \frac{5}{2}x^2.$$

If $G(x) = F(x) + c_0$ has to satisfy $G(1) = 0$, then $c_0 = -\frac{9}{2}$, whence

$$G(x) = 2x^3 + \frac{5}{2}x^2 - \frac{9}{2}.$$

The image of $x_1 = 3$ is $G(3) = 72$.

iii) Consider the piecewise-defined map

$$f(x) = \begin{cases} x & \text{if } x \le 1, \\ (x-2)^2 & \text{if } x \ge 1. \end{cases}$$

Mimicking Example 9.6 iv) we obtain

$$F(x) = \begin{cases} \frac{1}{2}x^2 + c_1 & \text{if } x < 1, \\ \frac{1}{3}(x-2)^3 + c_2 & \text{if } x > 1. \end{cases}$$

Continuity at $x = 1$ forces $\frac{1}{2} + c_1 = -\frac{1}{3} + c_2$. From this relation, writing $c_1 = c$ gives

$$F(x) = \begin{cases} \frac{1}{2}x^2 + c & \text{if } x < 1, \\ \frac{1}{3}(x-2)^3 + \frac{5}{6} + c & \text{if } x \ge 1. \end{cases}$$

Let us find the primitive of $f(x)$ with zero $x_0 = 3$. Since $x_0 > 1$, the second expression of $F(x)$

$$F(3) = \frac{1}{3}(3-2)^3 + \frac{5}{6} + c = 0$$

tells $c = -\frac{7}{6}$. It follows

$$F(x) = \begin{cases} \frac{1}{2}x^2 - \frac{7}{6} & \text{if } x < 1, \\ \frac{1}{3}(x-2)^3 - \frac{1}{3} & \text{if } x \ge 1. \end{cases}$$

Beware that it would have been wrong to make $\frac{1}{2}x^2 + c$ vanish at $x_0 = 3$, for this expression is a primitive *only when* $x < 1$ and not on the entire line.

Determining the primitive of $f(x)$ that is zero at $x_0 = 1$ does not depend on the choice of expression for $F(x)$, because of continuity. The solution is

$$F(x) = \begin{cases} \frac{1}{2}x^2 - \frac{1}{2} & \text{if } x < 1, \\ \frac{1}{3}(x-2)^3 + \frac{1}{3} & \text{if } x \ge 1. \end{cases}$$

□

9.2 Rules of indefinite integration

The integrals of the elementary functions are important for the determination of other indefinite integrals. The rules below provide basic tools for handling integrals.

Theorem 9.8 (Linearity of the integral) *Suppose $f(x), g(x)$ are integrable functions on the interval I. For any $\alpha, \beta \in \mathbb{R}$ the map $\alpha f(x) + \beta g(x)$ is still integrable on I, and*

$$\int \big(\alpha f(x) + \beta g(x)\big)\,\mathrm{d}x = \alpha \int f(x)\,\mathrm{d}x + \beta \int g(x)\,\mathrm{d}x. \qquad (9.2)$$

Proof. Suppose $F(x)$ is a primitive of $f(x)$ and $G(x)$ a primitive of $g(x)$. By linearity of the derivative

$$\Big(\alpha F(x) + \beta G(x)\Big)' = \alpha F'(x) + \beta G'(x) = \alpha f(x) + \beta g(x), \quad \forall x \in I.$$

This means $\alpha F(x) + \beta G(x)$ is a primitive of $\alpha f(x) + \beta g(x)$ on I, which is the same as (9.2). $\square$

The above property says that one can integrate a sum one summand at a time, and pull multiplicative constants out of the integral sign.

Examples 9.9

i) Integrate the polynomial $4x^2 + 3x - 5$. By (9.1) a)

$$\begin{aligned}\int (4x^2 + 3x - 5)\,\mathrm{d}x &= 4\int x^2\,\mathrm{d}x + 3\int x\,\mathrm{d}x - 5\int \mathrm{d}x \\ &= 4\left(\frac{1}{3}x^3 + c_1\right) + 3\left(\frac{1}{2}x^2 + c_2\right) - 5(x + c_3) \\ &= \frac{4}{3}x^3 + \frac{3}{2}x^2 - 5x + c.\end{aligned}$$

The numbers c_1, c_2, c_3 arising from the single integrals have been 'gathered' into one arbitrary constant c.

ii) Integrate $f(x) = \cos^2 x$. From

$$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

and $\mathrm{D}\sin 2x = 2\cos 2x$, it follows

$$\int \cos^2 x\,\mathrm{d}x = \frac{1}{2}\int \mathrm{d}x + \frac{1}{2}\int \cos 2x\,\mathrm{d}x = \frac{1}{2}x + \frac{1}{4}\sin 2x + c\,.$$

Similarly

$$\int \sin^2 x\,\mathrm{d}x = \frac{1}{2}x - \frac{1}{4}\sin 2x + c\,.$$

$\square$

Theorem 9.10 (Integration by parts) *Let $f(x), g(x)$ be differentiable over I. If the map $f'(x)g(x)$ is integrable on I, then so is $f(x)g'(x)$, and*

$$\int f(x)g'(x)\,\mathrm{d}x = f(x)g(x) - \int f'(x)g(x)\,\mathrm{d}x. \tag{9.3}$$

Proof. Let $H(x)$ be any primitive of $f'(x)g(x)$ on I. By formula (6.4)

$$\begin{aligned}[f(x)g(x) - H(x)]' &= \big(f(x)g(x)\big)' - H'(x) \\ &= f'(x)g(x) + f(x)g'(x) - f'(x)g(x) \\ &= f(x)g'(x).\end{aligned}$$

Therefore the map $f(x)g(x) - H(x)$ is a primitive of $f(x)g'(x)$, exactly what (9.3) claims. □

In practice, one integrates a product of functions by identifying first one factor with $f(x)$ and the other with $g'(x)$; then one determines a primitive $g(x)$ of $g'(x)$ and, at last, one finds the primitive of $f'(x)g(x)$ and uses (9.3).

Examples 9.11

i) Compute

$$\int x\mathrm{e}^x \,\mathrm{d}x.$$

Call $f(x) = x$ and $g'(x) = \mathrm{e}^x$. Then $f'(x) = 1$, and we conveniently choose e^x as primitive of itself. Formula (9.3) yields

$$\int x\mathrm{e}^x \,\mathrm{d}x = x\mathrm{e}^x - \int \mathrm{e}^x \,\mathrm{d}x = x\mathrm{e}^x - (\mathrm{e}^x + c) = (x-1)\mathrm{e}^x + c.$$

Since the constant of integration is completely arbitrary, in the last step the sign of c was flipped with no harm done.

Had we chosen $f(x) = \mathrm{e}^x$ and $g'(x) = x$ ($f'(x) = \mathrm{e}^x$ and $g(x) = \frac{1}{2}x^2$), we would have ended up with

$$\int x\mathrm{e}^x \,\mathrm{d}x = \frac{1}{2}x^2\mathrm{e}^x - \frac{1}{2}\int x^2\mathrm{e}^x \,\mathrm{d}x,$$

which is not particularly helpful (rather the opposite).

ii) Determine

$$\int \log x \,\mathrm{d}x.$$

Let us put $f(x) = \log x$ and $g'(x) = 1$, so that $f'(x) = \frac{1}{x}$, $g(x) = x$. Thus

$$\begin{aligned}\int \log x \,\mathrm{d}x &= x\log x - \int \frac{1}{x}\, x \,\mathrm{d}x = x\log x - \int \mathrm{d}x \\ &= x\log x - (x + c) = x(\log x - 1) + c,\end{aligned}$$

given that c is arbitrary.

iii) Find

$$S = \int \mathrm{e}^x \sin x \,\mathrm{d}x.$$

We start by defining $f(x) = \mathrm{e}^x$ and $g'(x) = \sin x$. Then $f'(x) = \mathrm{e}^x$, $g(x) = -\cos x$, and

$$S = -\mathrm{e}^x \cos x + \int \mathrm{e}^x \cos x \,\mathrm{d}x.$$

Let us integrate by parts once again, by putting $f(x) = \mathrm{e}^x$ and $g'(x) = \cos x$ this time. Since $f'(x) = \mathrm{e}^x$, $g(x) = \sin x$,

$$S = -\mathrm{e}^x \cos x + \mathrm{e}^x \sin x - \int \mathrm{e}^x \sin x \,\mathrm{d}x = \mathrm{e}^x(\sin x - \cos x) - S.$$

A primitive $F(x)$ of $\mathrm{e}^x \sin x$ may be written as

$$F(x) = \mathrm{e}^x(\sin x - \cos x) - G(x)\,,$$

$G(x)$ being another primitive of $\mathrm{e}^x \sin x$. By the characterisation of Theorem 9.4 then,

$$2S = \mathrm{e}^x(\sin x - \cos x) + c$$

hence

$$S = \frac{1}{2}\mathrm{e}^x(\sin x - \cos x) + c.$$ □

Theorem 9.12 (**Integration by substitution**) *Let $f(y)$ be integrable on the interval J and $F(y)$ a primitive. Suppose $\varphi(x)$ is a differentiable function from I to J. Then the map $f\big(\varphi(x)\big)\varphi'(x)$ is integrable on I and*

$$\int f\big(\varphi(x)\big)\varphi'(x)\,\mathrm{d}x = F\big(\varphi(x)\big) + c, \tag{9.4}$$

which is usually stated in the less formal yet simpler way

$$\int f\big(\varphi(x)\big)\varphi'(x)\,\mathrm{d}x = \int f(y)\,\mathrm{d}y. \tag{9.5}$$

Proof. Formula (6.7) for differentiating a composite map gives

$$\frac{\mathrm{d}}{\mathrm{d}x}F\big(\varphi(x)\big) = \frac{\mathrm{d}F}{\mathrm{d}y}\big(\varphi(x)\big)\,\frac{\mathrm{d}\varphi}{\mathrm{d}x}(x) = f\big(\varphi(x)\big)\varphi'(x).$$

Thus $F\big(\varphi(x)\big)$ integrates $f\big(\varphi(x)\big)\varphi'(x)$, i.e., (9.4) is proven. □

We insist on the fact that the correct meaning of (9.5) is expressed by (9.4): the integral on the left is found by integrating f with respect to y and then substituting to y the function $\varphi(x)$, so that the right-hand side too depends on the variable x. Formula (9.5) is easy to remember with a trick: differentiate $y = \varphi(x)$, so that $\frac{\mathrm{d}y}{\mathrm{d}x} = \varphi'(x)$. Viewing the right-hand side as a formal quotient (in Leibniz's notation), multiply it by $\mathrm{d}x$; substituting $\mathrm{d}y = \varphi'(x)\mathrm{d}x$ in one integral yields the other.

Examples 9.13

i) Determine

$$\int x\,\mathrm{e}^{x^2}\,\mathrm{d}x.$$

Let $y = \varphi(x) = x^2$, so $\varphi'(x) = 2x$. Then

$$\int x\,\mathrm{e}^{x^2}\,\mathrm{d}x = \frac{1}{2}\int \mathrm{e}^{x^2} 2x\,\mathrm{d}x = \frac{1}{2}\int \mathrm{e}^{y}\,\mathrm{d}y = \frac{1}{2}\mathrm{e}^{y} + c.$$

Going back to x,

$$\int x\,\mathrm{e}^{x^2}\,\mathrm{d}x = \frac{1}{2}\mathrm{e}^{x^2} + c.$$

ii) Compute

$$\int \tan x\,\mathrm{d}x.$$

First, recall $\tan x = \dfrac{\sin x}{\cos x}$ and $(\cos x)' = -\sin x$. Put $y = \varphi(x) = \cos x$:

$$\begin{aligned}\int \tan x\,\mathrm{d}x &= -\int \frac{1}{\cos x}(\cos x)'\,\mathrm{d}x = -\int \frac{1}{y}\,\mathrm{d}y \\ &= -\log|y| + c = -\log|\cos x| + c.\end{aligned}$$

iii) Find

$$\int \frac{1}{\sqrt{1+x^2}}\,\mathrm{d}x\,.$$

By (6.18) it follows directly

$$\int \frac{1}{\sqrt{1+x^2}}\,\mathrm{d}x = \sinh^{-1} x + c\,.$$

Alternatively, we may substitute $y = \varphi(x) = \sqrt{1+x^2} - x$:

$$\mathrm{d}y = \left(\frac{x}{\sqrt{1+x^2}} - 1\right)\mathrm{d}x = \frac{x - \sqrt{1+x^2}}{\sqrt{1+x^2}}\,\mathrm{d}x\,,$$

hence $\dfrac{1}{\sqrt{1+x^2}}\,\mathrm{d}x = -\dfrac{1}{y}\,\mathrm{d}y$. This gives

$$\int \frac{1}{\sqrt{1+x^2}}\,\mathrm{d}x = -\int \frac{1}{y}\,\mathrm{d}y = -\log|y| + c = -\log(\sqrt{1+x^2} - x) + c\,,$$

where the absolute value was removed, as $\sqrt{1+x^2} - x > 0$ for any x.
The two expressions are indeed the same, for

$$-\log(\sqrt{1+x^2} - x) = \log(\sqrt{1+x^2} + x) = \sinh^{-1} x\,.$$

iv) The integral

$$\int \frac{1}{\sqrt{x^2-1}}\,\mathrm{d}x$$

can be determined by the previous technique. The substitution $y = \varphi(x) = \sqrt{x^2-1} - x$ gives

$$\int \frac{1}{\sqrt{x^2-1}}\,\mathrm{d}x = \log|\sqrt{x^2-1} + x| + c\,.$$

v) The integral

$$S = \int \sqrt{1+x^2}\,\mathrm{d}x$$

is found as in example iii). Integrate by parts with $f(x) = \sqrt{1+x^2}$ and $g'(x) = 1$, so $f'(x) = \dfrac{x}{\sqrt{1+x^2}}$, $g(x) = x$ and

$$\begin{aligned} S &= x\sqrt{1+x^2} - \int \frac{x^2}{\sqrt{1+x^2}}\,\mathrm{d}x = x\sqrt{1+x^2} - \int \frac{x^2+1-1}{\sqrt{1+x^2}}\,\mathrm{d}x \\ &= x\sqrt{1+x^2} - \int \sqrt{1+x^2}\,\mathrm{d}x + \int \frac{1}{\sqrt{1+x^2}}\,\mathrm{d}x \\ &= x\sqrt{1+x^2} - S + \int \frac{1}{\sqrt{1+x^2}}\,\mathrm{d}x\,. \end{aligned}$$

Therefore

$$2S = x\sqrt{1+x^2} + \int \frac{1}{\sqrt{1+x^2}}\,\mathrm{d}x = x\sqrt{1+x^2} + \log(\sqrt{1+x^2}+x) + c,$$

and eventually

$$S = \frac{1}{2}x\sqrt{1+x^2} + \frac{1}{2}\log(\sqrt{1+x^2}+x) + c\,.$$

Similar story for $\displaystyle\int \sqrt{x^2-1}\,\mathrm{d}x$.

vi) Determine

$$S = \int \sqrt{1-x^2}\,\mathrm{d}x\,.$$

As above, we may integrate by parts remembering $\displaystyle\int \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x = \arcsin x + c$. Namely, with $f(x) = \sqrt{1-x^2}$, $g'(x) = 1$, we have $f'(x) = -\dfrac{x}{\sqrt{1-x^2}}$, $g(x) = x$, whence

$$S = x\sqrt{1-x^2} - \int \frac{-x^2}{\sqrt{1-x^2}}\,\mathrm{d}x = x\sqrt{1-x^2} - S + \int \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x\,.$$

So we have

$$2S = x\sqrt{1-x^2} + \int \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x\,,$$

i.e.,

$$S = \frac{1}{2}x\sqrt{1-x^2} + \frac{1}{2}\arcsin x + c\,.$$

Let us do this in a different way. Put $y = \arcsin x$, so $\mathrm{d}x = \cos y\,\mathrm{d}y$ and $\sqrt{1-x^2} = \cos y$. These give

$$\begin{aligned} S &= \int \cos^2 y\,\mathrm{d}y = \frac{1}{2}\int (\cos 2y + 1)\,\mathrm{d}y = \frac{1}{4}\sin 2y + \frac{1}{2}y + c \\ &= \frac{1}{2}\sin y\cos y + \frac{1}{2}y + c = \frac{1}{2}x\sqrt{1-x^2} + \frac{1}{2}\arcsin x + c\,. \end{aligned}$$

vii) Finally, let us determine

$$\int \frac{1}{e^x + e^{-x}}\, dx.$$

Change $y = e^x$, so $dy = e^x dx$, or $dx = \frac{1}{y} dy$. Then

$$\int \frac{1}{e^x + e^{-x}}\, dx = \int \frac{1}{y + \frac{1}{y}} \frac{1}{y}\, dy$$

$$= \int \frac{1}{1 + y^2}\, dy = \arctan y + c = \arctan e^x + c. \qquad \square$$

Example ii) is a special case of the following useful relation

$$\frac{\varphi'(x)}{\varphi(x)}\, dx = \log |\varphi(x)| + c \tag{9.6}$$

that descends from (9.5) by $f(y) = \frac{1}{y}$:

Hitherto all instances had one common feature: the maps f were built from a finite number of elementary functions by algebraic operations and compositions, and so were the primitives F. In such a case, one says that f is **integrable by elementary methods**. Unfortunately though, not all functions arising this way are integrable by elementary methods. Consider $f(x) = e^{-x^2}$, whose relevance in Probability Theory is paramount. It can be shown its primitives (which exist, for f is continuous on $\mathbb{R}$) cannot be expressed by elementary functions. The same holds for $f(x) = \dfrac{\sin x}{x}$.

The problem of finding an explicit primitive for a given function is highly non-trivial. A large class of maps which are integrable by elementary methods is that of *rational functions*.

9.2.1 Integrating rational maps

Consider maps of the general form

$$f(x) = \frac{P(x)}{Q(x)},$$

where $P(x)$ and $Q(x)$ denote polynomials of degrees n, m $(m \geq 1)$ respectively. We want to prove they admit primitives in terms of rational functions, logarithms and inverse tangent functions.

First of all note that if $n \geq m$, we may divide $P(x)$ by $Q(x)$

$$P(x) = Q(x)D(x) + R(x),$$

with $D(x)$ a polynomial of degree $n - m$ and $R(x)$ of degree $\leq m - 1$. Therefore

$$\int \frac{P(x)}{Q(x)}\,\mathrm{d}x = \int D(x)\,\mathrm{d}x + \int \frac{R(x)}{Q(x)}\,\mathrm{d}x.$$

The problem boils down to integrating a rational map $g(x) = \dfrac{R(x)}{Q(x)}$ in which the numerator's degree is less than the denominator's.

We discuss a few simple situations of this type, which will turn out to be fundamental for treating the generic integrand.

i) Let $g(x) = \dfrac{1}{x-\alpha}$, with $\alpha \in \mathbb{R}$; by (9.1) b)

$$\int \frac{1}{x-\alpha}\,\mathrm{d}x = \log|x-\alpha| + c. \tag{9.7}$$

ii) Take $g(x) = \dfrac{}{}$, where $r > 1$; using (9.1) a) yields

$$\int \frac{1}{(x-\alpha)^r}\,\mathrm{d}x = \frac{1}{1-r}\,\frac{1}{(x-\alpha)^{r-1}} + c. \tag{9.8}$$

iii) Let $g(x) = \dfrac{1}{x^2+2px+q}$, with $p^2 - q < 0$, so that the denominator has no real roots and is positive. Putting

$$s = \sqrt{q-p^2} > 0,$$

a little algebra shows

$$x^2+2px+q = x^2+2px+p^2+(q-p^2) = (x+p)^2+s^2 = s^2\left[1+\left(\frac{x+p}{s}\right)^2\right].$$

Now substitute $y = \varphi(x) = \dfrac{x+p}{s}$

$$\int \frac{1}{x^2+2px+q}\,\mathrm{d}x = \frac{1}{s^2}\int \frac{1}{1+y^2}\,s\,\mathrm{d}y.$$

Recalling (9.1) f) we may conclude

$$\int \frac{1}{x^2+2px+q}\,\mathrm{d}x = \frac{1}{s}\arctan\frac{x+p}{s} + c. \tag{9.9}$$

iv) Consider $g(x) = \dfrac{ax+b}{x^2+2px+q}$, with p^2-q still negative. Due to the identity

$$ax+b = ax+ap+b-ap = \frac{a}{2}(2x+2p)+(b-ap)$$

we write

$$\int \frac{ax+b}{x^2+2px+q}\,\mathrm{d}x = \frac{a}{2}\int \frac{2x+2p}{x^2+2px+q}\,\mathrm{d}x + (b-ap)\int \frac{1}{x^2+2px+q}\,\mathrm{d}x.$$

Now use (9.6) with $\varphi(x) = x^2+2px+q$, and (9.9):

$$\int \frac{ax+b}{x^2+2px+q}\,\mathrm{d}x = \frac{a}{2}\log(x^2+2px+q) + \frac{b-ap}{s}\arctan\frac{x+p}{s} + c. \tag{9.10}$$

v) More generally, let $g(x) = \dfrac{ax+b}{(x^2+2px+q)^r}$, with $p^2-q<0$ and $r>1$. Integrating by parts

$$\int \frac{1}{(x^2+2px+q)^{r-1}}\,\mathrm{d}x$$

and substituting $\varphi(x) = x^2+2px+q$, we end up writing the integral of g as sum of known terms, plus the integral of a map akin to g, but where the exponent is $r-1$. Thus the integrand to consider simplifies to one whose denominator is raised to a smaller power. From $r=1$, solved above, we find $r=2$, then $r=3$ et cetera up to the given r, one step at a time. The argument's details are left to the willing reader.

Examples 9.14

As direct application we compute

$$\int \frac{1}{2x-4}\,\mathrm{d}x = \frac{1}{2}\log|x-2| + c,$$

$$\int \frac{1}{(3x+5)^2}\,\mathrm{d}x = -\frac{1}{3(3x+5)} + c,$$

$$\begin{aligned}\int \frac{4x-5}{x^2-2x+10}\,\mathrm{d}x &= 2\int \frac{2x-2}{x^2-2x+10}\,\mathrm{d}x - \int \frac{1}{(x-1)^2+9}\,\mathrm{d}x \\ &= 2\log(x^2-2x+10) - \frac{1}{3}\arctan\frac{x-1}{3} + c.\end{aligned}$$

□

Reducing the integration of the general rational function $g(x) = \dfrac{R(x)}{Q(x)}$ to the previous special cases requires a factorisation of the denominator involving only terms like

$$(x-\alpha)^r \qquad \text{or} \qquad (x^2+2px+q)^s$$

with $p^2-q<0$. The existence of such a decomposition descends from a version of the *Fundamental Theorem of Algebra*.

Theorem 9.15 *A polynomial $Q(x)$ of degree m with real coefficients decomposes uniquely as a product*

$$Q(x) = d(x-\alpha_1)^{r_1}\cdots(x-\alpha_h)^{r_h}(x^2+2p_1x+q_1)^{s_1}\cdots(x^2+2p_kx+q_k)^{s_k}, \tag{9.11}$$

where d, α_i, p_j, q_j are real and r_i, s_j integers such that

$$r_1 + \cdots + r_h + 2s_1 + \cdots + 2s_k = m\,.$$

The α_i, all distinct, are the real roots of Q counted with multiplicity r_i. The factors $x^2+2p_jx+q_j$ are pairwise distinct and irreducible over $\mathbb{R}$, i.e., $p_j^2-q_j < 0$, and have two complex (-conjugate) roots $\beta_{j,\pm}$ of multiplicity s_j.

Using the factorisation (9.11) of $Q(x)$ we can now write $g(x)$ as sum of partial fractions

$$\frac{R(x)}{Q(x)} = \frac{1}{d}\left[F_1(x) + \cdots + F_h(x) + \bar{F}_1(x) + \cdots + \bar{F}_k(x)\right], \tag{9.12}$$

where each $F_i(x)$ takes the form

$$F_i(x) = \frac{A_{i1}}{x-\alpha_i} + \frac{A_{i2}}{(x-\alpha_i)^2} + \cdots + \frac{A_{ir_i}}{(x-\alpha_i)^{r_i}},$$

while $\bar{F}_j(x)$ are like

$$\bar{F}_j(x) = \frac{B_{j1}x + C_{j1}}{x^2+2p_jx+q_j} + \frac{B_{j2}x + C_{j2}}{(x^2+2p_jx+q_j)^2} + \cdots + \frac{B_{j\bar{r}_j}x + C_{j\bar{r}_j}}{(x^2+2p_jx+q_j)^{s_j}},$$

for suitable constants $A_{i\ell}, B_{j\mu}, C_{j\mu}$. Note the total number of constants is $r_1 + \cdots r_h + 2s_1 + \cdots + 2s_k = m$.

To recover the undetermined coefficients we can transform the right-hand side of (9.12) into one fraction, whose denominator is clearly $Q(x)$. The numerator $\mathcal{R}(x)$ is a polynomial of degree $\leq m-1$ that must coincide with $R(x)$, and its coefficients are linear combinations of the unknown constants we are after. To find these numbers, the following principle on identity of polynomials is at our disposal.

Theorem 9.16 *Two polynomials of degree $m-1$ coincide if and only if either of the next conditions holds*

a) the coefficients of corresponding monomials coincide;
b) the polynomials assume the same values at m distinct points.

The first equivalence is easily derived from Proposition 7.5.

Going back to the m unknowns $A_{i\ell}, B_{j\mu}, C_{j\mu}$, we could impose that the coefficients of each monomial in $\mathcal{R}(x)$ and $R(x)$ be the same, or else choose m values of x where the polynomials must agree. In the latter case the best choice falls on the real zeroes of $Q(x)$; should these be less than m in number, we could also take $x = 0$.

Once these coefficients have been determined, we can start integrating the right-hand side of (9.12) and rely on the fundamental cases i)–v) above.

As usual, the technique is best illustrated with a few examples.

Examples 9.17

i) Let us integrate

$$f(x) = \frac{2x^3 + x^2 - 4x + 7}{x^2 + x - 2}.$$

The numerator has greater degree than the denominator, so we divide the polynomials

$$f(x) = 2x - 1 + \frac{x+5}{x^2 + x - 2}.$$

The denominator factorises as $Q(x) = (x-1)(x+2)$. Therefore the coefficients to be found, $A_1 = A_{11}$ and $A_2 = A_{21}$, should satisfy

$$\frac{x+5}{x^2+x-2} = \frac{A_1}{x-1} + \frac{A_2}{x+2},$$

that is to say

$$x + 5 = A_1(x+2) + A_2(x-1), \tag{9.13}$$

hence

$$x + 5 = (A_1 + A_2)x + (2A_1 - A_2)\,.$$

Comparing coefficients yields the linear system

$$\begin{cases} A_1 + A_2 = 1, \\ 2A_1 - A_2 = 5, \end{cases}$$

solved by $A_1 = 2$, $A_2 = -1$. Another possibility is to compute (9.13) at the zeroes $x = 1$, $x = -2$ of $Q(x)$, obtaining $6 = 3A_1$ and $3 = -3A_2$, whence again $A_1 = 2, A_2 = -1$. Therefore,

$$\begin{aligned} \int f(x)\,\mathrm{d}x &= \int (2x-1)\,\mathrm{d}x + 2\int \frac{1}{x-1}\,\mathrm{d}x - \int \frac{1}{x+2}\,\mathrm{d}x \\ &= x^2 - x + 2\log|x-1| - \log|x+2| + c. \end{aligned}$$

ii) Determine a primitive of the function

$$f(x) = \frac{x^2 - 3x + 3}{x^3 - 2x^2 + x}.$$

The denominator splits as $Q(x) = x(x-1)^2$, so we must search for $A_1 = A_{11}$, A_{21} and A_{22} such that

$$\frac{x^2 - 3x + 3}{x^3 - 2x^2 + x} = \frac{A_1}{x} + \frac{A_{21}}{x-1} + \frac{A_{22}}{(x-1)^2},$$

or

$$x^2 - 3x + 3 = A_1(x-1)^2 + A_{21}x(x-1) + A_{22}x\,.$$

Putting $x = 0$ yields $A_1 = 3$, with $x = 1$ we find $A_{22} = 1$. The remaining A_{21} is determined by picking a third value $x \neq 0, 1$. For instance $x = -1$ gives $7 = 12 + 2A_{21} - 1$, so $A_{21} = -2$.

In conclusion,

$$\int f(x)\,\mathrm{d}x = 3\int \frac{1}{x}\,\mathrm{d}x - 2\int \frac{1}{x-1}\,\mathrm{d}x + \int \frac{1}{(x-1)^2}\,\mathrm{d}x$$
$$= 3\log|x| - 2\log|x-1| - \frac{1}{x-1} + c.$$

iii) Integrate

$$f(x) = \frac{3x^2+x-4}{x^3+5x^2+9x+5}.$$

The point $x = -1$ annihilates the denominator (the sum of the odd-degree coefficients equals those of even degree), so the denominator splits $Q(x) = (x+1)(x^2+4x+5)$ by Ruffini's rule. The unknown coefficients are $A = A_{11}$, $B = B_{11}$, $C = C_{11}$ so that

$$\frac{3x^2+x-4}{x^3+5x^2+9x+5} = \frac{A}{x+1} + \frac{Bx+C}{x^2+4x+5},$$

hence

$$3x^2+x-4 = A(x^2+4x+5) + (Bx+C)(x+1)\,.$$

Choosing $x = -1$, and then $x = 0$, produces $A = -1$ and $C = 1$. The last coefficient $B = 4$ is found by taking $x = -1$. Thus

$$\int f(x)\,\mathrm{d}x = -\int \frac{1}{x+1}\,\mathrm{d}x + \int \frac{4x+1}{x^2+4x+5}\,\mathrm{d}x$$
$$= -\int \frac{1}{x+1}\,\mathrm{d}x + 2\int \frac{2x+4}{x^2+4x+5}\,\mathrm{d}x - 7\int \frac{1}{1+(x+2)^2}\,\mathrm{d}x$$
$$= -\log|x+1| + 2\log(x^2+4x+5) - 7\arctan(x+2) + c. \qquad \square$$

Note that many functions $f(x)$ that are not rational in the variable x can be transformed – by an appropriate change $t = \varphi(x)$ – into a rational map in the new variable t. Special cases thereof include:

i) f is a rational function of $\sqrt[p]{x-a}$ for some integer p and a real. Then one lets

$$t = \sqrt[p]{x-a}, \qquad \text{whence} \;\; x = a + t^p \;\; \text{and} \;\; \mathrm{d}x = pt^{p-1}\mathrm{d}t.$$

ii) f is rational in e^{ax} for some real $a \neq 0$. The substitution

$$t = \mathrm{e}^{ax} \qquad \text{gives} \;\; x = \frac{1}{a}\log t \;\; \text{and} \;\; \mathrm{d}x = \frac{1}{at}\,\mathrm{d}t.$$

iii) f is rational in $\sin x$ and/or $\cos x$. In this case

$$t = \tan\frac{x}{2}\,,$$

together with the identities

$$\sin x = \frac{2t}{1+t^2}, \qquad \cos x = \frac{1-t^2}{1+t^2}\,, \tag{9.14}$$

does the job, because then $x = 2\arctan t$, hence

$$\mathrm{d}x = \frac{2}{1+t^2}\,\mathrm{d}t. \tag{9.15}$$

iv) If f is rational in $\sin^2 x$, $\cos^2 x$, $\tan x$, it is more convenient to set $t = \tan x$ and use

$$\sin^2 x = \frac{t^2}{1+t^2}, \qquad \cos^2 x = \frac{1}{1+t^2}; \tag{9.16}$$

from $x = \arctan t$, it follows

$$\mathrm{d}x = \frac{1}{1+t^2}\,\mathrm{d}t\,. \tag{9.17}$$

In the concluding examples we only indicate how to arrive at a rational expression in t, leaving it to the reader to integrate and return to the original variable x.

Examples 9.18

i) Consider

$$S = \int \frac{x}{1+\sqrt{x-1}}\,\mathrm{d}x.$$

We let $t = \sqrt{x-1}$, so $x = 1+t^2$ and $\mathrm{d}x = 2t\,\mathrm{d}t$. The substitution gives

$$S = 2\int \frac{(1+t^2)t}{1+t}\,\mathrm{d}t.$$

ii) The integral

$$S = \int \frac{\mathrm{e}^{-x}}{\mathrm{e}^{2x}-2\mathrm{e}^x+2}\,\mathrm{d}x$$

becomes, by $t = \mathrm{e}^x$, $\mathrm{d}x = \frac{1}{t}\,\mathrm{d}t$,

$$S = \int \frac{1}{t^2(t^2-2t+2)}\,\mathrm{d}t.$$

iii) Reduce the integrand in

$$S = \int \frac{\sin x}{1+\sin x}\,\mathrm{d}x$$

to a rational map.

Referring to (9.14) and (9.15),

$$S = 4\int \frac{t}{(1+t)^2(1+t^2)}\,\mathrm{d}t.$$

iv) At last, consider

$$S = \int \frac{1}{1+\sin^2 x}\,\mathrm{d}x.$$

Here we use (9.16) and (9.17):

$$S = \int \frac{1}{1+2t^2}\,\mathrm{d}t.$$

□

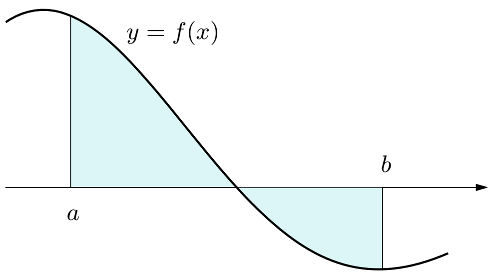


Figure 9.2. Trapezoidal region of f over $[a, b]$

9.3 Definite integrals

Let us consider a bounded map f defined on a bounded and closed interval $I = [a, b] \subset \mathbb{R}$. One suggestively calls **trapezoidal region of f over the interval** $[a, b]$, denoted by $\mathcal{T}(f; a, b)$, the part of plane enclosed within the interval $[a, b]$, the vertical lines passing through the end-points a, b and the graph of f (see Fig. 9.2)

$$\mathcal{T}(f; a, b) = \{(x, y) \in \mathbb{R}^2 \ : \ a \le x \le b, \ \ 0 \le y \le f(x) \text{ or } f(x) \le y \le 0\}$$

(which constraint on y clearly depending on the sign of $f(x)$).

Under suitable assumptions on f one can associate to the trapezoidal region of f over $[a, b]$ a number, the 'definite integral of f over $[a, b]$'. In case f is positive, this number is indeed the area of the region. In particular, when the region is particularly simple (a rectangle, a triangle, a trapezium and the like), the definite integral returns one of the classical formulas of elementary geometry.

The many notions of definite integral depend on what is demanded of the integrand. We shall present two types. The first one, normally linked to the name of Cauchy, deals with continuous or piecewise-continuous maps on $[a, b]$.

Definition 9.19 *A map $f : [a, b] \to \mathbb{R}$ is* **piecewise-continuous** *when it is continuous everywhere except at a finite number of points, at which the discontinuity is either removable or a jump.*

The second construction goes back to Riemann, and leads to a wider class of integrable functions[1].

[1] A further type, known as Lebesgue integral, defines yet another set of integrable functions, which turns out to be the most natural in modern applications. This theory though goes beyond the purposes of the present textbook.

9.4 The Cauchy integral

To start with, we assume f continuous on $[a,b]$, and generalise slighty at a successive step. The idea is to construct a sequence that approximates the trapezoidal region of f, and then take a limit-of-sorts. Let us see how.

Take n any positive integer. Divide $[a,b]$ in n equal parts of length $\Delta x = \frac{b-a}{n}$ and denote by $x_k = a + k\Delta x$, $k = 0, 1, \ldots, n$, the subdivision points; note that they are ordered increasingly by the index, as $a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b$. For $k = 1, \ldots, n$, we denote by I_k the interval $[x_{k-1}, x_k]$. The map f is continuous on $[a,b]$, hence by restriction on each I_k; Weierstrass's theorem 4.31 implies f assumes minimum and maximum on I_k, say

$$m_k = \min_{x \in I_k} f(x), \qquad M_k = \max_{x \in I_k} f(x).$$

Define now the quantities

$$s_n = \sum_{k=1}^{n} m_k \Delta x \qquad \text{and} \qquad S_n = \sum_{k=1}^{n} M_k \Delta x,$$

called respectively *lower sum* and *upper sum* of f for the above partition of $[a,b]$. By definition $m_k \le M_k$ and $\Delta x > 0$, so $s_n \le S_n$.

When f is positive on $[a,b]$, the meaning is immediate (Fig. 9.3): $m_k \Delta x$ represents the area of the rectangle $r_k = I_k \times [0, m_k]$, contained in the trapezoidal region of f over I_k. Thus, s_n is the total area of the rectangles r_k and approximates from below the area of $\mathcal{T}(f; a, b)$. For the same reasons, S_n is the area of the union of of rectangles $R_k = I_k \times [0, M_k]$, and it approximates $\mathcal{T}(f; a, b)$ from above.

Using properties of continuous maps defined on closed and bounded intervals, we can prove the following result (see Appendix A.5.1, p. 461).

Theorem 9.20 *The sequences $\{s_n\}$ and $\{S_n\}$ are convergent, and their limits coincide.*

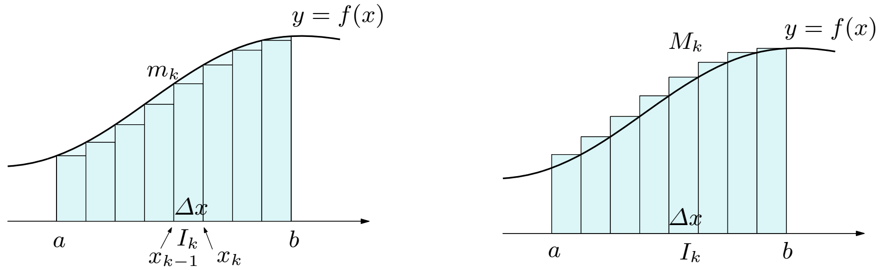


Figure 9.3. Lower sum (left) and upper sum (right) of f on $[a,b]$

Based on this fact, we can introduce the definite integral.

Definition 9.21 *One calls* **definite integral of f over** $[a, b]$ *the number*

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{n\to\infty} s_n = \lim_{n\to\infty} S_n$$

(*which we read* integral from a to b of $f(x)\mathrm{d}x$ *or just* integral from a to b of f).

Examples 9.22

i) Take a constant f on $[a, b]$. If c is its value, then $m_k = M_k = c$ for any k, so

$$s_n = S_n = c\sum_{k=1}^{n} \Delta x = c(b-a)$$

whichever n. Therefore $\displaystyle\int_a^b f(x)\,\mathrm{d}x = c(b-a)$.

ii) Consider $f(x) = x$ over $[0, 1]$. The region $\mathcal{T}(x; 0, 1)$ is the isosceles right triangle of vertices $A = (0, 0)$, $B = (1, 0)$, $C = (1, 1)$ that has area $\frac{1}{2}$. We want to check the definite integral of f over $[0, 1]$ gives the same result. Fix $n > 1$. Then $\Delta x = \frac{1}{n}$ and, for $k = 0, \dots, n$, $x_k = \frac{k}{n}$. Since f is increasing, $m_k = x_{k-1}$ and $M_k = x_k$, so

$$s_n = \sum_{k=1}^{n} x_{k-1}\Delta x = \frac{1}{n^2}\sum_{k=1}^{n}(k-1), \qquad S_n = \sum_{k=1}^{n} x_k \Delta x = \frac{1}{n^2}\sum_{k=1}^{n} k.$$

Now $\displaystyle\sum_{k=1}^{n} k$ is the sum of the first n natural numbers, hence $\frac{n(n+1)}{2}$, by (3.2). For analogous reasons $\displaystyle\sum_{k=1}^{n}(k-1)$ is the sum of natural numbers from 0 (or 1) to $n-1$, and equals $\frac{(n-1)n}{2}$, whence

$$s_n = \frac{n(n-1)}{2n^2}, \qquad S_n = \frac{n(n+1)}{2n^2}.$$

Taking the limit for $n \to \infty$ of these sequences, we find $\frac{1}{2}$ for both. □

This example shows that even for a function as harmless as $f(x) = x$, computing the definite integral using the definition is rather demanding. Obviously one would hope to have more efficient tools to calculate integrals of continuous maps. For that we shall have to wait until Sect. 9.8.

We discuss now the extension of the notion of definite integral. If f is continuous on $[a, b]$ and x^* denotes an interior point of the interval, it is possible to prove

$$\int_a^b f(x)\,\mathrm{d}x = \int_a^{x^*} f(x)\,\mathrm{d}x + \int_{x^*}^b f(x)\,\mathrm{d}x.$$

This formula's meaning is evident, and it suggests how to define integrals of piecewise-continuous maps. Let $x_0 = a < x_1 < \ldots < x_{m-1} < x_m = b$ be the points where f is not continuous, lying between a and b (assuming the latter might be discontinuity points too). Let f_i be the restriction of f to the interior of $[x_{i-1}, x_i]$ that extends f continuously at the boundary

$$f_i(x) = \begin{cases} \lim\limits_{x \to x_{i-1}^+} f(x), & \text{for } x = x_{i-1}, \\ f(x), & \text{for } x_{i-1} < x < x_i, \\ \lim\limits_{x \to x_i^-} f(x), & \text{for } x = x_i. \end{cases}$$

We define

$$\int_a^b f(x)\,\mathrm{d}x = \sum_{i=1}^m \int_{x_{i-1}}^{x_i} f_i(x)\,\mathrm{d}x.$$

If f is genuinely continuous on $[a, b]$, the above box coincides with Definition 9.21, because $m = 1$ and the map f_1 is f.

Moreover, it follows immediately that modifying a (piecewise-)continuous map at a finite number of points will not alter its definite integral.

The study of Cauchy's integral will be resumed with Sect. 9.6.

9.5 The Riemann integral

Throughout the section f will indicate a bounded map on $[a, b]$. Let us start from integrating some elementary functions (called step functions), and slowly proceed to more general maps, whose integral builds upon the former type by means of upper and lower bounds.

Choose $n + 1$ points of $[a, b]$ (not necessarily uniformly spread)

$$a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b\,.$$

They induce a *partition* of $[a, b]$ into sub-intervals $I_k = [x_{k-1}, x_k]$, $k = 1, \ldots, n$. Dividing further one of the I_k we obtain a so-called *finer partition*, also known as *refinement* of the initial partition. Step functions are constant on each subinterval of a partition of $[a, b]$, see Fig. 9.4. More precisely,

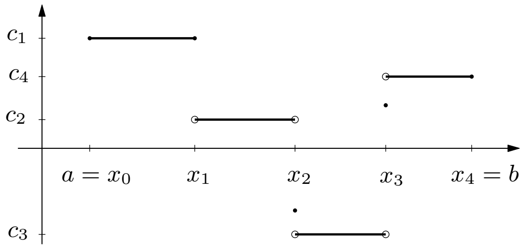


Figure 9.4. Graph of a step function on $[a, b]$

Definition 9.23 *A map $f : [a, b] \to \mathbb{R}$ is a* **step function** *if there exist a partition of $[a, b]$ by $\{x_0, x_1, \dots, x_n\}$ together with constants $c_1, c_2, \dots, c_n \in \mathbb{R}$ such that*

$$f(x) = c_k\,, \qquad \forall x \in (x_{k-1}, x_k), \quad k = 1, \dots, n.$$

We say that the partition is **adapted** to f if f is constant on each interval (x_{k-1}, x_k). Refinements of adapted partitions are still adapted. In particular if f and g are step functions on $[a, b]$, it is always possible to manifacture a partition that is adapted to both maps just by taking the union of the points of two partitions adapted to f and g, respectively.

From now on $\mathcal{S}([a, b])$ will denote the set of step functions on $[a, b]$.

Definition 9.24 *Let $f \in \mathcal{S}([a, b])$ and $\{x_0, x_1, \dots, x_n\}$ be an adapted partition. Call c_k the constant value of f on (x_{k-1}, x_k). Then the number*

$$\int_I f = \sum_{k=1}^{n} c_k (x_k - x_{k-1})$$

is called **definite integral** *of f on $I = [a, b]$.*

A few remarks are necessary.

i) The definition is independent of the chosen partition. In particular, if $f(x) = c$ is constant on $[a, b]$, $\displaystyle\int_I f = c(b - a)$.

ii) Redefining f at a finite number of places leaves the integral unchanged; in particular, the definite integral does not depend upon the values of f at points of discontinuity.

In case f is positive on I, the number $\int_I f$ represents precisely the area of the trapezoidal region of f over I: the latter is in fact the sum of rectangles with base $x_k - x_{k-1}$ and height c_k (Fig. 9.5).

The next result will play an important role.

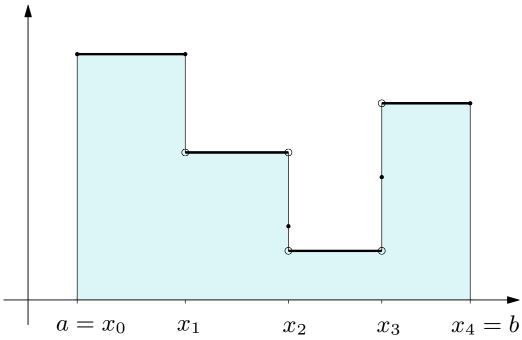


Figure 9.5. Region under a positive step function on the interval $[a, b]$

Property 9.25 *If $g, h \in \mathcal{S}([a,b])$ are such that $g(x) \le h(x)$, $\forall x \in [a,b]$, then*

$$\int_I g \le \int_I h\,.$$

Proof. Let $\{x_0, x_1, \ldots, x_n\}$ define a partition adapted to both maps (this exists by what said earlier). Call c_k and d_k the values assumed on (x_{k-1}, x_k) by g and h, respectively. By hypothesis $c_k \le d_k$, $k = 1, \ldots, n$, so

$$\int_I g = \sum_{k=1}^{n} c_k (x_k - x_{k-1}) \le \sum_{k=1}^{n} d_k (x_k - x_{k-1}) = \int_I h\,. \qquad \square$$

Now let $f : [a,b] \to \mathbb{R}$ be a generic bounded map, and put

$$s_f = \sup_{x \in [a,b]} f(x) \in \mathbb{R} \qquad \text{and} \qquad i_f = \inf_{x \in [a,b]} f(x) \in \mathbb{R}.$$

We introduce the sets of step functions bounding f from above or from below, namely

$$\mathcal{S}_f^+ = \Big\{ h \in \mathcal{S}([a,b]) : f(x) \le h(x),\ \forall x \in [a,b] \Big\}$$

contains all step functions bigger than f, while

$$\mathcal{S}_f^- = \Big\{ g \in \mathcal{S}([a,b]) : g(x) \le f(x),\ \forall x \in [a,b] \Big\}$$

contains all those smaller than f. These are not empty, for they contain at least the constant maps

$$h(x) = s_f \qquad \text{and} \qquad g(x) = i_f\,.$$

It then makes sense to look at the sets of definite integrals.

Definition 9.26 *The number*

$$\overline{\int_I} f = \inf \left\{ \int_I h : h \in \mathcal{S}_f^+ \right\}$$

is called the **upper integral** *of f on $I = [a, b]$, and*

$$\underline{\int_I} f = \sup \left\{ \int_I g : g \in \mathcal{S}_f^- \right\}$$

the **lower integral** *of f on $I = [a, b]$.*

As $\mathcal{S}_f^+ \neq \emptyset$, clearly $\overline{\int_I} f < +\infty$, and similarly $\underline{\int_I} f > -\infty$. The fact that such quantities are finite relies on the following.

Property 9.27 *Each bounded map f defined on $[a, b]$ satisfies*

$$\underline{\int_I} f \leq \overline{\int_I} f.$$

Proof. If $g \in \mathcal{S}_f^-$ and $h \in \mathcal{S}_f^+$, by definition

$$g(x) \leq f(x) \leq h(x)\,, \qquad \forall x \in [a, b],$$

so Property 9.25 implies

$$\int_I g \leq \int_I h.$$

Keeping g fixed and varying h we have

$$\int_I g \leq \overline{\int_I} f.$$

Now varying g in this inequality proves the claim. □

At this stage one could ask whether equality in (9.27) holds by any chance for all bounded maps. The answer is no, as the example tells.

Example 9.28

The *Dirichlet function* is defined as follows

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}\,, \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}\,. \end{cases}$$

Each interval (x_{k-1}, x_k) of a partition of $[0,1]$ contains rational and non-rational points. Step functions in $\mathcal{S}_f^+$ are all larger than one, whereas the maps in $\mathcal{S}_f^-$ will be non-positive (except at a finite number of places). In conclusion

$$\overline{\int_I} f = 1 \qquad \text{and} \qquad \underline{\int_I} f = 0\,. \qquad \square$$

Our observation motivates introducing the next term.

Definition 9.29 *A bounded map f on $I = [a,b]$ is said* **integrable** *(precisely: Riemann integrable) on I if*

$$\underline{\int_I} f = \overline{\int_I} f.$$

The common value is called **definite integral** *of f on $[a,b]$, and denoted with $\int_I f$ or $\int_a^b f(x)\,\mathrm{d}x$.*

When f is a positive map on $[a,b]$ the geometric meaning of the definite integral is quite clear: $\mathcal{T}(f;a,b)$ is a subset of $\mathcal{T}(h;a,b)$ for any function $h \in \mathcal{S}_f^+$, and contains $\mathcal{T}(g;a,b)$ relative to any $g \in \mathcal{S}_f^-$. The upper integral gives thus an estimate from above (i.e., larger) of the area of the trapezoidal region of f over I, and similarly, the lower integral represents an approximation from below. Essentially, f is integrable when these two coincide, hence when the integral 'is' the area of the trapezoidal region of f.

Step functions f are evidently integrable: denoting by $\int_I f$ the quantity of Definition 9.24, the fact that $f \in \mathcal{S}_f^-$ implies $\int_I f \leq \underline{\int_I} f$, and $\overline{\int_I} f \leq \int_I f$ is consequence of $f \in \mathcal{S}_f^+$. Therefore

$$\int_I f \leq \underline{\int_I} f \leq \overline{\int_I} f \leq \int_I f$$

and the upper integral must be equal to the lower.

Beyond step functions, the world of integrable maps is vast.

Example 9.30

Consider $f(x) = x$ on $[0,1]$. We verify by Riemann integration that the trapezoidal region of f measures indeed $1/2$. Divide $[0,1]$ into $n > 1$ equal parts, a partition corresponding to the points $\left\{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, 1\right\} = \left\{\frac{k}{n} : k = 0, \ldots, n\right\}$. Now take the step functions

$$h_n(x) = \begin{cases} \dfrac{k}{n} & \text{if } \dfrac{k-1}{n} < x \leq \dfrac{k}{n}, \quad k = 1, \ldots, n, \\ 0 & \text{if } x = 0, \end{cases}$$

and

$$g_n(x) = \begin{cases} \dfrac{k-1}{n} & \text{if } \dfrac{k-1}{n} < x \le \dfrac{k}{n}, \quad k = 1, \dots, n, \\ 0 & \text{if } x = 0. \end{cases}$$

Since $g_n(x) \le f(x) \le h_n(x)$, $\forall x \in [0,1]$, it follows $h_n \in \mathcal{S}_f^+$, $g_n \in \mathcal{S}_f^-$. Moreover by (3.2),

$$\int_I h_n = \sum_{k=1}^{n} \frac{k}{n}\left(\frac{k}{n} - \frac{k-1}{n}\right) = \sum_{k=1}^{n} \frac{k}{n^2} = \frac{1}{n^2}\sum_{k=1}^{n} k = \frac{1}{n^2}\frac{n(n+1)}{2} = \frac{1}{2} + \frac{1}{2n}$$

and similarly

$$\int_I g_n = \frac{1}{2} - \frac{1}{2n}.$$

These imply

$$\overline{\int_I} f \le \inf_n \int_I h_n = \frac{1}{2} \qquad \text{and} \qquad \underline{\int_I} f \ge \sup_n \int_I g_n = \frac{1}{2},$$

hence

$$\overline{\int_I} f \le \frac{1}{2} \le \underline{\int_I} f.$$

Recalling 9.27 we conclude $\int_I f = \frac{1}{2}$. □

Studying the integrability of a map by means of the definition is rather non-trivial, even when one deals with maps having simple expression. So it would be good on the one hand to know in advance a large class of integrable maps, on the other to have powerful methods for computing integrals. While the second point will be addressed in Sect. 9.8, the result we state next is a relatively broad answer to the former problem; its proof may be found in Appendix A.5.2, p. 463.

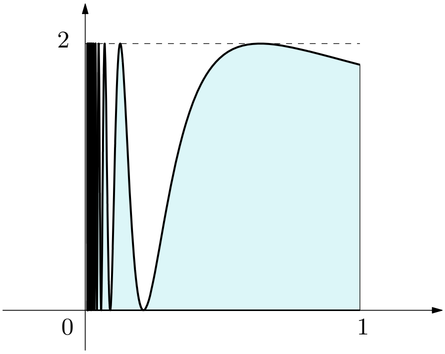
Theorem 9.31 *Among the class of integrable maps on $[a, b]$ are*

a) continuous maps on $[a, b]$;
b) piecewise-continuous maps on $[a, b]$;
c) continuous maps on (a, b) which are bounded on $[a, b]$;
d) monotone functions on $[a, b]$.

As an application of the theorem,

$$f(x) = \begin{cases} 1 + \sin\dfrac{1}{x} & \text{if } 0 < x \le 1, \\ 0 & \text{if } x = 0, \end{cases}$$

is integrable, for continuous on $(0, 1]$ and bounded (by 0 and 2) on $[0, 1]$.



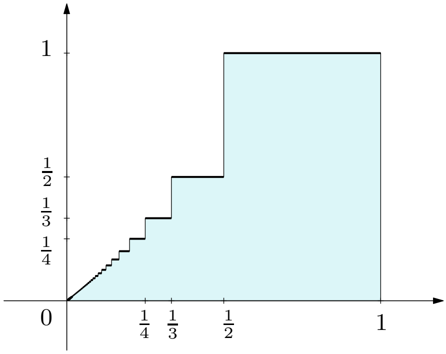


Figure 9.6. Integrable maps on $[0, 1]$

The same for

$$f(x) = \begin{cases} \dfrac{1}{n} & \text{if } \dfrac{1}{n+1} < x \leq \dfrac{1}{n}, \ n = 1, 2, \ldots, \\ 0 & \text{if } x = 0, \end{cases}$$

which is increasing (not strictly) on $[0, 1]$, see Fig. 9.6.

A couple more properties will be useful later; see Appendix A.5.2, p. 466, for their proof.

Proposition 9.32 *If f is integrable on $[a, b]$, then*

i) f *is integrable on any subinterval* $[c, d] \subset [a, b]$*;*

ii) $|f|$ *is integrable on* $[a, b]$*.*

9.6 Properties of definite integrals

A (piecewise-)continuous map is Cauchy integrable (Theorem 9.20) and at the same time integrable following Riemann (Theorem 9.31). The two types of definite integral always agree for such maps, as seen explicitly for $f(x) = x$ in Examples 9.22 ii), 9.30. We shall not prove this fact rigorously. Anyhow, that reason is good enough to use a unique symbol for both Riemann's and Cauchy's integrals. Henceforth $\mathcal{R}([a, b])$ shall be the set of integrable maps on $[a, b]$.

Recall $\int_a^b f(x)\,\mathrm{d}x$ is a *number*, depending only on f and the interval $[a, b]$; it certainly depends upon no variable. The letter x, present in the symbol for historical reasons essentially, is a 'virtual variable', and as such may be substituted by one's own preferred letter; writing $\int_a^b f(x)\,\mathrm{d}x$, rather than $\int_a^b f(s)\,\mathrm{d}s$ or $\int_a^b f(y)\,\mathrm{d}y$ is a matter of taste, for all three symbols represent the *same number*.

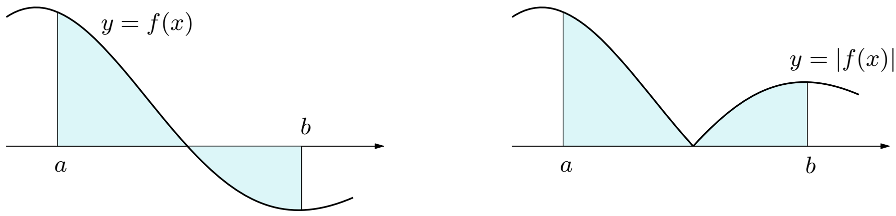


Figure 9.7. The area of the trapezoidal region of f on $[a, b]$ is $\int_a^b |f(x)|\,\mathrm{d}x$

If $f \in \mathcal{R}([a, b])$ is positive we have shown the definite integral expresses the area of the trapezoidal region of f over $[a, b]$. For negative f the same holds provided one changes sign to the value. When f has no fixed sign, the integral measures the difference of the positive regions (above the x-axis) and the negative regions (below it), so the area between f and the horizontal axis is also the integral of the map $|f|$

$$\text{Area of } \mathcal{T}(f; a, b) = \int_a^b |f(x)|\,\mathrm{d}x.$$

This is due to the symmetrising effect of the absolute value, which reflects the regions lying below the axis in a rigid way (as in Fig. 9.7).

Finally, let us slightly generalise the definite integral. Take $f \in \mathcal{R}([a, b])$. For $a \le c < d \le b$, set

$$\int_d^c f(x)\,\mathrm{d}x = -\int_c^d f(x)\,\mathrm{d}x \qquad \text{and} \qquad \int_c^c f(x)\,\mathrm{d}x = 0. \tag{9.18}$$

The symbol $\int_c^d f(x)\,\mathrm{d}x$ is now defined whichever limits c and d we consider in the integrability domain $[a, b]$.

The following five properties descend immediately from the definition.

Theorem 9.33 *Let f and g be integrable on a bounded interval I of the real line.*

i) (**Additivity with respect to the domain of integration**) *For any $a, b, c \in I$,*

$$\int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x.$$

ii) (**Linearity**) *For any $a, b \in I$ and $\alpha, \beta \in \mathbb{R}$,*

$$\int_a^b \Big(\alpha f(x) + \beta g(x)\Big)\,\mathrm{d}x = \alpha \int_a^b f(x)\,\mathrm{d}x + \beta \int_a^b g(x)\,\mathrm{d}x.$$

iii) (**Positivity**) *Let $a, b \in I$, with $a < b$. If $f \geq 0$ on $[a, b]$ then*

$$\int_a^b f(x)\,\mathrm{d}x \geq 0.$$

If f is additionally continuous, equality holds if and only if f is the zero map.

iv) (**Monotonicity**) *Let $a, b \in I$, $a < b$. If $f \leq g$ in $[a, b]$, then*

$$\int_a^b f(x)\,\mathrm{d}x \leq \int_a^b g(x)\,\mathrm{d}x.$$

v) (**Upper and lower bounds**) *Let $a, b \in I$, $a < b$. Then*

$$\left|\int_a^b f(x)\,\mathrm{d}x\right| \leq \int_a^b |f(x)|\,\mathrm{d}x.$$

Proof. See Appendix A.5.2, p. 467. □

9.7 Integral mean value

The definite integral of an integrable map f over the usual real interval $[a, b]$ furnishes a way of approximating the function's behaviour by a constant.

Definition 9.34 *By (integral)* **mean value** (*sometimes* **integral average**) **of** f **over the interval** $[a, b]$ *one understands the number*

$$\mathrm{m}(f; a, b) = \frac{1}{b-a} \int_a^b f(x)\,\mathrm{d}x.$$

The geometric meaning is clear when f is positive on $[a, b]$, for an equivalent version of the mean value reads

$$\int_a^b f(x)\,\mathrm{d}x = (b-a)\mathrm{m}(f; a, b).$$

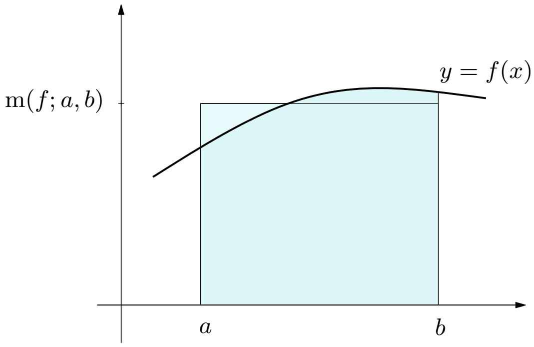


Figure 9.8. Integral average of f over $[a, b]$

In this case $\mathcal{T}(f; a, b)$ equals the area of the rectangle with base $[a, b]$ and having the integral average as height (Fig. 9.8).

The next statement formalises the relation between the integral mean value of a function and its range.

Theorem 9.35 (Mean Value Theorem) *Let f be integrable over $[a, b]$. The integral mean of f over $[a, b]$ satisfies*

$$\inf_{x\in[a,b]} f(x) \le \mathrm{m}(f; a, b) \le \sup_{x\in[a,b]} f(x). \tag{9.19}$$

If moreover f is continuous on $[a, b]$, there is at least one $z \in [a, b]$ such that

$$\mathrm{m}(f; a, b) = f(z). \tag{9.20}$$

Proof. Call $i_f = \inf_{x\in[a,b]} f(x)$ and $s_f = \sup_{x\in[a,b]} f(x)$, so for any $x \in [a, b]$

$$i_f \le f(x) \le s_f.$$

By property iv) of Theorem 9.33

$$(b-a)\, i_f = \int_a^b i_f \,\mathrm{d}x \le \int_a^b f(x)\,\mathrm{d}x \le \int_a^b s_f \,\mathrm{d}x = (b-a)\, s_f.$$

where we have used the expression for the integral of a constant. Now divide by $b - a$ to attain (9.19).

Supposing f continuous, Weierstrass's Theorem 4.31 yields

$$i_f = \min_{x\in[a,b]} f(x) \qquad \text{and} \qquad s_f = \max_{x\in[a,b]} f(x),$$

hence (9.19) tells that $\mathrm{m}(f; a, b)$ lies between the maximum and minimum of f on $[a, b]$. The existence of a point z for which (9.20) holds then follows from (4.16). □

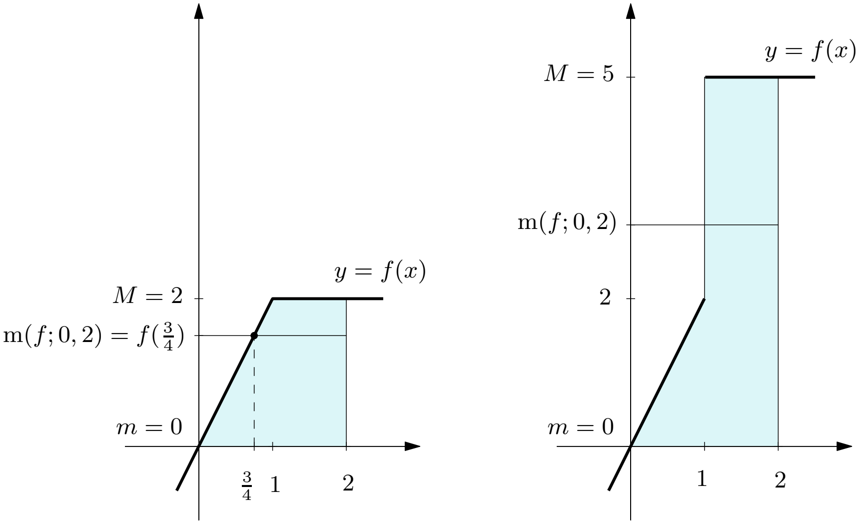


Figure 9.9. The Mean Value Theorem of integral calculus

Example 9.36

The integral mean of the continuous map

$$f(x)=\begin{cases}2x & \text{if } 0\le x\le 1,\\ 2 & \text{if } 1<x\le 2,\end{cases}$$

over $[0,2]$ is

$$\mathrm{m}(f;0,2)=\frac{1}{2}\int_0^2 f(x)\,\mathrm{d}x=\frac{1}{2}\left(\int_0^1 2x\,\mathrm{d}x+\int_1^2 2\,\mathrm{d}x\right)=\frac{1}{2}(1+2)=\frac{3}{2}.$$

In conformity with the statement, the mean value is indeed a value the function takes, in fact $\mathrm{m}(f;0,2)=f(\frac{3}{4})$ (Fig. 9.9, left).

Consider now the piecewise-continuous map

$$f(x)=\begin{cases}2x & \text{if } 0\le x\le 1,\\ 5 & \text{if } x>1.\end{cases}$$

The mean value over $[0,5/4]$ is $\mathrm{m}(f;0,5/4)=f(9/10)$ and belongs to the map's range; this is not so when we consider $[0,2]$, because $\mathrm{m}(f;0,2)=3$ (Fig. 9.9, right). This example shows that the continuity of f is just a sufficient condition for (9.20) to hold. □

A closing remark for the sequel. Taking (9.18) into account, we observe that the mean value formula stays valid if the limits of integration are interchanged, hence the theorem is correct also when $a>b$:

$$\mathrm{m}(f;a,b)=\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x=\frac{1}{a-b}\int_b^a f(x)\,\mathrm{d}x=\mathrm{m}(f;b,a). \tag{9.21}$$

9.8 The Fundamental Theorem of integral calculus

Let f be defined on the real interval I, which we do not assume bounded necessarily, but suppose f is integrable on every closed and bounded subinterval of I. This is the case if f is continuous. We call **integral function of f on I** any map of the form

$$F(x) = F_{x_0}(x) = \int_{x_0}^{x} f(s)\,\mathrm{d}s, \tag{9.22}$$

where $x_0 \in I$ is a *fixed* point and x varies in I. An integral function is thus obtained by integrating f on an interval in which one end-point is fixed while the other is variable. By (9.18) any integral function is defined on the whole I, and F_{x_0} has a zero at x_0.

The Fundamental Theorem of integral calculus establishes the basic inverse character of the operations of differentiation and integration, namely that any integral function of a given continuous map f over I is a primitive of f on that interval.

Theorem 9.37 (**fundamental of integral calculus**) *Let f be defined and continuous over a real interval I. Given $x_0 \in I$, let*

$$F(x) = \int_{x_0}^{x} f(s)\,\mathrm{d}s$$

denote an integral function of f on I. Then F is differentiable everywhere over I and

$$F'(x) = f(x), \qquad \forall x \in I.$$

Proof. Let us start by fixing an x inside I and calling Δx an increment (positive or negative) such that $x+\Delta x$ belongs to I. Consider the difference quotient of F

$$\frac{F(x+\Delta x) - F(x)}{\Delta x} = \frac{1}{\Delta x}\left(\int_{x_0}^{x+\Delta x} f(s)\,\mathrm{d}s - \int_{x_0}^{x} f(s)\,\mathrm{d}s\right).$$

By property i) in Theorem 9.33,

$$\int_{x_0}^{x+\Delta x} f(s)\,\mathrm{d}s = \int_{x_0}^{x} f(s)\,\mathrm{d}s + \int_{x}^{x+\Delta x} f(s)\,\mathrm{d}s$$

so

$$\frac{F(x+\Delta x) - F(x)}{\Delta x} = \frac{1}{\Delta x}\int_{x}^{x+\Delta x} f(s)\,\mathrm{d}s = \mathrm{m}(f; x, x+\Delta x).$$

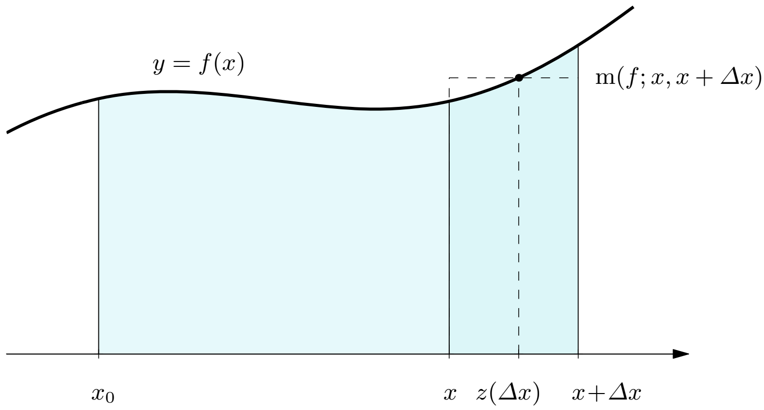


Figure 9.10. The Fundamental Theorem of integral calculus

Thus, the difference quotient of the integral function F between x and $x + \Delta x$ equals the mean value of f between x and $x + \Delta x$. Since f is continuous, the Mean Value Theorem 9.35 guarantees the existence in that interval of a $z = z(\Delta x)$ for which $\mathrm{m}(f; x, x + \Delta x) = f\big(z(\Delta x)\big)$, in other words

$$\frac{F\big(x + \Delta x\big) - F(x)}{\Delta x} = f(z(\Delta x)). \tag{9.23}$$

Take the limit for $\Delta x \to 0$. For simplicity we can assume $\Delta x > 0$. From

$$x \le z(\Delta x) \le x + \Delta x$$

and Theorem 4.5 we deduce that

$$\lim_{\Delta x \to 0^+} z(\Delta x) = x.$$

By similar arguments $\lim\limits_{\Delta x \to 0^-} z(\Delta x) = x$, so $\lim\limits_{\Delta x \to 0} z(\Delta x) = x$. But f is continuous at x, hence (4.11) implies

$$\lim_{\Delta x \to 0} f\big(z(\Delta x)\big) = f\big(\lim_{\Delta x \to 0} z(\Delta x)\big) = f(x).$$

Passing to the limit in (9.23), we find the thesis

$$F'(x) = \lim_{\Delta x \to 0} \frac{F(x + \Delta x) - F(x)}{\Delta x} = f(x).$$

In case x is a boundary point of I it suffices to take one-sided limits instead, and the same conclusion follows. $\square$

Corollary 9.38 *Let F_{x_0} be an integral function of a continuous f on I. Then*

$$F_{x_0}(x) = G(x) - G(x_0), \qquad \forall x \in I$$

for any primitive map G of f on I.

Proof. There exists a number c with $F_{x_0}(x) = G(x) - c$, $\forall x \in I$ by Theorem 9.4. The constant is fixed by the condition $F_{x_0}(x_0) = 0$. □

The next corollary has great importance, for it provides the definite integral by means of an arbitrary primitive of the integrand.

Corollary 9.39 *Let f be continuous on $[a, b]$ and G any primitive of f on that interval. Then*

$$\int_a^b f(x)\,\mathrm{d}x = G(b) - G(a). \tag{9.24}$$

Proof. Denoting F_a the integral map vanishing at a, one has

$$\int_a^b f(x)\,\mathrm{d}x = F_a(b).$$

The previous corollary proves the claim once we put $x_0 = a, x = b$. □

Very often the difference $G(b) - G(a)$ is written as

$$\big[G(x)\big]_a^b \qquad \text{or} \qquad G(x)\Big|_a^b.$$

Examples 9.40

The three integrals below are computed using (9.24).

$$\int_0^1 x^2\,\mathrm{d}x = \left[\frac{1}{3}x^3\right]_0^1 = \frac{1}{3}.$$

$$\int_0^\pi \sin x\,\mathrm{d}x = \big[-\cos x\big]_0^\pi = 2.$$

$$\int_2^6 \frac{1}{x}\,\mathrm{d}x = \big[\log x\big]_2^6 = \log 6 - \log 2 = \log 3.$$

□

Remark 9.41 There is a generalisation of the Fundamental Theorem of integral calculus to piecewise-continuous maps, which goes like this. If f is piecewise-continuous on all closed and bounded subintervals of I, then any integral function F on I is continuous on I, it is differentiable at all points where f is continuous, and $F'(x) = f(x)$. Jump discontinuities for f inside I correspond to corner points for F.

The integral F is called then a *generalised primitive* of f on I. □

Now we present an integral representation of a differentiable map, which turns out to be useful in many circumstances.

Corollary 9.42 *Given f differentiable on I with continuous first derivative, and any $x_0 \in I$,*

$$f(x) = f(x_0) + \int_{x_0}^{x} f'(s)\,\mathrm{d}s, \qquad \forall x \in I. \tag{9.25}$$

Proof. Obviously f is a primitive of its own derivative, so (9.24) gives

$$\int_{x_0}^{x} f'(s)\,\mathrm{d}s = f(x) - f(x_0),$$

whence the result follows. □

We illustrate this result by providing two applications. The first one is the justification of the Maclaurin expansion of $f(x) = \arcsin x$ and $f(x) = \arctan x$. First though, a technical lemma.

Lemma 9.43 *If φ is a continuous map around 0 such that $\varphi(x) = o(x^\alpha)$ for $x \to 0$, and $\alpha \geq 0$, then the primitive $\psi(x) = \int_0^x \varphi(s)\,\mathrm{d}s$ satisfies $\psi(x) = o(x^{\alpha+1})$ as $x \to 0$. This can be written as*

$$\int_0^x o(s^\alpha)\,\mathrm{d}s = o(x^{\alpha+1}) \qquad \textit{for } x \to 0. \tag{9.26}$$

Proof. From de l'Hôpital's Theorem 6.41,

$$\lim_{x\to 0} \frac{\psi(x)}{x^{\alpha+1}} = \lim_{x\to 0} \frac{\psi'(x)}{(\alpha+1)x^\alpha} = \frac{1}{\alpha+1} \lim_{x\to 0} \frac{\varphi(x)}{x^\alpha} = 0. \qquad \square$$

So now take $f(x) = \arctan x$. As its derivative reads $f'(x) = \dfrac{1}{1+x^2}$, (9.25) allows us to write

$$\arctan x = \int_0^x \frac{1}{1+s^2}\,\mathrm{d}s.$$

The Maclaurin expansion of $f'(s)$, obtained from (7.18) changing $x = s^2$, reads

$$\frac{1}{1+s^2} = 1 - s^2 + s^4 - \ldots + (-1)^m s^{2m} + o(s^{2m+1})$$
$$= \sum_{k=0}^{m} (-1)^k s^{2k} + o(s^{2m+1}).$$

Term-by-term integration together with (9.26) yields Maclaurin's expansion for $f(x)$:

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots + (-1)^m \frac{x^{2m+1}}{2m+1} + o(x^{2m+2})$$
$$= \sum_{k=0}^{m} (-1)^k \frac{x^{2k+1}}{2k+1} + o(x^{2m+2}).$$

As for the inverse sine, write

$$f(x) = \arcsin x = \int_0^x \frac{1}{\sqrt{1-s^2}}\, \mathrm{d}s.$$

Now use (7.17) with $\alpha = -\frac{1}{2}$ and change $x = -s^2$:

$$\frac{1}{\sqrt{1-s^2}} = 1 + \frac{1}{2}s^2 + \frac{3}{8}s^4 + \ldots + \left|\binom{-\frac{1}{2}}{m}\right| s^{2m} + o(s^{2m+1})$$
$$= \sum_{k=0}^{m} \left|\binom{-\frac{1}{2}}{k}\right| s^{2k} + o(s^{2m+1}).$$

Integrating the latter term-by-term and using (9.26) yields the expansion

$$\arcsin x = x + \frac{x^3}{6} + \frac{3x^5}{40} + \ldots + \left|\binom{-\frac{1}{2}}{m}\right| \frac{x^{2m+1}}{2m+1} + o(x^{2m+2})$$
$$= \sum_{k=0}^{m} \left|\binom{-\frac{1}{2}}{k}\right| \frac{x^{2k+1}}{2k+1} + o(x^{2m+2}).$$

As a second application of Corollary 9.42, we derive a new form for the remainder in a Taylor formula, which adds to the already known expressions due to Peano and Lagrange (recall formulas (7.6) and (7.8)). Such a form, called integral form, may provide more accurate information on the behaviour of the error than the previous ones, although under a stronger assumption on the function f. The proof of this result, that makes use of the Principle of Induction, is given in Appendix A.4.4, p. 458, where the reader may also find an example of application of the new form.

Theorem 9.44 (**Taylor formula with integral remainder**) *Let $n \geq 0$ be an arbitrary integer, f differentiable $n+1$ times around a point x_0, with continuous derivative of order $n+1$. Then*

$$f(x) - Tf_{n,x_0}(x) = \frac{1}{n!} \int_{x_0}^{x} f^{(n+1)}(t)(x-t)^n \, dt \, .$$

9.9 Rules of definite integration

The Fundamental Theorem of integral calculus and the rules that apply to indefinite integrals, presented in Sect. 9.2, furnish similar results for definite integrals.

Theorem 9.45 (**Integration by parts**) *Let f and g be differentiable with continuity on $[a,b]$. Then*

$$\int_a^b f(x)g'(x) \, \mathrm{d}x = [f(x)g(x)]_a^b - \int_a^b f'(x)g(x) \, \mathrm{d}x. \tag{9.27}$$

Proof. If $H(x)$ denotes any primitive of $f'(x)g(x)$ on $[a,b]$, the known result on integration by parts prescribes that $f(x)g(x) - H(x)$ is a primitive of $f(x)g'(x)$. Thus (9.24) implies

$$\int_a^b f(x)g'(x) \, \mathrm{d}x = \big[f(x)g(x)\big]_a^b - \big[H(x)\big]_a^b.$$

It then suffices to use (9.24) on the map $f'(x)g(x)$. □

Theorem 9.46 (**Integration by substitution**) *Let $f(y)$ be continuous on $[a,b]$. Take a map $\varphi(x)$ from $[\alpha,\beta]$ to $[a,b]$, differentiable with continuity. Then*

$$\int_\alpha^\beta f\big(\varphi(x)\big)\varphi'(x) \, \mathrm{d}x = \int_{\varphi(\alpha)}^{\varphi(\beta)} f(y) \, \mathrm{d}y. \tag{9.28}$$

If φ bijects $[\alpha,\beta]$ onto $[a,b]$, this formula may be written as

$$\int_a^b f(y) \, \mathrm{d}y = \int_{\varphi^{-1}(a)}^{\varphi^{-1}(b)} f\big(\varphi(x)\big)\varphi'(x) \, \mathrm{d}x. \tag{9.29}$$

Proof. Let $F(y)$ be a primitive of $f(y)$ on $[a,b]$. Formula (9.28) follows from (9.4) and Corollary 9.39. When φ is bijective, the two formulas are equivalent for $a=\varphi(\alpha),\ b=\varphi(\beta)$ if φ is strictly increasing, and $a=\varphi(\beta),\ b=\varphi(\alpha)$ if strictly decreasing. □

Both formulas are used in concrete applications.

Examples 9.47

i) Compute
$$\int_0^{\frac{3\pi}{4}} \sin^3 x \cos x \,\mathrm{d}x.$$
Set $y=\varphi(x)=\sin x$, so that $\varphi'(x)=\cos x$, $\varphi(0)=0$, $\varphi(\frac{3\pi}{4})=\frac{1}{\sqrt{2}}$. From (9.28) we obtain
$$\int_0^{\frac{3\pi}{4}} \sin^3 x \cos x \,\mathrm{d}x = \int_0^{\frac{1}{\sqrt{2}}} y^3 \,\mathrm{d}y = \left[\frac{1}{4}y^4\right]_0^{\frac{1}{\sqrt{2}}} = \frac{1}{16}.$$
Note φ is not injective on $[0,\frac{3\pi}{4}]$.

ii) To determine
$$S=\int_0^1 \arcsin\sqrt{1-y^2}\,\mathrm{d}y,$$
we change $y=\varphi(x)=\cos x$, with x varying in $[0,\frac{\pi}{2}]$. On this interval φ is strictly decreasing, hence one-to-one; moreover $\varphi(0)=1$ and $\varphi(\frac{\pi}{2})=0$, i.e., $\varphi^{-1}(0)=\frac{\pi}{2}$, $\varphi^{-1}(1)=0$. Note also
$$\arcsin\sqrt{1-\cos^2 x} = \arcsin\sqrt{\sin^2 x} = \arcsin(\sin x) = x.$$
Formula (9.29) gives
$$S=\int_{\pi/2}^0 (\arcsin\sqrt{1-\cos^2 x})\,(-\sin x)\,\mathrm{d}x = \int_0^{\pi/2} x\sin x\,\mathrm{d}x,$$
and eventually we may use (9.27)
$$S=\big[-x\cos x\big]_0^{\pi/2} + \int_0^{\pi/2} \cos x\,\mathrm{d}x = \big[\sin x\big]_0^{\pi/2} = 1.$$ □

Corollary 9.48 *Let f be integrable on the interval $[-a,a]$, $a>0$. If f is an even map,*
$$\int_{-a}^a f(x)\,\mathrm{d}x = 2\int_0^a f(x)\,\mathrm{d}x\,;$$
if f is odd,
$$\int_{-a}^a f(x)\,\mathrm{d}x = 0\,.$$

Proof. Theorem 9.33 *i)* gives

$$\int_{-a}^{a} f(x)\,\mathrm{d}x = \int_{-a}^{0} f(x)\,\mathrm{d}x + \int_{0}^{a} f(x)\,\mathrm{d}x\,.$$

Substitute $y = \varphi(x) = -x$ in the middle integral

$$\int_{-a}^{0} f(x)\,\mathrm{d}x = -\int_{a}^{0} f(-y)\,\mathrm{d}y = \int_{0}^{a} f(-y)\,\mathrm{d}y\,.$$

The right-most integral coincides with $\int_0^a f(y)\,\mathrm{d}y$ if f is even, with its opposite when f is odd. The claim follows because the variable of integration is a mute symbol. □

9.9.1 Application: computation of areas

This first chapter on integrals ends with a few examples of the use of the Fundamental Theorem to determine the area of planar regions.

i) Suppose we are asked to find the area A of the region enclosed by the graphs of the maps $y = f(x) = x^2$ and $y = g(x) = \sqrt{x}$, given in Fig. 9.11. The curves meet at points corresponding to $x = 0$ and $x = 1$, and the region in question can be seen as difference between the trapezoidal region of g and the trapezoidal region of f, both over the interval $[0, 1]$. Therefore

$$A = \int_0^1 g(x)\,\mathrm{d}x - \int_0^1 f(x)\,\mathrm{d}x = \int_0^1 [\sqrt{x} - x^2]\,\mathrm{d}x = \left[\frac{2}{3}x^{3/2} - \frac{1}{3}x^3\right]_0^1 = \frac{1}{3}.$$

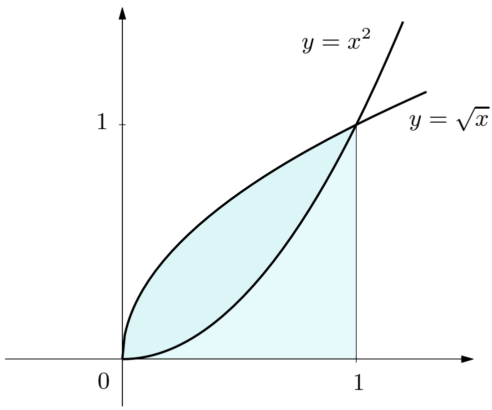


Figure 9.11. Region bounded by the graphs of $f(x) = x^2$ and $g(x) = \sqrt{x}$

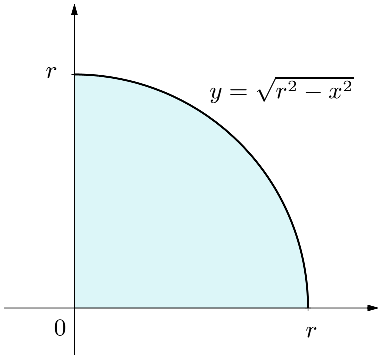


Figure 9.12. Area under $y = \sqrt{r^2 - x^2}$ in the first quadrant

ii) In the second example we check the known relation $A(r) = \pi r^2$ for the area of a disc in function of its radius r. The disc centred at the origin with radius r is the set of points (x, y) such that $x^2 + y^2 \le r^2$. The quarter is then the trapezoidal region of $y = \sqrt{r^2 - x^2}$ relative to $[0, r]$ (Fig. 9.12), so

$$A(r) = 4 \int_0^r \sqrt{r^2 - x^2}\, \mathrm{d}x.$$

Let us change variables by $x = \varphi(t) = rt$, so that $\mathrm{d}x = r\mathrm{d}t$ and $0 = \varphi(0)$, $r = \varphi(1)$. Because of (9.29), we have

$$A(r) = 4r^2 \int_0^1 \sqrt{1 - t^2}\, \mathrm{d}t. \tag{9.30}$$

From Example 9.13 vi), we already know a primitive of $f(t) = \sqrt{1 - t^2}$ is

$$F(t) = \frac{1}{2} t\sqrt{1 - t^2} + \frac{1}{2} \arcsin t\,.$$

Therefore

$$A(r) = 4r^2 \left[\frac{1}{2} t\sqrt{1 - t^2} + \frac{1}{2} \arcsin t \right]_0^1 = 4r^2 \frac{\pi}{4} = \pi r^2\,.$$

iii) We compute the area A of the part of plane bounded by the parabola $y = f(x) = x(1 - x)$ and the line $y = g(x) = -\frac{x}{2}$ (Fig. 9.13, left). The curves intersect at the origin and at $(\frac{3}{2}, -\frac{3}{4})$, plus on the interval $[0, \frac{3}{2}]$ we have $f(x) \ge g(x)$. Albeit part of the region overlaps the negative half-plane (where $y < 0$), the total area can still be calculated by

$$A = \int_0^{3/2} \big(f(x) - g(x)\big)\, \mathrm{d}x$$

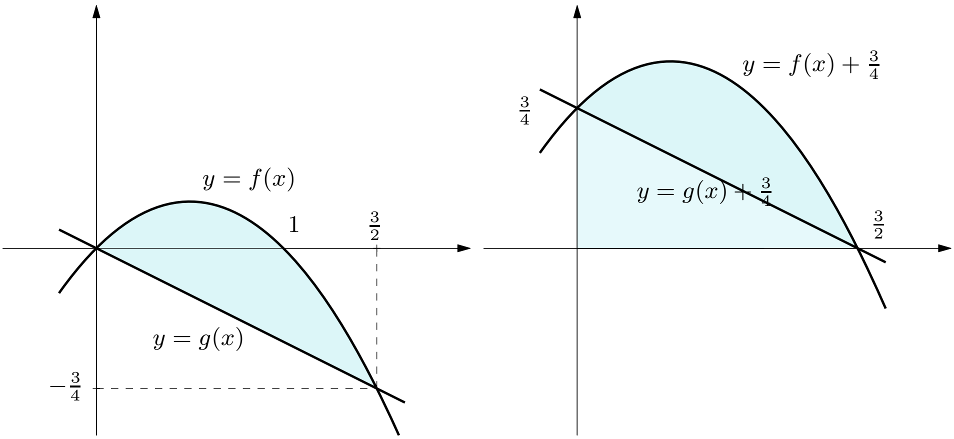


Figure 9.13. The area bounded by the graphs of $f(x)$ and $g(x)$ is translation-invariant

for the following reason. The number A is also the area of the region bounded by the graphs of the translated maps $f(x)+\frac{3}{4}$ and $g(x)+\frac{3}{4}$; shifting the x-axis vertically so that the origin goes to $(0,-\frac{3}{4})$, does not alter the surface area (Fig. 9.13, right). So,

$$A = \int_0^{3/2} \left(\frac{3}{2}x - x^2\right) \mathrm{d}x = \left[\frac{3}{4}x^2 - \frac{1}{3}x^3\right]_0^{3/2} = \frac{9}{16}\,.$$

9.10 Exercises

1. *Determine the general primitive of :*

   a) $f(x) = (x+1)^{27}$ b) $f(x) = \mathrm{e}^{-3x} - \mathrm{e}^{-5x}$

   c) $f(x) = \dfrac{x+1}{x^2+1}$ d) $f(x) = \dfrac{2-\sin x}{2x+\cos x}$

2. *Find the primitive map taking the value y_0 at x_0 of the functions:*

   a) $f(x) = x\mathrm{e}^{2x^2}$ $x_0 = \sqrt{2}$ $y_0 = 1$

   b) $f(x) = \dfrac{x^2}{1+x^6}$ $x_0 = 0$ $y_0 = 1$

   c) $f(x) = \dfrac{\log x}{x}$ $x_0 = \mathrm{e}$ $y_0 = 0$

   d) $f(x) = \cos x\, \mathrm{e}^{\sin x}$ $x_0 = \dfrac{\pi}{2}$ $y_0 = \mathrm{e}$

3. *Compute the indefinite integrals:*

a) $\displaystyle\int \frac{x}{x^2+7}\,\mathrm{d}x$ b) $\displaystyle\int (6x+3)^8\,\mathrm{d}x$

$\boxed{\text{c)}}$ $\displaystyle\int \frac{\mathrm{e}^{1/x^2}}{x^3}\,\mathrm{d}x$ d) $\displaystyle\int \frac{1}{x\log^2 x}\,\mathrm{d}x$

$\boxed{\text{e)}}$ $\displaystyle\int \mathrm{e}^x\sqrt{1+\mathrm{e}^x}\,\mathrm{d}x$ f) $\displaystyle\int \frac{x}{\sqrt{x^2+7}}\,\mathrm{d}x$

4. *Compute the indefinite integrals:*

a) $\displaystyle\int x^2\sin x\,\mathrm{d}x$ b) $\displaystyle\int x^2\log 2x\,\mathrm{d}x$

$\boxed{\text{c)}}$ $\displaystyle\int \log^2 x\,\mathrm{d}x$ $\boxed{\text{d)}}$ $\displaystyle\int x\arctan x\,\mathrm{d}x$

e) $\displaystyle\int \mathrm{e}^{2x}\cos x\,\mathrm{d}x$ $\boxed{\text{f)}}$ $\displaystyle\int \frac{1}{(1+x^2)^2}\,\mathrm{d}x$

5. *Compute the indefinite integrals:*

a) $\displaystyle\int \frac{2x}{x^2-4x+3}\,\mathrm{d}x$ b) $\displaystyle\int \frac{x^4-5x^3+8x^2-9x+11}{x^2-5x+6}\,\mathrm{d}x$

$\boxed{\text{c)}}$ $\displaystyle\int \frac{x}{x^3-1}\,\mathrm{d}x$ d) $\displaystyle\int \frac{17x^2-16x+60}{x^4-16}\,\mathrm{d}x$

$\boxed{\text{e)}}$ $\displaystyle\int \frac{x^4+1}{x^3-x^2}\,\mathrm{d}x$ $\boxed{\text{f)}}$ $\displaystyle\int \frac{2x^3-2x^2+7x+3}{(x^2+4)(x-1)^2}\,\mathrm{d}x$

6. *Compute the indefinite integrals:*

$\boxed{\text{a)}}$ $\displaystyle\int \frac{\mathrm{e}^{2x}}{\mathrm{e}^x+1}\,\mathrm{d}x$ b) $\displaystyle\int \frac{1}{(\mathrm{e}^x-2)^2}\,\mathrm{d}x$

$\boxed{\text{c)}}$ $\displaystyle\int \frac{1+\cos x}{1-\cos x}\,\mathrm{d}x$ d) $\displaystyle\int \frac{1}{1+\sin x}\,\mathrm{d}x$

e) $\displaystyle\int \frac{1}{\cos x}\,\mathrm{d}x$ $\boxed{\text{f)}}$ $\displaystyle\int \frac{\cos^2 x}{1-2\sin^2 x}\,\mathrm{d}x$

7. *Compute the indefinite integrals:*

a) $\displaystyle\int \frac{x}{\sqrt{2+x}}\,\mathrm{d}x$ b) $\displaystyle\int \frac{x}{(1+x^2)^2}\,\mathrm{d}x$

$\boxed{\text{c)}}$ $\displaystyle\int \frac{1}{x-3+\sqrt{3-x}}\,\mathrm{d}x$ $\boxed{\text{d)}}$ $\displaystyle\int \frac{1}{\sinh x}\,\mathrm{d}x$

e) $\displaystyle\int \cosh^2 x \,\mathrm{d}x$

$\boxed{\text{f)}}$ $\displaystyle\int \log \sqrt[3]{1+x^2}\,\mathrm{d}x$

g) $\displaystyle\int \frac{1}{1+\tan x}\,\mathrm{d}x$

$\boxed{\text{h)}}$ $\displaystyle\int \frac{1}{\mathrm{e}^{4x}+1}\,\mathrm{d}x$

$\boxed{\text{i)}}$ $\displaystyle\int \sin^5 x \,\mathrm{d}x$

$\boxed{\ell)}$ $\displaystyle\int \cos^4 x \,\mathrm{d}x$

$\boxed{8.}$ *Write the primitive of* $f(x) = |x|\log(2-x)$ *that has a zero at* $x = 1$.

$\boxed{9.}$ *Find the primitive* $F(x)$ *of* $f(x) = x\mathrm{e}^{-|x|}$ *with* $\lim_{x\to+\infty} F(x) = -5$.

$\boxed{10.}$ *What is the primitive, on the interval* $(-3, +\infty)$, *of*

$$f(x) = \frac{x+2}{(|x|+3)(x-3)}$$

that vanishes at $x = 0$?

$\boxed{11.}$ *Determine the generalised primitive of*

$$f(x) = \begin{cases} 2x^3 - 5x + 3 & \textit{if } x \geq 1, \\ 4x - 7 & \textit{if } x < 1 \end{cases}$$

vanishing at the origin.

$\boxed{12.}$ *Verify that*

$$\arctan\frac{1}{x} = \frac{\pi}{2} - \arctan x\,, \qquad \forall x > 0\,.$$

$\boxed{13.}$ *Write the Maclaurin expansion of order* 9 *for the generic primitive of the map* $f(x) = \cos 2x^2$.

$\boxed{14.}$ *Write the Maclaurin expansion of order* 4 *for the generic primitive of the map* $f(x) = \dfrac{2+\mathrm{e}^{-x}}{3+x^3}$.

15. *Determine the following definite integrals:*

a) $\displaystyle\int_0^{\pi} x\cos x\,\mathrm{d}x$

b) $\displaystyle\int_0^{1/2} \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x$

c) $\displaystyle\int_{\mathrm{e}}^{\mathrm{e}^2} x\log x\,\mathrm{d}x$

d) $\displaystyle\int_0^{\pi/2} \frac{1}{4\sin x + 3\cos x}\,\mathrm{d}x$

$\boxed{\text{e)}}$ $\displaystyle\int_1^{3} \frac{1}{[x]^2}\,\mathrm{d}x$

$\boxed{\text{f)}}$ $\displaystyle\int_0^{\sqrt{3}} M(x^2-1)\,\mathrm{d}x$

(Recall $[x]$ is the integer part of x and $M(x)$ denotes the mantissa.)

$\boxed{16.}$ *Compute the area of the trapezoidal region of* $f(x) = |\log x|$ *restricted to* $[e^{-1}, e]$.

17. *Find the area of the region enclosed by* $y = f(x)$ *and* $y = g(x)$*, where:*

$\boxed{\text{a)}}$ $f(x) = |x|\,,\qquad g(x) = \sqrt{1-x^2}$

b) $f(x) = x^2 - 2x\,,\qquad g(x) = -x^2 + x$

$\boxed{18.}$ *Determine*

$$F(x) = \int_{-1}^{x} (|t-1| + 2)\,dt\,.$$

9.10.1 Solutions

1. *Primitive functions:*

a) $F(x) = \frac{1}{28}(x+1)^{28} + c\,;$

b) $F(x) = \frac{1}{5}e^{-5x} - \frac{1}{3}e^{-3x} + c\,.$

c) Since we can write

$$\frac{x+1}{x^2+1} = \frac{1}{2}\frac{2x}{x^2+1} + \frac{1}{x^2+1},$$

it follows

$$F(x) = \frac{1}{2}\log(x^2+1) + \arctan x + c\,.$$

d) $F(x) = \log|2x + \cos x| + c\,.$

2. *Primitives:*

a) The general primitive $f(x)$ reads $F(x) = \frac{1}{4}e^{2x^2} + c$. Imposing $F(\sqrt{2}) = 1$, we get

$$1 = \frac{1}{4}e^4 + c \qquad \text{whence} \qquad c = 1 - \frac{1}{4}e^4,$$

so the required map is

$$F(x) = \frac{1}{4}e^{2x^2} + 1 - \frac{1}{4}e^4\,.$$

b) $F(x) = \frac{1}{3}\arctan x^3 + 1\,;$

c) $F(x) = \frac{1}{2}\log^2 x - \frac{1}{2}\,;$

d) $F(x) = e^{\sin x}\,.$

3. *Indefinite integrals:*

a) $S = \frac{1}{2}\log(x^2+7) + c\,;$

b) $S = \frac{1}{54}(6x+3)^9 + c\,.$

c) Changing $y = \frac{1}{x^2}$ gives $dy = -\frac{2}{x^3}\,dx$, hence

$$S = -\frac{1}{2}\int e^t\,dt = -\frac{1}{2}e^t + c = -\frac{1}{2}e^{1/x^2} + c\,.$$

d) $S = -\frac{1}{\log x} + c$.

e) Set $y = 1 + e^x$, so that $dy = e^x\, dx$ and

$$S = \int \sqrt{t}\, dt = \frac{2}{3} t^{3/2} + c = \frac{2}{3}\sqrt{(1+e^x)^3} + c\,.$$

f) $S = \sqrt{x^2+7} + c$.

4. *Indefinite integrals:*

a) $S = (2 - x^2)\cos x + 2x \sin x + c$;

b) $S = \frac{1}{3}x^3(\log 2x - \frac{1}{3}) + c$.

c) We integrate by parts choosing $f(x) = \log^2 x$ and $g'(x) = 1$. Then $f'(x) = \frac{2}{x}\log x$, $g(x) = x$, giving

$$S = x \log^2 x - 2\int \log x\, dx\,.$$

The integral on the right requires another integration by parts (as in Example 9.11 ii)) and eventually leads to

$$S = x\log^2 x - 2x(\log x - 1) + c = x(\log^2 x - 2\log x + 2) + c\,.$$

d) We take $f(x) = \arctan x, g'(x) = x$ and integrate by parts. Since $f'(x) = \frac{1}{1+x^2}$ and $g(x) = \frac{1}{2}x^2$,

$$\begin{aligned} S &= \frac{1}{2}x^2 \arctan x - \frac{1}{2}\int \frac{x^2}{1+x^2}\, dx \\ &= \frac{1}{2}x^2 \arctan x - \frac{1}{2}\int \left(1 - \frac{1}{1+x^2}\right) dx \\ &= \frac{1}{2}x^2 \arctan x - \frac{1}{2}x + \frac{1}{2}\arctan x + c\,. \end{aligned}$$

e) $S = \frac{1}{5}e^{2x}(\sin x + 2\cos x) + c$.

f) The remarks made on p. 314 v) suggest to integrate $S_1 = \int \frac{1}{1+x^2}\, dx$ by parts with $f(x) = \frac{1}{1+x^2}$ and $g'(x) = 1$. Then $f'(x) = -\frac{2x}{(1+x^2)^2}$, $g(x) = x$, and

$$\begin{aligned} S_1 &= \int \frac{1}{1+x^2}\, dx = \frac{x}{1+x^2} + 2\int \frac{x^2}{(1+x^2)^2}\, dx \\ &= \frac{x}{1+x^2} + 2\int \frac{x^2+1-1}{(1+x^2)^2}\, dx = \frac{x}{1+x^2} + 2S_1 - 2\int \frac{1}{(1+x^2)^2}\, dx\,. \end{aligned}$$

The solution is then

$$S = \frac{1}{2}\left(S_1 + \frac{x}{1+x^2}\right) = \frac{1}{2}\left(\arctan x + \frac{x}{1+x^2}\right) + c\,.$$

5. *Indefinite integrals:*

a) $S = 3\log|x-3| - \log|x-1| + c$.

b) $S = \frac{1}{3}x^3 + 2x + 2\log|x-3| - \log|x-2| + c$.

c) Splitting into partial fractions

$$\frac{x}{x^3-1} = \frac{x}{(x-1)(x^2+x+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1},$$

yields $A(x^2+x+1) + (Bx+C)(x-1) = x$. From $x=1$ and $x=0$ we find the constants $A = C = \frac{1}{3}$, while $x=-1$ determines $B = -\frac{1}{3}$. Therefore

$$\begin{aligned}\frac{x}{x^3-1} &= \frac{1}{3}\left(\frac{1}{x-1} - \frac{x-1}{x^2+x+1}\right)\\ &= \frac{1}{3}\left(\frac{1}{x-1} - \frac{1}{2}\frac{2x+1-3}{x^2+x+1}\right)\\ &= \frac{1}{3}\left(\frac{1}{x-1} - \frac{1}{2}\frac{2x+1}{x^2+x+1} + \frac{3}{2}\frac{1}{(x+\frac{1}{2})^2+\frac{3}{4}}\right).\end{aligned}$$

In conclusion,

$$S = \frac{1}{3}\left(\log|x-1| - \frac{1}{2}\log(x^2+x+1) + \sqrt{3}\arctan\frac{2}{\sqrt{3}}(x+\frac{1}{2})\right) + c.$$

d) $S = \log(x^2+4) + 3\log|x-2| - 5\log|x+2| + \frac{1}{2}\arctan\frac{x}{2} + c$.

e) We search for the undetermined coefficients in

$$\frac{x^4+1}{x^3-x^2} = x+1+\frac{x^2+1}{x^3-x^2} = x+1+\frac{A}{x-1}+\frac{B}{x}+\frac{C}{x^2}.$$

Choosing $x=1$ and $x=0$ for

$$Ax^2 + (Bx+C)(x-1) = x^2+1$$

produces $A=2, C=-1$, while $x=-1$ tells $B=-1$:

$$S = \int\left(x+1+\frac{2}{x-1}-\frac{1}{x}-\frac{1}{x^2}\right)\mathrm{d}x = \frac{1}{2}x^2+x+2\log|x-1|-\log|x|+\frac{1}{x}+c.$$

f) The integrand splits as

$$\frac{2x^3-2x^2+7x+3}{(x^2+4)(x-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+4}.$$

Imposing

$$A(x-1)(x^2+4) + B(x^2+4) + (Cx+D)(x-1)^2 = 2x^3-2x^2+7x+3,$$

leads to $A = 1,\ B = 2,\ C = 1, D = -1$, hence

$$S = \int \left(\frac{1}{x-1} + \frac{2}{(x-1)^2} + \frac{x-1}{x^2+4} \right) \mathrm{d}x$$
$$= \log|x-1| - \frac{2}{x-1} + \frac{1}{2}\log(x^2+4) - \frac{1}{2}\arctan\frac{x}{2} + c\,.$$

6. *Indefinite integrals:*

a) Put $y = \mathrm{e}^x$, then $\mathrm{d}y = \mathrm{e}^x\,\mathrm{d}x$, and

$$S = \int \frac{y}{y+1}\,\mathrm{d}y = \int \left(1 - \frac{1}{y+1}\right)\mathrm{d}y$$
$$= y - \log|y+1| + c$$
$$= \mathrm{e}^x - \log(\mathrm{e}^x+1) + c\,.$$

b) $S = \frac{1}{4}x - \frac{1}{4}\log|\mathrm{e}^x - 2| - \frac{1}{2}\frac{1}{\mathrm{e}^x-2} + c\,.$

c) Changing $t = \tan\frac{x}{2}$ we can write $\cos x = \frac{1-t^2}{1+t^2}$ and $\mathrm{d}x = \frac{2}{1+t^2}\,\mathrm{d}t$. Then

$$S = 2\int \frac{1}{t^2(1+t^2)}\,\mathrm{d}t = 2\int \left(\frac{1}{t^2} - \frac{1}{1+t^2}\right)\mathrm{d}t$$
$$= -\frac{2}{t} - 2\arctan t + c = -\frac{2}{\tan\frac{x}{2}} - x + c\,.$$

d) $S = -\dfrac{2}{1+\tan\frac{x}{2}} + c\,;$ e) $S = \log\left|\dfrac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}\right| + c\,.$

f) Set $t = \tan x$, so $\sin^2 x = \frac{t^2}{1+t^2}$, $\cos^2 t = \frac{1}{1+t^2}$ and $\mathrm{d}x = \frac{1}{1+t^2}\,\mathrm{d}t$. From that,

$$S = \int \frac{1}{(1+t^2)(1-t^2)}\,\mathrm{d}t = \int \left(\frac{A}{1+t} + \frac{B}{1-t} + \frac{Ct+D}{1+t^2}\right)\mathrm{d}t\,.$$

Now evaluate at $t=-1,\ t=1,\ t=0$ and $t=2$ the condition

$$A(1-t)(1+t^2) + B(1+t)(1+t^2) + (Ct+D)(1-t^2) = 1\,,$$

to obtain $A = \frac{1}{4}$, $B = \frac{1}{4}$, $C = 0, D = \frac{1}{2}$. Then

$$S = \int \left(\frac{1}{4}\frac{1}{1+t} + \frac{1}{4}\frac{1}{1-t} + \frac{1}{2}\frac{1}{1+t^2}\right)\mathrm{d}t$$
$$= \frac{1}{4}\log|1+t| - \frac{1}{4}\log|1-t| + \frac{1}{2}\arctan t + c$$
$$= \frac{1}{4}\log\left|\frac{1+t}{1-t}\right| + \frac{1}{2}\arctan t + c = \frac{1}{4}\log\left|\frac{\sin x + \cos x}{\sin x - \cos x}\right| + \frac{1}{2}x + c\,.$$

7. *Indefinite integrals:*

a) $S = \frac{2}{3}\sqrt{(2+x)^3} - 4\sqrt{2+x} + c$; b) $S = -\frac{1}{2(1+x^2)} + c$.

c) With $t^2 = 3 - x$ we have $x = 3 - t^2$ and $2t\,\mathrm{d}t = -\mathrm{d}x$, so

$$S = \int \frac{2t}{t^2 - t}\,\mathrm{d}t = 2\int \frac{1}{t-1}\,\mathrm{d}t = 2\log|t-1| + c = 2\log|\sqrt{3-x} - 1| + c\,.$$

d) By definition $\sinh x = \frac{\mathrm{e}^x - \mathrm{e}^{-x}}{2}$, so $y = \mathrm{e}^x$ yields

$$\begin{aligned} S &= \int \frac{2}{y^2-1}\,\mathrm{d}y = \int \left(\frac{1}{y-1} - \frac{1}{y+1}\right)\mathrm{d}y \\ &= \log|y-1| - \log|y+1| + c = \log\frac{|\mathrm{e}^x - 1|}{\mathrm{e}^x + 1} + c\,. \end{aligned}$$

e) $S = \frac{1}{4}\left(\frac{1}{2}\mathrm{e}^{2x} - \frac{1}{2}\mathrm{e}^{-2x} + 2x\right) + c = \frac{1}{4}\sinh 2x + \frac{1}{2}x + c\,.$

f) Observe $\log\sqrt[3]{1+x^2} = \frac{1}{3}\log(1+x^2)$. We integrate by parts putting $f(x) = \log(1+x^2), g'(x) = 1$, so $f'(x) = \frac{2x}{1+x^2}$ and $g(x) = x$. Then

$$\begin{aligned} S &= \frac{1}{3}\left(x\log(1+x^2) - 2\int \frac{x^2}{1+x^2}\,\mathrm{d}x\right) \\ &= \frac{1}{3}\left(x\log(1+x^2) - 2\int \left(1 - \frac{1}{1+x^2}\right)\mathrm{d}x\right) \\ &= \frac{1}{3}\left(x\log(1+x^2) - 2x + 2\arctan x\right) + c\,. \end{aligned}$$

g) $S = \frac{1}{2}\left(\log|1+\tan x| - \frac{1}{2}\log(1+\tan^2 x) + x\right) + c\,.$

h) Setting $y = \mathrm{e}^{4x}$ implies $\mathrm{d}y = 4\mathrm{e}^{4x}\,\mathrm{d}x$, $\mathrm{d}x = \frac{1}{4y}\,\mathrm{d}y$. Thus

$$\begin{aligned} S &= \frac{1}{4}\int \frac{1}{y(y+1)}\,\mathrm{d}y = \frac{1}{4}\int \left(\frac{1}{y} - \frac{1}{y+1}\right)\mathrm{d}y \\ &= \frac{1}{4}\left(\log|y| - \log|y+1|\right) + c = \frac{1}{4}(4x - \log(\mathrm{e}^{4x}+1)) + c \\ &= x - \frac{1}{4}\log(\mathrm{e}^{4x}+1) + c\,. \end{aligned}$$

i) Because

$$\sin^5 x = \sin x\,\sin^4 x = \sin x(1-\cos^2 x)^2\,,$$

choosing $y = \cos x$ has the effect that $\mathrm{d}y = -\sin x\,\mathrm{d}x$ and

$$\begin{aligned} \int \sin^5 x\,\mathrm{d}x &= -\int (1-y^2)^2\,\mathrm{d}y = \int (-1 + 2y^2 - y^4)\,\mathrm{d}y \\ &= -y + \frac{2}{3}y^3 - \frac{1}{5}y^5 + c = -\cos x + \frac{2}{3}\cos^3 x - \frac{1}{5}\cos^5 x + c\,. \end{aligned}$$

ℓ) Given that $\cos^4 x = \cos x \cos^3 x$, let us integrate by parts with $f(x) = \cos^3 x$ and $g'(x) = \cos x$, implying $f'(x) = -3\sin x \cos^2 x$ and $g(x) = \sin x$. Thus

$$\begin{aligned} S = \int \cos^4 x \,\mathrm{d}x &= \sin x \cos^3 x + 3\int \cos^2 x \sin^2 x \,\mathrm{d}x \\ &= \sin x \cos^3 x + 3\int \cos^2 x(1-\cos^2 x)\,\mathrm{d}x \\ &= \sin x \cos^3 x + 3\int \cos^2 x \,\mathrm{d}x - 3S\,. \end{aligned}$$

Now recalling Example 9.9 ii),

$$4S = \sin x \cos^3 x + 3\left(\frac{1}{2}x + \frac{1}{4}\sin 2x\right) + c\,.$$

Finally,

$$\int \cos^4 x \,\mathrm{d}x = \frac{1}{4}\sin x \cos^3 x + \frac{3}{8}x + \frac{3}{16}\sin 2x + c\,.$$

8. Note that $f(x)$ is defined on $(-\infty, 2)$, where

$$f(x) = \begin{cases} x\log(2-x) & \text{if } 0 \le x < 2, \\ -x\log(2-x) & \text{if } x < 0\,. \end{cases}$$

In order to find a primitive, compute the integral $\int x\log(2-x)\,\mathrm{d}x$ by parts. Put $g(x) = \log(2-x)$ and $h'(x) = x$, implying $g'(x) = \frac{1}{x-2}$, $h(x) = \frac{1}{2}x^2$, and

$$\begin{aligned} \int x\log(2-x)\,\mathrm{d}x &= \frac{1}{2}x^2\log(2-x) - \frac{1}{2}\int \frac{x^2}{x-2}\,\mathrm{d}x \\ &= \frac{1}{2}x^2\log(2-x) - \frac{1}{2}\int \left(x+2+\frac{4}{x-2}\right)\mathrm{d}x \\ &= \frac{1}{2}x^2\log(2-x) - \frac{1}{4}x^2 - x - 2\log(2-x) + c\,. \end{aligned}$$

Thus

$$F(x) = \begin{cases} \frac{1}{2}x^2\log(2-x) - \frac{1}{4}x^2 - x - 2\log(2-x) + c_1 & \text{if } 0 \le x < 2\,, \\ -\frac{1}{2}x^2\log(2-x) + \frac{1}{4}x^2 + x + 2\log(2-x) + c_2 & \text{if } x < 0\,. \end{cases}$$

The constraint $F(1) = 0$ forces $c_1 = \frac{5}{4}$, and since F must be continuous at $x = 0$ it follows

$$F(0^+) = -2\log 2 + \frac{5}{4} = F(0^-) = 2\log 2 + c_2\,.$$

This gives $c_2 = -4\log 2 + \frac{5}{4}$, and the primitive is

$$F(x) = \begin{cases} \frac{1}{2}x^2\log(2-x) - \frac{1}{4}x^2 - x - 2\log(2-x) + \frac{5}{4} & \text{if } 0 \le x < 2\,, \\ -\frac{1}{2}x^2\log(2-x) + \frac{1}{4}x^2 + x + 2\log(2-x) - 4\log 2 + \frac{5}{4} & \text{if } x < 0\,. \end{cases}$$

9. We write, equivalently,

$$f(x) = \begin{cases} x\mathrm{e}^{-x} & \text{if } x \geq 0\,, \\ x\mathrm{e}^{x} & \text{if } x < 0\,. \end{cases}$$

With Example 9.11 i) in mind,

$$F(x) = \begin{cases} -(x+1)\mathrm{e}^{-x} + c_1 & \text{if } x \geq 0\,, \\ (x-1)\mathrm{e}^{x} + c_2 & \text{if } x < 0\,. \end{cases}$$

Continuity at $x = 0$ implies $F(0) = F(0^+) = c_1 = F(0^-) = c_2$, so the generic primitive of f is

$$F(x) = \begin{cases} -(x+1)\mathrm{e}^{-x} + c & \text{if } x \geq 0\,, \\ (x-1)\mathrm{e}^{x} + c & \text{if } x < 0, \end{cases}$$

i.e., $F(x) = -(|x|+1)\mathrm{e}^{-|x|} + c$. Additionally,

$$\lim_{x\to+\infty} F(x) = \lim_{x\to+\infty} \big(-(x+1)\mathrm{e}^{-x} + c \big) = c\,,$$

meaning that the condition $\lim\limits_{x\to+\infty} F(x) = -5$ holds when $c = -5$. The required map is

$$F(x) = -(|x|+1)\mathrm{e}^{-|x|} - 5\,.$$

10. Integrate the two cases

$$f(x) = \begin{cases} \dfrac{x+2}{(x+3)(x-3)} & \text{if } x \geq 0\,, \\ -\dfrac{x+2}{(x-3)^2} & \text{if } -3 < x < 0\,, \end{cases}$$

separately, that is, determine

$$S_1 = \int \frac{x+2}{(x+3)(x-3)}\,\mathrm{d}x \qquad \text{and} \qquad S_2 = \int \frac{x+2}{(x-3)^2}\,\mathrm{d}x\,.$$

These are rational integrands, so we ought to find the partial fractions first. Rather easily one sees

$$\frac{x+2}{(x+3)(x-3)} = \frac{A}{x+3} + \frac{B}{x-3} = \frac{1}{6}\left(\frac{1}{x+3} + \frac{5}{x-3} \right)$$
$$\frac{x+2}{(x-3)^2} = \frac{A}{x-3} + \frac{B}{(x-3)^2} = \frac{1}{x-3} + \frac{5}{(x-3)^2}\,,$$

whence

$$S_1 = \frac{1}{6}\left(\log|x+3| + 5\log|x-3|\right) + c_1\,, \qquad S_2 = \log|x-3| - \frac{5}{x-3} + c_2\,.$$

A primitive of f has then the form

$$F(x) = \begin{cases} S_1 & \text{if } x \geq 0\,, \\ -S_2 & \text{if } -3 < x < 0 \end{cases}$$
$$= \begin{cases} \dfrac{1}{6}\left(\log|x+3| + 5\log|x-3|\right) + c_1 & \text{if } x \geq 0\,, \\ -\log|x-3| + \dfrac{5}{x-3} + c_2 & \text{if } -3 < x < 0\,. \end{cases}$$

Continuity and the vanishing at $x = 0$ tell

$$0 = F(0) = F(0^+) = \log 3 + c_1 = F(0^-) = -\log 3 - \frac{5}{3} + c_2\,.$$

Thus $c_1 = -\log 3$, $c_2 = \log 3 + \frac{5}{3}$, and

$$F(x) = \begin{cases} \dfrac{1}{6}\left(\log(x+3) + 5\log|x-3|\right) - \log 3 & \text{if } x \geq 0\,, \\ -\log(3-x) + \dfrac{5}{x-3} + \log 3 + \dfrac{5}{3} & \text{if } -3 < x < 0\,. \end{cases}$$

11. The generalised primitive $F(x)$ of $f(x)$ should be continuous and satisfy $F'(x) = f(x)$ at all points where $f(x)$ is continuous, in our case every $x \neq 1$. Therefore

$$F(x) = \begin{cases} \displaystyle\int (2x^3 - 5x + 3)\,\mathrm{d}x & \text{if } x \geq 1, \\ \displaystyle\int (4x - 7)\,\mathrm{d}x & \text{if } x < 1 \end{cases} = \begin{cases} \dfrac{1}{2}x^4 - \dfrac{5}{2}x^2 + 3x + c_1 & \text{if } x \geq 1, \\ 2x^2 - 7x + c_2 & \text{if } x < 1\,; \end{cases}$$

the relation of c_1, c_2 derives from imposing continuity at $x = 1$:

$$F(1) = F(1^+) = 1 + c_1 = F(1^-) = -5 + c_2\,.$$

Thus $c_2 = 6 + c_1$ and

$$F(x) = \begin{cases} \dfrac{1}{2}x^4 - \dfrac{5}{2}x^2 + 3x + c & \text{if } x \geq 1, \\ 2x^2 - 7x + 6 + c & \text{if } x < 1\,. \end{cases}$$

Let us demand $F(0) = 6 + c = 0$, i.e., $c = -6$. That implies

$$F(x) = \begin{cases} \dfrac{1}{2}x^4 - \dfrac{5}{2}x^2 + 3x - 6 & \text{if } x \geq 1, \\ 2x^2 - 7x & \text{if } x < 1\,. \end{cases}$$

Alternatively, notice the required map (cf. Remark 9.41) equals

$$F(x) = \int_0^x f(t)\,\mathrm{d}t\,,$$

from which we may then integrate $f(t)$.

12. Consider $F(x) = \arctan \frac{1}{x}$ and $G(x) = -\arctan x$. As

$$F'(x) = -\frac{1}{1+x^2} = G'(x)\,,$$

$F(x)$ and $G(x)$ are primitives of the same $f(x) = -\frac{1}{1+x^2}$. As such, Proposition 9.3 ensures they differ by a constant $c \in \mathbb{R}$

$$F(x) = G(x) + c\,.$$

The value $c = \frac{\pi}{2}$ is a consequence of $F(1) = \frac{\pi}{4}$, $G(1) = -\frac{\pi}{4}$.

13. The generic primitive for f is like

$$F(x) = c + \int_0^x \cos 2t^2 \,\mathrm{d}t\,.$$

By Lemma 9.43, if

$$\cos 2t^2 = 1 - 2t^4 + \frac{2}{3}t^8 + o(t^9)\,, \qquad t \to 0\,,$$

F expands, for $x \to 0$, as

$$F(x) = c + \int_0^x \left(1 - 2t^4 + \frac{2}{3}t^8 + o(t^9)\right) \mathrm{d}t = c + x - \frac{2}{5}x^5 + \frac{2}{27}x^9 + o(x^{10})\,.$$

14. As in Exercise 13, write first Maclaurin's polynomial up to degree 3:

$$\begin{aligned}
f(x) &= \frac{1}{3}\left(2 + \mathrm{e}^{-x}\right)\left(1 + \frac{x^3}{3}\right)^{-1} \\
&= \frac{1}{3}\left(3 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + o(x^3)\right)\left(1 - \frac{1}{3}x^3 + o(x^3)\right) \\
&= \frac{1}{3}\left(3 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3 - x^3 + o(x^3)\right) \\
&= 1 - \frac{1}{3}x + \frac{1}{6}x^2 - \frac{7}{18}x^3 + o(x^3)\,, \quad x \to 0\,.
\end{aligned}$$

Then

$$\begin{aligned}
F(x) &= c + \int_0^x f(t)\,\mathrm{d}t = c + \int_0^x \left(1 - \frac{1}{3}t + \frac{1}{6}t^2 - \frac{7}{18}t^3 + o(t^3)\right) \mathrm{d}t \\
&= c + x - \frac{1}{6}x^2 + \frac{1}{18}x^3 - \frac{7}{72}x^4 + o(x^4)\,, \quad x \to 0\,.
\end{aligned}$$

15. *Definite integrals:*

a) -2; b) $\frac{\pi}{6}$; c) $\frac{1}{4}\mathrm{e}^2(3\mathrm{e}^2 - 1)$; d) $\frac{1}{5}\log 6$.

e) Since

$$[x] = \begin{cases} 1 & \text{if } 1 \le x < 2\,, \\ 2 & \text{if } 2 \le x < 3\,, \\ 3 & \text{if } x = 3\,, \end{cases}$$

we have

$$S = \int_1^2 \mathrm{d}x + \int_2^3 \frac{1}{4}\,\mathrm{d}x = \frac{5}{4}\,.$$

f) The parabola $y = x^2 - 1$, on $0 \le x \le \sqrt{3}$, has the following range set:

$$\begin{array}{lll} -1 \le x^2 - 1 < 0 & \text{for} & x \in [0,1) \\ 0 \le x^2 - 1 < 1 & \text{for} & x \in [1,\sqrt{2}) \\ 1 \le x^2 - 1 < 2 & \text{for} & x \in [\sqrt{2},\sqrt{3}). \end{array}$$

Therefore

$$M(x^2-1) = \begin{cases} x^2 - 1 + 1 & \text{if } x \in [0,1)\,, \\ x^2 - 1 & \text{if } x \in [1,\sqrt{2})\,, \\ x^2 - 1 - 1 & \text{if } x \in [\sqrt{2},\sqrt{3})\,, \\ 0 & \text{if } x = \sqrt{3}\,, \end{cases}$$

and

$$S = \int_0^1 x^2\,\mathrm{d}x + \int_1^{\sqrt{2}} (x^2-1)\,\mathrm{d}x + \int_{\sqrt{2}}^{\sqrt{3}} (x^2-2)\,\mathrm{d}x = \sqrt{2} - \sqrt{3} + 1\,.$$

16. As (see Fig. 9.14)

$$|\log x| = \begin{cases} -\log x & \text{if } \mathrm{e}^{-1} \le x < 1\,, \\ \log x & \text{if } 1 \le x < \mathrm{e}\,, \end{cases}$$

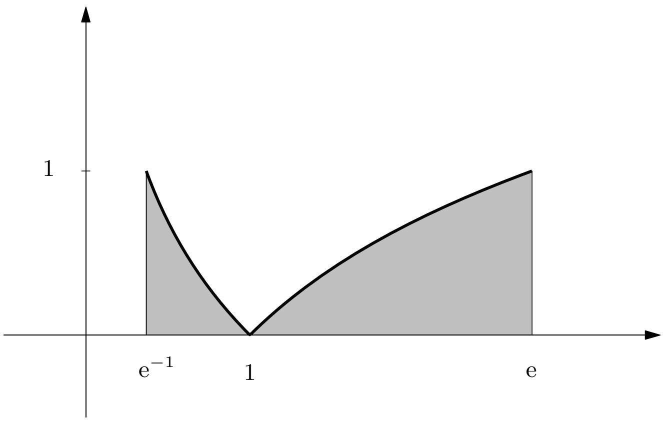


Figure 9.14. Trapezoidal region of the function $f(x) = |\log x|$

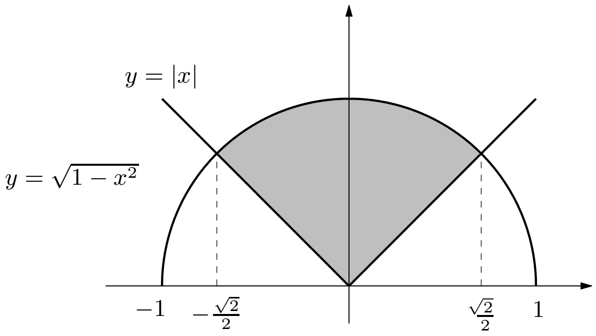


Figure 9.15. Region of Exercise 17 a)

from Example 9.11 ii) we infer

$$A = \int_{e^{-1}}^{e} |\log x| \, dx = -\int_{e^{-1}}^{1} \log x \, dx + \int_{1}^{e} \log x \, dx$$
$$= -\Big[x(\log x - 1)\Big]_{e^{-1}}^{1} + \Big[x(\log x - 1)\Big]_{1}^{e} = 2 - \frac{2}{e}\,.$$

17. *Computing areas:*

a) The region is symmetric with respect to the y-axis (Fig. 9.15). Comparing to the example of Sect. 9.9.1, we can say that the area will be

$$A = 2\left(\int_0^{\sqrt{2}/2} (\sqrt{1-x^2} - x) \, dx\right)$$
$$= \Big[x\sqrt{1-x^2} + \arcsin x\Big]_0^{\sqrt{2}/2} - \Big[x^2\Big]_0^{\sqrt{2}/2} = \frac{\pi}{4}\,.$$

The result agrees with the fact that the region is actually one quarter of a disc.

b) $\frac{9}{8}$.

18. From

$$|t-1| = \begin{cases} 1-t & \text{if } t < 1\,, \\ t-1 & \text{if } t \ge 1\,, \end{cases}$$

we write

$$F(x) = \begin{cases} \displaystyle\int_{-1}^{x} (1-t+2) \, dt & \text{if } x < 1\,, \\ \displaystyle\int_{-1}^{1} (1-t+2) \, dt + \int_{1}^{x} (t-1+2) \, dt & \text{if } x \ge 1 \end{cases}$$

$$= \begin{cases} -\frac{1}{2}x^2 + 3x + \frac{7}{2} & \text{if } x < 1\,, \\ \frac{1}{2}x^2 + x + \frac{9}{2} & \text{if } x \ge 1\,. \end{cases}$$

10

Integral calculus II

This second chapter on integral calculus consists roughly of two parts. In the first part we give a meaning to the term ‘improper’ integral, and thus extend the notion of area to include unbounded regions. The investigation relies on the tools developed when discussing limits.

The remaining part is devoted to the integration of functions of several variables along curves, which generalises the results on real intervals of Chap. 9.

10.1 Improper integrals

Hitherto integrals have been defined for bounded maps over closed bounded intervals of the real line. However, several applications induce one to consider unbounded intervals quite often, or functions tending to infinity. To cover such cases the notion of integral, be it Cauchy’s or Riemann’s, must be extended by means of limits.

We begin with improper integrals with unbounded domain of integration, and then treat infinite integrands.

10.1.1 Unbounded domains of integration

Let $\mathcal{R}_{\mathrm{loc}}([a,+\infty))$ be the set of maps defined on the ray $[a,+\infty)$ and integrable on every closed and bounded subinterval $[a,c]$ of the domain.

Taking $f \in \mathcal{R}_{\mathrm{loc}}([a,+\infty))$ we can introduce the integral function

$$F(c) = \int_a^c f(x)\,\mathrm{d}x$$

on $[a,+\infty)$. The natural question to answer concerns its behaviour when $c \to +\infty$.

Definition 10.1 *Let $f \in \mathcal{R}_{\text{loc}}([a,+\infty))$. We (formally) set*

$$\int_a^{+\infty} f(x)\,\mathrm{d}x = \lim_{c\to+\infty}\int_a^c f(x)\,\mathrm{d}x.$$

The symbol on the left is said **improper integral** *of f on $[a,+\infty)$.*

i) If the limit exists and is finite, we say that the map f is integrable over $[a,+\infty)$, or equivalently, that its improper **integral converges***.*
ii) If the limit exists but is infinite, we say that the improper **integral** *of f* **diverges***.*
iii) If the limit does not exist, we say that the improper **integral** *is* **indeterminate***.*

The class of integrable maps over $[a,+\infty)$ will be indicated $\mathcal{R}([a,+\infty))$.

Visualising the improper integral of a positive function is easy. Note first that the following holds.

Proposition 10.2 *Let $f \in \mathcal{R}_{\text{loc}}([a,+\infty))$ be such that $f(x) \geq 0$, for all $x \in [a,+\infty)$. Then the integral map $F(c)$ is increasing on $[a,+\infty)$.*

Proof. Take $c_1, c_2 \in [a,+\infty)$ with $c_1 < c_2$. By the property of additivity of the domain of integration (Theorem 9.33, i)),

$$\begin{aligned} F(c_2) = \int_a^{c_2} f(x)\,\mathrm{d}x &= \int_a^{c_1} f(x)\,\mathrm{d}x + \int_{c_1}^{c_2} f(x)\,\mathrm{d}x \\ &= F(c_1) + \int_{c_1}^{c_2} f(x)\,\mathrm{d}x. \end{aligned}$$

The last integral is ≥ 0 by Theorem 9.33, iii). Therefore $F(c_2) \geq F(c_1)$. □

Corollary 10.3 *The improper integral of a positive map belonging to $\mathcal{R}_{\text{loc}}([a,+\infty))$ is either convergent or divergent to $+\infty$.*

Proof. This descends from the proposition by applying Theorem 3.27 to F. □

Going back to the geometric picture, we can say that the improper integral of a positive function represents the area of the trapezoidal region of f over $[a,+\infty)$ (Fig. 10.1). This region is unbounded and may be viewed as the limit, for $c \to \infty$, of the regions defined over the subintervals $[a,c]$. The area of the trapezoidal region over the entire domain of integration $[a,+\infty)$ is finite if the improper integral converges, and one says that the area is infinite when the integral is divergent.

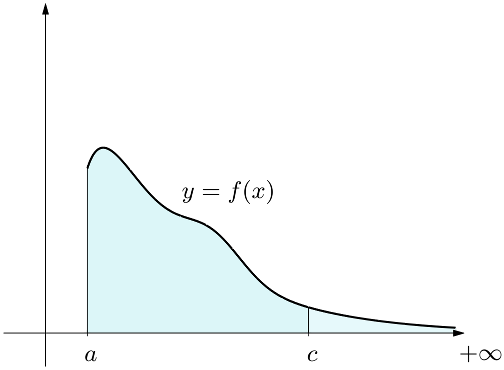


Figure 10.1. Trapezoidal region of f over the unbounded interval $[a, +\infty)$

Examples 10.4

i) We consider the integral over $[1, +\infty)$ of the family of functions $f(x) = \dfrac{1}{x^\alpha}$ for various $\alpha > 0$. Since

$$\int_1^c \frac{1}{x^\alpha}\,\mathrm{d}x = \begin{cases} \left.\dfrac{x^{1-\alpha}}{1-\alpha}\right|_1^c & \text{if } \alpha \neq 1, \\ \log x\big|_1^c & \text{if } \alpha = 1 \end{cases} = \begin{cases} \dfrac{c^{1-\alpha}-1}{1-\alpha} & \text{if } \alpha \neq 1, \\ \log c & \text{if } \alpha = 1, \end{cases}$$

when $\alpha \neq 1$, one has

$$\int_1^{+\infty} \frac{1}{x^\alpha}\,\mathrm{d}x = \lim_{c\to+\infty} \frac{c^{1-\alpha}-1}{1-\alpha} = \begin{cases} \dfrac{1}{\alpha-1} & \text{if } \alpha > 1, \\ +\infty & \text{if } \alpha < 1. \end{cases}$$

If $\alpha = 1$ instead,

$$\int_1^{+\infty} \frac{1}{x}\,\mathrm{d}x = \lim_{c\to+\infty} \log c = +\infty.$$

The integral behaves in the same manner whichever the lower limit of integration $a > 0$. Therefore

$$\int_a^{+\infty} \frac{1}{x^\alpha}\,\mathrm{d}x \ \begin{cases} \text{converges} & \text{if } \alpha > 1, \\ \text{diverges} & \text{if } \alpha \leq 1. \end{cases}$$

ii) Let $f(x) = \cos x$. The integral

$$F(c) = \int_0^c \cos x\,\mathrm{d}x = \sin c$$

does not admit limit for $c \to +\infty$, hence $\int_0^{+\infty} \cos x\,\mathrm{d}x$ is indeterminate. □

Improper integrals inherit some features of definite integrals. To be precise, if f, g belong to $\mathcal{R}([a, +\infty))$:

i) for any $c > a$

$$\int_a^{+\infty} f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^{+\infty} f(x)\,\mathrm{d}x\,;$$

ii) for any $\alpha, \beta \in \mathbb{R}$

$$\int_a^{+\infty} \Big(\alpha f(x) + \beta g(x)\Big)\,\mathrm{d}x = \alpha\int_a^{+\infty} f(x)\,\mathrm{d}x + \beta\int_a^{+\infty} g(x)\,\mathrm{d}x\,;$$

iii) supposing $f \geq 0$ on $[a, +\infty)$ then

$$\int_a^{+\infty} f(x)\,\mathrm{d}x \geq 0\,.$$

All are consequence of properties *i)-iii)* in Theorem 9.33 and the properties of limits.

Convergence criteria

The integrability of $f \in \mathcal{R}_{\mathrm{loc}}([a, +\infty))$ cannot always be established using just the definition. Indeed, we may not be able to find an integral function $F(c)$ explicitly. Thus, it becomes all the more important to have other ways to decide about convergence. When the integral is convergent, computing it might require techniques that are too sophisticated for this textbook, and which will not be discussed.

The first convergence test we present concerns positive functions.

Theorem 10.5 (Comparison test) *Let $f, g \in \mathcal{R}_{\mathrm{loc}}([a, +\infty))$ be such that $0 \leq f(x) \leq g(x)$ for all $x \in [a, +\infty)$. Then*

$$0 \leq \int_a^{+\infty} f(x)\,\mathrm{d}x \leq \int_a^{+\infty} g(x)\,\mathrm{d}x. \tag{10.1}$$

In particular,

i) if the integral of g converges, so does the integral of f;
ii) if the integral of f diverges, then the integral of g diverges, too.

Proof. The definite integral is monotone, and using $0 \leq f(x) \leq g(x)$ over $[a, +\infty)$, we have

$$F(c) = \int_a^c f(x)\,\mathrm{d}x \leq \int_a^c g(x)\,\mathrm{d}x = G(c).$$

By Corollary 10.3 the maps $F(c)$ and $G(c)$ admit limit for $c \to +\infty$; comparing the limits, with the help of Corollary 4.4, we obtain

$$0 \leq \lim_{c\to+\infty} F(c) \leq \lim_{c\to+\infty} G(c),$$

which is (10.1). The statements *i)* and *ii)* are straightforward consequences of (10.1). □

Example 10.6

Discuss the convergence of the integrals

$$\int_1^{+\infty} \frac{\arctan x}{x^2}\,\mathrm{d}x \qquad \text{and} \qquad \int_1^{+\infty} \frac{\arctan x}{x}\,\mathrm{d}x.$$

For all $x \in [1, +\infty)$

$$\frac{\pi}{4} \le \arctan x < \frac{\pi}{2},$$

so

$$\frac{\arctan x}{x^2} < \frac{\pi}{2x^2} \qquad \text{and} \qquad \frac{\pi}{4x} \le \frac{\arctan x}{x}.$$

Therefore

$$\int_1^{+\infty} \frac{\arctan x}{x^2}\,\mathrm{d}x < \int_1^{+\infty} \frac{\pi}{2x^2}\,\mathrm{d}x \qquad \text{and} \qquad \int_1^{+\infty} \frac{\pi}{4x}\,\mathrm{d}x \le \int_1^{+\infty} \frac{\arctan x}{x}\,\mathrm{d}x.$$

From Example 10.4 we know $\displaystyle\int_1^{+\infty} \frac{\pi}{2x^2}\,\mathrm{d}x$ converges, whereas $\displaystyle\int_1^{+\infty} \frac{\pi}{4x}\,\mathrm{d}x$ diverges. Because of Theorem 10.5, the implication of *i)* ensures that the integral $\displaystyle\int_1^{+\infty} \frac{\arctan x}{x^2}\,\mathrm{d}x$ converges, while *ii)* makes $\displaystyle\int_1^{+\infty} \frac{\arctan x}{x}\,\mathrm{d}x$ diverge. □

When the integrand has no fixed sign, we can rely on this criterion.

Theorem 10.7 (Absolute convergence test) *Suppose $f \in \mathcal{R}_{\mathrm{loc}}([a, +\infty))$ is such that $|f| \in \mathcal{R}([a, +\infty))$. Then $f \in \mathcal{R}([a, +\infty))$, and moreover*

$$\left| \int_a^{+\infty} f(x)\,\mathrm{d}x \right| \le \int_a^{+\infty} |f(x)|\,\mathrm{d}x.$$

Proof. We introduce f_+ and f_-, respectively called **positive** and **negative part** of f, as follows:

$$f_+(x) = \max(f(x), 0) = \begin{cases} f(x) & \text{if } f(x) \ge 0\,, \\ 0 & \text{if } f(x) < 0, \end{cases}$$

$$f_-(x) = \max(-f(x), 0) = \begin{cases} 0 & \text{if } f(x) \ge 0\,, \\ -f(x) & \text{if } f(x) < 0. \end{cases}$$

Both are non-negative, and allow to decompose f, $|f|$:

$$f(x) = f_+(x) - f_-(x), \qquad\qquad |f(x)| = f_+(x) + f_-(x) \tag{10.2}$$

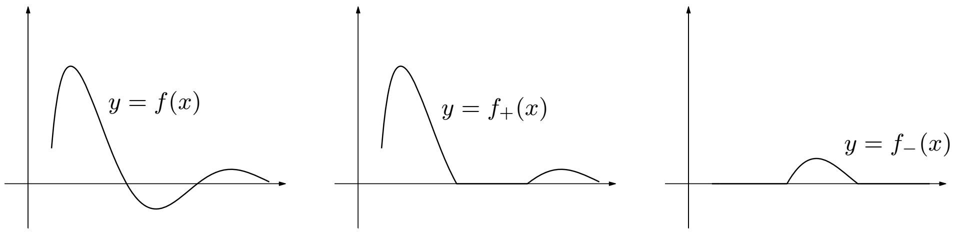


Figure 10.2. Graphs of a map f (left), its positive part (centre) and negative part (right)

(see Fig. 10.2). Adding and subtracting these relations leads to

$$f_+(x) = \frac{|f(x)| + f(x)}{2}, \qquad f_-(x) = \frac{|f(x)| - f(x)}{2},$$

which, together with Theorem 9.33 *ii)*, imply $f_+, f_- \in \mathcal{R}_{\text{loc}}([a, +\infty))$. Since $0 \le f_+(x), f_-(x) \le |f(x)|$ for any $x \ge a$, the Comparison test 10.5 yields that f_+ and f_- are integrable over $[a, +\infty)$. The first of (10.2) tells that also f satisfies the same.
Eventually, property *v)* of Theorem 9.33 implies, for all $c > a$,

$$\left| \int_a^c f(x)\,\mathrm{d}x \right| \le \int_a^c |f(x)|\,\mathrm{d}x;$$

Passing to the limit $c \to +\infty$ proves the claim. □

Example 10.8

Let us consider the integral

$$\int_1^{+\infty} \frac{\cos x}{x^2}\,\mathrm{d}x.$$

Since $\left|\dfrac{\cos x}{x^2}\right| \le \dfrac{1}{x^2}$, the function $|f(x)| = \left|\dfrac{\cos x}{x^2}\right|$ is integrable on $[1, +\infty)$ by Theorem 10.5 and Example 10.4. The above test guarantees integrability, and

$$\left| \int_1^{+\infty} \frac{\cos x}{x^2}\,\mathrm{d}x \right| \le \int_1^{+\infty} \left|\frac{\cos x}{x^2}\right| \mathrm{d}x \le \int_1^{+\infty} \frac{1}{x^2}\,\mathrm{d}x = 1.$$ □

Remark 10.9 The Absolute convergence test is a sufficient condition for integrability, not a necessary one. This is clarified by the following example

$$\int_1^{+\infty} \frac{\sin x}{x}\,\mathrm{d}x \quad \text{converges, but} \quad \int_1^{+\infty} \left|\frac{\sin x}{x}\right| \mathrm{d}x \quad \text{diverges.} \tag{10.3}$$

(For a proof we refer to Appendix A.5.3, p. 470.) □

A map f whose absolute value $|f|$ belongs to $\mathcal{R}([a, +\infty))$ is said **absolutely integrable** *on $[a, +\infty)$.*

Another useful result is based on the study of the order of infinitesimal of the integrand as $x \to +\infty$.

Theorem 10.10 (**Asymptotic comparison test**) *Suppose the function $f \in \mathcal{R}_{\mathrm{loc}}([a, +\infty))$ is infinitesimal of order α, for $x \to +\infty$, with respect to $\varphi(x) = \frac{1}{x}$. Then*

i) *if $\alpha > 1$, $f \in \mathcal{R}([a, +\infty))$;*

ii) *if $\alpha \le 1$, $\displaystyle\int_a^{+\infty} f(x)\,\mathrm{d}x$ diverges.*

Proof. See Appendix A.5.3, p. 471. □

Examples 10.11

i) Consider

$$\int_1^{+\infty} (\pi - 2\arctan x)\,\mathrm{d}x.$$

The map $f(x) = \pi - 2\arctan x$ is infinitesimal of first order for $x \to +\infty$: by de l'Hôpital's Theorem namely,

$$\lim_{x\to+\infty} \frac{\pi - 2\arctan x}{1/x} = \lim_{x\to+\infty} \frac{2x^2}{1+x^2} = 2.$$

The integral therefore diverges.

ii) Discuss the integral

$$\int_1^{+\infty} \frac{x + \cos x}{x^3 + \sin x}\,\mathrm{d}x.$$

As $\cos x = o(x)$, $\sin x = o(x^3)$ for $x \to +\infty$, it follows

$$\frac{x + \cos x}{x^3 + \sin x} \sim \frac{1}{x^2} \qquad x \to +\infty,$$

and the integral converges. □

Let us now consider a family of improper integrals generalising Example 10.4 i).

Example 10.12

We show how the convergence of

$$\int_2^{+\infty} \frac{1}{x^\alpha (\log x)^\beta}\,\mathrm{d}x$$

depends on the values of α, $\beta > 0$.

i) The case $\alpha = 1$ can be tackled by direct integration. Changing variables to $t = \log x$, one has

$$\int_2^{+\infty} \frac{1}{x(\log x)^\beta}\,\mathrm{d}x = \int_{\log 2}^{+\infty} \frac{1}{t^\beta}\,\mathrm{d}t,$$

so the integral converges if $\beta > 1$, diverges if $\beta \leq 1$.

ii) If $\alpha > 1$, we observe preliminarily that $x \geq 2$ implies $\log x \geq \log 2$ and hence

$$\frac{1}{x^\alpha(\log x)^\beta} \leq \frac{1}{x^\alpha(\log 2)^\beta}, \qquad \forall x \geq 2.$$

This is sufficient to conclude that the integral converges irrespective of β, by the Comparison test.

iii) When $\alpha < 1$, let us write

$$\frac{1}{x^\alpha(\log x)^\beta} = \frac{1}{x}\frac{x^{1-\alpha}}{(\log x)^\beta}.$$

The function $\dfrac{x^{1-\alpha}}{(\log x)^\beta}$ tends to $+\infty$, for any β. There is thus an $M > 0$ such that

$$\frac{1}{x^\alpha(\log x)^\beta} \geq \frac{M}{x}, \qquad \forall x \geq 2.$$

By comparison the integral diverges. □

If f is defined on $[k_0, +\infty)$, it could turn out useful, sometimes, to think of its value at $x = k$ as the general term a_k of a series. Under appropriate assumptions then, we can relate the behaviour of the series with that of the integral of f over $[k_0, +\infty)$, as shown hereby (a proof of this result may be found in Appendix A.5.3, p. 472).

Theorem 10.13 (**Integral test**) *Let f be continuous, positive and decreasing on $[k_0, +\infty)$, for $k_0 \in \mathbb{N}$. Then*

$$\sum_{k=k_0+1}^{\infty} f(k) \leq \int_{k_0}^{+\infty} f(x)\,\mathrm{d}x \leq \sum_{k=k_0}^{\infty} f(k)\,, \tag{10.4}$$

therefore the integral and the series share the same behaviour. Precisely:

a) $\displaystyle\int_{k_0}^{+\infty} f(x)\,\mathrm{d}x$ *converges* $\iff$ $\displaystyle\sum_{k=k_0}^{\infty} f(k)$ *converges;*

b) $\displaystyle\int_{k_0}^{+\infty} f(x)\,\mathrm{d}x$ *diverges* $\iff$ $\displaystyle\sum_{k=k_0}^{\infty} f(k)$ *diverges.*

Examples 10.14

i) The previous criterion tells for which values of the parameter α the **generalised harmonic series**

$$\sum_{k=1}^{\infty} \frac{1}{k^\alpha}$$

converges. Note in fact that $\frac{1}{x^\alpha}$, $\alpha > 0$, satisfies the theorem's hypotheses, and has convergent integral over $[1, +\infty)$ if and only if $\alpha > 1$. Therefore

$$\sum_{k=1}^{\infty} \frac{1}{k^\alpha} \quad \begin{cases} \text{converges for } \alpha > 1\,, \\ \text{diverges for } 0 < \alpha \le 1\,. \end{cases}$$

ii) In order to study

$$\sum_{k=2}^{\infty} \frac{1}{k \log k}$$

we take the map $f(x) = \dfrac{1}{x \log x}$; its integral over $[2, +\infty)$ diverges, by case i) of Example 10.12. Then the series $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k \log k}$ is divergent. □

A last remark to say that an integral can be defined over $(-\infty, b]$ by putting

$$\int_{-\infty}^{b} f(x)\, \mathrm{d}x = \lim_{c \to -\infty} \int_{c}^{b} f(x)\, \mathrm{d}x.$$

All properties and convergence results easily adapt.

10.1.2 Unbounded integrands

Consider the set $\mathcal{R}_{\text{loc}}([a, b))$ of functions defined on the bounded interval $[a, b)$ and integrable over each closed subinterval $[a, c]$, $a < c < b$.

If $f \in \mathcal{R}_{\text{loc}}([a, b))$ the integral function

$$F(c) = \int_a^c f(x)\, \mathrm{d}x$$

is thus defined over $[a, b)$. We wish to study the limiting behaviour of such, for $c \to b^-$.

Definition 10.15 *Let $f \in \mathcal{R}_{\text{loc}}([a, b))$ and define, formally,*

$$\int_a^b f(x)\, \mathrm{d}x = \lim_{c \to b^-} \int_a^c f(x)\, \mathrm{d}x; \tag{10.5}$$

as before, the left-hand side is called **improper integral** *of f over $[a, b)$.*

i) If the limit exists and is finite, one says f is (improperly) integrable on $[a,b)$, or that its **improper integral converges**.

ii) If the limit exists but infinite, one says that the **improper integral** *of f is* **divergent**.

iii) If the limit does not exist, one says that the **improper integral** *is* **indeterminate**.

As usual, integrable functions over $[a,b)$ shall be denoted by $\mathcal{R}([a,b))$.

If a map is bounded and integrable on $[a, b]$ (according to Cauchy or Riemann), it is also integrable on $[a, b)$ in the above sense. Its improper integral coincides with the definite integral. Indeed, letting $M = \sup_{x\in[a,b]} |f(x)|$, we have

$$\left| \int_a^b f(x)\,\mathrm{d}x - \int_a^c f(x)\,\mathrm{d}x \right| = \left| \int_c^b f(x)\,\mathrm{d}x \right| \le \int_c^b |f(x)|\,\mathrm{d}x \le M(b-c)\,.$$

In the limit for $c \to b^-$ we obtain (10.5). This is why the symbol is the same for definite and improper integrals. At the same time, (10.5) explains that the concept of improper integral over a bounded domain is especially relevant when the integrand is infinite in the neighbourhood of the point b.

Example 10.16

Take $f(x) = \dfrac{1}{(b-x)^\alpha}$ with $\alpha > 0$ (Fig. 10.3 shows one choice of the parameter), and study its integral over $[a,b)$:

$$\int_a^c \frac{1}{(b-x)^\alpha}\,\mathrm{d}x = \begin{cases} \left.\dfrac{(b-x)^{1-\alpha}}{\alpha-1}\right|_a^c & \text{if } \alpha \ne 1, \\ -\log(b-x)\Big|_a^c & \text{if } \alpha = 1 \end{cases}$$

$$= \begin{cases} \dfrac{(b-c)^{1-\alpha}-(b-a)^{1-\alpha}}{\alpha-1} & \text{if } \alpha \ne 1, \\ \log\dfrac{b-a}{b-c} & \text{if } \alpha = 1. \end{cases}$$

When $\alpha \ne 1$,

$$\int_a^b \frac{1}{(b-x)^\alpha}\,\mathrm{d}x = \lim_{c\to b^-} \frac{(b-c)^{1-\alpha}-(b-a)^{1-\alpha}}{\alpha-1} = \begin{cases} \dfrac{(b-a)^{1-\alpha}}{1-\alpha} & \text{if } \alpha < 1, \\ +\infty & \text{if } \alpha > 1. \end{cases}$$

For $\alpha = 1$,

$$\int_a^b \frac{1}{b-x}\,\mathrm{d}x = \lim_{c\to b^-} \log\frac{b-a}{b-c} = +\infty.$$

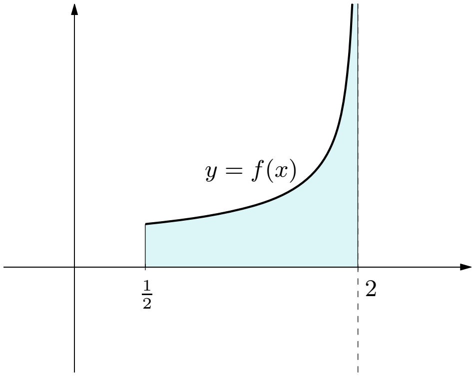


Figure 10.3. Trapezoidal region of the unbounded map $f(x) = \frac{1}{\sqrt{2-x}}$ over $[\frac{1}{2}, 2)$

Therefore

$$\int_a^b \frac{1}{(b-x)^\alpha}\,\mathrm{d}x \begin{cases} \text{converges} & \text{if } \alpha < 1, \\ \text{diverges} & \text{if } \alpha \geq 1. \end{cases}$$

□

In analogy to what seen previously, the integral of a positive f over $[a, b)$ can be proven to be either convergent or divergent to $+\infty$.

Convergence tests similar to those already mentioned hold in the present situation, so we just state a couple of results, without proofs.

Theorem 10.17 (**Comparison test**) *Let $f, g \in \mathcal{R}_{\mathrm{loc}}([a, b))$ be such that $0 \leq f(x) \leq g(x)$ for any $x \in [a, b)$. Then*

$$0 \leq \int_a^b f(x)\,\mathrm{d}x \leq \int_a^b g(x)\,\mathrm{d}x. \tag{10.6}$$

In particular,

i) if the integral of g converges, the integral of f converges;
ii) if the integral of f diverges, the integral of g diverges.

Theorem 10.18 (**Asymptotic comparison test**) *If $f \in \mathcal{R}_{\mathrm{loc}}([a, b))$ is infinite of order α for $x \to b^-$ with respect to $\varphi(x) = \frac{1}{b-x}$, then*

i) if $\alpha < 1$, $f \in \mathcal{R}([a, b))$;
ii) if $\alpha \geq 1$, $\displaystyle\int_a^b f(x)\,\mathrm{d}x$ diverges.

Integrals over $(a, b]$ are defined similarly:

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{c\to a^+} \int_c^b f(x)\,\mathrm{d}x.$$

With the obvious modifications all properties carry over.

Examples 10.19

i) Consider the integral

$$\int_1^3 \sqrt{\frac{7-x}{3-x}}\,\mathrm{d}x\,.$$

The function $f(x) = \sqrt{\frac{7-x}{3-x}}$ is defined and continuous on $[1, 3)$, but has a discontinuity for $x \to 3^-$. As $7 - x \le 6$ on $[1, 3)$, by the Comparison test we have

$$\int_1^3 \sqrt{\frac{7-x}{3-x}}\,\mathrm{d}x \le \int_1^3 \frac{\sqrt{6}}{\sqrt{3-x}}\,\mathrm{d}x < +\infty\,,$$

(recall Example 10.16). The integral therefore converges.

ii) Consider

$$\int_1^2 \frac{\mathrm{e}^x + 1}{(x-1)^2}\,\mathrm{d}x\,.$$

When $x \in (1, 2]$,

$$\frac{\mathrm{e}+1}{(x-1)^2} < \frac{\mathrm{e}^x+1}{(x-1)^2},$$

so by comparison the integral diverges to $+\infty$.

iii) Determine the behaviour of

$$\int_0^{\pi/2} \frac{\sqrt{x}}{\sin x}\,\mathrm{d}x\,.$$

For $x \to 0^+$, $f(x) = \dfrac{\sqrt{x}}{\sin x} \sim \dfrac{1}{\sqrt{x}}$, therefore the integral converges by the Asymptotic comparison test .

iv) The integral

$$\int_\pi^4 \frac{\log(x-3)}{x^3 - 8x^2 + 16x}\,\mathrm{d}x$$

has integrand f defined on $[\pi, 4)$; f tends to $+\infty$ for $x \to 4^-$ and

$$f(x) = \frac{\log(1 + (x-4))}{x(x-4)^2} \sim \frac{1}{4(x-4)}, \qquad x \to 4^-.$$

Thus Theorem 10.18 implies divergence to $-\infty$ ($f(x) = 1/(x-4)$ is negative at the left of $x = 4$). □

10.2 More improper integrals

Suppose we want to integrate a map with finitely many discontinuities in an interval I, bounded or not. Subdivide I into a finite number of intervals I_j, $j = 1, \dots, n$, so that the restricted map falls into one of the cases examined so far (see Fig. 10.4). Then formally define

$$\int_I f(x)\,\mathrm{d}x = \sum_{j=1}^{n} \int_{I_j} f(x)\,\mathrm{d}x\,.$$

One says that the **improper integral** of f on I **converges** if the integrals on the right all converge. It is not so hard to verify that the improper integral's behaviour and its value, if convergent, are independent of the chosen partition of I.

Examples 10.20

i) Suppose we want to study

$$S = \int_{-\infty}^{+\infty} \frac{1}{1+x^2}\,\mathrm{d}x\,.$$

If we split the real line at the origin we can write

$$S = \int_{-\infty}^{0} \frac{1}{1+x^2}\,\mathrm{d}x + \int_{0}^{+\infty} \frac{1}{1+x^2}\,\mathrm{d}x\,;$$

the two integrals converge, both to $\pi/2$, so $S = \pi$.

ii) The integrand of

$$S_1 = \int_{0}^{+\infty} \frac{\sin x}{x^2}\,\mathrm{d}x$$

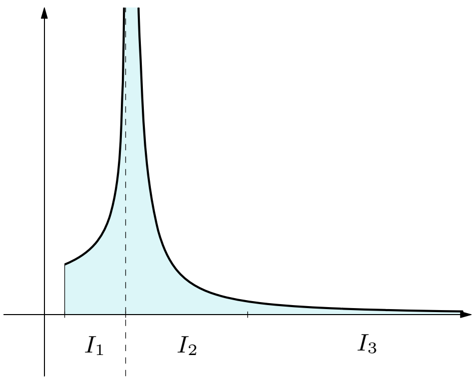


Figure 10.4. Trapezoidal region of an infinite map, over an unbounded interval

is infinite at the origin, so we divide the domain into $(0,1] \cup [1,+\infty)$, obtaining

$$S_1 = \int_0^1 \frac{\sin x}{x^2}\,\mathrm{d}x + \int_1^{+\infty} \frac{\sin x}{x^2}\,\mathrm{d}x\,;$$

but

$$\frac{\sin x}{x^2} \sim \frac{1}{x} \quad \text{for } x \to 0^+ \qquad \text{and} \qquad \left|\frac{\sin x}{x^2}\right| \le \frac{1}{x^2}\,,$$

so Theorem 10.18 forces the first integral to diverge, whereas the second converges by Theorem 10.5. In conclusion S_1 tends to $+\infty$.
For similar reasons

$$S_2 = \int_0^{+\infty} \frac{\sin x}{x^{3/2}}\,\mathrm{d}x$$

converges.

iii) Let S denote

$$\int_1^6 \frac{x-5}{(x+1)\sqrt[3]{x^2-6x+8}}\,\mathrm{d}x\,.$$

The integrand diverges at -1 (which lies outside the domain of integration), at 2 and also at 4. Hence we write

$$S = \left(\int_1^2 + \int_2^3 + \int_3^4 + \int_4^6\right) \frac{x-5}{(x+1)\sqrt[3]{(x-2)(x-4)}}\,\mathrm{d}x\,.$$

The function is infinite of order $1/3$ for $x \to 2^{\pm}$ and also for $x \to 4^{\pm}$, so the integral converges. □

10.3 Integrals along curves

The present and next sections deal with the problem of integrating over a curve, rather than just an interval (see Sect. 8.4). The concept of integral along a curve – or path integral as it is also known – has its origin in concrete applications, and is the first instance we encounter of an integral of a function of several real variables.

Let $\boldsymbol{\gamma} : [a,b] \to \mathbb{R}^d$ $(d = 2,3)$ be a regular arc and $C = \boldsymbol{\gamma}([a,b])$ its image, called a path. Take $f : \operatorname{dom} f \subseteq \mathbb{R}^d \to \mathbb{R}$ a function defined at least on C, hence such that $C \subseteq \operatorname{dom} f$. Suppose moreover that the composite map $f \circ \boldsymbol{\gamma} : [a,b] \to \mathbb{R}$, defined by $(f \circ \boldsymbol{\gamma})(t) = f\big(\boldsymbol{\gamma}(t)\big)$, is continuous on $[a,b]$.

Definition 10.21 *The* **line integral of f along $\boldsymbol{\gamma}$** *is the number*

$$\int_{\boldsymbol{\gamma}} f = \int_a^b f\big(\boldsymbol{\gamma}(t)\big)\, \|\boldsymbol{\gamma}'(t)\|\,\mathrm{d}t\,, \tag{10.7}$$

where $\|\boldsymbol{\gamma}'(t)\| = \sqrt{|x'(t)|^2 + |y'(t)|^2 + |z'(t)|^2}$ is the modulus (i.e., the Euclidean norm) of the vector $\boldsymbol{\gamma}'(t)$. Alternative expression are 'path integral of f along $\boldsymbol{\gamma}$', or simply, 'integral of f along $\boldsymbol{\gamma}$'.

The right-hand-side integral in (10.7) is well defined, for the map $f\big(\boldsymbol{\gamma}(t)\big)\,\|\boldsymbol{\gamma}'(t)\|$ is continuous on $[a,b]$. In fact $\boldsymbol{\gamma}$ is regular by hypothesis, its components' first derivatives are likewise continuous, and so is the norm $\|\boldsymbol{\gamma}'(t)\|$, by composition. And recall $f\big(\boldsymbol{\gamma}(t)\big)$ is continuous from the very beginning.

Integrals along curves have the following interpretation. Let $\boldsymbol{\gamma}$ be a simple arc in the plane with image C and f a non-negative function on C with graph

$$\Gamma(f) = \big\{(x,y,z) \in \mathbb{R}^3 : (x,y) \in \operatorname{dom} f,\ z = f(x,y)\big\}.$$

By

$$\Sigma = \big\{(x,y,z) \in \mathbb{R}^3 : (x,y) \in C,\ 0 \le z \le f(x,y)\big\}$$

we indicate the upright-standing surface bounded by C and by its image $f(C)$ lying on the graph of f, as in Fig. 10.5. One can prove that the value of the integral of f along $\boldsymbol{\gamma}$ equals the area of Σ. For example if f is constant on C, say equal to h, the area of Σ is the product of the height h times the base C. Accepting that the base measures $\ell(C) = \int_a^b \|\boldsymbol{\gamma}'(t)\|\,\mathrm{d}t$ (which we shall see in Sect. 10.3.1), we have

$$\text{Area}\,(\Sigma) = h\,\ell(C) = \int_a^b f\big(\boldsymbol{\gamma}(t)\big)\|\boldsymbol{\gamma}'(t)\|\,\mathrm{d}t = \int_{\boldsymbol{\gamma}} f\,.$$

Examples 10.22

i) Let $\boldsymbol{\gamma} : [0,1] \to \mathbb{R}^2$ be the regular arc $\boldsymbol{\gamma}(t) = (t, t^2)$ parametrising the parabola $y = x^2$ between $O = (0,0)$ and $A = (1,1)$. Then $\boldsymbol{\gamma}'(t) = (1, 2t)$ has length $\|\boldsymbol{\gamma}'(t)\| = \sqrt{1+4t^2}$. If $f : \mathbb{R} \times [0,+\infty) \to \mathbb{R}$ is defined by $f(x,y) = 3x + \sqrt{y}$, the composition $f \circ \boldsymbol{\gamma}$ reads $f\big(\boldsymbol{\gamma}(t)\big) = 3t + \sqrt{t^2} = 4t$ and therefore

$$\int_{\boldsymbol{\gamma}} f = \int_0^1 4t\sqrt{1+4t^2}\,\mathrm{d}t\,.$$

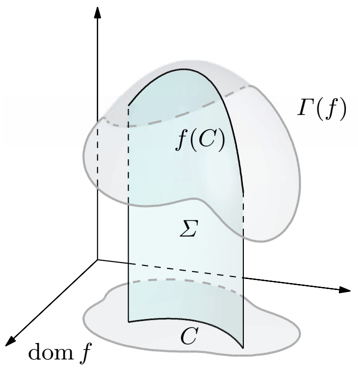


Figure 10.5. Geometric interpretation of the integral along a curve

Substituting $s = 1 + 4t^2$ we obtain

$$\int_{\boldsymbol{\gamma}} f = 2\int_1^5 \sqrt{s}\,\mathrm{d}s = 2\Big[\frac{2}{3}s^{3/2}\Big]_1^5 = \frac{4}{3}(5\sqrt{5}-1)\,.$$

ii) The curve $\boldsymbol{\gamma} : [0, 2\pi] \to \mathbb{R}^2$ parametrises the circle centred at $(2, 1)$ with radius 2, $\boldsymbol{\gamma}(t) = (2 + \cos t, 1 + \sin t)$, so $\|\boldsymbol{\gamma}'(t)\| = \sqrt{4\sin^2 t + 4\cos^2 t} = 2$ for all t. With the function $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x, y) = (x-2)(y-1) + 1$, we have $f\big(\boldsymbol{\gamma}(t)\big) = 4\sin t\cos t + 1$, and

$$\int_{\boldsymbol{\gamma}} f = 2\int_0^{2\pi} (4\sin t\cos t + 1)\,\mathrm{d}t = 2\big[2\sin 2t + t\big]_0^{2\pi} = 4\pi\,.$$

If we represent the circle by some $\overline{\boldsymbol{\gamma}}$ having the same components as $\boldsymbol{\gamma}$ but t varying in $[0, 2k\pi]$ (i.e., winding k times), then

$$\int_{\overline{\boldsymbol{\gamma}}} f = 2\int_0^{2k\pi} (4\sin t\cos t + 1)\,\mathrm{d}t = 4k\pi\,. \qquad \square$$

Example ii) shows that integrals along curves depend not only on the image of the curve along which one integrates, but upon the chosen parametric representation as well. That said, certain parametrisations give rise to the same integral.

Definition 10.23 *Two regular curves* $\boldsymbol{\gamma} : I \to \mathbb{R}^d$, $\boldsymbol{\delta} : J \to \mathbb{R}^d$ *are called* **equivalent** *if there is a bijection* $\varphi : J \to I$, *with continuous and strictly positive derivative, such that*

$$\boldsymbol{\delta} = \boldsymbol{\gamma} \circ \varphi\,,$$

i.e., $\boldsymbol{\delta}(\tau) = \boldsymbol{\gamma}\big(\varphi(\tau)\big)$ *for all* $\tau \in J$.

Definition 10.24 *Let* $\boldsymbol{\gamma} : I \to \mathbb{R}^d$ *be a regular curve. If* $-I$ *is the interval* $\{t \in \mathbb{R} : -t \in I\}$, *the curve* $-\boldsymbol{\gamma} : -I \to \mathbb{R}^d$ *defined by* $(-\boldsymbol{\gamma})(t) = \boldsymbol{\gamma}(-t)$ *is termed* **opposite to** $\boldsymbol{\gamma}$.

Flipping the parameter means we can write $(-\boldsymbol{\gamma}) = \boldsymbol{\gamma} \circ \varphi$, where $\varphi : -I \to I$ is the bijection $\varphi(t) = -t$ that reverts the orientation of the real line. If $\boldsymbol{\gamma} : [a, b] \to \mathbb{R}^d$ is a regular arc, so is $-\boldsymbol{\gamma}$ over $[-b, -a]$.

It is convenient to call **congruent** two curves $\boldsymbol{\gamma}$ e $\boldsymbol{\delta}$ that either are equivalent or one is equivalent to the opposite of the other. In other words, $\boldsymbol{\delta} = \boldsymbol{\gamma} \circ \varphi$ where φ is a strictly monotone bijection of class $\mathcal{C}^1$. Since the values of the parameter play the role of 'tags' for the points on the image C of $\boldsymbol{\gamma}$, all curves congruent to $\boldsymbol{\gamma}$ still have C as image. Furthermore, a curve congruent to a simple curve obviously remains simple.

Let f be a function defined on the image of a regular arc $\boldsymbol{\gamma} : [a,b] \to \mathbb{R}^d$ with $f \circ \boldsymbol{\gamma}$ continuous, so that the integral of f along $\boldsymbol{\gamma}$ exists. The map $f \circ \boldsymbol{\delta}$ (where $\boldsymbol{\delta}$ is an arc congruent to $\boldsymbol{\gamma}$) is continuous as well, for it arises by composing a continuous map between intervals with $f \circ \boldsymbol{\gamma}$.

Proposition 10.25 *Let $\boldsymbol{\gamma} : [a,b] \to \mathbb{R}^d$ be a regular arc with image C, f defined on C such that $f \circ \boldsymbol{\gamma}$ is continuous. Then*

$$\int_{\boldsymbol{\gamma}} f = \int_{\boldsymbol{\delta}} f\,,$$

for any arc $\boldsymbol{\delta}$ congruent to $\boldsymbol{\gamma}$.

Proof. Suppose $(-\boldsymbol{\gamma})'(t) = -\boldsymbol{\gamma}'(-t)$, so norms are preserved, i.e., $\|(-\boldsymbol{\gamma})'(t)\| = \|\boldsymbol{\gamma}'(-t)\|$, and

$$\begin{aligned}\int_{-\boldsymbol{\gamma}} f &= \int_{-b}^{-a} f\big((-\boldsymbol{\gamma})(t)\big)\,\|(-\boldsymbol{\gamma})'(t)\|\,\mathrm{d}t \\ &= \int_{-b}^{-a} f\big(\boldsymbol{\gamma}(-t)\big)\,\|\boldsymbol{\gamma}'(-t)\|\,\mathrm{d}t\,.\end{aligned}$$

With the change of variables $s = -t$, $\mathrm{d}s = -\mathrm{d}t$, we obtain

$$\begin{aligned}\int_{-\boldsymbol{\gamma}} f &= -\int_{b}^{a} f\big(\boldsymbol{\gamma}(s)\big)\,\|\boldsymbol{\gamma}'(s)\|\,\mathrm{d}s \\ &= \int_{a}^{b} f\big(\boldsymbol{\gamma}(s)\big)\,\|\boldsymbol{\gamma}'(s)\|\,\mathrm{d}s = \int_{\boldsymbol{\gamma}} f\,.\end{aligned}$$

Similarly, if $\boldsymbol{\delta} = \boldsymbol{\gamma} \circ \varphi$, where $\varphi : [c,d] \to [a,b]$, is an equivalent arc to $\boldsymbol{\gamma}$, then $\boldsymbol{\delta}'(\tau) = \boldsymbol{\gamma}'\big(\varphi(\tau)\big)\varphi'(\tau)$ with $\varphi'(\tau) > 0$. Thus

$$\begin{aligned}\int_{\boldsymbol{\delta}} f &= \int_{c}^{d} f\big(\boldsymbol{\delta}(\tau)\big)\,\|\boldsymbol{\delta}'(\tau)\|\,\mathrm{d}\tau \\ &= \int_{c}^{d} f\big(\boldsymbol{\gamma}(\varphi(\tau))\big)\,\|\boldsymbol{\gamma}'\big(\varphi(\tau)\big)\varphi'(\tau)\|\,\mathrm{d}\tau \\ &= \int_{c}^{d} f\big(\boldsymbol{\gamma}(\varphi(\tau))\big)\,\|\boldsymbol{\gamma}'\big(\varphi(\tau)\big)\|\,\varphi'(\tau)\,\mathrm{d}\tau\,.\end{aligned}$$

By $t = \varphi(\tau)$, hence $\mathrm{d}t = \varphi'(\tau)\mathrm{d}\tau$, we see that

$$\int_{\boldsymbol{\delta}} f = \int_{a}^{b} f\big(\boldsymbol{\gamma}(t)\big)\,\|\boldsymbol{\gamma}'(t)\|\,\mathrm{d}t = \int_{\boldsymbol{\gamma}} f\,.$$

□

The proposition immediately implies the following result.

Corollary 10.26 *The integral of a function along a curve does not change if the curve is replaced by another one, congruent to it.*

Next, let us note that naming c an arbitrary point in (a,b) and setting $\boldsymbol{\gamma}_1 = \boldsymbol{\gamma}_{|[a,c]}$, $\boldsymbol{\gamma}_2 = \boldsymbol{\gamma}_{|[c,b]}$, we have

$$\int_{\boldsymbol{\gamma}} f = \int_{\boldsymbol{\gamma}_1} f + \int_{\boldsymbol{\gamma}_2} f\,, \tag{10.8}$$

because integrals are additive with respect to their domain of integration.

Integrating along a curve extends automatically to piecewise-regular arcs. More precisely, we let $\boldsymbol{\gamma} : [a,b] \to \mathbb{R}^3$ be a piecewise-regular arc and take points $a = a_0 < a_1 < \ldots < a_n = b$ so that the arcs $\boldsymbol{\gamma}_i = \boldsymbol{\gamma}_{|[a_{i-1},a_i]}$, $i = 1,\ldots,n$, are regular. Suppose, as before, that f is a map with domain containing the image C of $\boldsymbol{\gamma}$ and such that $f \circ \boldsymbol{\gamma}$ is piecewise-continuous on $[a,b]$. Then we define

$$\int_{\boldsymbol{\gamma}} f = \sum_{i=1}^{n} \int_{\boldsymbol{\gamma}_i} f\,,$$

coherently with (10.8).

Remark 10.27 Finding an integral along a piecewise-regular arc might be easier if one uses Corollary 10.26. According to this,

$$\int_{\boldsymbol{\gamma}} f = \sum_{i=1}^{n} \int_{\boldsymbol{\delta}_i} f, \tag{10.9}$$

where $\boldsymbol{\delta}_i$ are suitable arcs congruent to $\boldsymbol{\gamma}_i$, $i = 1,\ldots,n$, chosen to simplify computations. □

Example 10.28

We want to calculate $\int_{\boldsymbol{\gamma}} x^2$, where $\boldsymbol{\gamma} : [0,4] \to \mathbb{R}^2$ is the following parametrisation of the boundary of the unit square $[0,1] \times [0,1]$:

$$\boldsymbol{\gamma}(t) = \begin{cases} \boldsymbol{\gamma}_1(t) = (t,0) & \text{if } 0 \le t < 1\,, \\ \boldsymbol{\gamma}_2(t) = (1,t-1) & \text{if } 1 \le t < 2\,, \\ \boldsymbol{\gamma}_3(t) = (3-t,1) & \text{if } 2 \le t < 3\,, \\ \boldsymbol{\gamma}_4(t) = (0,4-t) & \text{if } 3 \le t \le 4 \end{cases}$$

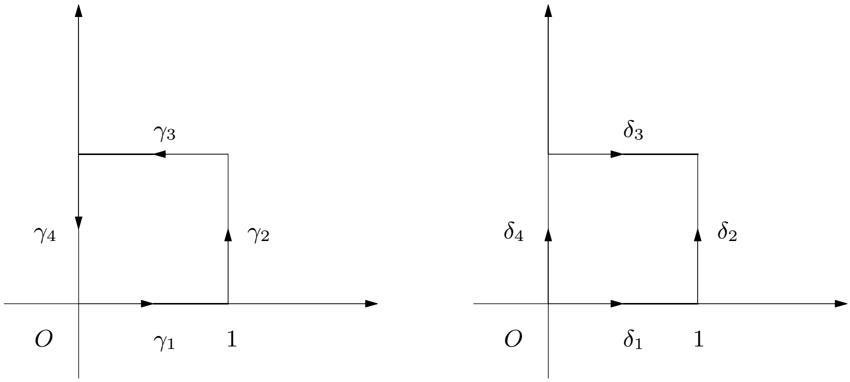


Figure 10.6. Parametrisation of the unit square, Example 10.28

(see Fig. 10.6, left). Let us represent the four sides by

$$\begin{array}{lll} \boldsymbol{\delta}_1(t) = \boldsymbol{\gamma}_1(t) & 0 \le t \le 1\,, & \boldsymbol{\delta}_1 = \boldsymbol{\gamma}_1\,, \\ \boldsymbol{\delta}_2(t) = (1,t) & 0 \le t \le 1\,, & \boldsymbol{\delta}_2 \sim \boldsymbol{\gamma}_2\,, \\ \boldsymbol{\delta}_3(t) = (t,1) & 0 \le t \le 1\,, & \boldsymbol{\delta}_3 \sim -\boldsymbol{\gamma}_3\,, \\ \boldsymbol{\delta}_4(t) = (0,t) & 0 \le t \le 1\,, & \boldsymbol{\delta}_4 \sim -\boldsymbol{\gamma}_4 \end{array}$$

(see Fig. 10.6, right). Then

$$\int_{\boldsymbol{\gamma}} x^2 = \int_0^1 t^2\,\mathrm{d}t + \int_0^1 1\,\mathrm{d}t + \int_0^1 t^2\,\mathrm{d}t + \int_0^1 0\,\mathrm{d}t = \frac{5}{3}\,. \qquad \square$$

10.3.1 Length of a curve and arc length

The **length** of a piecewise-regular curve $\boldsymbol{\gamma} : [a,b] \to \mathbb{R}^3$ is, by definition,

$$\ell(\boldsymbol{\gamma}) = \int_{\boldsymbol{\gamma}} 1\,. \tag{10.10}$$

In case of a regular arc, (10.10) reads

$$\ell(\boldsymbol{\gamma}) = \int_a^b \|\boldsymbol{\gamma}'(t)\|\,\mathrm{d}t = \int_a^b \sqrt{\big(x'(t)\big)^2 + \big(y'(t)\big)^2 + \big(z'(t)\big)^2}\,\mathrm{d}t\,. \tag{10.11}$$

The origin of the term is once again geometric. A fixed partition $a = t_0 < t_1 < \ldots, t_{n-1} < t_n = b$ of $[a,b]$ determines points $P_i = \boldsymbol{\gamma}(t_i) \in C$, $i = 0, \ldots, n$. These in turn give rise to a (possibly degenerate) polygonal path in $\mathbb{R}^3$ whose length is clearly

$$\ell(t_0, t_1, \ldots, t_n) = \sum_{i=1}^{n} \operatorname{dist}(P_{i-1}, P_i),$$

$\text{dist}\,(P_{i-1}, P_i) = \|P_i - P_{i-1}\|$ being the Euclidean distance of two consecutive points. If we let $\Delta t_i = t_i - t_{i-1}$, and

$$\left(\frac{\Delta x}{\Delta t}\right)_i = \left(\frac{x(t_i) - x(t_{i-1})}{t_i - t_{i-1}}\right)$$

(and similarly for the coordinates y and z), then

$$\begin{aligned}\|P_i - P_{i-1}\| &= \sqrt{\big(x(t_i) - x(t_{i-1})\big)^2 + \big(y(t_i) - y(t_{i-1})\big)^2 + \big(z(t_i) - z(t_{i-1})\big)^2} \\ &= \sqrt{\left(\frac{\Delta x}{\Delta t}\right)_i^2 + \left(\frac{\Delta y}{\Delta t}\right)_i^2 + \left(\frac{\Delta z}{\Delta t}\right)_i^2}\,\Delta t_i .\end{aligned}$$

Therefore

$$\ell(t_0, t_1, \ldots, t_n) = \sum_{i=1}^{n} \sqrt{\left(\frac{\Delta x}{\Delta t}\right)_i^2 + \left(\frac{\Delta y}{\Delta t}\right)_i^2 + \left(\frac{\Delta z}{\Delta t}\right)_i^2}\,\Delta t_i\,,$$

which ought to be considered an approximation of the integral appearing in (10.11). Provided the curve is sufficiently regular (piecewise-regular is enough), one can indeed prove that the supremum of $\ell(t_0, t_1, \ldots, t_n)$, taken over all possible partitions of $[a, b]$, is finite and equals $\ell(\boldsymbol{\gamma})$.

The length, as of (10.10), depends on the image C of the curve but also on the parametrisation. The circle $x^2 + y^2 = r^2$, parametrised by $\boldsymbol{\gamma}_1(t) = (r\cos t, r\sin t)$, $t \in [0, 2\pi]$, has length

$$\ell(\boldsymbol{\gamma}_1) = \int_0^{2\pi} r\,\mathrm{d}t = 2\pi r\,,$$

a well-known result in elementary geometry. But if we represent it using the curve $\boldsymbol{\gamma}_2(t) = (r\cos 2t, r\sin 2t)$, $t \in [0, 2\pi]$, we obtain

$$\ell(\boldsymbol{\gamma}_2) = \int_0^{2\pi} 2r\,\mathrm{d}t = 4\pi r\,,$$

because now the circle winds around twice. Proposition 10.25 says that congruent curves keep lengths fixed, and it is a fact that the length of a *simple* curve depends but on its image C (and not the parametrisation); it is called the **length** $\ell(C)$ of C. In the example, $\boldsymbol{\gamma}_1$ is simple, in contrast to $\boldsymbol{\gamma}_2$; as we have seen, $\ell(C) = \ell(\boldsymbol{\gamma}_1)$.

Let now $\boldsymbol{\gamma}$ be a regular curve on the interval I. We fix a point $t_0 \in I$ and define the map $s : I \to \mathbb{R}$

$$s(t) = \int_{t_0}^{t} \|\boldsymbol{\gamma}'(\tau)\|\,\mathrm{d}\tau\,. \tag{10.12}$$

Recalling (10.11), we have

$$s(t) = \begin{cases} \ell(\gamma_{|[t_0,t]}) & \text{if } t > t_0\,, \\ 0 & \text{if } t = t_0\,, \\ -\ell(\gamma_{|[t,t_0]}) & \text{if } t < t_0\,. \end{cases}$$

In practice the function s furnishes a reparametrisation of the image of $\boldsymbol{\gamma}$. As a matter of fact,

$$s'(t) = \|\boldsymbol{\gamma}'(t)\| > 0\,, \qquad \forall t \in I$$

by the Fundamental Theorem of integral calculus and by regularity. Therefore s is strictly increasing, hence invertible, on I. Letting $J = s(I)$ be the image interval under s, we denote by $t : J \to I \subseteq \mathbb{R}$ the inverse map to s. Otherwise said, we write $t = t(s)$ in terms of the new parameter s. The curve $\widetilde{\boldsymbol{\gamma}} : J \to \mathbb{R}^d$, $\widetilde{\boldsymbol{\gamma}}(s) = \boldsymbol{\gamma}\big(t(s)\big)$, is equivalent to $\boldsymbol{\gamma}$ (and as such it has the same image C). If $P_1 = \boldsymbol{\gamma}(t_1)$ is a point on C and t_1 corresponds to s_1 under the change of variable, then we also have $P_1 = \widetilde{\boldsymbol{\gamma}}(s_1)$. The number s_1 is called **arc length** of P_1.

Differentiating the inverse map,

$$\widetilde{\boldsymbol{\gamma}}'(s) = \frac{\mathrm{d}\widetilde{\boldsymbol{\gamma}}}{\mathrm{d}s}(s) = \frac{\mathrm{d}\boldsymbol{\gamma}}{\mathrm{d}t}\big(t(s)\big)\,\frac{\mathrm{d}t}{\mathrm{d}s}(s) = \frac{\boldsymbol{\gamma}'(t)}{\|\boldsymbol{\gamma}'(t)\|}\,,$$

whence

$$\|\widetilde{\boldsymbol{\gamma}}'(s)\| = 1\,, \qquad \forall s \in J\,.$$

This expresses the fact that the arc length parametrises the motion along a curve with constant ‘speed’ 1.

Remark 10.29 Take $\boldsymbol{\gamma} : [a,b] \to \mathbb{R}$ a regular curve and let s be the arc length as in (10.12), with $t_0 = a$; then $s(a) = 0$ and $s(b) = \int_a^b \|\boldsymbol{\gamma}'(\tau)\|\,\mathrm{d}\tau = \ell(\boldsymbol{\gamma})$. Using this special parameter, we have

$$\int_{\boldsymbol{\gamma}} f = \int_{\widetilde{\boldsymbol{\gamma}}} f = \int_0^{\ell(\boldsymbol{\gamma})} f\big(\widetilde{\boldsymbol{\gamma}}(s)\big)\,\mathrm{d}s = \int_0^{\ell(\boldsymbol{\gamma})} f\big(\widetilde{\boldsymbol{\gamma}}(t(s))\big)\,\mathrm{d}s\,. \qquad \square$$

The notion of arc length can be defined to cover in the obvious way piecewise-regular curves.

Example 10.30

The curve $\boldsymbol{\gamma} : \mathbb{R} \to \mathbb{R}^3$, $\boldsymbol{\gamma}(t) = (\cos t, \sin t, t)$ describes the circular helix (see Example 8.8 vi)). Since $\|\boldsymbol{\gamma}'(t)\| = \|(-\sin t, \cos t, 1)\| = (\sin^2 t + \cos^2 t + 1)^{1/2} = \sqrt{2}$, choosing $t_0 = 0$ we have

$$s(t) = \int_0^t \|\boldsymbol{\gamma}'(\tau)\|\,\mathrm{d}\tau = \sqrt{2}\int_0^t \mathrm{d}\tau = \sqrt{2}t\,.$$

It follows that $t = t(s) = \frac{\sqrt{2}}{2}s$, $s \in \mathbb{R}$, and the helix can be reparametrised by arc length

$$\widetilde{\boldsymbol{\gamma}}(s) = \left(\cos\frac{\sqrt{2}}{2}s, \sin\frac{\sqrt{2}}{2}s, \frac{\sqrt{2}}{2}s\right)\,. \qquad \square$$

10.4 Integral vector calculus

The last section deals with vector fields and their integration, and provides the correct mathematical framework for basic dynamical concepts such as force fields and the work done by a force.

Definition 10.31 *Let Ω indicate a non-empty subset of* $\mathbb{R}^d$, $d = 2, 3$. *A function* $\boldsymbol{F} : \Omega \to \mathbb{R}^d$ *is called a* **vector field** *on* Ω.

Conventionally $f_i : \Omega \to \mathbb{R}$, $i = 1, \ldots, d$, are the **components** of $\boldsymbol{F}$, written $\boldsymbol{F} = (f_1, \ldots, f_d)$. Using the unit vectors $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ introduced in Sect. 8.2.2, we can also write $\boldsymbol{F} = f_1\boldsymbol{i} + f_2\boldsymbol{j}$ if $d = 2$ and $\boldsymbol{F} = f_1\boldsymbol{i} + f_2\boldsymbol{j} + f_3\boldsymbol{k}$ if $d = 3$.

Vector fields may be integrated along curves, leading to a slightly more general notion of path integral. Take a regular arc $\boldsymbol{\gamma} : [a, b] \to \mathbb{R}^d$ whose image $C = \boldsymbol{\gamma}([a, b])$ is contained in Ω. In this fashion the composition $\boldsymbol{F} \circ \boldsymbol{\gamma} : t \mapsto \boldsymbol{F}\big(\boldsymbol{\gamma}(t)\big)$ maps $[a, b]$ to $\mathbb{R}^d$. We shall assume this composite is continuous, i.e., every $f_i\big(\boldsymbol{\gamma}(t)\big)$ from $[a, b]$ to $\mathbb{R}$ is a continuous map. For any $t \in [a, b]$ we denote by

$$\boldsymbol{\tau}(t) = \frac{\boldsymbol{\gamma}'(t)}{\|\boldsymbol{\gamma}'(t)\|}$$

the **unit tangent vector** to C at $P(t) = \boldsymbol{\gamma}(t)$. The scalar function $F_\tau = \boldsymbol{F} \cdot \boldsymbol{\tau}$,

$$F_\tau(t) = \big(\boldsymbol{F} \cdot \boldsymbol{\tau}\big)(t) = \boldsymbol{F}\big(\boldsymbol{\gamma}(t)\big) \cdot \boldsymbol{\tau}(t)$$

is the component of the field $\boldsymbol{F}$ along the unit tangent to $\boldsymbol{\gamma}$ at the point $P = \boldsymbol{\gamma}(t)$.

Definition 10.32 *The* **line integral or path integral of $\boldsymbol{F}$ along $\boldsymbol{\gamma}$** *is the integral along the curve $\boldsymbol{\gamma}$ of the map F_τ:*

$$\int_{\boldsymbol{\gamma}} \boldsymbol{F} \cdot \mathrm{d}P = \int_{\boldsymbol{\gamma}} F_\tau \,.$$

As the integral on the right equals

$$\int_{\boldsymbol{\gamma}} F_\tau = \int_{\boldsymbol{\gamma}} \boldsymbol{F} \cdot \boldsymbol{\tau} = \int_a^b \boldsymbol{F}\big(\boldsymbol{\gamma}(t)\big) \cdot \boldsymbol{\tau}(t)\, \|\boldsymbol{\gamma}'(t)\| \,\mathrm{d}t = \int_a^b \boldsymbol{F}\big(\boldsymbol{\gamma}(t)\big) \cdot \boldsymbol{\gamma}'(t) \,\mathrm{d}t \,,$$

the line integral of $\boldsymbol{F}$ on $\boldsymbol{\gamma}$ reads

$$\int_{\boldsymbol{\gamma}} \boldsymbol{F} \cdot \mathrm{d}P = \int_a^b \boldsymbol{F}\big(\boldsymbol{\gamma}(t)\big) \cdot \boldsymbol{\gamma}'(t) \,\mathrm{d}t \,. \tag{10.13}$$

Here the physical interpretation is paramount, and throws new light on the considerations made so far. If $\boldsymbol{F}$ describes a field of forces applied to C, the line integral becomes the work done by $\boldsymbol{F}$ during motion along the curve. The counterpart to Proposition 10.25 is

Proposition 10.33 *Let $\boldsymbol{\gamma} : [a, b] \to \mathbb{R}^d$ be a regular curve with image C, and $\boldsymbol{F}$ a vector field over C such that $\boldsymbol{F} \circ \boldsymbol{\gamma}$ is continuous. Then*

$$\int_{\boldsymbol{\gamma}} \boldsymbol{F} \cdot \mathrm{d}P = -\int_{-\boldsymbol{\gamma}} \boldsymbol{F} \cdot \mathrm{d}P \qquad \textit{and} \qquad \int_{\boldsymbol{\gamma}} \boldsymbol{F} \cdot \mathrm{d}P = \int_{\boldsymbol{\delta}} \boldsymbol{F} \cdot \mathrm{d}P\,,$$

over any curve $\boldsymbol{\delta}$ equivalent to $\boldsymbol{\gamma}$.

In mechanics this result would tell that the work is done by (resp. against) the force if the directions of force and motion are the same (opposite); once a direction of motion has been fixed, the work depends only on the path and not on how we move along it.

Examples 10.34

i) Consider the planar vector field $\boldsymbol{F} : \mathbb{R}^2 \to \mathbb{R}^2$ given by $\boldsymbol{F}(x, y) = (y, x)$. Take the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$, parametrised by $\boldsymbol{\gamma} : [0, 2\pi] \to \mathbb{R}^2$, $\boldsymbol{\gamma}(t) = (3\cos t, 2\sin t)$. Then $\boldsymbol{F}\big(\boldsymbol{\gamma}(t)\big) = (2\sin t, 3\cos t)$ and $\boldsymbol{\gamma}'(t) = (-3\sin t, 2\cos t)$. Therefore

$$\begin{aligned}\int_{\boldsymbol{\gamma}} \boldsymbol{F} \cdot \mathrm{d}P &= \int_0^{2\pi} (2\sin t, 3\cos t) \cdot (-3\sin t, 2\cos t)\,\mathrm{d}t \\ &= 6\int_0^{2\pi} (-\sin^2 t + \cos^2 t)\,\mathrm{d}t = 6\int_0^{2\pi} (2\cos^2 t - 1)\,\mathrm{d}t \\ &= 12\int_0^{2\pi} \cos^2 t\,\mathrm{d}t - 12\pi = 0\,,\end{aligned}$$

because

$$\int_0^{2\pi} \cos^2 t\,\mathrm{d}t = \left[\frac{1}{2}t + \frac{1}{4}\sin 2t\right]_0^{2\pi} = \pi$$

(see Example 9.9 ii)).

ii) Let $\boldsymbol{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be given by $\boldsymbol{F}(x, y, z) = (\mathrm{e}^x, x+y, y+z)$, and $\boldsymbol{\gamma} : [0, 1] \to \mathbb{R}^3$ by $\boldsymbol{\gamma}(t) = (t, t^2, t^3)$. The vector field along the path reads

$$\boldsymbol{F}\big(\boldsymbol{\gamma}(t)\big) = (\mathrm{e}^t, t + t^2, t^2 + t^3) \quad \text{and} \quad \boldsymbol{\gamma}'(t) = (1, 2t, 3t^2)\,.$$

Thus

$$\begin{aligned}\int_{\boldsymbol{\gamma}} \boldsymbol{F} \cdot \mathrm{d}P &= \int_0^1 (\mathrm{e}^t, t + t^2, t^2 + t^3) \cdot (1, 2t, 3t^2)\,\mathrm{d}t \\ &= \int_0^1 \big[\mathrm{e}^t + 2(t^2 + t^3) + 3(t^4 + t^5)\big]\,\mathrm{d}t = \mathrm{e} + \frac{19}{15}\,.\end{aligned}$$

□

10.5 Exercises

1. *Check the convergence of the following integrals and compute them explicitly:*

a) $\displaystyle\int_0^{+\infty} \frac{1}{x^2+3x+2}\,\mathrm{d}x$ b) $\displaystyle\int_0^{+\infty} \frac{x}{(x+1)^3}\,\mathrm{d}x$

c) $\displaystyle\int_2^{+\infty} \frac{1}{x\sqrt{x-2}}\,\mathrm{d}x$ d) $\displaystyle\int_{-1}^{1} \frac{1}{\sqrt{|x|}(x-4)}\,\mathrm{d}x$

2. *Discuss the convergence of the improper integrals:*

a) $\displaystyle\int_0^{+\infty} \frac{\sin x}{x\sqrt{x}}\,\mathrm{d}x$ b) $\displaystyle\int_0^{+\infty} \frac{1}{\log^2(2+\mathrm{e}^x)}\,\mathrm{d}x$

c) $\displaystyle\int_0^{+\infty} x\mathrm{e}^{-x}\,\mathrm{d}x$ d) $\displaystyle\int_{\mathrm{e}}^{+\infty} \frac{\log x}{\sqrt[3]{x^2}}\,\mathrm{d}x$

e) $\displaystyle\int_0^{1} \frac{\sqrt{x-x^2}}{\sin \pi x}\,\mathrm{d}x$ f) $\displaystyle\int_0^{\pi/2} \frac{1}{\sqrt{\sin x}}\,\mathrm{d}x$

g) $\displaystyle\int_0^{\pi} \frac{x-\pi/2}{\cos x\sqrt{\sin x}}\,\mathrm{d}x$ h) $\displaystyle\int_0^{\pi} \frac{(\pi-x)\log x}{\sqrt{|\log(1-\sin x)|}}\,\mathrm{d}x$

3. *Study the convergence of*

$$S_n = \int_2^{+\infty} \frac{x}{\sqrt{(x^2+3)^n}}\,\mathrm{d}x$$

for varying $n \in \mathbb{N}$. What is the smallest value of n for which S_n converges?

4. *Determine $\alpha \in \mathbb{R}$ such that the integrals below converge:*

a) $\displaystyle\int_{-\infty}^{+\infty} \frac{\arctan x}{|x|^\alpha}\,\mathrm{d}x$ b) $\displaystyle\int_{-\infty}^{+\infty} \frac{1}{|x^3+5x^2+8x+4|^\alpha}\,\mathrm{d}x$

c) $\displaystyle\int_0^{+\infty} \frac{1}{x^\alpha(4+9x)^2}\,\mathrm{d}x$ d) $\displaystyle\int_\alpha^{+\infty} \frac{1}{(x-2)\sqrt{|x-3|}}\,\mathrm{d}x$

5. *For which $\alpha \in \mathbb{R}$ does*

$$\int_2^3 \frac{x(\sin(x-2))^\alpha}{\sqrt{x^2-4}}\,\mathrm{d}x$$

converge? What is its value when $\alpha = 0$?

6. *Tell when the following integrals converge:*

a) $\displaystyle\int_1^{+\infty} (\log(x+1) - \log x)\,\mathrm{d}x$ b) $\displaystyle\int_0^{+\infty} \frac{\mathrm{e}^x - 1 - \sin x}{\mathrm{e}^{\pi x} - 1 - \sin \pi x}\,\mathrm{d}x$

$\boxed{\text{c)}}$ $\displaystyle\int_2^{+\infty} \frac{1}{\sqrt[3]{x-2}} \log\frac{x-2}{x+1}\,\mathrm{d}x$ $\boxed{\text{d)}}$ $\displaystyle\int_0^{+\infty} \frac{x}{\sin x - (x+x^2)\log(\mathrm{e}+x)}\,\mathrm{d}x$

$\boxed{7.}$ *Compute the integral of*

$$f(x,y,z) = \frac{x^2(1+8y)}{\sqrt{1+y+4x^2y}}$$

along the curve $\gamma(t) = (t, t^2, \log t)\,,\ t \in [1,2]$.

8. *Integrate the function* $f(x,y) = x$ *on the Jordan curve* γ *whose image consists of the parabolic arc of equation* $y = 4-x^2$ *going from* $A = (-2,0)$ *to* $C = (2,0)$*, and the circle* $x^2 + y^2 = 4$ *between* C *and* A.

$\boxed{9.}$ *Let* γ *be the curve in the first quadrant with image the union of the segment from* $O = (0,0)$ *to* $A = (1,0)$*, the arc of the ellipse* $4x^2 + y^2 = 4$ *between* A *and* $B = (\frac{\sqrt{2}}{2}, \sqrt{2})$*, and the segment joining* B *to the origin. Integrate* $f(x,y) = x + y$ *along the simple closed curve* γ.

10. *Integrate*

$$f(x,y) = \frac{1}{x^2+y^2+1}$$

along the simple closed curve γ *which is piecewise-defined by the segment from* O *to* $A = (\sqrt{2}, 0)$*, the arc of equation* $x^2+y^2 = 2$ *lying between* A *and* $B = (1,1)$*, and the segment joining* B *to the origin.*

$\boxed{11.}$ *Integrate the vector field* $\boldsymbol{F}(x,y) = (x^2, xy)$ *along the curve* $\gamma(t) = (t^2, t)$, $t \in [0,1]$.

12. *Compute the line integral of the field* $\boldsymbol{F}(x,y,z) = (z, y, 2x)$ *along* $\gamma(t) = (t, t^2, t^3)$, $t \in [0,1]$.

13. *Integrate* $\boldsymbol{F}(x,y,z) = (2\sqrt{z}, x, y)$ *along* $\gamma(t) = (-\sin t, \cos t, t^2)$, $t \in [0, \frac{\pi}{2}]$.

$\boxed{14.}$ *Integrate* $\boldsymbol{F}(x,y) = (xy^2, x^2y)$ *along the simple path* γ *consisting of the quadrilateral of vertices* $A = (0,1)$, $B = (1,1)$, $C = (0,2)$ *and* $D = (1,2)$.

15. *Integrate* $\boldsymbol{F}(x,y) = (0,y)$ *along the closed simple curve consisting of the segment from* O *to* $A = (1,0)$*, the arc of circumference* $x^2+y^2 = 1$ *between* A *and* $B = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$*, the segment from* B *back to* O.

10.5.1 Solutions

1. Convergence and computation of integrals:

a) $\log 2$; b) $\frac{1}{2}$.

c) The function $f(x) = \frac{1}{x\sqrt{x-2}}$ is unbounded at $x = 0$ and $x = 2$. The point $x = 0$ lies outside the domain of integration, hence can be ignored. We then split the integral as

$$\int_2^{+\infty} \frac{1}{x\sqrt{x-2}}\,\mathrm{d}x = \int_2^3 \frac{1}{x\sqrt{x-2}}\,\mathrm{d}x + \int_3^{+\infty} \frac{1}{x\sqrt{x-2}}\,\mathrm{d}x = S_1 + S_2\,.$$

For $x \to 2^+$, $f(x) \sim \frac{1}{2(x-2)^{1/2}}$, so f is infinite of order $\frac{1}{2} < 1$. By Asymptotic comparison test, i.e., Theorem 10.18, S_1 converges. As for S_2, let us consider f when $x \to +\infty$. Because

$$f(x) \sim \frac{1}{x \cdot x^{1/2}} = \frac{1}{x^{3/2}}\,, \qquad x \to +\infty\,,$$

Theorem 10.10 guarantees S_2 converges as well.
To compute the integral, let $t^2 = x - 2$, hence $2t\mathrm{d}t = \mathrm{d}x$ and $x = t^2 + 2$, by which

$$S = \int_0^{+\infty} \frac{2}{t^2+2}\,\mathrm{d}t = \frac{2}{\sqrt{2}} \arctan \frac{t}{\sqrt{2}}\Big|_0^{+\infty} = \frac{\sqrt{2}}{2}\pi\,.$$

d) The integrand is infinite at $x = 0$, $x = 4$. The latter point is irrelevant, for it does not belong to the domain of integration. At $x = 0$

$$f(x) \sim -\frac{1}{4\sqrt{|x|}} \quad \text{for} \quad x \to 0\,,$$

so the integral converges by applying Theorem 10.18 to

$$S_1 = \int_{-1}^0 \frac{1}{\sqrt{-x}(x-4)}\,\mathrm{d}x \qquad \text{and} \qquad S_2 = \int_0^1 \frac{1}{\sqrt{x}(x-4)}\,\mathrm{d}x$$

separately. For S_1, let us change $t^2 = -x$, so $2t\mathrm{d}t = -\mathrm{d}x$ and $x - 4 = -t^2 - 4$. Then

$$S_1 = -\int_0^1 \frac{2}{t^2+4}\,\mathrm{d}t = -\arctan \frac{t}{2}\Big|_0^1 = -\arctan \frac{1}{2}\,.$$

Putting $t^2 = x$ in S_2

$$S_2 = \int_0^1 \frac{2}{t^2-4}\,\mathrm{d}t = \frac{1}{2}\int_0^1 \left(\frac{1}{t-2} - \frac{1}{t+2}\right)\mathrm{d}t = \frac{1}{2}\left[\log\left|\frac{t-2}{t+2}\right|\right]_0^1 = \frac{1}{2}\log\frac{1}{3}\,.$$

Therefore $S = S_1 + S_2 = -\left(\arctan\frac{1}{2} + \frac{1}{2}\log 3\right)$.

2. *Convergence of improper integrals:*

a) Converges.

b) The map $f(x) = \frac{1}{\log^2(2+e^x)}$ has $\mathbb{R}$ as domain since $2 + e^x > 2$, $\forall x \in \mathbb{R}$. It is then sufficient to consider $x \to +\infty$. As

$$\log(2 + e^x) = \log e^x(1 + 2e^{-x}) = x + \log(1 + 2e^{-x}),$$

it follows

$$f(x) = \frac{1}{(x + \log(1 + 2e^{-x}))^2} \sim \frac{1}{x^2}, \qquad x \to +\infty.$$

The integral converges by Theorem 10.10.

c) Converges.

d) Over the integration domain the map is bounded. Moreover,

$$\frac{\log x}{\sqrt[3]{x^2}} \geq \frac{1}{\sqrt[3]{x^2}}, \qquad \forall x \geq e.$$

By the Comparison test (Theorem 10.5), the integral diverges.

e) Converges; f) converges.

g) The integrand is not defined at $x = 0$, $\frac{\pi}{2}$, nor at π. For $x = \frac{\pi}{2}$ though, the function admits a continuous prolongation mapping $\frac{\pi}{2}$ to -1, because if we put $t = x - \frac{\pi}{2}$, then

$$\cos x = \cos(t + \frac{\pi}{2}) = -\sin t = -\sin(x - \frac{\pi}{2})$$

and so

$$f(x) = \frac{x - \frac{\pi}{2}}{\cos x \sqrt{\sin x}} \sim -1, \quad x \to \frac{\pi}{2}.$$

Therefore the integral is 'proper' at $x = \frac{\pi}{2}$. From

$$f(x) \sim -\frac{\pi}{2\sqrt{x}}, \quad x \to 0^+, \qquad f(x) \sim -\frac{\pi}{2\sqrt{\pi - x}}, \quad x \to \pi^-,$$

we have convergence by asymptotic comparison (Theorem 10.18).

h) The map to be integrated is not defined for $x = 0$, $x = \frac{\pi}{2}$, $x = \pi$. In the limit $x \to 0^+$,

$$f(x) \sim \frac{\pi \log x}{|\log(1 - x)|^{1/2}} \sim \frac{\pi \log x}{\sqrt{x}}.$$

The map has no well-defined order of infinite with respect to the test function $\frac{1}{x}$; nevertheless, it is clearly infinite of smaller order than any power $\frac{1}{x^\alpha}$ with $\frac{1}{2} < \alpha < 1$, since the logarithm grows less than $\frac{1}{x^q}$ for any $q > 0$ when $x \to 0^+$. The Asymptotic comparison test (Theorem 10.18) forces the integral to converge around 0.

About the other points, for $x \to \frac{\pi}{2}$, the function tends to 0, so the integral in not improper at $\frac{\pi}{2}$; when $x \to \pi^-$, we have

$$f(x) \sim \frac{(\log \pi)(\pi - x)}{|\log(1 + \sin(x - \pi))|^{1/2}} \sim \frac{(\log \pi)(\pi - x)}{|\sin(x - \pi)|^{1/2}} \sim (\log \pi)(\pi - x)^{1/2},$$

so the integral in $x = \pi$ is proper because f goes to 0. Eventually then, the integral always converges.

3. The map is defined over all $\mathbb{R}$ with

$$f(x) \sim \frac{x}{x^n} = \frac{1}{x^{n-1}}, \qquad x \to +\infty.$$

Thus S converges if $n - 1 > 1$, i.e., the lowest n for which convergence occurs must be $n = 3$. Let us find

$$\int_2^{+\infty} \frac{x}{\sqrt{(x^2+3)^3}}\, \mathrm{d}x$$

then. Define $t = x^2 + 3$, so $\mathrm{d}t = 2x\mathrm{d}x$, and

$$\frac{1}{2}\int_7^{+\infty} t^{-3/2}\, \mathrm{d}t = \frac{1}{\sqrt{7}}.$$

4. *Interval of convergence of improper integrals:*

a) $\alpha \in (1, 2)$.

b) Having factorised $x^3 + 5x^2 + 8x + 4 = (x+2)^2(x+1)$, we can study the function for $x \to \pm\infty$, $x \to -2$ and $x \to -1$:

$$f(x) \sim \frac{1}{|x|^{3\alpha}}, \qquad x \to \pm\infty;$$
$$f(x) \sim \frac{1}{|x+2|^{2\alpha}}, \qquad x \to -2;$$
$$f(x) \sim \frac{1}{|x+1|^{\alpha}}, \qquad x \to -1.$$

In order to ensure convergence, we should impose $3\alpha > 1$, $2\alpha < 1$ plus $\alpha < 1$. Therefore $\alpha \in (\frac{1}{3}, \frac{1}{2})$.

c) $\alpha \in (-1, 1)$.

d) The integrand is infinite at $x = 2$ and $x = 3$. But

$$f(x) \sim \frac{1}{x^{3/2}}, \qquad x \to +\infty,$$
$$f(x) \sim \frac{1}{x-2}, \qquad x \to 2,$$
$$f(x) \sim \frac{1}{|x-3|^{1/2}}, \qquad x \to 3,$$

so everything is fine when $x \to +\infty$ or $x \to 3$. The point $x = 2$ is problematic only if included in the domain of integration, whence we should have $\alpha > 2$ to guarantee convergence.

5. $\alpha > -\frac{1}{2}$ and $S = \sqrt{5}$.

6. *Convergence of improper integrals:*

a) Diverges; b) converges.

c) Over $(2, +\infty)$ the map is not bounded in a neighbourhood of $x = 2$. Since

$$\log \frac{x-2}{x+1} \sim \log \frac{1}{3}(x-2)$$

is infinite of lower order than any positive power of $\frac{1}{x-2}$ when $x \to 2^+$, it follows f is infinite of lesser order than $\dfrac{1}{(x-2)^{1/3+\alpha}}$ (any $\alpha > 0$). This order, for a suitable choice of α (e.g., $\alpha = \frac{1}{2}$) is smaller than 1 . Therefore the integral converges at $x = 2$.
For $x \to +\infty$,

$$\log \frac{x-2}{x+1} \sim \log\left(1 - \frac{3}{x+1}\right) \sim -\frac{3}{x+1} \sim -\frac{3}{x},$$

whence

$$f(x) \sim -\frac{3}{x^{1/3} \cdot x} = -\frac{3}{x^{4/3}}, \qquad x \to +\infty\,.$$

Altogether, the integral converges.

d) Let us examine f at $x = 0$. As

$$\begin{aligned}\sin x - (x+x^2)\log(\mathrm{e}+x) &= x + o(x^2) - (x+x^2)\Big(1 + \log\Big(1+\frac{x}{\mathrm{e}}\Big)\Big)\\ &= -x^2 + o(x^2) - (x+x^2)\Big(\frac{x}{\mathrm{e}} + o(x)\Big)\\ &= -\left(1+\frac{1}{\mathrm{e}}\right)x^2 + o(x^2)\,, \quad x \to 0,\end{aligned}$$

we have

$$f(x) \sim -\frac{1}{(1+\frac{1}{\mathrm{e}})x}, \qquad x \to 0\,.$$

The integral then must diverge at $x = 0$.
Studying the behaviour for $x \to +\infty$ is unnecessary to conclude that the integral diverges (albeit a direct computation would establish the same).

7. When $t \in [1, 2]$,

$$f\big(\gamma(t)\big) = \frac{t^2(1+8t^2)}{\sqrt{1+t^2+4t^4}}, \qquad \gamma'(t) = \Big(1, 2t, \frac{1}{t}\Big),$$

whence

$$\int_{\boldsymbol{\gamma}} f = \int_1^2 \frac{t^2(1+8t^2)}{\sqrt{1+t^2+4t^4}} \, \frac{1}{t} \, \sqrt{1+t^2+4t^4} \, \mathrm{d}t = \int_1^2 t(1+8t^2)\,\mathrm{d}t = \frac{63}{2} \, .$$

8. 0.

9. First of all we find the coordinates of B, intersection in the first quadrant of the straight line $y = 2x$ and the ellipse $4x^2+y^2 = 4$, i.e., $B = (\frac{\sqrt{2}}{2}, \sqrt{2})$. The piecewise-regular curve $\boldsymbol{\gamma}$ can be divided into three regular arcs $\boldsymbol{\gamma}_1$, $\boldsymbol{\gamma}_2$, $\boldsymbol{\gamma}_3$, whose images are the segment OA, the elliptical path AB and the segment BO respectively. Let us reparametrise these arcs by calling them:

$$\begin{aligned}
&\boldsymbol{\delta}_1(t) = (t, 0) && 0 \le t \le 1\,, && \boldsymbol{\delta}_1 = \boldsymbol{\gamma}_1\,,\\
&\boldsymbol{\delta}_2(t) = (\cos t, 2\sin t) && 0 \le t \le \frac{\pi}{4}\,, && \boldsymbol{\delta}_2 \sim \boldsymbol{\gamma}_2\,,\\
&\boldsymbol{\delta}_3(t) = (t, 2t) && 0 \le t \le \frac{\sqrt{2}}{2}\,, && \boldsymbol{\delta}_3 \sim -\boldsymbol{\gamma}_3\,.
\end{aligned}$$

Then

$$\int_{\boldsymbol{\gamma}} f = \int_{\boldsymbol{\delta}_1} f + \int_{\boldsymbol{\delta}_2} f + \int_{\boldsymbol{\delta}_3} f \, .$$

Since

$$\begin{aligned}
&f\big(\boldsymbol{\delta}_1(t)\big) = t\,, && f\big(\boldsymbol{\delta}_2(t)\big) = \cos t + 2\sin t\,, && f\big(\boldsymbol{\delta}_3(t)\big) = 3t\,,\\
&\boldsymbol{\delta}_1(t) = (1,0)\,, && \boldsymbol{\delta}_2'(t) = (-\sin t, 2\cos t)\,, && \boldsymbol{\delta}_3'(t) = (1,2)\,,\\
&\|\boldsymbol{\delta}_1'(t)\| = 1\,, && \|\boldsymbol{\delta}_2'(t)\| = \sqrt{\sin^2 t + 4\cos^2 t}\,, && \|\boldsymbol{\delta}_3'(t)\| = \sqrt{5}\,,
\end{aligned}$$

we have

$$\begin{aligned}
\int_{\boldsymbol{\gamma}} f &= \int_0^1 t\,\mathrm{d}t + \int_0^{\pi/4} \big(\cos t + 2\sin t\big)\sqrt{\sin^2 t + 4\cos^2 t}\,\mathrm{d}t + \int_0^{\sqrt{2}/2} 3\sqrt{5}t\,\mathrm{d}t\\
&= \frac{1}{2} + \frac{3}{4}\sqrt{5} + \int_0^{\pi/4} \cos t\sqrt{4 - 3\sin^2 t}\,\mathrm{d}t + 2\int_0^{\pi/4} \sin t\sqrt{1 + 3\cos^2 t}\,\mathrm{d}t\\
&= \frac{1}{2} + \frac{3}{4}\sqrt{5} + I_1 + I_2 \, .
\end{aligned}$$

To compute I_1, put $u = \sqrt{3}\sin t$, so $\mathrm{d}u = \sqrt{3}\cos t\,\mathrm{d}t$, and

$$I_1 = \frac{1}{\sqrt{3}} \int_0^{\sqrt{6}/2} \sqrt{4-u^2}\,\mathrm{d}u \, .$$

With the substitution $v = \frac{u}{2}$, and recalling Example 9.13 vi),

$$I_1 = \frac{1}{\sqrt{3}} \left[\frac{1}{2}u\sqrt{4-u^2} + 2\arcsin\frac{u}{2}\right]_0^{\sqrt{6}/2} = \frac{\sqrt{5}}{4} + \frac{2}{\sqrt{3}}\arcsin\frac{\sqrt{6}}{4} \, .$$

For I_2 the story goes analogously: let $u = \sqrt{3}\cos t$, hence $\mathrm{d}u = -\sqrt{3}\sin t\,\mathrm{d}t$ and

$$I_2 = -\frac{2}{\sqrt{3}}\int_0^{\sqrt{6}/2} \sqrt{1+u^2}\,\mathrm{d}u\,.$$

By Example 9.13 v), we have

$$\begin{aligned} I_2 &= -\frac{2}{\sqrt{3}}\left[\frac{1}{2}u\sqrt{1+u^2} + \frac{1}{2}\log\left(\sqrt{1+u^2}+u\right)\right]_{\sqrt{3}}^{\sqrt{6}/2} \\ &= -\frac{\sqrt{5}}{2} + 2 + \frac{1}{\sqrt{3}}\left(\log(2+\sqrt{3}) - \log\frac{\sqrt{10}-\sqrt{6}}{2}\right)\,. \end{aligned}$$

Overall,

$$\int_{\gamma} f = \frac{5}{2} + \frac{\sqrt{5}}{2} + \frac{2}{\sqrt{3}}\arcsin\frac{\sqrt{6}}{4} + \frac{1}{\sqrt{3}}\left(\log(2+\sqrt{3}) - \log\frac{\sqrt{10}-\sqrt{6}}{2}\right)\,.$$

10. $2\arctan\sqrt{2} + \frac{\sqrt{2}}{12}\pi$.

11. Since $\boldsymbol{F}\big(\boldsymbol{\gamma}(t)\big) = (t^4, t^3)$ and $\boldsymbol{\gamma}'(t) = (2t, 1)$,

$$\int_{\gamma} \boldsymbol{F}\cdot \mathrm{d}P = \int_0^1 (t^4, t^3)\cdot(2t,1)\,\mathrm{d}t = \int_0^1 (2t^5 + t^3)\,\mathrm{d}t = \frac{7}{12}\,.$$

12. $\dfrac{9}{4}$; 13. $\dfrac{\pi}{4}$.

14. The arc $\boldsymbol{\gamma}$ is piecewise-regular, so we take the regular bits $\boldsymbol{\gamma}_1$, $\boldsymbol{\gamma}_2$, $\boldsymbol{\gamma}_3$ whose images are the segments AB, BC, CD. Define $\boldsymbol{\delta}_i$, reparametrisation of $\boldsymbol{\gamma}_i$, $\forall i = 1,2,3$, by

$$\begin{array}{lll} \boldsymbol{\delta}_1(t) = (t,1) & 0 \le t \le 1\,, & \boldsymbol{\delta}_1 \sim \boldsymbol{\gamma}_1\,, \\ \boldsymbol{\delta}_2(t) = (t, 2-t) & 0 \le t \le 1\,, & \boldsymbol{\delta}_2 \sim -\boldsymbol{\gamma}_2\,, \\ \boldsymbol{\delta}_3(t) = (t,2) & 0 \le t \le 1\,, & \boldsymbol{\delta}_3 \sim \boldsymbol{\gamma}_3\,. \end{array}$$

Since

$$\begin{array}{lll} \boldsymbol{F}\big(\boldsymbol{\delta}_1(t)\big) = (t,t^2)\,, & \boldsymbol{F}\big(\boldsymbol{\delta}_2(t)\big) = \big(t(2-t)^2, t^2(2-t)\big)\,, & \boldsymbol{F}\big(\boldsymbol{\delta}_3(t)\big) = (4t, 2t^2) \\ \boldsymbol{\delta}_1(t) = (1,0)\,, & \boldsymbol{\delta}_2'(t) = (1,-1)\,, & \boldsymbol{\delta}_3(t) = (1,0)\,, \end{array}$$

one has

$$\begin{aligned} \int_{\gamma} \boldsymbol{F}\cdot\mathrm{d}P &= \int_{\boldsymbol{\delta}_1} \boldsymbol{F}\cdot\mathrm{d}P - \int_{\boldsymbol{\delta}_2} \boldsymbol{F}\cdot\mathrm{d}P + \int_{\boldsymbol{\delta}_3} \boldsymbol{F}\cdot\mathrm{d}P \\ &= \int_0^1 (t,t^2)\cdot(1,0)\,\mathrm{d}t - \int_0^1 \big(t(2-t)^2, t^2(2-t)\big)\cdot(1,-1)\,\mathrm{d}t \\ &\qquad + \int_0^1 (4t, 2t^2)\cdot(1,0)\,\mathrm{d}t = 2\,. \end{aligned}$$

15. 0.

11

Ordinary differential equations

A large part of the natural phenomena occurring in physics, engineering and other applied sciences can be described by a *mathematical model*, a collection of relations involving a function and its derivatives. The example of uniformly accelerated motion is typical, the relation being

$$\frac{\mathrm{d}^2 s}{\mathrm{d}t^2} = g, \tag{11.1}$$

where $s = s(t)$ is the motion in function of time t, and g is the acceleration. Another example is radioactive decay. The rate of disintegration of a radioactive substance in time is proportional to the quantity of matter:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = -ky, \tag{11.2}$$

in which $y = y(t)$ is the mass of the element and $k > 0$ the decay constant. The above relations are instances of differential equations.

The present chapter aims at introducing the reader to some types of differential equations. Although we cannot afford to go into the general theory, we will present the basic notions and explain a few techniques for solving certain classes of differential equations (of first and second order) that we judge particularly significant.

11.1 General definitions

By an **ordinary differential equation**, abbreviated ODE, one understands a relation among an independent real variable, say x, an unknown function $y = y(x)$ and its derivatives $y^{(k)}$ up to a specified order n. It is indicated by

$$\mathcal{F}(x, y, y', ..., y^{(n)}) = 0, \tag{11.3}$$

where $\mathcal{F}$ is a real map depending on $n + 2$ real variables. The differential equation has **order** n, if n is the highest order of differentiation in (11.3). A **solution** (in the classical sense) of the ODE over a real interval I is a function $y : I \to \mathbb{R}$, differentiable n times on I, such that

$$\mathcal{F}\big(x, y(x), y'(x), ..., y^{(n)}(x)\big) = 0 \qquad \text{for all } x \in I.$$

It happens many times that the highest derivative $y^{(n)}$ in (11.3) can be expressed in terms of x and the remaining derivatives explicitly,

$$y^{(n)} = f(x, y, ..., y^{(n-1)}), \tag{11.4}$$

with f a real function of $n+1$ variables (in several concrete cases this is precisely the form in which the equation crops up). If so, the differential equation is written **in normal form**. It should also be clear what the term 'solution of an ordinary differential equation in normal form' means.

A differential equation is said **autonomous** if $\mathcal{F}$ (or f) does not depend on the variable x. Equations (11.1), (11.2) are autonomous differential equations in normal form, of order two and one respectively.

The rest of the chapter is committed to first order differential equations in normal form, together with a particularly important class of equations of the second order.

11.2 First order differential equations

Let f be a real-valued map defined on a subset of $\mathbb{R}^2$. A solution to the equation

$$y' = f(x, y) \tag{11.5}$$

over an interval I of $\mathbb{R}$ is a differentiable map $y = y(x)$ such that $y'(x) = f\big(x, y(x)\big)$ for any $x \in I$. The graph of a solution to (11.5) is called **integral curve** of the differential equation.

Relation (11.5) admits a significant geometric interpretation. For each point (x, y) in the domain of f, $f(x, y)$ is the slope of the tangent to the integral curve containing (x, y) – assuming the curve exists in the first place – so equation (11.5) is fittingly represented by a *field of directions* in the plane (see Fig. 11.1).

Remark 11.1 If we start to move from $(x, y) = (x_0, y_0)$ along the straight line with slope $f(x_0, y_0)$ (the tangent), we reach a point (x_1, y_1) in the proximity of the integral curve passing through (x_0, y_0). From there we can advance a little bit farther along the next tangent, reach (x_2, y_2) nearby the curve and so on, progressively building a polygonal path close to the integral curve issuing from (x_0, y_0). This is the so-called *explicit Euler method* which is the simplest numerical procedure for approximating the solution of a differential equation when no analytical tools are available. This and other techniques are the content of the lecture course on Numerical Analysis. □

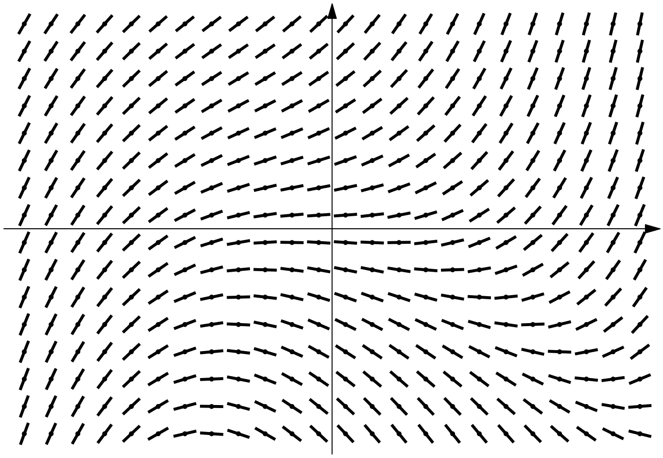


Figure 11.1. Field of directions representing $y' = (1 + x)y + x^2$

Solving (11.5) generalises the problem of finding the primitives of a given map. If f depends on x and not on y, (11.5) reads

$$y' = f(x); \tag{11.6}$$

assuming f continuous on I, the solutions are precisely the primitives $y(x) = F(x) + C$ of f over I, with F a particular primitive and C an arbitrary constant. This shows that, at least in the case where f does not depend upon y, (11.5) admits infinitely many distinct solutions, which depend on one constant. Note that any chosen integral curve is the vertical translate of another.

Actually, equation (11.6) plays a fundamental role, because in several circumstances, suitable manipulations show that solving (11.5) boils down to the quest for primitives of known functions. Furthermore, under fairly general hypotheses one can prove that (11.5) always admits a one-parameter family of distinct solutions, depending on an arbitrary constant of integration C. We shall write solutions in the form

$$y = y(x; C) \tag{11.7}$$

with C varying in (an interval of) $\mathbb{R}$. An expression like (11.7) is the **general integral** of equation (11.5), while any solution corresponding to a particular choice of C shall be a **particular integral**.

Example 11.2

Solving the differential equation

$$y' = y \tag{11.8}$$

amounts to locating the maps that coincide with their first derivative. We have already remarked that the exponential $y(x) = \mathrm{e}^x$ enjoys this important property.

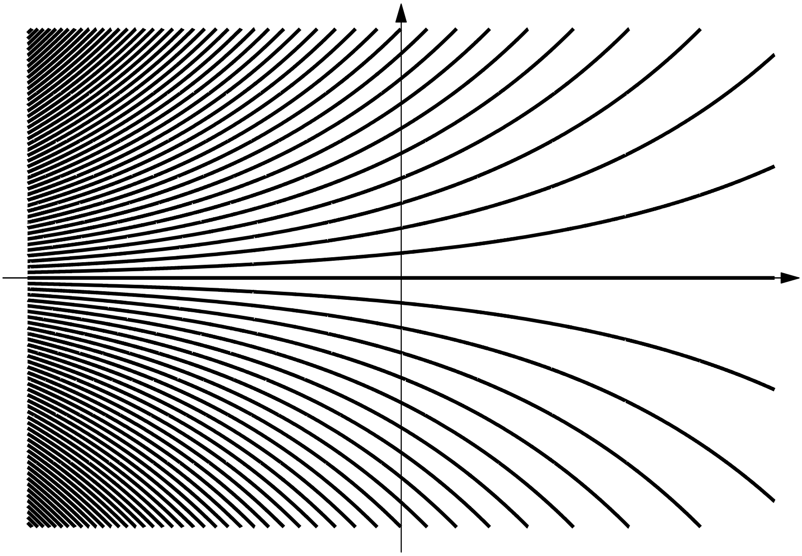


Figure 11.2. Integral curves of $y' = y$

Since differentiating is a linear operation, any function $y(x) = Ce^x$, $C \in \mathbb{R}$ possesses this feature. Further on we will prove there are no other maps doing the same, so we can conclude that the solutions to (11.8) belong to the family

$$y(x; C) = C\mathrm{e}^x, \qquad C \in \mathbb{R}.$$

The integral curves are drawn in Fig. 11.2. □

In order to get hold of a particular integral of (11.5), one should tell how to select one value of the constant of integration. A customary way to do so is to ask that the solution assume a specific value at a point x fixed in advance. More explicitly, we impose $y(x_0; C) = y_0$, where x_0 and y_0 are given, corresponding to the geometric constraint that the integral curve passes through (x_0, y_0). Essentially, we have solved a so-called initial value problem. More precisely, an **initial value problem**, or a **Cauchy problem**, for (11.5) on the interval I consists in determining a differentiable function $y = y(x)$ such that

$$\begin{cases} y' = f(x, y) & \text{on } I, \\ y(x_0) = y_0 \end{cases} \tag{11.9}$$

with given points $x_0 \in I$, $y_0 \in \mathbb{R}$. The understated reference to time in the words 'initial value' is due to the fact that many instances of (11.9) model the evolution of a physical system, which is in the state y_0 at the time x_0 in which simulation starts.

Example 11.3

The initial value problem

$$\begin{cases} y' = y, & \text{on } I = [0, +\infty)\,, \\ y(0) = 2, \end{cases}$$

is solved by the function $y(x) = 2\mathrm{e}^x$. □

Remark 11.4 The prescription of an initial condition, albeit rather common, is not the sole possibility to pin down a particular solution of a differential equation. Take the following problem as example: *find the solution of* $y' = y$ *having mean value* 1 *on the interval* $I = [0, 2]$. The general solution $y = C\mathrm{e}^x$ has to satisfy the constraint

$$\frac{1}{2}\int_0^2 y(x)\,\mathrm{d}x = 1\,,$$

which easily yields $C = \frac{2}{\mathrm{e}^2 - 1}$. □

Remark 11.5 Let us return to equations of order n for the moment. With the proper assumptions, the general integral of such an equation depends upon n real, arbitrary constants of integration C_k $(k = 1, 2, ..., n)$

$$y = y(x; C_1, C_2, ..., C_n).$$

The initial value problem supplies the values of y and its $n-1$ derivatives at a given $x_0 \in I$

$$\begin{cases} y^{(n)} = f(x, y, ..., y^{(n-1)}) & \text{on } I, \\ y(x_0) = y_{00}, \\ y'(x_0) = y_{01}, \\ \dots \\ y^{(n-1)}(x_0) = y_{0,n-1}, \end{cases}$$

where $y_{00}, y_{01}, ..., y_{0,n-1}$ are n fixed real numbers. For instance, the trajectory of the particle described by equation (11.1) is uniquely determined by the initial position $s(0)$ and initial velocity $s'(0)$.

Besides initial value problems, a particular solution to a higher order equation can be found by assigning values to the solution (and/or some derivatives) at the end-points of the interval. In this case one speaks of a **boundary value problem**. For instance, the problem of the second order

$$\begin{cases} y'' = k \sin y & \text{on the interval } (a, b), \\ y(a) = 0,\ y(b) = 0\,, \end{cases}$$

models the sag from the rest position of a thin elastic beam subject to a small load acting in the direction of the x-axis. □

We focus now on three special kinds of first order differential equations, which can be solved by finding few primitive functions.

11.2.1 Equations with separable variables

The variables are said "separable" in differential equations of type

$$y' = g(x)h(y), \tag{11.10}$$

where the $f(x, y)$ of (11.5) is the product of a continuous g depending only on x, and a continuous function h of y alone.

If $\bar{y} \in \mathbb{R}$ annihilates h, i.e., $h(\bar{y}) = 0$, the constant map $y(x) = \bar{y}$ is a particular integral of (11.10), for the equation becomes $0 = 0$. Therefore an equation with separable variables has, to start with, as many particular solutions $y(x) = constant$ as the number of distinct zeroes of h. These are called *singular integrals* of the differential equation.

On each interval J where $h(y)$ does not vanish we can write (11.10) as

$$\frac{1}{h(y)}\frac{\mathrm{d}y}{\mathrm{d}x} = g(x).$$

Let $H(y)$ be a primitive of $\dfrac{1}{h(y)}$ (with respect to y). By the Chain rule (Theorem 6.7)

$$\frac{\mathrm{d}}{\mathrm{d}x}H\big(y(x)\big) = \frac{\mathrm{d}H}{\mathrm{d}y}\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{h(y)}\frac{\mathrm{d}y}{\mathrm{d}x} = g(x)\,,$$

so $H\big(y(x)\big)$ is a primitive function of $g(x)$. Therefore, given an arbitrary primitive $G(x)$ of $g(x)$, we have

$$H\big(y(x)\big) = G(x) + C, \qquad\qquad C \in \mathbb{R}. \tag{11.11}$$

But we assumed $\dfrac{1}{h(y)} = \dfrac{\mathrm{d}H}{\mathrm{d}y}$ had no zeroes on J, hence it must have constant sign (being continuous). This implies that $H(y)$ is strictly monotone on J, i.e., invertible by Theorem 2.8. We are then allowed to make $y(x)$ explicit in (11.11):

$$y(x) = H^{-1}\big(G(x) + C\big), \tag{11.12}$$

where H^{-1} is the inverse of H. This expression is the general integral of equation (11.10) over every interval where $h\big(y(x)\big)$ is never zero. But, should we not be able to attain the analytic expression of $H^{-1}(x)$, formula (11.12) would have a paltry theoretical meaning. In such an event one is entitled to stop at the implicit form (11.11).

If equation (11.10) has singular solutions, these might admit the form (11.12) for special values of C. Sometimes, taking the limit for $C \to \pm\infty$ in (11.12) furnishes singular integrals.

Formula (11.11) is best remembered by interpreting the derivative $\frac{\mathrm{d}y}{\mathrm{d}x}$ as a formal ratio, following Leibniz. Namely, dividing (11.10) by $h(y)$ and 'multiplying' by $\mathrm{d}x$ gives

$$\frac{\mathrm{d}y}{h(y)} = g(x)\mathrm{d}x,$$

which can be then integrated

$$\int \frac{\mathrm{d}y}{h(y)} = \int g(x)\,\mathrm{d}x.$$

This corresponds exactly to (11.11). The reader must not forget though that the correct proof of the formula is the one showed earlier!

Examples 11.6

i) Solve the differential equation $y' = y(1-y)$. Let us put $g(x) = 1$ and $h(y) = y(1-y)$. The zeroes of h produce two singular integrals $y_1(x) = 0$ and $y_2(x) = 1$.
Suppose now $h(y)$ is not 0. We write the equation as

$$\int \frac{\mathrm{d}y}{y(1-y)} = \int \mathrm{d}x,$$

then integrate with respect to y on the left, and on the right with respect to x

$$\log\left|\frac{y}{1-y}\right| = x + C.$$

Exponentiating, we obtain

$$\left|\frac{y}{1-y}\right| = \mathrm{e}^{x+C} = k\mathrm{e}^x,$$

where $k = \mathrm{e}^C$ is an arbitrary positive constant. Therefore

$$\frac{y}{1-y} = \pm k\mathrm{e}^x = K\mathrm{e}^x,$$

K being any non-zero constant. Writing y in function of x, we get

$$y(x) = \frac{K\mathrm{e}^x}{1+K\mathrm{e}^x}.$$

Note the singular solution $y_1(x) = 0$ belongs to the above family for $K = 0$, a value K was originally prevented to take. The other singular integral, $y_2(x) = 1$, formally arises by letting K go to infinity.

ii) Consider the equation

$$y' = \sqrt{y}.$$

At first glance we spot the singular solution $y_1(x) = 0$. That apart, by separating variables we have

$$\int \frac{\mathrm{d}y}{\sqrt{y}} = \int \mathrm{d}x \qquad \text{hence} \qquad 2\sqrt{y} = x + C\,,$$

and so

$$y(x) = \left(\frac{x}{2} + C\right)^2, \qquad C \in \mathbb{R}$$

(where $C/2$ has become C).

iii) Solve

$$y' = \frac{e^x + 1}{e^y + 1}.$$

Let $g(x) = e^x + 1$ and $h(y) = \dfrac{1}{e^y + 1} > 0$ for any y; there are no singular integrals. The separation of x and y yields

$$\int (e^y + 1)\, dy = \int (e^x + 1)\, dx\,,$$

so

$$e^y + y = e^x + x + C, \qquad C \in \mathbb{R}.$$

But now we are stuck, for it is not possible to explicitly write y as function of the variable x. □

11.2.2 Linear equations

A differential equation akin to

$$y' + a(x)y = b(x)\,, \tag{11.13}$$

where a and b are continuous on I, is called linear, because the function $f(x, y) = -a(x)y + b(x)$ is a linear polynomial in y with coefficients in the variable x. This equation is said **homogeneous** if the source term vanishes, $b(x) = 0$, **non-homogeneous** otherwise.

We begin by solving the homogeneous case

$$y' = -a(x)y. \tag{11.14}$$

This is a particular example of equation with separable variables. So referring to (11.10) we have $g(x) = -a(x)$ and $h(y) = y$. The constant $y(x) = 0$ is a solution. Excluding this possibility, we can write

$$\int \frac{1}{y}\, dy = -\int a(x)\, dx.$$

If $A(x)$ denotes a primitive of $a(x)$, i.e., if

$$\int a(x)\, dx = A(x) + C, \qquad C \in \mathbb{R}, \tag{11.15}$$

then

$$\log |y| = -A(x) - C,$$

or, equivalently,

$$|y(x)| = \mathrm{e}^{-C}\mathrm{e}^{-A(x)}, \qquad \text{hence} \qquad y(x) = \pm K\mathrm{e}^{-A(x)},$$

where $K = \mathrm{e}^{-C} > 0$. The particular integral $y(x) = 0$ is included if we allow K to become 0. The solutions of the homogeneous linear equation (11.14) are

$$y(x) = K\mathrm{e}^{-A(x)}, \qquad K \in \mathbb{R},$$

with $A(x)$ defined by (11.15).

Now let us assess the case $b \neq 0$. We make use of the *method of variation of parameters*, which consists in searching for solutions of the form

$$y(x) = K(x)\,\mathrm{e}^{-A(x)},$$

where $K(x)$, a function of x, is unknown. Such a representation for $y(x)$ always exists, since $\mathrm{e}^{-A(x)} > 0$. Substituting in (11.13), we obtain

$$K'(x)\mathrm{e}^{-A(x)} + K(x)\mathrm{e}^{-A(x)}\big(-a(x)\big) + a(x)K(x)\mathrm{e}^{-A(x)} = b(x),$$

or

$$K'(x) = \mathrm{e}^{A(x)}b(x).$$

Calling $B(x)$ a primitive of $\mathrm{e}^{A(x)}b(x)$,

$$\int \mathrm{e}^{A(x)}b(x)\,\mathrm{d}x = B(x) + C, \qquad C \in \mathbb{R}, \tag{11.16}$$

we have

$$K(x) = B(x) + C,$$

so the general solution to (11.13) reads

$$y(x) = \mathrm{e}^{-A(x)}\big(B(x) + C\big), \tag{11.17}$$

where $A(x)$ and $B(x)$ are defined by (11.15) and (11.16). The integral is more often than not found in the form

$$y(x) = \mathrm{e}^{-\int a(x)\,\mathrm{d}x} \int \mathrm{e}^{\int a(x)\,\mathrm{d}x}\, b(x)\,\mathrm{d}x. \tag{11.18}$$

The expression highlights the various steps involved in the solution of a non-homogeneous linear equation: one has to integrate twice, in succession.

If we are asked to solve the initial value problem

$$\begin{cases} y' + a(x)y = b(x) & \text{on the interval } I, \\ y(x_0) = y_0, & \text{with } x_0 \in I \text{ and } y_0 \in \mathbb{R}, \end{cases} \tag{11.19}$$

we might want to choose, as primitive for $a(x)$, the one vanishing at x_0, which we write $A(x) = \displaystyle\int_{x_0}^{x} a(s)\,\mathrm{d}s$ by the Fundamental Theorem of integral calculus; the same we do for

$$B(x) = \int_{x_0}^{x} \mathrm{e}^{\int_{x_0}^{t} a(s)\,\mathrm{d}s}\, b(t)\,\mathrm{d}t$$

(recall the variables in the definite integral are arbitrary symbols). Substituting these expressions in (11.17) we obtain $y(x_0) = C$, hence the solution to (11.19) will satisfy $C = y_0$, namely

$$y(x) = \mathrm{e}^{-\int_{x_0}^{x} a(s)\,\mathrm{d}s}\left(y_0 + \int_{x_0}^{x} \mathrm{e}^{\int_{x_0}^{t} a(s)\,\mathrm{d}s}\, b(t)\,\mathrm{d}t\right). \tag{11.20}$$

Examples 11.7

i) Determine the general integral of the linear equation

$$y' + ay = b,$$

where $a \neq 0$ and b are real numbers. By choosing $A(x) = ax$, $B(x) = \dfrac{b}{a}\mathrm{e}^{ax}$ we find the general solution

$$y(x) = C\mathrm{e}^{-ax} + \frac{b}{a}.$$

If $a = -1$, $b = 0$, the formula provides the announced result that every solution of $y' = y$ has the form $y(x) = C\mathrm{e}^{x}$.

For the initial value problem

$$\begin{cases} y' + ay = b \quad \text{on } [1, +\infty), \\ y(1) = y_0, \end{cases}$$

it is convenient to have $A(x) = a(x-1)$, $B(x) = \dfrac{b}{a}\left(\mathrm{e}^{a(x-1)} - 1\right)$, so that

$$y(x) = \left(y_0 - \frac{b}{a}\right)\mathrm{e}^{-a(x-1)} + \frac{b}{a}.$$

Note that if $a > 0$ the solution converges to $\dfrac{b}{a}$ for $x \to +\infty$ (independent of the initial datum y_0).

ii) Determine the integral curves of

$$xy' + y = x^2$$

that lie in the first quadrant of the (x, y)-plane. Written as (11.13), the equation is

$$y' + \frac{1}{x}y = x,$$

so $a(x) = \frac{1}{x}$, $b(x) = x$. With $A(x) = \log x$ we have $\mathrm{e}^{A(x)} = x$ and $\mathrm{e}^{-A(x)} = \frac{1}{x}$. Consequently,

$$\int \mathrm{e}^{A(x)} b(x)\,\mathrm{d}x = \int x^2\,\mathrm{d}x = \frac{1}{3}x^3 + C.$$

Therefore, when $x > 0$ the general integral is

$$y(x) = \frac{1}{x}\left(\frac{1}{3}x^3 + C\right) = \frac{1}{3}x^2 + \frac{C}{x}.$$

For $C \geq 0$, $y(x) > 0$ for any $x > 0$, whereas $C < 0$ implies $y(x) > 0$ for $x > \sqrt[3]{3|C|}$. □

11.2.3 Homogeneous equations

Homogeneity refers to the form

$$y' = \varphi\left(\frac{y}{x}\right), \tag{11.21}$$

in which $\varphi = \varphi(z)$ is continuous in the variable z. Thus, $f(x, y)$ depends on x, y only in terms of their ratio $\frac{y}{x}$; we can equivalently say that $f(\lambda x, \lambda y) = f(x, y)$ for any $\lambda > 0$.

A homogeneous equation can be solved by separation of variables, in that one puts $z = \frac{y}{x}$, to be understood as $z(x) = \frac{y(x)}{x}$. In this manner $y(x) = xz(x)$ and $y'(x) = z(x) + xz'(x)$. Substituting in (11.21) yields

$$z' = \frac{\varphi(z) - z}{x},$$

an equation in z where the variables are separated. We can apply the strategy of Sect. 11.2.1. Every solution $\bar{z}$ of $\varphi(z) = z$ gives rise to a singular integral $z(x) = \bar{z}$, i.e., $y(x) = \bar{z}x$. Supposing instead $\varphi(z)$ different from z, we have

$$\int \frac{\mathrm{d}z}{\varphi(z) - z} = \int \frac{\mathrm{d}x}{x},$$

giving

$$H(z) = \log|x| + C,$$

where $H(z)$ is a primitive of $\frac{1}{\varphi(z) - z}$. Indicating by H^{-1} the inverse map, we have

$$z(x) = H^{-1}(\log|x| + C),$$

so the general integral of (11.21) reads (returning to y)

$$y(x) = x\, H^{-1}(\log|x| + C).$$

Example 11.8

Solve

$$x^2y' = y^2 + xy + x^2. \tag{11.22}$$

We can put the equation in normal form

$$y' = \left(\frac{y}{x}\right)^2 + \frac{y}{x} + 1,$$

which is homogeneous for $\varphi(z) = z^2 + z + 1$. Substituting $y = xz$, we arrive at

$$z' = \frac{z^2+1}{x},$$

whose variables are separated.

As $z^2 + 1$ is positive, there are no singular solutions. Integrating we obtain

$$\arctan z = \log|x| + C$$

and the general solution to (11.22) is

$$y(x) = x\tan(\log|x| + C).$$

We remark that C can be chosen either in $(-\infty, 0)$ or in $(0, +\infty)$, because of the singularity at $x = 0$. Moreover, the domain of existence of each solution depends on the value of C. □

11.2.4 Second order equations reducible to first order

Suppose an equation of second order does not contain the variable y explicitly, that is,

$$y'' = f(y', x). \tag{11.23}$$

Then the substitution $z = y'$ transforms it into a first order equation

$$z' = f(z, x)$$

in the unknown $z = z(x)$. If the latter has general solution $z(x; C_1)$, we can recover the integrals of (11.23) by solving

$$y' = z,$$

hence by finding the primitives of $z(x; C_1)$. This will generate a new constant of integration C_2. The general solution to (11.23) will have the form

$$y(x; C_1, C_2) = \int z(x; C_1)\,\mathrm{d}x = Z(x; C_1) + C_2,$$

where $Z(x; C_1)$ is a particular primitive of $z(x; C_1)$.

Example 11.9

Solve

$$y'' - (y')^2 = 1.$$

Put $z = y'$ so that the equation becomes

$$z' = z^2 + 1,$$

The variables are separated and the integral is $\arctan z = x + C_1$, i.e.,

$$z(x, C_1) = \tan(x + C_1).$$

Integrating once again,

$$\begin{aligned} y(x; C_1, C_2) &= \int \tan(x + C_1)\,\mathrm{d}x \\ &= \int \frac{\sin(x + C_1)}{\cos(x + C_1)}\,\mathrm{d}x \\ &= -\log\big(\cos(x + C_1)\big) + C_2\,, \qquad C_1, C_2 \in \mathbb{R}\,. \end{aligned}$$

□

11.3 Initial value problems for equations of the first order

Hitherto we have surveyed families of differential equations of the first order, and shown ways to express the general solution in terms of indefinite integrals of known functions. These examples do not exhaust the class of equations which can be solved analytically, and various other devices have been developed to furnish exact solutions to equations with particularly interesting applications. That said, analytical tools are not available for any conceivable equation, and even when so, they might be unpractical. In these cases it is necessary to adopt approximations, often numerical ones. Most of the times one can really only hope to approximate an integral stemming, for instance, from an initial value problem. The use of such techniques must in any case follow a qualitative investigation of the ODE, to make sure at least that a solution exists. A qualitative study of this kind has its own interest, regardless of subsequent approximations, for it allows to understand in which way the solution of an initial value problem depends upon the initial datum, among other things.

Let us analyse the problem (11.9) and talk about a simple constraint on f that has a series of consequences: in the first place it guarantees that the problem admits a solution in a neighbourhood of x_0; secondly, that such solution is unique, and thirdly, that the latter depends on y_0 with continuity. Should all this happen, we say that the initial value problem (11.9) is **well posed** (in the sense of Hadamard).

11.3.1 Lipschitz functions

Before getting going, we present a remarkable way in which functions can depend on their variables.

Definition 11.10 *A real-valued map of one real variable $f : J \to \mathbb{R}$, J interval, is said* **Lipschitz continuous** *on J if there exists a constant $L \geq 0$ such that*

$$|f(y_1) - f(y_2)| \leq L|y_1 - y_2|\,, \qquad \forall y_1, y_2 \in J\,. \tag{11.24}$$

Another way to write the same is

$$\frac{|f(y_1) - f(y_2)|}{|y_1 - y_2|} \leq L\,, \qquad \forall y_1, y_2 \in J\,,\ y_1 \neq y_2\,, \tag{11.25}$$

which means the difference quotient of f is bounded as $y_1 \neq y_2$ vary in J.

If (11.24) holds for a certain constant L, it is valid for bigger numbers too. The smallest constant fulfilling (11.24) is called **Lipschitz constant** of f on J. The Lipschitz constant is nothing else but the supremum of the left-hand side of (11.25), when the variables vary in J. This number is far from being easy to determine, but normally one makes do with an approximation from above.

A Lipschitz-continuous map on J is necessarily continuous everywhere on J (actually, it is uniformly continuous on J, according to the definition given in Appendix A.3.3, p. 447), for condition (3.6) works with $\delta = \varepsilon/L$. Continuous maps that fail (11.25) do exist nevertheless, like $f(y) = \sqrt{y}$ over $J = [0, +\infty)$; choosing $y_2 = 0$ we have

$$\frac{|f(y_1) - f(y_2)|}{|y_1 - y_2|} = \frac{\sqrt{y_1}}{y_1} = \frac{1}{\sqrt{y_1}}\,, \qquad \forall y_1 > 0\,,$$

and in the limit for $y_1 \to 0$ the ratio on the left exceeds any constant. Note that the function has infinite (backward) derivative at $y = 0$.

The forthcoming result is the quickest to adopt, among those testing Lipschitz continuity.

Proposition 11.11 *Let $f : J \to \mathbb{R}$ be differentiable on J with bounded derivative, and $L = \sup\limits_{y \in J} |f'(y)| < +\infty$. Then f is Lipschitz continuous on J with Lipschitz constant L.*

Proof. For (11.24) it is enough to employ the second formula of the finite increment (6.13) to f between y_1, y_2, so that

$$f(y_1) - f(y_2) = f'(\bar{y})(y_1 - y_2)$$

for some $\bar{y}$ between y_1 and y_2. Therefore

$$|f(y_1) - f(y_2)| = |f'(\bar{y})|\,|y_1 - y_2| \leq L|y_1 - y_2|\,.$$

This proves the Lipschitz constant L^* of f is $\leq L$.

Vice versa, take any $y_0 \in J$. By (11.25)

$$\left| \frac{f(y) - f(y_0)}{y - y_0} \right| \le L^* , \qquad \forall y \in J ,$$

so

$$|f'(y_0)| = \left| \lim_{y \to y_0} \frac{f(y) - f(y_0)}{y - y_0} \right| = \lim_{y \to y_0} \left| \frac{f(y) - f(y_0)}{y - y_0} \right| \le L^* ,$$

and then $L \le L^*$. □

Let us see some examples of Lipschitz-continuous maps.

Examples 11.12

i) The function $f(y) = \sqrt{y}$ is Lipschitz continuous on every interval $[a, +\infty)$ with $a > 0$, because

$$0 < f'(y) = \frac{1}{\sqrt{2y}} \le \frac{1}{\sqrt{2a}}$$

on said intervals; the Lipschitz constant is $L = \frac{1}{\sqrt{2a}}$.

ii) The trigonometric maps $f(y) = \sin y$, $f(y) = \cos y$ are Lipschitz continuous on the whole $\mathbb{R}$ with $L = 1$, since $|f'(y)| \le 1$, $\forall y \in \mathbb{R}$ and there exist $y \in \mathbb{R}$ at which $|f'(y)| = 1$.

ii) The exponential $f(y) = \mathrm{e}^y$ is Lipschitz continuous on all intervals $(-\infty, b]$, $b \in \mathbb{R}$, with constant $L = \mathrm{e}^b$; it is not globally Lipschitz continuous, for $\sup\limits_{y \in \mathbb{R}} f'(y) = +\infty$. □

Proposition 11.11 gives a sufficient condition for Lipschitz continuity. A function can in fact be Lipschitz continuous on an interval without being differentiable: $f(y) = |y|$ is not differentiable at the origin, yet has Lipschitz constant 1 everywhere on $\mathbb{R}$, because

$$\big| |y_1| - |y_2| \big| \le |y_1 - y_2| , \qquad \forall y_1, y_2 \in \mathbb{R} .$$

Now to several variables. A function $f : \Omega \subseteq \mathbb{R}^d \to \mathbb{R}$ is **Lipschitz continuous** on Ω if there is a constant $L \ge 0$ such that

$$|f(\boldsymbol{y}_1) - f(\boldsymbol{y}_2)| \le L \|\boldsymbol{y}_1 - \boldsymbol{y}_2\| , \qquad \forall \boldsymbol{y}_1, \boldsymbol{y}_2 \in \Omega .$$

We say a map $f : I \times J \subseteq \mathbb{R}^2 \to \mathbb{R}$, with I, J real intervals, is **Lipschitz continuous on $\Omega = I \times J$ in y, uniformly in** x, if there is a constant $L \ge 0$ such that

$$|f(x, y_1) - f(x, y_2)| \le L |y_1 - y_2| , \qquad \forall y_1, y_2 \in J, \forall x \in I . \tag{11.26}$$

This condition holds if f has bounded partial y-derivative on Ω, i.e., $L = \sup\limits_{(x,y) \in \Omega} \left| \frac{\partial f}{\partial y}(x, y) \right| < +\infty$, because Proposition 11.11 can be applied for every $x \in I$.

Example 11.13

Consider

$$f(x,y) = \sqrt[3]{x}\sin(x+y)$$

on $\Omega = [-8,8] \times \mathbb{R}$. Since

$$\frac{\partial f}{\partial y}(x,y) = \sqrt[3]{x}\cos(x+y)\,,$$

for any $(x,y) \in \Omega$

$$\left|\frac{\partial f}{\partial y}(x,y)\right| = |\sqrt[3]{x}\,|\,|\cos(x+y)| \le \sqrt[3]{8}\cdot 1 = 2\,.$$

Thus (11.26) holds with $L = 2$. □

11.3.2 A criterion for solving initial value problems

After the intermezzo on Lipschitz-continuous functions, we are ready to state the main result concerning the initial value problem (11.9).

Theorem 11.14 *Let I, J be non-empty real intervals, J additionally open. Suppose $f : \Omega = I \times J \subseteq \mathbb{R}^2 \to \mathbb{R}$ is continuous on Ω and Lipschitz continuous on Ω in y, uniformly in x.*
For any $(x_0, y_0) \in \Omega$, the initial value problem (11.9) *admits one, and only one, solution $y = y(x)$, defined and differentiable with continuity on an interval $I' \subseteq I$ containing x_0 and bigger than a singlet, such that $\big(x, y(x)\big) \in \Omega$ for any $x \in I'$.*
If $\tilde{y} = \tilde{y}(x)$ denotes the solution on an interval $I'' \subseteq I$ to the problem with initial value $(x_0, \tilde{y}_0) \in \Omega$, then

$$|y(x) - \tilde{y}(x)| \le \mathrm{e}^{L|x-x_0|}|y_0 - \tilde{y}_0|\,, \qquad \forall x \in I' \cap I''\,, \tag{11.27}$$

where L is the constant of (11.26).

The theorem ensures existence and uniqueness of a "local" solution, a solution defined in a neighbourhood of x_0. The point is, the solution might be defined not everywhere on I, because the integral curve $\big(x, y(x)\big)$, also known as trajectory, could leave the region Ω before x has run over the entire I. For example, $f(y) = y^2$ is Lipschitz continuous on every bounded interval $J_a = (-a,a)$, $a > 0$, because

$$\sup_{y\in J_a} |f'(y)| = \sup_{|y|<a} |2y| = 2a\,,$$

but is not Lipschitz continuous on $\mathbb{R}$. The initial value problem

$$\begin{cases} y' = y^2\,, \\ y(0) = \dfrac{1}{2}\,, \end{cases} \tag{11.28}$$

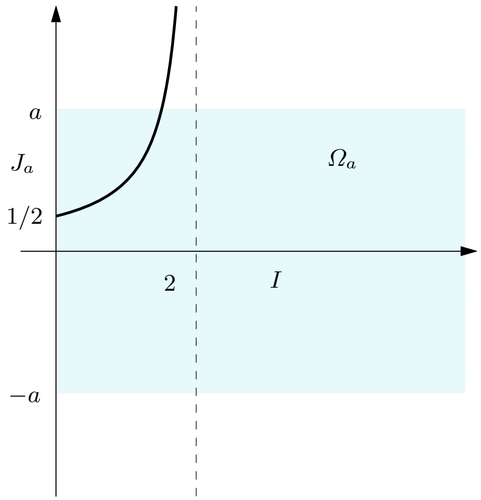


Figure 11.3. The solution of (11.28) is not defined on $I = [0, +\infty)$

has no solution over all of $I = [0, +\infty)$: separating variables we discover

$$y(x) = \frac{1}{2-x},$$

showing that the trajectory $\big(x, y(x)\big)$ leaves every strip $\Omega_a = I \times J_a$, $a > 1$, before x can reach 2 (see Fig. 11.3).

When the theorem is true with $J = \mathbb{R}$, we can prove the solution exists over all of I.

The uniqueness of the solution to (11.9) follows immediately from (11.27): if $y(x)$ and $\tilde{y}(x)$ are solutions corresponding to the same initial datum $y_0 = \tilde{y}_0$ at x_0, then $y(x) = \tilde{y}(x)$ for any x.

Observe that if f is not Lipschitz continuous in the second variable around (x_0, y_0), the initial value problem may have many solutions. The problem

$$\begin{cases} y' = \sqrt{y}\,, \\ y(0) = 0 \end{cases}$$

is solvable by separation of variables, and admits the constant $y(x) = 0$ (the singular integral), as well as $y(x) = \frac{1}{4}x^2$ as solutions. As a matter of fact there are infinitely many solutions

$$y(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq c\,, \\ \frac{1}{4}(x-c)^2 & \text{if } x > c\,, \end{cases} \qquad c \geq 0\,,$$

obtained by 'gluing' in the right way the aforementioned integrals.

Finally, (11.27) expresses the continuous dependency of the solution to (11.9) upon y_0: an ε-deviation of the initial datum affects at most by $\mathrm{e}^{L|x-x_0|}\varepsilon$ the solution at $x \neq x_0$. Otherwise said, when two solutions evolve the distance of the corresponding trajectories can grow at most by the factor $\mathrm{e}^{L|x-x_0|}$ in going from x_0 to x. In any case the factor $\mathrm{e}^{L|x-x_0|}\varepsilon$ is an exponential in x, so its impact depends on the distance $|x-x_0|$ and on the Lipschitz constant.

11.4 Linear second order equations with constant coefficients

A linear equation of order two with constant coefficients has the form

$$y'' + ay' + by = g, \tag{11.29}$$

where a, b are real constants and $g = g(x)$ is a continuous map. We shall prove that the general integral can be computed without too big an effort in case $g = 0$, hence when the equation is **homogeneous**. We will show, moreover, how to find the explicit solutions when g is a product of exponentials, algebraic polynomials, sine- and cosine-type functions or, in general, a sum of these.

To study equation (11.29) we let the map $y = y(x)$ be complex-valued, for convenience. The function $y : I \subseteq \mathbb{R} \to \mathbb{C}$ is (n times) differentiable if $y_r = \mathcal{R}e\, y : I \to \mathbb{R}$ and $y_i = \mathcal{I}m\, y : I \to \mathbb{R}$ are (n times) differentiable, in which case $y^{(n)}(x) = y_r^{(n)}(x) + i y_i^{(n)}(x)$.

A special case of this situation goes as follows. Let $\lambda = \lambda_r + i\lambda_i \in \mathbb{C}$ be an arbitrary complex number. With (8.39) in mind, we consider the complex-valued map of one real variable $x \mapsto \mathrm{e}^{\lambda x} = \mathrm{e}^{\lambda_r x}(\cos\lambda_i x + i\sin\lambda_i x)$. Then

$$\frac{\mathrm{d}}{\mathrm{d}x}\,\mathrm{e}^{\lambda x} = \lambda \mathrm{e}^{\lambda x}, \tag{11.30}$$

precisely as if λ were real. In fact,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\,\mathrm{e}^{\lambda x} &= \frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^{\lambda_r x}\cos\lambda_i x) + i\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^{\lambda_r x}\sin\lambda_i x) \\
&= \lambda_r \mathrm{e}^{\lambda_r x}\cos\lambda_i x - \lambda_i \mathrm{e}^{\lambda_r x}\sin\lambda_i x + i(\lambda_r \mathrm{e}^{\lambda_r x}\sin\lambda_i x + \lambda_i \mathrm{e}^{\lambda_r x}\cos\lambda_i x) \\
&= \lambda_r \mathrm{e}^{\lambda_r x}(\cos\lambda_i x + i\sin\lambda_i x) + i\lambda_i \mathrm{e}^{\lambda_r x}(\cos\lambda_i x + i\lambda_i\sin\lambda_i x) \\
&= (\lambda_r + i\lambda_i)\mathrm{e}^{\lambda x} = \lambda \mathrm{e}^{\lambda x}\,.
\end{aligned}$$

Let us indicate by $\mathcal{L}y = y'' + ay' + by$ the left-hand side of (11.29). Differentiating is a linear operation, so

$$\mathcal{L}(\alpha y + \beta z) = \alpha\mathcal{L}y + \beta\mathcal{L}z \tag{11.31}$$

for any $\alpha, \beta \in \mathbb{R}$ and any twice-differentiable real functions $y = y(x)$, $z = z(x)$. Furthermore, the result holds also for $\alpha, \beta \in \mathbb{C}$ and $y = y(x)$, $z = z(x)$ complex-valued. This sort of linearity of the differential equation will be crucial in the study.

We are ready to tackle (11.29). Let us begin with the homogeneous case

$$\mathcal{L}y = y'' + ay' + by = 0\,, \tag{11.32}$$

and denote by

$$\chi(\lambda) = \lambda^2 + a\lambda + b$$

the **characteristic polynomial** of the differential equation, obtained by replacing kth derivatives by the power λ^k, for every $k \geq 0$. Equation (11.30) suggests to look for a solution of the form $y(x) = \mathrm{e}^{\lambda x}$ for a suitable λ. If we do so,

$$\mathcal{L}(\mathrm{e}^{\lambda x}) = \lambda^2 \mathrm{e}^{\lambda x} + a\lambda \mathrm{e}^{\lambda x} + b\mathrm{e}^{\lambda x} = \chi(\lambda)\mathrm{e}^{\lambda x},$$

and the equation holds if and only if λ is a root of the **characteristic equation**

$$\lambda^2 + a\lambda + b = 0\,.$$

When the discriminant $\Delta = a^2 - 4b$ is non-zero, there are two distinct roots λ_1, λ_2, to whom correspond distinct solutions $y_1(x) = \mathrm{e}^{\lambda_1 x}$ and $y_2(x) = \mathrm{e}^{\lambda_2 x}$; roots and relative solutions are real if $\Delta > 0$, complex-conjugate if $\Delta < 0$. When $\Delta = 0$, there is a double root λ, hence one solution $y_1(x) = \mathrm{e}^{\lambda x}$. Multiplicity two implies $\chi'(\lambda) = 0$; letting $y_2(x) = x\mathrm{e}^{\lambda x}$, we have

$$y_2(x) = (1 + \lambda x)\,\mathrm{e}^{\lambda x} \qquad \text{and} \qquad y_2''(x) = (2\lambda + \lambda^2 x)\,\mathrm{e}^{\lambda x}\,.$$

Substituting back into the equation we obtain

$$\mathcal{L}(y_2) = \chi(\lambda)\,x\,\mathrm{e}^{\lambda x} + \chi'(\lambda)\,\mathrm{e}^{\lambda x} = 0$$

after a few algebraic steps. Therefore the function y_2 solves the equation, and is other than y_1. In all cases, we have found two distinct solutions y_1, y_2 of (11.32).

Since (11.31) is linear, if y_1, y_2 solve (11.32) and C_1, C_2 are constants, then

$$\mathcal{L}(C_1 y_1 + C_2 y_2) = C_1 \mathcal{L}(y_1) + C_2 \mathcal{L}(y_2) = C_1 0 + C_2 0 = 0\,,$$

hence the linear combination $C_1 y_1 + C_2 y_2$ is yet another solution of the homogeneous equation. Moreover, if y denotes a solution, one can prove that there exist two constants C_1, C_2 such that $y = C_1 y_1 + C_2 y_2$, where y_1, y_2 are the solutions found earlier.

In conclusion, the general integral of the homogeneous equation (11.32) takes the form

$$y(x; C_1, C_2) = C_1 y_1(x) + C_2\, y_2(x),$$

with C_1, C_2 constants and $y_1(x)$, $y_2(x)$ defined by the recipe:

if $\Delta \neq 0,\ y_1(x) = \mathrm{e}^{\lambda_1 x}$ and $y_2(x) = \mathrm{e}^{\lambda_2 x}$ with λ_1, λ_2 distinct roots of the characteristic equation $\chi(\lambda) = 0$;

if $\Delta = 0,\ y_1(x) = \mathrm{e}^{\lambda x}$ and $y_2(x) = x\mathrm{e}^{\lambda x}$, where λ is the double root of $\chi(\lambda) = 0$.

When $\Delta < 0$, the solution can be written using real functions, instead of complex-conjugate ones as above. It is enough to substitute to $y_1(x)$, $y_2(x)$ the real part $e^{\lambda_r x} \cos \lambda_i x$ and the imaginary part $e^{\lambda_r x} \sin \lambda_i x$ of $y_1(x)$ respectively, where $\lambda_1 = \bar{\lambda}_2 = \lambda_r + i\lambda_i$. In fact, if y is a solution of the homogeneous equation,

$$\mathcal{L}(\mathcal{R}e\, y) = \mathcal{R}e\,(\mathcal{L}y) = \mathcal{R}e\, 0 = 0\,, \qquad \mathcal{L}(\mathcal{I}m\, y) = \mathcal{I}m\,(\mathcal{L}y) = \mathcal{I}m\, 0 = 0$$

since the coefficients are real, so $\mathcal{R}e\, y$ and $\mathcal{I}m\, y$ are solutions too.

Summarising, the general integral of the homogeneous equation (11.32) can be expressed in terms of real functions as follows.

The case $\Delta > 0$. The characteristic equation has two distinct real roots

$$\lambda_{1,2} = \frac{-a \pm \sqrt{\Delta}}{2}$$

and the general integral reads

$$y(x; C_1, C_2) = C_1\, e^{\lambda_1 x} + C_2\, e^{\lambda_2 x},$$

with C_1, C_2 arbitrary constants.

The case $\Delta = 0$. The characteristic equation has a double root

$$\lambda = -\frac{a}{2}\,,$$

and the general integral reads

$$y(x; C_1, C_2) = (C_1 + C_2 x)\, e^{\lambda x}\,, \qquad\qquad C_1,\ C_2 \in \mathbb{R}\,.$$

The case $\Delta < 0$. The characteristic equation has no real roots. Defining

$$\sigma = \lambda_r = -\frac{a}{2}, \qquad\qquad \omega = \lambda_i = \frac{\sqrt{|\Delta|}}{2},$$

the general integral reads

$$y(x; C_1, C_2) = e^{\sigma x}(C_1 \cos \omega x + C_2 \sin \omega x)\,, \qquad\qquad C_1,\ C_2 \in \mathbb{R}.$$

Now we are ready for the non-homogeneous equation (11.29). The general integral can be written like

$$y(x; C_1, C_2) = y_0(x; C_1, C_2) + y_p(x), \tag{11.33}$$

where $y_0(x; C_1, C_2)$ is the general solution of the associated homogeneous equation (11.32), while $y_p(x)$ denotes an arbitrary particular integral of (11.29). Based on linearity in fact,

$$\mathcal{L}(y_0 + y_p) = \mathcal{L}(y_0) + \mathcal{L}(y_p) = 0 + g = g\,,$$

so the right-hand side of (11.33) solves (11.29). Vice versa, if $y(x)$ is a generic solution of (11.29), the function $y(x) - y_p(x)$ satisfies

$$\mathcal{L}(y - y_p) = \mathcal{L}(y) - \mathcal{L}(y_p) = g - g = 0\,,$$

so it will be of the form $y_0(x; C_1, C_2)$ for some C_1 and C_2.

Should the source term g be a mixture of products of algebraic polynomials, trigonometric and exponentials functions, we can find a particular integral of the same sort. To understand better, we start with $g(x) = p_n(x)\,\mathrm{e}^{\alpha x}$, where $\alpha \in \mathbb{C}$ and $p_n(x)$ is a polynomial of degree $n \geq 0$. We look for a particular solution of the form $y_p(x) = q_N(x)\,\mathrm{e}^{\alpha x}$, with q_N unknown polynomial of degree $N \geq n$. Substituting the latter and its derivatives in the equation, we obtain

$$\mathcal{L}(q_N(x)\,\mathrm{e}^{\alpha x}) = \big(\chi(\alpha)q_N(x) + \chi'(\alpha)q_N'(x) + q_N''(x)\big)\,\mathrm{e}^{\alpha x} = p_n(x)\,\mathrm{e}^{\alpha x}\,,$$

whence

$$\chi(\alpha)q_N(x) + \chi'(\alpha)q_N'(x) + q_N''(x) = p_n(x)\,.$$

If α is not a characteristic root, it suffices to choose $N = n$ and determine the unknown coefficients of q_n by comparing the polynomials on either side of the equation; it is better to begin from the leading term and proceed to the lower-degree monomials.

If α is a simple root, $\chi(\alpha) = 0$ and $\chi'(\alpha) \neq 0$; we choose $N = n + 1$ and hunt for a polynomial solution of $\chi'(\alpha)q_N'(x) + q_N''(x) = p_n(x)$. Since the coefficient of q_{n+1} of degree 0 is not involved in the expression, we limit ourselves to q_{n+1} of the form $q_{n+1}(x) = xq_n(x)$, with q_n an arbitrary polynomial of degree n.

Eventually, if α is a multiple root, we put $N = n+2$ and solve $q_{n+2}''(x) = p_n(x)$, seeking q_{n+2} in the form $q_{n+2}(x) = x^2 q_n(x)$, where q_n is arbitrary and of degree n. In the second and third cases one speaks of **resonance**.

When α is complex, $\chi(\alpha)$ and $\chi'(\alpha)$ are complex expressions, so $q_N(x)$ has to be found among polynomials over $\mathbb{C}$, generally speaking. But as in the homogeneous case, we can eschew complex variables by inspecting the real and imaginary parts of $p_n(x)\,\mathrm{e}^{\alpha x}$; with $\alpha = \mu + i\vartheta$, they are $p_n(x)\,\mathrm{e}^{\mu x}\cos\vartheta x$ and $p_n(x)\,\mathrm{e}^{\mu x}\sin\vartheta x$.

Our analysis has shown that if the source term g is real and of the form

$$g(x) = p_n(x)\,\mathrm{e}^{\mu x}\cos\vartheta x \qquad \text{or} \qquad g(x) = p_n(x)\,\mathrm{e}^{\mu x}\sin\vartheta x\,, \tag{11.34}$$

we can attempt to find a particular solution

$$y_p(x) = x^m \mathrm{e}^{\mu x}(q_{1,n}(x)\cos\vartheta x + q_{2,n}(x)\sin\vartheta x), \tag{11.35}$$

where $q_{i,n}(x)$ are algebraic polynomials of degree n, and m is generically 0 except in case of *resonance*:

i) for $\Delta > 0$: set $m = 1$ if $\vartheta = 0$ and if μ coincides with either root λ_1, λ_2 of the characteristic polynomial;

ii) for $\Delta = 0$: set $m = 2$ if $\vartheta = 0$ and μ coincides with the (double) root λ of the characteristic polynomial;

iii) for $\Delta < 0$: set $m = 1$ if $\mu = \sigma$ and $\vartheta = \omega$.

Substituting the particular integral (11.35) in (11.29), and comparing the terms $x^k e^{\mu x} \sin \vartheta x$ and $x^k e^{\mu x} \cos \vartheta x$ for all $k = 0, \ldots, n$, we can determine y_p.

At last, if g is a sum of pieces of the form (11.34), y_p will be the sum of the particular solutions corresponding to the single source terms: suppose that $g = g_1 + g_2 + \ldots + g_K$ and y_{pk} solves $\mathcal{L}(y) = g_k$ for all $k = 1, \ldots, K$. Then $y_p = y_{p1} + \ldots + y_{pK}$ satisfies

$$\mathcal{L}(y_p) = \mathcal{L}(y_{p1}) + \ldots + \mathcal{L}(y_{pK}) = g_1 + \ldots + g_K = g\,,$$

and as such it solves $\mathcal{L}(y) = g$ as well. This is the so-called *principle of superposition.*

With the help of a few examples the procedure will result much clearer.

Examples 11.15

i) Consider

$$y'' + y' - 6y = g. \tag{11.36}$$

First of all, we find the general integral of the associated homogeneous equation

$$y'' + y' - 6y = 0. \tag{11.37}$$

The characteristic equation

$$\lambda^2 + \lambda - 6 = 0$$

has distinct roots $\lambda_1 = -3$, $\lambda_2 = 2$, so the general integral of (11.37) is

$$y_0(x; C_1, C_2) = C_1\, e^{-3x} + C_2\, e^{2x}.$$

Now we determine a particular solution to (11.36), assuming that $g(x) = 3x^2 - x + 2$. By (11.34), $p_2(x) = 3x^2 - x + 2$ and $\mu = \vartheta = 0$. Since μ is neither λ_1 nor λ_2, y_p will have the form $y_p(x) = \alpha x^2 + \beta x + \gamma$. Substituting y_p', y_p'' in (11.36) yields

$$-6\alpha x^2 + (2\alpha - 6\beta)x + (2\alpha + \beta - 6\gamma) = 3x^2 - x + 2.$$

The comparison of coefficients implies

$$y_p(x) = -\frac{1}{2}(x^2 + 1).$$

Therefore, the general integral of (11.36) reads

$$y(x; C_1, C_2) = C_1\, e^{-3x} + C_2\, e^{2x} - \frac{1}{2}(x^2 + 1).$$

Assume instead that $g(x) = \mathrm{e}^{2x}$. In (11.34) we have $p_0(x) = 1$, $\mu = \lambda_2 = 2$, $\vartheta = 0$. We need a y_p written as $y_p(x) = \alpha x \mathrm{e}^{2x}$. Substituting in (11.36) gives

$$5\alpha \mathrm{e}^{2x} = \mathrm{e}^{2x}\,,$$

hence $\alpha = \frac{1}{5}$. The general solution is then

$$y(x; C_1, C_2) = C_1 \mathrm{e}^{-3x} + \left(C_2 + \frac{1}{5}x\right)\mathrm{e}^{2x}.$$

ii) Examine the equation

$$y'' - 2y' + y = g. \tag{11.38}$$

The characteristic polynomial $\lambda^2 - 2\lambda + 1$ has a root $\lambda = 1$ of multiplicity two. The general integral of the homogeneous equation is thus

$$y_0(x; C_1, C_2) = (C_1 + C_2 x)\,\mathrm{e}^x.$$

Suppose $g(x) = x\mathrm{e}^{3x}$. As $\mu = 3$ is not $\lambda = 1$, we search for a particular solution $y_p(x) = (\alpha x + \beta)\,\mathrm{e}^{3x}$. As before, the substitution of the latter back into the equation yields

$$4(\alpha x + \alpha + \beta)\,\mathrm{e}^{3x} = x\,\mathrm{e}^{3x},$$

giving

$$y_p(x) = \frac{1}{4}(x-1)\,\mathrm{e}^{3x}.$$

We conclude that the general integral is

$$y(x; C_1, C_2) = (C_1 + C_2 x)\,\mathrm{e}^x + \frac{1}{4}(x-1)\,\mathrm{e}^{3x}.$$

Taking $g(x) = -4\mathrm{e}^x$, instead, calls for a y_p of type $y_p(x) = \alpha x^2 \mathrm{e}^x$. Then

$$2\alpha \mathrm{e}^x = -4\mathrm{e}^x$$

implies $\alpha = -2$, and the general solution reads

$$y(x; C_1, C_2) = (C_1 + C_2 x - 2x^2)\,\mathrm{e}^x\,.$$

iii) The last example is

$$y'' + 2y' + 5y = g. \tag{11.39}$$

This ODE has characteristic equation $\lambda^2 + 2\lambda + 5 = 0$ with negative discriminant $\Delta = -16$. From $\sigma = -1$, $\omega = 2$, the general integral of the homogeneous equation is

$$y_0(x; C_1, C_2) = \mathrm{e}^{-x}(C_1 \cos 2x + C_2 \sin 2x).$$

Take $g(x) = \sin x$. Referring to the left term in (11.34), we have $p_0(x) = 1$, $\mu = 0$, $\vartheta = 1$. We want a particular integral $y_p(x) = \alpha\cos x + \beta \sin x$. Rewrite (11.39) using the derivatives y_p', y_p'' and y_p, so that

$$(4\alpha + 2\beta)\cos x + (4\beta - 2\alpha)\sin x = \sin x.$$

Compare the coefficients of $\sin x$ and $\cos x$, so $\alpha = -\frac{1}{10}$ and $\beta = \frac{1}{5}$, i.e.,

$$y_p(x) = -\frac{1}{10}\cos x + \frac{1}{5}\sin x.$$

The general solution is

$$y(x) = \mathrm{e}^{-x}(C_1 \cos 2x + C_2 \sin 2x) - \frac{1}{10}\cos x + \frac{1}{5}\sin x\,.$$

Another variant is to suppose $g(x) = \mathrm{e}^{-x}\sin 2x$. Using the first of (11.34), $\mu = \sigma = -1$ and $\vartheta = \omega = 2$, so we look for the particular integral $y_p(x) = x\mathrm{e}^{-x}(\alpha\cos 2x + \beta\sin 2x)$. The substitution yields

$$\mathrm{e}^{-x}(4\beta\cos 2x - 4\alpha\sin 2x) = \mathrm{e}^{-x}\sin 2x,$$

hence $\alpha = -\frac{1}{4}$, $\beta = 0$, and the general solution reads

$$y(x) = \mathrm{e}^{-x}\left(\left(C_1 - \frac{1}{4}x\right)\cos 2x + C_2 \sin 2x\right). \qquad \square$$

11.5 Exercises

1. *Determine the general integral of these ODEs with separable variables:*

   a) $y' = x\log(1+x^2)$ $\boxed{\text{b)}}$ $y' = \dfrac{(x+2)y}{x(x+1)}$

   $\boxed{\text{c)}}$ $y' = \dfrac{y^2}{x\log x} - \dfrac{1}{x\log x}$ d) $y' = \sqrt[3]{2y+3}\,\tan^2 x$

2. *Find the general solution of the homogeneous ODEs:*

   $\boxed{\text{a)}}$ $4x^2y' = y^2 + 6xy - 3x^2$ b) $x^2y' = x^2 + 4y^2 + yx$

   c) $xyy' = x^2 + y^2$ $\boxed{\text{d)}}$ $x^2y' - y^2\mathrm{e}^{x/y} = xy$

3. *Solve in full generality the linear ODEs:*

   a) $y' + 3xy = x^3$ $\boxed{\text{b)}}$ $y' = \dfrac{1}{x}y - \dfrac{3x+2}{x^3}$

   $\boxed{\text{c)}}$ $y' = \dfrac{2x-y}{x-1}$ d) $xy' = y + \dfrac{2x^2}{1+x^2}$

$\boxed{4.}$ *Write the particular solution of the equation*

$$y' = \frac{1-\mathrm{e}^{-y}}{2x+1}$$

such that $y(0) = 1$.

$\boxed{5.}$ *Establish whether the differential equation*

$$y' = -2y + \mathrm{e}^{-2x}$$

has solutions with vanishing derivative at the origin.

6. *Solve, over the ray* $[\sqrt[4]{\mathrm{e}}, +\infty)$, *the initial value problem*

$$\begin{cases} \mathrm{e}^y y' = 4x^3 \log x(1 + \mathrm{e}^y) \\ y(\sqrt[4]{\mathrm{e}}) = 0. \end{cases}$$

7. *Find the solutions of*

$$\begin{cases} y' = \dfrac{3x}{x^2 - 4}|y| \\ y(0) = -1 \end{cases}$$

that are defined on the interval $(-2, 2)$.

8. *Determine the general integral of*

$$y' \sin 2x - 2(y + \cos x) = 0, \qquad x \in \left(0, \frac{\pi}{2}\right),$$

and indicate the solution that stays bounded when $x \to \frac{\pi}{2}^-$.

9. *For* $\alpha \in \mathbb{R}$, *solve the ODE*

$$y' = (2 + \alpha)y - 2\mathrm{e}^{\alpha x}$$

so that $y(0) = 3$. *Tell which values of* α *make the improper integral* $\displaystyle\int_0^{+\infty} y(x)\,\mathrm{d}x$ *converge.*

10. *Let* a, b *be real numbers. Solve the initial value problem*

$$\begin{cases} y' = a\dfrac{y}{x} + 3x^b \\ y(2) = 1 \end{cases}$$

restricted to the half-line $[2, +\infty)$.

11. *Consider the parametric differential equation*

$$y'(x) = -3xy(x) + kx$$

depending on $k \in \mathbb{R}$.
a) *Find the solution with a zero at the origin.*
b) *For such solution* y *determine* k *so that* $y(x) \sim x^2$ *as* $x \to 0$.

12. *Given the ODE*

$$y' = \frac{y^2 - 2y - 3}{2(1 + 4x)},$$

determine:
a) *the general integral;*
b) *the particular integral* $y_0(x)$ *satisfying* $y_0(0) = 1$;
c) *Maclaurin's expansion of* $y_0(x)$ *up to second order.*

13. *Work out the general solution to the following second order ODEs by reducing them to the first order:*

a) $y'' = 2\mathrm{e}^x$

$\boxed{\text{b)}}$ $y'' + y' - x^2 = 0$

14. *Compute the general integral of the linear ODEs of the second order:*

a) $y'' + 3y' + 2y = x^2 + 1$

$\boxed{\text{b)}}$ $y'' - 4y' + 4y = \mathrm{e}^{2x}$

$\boxed{\text{c)}}$ $y'' + y = 3\cos x$

d) $y'' - 3y' + 2y = \mathrm{e}^x$

$\boxed{\text{e)}}$ $y'' - 9y = \mathrm{e}^{-3x}$

f) $y'' - 2y' - 3y = \sin x$

15. *Solve the initial value problems:*

a) $\begin{cases} y'' + 2y' + 5y = 0 \\ y(0) = 0 \\ y'(0) = 2 \end{cases}$

$\boxed{\text{b)}}$ $\begin{cases} y'' - 5y' + 4y = 2x + 1 \\ y(0) = \frac{7}{8} \\ y'(0) = 0 \end{cases}$

11.5.1 Solutions

1. *ODEs with separable variables:*

a) $y = \frac{1}{2}(1 + x^2)\log(1 + x^2) - \frac{1}{2}x^2 + C$.

b) The map $h(y) = y$ has a zero at $y = 0$, which is consequently a singular integral. Suppose then $y \neq 0$ and separate the variables:

$$\int \frac{1}{y}\,\mathrm{d}y = \int \frac{x+2}{x(x+1)}\,\mathrm{d}x .$$

We compute the integral on the right by partial fractions:

$$\frac{x+2}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1} = \frac{2}{x} - \frac{1}{x+1}$$

implies

$$\begin{aligned}\int \frac{x+2}{x(x+1)}\,\mathrm{d}x &= \int \left(\frac{2}{x} - \frac{1}{x+1}\right)\mathrm{d}x = 2\log|x| - \log|x+1| + \log C \\ &= \log\frac{Cx^2}{|x+1|}, \qquad C > 0 .\end{aligned}$$

Thus

$$\log|y| = \log\frac{Cx^2}{|x+1|}, \qquad C > 0 ,$$

$$|y| = C\,\frac{x^2}{|x+1|}, \qquad C > 0 ,$$

$$y = C\,\frac{x^2}{x+1}\,, \qquad C \neq 0\,.$$

The singular integral $y = 0$ belongs to this family if we allow the value $C = 0$.

c) The problem requires $x > 0$ due to the presence of the logarithm. Rearranging the equation as

$$y' = \frac{y^2 - 1}{x \log x}$$

yields $h(y) = y^2 - 1$. Thus the constant maps $y = 1$, $y = -1$ are singular solutions. Let now $y \neq \pm 1$; separating the variables returns

$$\int \frac{1}{y^2 - 1}\,\mathrm{d}y = \int \frac{1}{x \log x}\,\mathrm{d}x\,.$$

The method of partial fractions in the left integral and the substitution $t = \log x$ on the right give

$$\frac{1}{2}\log\left|\frac{y-1}{y+1}\right| = \log|\log x| + \log C = \log C|\log x|\,, \qquad C > 0\,,$$

equivalent to

$$\log\left|\frac{y-1}{y+1}\right| = \log C \log^2 x\,, \qquad C > 0\,,$$

or

$$\frac{y-1}{y+1} = C\log^2 x\,, \qquad C \neq 0\,;$$

Altogether, the general integral is

$$y = \frac{1 + C\log^2 x}{1 - C\log^2 x}\,, \qquad C \in \mathbb{R}\,,$$

which includes the singular solution $y = 1$ for $C = 0$.

d) $y = -\frac{3}{2} \pm \frac{1}{2}\left[\frac{4}{3}(\tan x - x + C)\right]^{3/2}$ plus the constant solution $y = -\frac{3}{2}$.

2. *Homogeneous differential equations:*

a) Supposing $x \neq 0$ and dividing by $4x^2$ gives

$$y' = \frac{1}{4}\frac{y^2}{x^2} + \frac{3}{2}\frac{y}{x} - \frac{3}{4}\,.$$

By renaming $z = \frac{y}{x}$ we have $y' = z + xz'$, hence

$$z + xz' = \frac{1}{4}z^2 + \frac{3}{2}z - \frac{3}{4}\,,$$

$$4xz' = (z-1)(z+3)\,.$$

Because $\varphi(z) = (z-1)(z+3)$ has zeroes $z = 1$, $z = -3$, the maps $y = x$ $y = -3x$ are singular integrals. For the general solution let us separate the variables
$$\int \frac{4}{(z-1)(z+3)}\,\mathrm{d}z = \int \frac{1}{x}\,\mathrm{d}x\,.$$
Decomposing
$$\frac{4}{(z-1)(z+3)} = \frac{A}{z-1} + \frac{B}{z+3} = \frac{1}{z-1} - \frac{1}{z+3}\,,$$
the right-hand-side integral is
$$\int \frac{4}{(z-1)(z+3)}\,\mathrm{d}z = \int \left(\frac{1}{z-1} - \frac{1}{z+3}\right)\mathrm{d}z = \log\left|\frac{z-1}{z+3}\right| + c\,.$$
Therefore
$$\log\left|\frac{z-1}{z+3}\right| = \log C|x|\,, \qquad C > 0\,,$$
$$\frac{z-1}{z+3} = Cx\,, \qquad C \neq 0\,,$$
$$z = \frac{1+3Cx}{1-Cx}\,, \qquad C \in \mathbb{R}\,;$$
this incorporates the singular solution $z = 1$ as well. Returning to the variable y, the general integral reads
$$y = \frac{x+3Cx^2}{1-Cx}\,, \qquad C \in \mathbb{R}\,.$$

b) $y = \frac{1}{2}x\tan\left(2\log C|x|\right)\,,\ C > 0\,;$ c) $y = \pm x\sqrt{2\log C|x|}\,,\ C > 0\,.$

d) If $x \neq 0$ we divide by x^2
$$y' = \frac{y^2}{x^2}\mathrm{e}^{x/y} + \frac{y}{x}\,.$$
Changing $z = \frac{y}{x}$ gives $y' = z + xz'$, so
$$z + xz' = z^2\mathrm{e}^{1/z} + z\,,$$
whence
$$xz' = z^2\mathrm{e}^{1/z}\,.$$
The function $z = 0$, corresponding to $y = 0$, is a singular integral of the ODE. By separation of the variables
$$\int \frac{\mathrm{e}^{-1/z}}{z^2}\,\mathrm{d}z = \int \frac{1}{x}\,\mathrm{d}x\,,$$
integrating which,

$$e^{-1/z} = \log C|x|\,, \qquad C > 0\,,$$

$$-\frac{1}{z} = \log\log C|x|\,, \qquad C > 0\,,$$

$$z = -\frac{1}{\log\log C|x|}\,, \qquad C > 0\,,$$

i.e.,

$$y = -\frac{x}{\log\log C|x|}\,, \qquad C > 0\,.$$

in terms of y.

3. *Linear ODEs:*

a) $y = \frac{1}{3}\left(x^2 - \frac{2}{3}\right) + Ce^{-\frac{3}{2}x^2}$.

b) Using (11.18) with $a(x) = -\frac{1}{x}$ and $b(x) = -\frac{3x+2}{x^3}$ produces

$$\begin{aligned} y &= e^{\int \frac{1}{x}\,dx} \int e^{-\int \frac{1}{x}\,dx}\left(-\frac{3x+2}{x^3}\right) dx = e^{\log|x|} \int e^{\log\frac{1}{|x|}}\left(-\frac{3x+2}{x^3}\right) dx \\ &= |x| \int \frac{-(3x+2)}{|x|x^3}\,dx = x \int \frac{-(3x+2)}{xx^3}\,dx \\ &= x \int \left(-\frac{3}{x^3} - \frac{2}{x^4}\right) dx = x\left(\frac{3}{2x^2} + \frac{2}{3x^3} + C\right) \\ &= \frac{3}{2x} + \frac{2}{3x^2} + Cx\,, \qquad C \in \mathbb{R}\,. \end{aligned}$$

c) By writing

$$y' + \frac{1}{x-1}y = \frac{2x}{x-1}$$

we recognise formula (11.18) where $a(x) = \frac{1}{x-1}$, $b(x) = \frac{2x}{x-1}$. Then

$$\begin{aligned} y &= e^{-\int \frac{1}{x-1}\,dx} \int e^{\int \frac{1}{x-1}\,dx} \frac{2x}{x-1}\,dx = e^{-\log|x-1|} \int e^{\log|x-1|} \frac{2x}{x-1}\,dx \\ &= \frac{1}{|x-1|} \int |x-1| \frac{2x}{x-1}\,dx = \frac{1}{x-1} \int 2x\,dx = \frac{1}{x-1}(x^2 + C)\,, \quad C \in \mathbb{R}\,. \end{aligned}$$

d) $y = 2x \arctan x + Cx\,,\ C \in \mathbb{R}\,.$

4. The equation has separable variables, but the constant solution $y = 0$ is not admissible, for it does not meet the condition $y(0) = 1$. So we write

$$\int \frac{1}{1 - e^{-y}}\,dy = \int \frac{1}{2x+1}\,dx.$$

Renaming $t = e^{-y}$ (hence $dt = -e^{-y}dy$, $-\frac{1}{t}\,dt = dy$) implies

$$\int \frac{1}{1-\mathrm{e}^{-y}}\,\mathrm{d}y = \int \frac{1}{t(t-1)}\,\mathrm{d}t = \int \left(\frac{1}{t-1} - \frac{1}{t}\right)\mathrm{d}t$$
$$= \log\left|\frac{t-1}{t}\right| + c = \log\left|1-\frac{1}{t}\right| + c = \log|1-\mathrm{e}^{y}| + c.$$

Then

$$\log|1-\mathrm{e}^{y}| = \frac{1}{2}\log|2x+1| + \log C\ , \qquad C>0,$$
$$\log|1-\mathrm{e}^{y}| = \log C\sqrt{|2x+1|}\,, \qquad C>0,$$
$$|1-\mathrm{e}^{y}| = C\sqrt{|2x+1|}\,, \quad C>0\,,$$
$$1-\mathrm{e}^{y} = C\sqrt{|2x+1|}\,, \quad C\neq 0.$$

In conclusion, the general integral

$$y = \log\left(1 - C\sqrt{|2x+1|}\right)\,, \qquad C\in\mathbb{R}\,,$$

also takes into account $y=0$, corresponding to $C=0$.

Now to the condition $y(0)=1$: from $C=1-\mathrm{e}$ the required solution is

$$y = \log\left(1+(\mathrm{e}-1)\sqrt{|2x+1|}\right).$$

5. The general integral of the linear equation reads

$$y = \mathrm{e}^{-\int 2\,\mathrm{d}x}\int \mathrm{e}^{\int 2\,\mathrm{d}x}\mathrm{e}^{-2x}\,\mathrm{d}x = \mathrm{e}^{-2x}(x+C)\,, \qquad C\in\mathbb{R}\,.$$

The constraint is $y'(0)=0$. Putting $x=0$ in $y'(x)=-2y(x)+\mathrm{e}^{-2x}$, that becomes $y(0)=\frac{1}{2}$, and implies $C=\frac{1}{2}$. The final solution is thus

$$y = \mathrm{e}^{-2x}\left(x+\frac{1}{2}\right).$$

6. $y = \log\left(2\mathrm{e}^{x^4(\log x - \frac{1}{4})} - 1\right).$

7. When $x\in(-2,2)$, $x^2-4<0$. The initial condition $y(0)=-1$ allows moreover to restrict to $y(x)<0$ in a neighbourhood of $x=0$. That said, we separate variables:

$$-\int \frac{1}{y}\,\mathrm{d}y = \int \frac{3x}{x^2-4}\,\mathrm{d}x,$$
$$-\log|y| = -\log(-y) = \frac{3}{2}\log|x^2-4| + C, \qquad C\in\mathbb{R},$$
$$-\frac{1}{y} = C(4-x^2)^{3/2}, \qquad C>0,$$

$$y = C(4 - x^2)^{-3/2}, \qquad C < 0.$$

Since $y(0) = -1$, C must be -8 and the solution reads

$$y = -\frac{8}{(4 - x^2)^{3/2}}.$$

Notice that the constant function $y = 0$ was disregarded because it fails to meet $y(0) = -1$.

8. The duplication formula $\sin 2x = 2 \sin x \cos x$ bestows

$$y' \sin x \cos x = y + \cos x.$$

For $x \in \left(0, \frac{\pi}{2}\right)$ we have $\sin x \cos x \neq 0$, and so

$$y' = \frac{1}{\sin x \cos x}\, y + \frac{1}{\sin x}.$$

This is a linear equation, with integral

$$y = \mathrm{e}^{\int \frac{1}{\sin x \cos x}\, \mathrm{d}x} \int \mathrm{e}^{-\int \frac{1}{\sin x \cos x}\, \mathrm{d}x} \cdot \frac{1}{\sin x}\, \mathrm{d}x.$$

Let us compute

$$S = \int \frac{1}{\sin x \cos x}\, \mathrm{d}x$$

by setting $t = \sin x$ ($\mathrm{d}t = \cos x\, \mathrm{d}x$, $\cos^2 x = 1 - t^2$) and integrating the rational function thus obtained:

$$\begin{aligned} S &= \int \frac{1}{t(1-t^2)}\, \mathrm{d}t = \int \left(\frac{1}{t} + \frac{1}{2(1-t)} - \frac{1}{2(1+t)} \right) \mathrm{d}t \\ &= \log|t| - \frac{1}{2}|1-t| - \frac{1}{2}\log|1+t| + c \\ &= \log \frac{|t|}{\sqrt{|1-t^2|}} + c = \log \frac{\sin x}{\cos x} + c\,, \qquad x \in \left(0, \frac{\pi}{2}\right). \end{aligned}$$

Then

$$y = \frac{\sin x}{\cos x} \int \frac{\cos x}{\sin^2 x}\, \mathrm{d}x = \frac{\sin x}{\cos x} \left(-\frac{1}{\sin x} + C \right), \qquad C \in \mathbb{R}\,,$$

and the solution to the ODE is

$$y = \frac{C \sin x - 1}{\cos x}\,, \qquad C \in \mathbb{R}\,.$$

We need to find a bounded solution around $\frac{\pi}{2}^-$:

$$\lim_{x \to \frac{\pi}{2}^-} \frac{C \sin x - 1}{\cos x} \in \mathbb{R}.$$

But

$$\lim_{x\to\frac{\pi}{2}-} \frac{C\sin x - 1}{\cos x} = \lim_{t\to 0^-} \frac{1 - C\cos t}{\sin t} = \lim_{t\to 0^-} \frac{1 - C(1 + o(t^2))}{t + o(t^2)} = 0$$

if and only if $C = 1$, so the desired solution is

$$y = \frac{\sin x - 1}{\cos x}.$$

9. The equation is linear, so at once

$$\begin{aligned} y &= \mathrm{e}^{\int (2+\alpha)\,\mathrm{d}x} \int \mathrm{e}^{-\int (2+\alpha)\,\mathrm{d}x} (-2\mathrm{e}^{\alpha x})\,\mathrm{d}x \\ &= \mathrm{e}^{(2+\alpha)x}(\mathrm{e}^{-2x} + C) = \mathrm{e}^{\alpha x}(1 + C\,\mathrm{e}^{2x}), \quad C \in \mathbb{R}. \end{aligned}$$

From $y(0) = 3$ it follows $3 = 1 + C$, so $C = 2$. The solution we want is thus

$$y = \mathrm{e}^{\alpha x}(1 + 2\mathrm{e}^{2x}).$$

The improper integral

$$\int_0^{+\infty} (\mathrm{e}^{\alpha x} + 2\mathrm{e}^{(\alpha+2)x})\,\mathrm{d}x$$

converges precisely when the exponential of bigger magnitude has negative exponent. Therefore the integral converges if $\alpha < -2$.

10. Directly from the formula for linear equations,

$$\begin{aligned} y &= e^{a\int \frac{1}{x}\,\mathrm{d}x} \left(3 \int e^{-a\int \frac{1}{x}\,\mathrm{d}x} x^b\,\mathrm{d}x\right) = x^a \left(3 \int x^{b-a}\,\mathrm{d}x\right) \\ &= \begin{cases} x^a \left(\dfrac{3}{b-a+1}\, x^{b-a+1} + C\right) & \text{if } b-a \neq -1, \\ x^a\,(3\log x + C) & \text{if } b-a = -1, \end{cases} \\ &= \begin{cases} \dfrac{3}{b-a+1} x^{b+1} + C\,x^a & \text{if } b-a \neq -1, \\ 3x^a \log x + Cx^a & \text{if } b-a = -1. \end{cases} \end{aligned}$$

Imposing $y(2) = 1$,

$$\begin{cases} \dfrac{3}{b-a+1}\, 2^{b+1} + C\,2^a = 1 & \text{if } b-a \neq -1, \\ 3 \cdot 2^a \log 2 + C\,2^a = 1 & \text{if } b-a = -1, \end{cases}$$

whence the constant is respectively

$$\begin{cases} C = 2^{-a}\left(1 - \dfrac{3}{b-a+1}\, 2^{b+1}\right) & \text{if } b-a \neq -1, \\ C = 2^{-a} - 3\log 2 & \text{if } b-a = -1. \end{cases}$$

In conclusion,

$$y = \begin{cases} \dfrac{3}{b-a+1}\,x^{b+1} + 2^{-a}\left(1 - \dfrac{3}{b-a+1}\,2^{b+1}\right)x^a & \text{if } b-a \neq -1, \\ 3x^a \log x + \left(2^{-a} - 3\log 2\right)x^a & \text{if } b-a = -1. \end{cases}$$

11. *The ODE $y'(x) = -3xy(x) + kx$:*

a) The equation is linear, with integral

$$\begin{aligned} y &= \mathrm{e}^{-3\int x\,\mathrm{d}x} \int \mathrm{e}^{3\int x\,\mathrm{d}x}\, kx\;\mathrm{d}x \\ &= \mathrm{e}^{-\frac{3}{2}x^2}\left(\frac{k}{3}\,\mathrm{e}^{\frac{3}{2}x^2} + C\right) = \frac{k}{3} + C\,\mathrm{e}^{-\frac{3}{2}x^2}, \qquad C \in \mathbb{R}. \end{aligned}$$

The condition $y(0) = 0$ forces $0 = \frac{k}{3} + C$, so $C = -\frac{k}{3}$, and the solution is

$$y = \frac{k}{3}\left(1 - \mathrm{e}^{-\frac{3}{2}x^2}\right).$$

b) The solution must now fulfill

$$\frac{k}{3}\left(1 - \mathrm{e}^{-\frac{3}{2}x^2}\right) \sim x^2 \qquad \text{as } x \to 0.$$

But

$$\mathrm{e}^{-\frac{3}{2}x^2} = 1 - \frac{3}{2}x^2 + o(x^2) \qquad \text{for } x \to 0$$

implies

$$y(x) = \frac{k}{3}\left(1 - 1 + \frac{3}{2}x^2 + o(x^2)\right) = \frac{k}{2}x^2 + o(x^2) \qquad \text{for } x \to 0.$$

Therefore y is fixed by $\frac{k}{2} = 1$, i.e., $k = 2$.

12. *Solution of $y' = \frac{y^2-2y-3}{2(1+4x)}$:*

a) $y(x) = \dfrac{3 + C\sqrt{|1+4x|}}{1 - C\sqrt{|1+4x|}}$ with $C \in \mathbb{R}$, and the constant $y(x) = -1$.

b) $y_0(x) = \dfrac{3 - \sqrt{|1+4x|}}{1 + \sqrt{|1+4x|}}$;

c) $T_2(x) = 1 - 2x + 4x^2 + o(x^2)$.

13. *Second order linear ODEs reducible to first order:*

a) $y = 2\mathrm{e}^x + C_1 x + C_2\,,\ C_1, C_2 \in \mathbb{R}\,.$

b) We define $z = y'$ so that to obtain the linear equation of the first order

$$z' + z = x^2 ,$$

solved by

$$z = \mathrm{e}^{-\int \mathrm{d}x} \int \mathrm{e}^{\int \mathrm{d}x} x^2 \, \mathrm{d}x = \mathrm{e}^{-x} \int x^2 \mathrm{e}^{x} \, \mathrm{d}x .$$

By integration by parts (twice),

$$z = \mathrm{e}^{-x} \left(x^2 \mathrm{e}^{x} - 2x\mathrm{e}^{x} + 2\mathrm{e}^{x} + C_1 \right) = x^2 - 2x + 2 + C_1 \mathrm{e}^{-x} , \quad C_1 \in \mathbb{R} .$$

Integrating once more to go back to y gives

$$y = \frac{1}{3}x^3 - x^2 + 2x + C_1 \mathrm{e}^{-x} + C_2 , \quad C_1, C_2 \in \mathbb{R} .$$

14. *Linear ODEs of the second order:*

a) $y = C_1 \mathrm{e}^{-x} + C_2 \mathrm{e}^{-2x} + \frac{1}{2}x^2 - \frac{3}{2}x + \frac{9}{4}$, $C_1, C_2 \in \mathbb{R}$.

b) Let us solve first the homogeneous equation. The characteristic polynomial $\lambda^2 - 4\lambda + 4\lambda = 0$ admits a unique root $\lambda = 2$ with multiplicity two; the integral is then

$$y_0(x; C_1, C_2) = (C_1 + C_2 x)\mathrm{e}^{2x} , \quad C_1, C_2 \in \mathbb{R} .$$

As $\mu = \lambda = 2$, we require the particular integral to resemble $y_p(x) = \alpha x^2 \mathrm{e}^{2x}$. Differentiating and substituting,

$$2\alpha \mathrm{e}^{2x} = \mathrm{e}^{2x}$$

forces $\alpha = \frac{1}{2}$. Thus $y_p(x) = \frac{1}{2}x^2 \mathrm{e}^{2x}$, and the general solution to the ODE is

$$y(x; C_1, C_2) = (C_1 + C_2 x)\mathrm{e}^{2x} + \frac{1}{2}x^2 \mathrm{e}^{2x} , \quad C_1, C_2 \in \mathbb{R} .$$

c) The characteristic equation $\lambda^2 + 1 = 0$ has discriminant $\Delta = -4$, hence $\sigma = 0$, $\omega = 1$, making

$$y_0(x; C_1, C_2) = C_1 \cos x + C_2 \sin x , \quad C_1, C_2 \in \mathbb{R} ,$$

the general solution of the homogeneous case. Since $\mu = \sigma = 0$ we want a particular integral $y_p(x) = x(\alpha \cos x + \beta \sin x)$. This gives

$$-2\alpha \sin x + 2\beta \cos x = 3 \cos x ,$$

hence $\alpha = 0$ and $\beta = \frac{3}{2}$, and in turn $y_p(x) = \frac{3}{2}x \sin x$. Thus

$$y(x; C_1, C_2) = C_1 \cos x + C_2 \sin x + \frac{3}{2}x \sin x , \quad C_1, C_2 \in \mathbb{R} .$$

d) $y = C_1 e^x + C_2 e^{2x} - x e^x\,,\ C_1, C_2 \in \mathbb{R}\,.$

e) $\lambda = \pm 3$ solve the characteristic equation $\lambda^2 - 9 = 0$, so

$$y_0(x; C_1, C_2) = C_1 e^{-3x} + C_2 e^{3x}\,, \quad C_1, C_2 \in \mathbb{R}\,,$$

is how the integral of the homogeneous equation looks like. We are seeking a particular integral $y_p(x) = \alpha x e^{-3x}$. In the usual way

$$-6\alpha e^{-3x} = e^{-3x}\,,$$

from which $\alpha = -\frac{1}{6}$ follows. The particular solution $y_p(x) = -\frac{1}{6} x e^{-3x}$ is assimilated into the general integral

$$y(x; C_1, C_2) = C_1 e^{-3x} + C_2 e^{3x} - \frac{1}{6} x e^{-3x}\,, \quad C_1, C_2 \in \mathbb{R}\,.$$

f) $y = C_1 e^{-x} + C_2 e^{3x} + \frac{1}{10}\cos x - \frac{1}{5}\sin x\,,\ C_1, C_2 \in \mathbb{R}\,.$

15. *Initial value problems:*

a) $y = e^{-x}\sin 2x\,.$

b) We start from the homogeneous ODE, and solve the characteristic equation $\lambda^2 - 5\lambda + 4 = 0$, which has roots $\lambda = 1$, $\lambda = 4$. In this way

$$y_0(x; C_1, C_2) = C_1 e^{x} + C_2 e^{4x}\,, \quad C_1, C_2 \in \mathbb{R}\,,$$

is the general expression for the integral. A particular solution $y_p(x) = \alpha x + \beta$ furnishes

$$-5\alpha + 4\alpha x + 4\beta = 2x + 1\,,$$

hence $\alpha = \frac{1}{2}$, $\beta = \frac{7}{8}$. In this way we subsume $y_p(x) = \frac{1}{2}x + \frac{7}{8}$ into the general integral

$$y(x; C_1, C_2) = C_1 e^{x} + C_2 e^{4x} + \frac{1}{2}x + \frac{7}{8}\,, \quad C_1, C_2 \in \mathbb{R}\,.$$

The initial conditions lead to the system

$$\begin{cases} C_1 + C_2 = 0 \\ C_1 + 4C_2 + \dfrac{1}{2} = 0\,. \end{cases}$$

Its solutions $C_1 = \frac{1}{6}$, $C_2 = -\frac{1}{6}$ now give

$$y = \frac{1}{6}e^{x} - \frac{1}{6}e^{4x} + \frac{1}{2}x + \frac{7}{8}\,.$$

Appendices

A.1

The Principle of Mathematical Induction

The Principle of Induction represents a useful *rule for proving* properties that hold for any integer number n, or possibly from a certain integer $n_0 \in \mathbb{N}$ onwards.

Theorem A.1.1 (**Principle of Mathematical Induction**) *Let $n_0 \geq 0$ be an integer and denote by $P(n)$ a predicate defined for every integer $n \geq n_0$. Suppose the following conditions hold:*

i) $P(n_0)$ is true;
ii) for any $n \geq n_0$, if $P(n)$ is true then $P(n+1)$ is true.

Then $P(n)$ is true for all integers $n \geq n_0$.

Proof. The proof relies on the fact that each non-empty subset of $\mathbb{N}$ admits a minimum element; this property, which is self-evident, may be deduced from the axioms that define the set $\mathbb{N}$.

Let us proceed by contradiction, and assume there is an integer $n \geq n_0$ such that $P(n)$ is false. This is the same as saying that the set

$$F = \{n \in \mathbb{N} : n \geq n_0 \text{ and } P(n) \text{ is false}\}$$

is not empty. Define $\bar{n} = \min F$. As $P(\bar{n})$ is false, condition *i)* prevents $\bar{n}$ from being equal to n_0, so $\bar{n} > n_0$. Therefore $\bar{n} - 1 \geq n_0$, and $P(\bar{n}-1)$ is true by definition of the minimum. But applying *ii)* with $n = \bar{n} - 1$ implies that $P(\bar{n})$ is true, that is, $\bar{n} \notin F$. This contradicts the fact that $\bar{n}$ is the minimum of F. □

In practice, the Principle of Induction is employed as follows: one checks first that $P(n_0)$ is true; then one assumes that $P(n)$ is true for a generic n, and proves that $P(n+1)$ is true as well.

As a first application of the Principle of Induction, let us prove **Bernoulli's inequality**: For all $r \geq -1$,

$$(1+r)^n \geq 1 + nr\,, \qquad \forall n \geq 0\,.$$

In this case, the predicate $P(n)$ is given by "$(1+r)^n \geq 1+nr$". For $n=0$ we have $(1+r)^0 = 1 = 1+0r$, hence $P(0)$ holds.

Assume the inequality is true for a given n and let us show it holds for $n+1$. Observing that $1+r \geq 0$, we have

$$\begin{aligned}(1+r)^{n+1} &= (1+r)(1+r)^n \geq (1+r)(1+nr) \\ &= 1+r+nr+nr^2 = 1+(n+1)r+nr^2 \\ &\geq 1+(n+1)r\,,\end{aligned}$$

and thus the result.

Recall that this inequality has been already established in Example 5.18 with another proof, which however is restricted to the case $r > 0$.

The Principle of Induction allows us to prove various results given in previous chapters. Hereafter, we repeat their statements and add the corresponding proofs.

▶ Proof of the Newton binomial expansion, p. 20

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k\,, \qquad n \geq 0\,.$$

Proof. For $n = 0$ we have

$$(a+b)^0 = 1 \quad \text{and} \quad \sum_{k=0}^{0} \binom{0}{0} a^0 b^0 = a^0 b^0 = 1\,,$$

so the relation holds.

Let us now suppose that the formula is true for a generic n and verify that it remains true for the successive integer; the claim is thus

$$(a+b)^{n+1} = \sum_{k=0}^{n+1} \binom{n+1}{k} a^{n+1-k} b^k\,.$$

We expand the $n+1$-term

$$\begin{aligned}(a+b)^{n+1} &= (a+b)(a+b)^n = (a+b)\sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k \\ &= \sum_{k=0}^{n} \binom{n}{k} a^{n+1-k} b^k + \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^{k+1}\,;\end{aligned}$$

by putting $k+1 = h$ in the second sum we obtain

$$\sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^{k+1} = \sum_{h=1}^{n+1} \binom{n}{h-1} a^{n+1-h} b^h$$

and, going back to the original variable k, since h is merely a symbol, we obtain

$$\sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^{k+1} = \sum_{k=1}^{n+1} \binom{n}{k-1} a^{n+1-k} b^{k} .$$

Therefore

$$\begin{aligned} (a+b)^{n+1} &= \sum_{k=0}^{n} \binom{n}{k} a^{n+1-k} b^{k} + \sum_{k=1}^{n+1} \binom{n}{k-1} a^{n+1-k} b^{k} \\ &= \binom{n}{0} a^{n+1} b^{0} + \sum_{k=1}^{n} \left[\binom{n}{k} + \binom{n}{k-1} \right] a^{n+1-k} b^{k} + \binom{n}{n} a^{0} b^{n+1} . \end{aligned}$$

Using (1.12), with n replaced by $n+1$, and recalling that

$$\binom{n}{0} = 1 = \binom{n+1}{0} \quad \text{e} \quad \binom{n}{n} = 1 = \binom{n+1}{n+1}$$

we eventually find

$$\begin{aligned} (a+b)^{n+1} &= \binom{n+1}{0} a^{n+1} b^{0} + \sum_{k=1}^{n} \binom{n+1}{k} a^{n+1-k} b^{k} + \binom{n+1}{n+1} a^{0} b^{n+1} \\ &= \sum_{k=0}^{n+1} \binom{n+1}{k} a^{n+1-k} b^{k} , \end{aligned}$$

i.e., the claim. □

▶ Proof of the Theorem of existence of zeroes, p. 109

Theorem 4.23 (Existence of zeroes) *Let f be a continuous map on a closed, bounded interval $[a, b]$. If $f(a)f(b) < 0$, i.e., if the images of the end-points under f have different signs, f admits a zero within the open interval (a, b).*
If moreover f is strictly monotone on $[a, b]$, the zero is unique.

Proof. We refer to the proof given on p. 109. Therein, it is enough to justify the existence of two sequences $\{a_n\}$ and $\{b_n\}$, finite or infinite, that fulfill the predicate $P(n)$:

$$\begin{aligned} &[a_0, b_0] \supset [a_1, b_1] \supset \ldots \supset [a_n, b_n] \\ &f(a_n) < 0 < f(b_n) \quad \text{and} \quad b_n - a_n = \frac{b_0 - a_0}{2^n} . \end{aligned}$$

When $n = 0$, by assumption $f(a_0) = f(a) < 0 < f(b) = f(b_0)$, so trivially we have $b_0 - a_0 = \dfrac{b_0 - a_0}{2^0}$.

Assume the above relations hold up to a certain n. Let $c_n = \dfrac{a_n + b_n}{2}$ be the mid-point of the interval $[a_n, b_n]$. If $f(c_n) = 0$, the construction of the sequences terminates, since a zero of the function is found. If $f(c_n) \neq 0$, let us verify $P(n+1)$. If $f(c_n) > 0$, we set $a_{n+1} = a_n$ and $b_{n+1} = c_n$, whereas if $f(c_n) < 0$, we set $a_{n+1} = c_n$ and $b_{n+1} = b_n$. The interval $[a_{n+1}, b_{n+1}]$ is a sub-interval of $[a_n, b_n]$, and

$$f(a_{n+1}) < 0 < f(b_{n+1}) \quad \text{and} \quad b_{n+1} - a_{n+1} = \frac{b_n - a_n}{2} = \frac{b_0 - a_0}{2^{n+1}} . \qquad \square$$

▶ Proof of inequality (5.16), p. 140

Let us begin by establishing the following general property.

Property A.1.2 *Let $\{b_m\}_{m\geq 0}$ be a sequence with non-negative terms. Assume there exists a number $r > 0$ for which the following inequalities hold:*

$$b_{m+1} \leq r\, b_m\,, \qquad \forall m \geq 0\,.$$

Then one has

$$b_m \leq r^m b_0\,, \qquad \forall m \geq 0\,.$$

Proof. We apply the Principle of Induction. For $m = 0$, the inequality is trivially true, since $b_0 \leq r^0 b_0 = b_0$.

Let us assume the inequality to be true for m and let us check it for $m + 1$. Using the assumption, one has

$$b_{m+1} \leq r\, b_m \leq r\, r^m b_0 = r^{m+1} b_0\,. \qquad \square$$

If all terms of the sequence $\{b_m\}_{m\geq 0}$ are strictly positive, a similar statement holds with strict inequalities, i.e., with $\leq$ replaced by $<$.

Next, consider inequality (5.16). In order to derive the implication

$$a_{n+1} < r\, a_n \qquad \Rightarrow \qquad a_{n+1} < r^{n-n_\varepsilon} a_{n_\varepsilon+1}\,,$$

let us set $b_m = a_{m+n_\varepsilon+1}$ and observe that the assumption

$$a_{n+1} < r\, a_n\,, \qquad \forall n > n_\varepsilon$$

is equivalent to

$$b_{m+1} < r\, b_m\,, \qquad \forall m \geq 0\,.$$

Thus, the previous property yields $b_m < r^m b_0$, whence we get (5.16) by choosing $m = n - n_\varepsilon$.

A.2

Complements on limits and continuity

In this appendix, we first state and prove some results about limits, that are used in the subsequent proof of Theorem 4.10 concerning the algebra of limits. Next, we rigorously justify the limit behaviour of the most important elementary functions at the extrema of their domain, and we check the continuity of these functions at all points they are defined. At last, we provide the proofs of several properties of Napier's number stated in the text, and we show that this important number is irrational.

A.2.1 Limits

Now we discuss few results that will be useful later.

Theorem A.2.1 (**local boundedness**) *If a map f admits a finite limit for $x \to c$, there exist a neighbourhood $I(c)$ of c and a constant $M_f > 0$ such that*

$$\forall x \in \operatorname{dom} f \cap I(c) \setminus \{c\}\ ,\ \ |f(x)| \le M_f\ .$$

Proof. Let $\ell = \lim\limits_{x \to c} f(x) \in \mathbb{R}$; the definition of limit with, say, $\varepsilon = 1$, implies the existence of a neighbourhood $I(c)$ of c such that

$$\forall x \in \operatorname{dom} f, \qquad x \in I(c) \setminus \{c\} \quad \Rightarrow \quad |f(x) - \ell| < 1\,.$$

By the triangle inequality (1.1), on such set

$$|f(x)| = |f(x) - \ell + \ell| \le |f(x) - \ell| + |\ell| < 1 + |\ell|\,.$$

Therefore it is enough to choose $M_f = 1 + |\ell|$. □

Theorem A.2.2 (**Theorem 4.2, strong form**) *If f admits non-zero limit (finite or infinite) for $x \to c$, then there are a neighbourhood $I(c)$ of c and a constant $K_f > 0$ for which*

$$\forall x \in \operatorname{dom} f \cap I(c) \setminus \{c\}, \qquad |f(x)| > K_f . \tag{A.2.1}$$

Proof. Let $\ell = \lim\limits_{x\to c} f(x)$. If $\ell \in \mathbb{R} \setminus \{0\}$, and given for instance $\varepsilon = |\ell|/2$ in the definition of limit for f, there exists a neighbourhood $I(c)$ with $\forall x \in \operatorname{dom} f \cap I(c) \setminus \{c\}$, $|f(x) - \ell| < |\ell|/2$. Thus we have

$$|\ell| = |f(x) + \ell - f(x)| \le |f(x)| + |f(x) - \ell| < |f(x)| + \frac{|\ell|}{2}$$

hence

$$|f(x)| > |\ell| - \frac{|\ell|}{2} = \frac{|\ell|}{2} .$$

The claim follows by taking $K_f = \frac{|\ell|}{2}$.

If $\ell \pm \infty$, then $\lim\limits_{x\to c} |f(x)| = +\infty$ and it is sufficient to take $A = 1$ in the definition of limit to have $|f(x)| > 1$ in a neighbourhood $I(c)$ of c; in this case we may take in fact $K_f = 1$. □

Remark A.2.3 Notice that if $\ell > 0$, Theorem 4.2 ensures that on a suitable neighbourhood of c, possibly excluding c itself, the function is positive. Therefore the inequality in (A.2.1) becomes the more precise $f(x) > K_f$. Similarly for $\ell < 0$, in which case (A.2.1) reads $f(x) < -K_f$. In this sense Theorem A.2.2 is stronger than Theorem 4.2. □

The next property makes checking a limit an easier task.

Property A.2.4 *In order to prove that $\lim\limits_{x\to c} f(x) = \ell \in \mathbb{R}$ it is enough to find a constant $C > 0$ such that for every $\varepsilon > 0$ there is a neighbourhood $I(c)$ with*

$$\forall x \in \operatorname{dom} f, \qquad x \in I(c) \setminus \{c\} \quad \Rightarrow \quad |f(x) - \ell| < C\varepsilon . \tag{A.2.2}$$

Proof. Condition (3.8) follows indeed from (A.2.2) by choosing ε/C instead of ε. □

▶ Proof of Theorem 4.10, p. 96

Teorema 4.10 *Suppose f admits limit ℓ (finite or infinite) and g admits limit m (finite or infinite) for $x \to c$. Then*

$$\lim_{x\to c} [f(x) \pm g(x)] = \ell \pm m,$$
$$\lim_{x\to c} [f(x)\, g(x)] = \ell\, m,$$
$$\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{\ell}{m}$$

provided the right-hand-side expressions make sense. (In the last case one assumes $g(x) \neq 0$ on some $I(c)\setminus\{c\}$.)

Proof. The cases we shall prove are:

a) if $\ell \in \mathbb{R}$ and $m = -\infty$, then $\lim\limits_{x\to c} \big(f(x) - g(x)\big) = +\infty$;
b) if $\ell, m \in \mathbb{R}$, then $\lim\limits_{x\to c} f(x)g(x) \; = \ell\, m \in \mathbb{R}$;
c) if $\ell, m \in \mathbb{R}$ and $m \neq 0$, then $\lim\limits_{x\to c} \dfrac{f(x)}{g(x)} = \dfrac{\ell}{m} \in \mathbb{R}$;
d) if $\ell \in \mathbb{R} \setminus \{0\}$ or $\ell \pm \infty$, and $m = 0$, then $\lim\limits_{x\to c} \dfrac{f(x)}{g(x)} = \infty$.

All remaining possibilities are left to the reader as exercise.

a) Let $A > 0$ be arbitrarily fixed.
By Theorem A.2.1 applied to f, there is a neighbourhood $I'(c)$ of c and there is a constant $M_f > 0$ such that for each $x \in \operatorname{dom} f \cap I'(c) \setminus \{c\}$, $|f(x)| \leq M_f$.
Moreover, $\lim\limits_{x\to c} g(x) = -\infty$ is the same as saying that for any $B > 0$ there is an $I''(c)$ such that $g(x) < -B$ for every $x \in \operatorname{dom} g \cap I''(c)\setminus\{c\}$, i.e, $-g(x) > B$. Choose $B = A + M_f$ and set $I(c) = I'(c) \cap I''(c)$: then for all $x \in \operatorname{dom} f \cap \operatorname{dom} g \cap I(c) \setminus \{c\}$,

$$f(x) - g(x) > -M_f + B \geq A\,.$$

This proves that $\lim\limits_{x\to c} \big(f(x) - g(x)\big) = +\infty$.

b) Fix $\varepsilon > 0$.
Assuming $\lim\limits_{x\to c} f(x) = \ell \in \mathbb{R}$, as we have, tells

$$\exists I'(c) : \forall x \in \operatorname{dom} f,\ x \in I'(c) \setminus \{c\} \quad \Rightarrow \quad |f(x) - \ell| < \varepsilon\,,$$

while Theorem A.2.1 gives

$$\exists I''(c),\ \exists M_f > 0 : \forall x \in \operatorname{dom} f,\ x \in I''(c) \setminus \{c\} \quad \Rightarrow \quad |f(x)| < M_f\,.$$

Analogously, $\lim\limits_{x\to c} g(x) = m \in \mathbb{R}$ implies

$$\exists I'''(c) : \forall x \in \operatorname{dom} g,\ x \in I'''(c) \setminus \{c\} \quad \Rightarrow \quad |g(x) - m| < \varepsilon\,.$$

Set now $I(c) = I'(c) \cap I''(c) \cap I'''(c)$; for all $x \in \operatorname{dom} f \cap \operatorname{dom} g \cap I(c) \setminus \{c\}$ we have

$$\begin{aligned}
|f(x)g(x) - \ell m| &= |f(x)g(x) - f(x)m + f(x)m - \ell m| \\
&= |f(x)\big(g(x) - m\big) + \big(f(x) - \ell\big)m| \\
&\le |f(x)||g(x) - m| + |f(x) - \ell|\,|m| < (M_f + |m|)\varepsilon\,.
\end{aligned}$$

This means that (A.2.2) holds with $C = M_f + |m|\,$.

c) Let $\varepsilon > 0$ be given.
From $\lim\limits_{x\to c} f(x) = \ell \in \mathbb{R}$ and $\lim\limits_{x\to c} g(x) = m \in \mathbb{R}$ it follows

$$\exists I'(c) : \forall x \in \operatorname{dom} f,\ x \in I'(c) \setminus \{c\} \quad \Rightarrow \quad |f(x) - \ell| < \varepsilon$$

and

$$\exists I''(c) : \forall x \in \operatorname{dom} g,\ x \in I''(c) \setminus \{c\} \quad \Rightarrow \quad |g(x) - m| < \varepsilon\,.$$

Since $m \ne 0$ moreover, Theorem A.2.2 guarantees there is a neighbourhood $I'''(c)$ of c together with a constant $K_g > 0$ such that $|g(x)| > K_g$, $\forall x \in \operatorname{dom} g,\ x \in I'''(c) \setminus \{c\}$.
Set $I(c) = I'(c) \cap I''(c) \cap I'''(c)$; then for all $x \in \operatorname{dom} f \cap \operatorname{dom} g,\ x \in I(c) \setminus \{c\}$

$$\begin{aligned}
\left|\frac{f(x)}{g(x)} - \frac{\ell}{m}\right| &= \left|\frac{f(x)m - \ell g(x)}{mg(x)}\right| = \frac{|f(x)m - \ell m + \ell m - \ell g(x)|}{|m||g(x)|} \\
&= \frac{\big|\big(f(x) - \ell\big)m + \ell\big(m - g(x)\big)\big|}{|m||g(x)|} \le \frac{|f(x) - \ell|\,|m| + |\ell|\,|g(x) - m|}{|m||g(x)|} \\
&< \frac{|m| + |\ell|}{|m|K_g}\,\varepsilon\,.
\end{aligned}$$

Hence (A.2.2) holds for $C = \dfrac{|m| + |\ell|}{|m|K_g}\,.$

d) Fix a constant $A > 0$.
Using Theorem A.2.2 on f we know there are a neighbourhood $I'(c)$ of c and a $K_f > 0$ such that $\forall x \in \operatorname{dom} f \cap \in I'(c) \setminus \{c\},\ |f(x)| > K_f$.
By hypothesis $\lim\limits_{x\to c} g(x) = 0$, so choosing $\varepsilon = K_f/A$ ensures that there exists a neighbourhood $I''(c)$ of c with $|g(x)| < K_f/A$, for any $x \in \operatorname{dom} g \cap I''(c) \setminus \{c\}$.
Now let $I(c) = I'(c) \cap I''(c)$, so that for all $x \in \operatorname{dom} f \cap \operatorname{dom} g \cap I(c) \setminus \{c\}$

$$\left|\frac{f(x)}{g(x)}\right| > K_f\,\frac{A}{K_f} = A\,.$$

This shows that $\lim\limits_{x\to c}\left|\dfrac{f(x)}{g(x)}\right| = +\infty$, which was our claim. $\square$

A.2.2 Elementary functions

▶ Check of the limits in the table on p. 101

a) $\lim_{x\to+\infty} x^\alpha = +\infty\,, \qquad \lim_{x\to 0^+} x^\alpha = 0 \qquad \alpha > 0$

b) $\lim_{x\to+\infty} x^\alpha = 0\,, \qquad \lim_{x\to 0^+} x^\alpha = +\infty \qquad \alpha < 0$

c) $\lim_{x\to\pm\infty} \frac{a_n x^n + \ldots + a_1 x + a_0}{b_m x^m + \ldots + b_1 x + b_0} = \frac{a_n}{b_m} \lim_{x\to\pm\infty} x^{n-m}$

d) $\lim_{x\to+\infty} a^x = +\infty\,, \qquad \lim_{x\to-\infty} a^x = 0 \qquad a > 1$

e) $\lim_{x\to+\infty} a^x = 0\,, \qquad \lim_{x\to-\infty} a^x = +\infty \qquad a < 1$

f) $\lim_{x\to+\infty} \log_a x = +\infty\,, \qquad \lim_{x\to 0^+} \log_a x = -\infty \qquad a > 1$

g) $\lim_{x\to+\infty} \log_a x = -\infty\,, \qquad \lim_{x\to 0^+} \log_a x = +\infty \qquad a < 1$

h) $\lim_{x\to\pm\infty} \sin x\,, \qquad \lim_{x\to\pm\infty} \cos x\,, \qquad \lim_{x\to\pm\infty} \tan x$ do not exist

i) $\lim_{x\to\left(\frac{\pi}{2}+k\pi\right)^\pm} \tan x = \mp\infty\,, \quad \forall k \in \mathbb{Z}$

l) $\lim_{x\to\pm 1} \arcsin x = \pm\frac{\pi}{2} = \arcsin(\pm 1)$

m) $\lim_{x\to+1} \arccos x = 0 = \arccos 1\,, \qquad \lim_{x\to-1} \arccos x = \pi = \arccos(-1)$

n) $\lim_{x\to\pm\infty} \arctan x = \pm\frac{\pi}{2}$

Proof.

a) Take the first limit. Fix $A > 0$ and set $B = A^{1/\alpha} > 0$. As power functions are monotone,
$$\forall x \in \mathbb{R}_+, \quad x > B \quad \Rightarrow \quad x^\alpha > B^\alpha = A\,,$$
so the requirement for the limit to hold (Definition 3.12) is satisfied.
As for the second limit, with a given $\varepsilon > 0$ we let $\delta = \varepsilon^{1/\alpha}$; again by monotonicity we have
$$\forall x \in \mathbb{R}_+, \quad x < \delta \quad \Rightarrow \quad x^\alpha > \delta^\alpha = \varepsilon\,.$$
The condition of Definition 3.15 holds.

b) These limits follow from a) by substituting $z = \frac{1}{x}$, which gives $x^\alpha = \frac{1}{z^{|\alpha|}}$. The algebra of limits and the Substitution theorem 4.15 allow to conclude.
c) The formula was proved in Example 4.14 iii).
d) Put $a = 1 + b$, with $b > 0$, in the first limit and use Bernoulli's inequality $a^n = (1 + b)^n \geq 1 + nb$. Fix an arbitrary $A > 0$ and let $n \in \mathbb{N}$ be such that $1 + nb > A$. Since the exponential is monotone we obtain

$$\forall x \in \mathbb{R}, \quad x > n \quad \Rightarrow \quad a^x > a^n \geq 1 + nb > A\,,$$

hence the condition of Definition 3.12 holds for $B = n$.
The second limit is a consequence of the first, for

$$\lim_{x \to -\infty} a^x = \lim_{x \to -\infty} \frac{1}{a^{-x}} = \frac{1}{\lim\limits_{z \to +\infty} a^z} = 0\,.$$

e) These descend from d) using the identity $a^x = \dfrac{1}{(1/a)^x}$.
f) The limits of d) and Corollary 4.30 imply that the range of $y = a^x$ is the interval $(0, +\infty)$. Therefore the inverse $y = \log_a x$ is well defined on $(0, +\infty)$, and strictly increasing because inverse of a likewise map; its range is $(-\infty, +\infty)$. The claim then follows from Theorem 3.27.
g) A consequence of e), for the same reason as above.
h) That the first limit does not exist was already observed in Remark 4.19. In a similar way one can discuss the remaining cases.
More generally, notice that a non-constant periodic function does not admit limit for $x \to \pm\infty$.
i) Follows from the algebra of limits.
l)-m) The functions are continuous at the limit points (Theorem 4.33), making the results clear.
n) We can argue as in f) relatively to $y = \tan x$, restricted to $(-\frac{\pi}{2}, \frac{\pi}{2})$, and its inverse map $y = \arctan x$. □

▶ Proof of Proposition 3.20, p. 81

Proposition 3.20 *All elementary functions (polynomials, rational functions, powers, trigonometric functions, exponentials and their inverses) are continuous over their entire domains.*

Proof. The continuity of rational functions was established in Corollary 4.12. That, together with Theorems 4.17 and 4.33 on composites and inverses, implies in particular that power functions with positive rational exponent $y = x^{m/n} = \sqrt[m]{x^n}$ are continuous; the same holds for negative rational exponent $x^q = \frac{1}{x^{-q}}$ by using the algebra of limits. At last, powers with irrational exponent are continuous by definition $x^\alpha = \mathrm{e}^{\alpha \log x}$ and because of Theorem 4.17, once the logarithm and the exponential function have been proven continuous.

As for sine and cosine, their continuity was ascertained in Example 3.17 iii), so the algebra of limits warrants continuity to the tangent and cotangent functions; by Theorem 4.33 we infer that the inverse trigonometric functions arcsine, arccosine, arctangent and arccotangent are continuous as well.

What remains to show is then the continuity of the exponential map only, because that of the logarithm will follow from Theorem 4.33. Let us consider the case $a > 1$, for if $0 < a < 1$, we can write $a^x = \frac{1}{(1/a)^x}$ and use the same argument. The identities

$$a^{x_1+x_2} = a^{x_1}a^{x_2}\,, \qquad a^{-x} = \frac{1}{a^x}$$

and the monotonicity

$$x_1 < x_2 \quad \Rightarrow \quad a^{x_1} < a^{x_2}$$

follow easily from the properties of integer powers and their inverses when the exponents are rational; for real exponents, we can apply the same argument using the definitions of exponential function and supremum.

First of all let us prove that $y = a^x$ is continuous at the right of the origin

$$\lim_{x\to 0^+} a^x = 1\,. \tag{A.2.3}$$

With $\varepsilon > 0$ fixed, we shall determine a $\delta > 0$ such that

$$0 \le x < \delta \quad \Rightarrow \quad 0 \le a^x - 1 < \varepsilon\,.$$

The exponential map being monotone, it suffices to find δ with $a^\delta - 1 < \varepsilon$, i.e., $a^\delta < 1 + \varepsilon$. Searching for δ of the form $\delta = \frac{1}{n}$, with n integer, the condition becomes $a < (1+\varepsilon)^n$. Bernoulli's inequality (5.15) implies $(1+\varepsilon)^n \ge 1 + n\varepsilon$. It is therefore enough to pick n so that $1 + n\varepsilon > a$, or $n > \frac{a-1}{\varepsilon}$. Thus (A.2.3) holds. Left-continuity at the origin is a consequence of

$$\lim_{x\to 0^-} a^x = \lim_{x\to 0^-} a^{-(-x)} = \lim_{x\to 0^-} \frac{1}{a^{-x}} = \frac{1}{\lim\limits_{z\to 0^+} a^z} = 1\,,$$

so the exponential map is indeed continuous at the origin. Eventually, from

$$\lim_{x\to x_0} a^x = \lim_{x\to x_0} a^{x_0+(x-x_0)} = a^{x_0} \lim_{x\to x_0} a^{x-x_0} = a^{x_0} \lim_{z\to 0} a^z = a^{x_0}\,,$$

we deduce that the function is continuous at every point $x_0 \in \mathbb{R}$. □

A.2.3 Napier's number

We shall prove some properties of the sequence $a_n = \left(1 + \dfrac{1}{n}\right)^n$, $n > 0$, defining the Napier's number e (p. 72).

Property A.2.5 *The sequence $\{a_n\}$ is strictly increasing.*

Proof. Using Newton's formula (1.13) and (1.11), we may write

$$\begin{aligned} a_n &= \left(1+\frac{1}{n}\right)^n = \sum_{k=0}^{n} \binom{n}{k} \frac{1}{n^k} = \sum_{k=0}^{n} \frac{1}{k!} \frac{n(n-1)\cdots(n-k+1)}{n^k} \\ &= \sum_{k=0}^{n} \frac{1}{k!} \frac{n}{n} \frac{n-1}{n} \cdots \frac{n-k+1}{n} \\ &= \sum_{k=0}^{n} \frac{1}{k!} \cdot 1 \cdot \left(1-\frac{1}{n}\right) \cdots \left(1-\frac{k-1}{n}\right) ; \end{aligned} \qquad \text{(A.2.4)}$$

similarly,

$$a_{n+1} = \sum_{k=0}^{n+1} \frac{1}{k!} \cdot 1 \cdot \left(1-\frac{1}{n+1}\right) \cdots \left(1-\frac{k-1}{n+1}\right) . \qquad \text{(A.2.5)}$$

We note that

$$\left(1-\frac{1}{n}\right) < \left(1-\frac{1}{n+1}\right), \quad \cdots \quad , \left(1-\frac{k-1}{n}\right) < \left(1-\frac{k-1}{n+1}\right)$$

so each summand of (A.2.4) is smaller than the corresponding term in (A.2.5). The latter sum, moreover, contains an additional positive summand labelled by $k = n+1$. Therefore $a_n < a_{n+1}$ for each n. □

Property A.2.6 *The sequence $\{a_n\}$ is bounded; precisely,*

$$2 < a_n < 3 , \qquad \forall n > 1 .$$

Proof. Since $a_1 = 2$, and the sequence is strictly monotone by the previous property, we have $a_n > 2$, $\forall n > 1$. Let us show that $a_n < 3$, $\forall n > 1$. By (A.2.4), and observing that $k! = 1 \cdot 2 \cdot 3 \cdots k \geq 1 \cdot 2 \cdot 2 \cdots 2 = 2^{k-1}$, it follows

$$\begin{aligned} a_n &= \sum_{k=0}^{n} \frac{1}{k!} \cdot 1 \cdot \left(1-\frac{1}{n}\right) \cdots \left(1-\frac{k-1}{n}\right) < \sum_{k=0}^{n} \frac{1}{k!} \\ &= 1 + \sum_{k=1}^{n} \frac{1}{k!} \leq 1 + \sum_{k=1}^{n} \frac{1}{2^{k-1}} = 1 + \sum_{k=0}^{n-1} \frac{1}{2^k} . \end{aligned}$$

Example 5.27 will tell us that

$$\sum_{k=0}^{n-1} \frac{1}{2^k} = \frac{1-\frac{1}{2^n}}{1-\frac{1}{2}} = 2\left(1-\frac{1}{2^n}\right) < 2 .$$

We conclude that $a_n < 3$. □

▶ Proof of Property 4.20, p. 105

Property 4.20 *The following limit holds*

$$\lim_{x\to\pm\infty}\left(1+\frac{1}{x}\right)^x = \mathrm{e}.$$

Proof. We start by considering the limit for $x \to +\infty$. Denoting by $n = [x]$ the integer part of x (see Examples 2.1), by definition $n \le x < n+1$; from that it follows $\dfrac{1}{n+1} < \dfrac{1}{x} \le \dfrac{1}{n}$, in other words $1+\dfrac{1}{n+1} < 1+\dfrac{1}{x} \le 1+\dfrac{1}{n}$. The familiar features of power functions yield

$$\left(1+\frac{1}{n+1}\right)^n \le \left(1+\frac{1}{n+1}\right)^x < \left(1+\frac{1}{x}\right)^x \le \left(1+\frac{1}{n}\right)^x < \left(1+\frac{1}{n}\right)^{n+1}$$

hence

$$\left(1+\frac{1}{n+1}\right)^{n+1}\left(1+\frac{1}{n+1}\right)^{-1} < \left(1+\frac{1}{x}\right)^x < \left(1+\frac{1}{n}\right)^n\left(1+\frac{1}{n}\right). \tag{A.2.6}$$

When x tends to $+\infty$, n does the same. Using (3.3) we have

$$\lim_{n\to+\infty}\left(1+\frac{1}{n}\right)^n\left(1+\frac{1}{n}\right) = \lim_{n\to+\infty}\left(1+\frac{1}{n}\right)^n \lim_{n\to+\infty}\left(1+\frac{1}{n}\right) = \mathrm{e};$$

the substitution $m = n+1$ similarly gives

$$\lim_{n\to+\infty}\left(1+\frac{1}{n+1}\right)^{n+1}\left(1+\frac{1}{n+1}\right)^{-1} = \mathrm{e}.$$

Applying the Second comparison theorem 4.5 to the three functions in (A.2.6) proves the claim for $x \to +\infty$. Now let us look at the case when x tends to $-\infty$. If $x < 0$ we can write $x = -|x|$, so

$$\left(1+\frac{1}{x}\right)^x = \left(1-\frac{1}{|x|}\right)^{-|x|} = \left(\frac{|x|-1}{|x|}\right)^{-|x|} = \left(\frac{|x|}{|x|-1}\right)^{|x|} = \left(1+\frac{1}{|x|-1}\right)^{|x|}.$$

Set $y = |x|-1$ and note y tends to $+\infty$ as x goes to $-\infty$. Therefore,

$$\lim_{x\to-\infty}\left(1+\frac{1}{x}\right)^x = \lim_{y\to+\infty}\left(1+\frac{1}{y}\right)^{y+1} = \lim_{y\to+\infty}\left(1+\frac{1}{y}\right)^y \lim_{y\to+\infty}\left(1+\frac{1}{y}\right) = \mathrm{e}.$$

This concludes the proof. □

▶ Proof of the irrationality of Napier's number, p. 72

Property A.2.7 *Napier's number* e *is irrational, and lies between* 2 *and* 3*.*

Proof. Based on the First comparison theorem for sequences (p. 137, Theorem 4), from the previous property we quickly deduce

$$2 < \mathrm{e} \le 3\,. \tag{A.2.7}$$

Suppose, by contradiction, that e is a rational number, so that there exist two integers m_0 and $n_0 \neq 0$ such that $\mathrm{e} = \dfrac{m_0}{n_0}$. Recall that for any $n \geq 0$

$$\mathrm{e} = \sum_{k=0}^{n} \frac{1}{k!} + \frac{\mathrm{e}^{\bar{x}_n}}{(n+1)!}\,, \qquad 0 < \bar{x}_n < 1$$

(see Remark 7.4). From this,

$$n!\mathrm{e} = n!\frac{m_0}{n_0} = \sum_{k=0}^{n} \frac{n!}{k!} + \frac{\mathrm{e}^{\bar{x}_n}}{n+1}\,. \tag{A.2.8}$$

As the exponential map is monotone, and using (A.2.7), we deduce

$$1 = \mathrm{e}^0 < \mathrm{e}^{\bar{x}_n} < \mathrm{e} < 3\,.$$

Choosing now $n \geq \max(3, n_0)$, the numbers $n!\dfrac{m_0}{n_0}$ and $\displaystyle\sum_{k=0}^{n} \frac{n!}{k!}$ are integers, whereas $\dfrac{\mathrm{e}^{\bar{x}_n}}{n+1}$ lies in the open interval between 0 and 1. The identity (A.2.8) then must be false, so e is irrational and equality in (A.2.7) never occurs. □

A.3

Complements on the global features of continuous maps

We first introduce the concept of subsequence of a given sequence, and establish a number of related properties. Among them, the Theorem of Bolzano-Weierstrass, which is a fundamental ingredient in the subsequent proof of the Theorem of Weierstrass concerning continuous functions on an interval of the real line; the proofs of other results for such functions are also provided. The appendix ends with the definition of uniform continuity and the discussion of some of its properties; these concepts will find application to the study of integral calculus and differential equations.

A.3.1 Subsequences

Theorem 2 on p. 137 states that every converging sequence is bounded. In general, though, the opposite implication is false. In fact, the sequence $a_n = (-1)^n$ does not converge despite being bounded ($|a_n| = 1,\ \forall n$). But if we take just the elements with even subscript, we obtain the constant sequence $\{b_k\}_{k\geq 0}$ where $b_k = a_{2k} = 1,\ k \geq 0$, which is patently convergent. Similarly if we take odd indexes only: the constant sequence $\{c_k\}_{k\geq 0}$ with $c_k = a_{2k+1} = -1,\ k \geq 0$, converges. Such sequences have been *extracted*, so to say, from the initial sequence $\{a_n\}_{n\geq 0}$, in the sense formalised below.

Definition A.3.1 *Let $\{a_n\}_{n\geq n^*}$ be a sequence and $\{n_k\}_{k\geq 0}$ a strictly increasing sequence of integers $\geq n^*$. The sequence $\{a_{n_k}\}_{k\geq 0}$ is said* **subsequence** *of $\{a_n\}_{n\geq n^*}$.*

Observe that the sequence $\{a_{n_k}\}_{k\geq 0}$ is a composite function, for it is obtained by composing the map $k \mapsto n_k$ with $n \mapsto a_n$.

Any subsequence of a converging or diverging sequence preserves the limit behaviour of the 'mother' sequence:

Proposition A.3.2 *Let the sequence* $\{a_n\}_{n\geq n^*}$ *admit limit* $\lim\limits_{n\to+\infty} a_n = \ell$*, finite or infinite. Then for any subsequence* $\{a_{n_k}\}_{k\geq 0}$

$$\lim_{k\to+\infty} a_{n_k} = \ell\,.$$

Proof. It is not that difficult to see, by induction, that

$$n_k \geq k\,, \qquad \forall k \geq 0\,. \tag{A.3.1}$$

Clearly, $n_0 \geq 0$; supposing $n_k \geq k$ we have $n_{k+1} > n_k$ because the sequence is strictly increasing. That in turn implies $n_{k+1} \geq k+1$, whence the claim follows.

Due to the First comparison theorem (p. 137, Theorem 4), the inequality (A.3.1) tells that the sequence $\{n_k\}$ diverges to $+\infty$. The result then follows from Theorem 4.15 adapted to sequences (whose proof is similar to the one given on p. 102). □

The fact that one can extract a converging subsequence from a bounded sequence, as we showed with the example $a_n = (-1)^n$, is a general and deep result. This is how it goes.

Theorem A.3.3 (Bolzano-Weierstrass) *A bounded sequence always admits a converging subsequence.*

Proof. Suppose $\{x_n\}_{n\geq n^*}$ is a bounded sequence by assuming

$$a \leq x_n \leq b\,, \qquad \forall n \geq n^*\,,$$

for suitable $a, b \in \mathbb{R}$. We shall bisect the interval $[a, b]$ over and over, as in the proof of Theorem 4.23 of existence of zeroes. Set

$$a_0 = a\,, \qquad b_0 = b\,, \qquad \mathcal{N}_0 = \{n \geq n^*\}\,, \qquad n_0 = n^*.$$

Call c_0 the midpoint of $[a_0, b_0]$ and define the sets

$$\mathcal{N}_0^- = \{n \in \mathcal{N}_0 : x_n \in [a_0, c_0]\}\,, \qquad \mathcal{N}_0^+ = \{n \in \mathcal{N}_0 : x_n \in [c_0, b_0]\}\,.$$

Note $\mathcal{N}_0 = \mathcal{N}_0^- \cup \mathcal{N}_0^+$, where at least one of $\mathcal{N}_0^-$, $\mathcal{N}_0^+$ must be infinite because $\mathcal{N}_0$ is. If $\mathcal{N}_0^-$ is infinite, set

$$a_1 = a_0\,, \qquad b_1 = c_0\,, \qquad \mathcal{N}_1 = \mathcal{N}_0^-\,;$$

otherwise,

$$a_1 = c_0\,, \qquad b_1 = b_0\,, \qquad \mathcal{N}_1 = \mathcal{N}_0^+\,.$$

Now let n_1 be the first index $> n_0$ contained $\mathcal{N}_1$; we can make such a choice since $\mathcal{N}_1$ is infinite. Iterating the procedure (as always, in these situations, the Principle of Induction A.1.1 is required in order to make things formal), we can build a sequence of intervals

$$[a_0, b_0] \supset [a_1, b_1] \supset \ldots \supset [a_k, b_k] \supset \ldots , \qquad \text{with} \quad b_k - a_k = \frac{b_0 - a_0}{2^k},$$

a sequence of infinite sets

$$\mathcal{N}_0 \supseteq \mathcal{N}_1 \supseteq \ldots \supseteq \mathcal{N}_k \supseteq \ldots$$

and a strictly increasing sequence of indices $\{n_k\}_{k\geq 0}$, $n_k \in \mathcal{N}_k$, such that

$$a_k \leq x_{n_k} \leq b_k , \qquad \forall k \geq 0 .$$

Then just as in the proof of Theorem 4.23, there will be a unique $\ell \in [a, b]$ satisfying

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty} b_k = \ell .$$

From the Second comparison theorem (Theorem 5 on p. 137) we deduce that the sequence $\{x_{n_k}\}_{k\geq 0}$, extracted from $\{x_n\}_{n\geq n^*}$, converges to ℓ. □

A.3.2 Continuous functions on an interval

▶ Proof of the Theorem of Weierstrass, p. 114

Theorem 4.31 (Weierstrass) *A continuous map f on a closed and bounded interval $[a, b]$ is bounded and admits minimum and maximum*

$$m = \min_{x\in[a,b]} f(x) \qquad \textit{and} \qquad M = \max_{x\in[a,b]} f(x).$$

Consequently,

$$f([a, b]) = [m, M].$$

Proof. We will show first that f admits a maximum in $[a, b]$, in other words there exists $\xi \in [a, b]$ such that $f(x) \leq f(\xi)$, $\forall x \in [a, b]$. For this, let

$$M = \sup f([a, b]),$$

which is allowed to be both real or $+\infty$. In the former case the characterisation of the supremum, (1.7) *ii)*, tells that for any $n \geq 1$ there is $x_n \in [a, b]$ with

$$M - \frac{1}{n} < f(x_n) \leq d.$$

Letting n go to $+\infty$, from the Second comparison theorem (Theorem 5 on p. 137) we infer
$$\lim_{n\to\infty} f(x_n) = M.$$
In the other case, by definition of unbounded set (from above) we deduce that for any $n \geq 1$ there is $x_n \in [a,b]$ such that
$$f(x_n) \geq n.$$
The Second comparison theorem implies
$$\lim_{n\to\infty} f(x_n) = +\infty = M.$$
In either situation, the sequence $\{x_n\}_{n\geq 1}$ thus defined is bounded (it is contained in $[a,b]$). We are then entitled to use Theorem of Bolzano-Weierstrass and call $\{x_{n_k}\}_{k\geq 0}$ a convergent subsequence. Let ξ be its limit; since all x_{n_k} belong to $[a,b]$, necessarily $\xi \in [a,b]$. But $\{f(x_{n_k})\}_{k\geq 0}$ is a subsequence of $\{f(x_n)\}_{n\geq 0}$, so by Proposition A.3.2
$$\lim_{k\to\infty} f(x_{n_k}) = M.$$
The continuity of f at ξ implies
$$f(\xi) = f(\lim_{k\to\infty} x_{n_k}) = \lim_{k\to\infty} f(x_{n_k}) = M,$$
which tells us that M cannot be $+\infty$. Moreover, M belongs to the range of f, hence
$$M = \max f([a,b]).$$
Arguing in a similar fashion one proves that the number
$$m = \min f([a,b])$$
exists and is finite. The final claim is a consequence of Corollary 4.30. □

▶ Proof of Corollary 4.25, p. 111

Corollary 4.25 *Let f be continuous on the interval I and suppose it admits non-zero limits (finite or infinite) that are different in sign for x tending to the end-points of I. Then f has a zero in I, which is unique if f is strictly monotone on I.*

Proof. We indicate by α, β (finite or not) the end-points of I and call
$$\lim_{x\to\alpha^+} f(x) = \ell_\alpha \qquad \text{and} \qquad \lim_{x\to\beta^-} f(x) = \ell_\beta\,.$$

Should one end-point, or both, be infinite, these writings denote the usual limits at infinity.

We suppose $\ell_\alpha < 0 < \ell_\beta$, for otherwise we can swap the roles of ℓ_α and ℓ_β. By Theorem 4.2 there exist a right neighbourhood $I^+(\alpha)$ of α and a left neighbourhood $I^-(\beta)$ of β such that

$$\forall x \in I^+(\alpha)\,,\ f(x) < 0 \qquad \text{and} \qquad \forall x \in I^-(\beta)\,,\ f(x) > 0\,.$$

Let us fix points $a \in I^+(\alpha)$ and $b \in I^-(\beta)$ with $\alpha < a$, $b < \beta$. The interval $[a, b]$ is contained in I, hence f is continuous on it, and by construction $f(a) < 0 < f(b)$. Therefore f will vanish somewhere, in virtue of Theorem 4.23 (existence of zeroes) applied to $[a, b]$.

If f is strictly monotone, uniqueness follows from Proposition 2.8 on the interval I. □

▶ Proof of Theorems 4.32 and 4.33, p. 114

Let us prove a preliminary result before proceeding.

Lemma A.3.4 *Let f be continuous and invertible on an interval I. For any chosen points $x_1 < x_2 < x_3$ in I, then one, and only one, of*

(i) $$f(x_1) < f(x_2) < f(x_3)$$

or

(ii) $$f(x_1) > f(x_2) > f(x_3)$$

holds.

Proof. As f is invertible, hence one-to-one, the images $f(x_1)$ and $f(x_3)$ cannot coincide. Then either $f(x_1) < f(x_3)$ or $f(x_1) > f(x_3)$, and we claim that these cases imply *(i)* or *(ii)*, respectively.

Suppose $f(x_1) < f(x_3)$, and assume by contradiction that *(i)* is false, so $f(x_2)$ does not lie strictly between $f(x_1)$ and $f(x_3)$. For instance,

$$f(x_1) < f(x_3) < f(x_2)$$

(if $f(x_2) < f(x_1) < f(x_3)$ the argument is the same). As f is continuous on the closed interval $[x_1, x_2] \subseteq I$, the Intermediate value theorem 4.29 prescribes that it will assume every value between $f(x_1)$ and $f(x_2)$ on $[x_1, x_2]$. In particular, there will be a point $\bar{x} \in (x_1, x_2)$ such that

$$f(\bar{x}) = f(x_3),$$

in contradiction to injectivity: $\bar{x}$ and x_3 are in fact distinct, because separated by x_2. □

Theorem 4.32 *A continuous function f on an interval I is one-to-one if and only if it is strictly monotone.*

Proof. Thanks to Proposition 2.8 we only need to prove the implication

$$f \text{ invertible on} I \quad \Rightarrow \quad f \ \text{ strictly monotone on } \ I\,.$$

Letting $x_1 < x_2$ be arbitrary points of I, we claim that if $f(x_1) < f(x_2)$ then f is strictly increasing on I ($f(x_1) > f(x_2)$ will similarly imply f strictly decreases on I).

Let $z_1 < z_2$ be points in I, and suppose both lie within (x_1, x_2); the other possibilities are dealt with in the same way. Hence we have

$$x_1 < z_1 < z_2 < x_2.$$

Let us use Lemma A.3.4 on the triple $x_1,\ z_1,\ x_2$: since we have assumed $f(x_1) < f(x_2)$, it follows

$$f(x_1) < f(z_1) < f(x_2).$$

Now we employ the triple $z_1,\ z_2,\ x_2$, to the effect that

$$f(z_1) < f(z_2) < f(x_2).$$

The first inequality in the above line tells f is strictly increasing, proving Theorem 4.32. □

Theorem 4.33 *Let f be continuous and invertible on an interval I. Then the inverse f^{-1} is continuous on the interval $J = f(I)$.*

Proof. The first remark is that J is indeed an interval, by Corollary 4.30. Using Theorem 4.32 we deduce f is strictly monotone on I: to fix ideas, suppose it is strictly increasing (having f strictly decreasing would not change the proof). By definition of a monotone map we have that f^{-1} is strictly increasing on J as well. But it is known that a monotone map admits at most discontinuities of the first kind (Corollary 3.28). We will show that f^{-1} cannot have this type either. By contradiction, suppose there is a jump point $y_0 = f(x_0) \in J = f(I)$ for f^{-1}. Equivalently, let

$$z_0^- = \sup_{y<y_0} f^{-1}(y) = \lim_{y \to y_0^-} f^{-1}(y)\,,$$

$$z_0^+ = \inf_{y>y_0} f^{-1}(y) = \lim_{y \to y_0^+} f^{-1}(y)\,,$$

and suppose $z_0^- < z_0^+$. Then inside (z_0^-, z_0^+) there will be at most one element $x_0 = f^{-1}(y_0)$ of the range $f^{-1}(J)$. Thus $f^{-1}(J)$ is not an interval. By definition of J, on the other hand, $f^{-1}(J) = I$ is an interval by hypothesis. In conclusion, f^{-1} must be continuous at each point of J. □

A.3.3 Uniform continuity

Let the map f be defined on the real interval I. Recall f is called continuous on I if it is continuous at each point $x_0 \in I$, i.e., for any $x_0 \in I$ and any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\forall x \in I, \qquad |x - x_0| < \delta \quad \Rightarrow \quad |f(x) - f(x_0)| < \varepsilon\,.$$

In general $\delta = \delta(\varepsilon, x_0)$, meaning that δ depends on x_0, too. But if, for fixed $\varepsilon > 0$, we find $\delta = \delta(\varepsilon)$ independent of $x_0 \in I$, we say f is uniformly continuous on I. More precisely,

Definition A.3.5 *A function f is called* **uniformly continuous on** I *if, for any $\varepsilon > 0$, there is a $\delta > 0$ satisfying*

$$\forall x', x'' \in I, \qquad |x' - x''| < \delta \quad \Rightarrow \quad |f(x') - f(x'')| < \varepsilon\,. \qquad \text{(A.3.2)}$$

Examples A.3.6

i) Let $f(x) = x^2$, defined on $I = [0, 1]$. Then

$$|f(x') - f(x'')| = |(x')^2 - (x'')^2| = |x' + x''|\,|x' - x''| \le 2|x' - x''|\,.$$

If $|x' - x''| < \frac{\varepsilon}{2}$ we see $|f(x') - f(x'')| < \varepsilon$, hence $\delta = \frac{\varepsilon}{2}$ fulfills (A.3.2) on I, rendering f uniformly continuous on I.

ii) Take $f(x) = x^2$ on $I = [0, +\infty)$. We want to prove by contradiction that f is not uniformly continuous on I. If it were, with $\varepsilon = 1$ for example, there would be a $\delta > 0$ satisfying (A.3.2). Choose $x' \in I$ and let $x'' = x' + \frac{\delta}{2}$, so that $|x' - x''| = \frac{\delta}{2} < \delta$; then

$$|f(x') - f(x'')| = |x' + x''|\,|x' - x''| < 1\,,$$

or

$$\Big(2x' + \frac{\delta}{2}\Big)\frac{\delta}{2} < 1\,.$$

Now letting x' tend to $+\infty$ we obtain a contradiction.

iii) Consider $f(x) = \sin x$ on $I = \mathbb{R}$. From

$$\sin x' - \sin x'' = 2 \sin \frac{x' - x''}{2} \cos \frac{x' + x''}{2}$$

we have

$$|\sin x' - \sin x''| \le |x' - x''|\,, \qquad \forall x', x'' \in \mathbb{R}\,.$$

With a fixed $\varepsilon > 0$, $\delta = \varepsilon$ satisfies the requirement for uniform continuity.

iv) Let $f(x) = \frac{1}{x}$ on $I = (0, +\infty)$. Note that

$$|f(x') - f(x'')| = \left| \frac{1}{x'} - \frac{1}{x''} \right| = \frac{|x' - x''|}{x'x''} \,.$$

By letting x', x'' tend to 0, one easily verifies that f cannot be uniformly continuous on I.

But if we consider only $I_a = [a, +\infty)$, where $a > 0$ is fixed, then

$$|f(x') - f(x'')| \le \frac{|x' - x''|}{a^2},$$

so $\delta = a^2\varepsilon$ satisfies the requirement on I_a, for any given $\varepsilon > 0$. □

Are there conditions guaranteeing uniform continuity? One answer is provided by the following result.

Theorem A.3.7 (**Heine-Cantor**) *Let f be a continuous map on the closed and bounded interval $I = [a, b]$. Then f is uniformly continuous on I.*

Proof. Let us suppose f is not uniformly continuous on I. This means that there exists an $\varepsilon > 0$ such that, for any $\delta > 0$, there are $x', x'' \in I$ with $|x' - x''| < \delta$ and $|f(x') - f(x'')| \ge \varepsilon$. Choosing $\delta = \frac{1}{n}$, $n \ge 1$, we find two sequences of points $\{x'_n\}_{n\ge1}$, $\{x''_n\}_{n\ge1}$ inside I such that

$$|x'_n - x''_n| < \frac{1}{n} \qquad \text{and} \qquad |f(x'_n) - f(x''_n)| \ge \varepsilon \,.$$

The condition on the left implies

$$\lim_{n\to\infty} (x'_n - x''_n) = 0 \,,$$

while the one on the right implies that $f(x'_n) - f(x''_n)$ will not tend to 0 for $n \to \infty$. On the other hand the sequence $\{x'_n\}_{n\ge1}$ is bounded, $a \le x'_n \le b$ for all n, so Theorem A.3.3, of Bolzano-Weierstrass, will give a subsequence $\{x'_{n_k}\}_{k\ge0}$ converging to a certain $\bar{x} \in I$:

$$\lim_{k\to\infty} x_{n_k} = \bar{x} \,.$$

Also the subsequence $\{x''_{n_k}\}_{k\ge0}$ converges to $\bar{x}$, for

$$\lim_{k\to\infty} x''_{n_k} = \lim_{k\to\infty} [x'_{n_k} + (x''_{n_k} - x'_{n_k})] = \lim_{k\to\infty} x_{n_k} + \lim_{k\to\infty} (x''_{n_k} - x'_{n_k}) = \bar{x} + 0 = \bar{x} \,.$$

Now, f being continuous at $\bar{x}$, we have

$$\lim_{k\to\infty} f(x'_{n_k}) = f\big(\lim_{k\to\infty} x_{n_k}\big) = f(\bar{x}) \quad \text{and} \quad \lim_{k\to\infty} f(x''_{n_k}) = f\big(\lim_{k\to\infty} x''_{n_k}\big) = f(\bar{x}) \,.$$

Then

$$\lim_{k\to\infty} \big(f(x'_{n_k}) - f(x''_{n_k})\big) = f(\bar{x}) - f(\bar{x}) = 0 \,,$$

contradicting the fact that

$$|f(x'_{n_k}) - f(x''_{n_k})| \ge \varepsilon > 0 \,, \qquad \forall k \ge 0 \,.$$ □

A.4

Complements on differential calculus

This appendix is entirely devoted to the proof of important results of differential calculus. We first justify the main derivation formulas, then we prove the Theorem of de l'Hôpital. Our next argument is the study of differentiable and convex functions, for which we highlight logical links between convexity and certain properties of the first derivative. At last, we establish Taylor formulas with three forms of the remainder, i.e., Peano's, Lagrange's and the integral form.

A.4.1 Derivation formulas

▶ Proof of Theorem 6.4, p. 174

Theorem 6.4 (**Algebraic operations**) *Let $f(x), g(x)$ be differentiable maps at $x_0 \in \mathbb{R}$. Then the maps $f(x) \pm g(x)$, $f(x)g(x)$ and, if $g(x_0) \neq 0$, $\dfrac{f(x)}{g(x)}$ are differentiable at x_0. To be precise,*

$$
\begin{aligned}
(f \pm g)'(x_0) &= f'(x_0) \pm g'(x_0), \\
(f\,g)'(x_0) &= f'(x_0)g(x_0) + f(x_0)g'(x_0), \\
\left(\frac{f}{g}\right)'(x_0) &= \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{[g(x_0)]^2}.
\end{aligned}
$$

Proof. Let us start by (6.3). By Theorem 4.10 we have

$$
\begin{aligned}
\lim_{x\to x_0} \frac{\big(f(x) \pm g(x)\big) - \big(f(x_0) \pm g(x_0)\big)}{x - x_0} &= \lim_{x\to x_0}\left(\frac{f(x) - f(x_0)}{x - x_0} \pm \frac{g(x) - g(x_0)}{x - x_0}\right) \\
&= \lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0} \pm \lim_{x\to x_0} \frac{g(x) - g(x_0)}{x - x_0} = f'(x_0) \pm g'(x_0)\,.
\end{aligned}
$$

Next we prove (6.4). For this, recall that a differentiable map is continuous (Proposition 6.3), so $\lim_{x\to x_0} g(x) = g(x_0)$. Therefore

$$\begin{aligned}
\lim_{x\to x_0} & \frac{f(x)\,g(x) - f(x_0)\,g(x_0)}{x - x_0} \\
& \lim_{x\to x_0} \frac{f(x)\,g(x) - f(x_0)\,g(x) + f(x_0)\,g(x) - f(x_0)\,g(x_0)}{x - x_0} \\
& = \lim_{x\to x_0} \left(\frac{f(x) - f(x_0)}{x - x_0}\, g(x) + f(x_0)\, \frac{g(x) - g(x_0)}{x - x_0} \right) \\
& = \lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0} \lim_{x\to x_0} g(x) + f(x_0) \lim_{x\to x_0} \frac{g(x) - g(x_0)}{x - x_0} \\
& = f'(x_0)\,g(x_0) + f(x_0)\,g'(x_0)\,.
\end{aligned}$$

Eventually, we show (6.5). Since $\lim_{x\to x_0} g(x) = g(x_0) \neq 0$, Theorem 4.2 ensures there is a neighbourhood of x_0 where $g(x) \neq 0$. Then the function $\dfrac{f(x)}{g(x)}$ is well defined on such neighbourhood and we can consider its difference quotient

$$\begin{aligned}
\lim_{x\to x_0} \frac{\frac{f(x)}{g(x)} - \frac{f(x_0)}{g(x_0)}}{x - x_0} &= \lim_{x\to x_0} \frac{f(x)\,g(x_0) - f(x_0)\,g(x)}{g(x)\,g(x_0)\,(x - x_0)} \\
&= \lim_{x\to x_0} \frac{f(x)\,g(x_0) - f(x_0)\,g(x_0) + f(x_0)\,g(x_0) - f(x_0)\,g(x)}{g(x)\,g(x_0)\,(x - x_0)} \\
&= \lim_{x\to x_0} \frac{1}{g(x)\,g(x_0)} \lim_{x\to x_0} \left(\frac{f(x) - f(x_0)}{x - x_0} g(x_0) - f(x_0) \frac{g(x) - g(x_0)}{x - x_0} \right) \\
&= \frac{1}{\big(g(x_0)\big)^2} \left(\lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0} g(x_0) - \lim_{x\to x_0} f(x_0) \frac{g(x) - g(x_0)}{x - x_0} \right) \\
&= \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{\big(g(x_0)\big)^2}\,.
\end{aligned}$$

□

▶ Proof of Theorem 6.7, p. 175

Theorem 6.7 ("Chain rule") *Let $f(x)$ be differentiable at $x_0 \in \mathbb{R}$ and $g(y)$ a differentiable map at $y_0 = f(x_0)$. Then the composition $g \circ f(x) = g\big(f(x)\big)$ is differentiable at x_0 and*

$$(g \circ f)'(x_0) = g'(y_0) f'(x_0) = g'\big(f(x_0)\big) f'(x_0).$$

Proof. Let us use the first formula of the finite increment (6.11) on g at the point $y_0 = f(x_0)$:

$$g(y) - g(y_0) = g'(y_0)(y - y_0) + o(y - y_0), \quad y \to y_0 .$$

By definition of 'little o', the above means that there exists a map φ such that $\lim_{y \to y_0} \varphi(y) = 0 = \varphi(y_0)$ satisfying

$$g(y) - g(y_0) = g'(y_0)(y - y_0) + \varphi(y)(y - y_0), \quad \text{on a neighbourhood } I(y_0) \text{ of } y_0 .$$

As f is continuous at x_0 (Proposition 6.3), there is a neighbourhood $I(x_0)$ of x_0 such that $f(x) \in I(y_0)$ for all $x \in I(x_0)$. If we put $y = f(x)$ in the displayed relation, this becomes

$$\frac{g(f(x)) - g(f(x_0))}{x - x_0} = g'(f(x_0)) \frac{f(x) - f(x_0)}{x - x_0} + \varphi(f(x)) \frac{f(x) - f(x_0)}{x - x_0} .$$

Observing that

$$\lim_{x \to x_0} \varphi(f(x)) = \lim_{y \to y_0} \varphi(y) = 0$$

by the Substitution theorem 4.15, we can pass to the limit and conclude

$$\begin{aligned} \lim_{x \to x_0} & \frac{g(f(x)) - g(f(x_0))}{x - x_0} \\ &= g'(f(x_0)) \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} + \lim_{x \to x_0} \varphi(f(x)) \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \\ &= g'(f(x_0)) f'(x_0) . \end{aligned}$$

□

▶ Proof of Theorem 6.9, p. 175

Theorem 6.9 (Derivative of the inverse function) *Suppose $f(x)$ is a continuous, invertible map on a neighbourhood of $x_0 \in \mathbb{R}$, and differentiable at x_0, with $f'(x_0) \neq 0$. Then the inverse map $f^{-1}(y)$ is differentiable at $y_0 = f(x_0)$, and*

$$(f^{-1})'(y_0) = \frac{1}{f'(x_0)} = \frac{1}{f'(f^{-1}(y_0))} .$$

Proof. The inverse map is continuous on a neighbourhood of y_0 in virtue of Theorem 4.33. Write $x = f^{-1}(y)$, so that on the same neighbourhood

$$\frac{f^{-1}(y) - f^{-1}(y_0)}{y - y_0} = \frac{x - x_0}{f(x) - f(x_0)} = \frac{1}{\frac{f(x) - f(x_0)}{x - x_0}} .$$

By the Substitution theorem 4.15, with $x = f^{-1}(y)$, we have

$$\lim_{y \to y_0} \frac{f^{-1}(y) - f^{-1}(y_0)}{y - y_0} = \lim_{x \to x_0} \frac{1}{\frac{f(x) - f(x_0)}{x - x_0}} = \frac{1}{f'(x_0)} .$$

□

A.4.2 De l'Hôpital's Theorem

▶ Proof of de l'Hôpital's Theorem, p. 200

Theorem 6.41 (**de l'Hôpital**) *Let f, g be maps defined on a neighbourhood of c, except possibly at c, and such that*

$$\lim_{x \to c} f(x) = \lim_{x \to c} g(x) = L,$$

where $L = 0, +\infty$ or $-\infty$. If f and g are differentiable around c, except possibly at c, with $g' \neq 0$, and if

$$\lim_{x \to c} \frac{f'(x)}{g'(x)}$$

exists (finite or not), then also

$$\lim_{x \to c} \frac{f(x)}{g(x)}$$

exists and equals the previous limit.

Proof. The theorem includes several statements corresponding to the values assumed by L and c, and the arguments used for the proofs vary accordingly. For this reason we have grouped the proofs together into cases.

a) The cases $L = 0$, $c = x_0^+, x_0^-, x_0$.
Let us suppose $c = x_0^+$. By assumption $\lim_{x \to x_0^+} f(x) = \lim_{x \to x_0^+} g(x) = 0$, so we may extend both functions to x_0 (re-defining their values if necessary) by putting $f(x_0) = g(x_0) = 0$; thus f and g become right-continuous at x_0. Let $I^+(x_0)$ denote the right neighbourhood of x_0 where f, g satisfy the theorem, and take $x \in I^+(x_0)$. On the interval $[x_0, x]$ Theorem 6.25 is valid, so there is $t = t(x) \in (x_0, x)$ such that

$$\frac{f(x)}{g(x)} = \frac{f(x) - f(x_0)}{g(x) - g(x_0)} = \frac{f'(t)}{g'(t)}.$$

As $x_0 < t(x) < x$, the Second comparison theorem 4.5 guarantees that for x tending to x_0 also $t = t(x)$ approaches x_0. Now the Substitution theorem 4.15 yields

$$\lim_{x \to x_0^+} \frac{f(x)}{g(x)} = \lim_{x \to x_0^+} \frac{f'(t(x))}{g'(t(x))} = \lim_{t \to x_0^+} \frac{f'(t)}{g'(t)},$$

and the proof ends.
We proceed similarly when $c = x_0^-$; the remaining case $c = x_0$ descends from the two one-sided limits.

b) The cases $L = 0$, $c = \pm\infty$.

Suppose $c = +\infty$. The substitution $z = \frac{1}{x}$ leads to consider the limit of the quotient $\dfrac{f(\frac{1}{z})}{g(\frac{1}{z})}$ for $z \to 0^+$. Because $\dfrac{\mathrm{d}}{\mathrm{d}z} f\left(\dfrac{1}{z}\right) = -\dfrac{1}{z^2} f'\left(\dfrac{1}{z}\right)$, and similarly for the map g, it follows

$$\lim_{z\to 0^+} \frac{\frac{\mathrm{d}}{\mathrm{d}z} f\left(\frac{1}{z}\right)}{\frac{\mathrm{d}}{\mathrm{d}z} g\left(\frac{1}{z}\right)} = \lim_{z\to 0^+} \frac{f'\left(\frac{1}{z}\right)}{g'\left(\frac{1}{z}\right)} = \lim_{x\to +\infty} \frac{f'(x)}{g'(x)} .$$

In this way we return to the previous case $c = 0^+$, and the result is proved. The same for $c = -\infty$.

c) The cases $L = \pm\infty$, $c = x_0^+, x_0^-, x_0$.

Assume $c = x_0^+$ and put $\displaystyle\lim_{x\to x_0^+} \frac{f'(x)}{g'(x)} = \ell$. When $\ell \in \mathbb{R}$, let $I^+(x_0)$ be the right neighbourhood of x_0 on which f and g satisfy the theorem. For every $\varepsilon > 0$ there exists $\delta_1 > 0$ with $x_0 + \delta_1 \in I^+(x_0)$ so that for all $x \in (x_0, x_0 + \delta_1)$ we have $\left|\dfrac{f'(x)}{g'(x)} - \ell\right| < \varepsilon$. On $[x, x_0 + \delta_1]$ Theorem 6.25 holds, hence there is $t = t(x) \in (x, x_0 + \delta_1)$ such that

$$\frac{f(x) - f(x_0 + \delta_1)}{g(x) - g(x_0 + \delta_1)} = \frac{f'(t)}{g'(t)} . \tag{A.4.1}$$

Write the ratio $\dfrac{f(x)}{g(x)}$ as

$$\frac{f(x)}{g(x)} = \psi(x) \frac{f'(t)}{g'(t)} ,$$

where, by (A.4.1),

$$\psi(x) = \frac{1 - \frac{g(x_0+\delta_1)}{g(x)}}{1 - \frac{f(x_0+\delta_1)}{f(x)}} , \qquad \text{with} \qquad \lim_{x\to x_0^+} \psi(x) = 1 ,$$

because $L = \pm\infty$. The last limit implies that there is a $\delta_2 > 0$, with $\delta_2 < \delta_1$, such that

$$|\psi(x)| \le 2 \qquad \text{and} \qquad |\psi(x) - 1| < \varepsilon$$

for every $x \in (x_0, x_0 + \delta_2)$. Therefore, for all $x \in (x_0, x_0 + \delta_2)$,

$$\begin{aligned} \left|\frac{f(x)}{g(x)} - \ell\right| &= \left|\psi(x)\frac{f'(t)}{g'(t)} - \psi(x)\ell + \psi(x)\ell - \ell\right| \\ &= |\psi(x)| \left|\frac{f'(t)}{g'(t)} - \ell\right| + |\psi(x) - 1||\ell| < \big(2 + |\ell|\big)\varepsilon . \end{aligned}$$

We conclude

$$\lim_{x\to x_0^+} \frac{f(x)}{g(x)} = \ell .$$

Let now $\ell = +\infty$; for all $A > 0$ there is $\delta_1 > 0$, with $x_0 + \delta_1 \in I^+(x_0)$, such that for all $x \in (x_0, x_0 + \delta_1)$ we have $\dfrac{f'(x)}{g'(x)} > A$. As before, using Theorem A.2.2 we observe that $\lim\limits_{x\to x_0^+} \psi(x) = 1$ implies the existence of a $\delta_2 > 0$, with $\delta_2 < \delta_1$, such that $\psi(x) \geq \frac{1}{2}$ for all $x \in (x_0, x_0 + \delta_2)$. Therefore, for every $x \in (x_0, x_0 + \delta_2)$,

$$\frac{f(x)}{g(x)} = \psi(x)\frac{f'(t)}{g'(t)} \geq \frac{1}{2}A\,,$$

proving the claim. The procedure is the same for $\ell = -\infty$.

An analogous proof holds for $c = x_0^-$, and $c = x_0$ is dealt with by putting the two arguments together.

d) The cases $L = \pm\infty$, $c = \pm\infty$.

As in b), we may substitute $z = \frac{1}{x}$ and use the previous argument. □

A.4.3 Convex functions

We begin with a lemma, which tells that local convexity (i.e., convexity on a neighbourhood of every point of I) is in fact a global condition (valid on all of I).

Lemma A.4.1 *Let f be differentiable on the interval I. Then f is convex on I if and only if for every $x_0 \in I$*

$$f(x) \geq f(x_0) + f'(x_0)(x - x_0) \qquad \forall x \in I\,. \tag{A.4.2}$$

Proof. Obviously, it is enough to show that if f is convex according to Definition 6.33 on I, then also (A.4.2) holds. To this end, one usefully notes that f is convex on I if and only if the map $g(x) = f(x)+ax+b$, $a, b \in \mathbb{R}$ is convex; in fact, requiring $f(x) \geq f(x_0)+f'(x_0)(x-x_0)$ is equivalent to $g(x) \geq g(x_0)+g'(x_0)(x-x_0)$.

Let then $x_0 \in I$ be fixed arbitrarily and consider the convex map $g(x) = f(x) - f(x_0) - f'(x_0)(x - x_0)$, which satisfies $g(x_0) = g'(x_0) = 0$. We have to prove that $g(x) \geq 0,\ \forall x \in I$. Suppose x_0 is not the right end-point of I and let us show $g(x) \geq 0,\ \forall x \in I,\ x > x_0$; a 'symmetry' argument will complete the proof.

Being g convex at x_0, we have $g(x) \geq 0$ on a (right) neighbourhood of x_0. It makes then sense to define

$$P = \{x > x_0 : g(s) \geq 0,\ \forall s \in [x_0, x]\}$$

and $x_1 = \sup P$.

If x_1 coincides with the right end-point of I, the assertion follows. Let us assume, by contradiction, x_1 lies inside I; By definition $g(x) \geq 0,\ \forall x \in [x_0, x_1)$, while in each (right) neighbourhood of x_1 there exist points $x \in I$ at which $g(x) < 0$. From this and the continuity of g at x_1 we deduce that necessarily $g(x_1) = 0$ (so, in particular, $x_1 = \max P$). We want to prove $g(x) = 0$, $\forall x \in [x_0, x_1]$. Once we have done that, then $g'(x_1) = 0$ (as g is differentiable at x_1 and constant on a left neighbourhood of the same point). Therefore the convexity of g at x_1 implies the existence of a neighbourhood of x_1 where $g(x) \geq 0$, against the definition of x_1.

It remains to prove $g(x) = 0$ in $[x_0, x_1]$. As $g(x) \geq 0$ on $[x_0, x_1]$ by definition, we assume, again by contradiction, that $M = \max\{g(x) : x \in [x_0, x_1]\} > 0$, and let $\bar{x} \in (x_0, x_1)$ be a pre-image of $g(\bar{x}) = M$. By Fermat's Theorem 6.21 $g'(\bar{x}) = 0$, so the convexity at $\bar{x}$ yields a neighbourhood of $\bar{x}$ on which $g(x) \geq g(\bar{x}) = M$; but M is the maximum of g on $[x_0, x_1]$, so $g(x) = M$ on said neighbourhood. Now define

$$Q = \{x > \bar{x} : g(s) = M, \ \forall s \in [\bar{x}, x]\}$$

and $x_2 = \sup Q$. The map g is continuous, hence $x_2 = \max Q$, and moreover $x_2 < x_1$ because $g(x_1) = 0$. As before, the hypothesis of convexity at x_2 leads to a contradiction. □

► Proof of Theorem 6.37, p. 193

Theorem 6.37 *Given a differentiable map f on the interval I,*

a) if f is convex on I, then f' is increasing on I.

b1) If f' is increasing on I, then f is convex on I;

b2) if f' is strictly increasing on I, then f is strictly convex on I.

Proof.
a) Take $x_1 < x_2$ two points in I. From (A.4.2) with $x_0 = x_1$ and $x = x_2$ we obtain

$$f'(x_1) \leq \frac{f(x_2) - f(x_1)}{x_2 - x_1},$$

while putting $x_0 = x_2$, $x = x_1$ gives

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq f'(x_2).$$

Combining the two inequalities yields the result.

b1) Let $x > x_0$ be chosen in I. The second formula of the finite increment of f on $[x_0, x]$ prescribes the existence of a point $\bar{x} \in (x_0, x)$ such that

$$f(x) = f(x_0) + f'(\bar{x})(x - x_0).$$

The map f' is monotone, so $f'(\bar{x}) \geq f'(x_0)$ hence (A.4.2). When $x < x_0$ the argument is analogous.

b2) In the proof for *b1)* we now have $f'(\bar{x}) > f'(x_0)$, whence (A.4.2) is strict (for $x \neq x_0$). □

A.4.4 Taylor formulas

We open this section by describing an interesting property of Taylor expansions. Observe to this end that if a map g is defined only at one point x_0, its Taylor polynomial of degree 0 can still be defined, by letting $Tg_{0,x_0}(x) = g(x_0)$.

Lemma A.4.2 *Let f be n times differentiable at x_0. The derivative of order h, $0 \le h \le n$, of the Taylor polynomial of f of degree n at x_0 coincides with the Taylor polynomial of $f^{(h)}$ of order $n-h$ at x_0:*

$$D^h Tf_{n,x_0}(x) = Tf^{(h)}_{n-h,x_0}(x)\,. \tag{A.4.3}$$

In particular,

$$D^h Tf_{n,x_0}(x_0) = f^{(h)}(x_0)\,, \qquad \forall h = 0, \dots, n\,. \tag{A.4.4}$$

Proof. From Example 6.31 i) we know that

$$D^h (x-x_0)^k = \begin{cases} 0 & \text{if } h > k \\ \dfrac{k!}{(k-h)!}(x-x_0)^{k-h} & \text{if } h \le k\,. \end{cases}$$

Therefore

$$\begin{aligned} D^h Tf_{n,x_0}(x) &= \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} D^h (x-x_0)^k \\ &= \sum_{k=h}^{n} \frac{f^{(k)}(x_0)}{(k-h)!} (x-x_0)^{k-h}\,. \end{aligned}$$

Note also that

$$f^{(k)}(x_0) = f^{(h+k-h)}(x_0) = \big(f^{(h)}\big)^{(k-h)}(x_0)\,,$$

in other words differentiating $k-h$ times the derivative of order h produces the kth derivative. In this way, putting $\ell = k-h$ gives

$$\begin{aligned} D^h Tf_{n,x_0}(x) &= \sum_{k=h}^{n} \frac{\big(f^{(h)}\big)^{(k-h)}(x_0)}{(k-h)!} (x-x_0)^{k-h} \\ &= \sum_{\ell=0}^{n-h} \frac{\big(f^{(h)}\big)^{(\ell)}(x_0)}{\ell!} (x-x_0)^{\ell} = Tf^{(h)}_{n-h,x_0}(x)\,, \end{aligned}$$

which is $(A.4.3)$. Formula $(A.4.4)$ follows by recalling that the Taylor expansion at a point x_0 of a function coincides with the function itself at that point. $\square$

▶ Proof of Theorem 7.1, p. 228

Theorem 7.1 (**Taylor formula with Peano's remainder**) *Let $n \geq 0$ and f be n times differentiable at x_0. Then the* **Taylor formula** *holds*

$$f(x) = Tf_{n,x_0}(x) + o\big((x-x_0)^n\big), \quad x \to x_0,$$

where

$$\begin{aligned} Tf_{n,x_0}(x) &= \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(x_0)(x-x_0)^k \\ &= f(x_0) + f'(x_0)(x-x_0) + \ldots + \frac{1}{n!} f^{(n)}(x_0)(x-x_0)^n. \end{aligned}$$

Proof. We need to show that

$$L = \lim_{x \to x_0} \frac{f(x) - Tf_{n,x_0}(x)}{(x-x_0)^n} = 0\,.$$

The limit is an indeterminate form of type $\frac{0}{0}$; in order to apply de l'Hôpital's Theorem 6.41 we are lead to consider this

$$\lim_{x \to x_0} \frac{f'(x) - (Tf_{n,x_0})'(x)}{n(x-x_0)^{n-1}} = \lim_{x \to x_0} \frac{f'(x) - Tf'_{n-1,x_0}(x)}{n(x-x_0)^{n-1}}\,,$$

(in which Lemma A.4.2, with $h = 1$, was used); note that the other requirements of 6.41 are fulfilled.

For $n > 1$ we are still in presence of an indeterminate form $\frac{0}{0}$, so repeating $n-1$ times the argument above brings us to the limit

$$\begin{aligned} &\lim_{x \to x_0} \frac{f^{(n-1)}(x) - Tf^{(n-1)}_{1,x_0}(x)}{n!(x-x_0)} \\ &\qquad = \lim_{x \to x_0} \frac{f^{(n-1)}(x) - f^{(n-1)}(x_0) - f^{(n)}(x_0)(x-x_0)}{n!(x-x_0)} \\ &\qquad = \frac{1}{n!} \lim_{x \to x_0} \left(\frac{f^{(n-1)}(x) - f^{(n-1)}(x_0)}{x - x_0} - f^{(n)}(x_0) \right) = 0\,, \end{aligned}$$

by definition of nth derivative at x_0. This grants the green light to the use of de l'Hôpital's Theorem, and $L = 0$. □

▶ Proof of Theorem 7.2, p. 228

Theorem 7.2 (**Taylor formula with Lagrange's remainder**) *Let $n \geq 0$ and f differentiable n times at x_0, with continuous nth derivative, be given; suppose f is differentiable $n+1$ times around x_0, except possibly at x_0. Then the Taylor formula*

$$f(x) = Tf_{n,x_0}(x) + \frac{1}{(n+1)!} f^{(n+1)}(\bar{x})(x-x_0)^{n+1},$$

holds, for a suitable $\bar{x}$ between x_0 and x.

Proof. Let $\varphi(x) = f(x) - Tf_{n,x_0}(x)$ and $\psi(x) = (x-x_0)^{n+1}$. Using (A.4.4), for $h = 0, \dots, n$ we have

$$\varphi^{(h)}(x_0) = 0\,;$$

moreover, $\psi^{(h)}(x_0) = 0$ and $\psi^{(h)}(x) \neq 0$ for any $x \neq x_0$. Applying Theorem 6.25 to φ, ψ on the interval I_0 between x_0 and x, we know there is a point $x_1 \in I_0$ such that

$$\frac{\varphi(x)}{\psi(x)} = \frac{\varphi(x) - \varphi(x_0)}{\psi(x) - \psi(x_0)} = \frac{\varphi'(x_1)}{\psi'(x_1)}\,.$$

The same recipe used on the maps $\varphi'(x)$, $\psi'(x)$ on the interval I_1 between x_0, x_1 produces a point $x_2 \in I_1 \subset I_0$ satisfying

$$\frac{\varphi'(x_1)}{\psi'(x_1)} = \frac{\varphi'(x_1) - \varphi'(x_0)}{\psi'(x_1) - \psi'(x_0)} = \frac{\varphi''(x_2)}{\psi''(x_2)}\,.$$

Iterating the argument eventually gives a $x_{n+1} \in I_0$ such that

$$\frac{\varphi(x)}{\psi(x)} = \cdots = \frac{\varphi^{(n+1)}(x_{n+1})}{\psi^{(n+1)}(x_{n+1})}\,.$$

But $\varphi^{(n+1)}(x) = f^{(n+1)}(x)$ and $\psi^{(n+1)}(x) = (n+1)!$, putting $\bar{x} = x_{n+1}$ in which yields the assertion. □

▶ Proof of Theorem 9.44, p. 338

Theorem 9.44 (**Taylor formula with integral remainder**) *Let $n \geq 0$ be an arbitrary integer, f differentiable $n+1$ times around a point x_0, with continuous derivative of order $n+1$. Then*

$$f(x) - Tf_{n,x_0}(x) = \frac{1}{n!} \int_{x_0}^{x} f^{(n+1)}(t)(x-t)^n \,\mathrm{d}t\,.$$

Proof. We shall use the induction principle (see Appendix A.1). When $n = 0$, the formula reduces to the identity

$$f(x) - f(x_0) = \int_{x_0}^{x} f'(t)\,dt\,,$$

established in Corollary 9.42.

Supposing the statement true for a certain n, let us prove it for $n+1$. Integrating by parts and using the hypothesis,

$$\begin{aligned}
&\frac{1}{(n+1)!}\int_{x_0}^{x} f^{(n+2)}(t)(x-t)^{n+1}\,dt \\
&\quad= \frac{1}{(n+1)!}\left[f^{(n+1)}(t)(x-t)^{n+1}\Big|_{x_0}^{x} + (n+1)\int_{x_0}^{x} f^{(n+1)}(t)(x-t)^{n}\,dt\right] \\
&\quad= -\frac{f^{(n+1)}(x_0)}{(n+1)!}(x-x_0)^{n+1} + \frac{1}{n!}\int_{x_0}^{x} f^{(n+1)}(t)(x-t)^{n}\,dt \\
&\quad= -\frac{f^{(n+1)}(x_0)}{(n+1)!}(x-x_0)^{n+1} + f(x) - Tf_{n,x_0}(x) \\
&\quad= f(x) - Tf_{n+1,x_0}(x)\,.
\end{aligned}$$

□

At last, we provide an example that illustrates how a more accurate piece of information may be extracted from the integral form of the remainder, as opposed to the Lagrange form.

Example A.4.3 Consider the MacLaurin expansion of the exponential function $f(x) = \mathrm{e}^x$ with remainder of order 1, both in Lagrange's form and in integral form. Assuming $x > 0$, if we use the former form we have for a suitable $\bar{x} \in (0, x)$

$$\mathrm{e}^x = 1 + x + \frac{1}{2}\mathrm{e}^{\bar{x}}x^2\,, \tag{A.4.5}$$

whereas with the latter form we obtain

$$\mathrm{e}^x = 1 + x + \int_0^x \mathrm{e}^t(x-t)\,\mathrm{d}t\,. \tag{A.4.6}$$

Since the exponential function is strictly increasing, it holds $\mathrm{e}^{\bar{x}} < \mathrm{e}^x$, hence, we deduce from (A.4.5) that the error due to approximating e^x by the polynomial $1 + x$ satisfies

$$0 < \mathrm{e}^x - (1+x) < \frac{1}{2}x^2\mathrm{e}^x\,. \tag{A.4.7}$$

On the other hand, if we look at the integral remainder, we easily check that the function $g(t) = \mathrm{e}^t(x-t)$ under the integral sign admits for $x \geq 1$ a strict maximum at $t = x - 1$, where it takes the value e^{x-1}. Hence,

$$0 < \int_0^x \mathrm{e}^t (x-t)\,\mathrm{d}t < \mathrm{e}^{x-1} \int_0^x \mathrm{d}t = \mathrm{e}^{x-1} x \,.$$

Therefore, we deduce from (A.4.6) that

$$0 < \mathrm{e}^x - (1+x) < \frac{1}{\mathrm{e}} x \mathrm{e}^x \,, \qquad x \geq 1 \,. \tag{A.4.8}$$

Since it is trivially seen that $\frac{1}{\mathrm{e}} x\mathrm{e}^x < \frac{1}{2} x^2 \mathrm{e}^x$ for $x \geq 1$, we conclude that (A.4.8) provides a more accurate estimate of the approximation error than (A.4.7) does. For instance, for $x = 1$ the error is

$$\mathrm{e}^1 - (1+1) = \mathrm{e} - 2 = 0.71828\ldots;$$

inequality (A.4.7) gives the upper bound $0.71828\ldots < \frac{1}{2}\mathrm{e} = 1.35914\ldots$, whereas (A.4.8) gives the bound $0.71828\ldots < \frac{1}{\mathrm{e}}\mathrm{e} = 1$, which is sharper. □

A.5

Complements on integral calculus

We begin this appendix by checking the convergence of the two sequences that enter the definition of the Cauchy integral. We then consider the Riemann integral; we justify the integrability of relevant classes of functions, and we establish several properties of integrable functions and of the definite integral. We conclude by proving a few results concerning improper integrals.

A.5.1 The Cauchy integral

▶ Proof of Theorem 9.20, p. 320

Theorem 9.20 *The sequences $\{s_n\}$ and $\{S_n\}$ are convergent, and their limits coincide.*

Proof. We claim that for any $p \geq 1$

$$s_n \leq s_{pn}\,, \qquad S_{pn} \leq S_n\,.$$

In fact, subdividing the interval I_k in p subintervals I_{ki} $(1 \leq i \leq p)$ of equal width $\Delta x/p$, and letting

$$m_{ki} = \min_{x \in I_{ki}} f(x)\,,$$

it follows $m_k \leq m_{ki}$ for each i, hence

$$m_k \Delta x \leq \sum_{i=1}^{p} m_{ki} \frac{\Delta x}{p}\,.$$

Summing over k we obtain $s_n \leq s_{pn}$. The second inequality is similar.

Let now s_n, S_m be arbitrary sums. Since

$$s_n \leq s_{nm} \leq S_{nm} \leq S_m$$

any lower sum is less or equal than any upper sum. Define

$$s = \sup_n s_n \qquad \text{and} \qquad S = \inf_n S_n \, .$$

We know $s \le S_m$ holds for any m, so $s \le S$. We wish to prove that $s = S$, and that such number is indeed the required limit. By the Heine-Cantor's Theorem A.3.7 the map f is uniformly continuous: given $\varepsilon > 0$, there is $\delta > 0$ such that if $x', x'' \in [a, b]$ then

$$|x' - x''| < \delta \qquad \text{implies} \qquad |f(x') - f(x'')| < \varepsilon \, .$$

Let n_ε be the integer such that $\frac{b-a}{n_\varepsilon} < \delta$. Take any $n \ge n_\varepsilon$; in each subinterval I_k of $[a, b]$ of width $\Delta x = \frac{b-a}{n}$ there exist points ξ_k and η_k such that

$$f(\xi_k) = m_k = \min_{x \in I_k} f(x) \qquad \text{and} \qquad f(\eta_k) = M_k = \max_{x \in I_k} f(x) \, .$$

As $|\eta_k - \xi_k| \le \frac{b-a}{n} \le \frac{b-a}{n_\varepsilon} < \delta$, it follows

$$M_k - m_k = f(\eta_k) - f(\xi_k) < \varepsilon \, .$$

Therefore

$$\begin{aligned} S_n - s_n &= \sum_{k=1}^{n} M_k \Delta x - \sum_{k=1}^{n} m_k \Delta x \\ &= \sum_{k=1}^{n} \big(M_k - m_k\big) \Delta x < \varepsilon \sum_{k=1}^{n} \Delta x = \varepsilon (b-a) \, . \end{aligned}$$

In other words, given $\varepsilon > 0$ there is an $n_\varepsilon > 0$ so that for all $n \ge n_\varepsilon$ we have $0 \le S_n - s_n < \varepsilon(b-a)$. This implies

$$S - s \le S_n - s_n < \varepsilon(b-a) \, .$$

Letting ε tend to 0, $S = s$ follows. In addition,

$$S - s_n \le S_n - s_n < \varepsilon \qquad \text{if} \quad n \ge n_\varepsilon \, ,$$

that is,

$$\lim_{n \to \infty} s_n = S \, .$$

The same arguments may be adapted to show $\lim_{n \to \infty} S_n = S$. □

A.5.2 The Riemann integral

Throughout the section we shall repeatedly use the following result.

Lemma A.5.1 *Let f be a bounded function on $I = [a, b]$. Then f is integrable if and only if for any $\varepsilon > 0$ there exist two maps $h_\varepsilon \in \mathcal{S}_f^+$ and $g_\varepsilon \in \mathcal{S}_f^-$ such that*

$$\int_I h_\varepsilon - \int_I g_\varepsilon < \varepsilon\,.$$

Proof. According to the definition, f is integrable if and only if

$$\int_I f = \inf\left\{\int_I h : h \in \mathcal{S}_f^+\right\} = \sup\left\{\int_I g : g \in \mathcal{S}_f^-\right\}.$$

Let then f be integrable. Given $\varepsilon > 0$, by definition of lower and upper bound one can find a map $h_\varepsilon \in \mathcal{S}_f^+$ satisfying $\displaystyle\int_I h_\varepsilon - \int_I f < \varepsilon/2$ and, similarly, a function $g_\varepsilon \in \mathcal{S}_f^-$ such that $\displaystyle\int_I f - \int_I g_\varepsilon < \varepsilon/2$. Hence

$$\int_I h_\varepsilon - \int_I g_\varepsilon = \int_I h_\varepsilon - \int_I f + \int_I f - \int_I g_\varepsilon < \varepsilon\,.$$

Vice versa, using Definition 9.26 together with Property 9.27, one has

$$\int_I g_\varepsilon \le \underline{\int_I} f \le \overline{\int_I} f \le \int_I h_\varepsilon\,,$$

hence

$$\overline{\int_I} f - \underline{\int_I} f \le \int_I h_\varepsilon - \int_I g_\varepsilon < \varepsilon\,.$$

But ε is completely arbitrary, so $\displaystyle\underline{\int_I} f = \overline{\int_I} f$. In other words, f is integrable on $[a, b]$.

□

▶ Proof of Theorem 9.31, p. 327

Theorem 9.31 *Among the class of integrable maps on $[a, b]$ are*

a) continuous maps on $[a, b]$;
b) piecewise-continuous maps on $[a, b]$;
c) continuous maps on (a, b) which are bounded on $[a, b]$;
d) monotone functions on $[a, b]$.

Proof.
a) The Theorem of Weierstrass tells that f is bounded over $[a, b]$, and by Heine-Cantor's Theorem A.3.7 f is uniformly continuous on $[a, b]$. Thus for any given

$\varepsilon > 0$, there exists a $\delta > 0$ such that if $x', x'' \in [a, b]$ with $|x' - x''| < \delta$ then $|f(x') - f(x'')| < \varepsilon$. Let us consider a partition $\{x_0, x_1, \ldots, x_n\}$ of $[a, b]$ such that each interval $[x_{k-1}, x_k]$ has width $< \delta$ $(k = 1, \ldots, n)$. We apply Weierstrass' Theorem 4.31 to each one of them: for every $k = 1, \ldots, n$, there are points $\xi_k, \eta_k \in [x_{k-1}, x_k]$ such that

$$f(\xi_k) = m_k = \min_{x \in [x_{k-1}, x_k]} f(x) \qquad \text{and} \qquad f(\eta_k) = M_k = \max_{x \in [x_{k-1}, x_k]} f(x)\,.$$

Since $|\eta_k - \xi_k| < \delta$,

$$M_k - m_k = f(\eta_k) - f(\xi_k) < \varepsilon\,.$$

Let $h_\varepsilon \in \mathcal{S}_f^+$ and $g_\varepsilon \in \mathcal{S}_f^-$ be defined by

$$h_\varepsilon(x) = \begin{cases} M_k & \text{if } x \in (x_{k-1}, x_k],\ k = 1, \ldots, n, \\ f(a) & \text{if } x = a, \end{cases}$$

$$g_\varepsilon(x) = \begin{cases} m_k & \text{if } x \in (x_{k-1}, x_k],\ k = 1, \ldots, n, \\ f(a) & \text{if } x = a. \end{cases}$$

For any $x \in [a, b]$ we have $h_\varepsilon(x) - g_\varepsilon(x) < \varepsilon$, hence

$$\int_I h_\varepsilon - \int_I g_\varepsilon = \int_I (h_\varepsilon - g_\varepsilon) < \int_I \varepsilon = (b - a)\varepsilon\,.$$

Given that ε is arbitrary, Lemma A.5.1 yields the result.

b) Call $\{x_1, x_2, \ldots, x_{n-1}\}$ the discontinuity points of f inside $[a, b]$, with $x_{k-1} < x_k$, and set $x_0 = a$ and $x_n = b$. For $k = 1, \ldots, n$, consider the continuous maps on $[x_{k-1}, x_k]$ defined as follows:

$$f_k(x) = \begin{cases} f(x) & \text{if } x \in (x_{k-1}, x_k), \\ \lim\limits_{x \to x_{k-1}^+} f(x) & \text{if } x = x_{k-1}, \\ \lim\limits_{x \to x_k^-} f(x) & \text{if } x = x_k\,. \end{cases}$$

Mimicking the proof of part *a)*, given $\varepsilon > 0$ there exist $h_{\varepsilon,k} \in \mathcal{S}_{f_k}^+$, $g_{\varepsilon,k} \in \mathcal{S}_{f_k}^-$ such that

$$h_{\varepsilon,k}(x) - g_{\varepsilon,k}(x) < \varepsilon\,, \qquad \forall x \in [x_{k-1}, x_k]\,.$$

Define $h_\varepsilon \in \mathcal{S}_f^+$ and $g_\varepsilon \in \mathcal{S}_f^-$ by

$$h_\varepsilon(x) = \begin{cases} h_{\varepsilon,k}(x) & \text{if } x \in (x_{k-1}, x_k],\ k = 1, \ldots, n, \\ f(a) & \text{if } x = a, \end{cases}$$

$$g_\varepsilon(x) = \begin{cases} g_{\varepsilon,k}(x) & \text{if } x \in (x_{k-1}, x_k],\ k = 1, \ldots, n, \\ f(a) & \text{if } x = a. \end{cases}$$

For any $x \in [a, b]$ then, $h_\varepsilon(x) - g_\varepsilon(x) < \varepsilon$; as before, Lemma A.5.1 ends the proof.

c) Fix $\varepsilon > 0$ so that $I_\varepsilon = [a+\varepsilon, b-\varepsilon] \subset [a,b]$. The map f is continuous on I_ε and we may find – as in part *a)* – two step functions defined on I_ε, say φ_ε and ψ_ε, such that

$$\varphi_\varepsilon(x) \le f(x) \le \psi_\varepsilon(x) \qquad \text{and} \qquad \psi_\varepsilon(x) - \varphi_\varepsilon(x) < \varepsilon\,, \quad \forall x \in I_\varepsilon\,.$$

Name $M = \sup\limits_{x\in I} f(x)$ and $m = \inf\limits_{x\in I} f(x)$ the supremum and infimum of f. Consider the step functions $h_\varepsilon \in \mathcal{S}_f^+$, $g_\varepsilon \in \mathcal{S}_f^-$ given by

$$h_\varepsilon(x) = \begin{cases} \psi_\varepsilon(x) & \text{if } x \in I_\varepsilon, \\ M & \text{if } x \notin I_\varepsilon, \end{cases} \qquad\qquad g_\varepsilon(x) = \begin{cases} \varphi_\varepsilon(x) & \text{if } x \in I_\varepsilon, \\ m & \text{if } x \notin I_\varepsilon. \end{cases}$$

Theorem 9.33 *i)* implies

$$\begin{aligned} \int_I h_\varepsilon - \int_I g_\varepsilon &= \int_{[a,a+\varepsilon]} (h_\varepsilon - g_\varepsilon) + \int_{I\varepsilon} (h_\varepsilon - g_\varepsilon) + \int_{[b-\varepsilon,b]} (h_\varepsilon - g_\varepsilon) \\ &= 2(M-m)\varepsilon + \int_{I\varepsilon} (h_\varepsilon - g_\varepsilon) \\ &< 2(M-m)\varepsilon + (b-a-2\varepsilon)\varepsilon < \big(2(M-m) + b - a\big)\varepsilon\,. \end{aligned}$$

Now Lemma A.5.1 allows to conclude.

d) Assume f is increasing. (In case f is decreasing, the proof is analogous.) Note first that f is bounded on $[a,b]$, for $f(a) \le f(x) \le f(b)$, $\forall x \in [a,b]$.

Given $\varepsilon > 0$, let n be a natural number such that $n > \dfrac{b-a}{n}$; split the interval into n parts, each $\dfrac{b-a}{n} < \varepsilon$ wide, and let $\{x_0, x_1, \ldots, x_n\}$ indicate the partition points. Introduce the step maps $h_n \in \mathcal{S}_f^+$, $g_n \in \mathcal{S}_f^-$ by

$$h_n(x) = \begin{cases} f(x_k) & \text{if } x \in (x_{k-1}, x_k],\ k = 1, \ldots, n, \\ f(a) & \text{if } x = a, \end{cases}$$

$$g_n(x) = \begin{cases} f(x_{k-1}) & \text{if } x \in (x_{k-1}, x_k],\ k = 1, \ldots, n, \\ f(a) & \text{if } x = a. \end{cases}$$

Then

$$\begin{aligned} \int_I h_n - \int_I g_n &= \sum_{k=1}^n f(x_k)(x_k - x_{k-1}) - \sum_{k=1}^n f(x_{k-1})(x_k - x_{k-1}) \\ &= \frac{b-a}{n} \sum_{k=1}^n \big(f(x_k) - f(x_{k-1})\big) = \frac{b-a}{n}\big(f(b) - f(a)\big) \\ &= \varepsilon\big(f(b) - f(a)\big)\,. \end{aligned}$$

Once again, the result follows from Lemma A.5.1. □

▶ Proof of Proposition 9.32, p. 328

Proposition 9.32 *If f is integrable on $[a, b]$, then*

i) *f is integrable on any subinterval $[c, d] \subset [a, b]$;*

ii) *$|f|$ is integrable on $[a, b]$.*

Proof.

i) If f is a step function the statement is immediate. More generally, let f be integrable over $[a, b]$; for $\varepsilon > 0$, Lemma A.5.1 yields maps $h_\varepsilon \in \mathcal{S}_f^+$, $g_\varepsilon \in \mathcal{S}_f^-$ such that

$$\int_a^b h_\varepsilon - \int_a^b g_\varepsilon = \int_a^b \big(h_\varepsilon - g_\varepsilon\big) < \varepsilon\,.$$

As

$$\int_c^d \big(h_\varepsilon - g_\varepsilon\big) \leq \int_a^b \big(h_\varepsilon - g_\varepsilon\big) < \varepsilon\,,$$

the result is a consequence of Lemma A.5.1 applied to the function f restricted to $[c, d]$.

ii) Recall $|f| = f_+ + f_-$, where f_+ and f_- denote the positive and negative parts of f respectively. Thus it is enough to show that f_+ and f_- are integrable, for then we can use Theorem 9.33 *ii)*.

Let us prove f_+ is integrable. Given $\varepsilon > 0$, by Lemma A.5.1 there exist $h_\varepsilon \in \mathcal{S}_f^+$ and $g_\varepsilon \in \mathcal{S}_f^-$ such that $\int_a^b h_\varepsilon - \int_a^b g_\varepsilon < \varepsilon$. Let $\{x_0, x_1, \ldots, x_n\}$ be a partition of $I = [a, b]$ adapted to both maps h_ε, g_ε. Consider the positive parts $h_{\varepsilon,+}$, $g_{\varepsilon,+}$ of the step functions. Having fixed an interval $I_k = [x_{k-1}, x_k]$, we may examine the three possible occurrencies $0 \leq g_\varepsilon \leq h_\varepsilon$, $g_\varepsilon \leq 0 \leq h_\varepsilon$ or $g_\varepsilon \leq h_\varepsilon \leq 0$. It is easy to check

$$g_{\varepsilon,+} \leq f_+ \leq h_{\varepsilon,+}$$

and

$$\int_{I_k} h_{\varepsilon,+} - \int_{I_k} g_{\varepsilon,+} \leq \int_{I_k} h_\varepsilon - \int_{I_k} g_\varepsilon < \varepsilon\,.$$

Consequently, $h_{\varepsilon,+} \in \mathcal{S}_{f_+}^+$, $g_{\varepsilon,+} \in \mathcal{S}_{f_+}^-$, and

$$\int_I h_{\varepsilon,+} - \int_I g_{\varepsilon,+} = \sum_{k=1}^n \Big(\int_{I_k} h_{\varepsilon,+} - \int_{I_k} g_{\varepsilon,+}\Big) \leq \sum_{k=1}^n \Big(\int_{I_k} h_\varepsilon - \int_{I_k} g_\varepsilon\Big) < \varepsilon\,.$$

Lemma A.5.1 yields then integrability for f_+.
A similar proof would tell that f_- is integrable as well. □

▶ **Proof of Theorem 9.33, p. 329**

Theorem 9.33 *Let f and g be integrable on a bounded interval I of the real line.*

i) (**Additivity with respect to the domain of integration**) *For any $a, b, c \in I$,*

$$\int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x.$$

ii) (**Linearity**) *For any $a, b \in I$ and $\alpha, \beta \in \mathbb{R}$,*

$$\int_a^b \Big(\alpha f(x) + \beta g(x)\Big)\,\mathrm{d}x = \alpha \int_a^b f(x)\,\mathrm{d}x + \beta \int_a^b g(x)\,\mathrm{d}x.$$

iii) (**Positivity**) *Let $a, b \in I$, with $a < b$. If $f \geq 0$ on $[a, b]$ then*

$$\int_a^b f(x)\,\mathrm{d}x \geq 0.$$

If f is additionally continuous, equality holds if and only if f is the zero map.

iv) (**Monotonicity**) *Let $a, b \in I$, $a < b$. If $f \leq g$ in $[a, b]$, then*

$$\int_a^b f(x)\,\mathrm{d}x \leq \int_a^b g(x)\,\mathrm{d}x.$$

v) (**Upper and lower bounds**) *Let $a, b \in I$, $a < b$. Then*

$$\left|\int_a^b f(x)\,\mathrm{d}x\right| \leq \int_a^b |f(x)|\,\mathrm{d}x.$$

Proof. We shall directly prove statements *i)* -*v)* for generic integrable maps, for the case of step functions is fairly straightforward.

i) We shall suppose $a < c < b$, for the other instances descend from this and (9.18). By Proposition 9.32 *i)* f is integrable on the intervals $[a, b]$, $[a, c]$, $[c, b]$. Given $\varepsilon > 0$ moreover, let $g_\varepsilon \in \mathcal{S}_f^-$, $h_\varepsilon \in \mathcal{S}_f^+$ be such that

$$\int_a^b h_\varepsilon - \int_a^b g_\varepsilon < \varepsilon \qquad \text{and} \qquad \int_a^b g_\varepsilon \leq \int_a^b f \leq \int_a^b h_\varepsilon\,.$$

The property holds for step functions, so

$$\int_a^b g_\varepsilon = \int_a^c g_\varepsilon + \int_c^b g_\varepsilon \leq \int_a^c f + \int_c^b f \leq \int_a^c h_\varepsilon + \int_c^b h_\varepsilon = \int_a^b h_\varepsilon$$

and hence
$$\left| \int_a^b f - \int_a^c f - \int_c^b f \right| \le \int_a^b h_\varepsilon - \int_a^b g_\varepsilon < \varepsilon\,.$$
The claim follows because ε is arbitrary.

ii) We split the proof in two, and prove that

a) $\displaystyle\int_a^b \alpha f(x)\,\mathrm{d}x = \alpha \int_a^b f(x)\,\mathrm{d}x$

b) $\displaystyle\int_a^b \big(f(x)+g(x)\big)\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x + \int_a^b g(x)\,\mathrm{d}x\,.$

We start from a), and suppose $a<b$ for simplicity. When $\alpha = 0$ the result is clear, so let $\alpha>0$. If $g \in \mathcal{S}_f^-$, $h \in \mathcal{S}_f^+$ then $\alpha g \in \mathcal{S}_{\alpha f}^-$ and $\alpha h \in \mathcal{S}_{\alpha f}^+$; thus
$$\begin{aligned}\alpha \int_a^b g(x)\,\mathrm{d}x = \int_a^b \alpha g(x)\,\mathrm{d}x &\le \underline{\int_a^b} \alpha f(x)\,\mathrm{d}x \\ &\le \overline{\int_a^b} \alpha f(x)\,\mathrm{d}x \le \int_a^b \alpha h(x)\,\mathrm{d}x = \alpha \int_a^b h(x)\,\mathrm{d}x\,.\end{aligned}$$

From $\displaystyle\alpha \int_a^b g(x)\,\mathrm{d}x \le \underline{\int_a^b} \alpha f(x)\,\mathrm{d}x$, taking the upper bound of the integrals $\int_a^b g$ as g varies in $\mathcal{S}_f^-$, and using the integrability of f on $[a,b]$, we obtain
$$\alpha \underline{\int_a^b} f(x)\,\mathrm{d}x = \alpha \int_a^b f(x)\,\mathrm{d}x \le \underline{\int_a^b} \alpha f(x)\,\mathrm{d}x\,;$$

similarly from $\displaystyle\overline{\int_a^b} \alpha f(x)\,\mathrm{d}x \le \alpha \int_a^b h(x)\,\mathrm{d}x$ we get
$$\overline{\int_a^b} \alpha f(x)\,\mathrm{d}x \le \alpha \overline{\int_a^b} f(x)\,\mathrm{d}x = \alpha \int_a^b f(x)\,\mathrm{d}x\,.$$

In conclusion,
$$\alpha \int_a^b f(x)\,\mathrm{d}x \le \underline{\int_a^b} \alpha f(x)\,\mathrm{d}x \le \overline{\int_a^b} \alpha f(x)\,\mathrm{d}x \le \alpha \int_a^b f(x)\,\mathrm{d}x$$

hence $\displaystyle\alpha \int_a^b f(x)\,\mathrm{d}x = \int_a^b \alpha f(x)\,\mathrm{d}x$.
When $\alpha<0$, the proof is the same because $g \in \mathcal{S}_f^-$, $h \in \mathcal{S}_f^+$ satisfy $\alpha g \in \mathcal{S}_{\alpha f}^+$ and $\alpha h \in \mathcal{S}_{\alpha f}^-$.

Now part b). Take $f_1 \in \mathcal{S}_f^-$, $f_2 \in \mathcal{S}_f^+$, $g_1 \in \mathcal{S}_g^-$, $g_2 \in \mathcal{S}_g^+$; then $f_1+g_1 \in \mathcal{S}_{f+g}^-$, $f_2+g_2 \in \mathcal{S}_{f+g}^+$, and

$$\begin{aligned}\int_a^b f_1(x)\,\mathrm{d}x + \int_a^b g_1(x)\,\mathrm{d}x &= \int_a^b \big(f_1(x)+g_1(x)\big)\,\mathrm{d}x \le \underline{\int_a^b}\big(f(x)+g(x)\big)\,\mathrm{d}x \\ &\le \overline{\int_a^b}\big(f(x)+g(x)\big)\,\mathrm{d}x \le \int_a^b \big(f_2(x)+g_2(x)\big)\,\mathrm{d}x \\ &= \int_a^b f_2(x)\,\mathrm{d}x + \int_a^b g_2(x)\,\mathrm{d}x\,.\end{aligned}$$

Fix g_1, f_2 and g_2, and take the upper bound of the integrals $\displaystyle\int_a^b f_1(x)\,\mathrm{d}x$ as $f_1 \in \mathcal{S}_f^-$ varies:

$$\begin{aligned}\int_a^b f(x)\,\mathrm{d}x + \int_a^b g_1(x)\,\mathrm{d}x &\le \underline{\int_a^b}\big(f(x)+g(x)\big)\,\mathrm{d}x \\ &\le \overline{\int_a^b}\big(f(x)+g(x)\big)\,\mathrm{d}x \le \int_a^b f_2(x)\,\mathrm{d}x + \int_a^b g_2(x)\,\mathrm{d}x\,;\end{aligned}$$

varying g_1 in $\mathcal{S}_g^-$ and taking the upper bound of the integrals $\displaystyle\int_a^b g_1(x)\,\mathrm{d}x$ we find

$$\begin{aligned}\int_a^b f(x)\,\mathrm{d}x + \int_a^b g(x)\,\mathrm{d}x &\le \underline{\int_a^b}\big(f(x)+g(x)\big)\,\mathrm{d}x \\ &\le \overline{\int_a^b}\big(f(x)+g(x)\big)\,\mathrm{d}x \le \int_a^b f_2(x)\,\mathrm{d}x + \int_a^b g_2(x)\,\mathrm{d}x\,.\end{aligned}$$

Now we may repeat the argument fixing g_2 and varying $f_2 \in \mathcal{S}_f^+$ first, then varying $g_2 \in \mathcal{S}_g^+$, to obtain

$$\begin{aligned}\int_a^b f(x)\,\mathrm{d}x + \int_a^b g(x)\,\mathrm{d}x &\le \underline{\int_a^b}\big(f(x)+g(x)\big)\,\mathrm{d}x \\ &\le \overline{\int_a^b}\big(f(x)+g(x)\big)\,\mathrm{d}x \le \int_a^b f(x)\,\mathrm{d}x + \int_a^b g(x)\,\mathrm{d}x\,.\end{aligned}$$

iii) The zero map g belongs in $\mathcal{S}_f^-$ (it is constant), hence

$$0 = \int_a^b g(x)\,\mathrm{d}x \le \int_a^b f(x)\,\mathrm{d}x\,.$$

Suppose f continuous; clearly $f(x) = 0$ forces $\displaystyle\int_a^b f(x)\,\mathrm{d}x = 0$. We shall prove the opposite implication: $\displaystyle\int_a^b f(x)\,\mathrm{d}x = 0$ implies $f(x) = 0$. If, by contradiction, $f(\bar{x}) \neq 0$ for a certain $\bar{x} \in (a,b)$, Theorem A.2.2 would give a neighbourhood

$I_\delta(\bar{x}) = (\bar{x}-\delta, \bar{x}+\delta) \subset [a,b]$ and a constant $K_f > 0$, for any $x \in I_\delta(\bar{x})$. The step function

$$g(x) = \begin{cases} K_f & \text{if } x \in I_\delta(\bar{x}) \\ 0 & \text{if } x \notin I_\delta(\bar{x}) \end{cases}$$

would belong to $\mathcal{S}_f^-$, and

$$\int_a^b f(x)\, \mathrm{d}x \geq \int_a^b g(x)\, \mathrm{d}x = \delta K_f > 0\,,$$

a contradiction. Therefore $f(x) = 0$ for all $x \in (a,b)$, and by continuity f must vanish also at the end-points a, b.

iv) This follows directly from *iii)*, noting $h(x) = g(x) - f(x) \geq 0$.

v) Proposition 9.32 *ii)* says that $|f|$ is integrable over $[a,b]$. But $f = f_+ - f_-$ (f_+ and f_- are the positive and negative parts of f respectively), so the linearity proven in part *ii)* yields

$$\int_a^b f(x)\, \mathrm{d}x = \int_a^b f_+(x)\, \mathrm{d}x - \int_a^b f_-(x)\, \mathrm{d}x\,.$$

Using the triangle inequality, property *iii)* ($f_+, f_- \geq 0$) and the relation $|f| = f_+ + f_-$, we eventually have

$$\left|\int_a^b f(x)\, \mathrm{d}x\right| \leq \left|\int_a^b f_+(x)\, \mathrm{d}x\right| + \left|\int_a^b f_-(x)\, \mathrm{d}x\right| = \int_a^b f_+(x)\, \mathrm{d}x + \int_a^b f_-(x)\, \mathrm{d}x$$
$$= \int_a^b \big(f_+(x) + f_-(x)\big)\, \mathrm{d}x = \int_a^b |f(x)|\, \mathrm{d}x\,. \qquad \square$$

A.5.3 Improper integrals

▶ Check of property (10.3), p. 362

$$\int_1^{+\infty} \frac{\sin x}{x}\, \mathrm{d}x \quad \text{converges, but} \quad \int_1^{+\infty} \left|\frac{\sin x}{x}\right| \mathrm{d}x \quad \text{diverges.}$$

Proof. We explain first why $\displaystyle\int_1^{+\infty} \frac{\sin x}{x}\, \mathrm{d}x$ converges. Let us integrate by parts over each interval $[1,a]$ with $a > 1$, by putting $f(x) = \frac{1}{x}$ and $g'(x) = \sin x$; since $f'(x) = -\frac{1}{x^2}$, $g(x) = -\cos x$, it follows

$$\int_1^a \frac{\sin x}{x}\, \mathrm{d}x = -\left.\frac{\cos x}{x}\right|_1^a - \int_1^a \frac{\cos x}{x^2}\, \mathrm{d}x\,;$$

the last integral is known to converge from Example 10.8. Thus the map $\dfrac{\sin x}{x}$ has a well-defined improper integral over $[1, +\infty)$.

Now let us convince ourselves that $\dfrac{\sin x}{x}$ is not absolutely integrable on $[1, +\infty)$. Since $|\sin x| \leq 1$ for any x, we have

$$\left|\frac{\sin x}{x}\right| \geq \frac{\sin^2 x}{x} = \frac{1}{2}\frac{1-\cos 2x}{x}.$$

We claim the integral $\displaystyle\int_1^{+\infty} \frac{1-\cos 2x}{x}\,\mathrm{d}x$ diverges, hence the Comparison test (Theorem 10.5) forces $\displaystyle\int_1^{+\infty} \left|\frac{\sin x}{x}\right|\,\mathrm{d}x$ to diverge as well. In fact,

$$\int_1^{+\infty} \frac{1-\cos 2x}{x}\,\mathrm{d}x = \int_1^{+\infty} \frac{1}{x}\,\mathrm{d}x - \int_1^{+\infty} \frac{\cos 2x}{x}\,\mathrm{d}x\,.$$

While the first integral on the right-hand side diverges, the second one converges, as can be proved by the same procedure as above. Therefore $\displaystyle\int_1^{+\infty} \frac{1-\cos 2x}{x}\,\mathrm{d}t$ diverges, and the function $\dfrac{\sin x}{x}$ cannot be absolutely integrable. □

▶ Proof of Theorem 10.10, p. 363

Theorem 10.10 (Asymptotic comparison test) *Suppose the function $f \in \mathcal{R}_{\mathrm{loc}}([a, +\infty))$ is infinitesimal of order α, for $x \to +\infty$, with respect to $\varphi(x) = \frac{1}{x}$. Then*

i) *if $\alpha > 1$, $f \in \mathcal{R}([a, +\infty))$;*

ii) *if $\alpha \leq 1$, $\displaystyle\int_a^{+\infty} f(x)\,\mathrm{d}x$ diverges.*

Proof. Since $f(x) \sim \frac{1}{x^\alpha}$ for $x \to +\infty$, we may assume the map f has constant sign for x sufficiently large, for instance when $x > A > 0$. Without loss of generality we may also take f strictly positive, for otherwise we could just change sign. Moreover, for $x \to +\infty$,

$$f(x) \sim \frac{1}{x^\alpha} \qquad \Rightarrow \qquad f(x) = O\Big(\frac{1}{x^\alpha}\Big) \quad \text{and} \quad \frac{1}{x^\alpha} = O\big(f(x)\big)\,;$$

otherwise said, there exist positive constants c_1, c_2 such that

$$\frac{c_1}{x^\alpha} \leq f(x) \leq \frac{c_2}{x^\alpha}\,, \qquad \forall x > A\,.$$

In order to conclude, it suffices to use the Comparison test (Theorem 10.5) jointly with Example 10.4. □

▶ Proof of Theorem 10.13, p. 364

Theorem 10.13 (Integral test) *Let f be continuous, positive and decreasing on $[k_0, +\infty)$, for $k_0 \in \mathbb{N}$. Then*

$$\sum_{k=k_0+1}^{\infty} f(k) \le \int_{k_0}^{+\infty} f(x)\,\mathrm{d}x \le \sum_{k=k_0}^{\infty} f(k)\,.$$

Therefore the integral and the series share the same behaviour:

a) $\displaystyle\int_{k_0}^{+\infty} f(x)\,\mathrm{d}x$ *converges* $\iff$ $\displaystyle\sum_{k=k_0}^{\infty} f(k)$ *converges;*

b) $\displaystyle\int_{k_0}^{+\infty} f(x)\,\mathrm{d}x$ *diverges* $\iff$ $\displaystyle\sum_{k=k_0}^{\infty} f(k)$ *diverges.*

Proof. Since f decreases, for any $k \ge k_0$ we have

$$f(k+1) \le f(x) \le f(k)\,, \qquad \forall x \in [k, k+1]\,,$$

and as the integral is monotone,

$$f(k+1) \le \int_{k}^{k+1} f(x)\,\mathrm{d}x \le f(k)\,.$$

Then for all $n \in \mathbb{N}$ with $n > k_0$ we obtain

$$\sum_{k=k_0+1}^{n+1} f(k) \le \int_{k_0}^{n} f(x)\,\mathrm{d}x \le \sum_{k=k_0}^{n} f(k)$$

(after re-indexing the first series). Passing to the limit for $n \to +\infty$ and recalling f is positive and continuous, we conclude. □

Tables and Formulas

Recurrent formulas

$$\cos^2 x + \sin^2 x = 1, \qquad \forall x \in \mathbb{R}$$

$$\sin x = 0 \quad \text{se } x = k\pi, \quad \forall k \in \mathbb{Z}\,, \qquad \cos x = 0 \quad \text{se} \ \ x = \frac{\pi}{2} + k\pi$$

$$\sin x = 1 \quad \text{se} \ \ x = \frac{\pi}{2} + 2k\pi, \qquad \cos x = 1 \quad \text{se} \ \ x = 2k\pi$$

$$\sin x = -1 \quad \text{se} \ \ x = -\frac{\pi}{2} + 2k\pi, \qquad \cos x = -1 \quad \text{se} \ \ x = \pi + 2k\pi$$

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

$$\sin 2x = 2\sin x\cos x\,, \qquad \cos 2x = 2\cos^2 x - 1$$

$$\sin x - \sin y = 2\sin\frac{x-y}{2}\cos\frac{x+y}{2}$$

$$\cos x - \cos y = -2\sin\frac{x-y}{2}\sin\frac{x+y}{2}$$

$$\sin(x+\pi) = -\sin x\,, \qquad \cos(x+\pi) = -\cos x$$

$$\sin(x+\frac{\pi}{2}) = \cos x\,, \qquad \cos(x+\tfrac{\pi}{2}) = -\sin x$$

$$a^{x+y} = a^x a^y\,, \qquad a^{x-y} = \frac{a^x}{a^y}\,, \qquad (a^x)^y = a^{xy}$$

$$\log_a(xy) = \log_a x + \log_a y\,, \quad \forall x, y > 0$$

$$\log_a \frac{x}{y} = \log_a x - \log_a y, \quad \forall x, y > 0$$

$$\log_a(x^y) = y\log_a x, \quad \forall x > 0\,,\ \forall y \in \mathbb{R}$$

Fundamental limits

$$\lim_{x\to+\infty} x^\alpha = +\infty\,, \qquad \lim_{x\to 0^+} x^\alpha = 0\,, \qquad \alpha > 0$$

$$\lim_{x\to+\infty} x^\alpha = 0\,, \qquad \lim_{x\to 0^+} x^\alpha = +\infty\,, \qquad \alpha < 0$$

$$\lim_{x\to\pm\infty} \frac{a_n x^n + \ldots + a_1 x + a_0}{b_m x^m + \ldots + b_1 x + b_0} = \frac{a_n}{b_m} \lim_{x\to\pm\infty} x^{n-m}$$

$$\lim_{x\to+\infty} a^x = +\infty\,, \qquad \lim_{x\to-\infty} a^x = 0\,, \qquad a > 1$$

$$\lim_{x\to+\infty} a^x = 0\,, \qquad \lim_{x\to-\infty} a^x = +\infty\,, \qquad a < 1$$

$$\lim_{x\to+\infty} \log_a x = +\infty\,, \qquad \lim_{x\to 0^+} \log_a x = -\infty\,, \qquad a > 1$$

$$\lim_{x\to+\infty} \log_a x = -\infty\,, \qquad \lim_{x\to 0^+} \log_a x = +\infty\,, \qquad a < 1$$

$$\lim_{x\to\pm\infty} \sin x\,, \qquad \lim_{x\to\pm\infty} \cos x\,, \qquad \lim_{x\to\pm\infty} \tan x \qquad \text{do not exist}$$

$$\lim_{x\to\left(\frac{\pi}{2}+k\pi\right)^\pm} \tan x = \mp\infty\,,\ \forall k \in \mathbb{Z}\,, \quad \lim_{x\to\pm\infty} \arctan x = \pm\frac{\pi}{2}$$

$$\lim_{x\to\pm 1} \arcsin x = \pm\frac{\pi}{2} = \arcsin(\pm 1)$$

$$\lim_{x\to+1} \arccos x = 0 = \arccos 1\,, \qquad \lim_{x\to-1} \arccos x = \pi = \arccos(-1)$$

$$\lim_{x\to 0} \frac{\sin x}{x} = 1\,, \qquad \lim_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}$$

$$\lim_{x\to\pm\infty} \left(1+\frac{a}{x}\right)^x = \mathrm{e}^a\,, \quad a \in \mathbb{R}\,, \quad \lim_{x\to 0} (1+x)^{\frac{1}{x}} = \mathrm{e}$$

$$\lim_{x\to 0} \frac{\log_a(1+x)}{x} = \frac{1}{\log a}\,,\ a > 0; \quad \text{in particular,} \quad \lim_{x\to 0} \frac{\log(1+x)}{x} = 1$$

$$\lim_{x\to 0} \frac{a^x - 1}{x} = \log a\,, \quad a > 0; \qquad \text{in particular,} \quad \lim_{x\to 0} \frac{\mathrm{e}^x - 1}{x} = 1$$

$$\lim_{x\to 0} \frac{(1+x)^\alpha - 1}{x} = \alpha\,, \quad \alpha \in \mathbb{R}$$

Derivatives of elementary functions

$f(x)$	$f'(x)$
x^α	$\alpha x^{\alpha-1}\,, \qquad \forall \alpha \in \mathbb{R}$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$1+\tan^2 x = \dfrac{1}{\cos^2 x}$
$\arcsin x$	$\dfrac{1}{\sqrt{1-x^2}}$
$\arccos x$	$-\dfrac{1}{\sqrt{1-x^2}}$
$\arctan x$	$\dfrac{1}{1+x^2}$
a^x	$(\log a)\, a^x$
$\log_a \lvert x\rvert$	$\dfrac{1}{(\log a)\, x}$
$\sinh x$	$\cosh x$
$\cosh x$	$\sinh x$

Differentiation rules

$$\Big(\alpha f(x) + \beta g(x)\Big)' = \alpha f'(x) + \beta g'(x)$$

$$\Big(f(x)g(x)\Big)' = f'(x)g(x) + f(x)g'(x)$$

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{\big(g(x)\big)^2}$$

$$\Big(g(f(x))\Big)' = g'(f(x))f'(x)$$

Maclaurin's expansions

$$\mathrm{e}^x = 1 + x + \frac{x^2}{2} + \ldots + \frac{x^k}{k!} + \ldots + \frac{x^n}{n!} + o(x^n)$$

$$\log(1+x) = x - \frac{x^2}{2} + \ldots + (-1)^{n-1}\frac{x^n}{n} + o(x^n)$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^m \frac{x^{2m+1}}{(2m+1)!} + o(x^{2m+2})$$

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \ldots + (-1)^m \frac{x^{2m}}{(2m)!} + o(x^{2m+1})$$

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots + \frac{x^{2m+1}}{(2m+1)!} + o(x^{2m+2})$$

$$\cosh x = 1 + \frac{x^2}{2} + \frac{x^4}{4!} + \ldots + \frac{x^{2m}}{(2m)!} + o(x^{2m+1})$$

$$\arcsin x = x + \frac{x^3}{6} + \frac{3x^5}{40} + \ldots + \left|\binom{-\frac{1}{2}}{m}\right| \frac{x^{2m+1}}{2m+1} + o(x^{2m+2})$$

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots + (-1)^m \frac{x^{2m+1}}{2m+1} + o(x^{2m+2})$$

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2}x^2 + \ldots + \binom{\alpha}{n}x^n + o(x^n)$$

$$\frac{1}{1+x} = 1 - x + x^2 - \ldots + (-1)^n x^n + o(x^n)$$

$$\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + o(x^3)$$

Integrals of elementary functions

$f(x)$	$\int f(x)\,\mathrm{d}x$
x^α	$\dfrac{x^{\alpha+1}}{\alpha+1}+c\,, \qquad \alpha \neq -1$
$\dfrac{1}{x}$	$\log\lvert x\rvert + c$
$\sin x$	$-\cos x + c$
$\cos x$	$\sin x + c$
e^x	$\mathrm{e}^x + c$
$\sinh x$	$\cosh x + c$
$\cosh x$	$\sinh x + c$
$\dfrac{1}{1+x^2}$	$\arctan x + c$
$\dfrac{1}{\sqrt{1-x^2}}$	$\arcsin x + c$
$\dfrac{1}{\sqrt{1+x^2}}$	$\log(x+\sqrt{x^2+1})+c = \operatorname{sett}\sinh x + c$
$\dfrac{1}{\sqrt{x^2-1}}$	$\log(x+\sqrt{x^2-1})+c = \operatorname{sett}\cosh x + c$

Integration rules

$$\int \Big(\alpha f(x) + \beta g(x)\Big)\,\mathrm{d}x = \alpha \int f(x)\,\mathrm{d}x + \beta \int g(x)\,\mathrm{d}x$$

$$\int f(x) g'(x)\,\mathrm{d}x = f(x)g(x) - \int f'(x) g(x)\,\mathrm{d}x$$

$$\int \frac{\varphi'(x)}{\varphi(x)}\,\mathrm{d}x = \log\lvert\varphi(x)\rvert + c$$

$$\int f(\varphi(x))\varphi'(x)\,\mathrm{d}x = \int f(y)\,\mathrm{d}y \quad \text{where} \quad y = \varphi(x)$$

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